Pacific Journal of Mathematics

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Volume 268 No. 1 March 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 268, No. 1, 2014
dx.doi.org/10.2140/pjm.2014.268.79

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In 2003 Bieri and Geoghegan generalized the Bieri–Neumann–Strebel invariant Σ^1 by defining $\Sigma^1(\rho)$, ρ an isometric action by a finitely generated group G on a proper CAT(0) space M. In this paper, we show how the natural and well-known connection between Bass–Serre theory and covering space theory provides a framework for the calculation of $\Sigma^1(\rho)$ when ρ is a cocompact action by $G = B \rtimes A$, A a finitely generated group, on a locally finite Bass–Serre tree T for A. This framework leads to a theorem providing conditions for including an endpoint in, or excluding an endpoint from, $\Sigma^1(\rho)$. When A is a finitely generated free group acting on its Cayley graph, we can restate this theorem from a more algebraic perspective, which leads to some general results on Σ^1 for such actions.

1. Introduction

In [Bieri and Geoghegan 2003b], the authors begin with the following:

> Given a group G and a contractible metric space M, consider the set $\mathrm{Hom}(G, \mathrm{Isom}(M))$ of all actions by G on M by isometries. Are there invariants of such actions which distinguish one from another? Are there topological properties which one such action might possess while another might not?

The tool they apply to draw distinctions between such actions is controlled n-connectivity, which is developed in [Bieri and Geoghegan 2003a], and which we briefly describe here. Suppose ρ is an isometric action by a group G having type F_n on a proper CAT(0) metric space M. Fixing a basepoint $b \in M$, the CAT(0) boundary, ∂M, can be thought of as the set of geodesic rays τ emanating from b.[1] For an end point $e \in \partial M$ represented by a ray τ, there is a nested family of subsets

MSC2010: primary 20E08, 20F65, 57M07; secondary 05C05, 05C25.
Keywords: controlled connectivity, BNS, sigma invariants, tree actions, semidirect products.

[1] For background on the topological finiteness property "type F_n", see [Geoghegan 2008, §7.2], and for background on CAT(0) metric spaces and their boundaries see [Bridson and Haefliger 1999, II.1 and II.8]. A metric space is *proper* if each closed metric ball is compact.

$\mathrm{HB}_k(\tau)$, $k \in \mathbb{R}$, called “horoballs” which serve as metric balls (in M) “at e.”[2] This provides a sense of direction in M, which can be “lifted to G” by ρ via a G-equivariant map from the n-skeleton of the universal cover of a $K(G, 1)$. For a point $e \in \partial M$, if the lifts of the horoballs about e are (roughly) $(n-1)$-connected, then we say ρ is controlled $(n-1)$-connected over e. Bieri and Geoghegan show that this is independent of choice of $K(G, 1)$ or equivariant map. The invariant $\Sigma^n(\rho)$ is the subset of ∂M consisting of points over which ρ is controlled $(n-1)$-connected. This definition generalizes the Bieri–Neumann–Strebel–Renz (BNSR) invariants $\Sigma^n(G)$, which are open subsets of the CAT(0) boundary of the vector space $G_{ab} \otimes \mathbb{R}$. A key difference between $\Sigma^1(\rho)$ and the BNSR invariant $\Sigma^1(G)$ is that $\Sigma^1(\rho)$ is in general *not* an open subset of ∂M.

Apart from enabling one to draw geometric distinctions between isometric actions by a group on a proper CAT(0) space, the invariant can also provide group theoretical information: if the orbits under an action ρ are discrete, then the point stabilizers are finitely generated if and only if $\Sigma^1(\rho) = \partial M$ [Bieri and Geoghegan 2003a, Theorem A and Boundary Criterion].

When $M = T$ is a locally finite (simplicial) tree, the CAT(0) boundary is a metric Cantor set. Initial results in [Bieri and Geoghegan 2003a] led the authors to ask whether in this case $\Sigma^1(\rho)$ might always be one of $\varnothing$, a singleton, or the entire boundary ∂T. Work in [Jones 2012] establishes a class of actions for which this is the case. However, work by Ralf Lehnert in his diploma thesis demonstrates that other subsets of ∂T can be realized as $\Sigma^1(\rho)$ for certain actions [Lehnert 2009]. This hints at a potentially rich world of Σ^1 invariants, which we further explore here.

1.1. ***Statement of results.*** We restrict our attention to Σ^1, and study only the following scenario:

Definition 1 (actions of interest). Let A be a finitely generated group with finite generating set R, and let T be a locally finite[3] simplicial tree on which A acts cocompactly and with finitely generated stabilizers. For a group B, suppose we have a homomorphism $\varphi : A \to \mathrm{Aut}(B)$, and let $G = B \rtimes_\varphi A$ be the resulting semidirect product. Elements of G are of the form (b, a), where $a \in A$, $b \in B$, and multiplication in G operates under the rule

$$(b_1, a_1)(b_2, a_2) = (b_1 a_1 b_2 a_1^{-1}, a_1 a_2) = (b_1 \varphi_{a_1}(b_2), a_1 a_2).$$

[2]For background on horoballs, see [Bieri and Geoghegan 2003a, §10.1]. The convention followed there and in this paper is that as k increases, we approach e, the reverse of the convention in [Bridson and Haefliger 1999].

[3]A simplicial tree is a proper metric space if and only if it is locally finite.

Suppose G is finitely generated. Then it follows that B is finitely generated as an A-group. By this, we mean there is a finite subset $S \subset B$ such that the set $\{\varphi_a(S) \mid a \in A\}$ generates B, and so G is generated by $S \cup R$.

The natural projection $G \twoheadrightarrow A$ induces an action ρ by G on T which contains the normal subgroup B in its kernel. We investigate $\Sigma^1(\rho)$.

Remark 2. As mentioned, if the point stabilizers under ρ are finitely generated, then $\Sigma^1(\rho) = \partial T$ [Bieri and Geoghegan 2003a, Theorem A and Boundary Criterion]. Moreover, since T is locally finite, all point stabilizes are commensurable, so if any one is finitely generated, then all are. Thus with the assumption that the stabilizers under the A action on T are finitely generated, in order to obtain "interesting" invariants (those with $\Sigma^1(\rho) \neq \partial T$), one must assume that B is not finitely generated, since the stabilizers under ρ are simply semidirect products of B with the stabilizers in A.

Main result. With the action $\rho : G \to \operatorname{Isom}(T)$ as defined above, we apply the relationship between Bass–Serre theory and covering space theory to construct a commutative diagram of G-equivariant cellular maps between CW-complexes:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{r}"] \arrow[d, "p"] & T \arrow[d, "\mathrm{id}"] \\
\bar{X} \arrow[r, "\bar{r}"] \arrow[d, "q"] & T \arrow[d, "\bmod G"] \\
X \arrow[r, "r"] & V = T\backslash G
\end{tikzcd}$$

where X is a $K(G,1)$, $\bar{X}$ is a $K(B,1)$, $\tilde{X}$ is a contractible universal cover, p and q are covering projections, and r, $\bar{r}$, and $\tilde{r}$ are retracts.[4] For a geodesic ray τ in T and $k \in \mathbb{Z}$, consider the horoball $\mathrm{HB}_k(\tau)$.[5] For $W \subset X$ a finite subcomplex, set

$$\bar{X}_{(\tau,k,W)} = \bar{r}^{-1}(\mathrm{HB}_k(\tau)) \cap q^{-1}(W) \subset \bar{X}.$$

Theorem 3. *Let $e \in \partial T$ be represented by a geodesic ray τ.*

(i) *If there exists a finite subcomplex $W \subset X$ such that for every $k \in \mathbb{Z}$, $\bar{X}_{(\tau,k,W)}$ is connected and the map on π_1 induced by the inclusion $\bar{X}_{(\tau,k,W)} \hookrightarrow \bar{X}$ is surjective, then $e \in \Sigma^1(\rho)$.*

(ii) *If for every $k \in \mathbb{Z}$ and every finite subcomplex $W \subset X$ such that $\bar{X}_{(\tau,k,W)}$ is connected, the induced map on π_1 is not surjective, then $e \notin \Sigma^1(\rho)$.*

[4] This is the topological construction of the Bass–Serre tree [Geoghegan 2008, §6.2; Scott and Wall 1979], discussed further in Section 2.2.

[5] A precise description of $\mathrm{HB}_k(\tau)$ is given in Equation (1-1).

Consequences and examples. Theorem 3 has a number of consequences in the case where A is a free group and T is its Cayley graph. In this case, the vertices of T are the elements of A. Let e be an endpoint of T and suppose the geodesic ray τ represents e. For an integer k, let $A_k(\tau)$ be the elements of A that form the vertex set of the horoball $\mathrm{HB}_k(\tau)$. To avoid confusion with the group B, we will use the notation $\mathrm{Ball}_r(X, p)$ to refer to the metric ball of radius r in the space X about the point p. Just as the horoball $\mathrm{HB}_k(\tau)$ can be written as the nested union of closed metric balls in T:

$$\mathrm{HB}_k(\tau) = \bigcup_{l \geq \max\{0,k\}} \overline{\mathrm{Ball}_{l-k}(T, \tau(l))}, \tag{1-1}$$

the set $A_k(\tau)$ can be written as a nested union of closed metric balls in the word metric on A:

$$A_k(\tau) = \bigcup_{l \geq \max\{0,k\}} \overline{\mathrm{Ball}_{l-k}(A, \tau(l))}. \tag{1-2}$$

We will say B is finitely generated over a *subset* $A' \subseteq A$ if there is a finite subset $S \subseteq B$ such that $\{asa^{-1} \mid s \in S, a \in A'\}$ generates B.

In Section 4, we show that in this context Theorem 3 can be restated as follows:

Theorem 4. *Let A be a finitely generated free group, and let T be its Cayley graph with respect to a free basis. For the action ρ as in Theorem 3, and for $e \in \partial T$ represented by geodesic ray τ:*

(i) *If there is a finite set $S \subseteq B$ such that for each $k \in \mathbb{Z}_{\geq 0}$, S generates B over $A_k(\tau)$, then $e \in \Sigma^1(\rho)$.*

(ii) *If for each $k \in \mathbb{Z}_{\leq 0}$, B is not finitely generated over $A_k(\tau)$, then $e \notin \Sigma^1(\rho)$.*

This is reminiscent of the invariant $\Sigma_B(A)$ of [Bieri et al. 1987] and [Bieri and Strebel 1980], but whereas $\Sigma_B(A)$ is determined by the algebraic structure of G, our sets $A_k(\tau)$ are given by the geometry of T; in particular, they are not monoids.

Since B is finitely generated over A, we have:

Corollary 5. *If for each $k \in \mathbb{Z}_{\geq 0}$, $\varphi(A_k(\tau)) = \varphi(A)$, then $e \in \Sigma^1(\rho)$.*

Let $\{a_1, \dots, a_n\}$ freely generate A. For a generator a_i, let the function expsum_{a_i} map a reduced word w in $\{a_1, \dots, a_n\}^{\pm}$ to the corresponding exponent sum of a_i in w, and define the function $\mathrm{expsum}_{a_i^{-1}}$ to be $-\mathrm{expsum}_{a_i}$.

Corollary 6. *Let $t \in \{a_1, \dots, a_n\}^{\pm}$. Suppose there does not exist $m \in \mathbb{Z}$ such that B is finitely generated over $A - \mathrm{expsum}_t^{-1}([m, \infty))$, i.e any subset $A' \subseteq A$ must have reduced words with arbitrarily large exponent sum of t in order for B to be finitely generated over A'. Then any endpoint represented by a word eventually consisting of only t^{-1} does not lie in $\Sigma^1(\rho)$.*

Example 7. This is a generalization of an example calculated by Ralf Lehnert, although the methods used here are different from his. Consider the semidirect product $G = B \rtimes A$, where $B = \mathbb{Z}[1/(p_1 p_2 \dots p_n)]$, where the p_i are prime with $p_i \neq p_j$ for $1 \leq i, j \leq n$, and A is free on $\{a_1, \dots, a_n\}$. The action is given by a_i acting by multiplication by $1/p_i$. For $A' \subseteq A$, B is finitely generated over A' if and only if A' contains reduced words with arbitrarily large exponent sum of each a_i. One can show that for any $k \in \mathbb{Z}$, this will always be the case for $A' = A_k(\tau)$ unless τ eventually consists of only a_i^{-1} (see Lemma 24). Thus, by Corollary 6, any endpoint corresponding to an infinite word eventually consisting of a_i^{-1} for some i is not in Σ^1. By Theorem 4, any other endpoint is in Σ^1.

Example 8. Let $G = \mathbb{Z} \wr \mathbb{Z} = \oplus_{i \in \mathbb{Z}} \langle b_i \rangle \rtimes \langle t \rangle$. The action is by shifting: ${}^t b_i = b_{i+1}$. Let T be the Cayley graph of $\langle t \rangle$, a simplicial line. The action $\langle t \rangle \curvearrowright T$ induces an action $G \overset{\rho}{\curvearrowright} T$. It is known from previous work that $\Sigma^1(\rho)$ is empty, as follows. Because the endpoints of the action are fixed, we can relate ∂T to homomorphisms $G \twoheadrightarrow \mathbb{Z}$, and an end point lies in $\Sigma^1(\rho)$ if and only if the corresponding homomorphism represents a point of the BNSR invariant $\Sigma^1(G)$ [Bieri and Geoghegan 2003a, §10.6]. These homomorphisms do not represent points of $\Sigma^1(G)$ because they are not homomorphisms associated to HNN extension decompositions of G over finitely generated base groups [Brown 1987, Proposition 3.1]. Here it follows from Theorem 4, because B is not finitely generated over any proper subset of $\langle t \rangle$.

Corollary 5 can be applied to determine a nice criterion for finding endpoints of T lying in $\Sigma^1(\rho)$.

Theorem 9. *With notation as in Theorem 4, viewing endpoints of T as infinite words in the generators of A, $\Sigma^1(\rho)$ contains any endpoint represented by an infinite word containing infinitely many mutually distinct subwords lying in* $\ker \varphi$.

Corollary 10. *If $\varphi(A) \leq Aut(B)$ is abelian and A has rank $n \geq 2$, then $\Sigma^1(\rho)$ is nonempty.*

For example, any endpoint represented by an infinite word containing infinitely many commutators will be contained in $\Sigma^1(\rho)$.

Example 11. Let m and n be positive integers with $m \geq n$. Let $C = \langle a_1, \dots, a_n \rangle$ and $D = \langle a_{n+1}, \dots, a_m \rangle$ be free groups, and set $A = C * D$. For a finitely generated group K, let G be the restricted wreath product $K \operatorname{wr}_C A$, where the A-action on the indexing set C is defined by the composition of the natural projection $\pi : A \to C$ and left multiplication. In other words, $G = B \rtimes_\varphi A$, where $B = \oplus_{\omega \in C} K_\omega$ with each K_ω a copy of K. The elements of B are sequences (x_ω), $x_\omega \in K_\omega$, $\omega \in C$, with only finitely many x_ω nontrivial, and C acts on B by permuting the indices (by left multiplication on itself) while $D \leq \ker \varphi$. The projection $G \twoheadrightarrow A$ followed by the natural action by A on its Cayley graph $T = \Gamma(A, \{a_1, a_2, \dots, a_m\})$ induces an action ρ on T.

By Theorem 9, any endpoint containing infinitely many letters $a_i^{\pm}$, $n < i \leq m$ will lie in $\Sigma^1(\rho)$, while Corollary 6 ensures that any endpoint eventually consisting of a single letter $a_j^{\pm}$, $1 \leq j \leq n$, will not lie in $\Sigma^1(\rho)$. In fact, any end point represented by a geodesic ray that eventually consists of only letters from C lies outside $\Sigma^1(\rho)$, as is argued in Section 4.2. So an endpoint lies in $\Sigma^1(\rho)$ if and only if a representative geodesic ray contains infinitely many letters from D.

Example 12. One can also perform calculations in the case where A is not free. For example, let H be any finitely generated group and consider the group

$$G = B \rtimes_{\varphi} A, \quad \text{where } B = \prod_{i \in \mathbb{Z}} H \text{ and } A = \langle a \mid a^4 \rangle * \langle b \mid b^4 \rangle, \tag{1-3}$$

where $\varphi : A \to \mathrm{Aut}(B)$ consists first of the projection onto D_∞ collapsing a^2 and b^2 to the identity, followed by permutation of the indices $i \in \mathbb{Z}$ given by the natural action by D_∞ on $\mathbb{Z}$. Let ρ be the action by G on the regular 4-valent Bass–Serre tree T_4 corresponding to the free product structure of A. Notice, since this is the Bass–Serre tree corresponding to a free product, any point $e \in \partial T_4$ corresponds to a word in the normal form for the free product. One can apply Theorem 3 to calculate $\Sigma^1(\rho)$ directly to determine that a given $e \in \partial T_4$ if and only if it corresponds to an infinite normal form word containing infinitely many subwords of the form a^2 or b^2.

There is a stark similarity between this result and Theorem 9, and indeed a statement similar to Theorem 9 can be made in the case where A is a free product. However, only when A is a free product of *finite* groups will its corresponding Bass–Serre tree be locally finite (and hence proper); in this case the Kurosh subgroup theorem implies that A has a free subgroup A' of finite index. If $G = B \rtimes A$, then $G' = B \rtimes A'$ is a finite index subgroup of G, and the action ρ by G on the Bass–Serre tree corresponding to the free product decomposition of A restricts to an action by G' on the same tree. It follows from Theorem 12.1 of [Bieri and Geoghegan 2003a] that the invariant is the same for both actions. Hence, it is not clear that such an endeavor will add anything new to the discussion.

1.2. *Defining* Σ^1.

In general, there is a family of invariants Σ^n, $n \geq 0$, corresponding to the notion of controlled $(n-1)$-connectivity. The discussion below refers only to Σ^1 and controlled connectivity, but a similar discussion can be had in full generality.

We start with Bieri and Geoghegan's original definition of controlled connectivity.

Definition 13 [Bieri and Geoghegan 2003a]. Let ρ be an action by a finitely generated group G on a proper CAT(0) metric space (M, d). Choose a $K(G, 1)$

complex X whose universal cover $\tilde{X}$ has a cocompact 1-skeleton $(\tilde{X})^{(1)}$, and a continuous G-map $h : (\tilde{X})^{(1)} \to M$. Given a geodesic ray τ in M, $\tau(\infty)$ denotes the point of ∂M represented by τ. For $t \in \mathbb{R}$, let $\tilde{X}_{(\tau,t)}$ denote the largest subcomplex contained in $h^{-1}(\mathrm{HB}_t(\tau))$. Then h is *controlled connected over* $\tau(\infty)$ if there exists $\lambda : \mathbb{R} \to [0, \infty)$ such that for all $t \in \mathbb{R}$, any two points of $\tilde{X}_{(\tau,t)}$ can be connected by a path in $\tilde{X}_{(\tau,t-\lambda(t))}$, and $t - \lambda(t) \to \infty$ as $t \to \infty$.

The same authors also gave an "extended" definition, which we will show coincides with Definition 13 when G is finitely generated.

Definition 14 [Bieri and Geoghegan 2003b, p. 143]. Let ρ be an action by a (not necessarily finitely generated!) group G on a proper CAT(0) metric space (M, d). Choose a nonempty free contractible G-CW-complex $\tilde{X}$ and a continuous G-map $h : \tilde{X} \to M$. Fix a geodesic ray τ in M. For $t \in \mathbb{R}$, define $\tilde{X}_{(\tau,t)}$ to be the largest subcomplex of $h^{-1}(\mathrm{HB}_t(\tau))$. Then h is *controlled connected over* $\tau(\infty)$ if for every cocompact G-subspace $\widetilde{W} \subseteq \tilde{X}$, there exists a cocompact G-subspace $\widetilde{W}'$ containing $\widetilde{W}$ such that for all $t \in \mathbb{R}$, there exists $\lambda(t) \geq 0$ satisfying:

($*$) Any two points of $\tilde{X}_{(\tau,t)} \cap \widetilde{W}$ can be connected by a path through $\tilde{X}_{(\tau,t-\lambda(t))} \cap \widetilde{W}'$.

($**$) Any two points of $\tilde{X}_{(\tau,t+\lambda(t))} \cap \widetilde{W}$ can be connected by a path through $\tilde{X}_{(\tau,t)} \cap \widetilde{W}'$.

Both Definitions 13 and 14 are independent of choice of G-space $\tilde{X}$ or G-map $h : \tilde{X} \to M$, as is proved in [Bieri and Geoghegan 2003a; 2003b], respectively, in what the authors commonly refer to as the *invariance theorem*. For Definition 14, this is proved for the related concept of controlled connectivity over $a \in M$ [Bieri and Geoghegan 2003b, Theorem 2.3]; the proof carries over to controlled connectivity over an end point [ibid., p. 143].

The parameter $\lambda(t)$ is called a *lag*. In nice cases, λ may be constant, or even 0. A lag is necessary for invariance, but an arbitrarily generous lag would defeat the point. In Definition 14, condition ($**$) effectively replaces the condition that $t - \lambda(t) \to \infty$ found in Definition 13.

Suppose now that G is finitely generated and $h : \tilde{X} \to M$ satisfies Definition 14, but $\tilde{X}$ has noncocompact 1-skeleton. There is $h' : \tilde{X}' \to M$, where $\tilde{X}'$ has cocompact 1-skeleton, which by the invariance theorem also satisfies Definition 14. We now show that Definition 13 is satisfied by $h'|_{(\tilde{X})^{(1)}}$.

Proposition 15. *Let G be a finitely generated group, $\tilde{X}$ a contractible free G-complex with cocompact 1-skeleton $(\tilde{X})^{(1)}$, and geodesic ray τ in a proper CAT(0) space M. A G-map $h : \tilde{X} \to M$ satisfies Definition 14 if and only if the restriction $h| : (\tilde{X})^{(1)} \to M$ satisfies Definition 13.*

Proof. If $h|$ satisfies Definition 13 over $\tau(\infty)$, then there is a lag $\lambda(t)$ satisfying $t - \lambda(t) \to \infty$ as $t \to \infty$ such that for each t, any two points in $(\tilde{X})^{(1)}_{(\tau,t)}$ may be joined in $(\tilde{X})^{(1)}_{(\tau,t-\lambda(t))}$. Let $\widetilde{W}$ be any cocompact G-subset of $\tilde{X}$. Let Y be the smallest subcomplex of $\tilde{X}$ containing $\widetilde{W}$. Then Y is still a cocompact G-set. Take $\widetilde{W}' = Y \cup (\tilde{X})^{(1)}$. Then any two points of $\tilde{X}_{(\tau,t)} \cap \widetilde{W}$ may be joined in $\tilde{X}_{(\tau,t-\lambda(t))} \cap \widetilde{W}'$ by first moving into the 1-skeleton of $\tilde{X}_{(\tau,t)} \cap Y$. We now replace $\lambda(t)$ with a lag function $\lambda'(t)$ satisfying both $(*)$ and $(**)$. For any t, there exists $r > t$ such that for all $s \geq r$, $s - \lambda(s) > t$. (So points of $(\tilde{X})^{(1)}_{[\tau,s)}$ can be connected through a path in $(\tilde{X})^{(1)}_{[\tau,t)}$.) Let $\lambda'(t) = \max\{\lambda(t), r - t\}$.

Now suppose h satisfies Definition 14 over $\tau(\infty)$. For $\widetilde{W} = \widetilde{W}' = (\tilde{X})^{(1)}$, there is $\lambda : \mathbb{R} \to [0, \infty)$ such that by $(*)$, any two points of $\tilde{X}_{(\tau,t)} \cap (\tilde{X})^{(1)}$ may be joined through a path in $\tilde{X}_{(\tau,t-\lambda(t))} \cap (\tilde{X})^{(1)}$, since a path may be chosen which does not leave $(\tilde{X})^{(1)}$. We now find a lag $\lambda'(t)$ satisfying $t - \lambda'(t) \to \infty$. Since $\mathrm{HB}_s(\tau) \subseteq \mathrm{HB}_r(\tau)$ when $s > r$, $(**)$ says that for all $r \in \mathbb{R}$, for all $t > r + \lambda(r)$, a lag of $(t - r)$ suffices for $\mathrm{HB}_t(\tau)$. Hence, we may choose a real-valued sequence $s_1 < s_2 < \cdots$ satisfying $s_n \to \infty$ and for $t \in [s_n, s_{n+1})$ a lag of $t - n$ suffices. Define $\lambda'(t)$ by:

$$\lambda'(t) = \begin{cases} \lambda(t) & \text{if } t < s_1, \\ t - n & \text{if } s_n \leq t < s_{n+1},\ n = 1, 2, \ldots \end{cases}$$

Then $t - \lambda'(t) = n$ when $s_n \leq t < s_{n+1}$, so $t - \lambda'(t) \to \infty$ as $t \to \infty$. □

This means that one may test for controlled connectivity of a finitely generated group in the traditional sense by applying the more general definition with a space $\tilde{X}$, even when $(\tilde{X})^{(1)}$ is not cocompact.

Definition 16 (Σ^1). the invariance theorem ensures controlled connectivity is a property of the action ρ, so we define

$$\Sigma^1(\rho) = \{e \in \partial M \mid \rho \text{ is controlled connected over } e\}.$$

The action ρ induces an action on ∂M, and under this action $\Sigma^1(\rho)$ is a G-invariant set.

2. Covering spaces and Bass–Serre theory

2.1. *Some facts about covering spaces.* The following proposition counts the number of components over a connected subset in a covering projection.

Proposition 17 [Geoghegan 2008, Theorem 3.4.10]. *Let (X, Z) be a pair of path connected CW complexes, both containing a point z. Let $i : (Z, z) \to (X, z)$ be the inclusion map, and let $p : (\bar{X}, \bar{z}) \to (X, z)$ be a covering projection. Let $H_1 = \operatorname{im} p_\#$ and $H_2 = \operatorname{im} i_\#$. Then the number of path components of $p^{-1}(Z)$ equals the order of the set of double cosets*

$$\{H_1 g H_2 \mid g \in \pi_1(X, z)\}.$$

In particular, if $\bar{X} = \tilde{X}$ is the universal cover of X, then the number of components of $p^{-1}(Z)$ is the index of H_2 in $\pi_1(X, z)$.

For us the interesting case for us will be when Z has connected preimage in $\tilde{X}$. With this in mind, we will say Z is π_1-*surjective* when the inclusion $(Z, z) \hookrightarrow (X, z)$ induces a surjection on π_1.

A second fact we will need is a consequence of path lifting:

Proposition 18. *Let (X, Z) be a pair of path connected CW complexes. Let $p : \bar{X} \to X$ be a covering projection. Then each component of $p^{-1}(Z)$ surjects onto Z.*

2.2. ***Bass–Serre theory via covering spaces.*** We are concerned with cocompact actions by finitely generated groups on locally finite simplicial trees, particularly those without global fixed points. Thus all actions we consider can be understood though Bass–Serre theory [Bass 1993; Serre 1980]. There is a beautiful connection between Bass–Serre theory and covering space theory [Geoghegan 2008, §6.2; Scott and Wall 1979], which we take advantage of in order to calculate Σ^1 for actions as described by Definition 1. Here we briefly recount this topological construction of the Bass–Serre tree in the context of such actions, and in the process introduce an intermediary covering space which will be important for calculations.

Given an action ρ as in Definition 1, set $V = G\backslash T$, a finite graph since ρ is cocompact. Fix a base vertex v_0 of V. Choose a connected fundamental domain F for ρ, and let $\mathcal{V}$ be the system of stabilizers for F. (Here a fundamental domain is not a subgraph if V has loops.) Let $\bar{v}_0$ be the vertex of F over v_0. Let $\mathbb{V} = (V, \mathcal{V}, v_0)$ be the corresponding graph of groups associated with ρ.

For a cell (vertex or edge) c of V, the stabilizer $G_c \in \mathcal{V}$ is of the form $B \rtimes A_c$ (where $A_c \leq A$ is the stabilizer of c under the action by A). Following Remark 2, we assume G_c is not finitely generated. Let R_c be a finite generating set for A_c, and let S_c be an infinite generating set of B which contains a finite set S such that S generates B over A, as described in Definition 1. Let X_c be a $K(G_c, 1)$-complex having a single 0-cell and 1-cells in correspondence with $R_c \cup S_c$ [Geoghegan 2008, Chapter 7]; this is called a "vertex (or edge) space," depending on whether c is a vertex or edge. There is covering space $\bar{X}_c \twoheadrightarrow X_c$ which is a $K(B, 1)$, since $B \leq G_c$.

As in [Geoghegan 2008, Theorem 7.1.9], we assemble a $K(G, 1)$-complex (X, x_0) as a total space for the graph of groups $(V, \mathcal{V}, v_0)$. This is formed as a disjoint union of the vertex spaces X_v, to which we attach $X_e \times I$ for each edge e. The attaching maps are such that the induced maps on π_1 induce inclusions $G_e \hookrightarrow G_v$ when v is an endpoint of e. There is a retraction $r : (X, x_0) \to (V, v_0)$ collapsing X_c (or $X_c \times I$ if c is an edge) to c for each cell c of V. There is a covering space $q : (\bar{X}, \bar{x}_0) \to (X, x_0)$ corresponding to B. This, too, can be described as a total space of a graph of groups where the graph is the tree T itself, and each stabilizer is isomorphic to B, since $T = B\backslash T$.

We then have the universal cover $p : (\tilde{X}, \tilde{x}) \twoheadrightarrow (\bar{X}, \bar{x})$. Above the map r are maps $\bar{r} : (\bar{X}, \bar{x}_0) \to (T, \bar{v}_0)$ and $\tilde{r} : (\tilde{X}, \tilde{x}_0) \to (T, \bar{v}_0)$.

All maps are G-equivariant and continuous. We arrive at the commutative diagram given before the statement of Theorem 3.

3. Analysis of Σ^1 via subcomplexes of $\bar{X}$

We continue using the notation of the previous section.

Remark 19. Let the end point e be represented by the geodesic ray τ. Because τ emanates from a vertex, the horoball $\mathrm{HB}_t(\tau)$ is a subtree of T if and only if $t \in \mathbb{Z}$. We are interested in $\tilde{X}_{(\tau,t)} \subset \tilde{X}$, which is by definition the largest subcomplex of $(\bar{r} \circ p)^{-1}(\mathrm{HB}_t(\tau))$; by choice of τ, X, $\bar{r}$, and p, $\tilde{X}_{(\tau,t)} = (\bar{r} \circ p)^{-1}(\mathrm{HB}_t(\tau))$ exactly when $t \in \mathbb{Z}$. (There are no 0-cells of $\bar{X}$ mapped by $\bar{r}$ to the interior of an edge of T.) Hence, it is enough to look at horoballs of the form $\mathrm{HB}_k(\tau)$, $k \in \mathbb{Z}$. Similarly, the lag λ can always be taken to be in $\mathbb{Z}$, so that all horoballs under consideration are subtrees of T.

Definition 20. A finite subcomplex W will be called *suitable* if for each subtree U of T, the set $\bar{r}^{-1}(U) \cap q^{-1}(W) \subset \bar{X}$ is connected. By Remark 19, it follows that if W is suitable, then the set $\bar{X}_{(\tau,k,W)} = \bar{r}^{-1}(\mathrm{HB}_k(\tau)) \cap q^{-1}(W)$ is connected for any horoball $\mathrm{HB}_k(\tau)$.

Lemma 21. *Suppose W is a connected subcomplex of X such that for each vertex v of F, W contains the 1-cells of $X_v \subset X$ corresponding to R_v. Moreover, for each edge e of F, let $x_e \in X_e$ be the basepoint, and suppose W contains the 1-cell $\{x_e\} \times [0, 1] \in X$. Then W is suitable.*

Proof. Let U be a subtree of T. We show that $q^{-1}(W) \cap \bar{r}^{-1}(U)$ is connected. For a given vertex v of F, W contains loops generating A_v, and the image of the map $\bar{X}_v \hookrightarrow X_v$ is B. By Proposition 17 (with $H_1 \geq A_v$ and $H_2 = B$), $q^{-1}(W) \cap \bar{X}_v$ is connected. Hence the lemma holds if U is any vertex of T. If U contains edges, then since W contains all edges of X corresponding to base points of X_e, $e \in F$, there must be a path in $q^{-1}(W) \cap \bar{r}^{-1}(U)$ from the $\bar{r}$-preimage of any one vertex of U to any other. Furthermore, the fact that there is no cell of $\bar{X}$ lying completely over the interior of an edge of T ensures that there can be no components of $q^{-1}(W) \cap \bar{r}^{-1}(U)$ over the interior of an edge. □

Because each stabilizer A_v is finitely generated and V is finite, the following observation follows from Lemma 21.

Observation 22. If $W \subseteq X$ is compact, then there exists a suitable subcomplex $W' \subseteq X$ such that $W \subseteq W'$.

For convenience, we restate Theorem 3 before proving it. Recall that $\bar{X}_{(\tau,k,W)}$ denotes $\bar{r}^{-1}(\mathrm{HB}_k(\tau)) \cap q^{-1}(W) \subset \bar{X}$.

Theorem. *Let $e \in \partial T$ be represented by a geodesic ray τ.*

(i) *If there exists a finite subcomplex $W \subset X$ such that for every $k \in \mathbb{Z}$, $\bar{X}_{(\tau,k,W)}$ is connected and the map on π_1 induced by the inclusion $\bar{X}_{(\tau,k,W)} \hookrightarrow \bar{X}$ is surjective, then $e \in \Sigma^1(\rho)$.*

(ii) *If for every $k \in \mathbb{Z}$ and every finite subcomplex $W \subset X$ such that $\bar{X}_{(\tau,k,W)}$ is connected, the induced map on π_1 is not surjective, then $e \notin \Sigma^1(\rho)$.*

Proof. (i) We show that Definition 14 is satisfied with lag $\lambda = 0$; in this case, conditions $(*)$ and $(**)$ are the same. Let $\tilde{L} \subseteq \tilde{X}$ be a cocompact G-subcomplex and set $L = q(p(\tilde{L}))$. Let $k \in \mathbb{Z}$. By Observation 22, there is a suitable subcomplex $W' \subseteq X$ with $L \cup W \subseteq W'$. Since $\bar{X}_{(\tau,k,W)}$ is π_1-surjective onto $\bar{X}$, it follows that $\bar{X}_{(\tau,k,W')}$ is as well. Because W' is suitable, Proposition 17 applies to $\bar{X}_{(\tau,k,W')} \subset \bar{X}$ to ensure that $p^{-1}(q^{-1}(W')) \cap \tilde{X}_{(\tau,k)}$ is connected. Moreover this contains $L \cap \tilde{X}_{(\tau,k)}$, so condition $(*)$ is satisfied.

(ii) Let $\tilde{L}$ be a cocompact G-subcomplex of $\tilde{X}$, and let $\tilde{L}'$ be any cocompact G-subcomplex of $\tilde{X}$ containing $\tilde{L}$. We show that for any lag $k \geq 0 \in \mathbb{Z}$, there exist points of $\tilde{L} \cap \tilde{X}_{(\tau,0)}$ lying in distinct components of $\tilde{L}' \cap \tilde{X}_{(\tau,-k)}$.

Let $L = p(q(\tilde{L}))$ and $L' = p(q(\tilde{L}'))$. By Observation 22 there exists a suitable complex $W \subseteq X$ with $L' \subseteq W$. Then $\bar{X}_{(\tau,-k,W)}$ is connected, and by assumption it is not π_1-surjective. Set $\tilde{\tilde{W}} = q^{-1}(p^{-1}(W))$. Then $\tilde{\tilde{W}} \cap \tilde{X}_{(\tau,-k)}$ is disconnected by Proposition 17. Furthermore, Proposition 18 ensures that each of its components contains components of $\tilde{L}' \cap \tilde{X}_{(\tau,-k)}$, which in turn contain points of $\tilde{L} \cap \tilde{X}_{(\tau,-k)}$. □

4. A a free group

Let the action ρ by G on T be as defined in Definition 1, with the additional restriction that A is a free group on the set $\{a_1, \ldots, a_n\}$ and T is its Cayley graph with respect to this set. Then the vertices of T are the elements of A. Let X, $q : (\bar{X}, \bar{x}_0) \to (X, \bar{x})$, $p : (\tilde{X}, \tilde{x}_0) \to (\bar{X}, \bar{x}_0)$, $r : X \to V$, and $\bar{r} : \bar{X} \to T$ be as defined in Section 2.2. The graph $V = A \backslash T$ has a unique vertex v_0, so the $K(G, 1)$-complex X can be chosen to have a unique 0-cell x_0, which we naturally choose as basepoint for X. In this case, for any cell c of V, X_c and $\bar{X}_c$ are both $K(B, 1)$-complexes. In fact, we can take $\bar{X}_c = X_c = X_{v_0}$ for all c, since passing from X to $\bar{X}$ simply "unwraps" loops in $A \subseteq G = \pi_1(X, x_0)$. Choose the base point $\bar{x}_0$ of $\bar{X}$ to be the unique 0-cell of $\bar{X}$ mapped to $1 \in A = \operatorname{vert} T$.

We uniquely represent ∂T by geodesic rays τ, with $\tau(0) = 1 \in A$ and $\tau(n)$ a freely reduced word on n letters. Thus each geodesic ray τ corresponds to a unique infinite freely reduced word $\prod_{i \in \mathbb{Z}_{\geq 0}} c_i$.

4.1. ***From suitable complexes to subgroups.*** From here on, we identify B with $\pi_1(\bar{X}, \bar{x}_0)$. Let W be a suitable subcomplex of X. Since W is finite, the subgroup

$$B(W) = \text{inclusion}_{\#}(q^{-1}(W) \cap \bar{r}^{-1}(1), \bar{x}_0) \leq B$$

is finitely generated. Let $S(W)$ be a finite generating set for $B(W)$. Let T' be a subtree of T. Fix $v \in \text{vert } T' \subseteq A$. Then $\bar{r}^{-1}(v) \cong \bar{X}_c$ has a single 0-cell; call it x'. Let $B(W, T', v)$ be the image of $\pi_1\big(q^{-1}(W) \cap \bar{r}^{-1}(T'), x'\big)$ in $\pi_1(\bar{X}, x')$. Let

$$\Psi_v : \pi_1(\bar{X}, x') \to \pi_1(\bar{X}, \bar{x}_0) = B$$

be the change-of-basepoint isomorphism. Then for $g \in \pi_1(\bar{X}, x')$, $\Psi_v(g) = vgv^{-1}$.

Lemma 23. *The subgroup of B generated by $\{usu^{-1} \mid s \in S(W), u \in T'\}$ is $\Psi_v(B(W, T', v))$.*

Proof. Any element $h \in B(W, T', v)$ can represented by a loop σ_h in the 1-skeleton of $q^{-1}(W) \cap \bar{r}^{-1}(T')$ based at x'. Because $\bar{X}$ has no 0-cells over the interiors of edges of T, and because each vertex space is a copy of X_{v_0} and each edge space a copy of $X_{v_0} \times [0, 1]$, the loop σ_h may be decomposed as concatenation of subpaths $\sigma_h^0, \sigma_h^1, \ldots, \sigma_h^m$, $m \in \mathbb{N}$, where each σ_h^i, $0 \leq i \leq m$, is either a 1-cell joining one vertex space to another (a “base edge” for an edge space) or a loop contained entirely in a vertex space and corresponding to some $s \in S(W)$. Between each pair of subpaths, we may introduce a path which returns straight back to x' (i.e., via 1-cells over lying over edges of T' exclusively). This process rewrites h as a product of conjugates of the form $v^{-1}usu^{-1}v$, $s \in S(W)$, $u \in T'$. □

Combining Theorem 3 with Lemma 23, we obtain a purely algebraic condition for determining whether an endpoint lies in $\Sigma^1(\rho)$. For a geodesic ray τ corresponding to the infinite word $\prod_i c_i$ and $k \in \mathbb{Z}$, define $A_k(\tau) = \text{vert}(\text{HB}_k(\tau))$ and $w_k = \tau(k) = c_1 c_2 \ldots c_k$. Then

$$\pi_1\big(q^{-1}(W) \cap \bar{r}^{-1}(\text{HB}_k(\tau)), w_k\big) = B(W, \text{HB}_k(\tau), w_k).$$

Theorem. *Let A be a finitely generated free group, and let T be its Cayley graph with respect to a free basis. For the action ρ as in Theorem 3, and for $e \in \partial T$ represented by a geodesic ray τ,*

(i) *If there is a finite set $S \subseteq B$ such that for each $k \in \mathbb{Z}_{\geq 0}$, S generates B over $A_k(\tau)$, then $e \in \Sigma^1(\rho)$.*

(ii) *If for each $k \in \mathbb{Z}_{\leq 0}$, B is not finitely generated over $A_k(\tau)$, then $e \notin \Sigma^1(\rho)$.*

Proof. (i) If there is such a finite set S, then we can choose a suitable subcomplex W containing loops corresponding to S. For any $k \in \mathbb{Z}_{\geq 0}$, let x' be the unique vertex of $\bar{r}^{-1}(w_k)$, and we have

$$B(W, \text{HB}_k(\tau), w_k) = \Psi_{w_k}^{-1}(B) = \pi_1(\bar{X}, x').$$

Thus by Theorem 3(i) we obtain $e \in \Sigma^1(\rho)$.

(ii) Given a suitable subcomplex W of X and $k \in \mathbb{Z}_{\leq 0}$, by assumption the subgroup $\Psi(B(W, \mathrm{HB}_k(\tau), w_k))$ is a proper subgroup of B. Hence, $B(W, \mathrm{HB}_k(\tau), w_k)$ is a proper subgroup of $\pi_1(\bar{X}, x')$. Thus, by part (ii) of Theorem 3, $e \notin \Sigma^1(\rho)$. $\square$

Recall that for $t \in \{a_1, \ldots, a_n\}^{\pm}$, the function expsum_t maps a reduced word w in $\{a_1, \ldots, a_n\}^{\pm}$ to the corresponding exponent sum of t in w. Also, recall we use the notation $\mathrm{Ball}_r(A, v)$ to refer to the r-ball around v in A (in the word metric), to avoid confusion with the subgroup B.

Lemma 24. *For an endpoint e represented by the geodesic ray τ, let*

$$Q_{t,k}(\tau) = \{\mathrm{expsum}_t(v) \mid v \in A_k(\tau)\} \subseteq \mathbb{Z}.$$

Then $Q_{t,k}(\tau)$ is bounded above if and only if τ eventually consists of only t^{-1}. Moreover, $Q_{t,k}(\tau)$ contains every integer within its bounds.

Proof. Let τ be represented by the infinite word $c_1c_2\ldots$, and fix $k \in \mathbb{Z}$. Recall that $A_k(\tau) = \bigcup_{l \geq \max\{0,k\}} \overline{\mathrm{Ball}_{l-k}(A, c_1c_2\ldots c_l)}$.

Suppose for $N \in \mathbb{Z}$, $c_i = t^{-1}$ for all $i > N$. For $j = 0, 1, 2, \ldots,$ the words $g_j = c_1c_2\ldots c_{N+j}t^{N+j-k}$ all represent the same element of A, and g_j has maximal expsum_t among elements of $\overline{\mathrm{Ball}_{N+j-k}(A, c_1c_2\ldots c_{N+j})}$. Since $A_k(\tau)$ is the union of these subsets, it follows that $Q_{t,k}(\tau)$ is bounded above.

On the other hand, suppose that there are infinitely many $i \in \mathbb{Z}$ such that $c_i \neq t^{-1}$. For $j \in \mathbb{Z}$, $j \geq \max\{0, k\}$, let $m(j)$ be the number of letters c_i in $c_1c_2\ldots c_j$ with $c_i \neq t^{-1}$. By assumption $m(j) \to \infty$ as $j \to \infty$. Let $g_j = c_1\ldots c_jt^{j-k}$. Then

$$g_j \in \overline{\mathrm{Ball}_{j-k}(A, c_1c_2\ldots c_j)} \subseteq A_k(\tau).$$

Since $\mathrm{expsum}_t(c_1c_2\ldots c_j) \geq -(j - m(j))$,

$$\mathrm{expsum}_t(g_j) = \mathrm{expsum}_t(c_1c_2\ldots c_j) + j - k \geq m(j) - k.$$

Letting $j \to \infty$, we have that $Q_{t,k}(\tau)$ is not bounded above.

The fact that $Q_{t,k}(\tau)$ contains every integer within its bounds follows from the observation that for $v, w \in A_k(\tau)$, if

$$\mathrm{expsum}_t(v) < m < \mathrm{expsum}_t(w),$$

the path connecting v to w contains a vertex u with $\mathrm{expsum}_t(u) = m$. $\square$

Proof of Corollary 6. Let $t \in \{a_1, \ldots, a_n\}^{\pm}$. Suppose $e \in \partial T$ is represented by an infinite word eventually consisting of only t^{-1}, and suppose there exists no $m \in \mathbb{Z}$ such that B is finitely generated over $A - \mathrm{expsum}_t^{-1}([m, \infty))$. By Lemma 24, $\{\mathrm{expsum}_t(a) \mid a \in A_k(\tau)\}$ is bounded above. Hence, B cannot be finitely generated over $A_k(\tau)$, and so by Theorem 4, part (ii), $e \notin \Sigma^1(\rho)$. $\square$

Proof of Theorem 9. Let $e = \tau(\infty)$, with τ corresponding to the infinite word $\prod_i c_i$. By Corollary 5, it is enough to show that $\varphi(A_k(\tau)) = \varphi(A)$ for each $k \geq 0 \in \mathbb{Z}$.

Let $w \in \mathcal{A}^*$ be a freely reduced word, and let l be the reduced length of w. We will find $w' \in A_k(\tau)$ with $\varphi(w') = \varphi(w)$. Choose $m \in \mathbb{Z}_{\geq 0}$ large enough to ensure that the word $c_1 \dots c_m$ has $k+l$ distinct subwords in $\ker \varphi$. Call these subwords ζ_i, $1 \leq i \leq k+l$, and let the remaining letters form subwords χ_i, $1 \leq i \leq k+l$, so that we have the decomposition

$$c_1 \dots c_m = \chi_1 \zeta_1 \chi_2 \zeta_2 \dots \chi_{k+l} \zeta_{k+l},$$

where each $\varphi(\zeta_i)$ is trivial, and each χ_i is possibly empty.

Now

$$\varphi(c_1 c_2 \dots c_m) = \varphi(\chi_1 \chi_2 \dots \chi_{k+l}),$$

and the reduced length of $\chi_1 \chi_2 \dots \chi_{k+l}$ is no greater than $m - l - k$. Thus the word $\xi = c_1 c_2 \dots c_m \chi_{k+l}^{-1} \dots \chi_2^{-1} \chi_1^{-1}$ is in both $\ker \varphi$ and $\overline{\mathrm{Ball}_{m-l-k}(A, c_1 \dots c_m)}$; moreover

$$\xi w \in \overline{\mathrm{Ball}_{m-k}(A, c_1 \dots c_m)} \subseteq A_k(\tau) \quad \text{and} \quad \varphi(w) = \varphi(\xi w). \qquad \square$$

4.2. *Argument for Example 11.* In Example 11, $G = B \rtimes_\varphi A$, where $A = C * D$ for free groups $C = \langle a_1, \dots, a_n \rangle$, $D = \langle a_{n+1}, \dots, a_m \rangle$, and $B = \oplus_{\omega \in C} K_\omega$ for some finitely generated group K. The claim is made that any endpoint of $T = \Gamma(A, \{a_1, a_2, \dots, a_m\})$ represented by a ray τ whose letters are eventually selected only from C does not lie in Σ^1. Since Σ^1 is G-invariant, we can assume τ consists of letters entirely in C. Then $\pi : A \twoheadrightarrow C$ fixes each vertex of τ. Moreover, it makes sense to discuss the subset $C_k(\tau) \subseteq C$.

Let $k \in \mathbb{Z}_{\leq 0}$ be given, and let S be any finite subset of B. We will show that the set $S' = \{\varphi_a(s) \mid s \in S,\ a \in A_k(\tau)\}$ does not generate B. Part (ii) of Theorem 4 thereby ensures that $\tau(\infty) \notin \Sigma^1(\rho)$.

To show that S' does not generate B, we will find an index $\psi \in C$ such that every $s \in S'$ is trivial at index ψ.

Observation 25. If $a \in A$ is in $A_k(\tau)$, then $\pi(a)$ is in $C_k(\tau)$.

Proof. Since $a \in A_k(\tau)$ and $k \leq 0$, there exists $l \geq 0$ such that $a \in \overline{\mathrm{Ball}_{l-k}(A, \tau(l))}$ by (1-2), so $\pi(a) \in \overline{\mathrm{Ball}_{l-k}(C, \tau(l))}$. But this is contained in $C_k(\tau)$, again by (1-2). $\square$

Define the set

$$\mathcal{I}(S) = \{\omega \in C \mid \exists\, s \in S \text{ such that } s \text{ is nontrivial at index } \omega\}.$$

Note that $\mathcal{I}(S)$ is a finite set, since S is finite and each $s \in S$ is nontrivial at only finitely many indices. Define

$$\mathcal{R}(S) = \max\{\text{reduced length of } \omega \mid \omega \in \mathcal{I}(S)\}.$$

Since $\mathcal{I}(S)$ is finite, $\mathcal{R}(S)$ is a nonnegative integer representing the maximum distance (in C) from any index of any nontrivial component of any element of S to the identity index $1 \in C$.

Since left multiplication by $c \in C$ is an isometry on C, it follows that the maximal distance in C from any nontrivial index of any element of $\varphi_c(S)$ to c is also $\mathcal{R}(S)$. Observation 25 therefore ensures that the set of nontrivial indices of elements of S' is a subset of the closed $\mathcal{R}(S)$-neighborhood of $C_k(\tau)$ in C. In fact, this neighborhood is the set $C_{k-\mathcal{R}(S)}(\tau)$. This is a proper subset of C (simply choose any geodesic ray other than τ and follow it far enough). For any $\psi \in C$ with $\psi \notin C_{k-\mathcal{R}(S)}(\tau)$, all $s \in S'$ will be trivial at index ψ. So S' can not generate B.

Acknowledgments

This paper extends results from the author's Ph.D dissertation at Binghamton University. It would not have been possible without the continuous support and encouragement of Ross Geoghegan. The author also wishes to thank Robert Bieri for his generosity with his time in discussing some of the ideas involved in this work. Additionally, the author is indebted to the anonymous referee, whose efforts have led to a more interesting and clearer paper.

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Received September 20, 2012. Revised November 29, 2013.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 268 No. 1 March 2014

0030-8730(2014)268:1;1-6