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COMPOSITION OPERATORS ON STRICTLY PSEUDOCONVEX DOMAINS WITH SMOOTH SYMBOL

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It is well known that the composition operator C_ϕ is unbounded on Hardy and Bergman spaces on the unit ball B_n in $\mathbb{C}^n$ when $n > 1$ for a linear holomorphic self-map ϕ of B_n. We find a sufficient and necessary condition for a composition operator with smooth symbol to be bounded on Hardy or Bergman spaces over a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Moreover, we show that this condition is equivalent to the compactness of the composition operator from a Hardy or Bergman space into the Bergman space whose weight is $\frac{1}{4}$ bigger. We also prove that a certain jump phenomenon occurs when the composition operator is not bounded. Our results generalize known results on the unit ball to strictly pseudoconvex domains.

1. Introduction

Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with a smooth boundary and let $d(z)$ be the distance from $z \in D$ to ∂D. Let $H(D)$ be the set of all holomorphic functions on D. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A^p_\alpha(D)$ is the space of all $f \in H(D)$ for which

$$\|f\|^p_{A^p_\alpha} = \int_D |f(z)|^p \, dV_\alpha(z) < \infty,$$

where $dV_\alpha(z) = d(z)^\alpha \, dV(z)$ and dV is the Lebesgue measure on D. Also, for $0 < p < \infty$, the Hardy space $H^p(D)$ is the space of all $f \in H(D)$ for which

$$\|f\|^p_{H^p} = \lim_{\epsilon \to 0} \int_{\partial D_\epsilon} |f(\zeta)|^p \, d\sigma_\epsilon(\zeta) < \infty,$$

where σ_ϵ is the surface measure on $\partial D_\epsilon = \{z \in D : d(z) = \epsilon\}$. It is well known

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(see [Krantz 2001]) that the admissible limit $f^*(\zeta)$ exists for almost every $\zeta \in \partial D$ when $f \in H^p(D)$ and

$$\|f\|_{H^p}^p = \int_{\partial D} |f^*(\zeta)|^p \, d\sigma_\epsilon(\zeta) < \infty,$$

where σ is the surface area measure on ∂D. For notational convenience we may view $H^p(D)$ as $A^p_{-1}(D)$.

Let $\phi = (\phi_1, \ldots, \phi_n) : D \to D$ be a holomorphic self-map on D. Then ϕ induces the composition operator, C_ϕ, defined on $H(D)$ by

$$C_\phi(f) = f \circ \phi.$$

When D is the unit disk, Δ, in $\mathbb{C}$, every composition operator is bounded on the weighted Bergman spaces and the Hardy spaces by Littlewood's subordination principle. On the other hand, when D is the unit ball, B_n, in $\mathbb{C}^n$ with $n \geq 2$, it is known that not every composition is bounded on the weighted Bergman spaces or the Hardy spaces. Among the early examples of unbounded composition operators on $H^p(B_2)$, the example $\phi(z_1, z_2) = (2z_1z_2, 0)$ is due to J.H. Shapiro and the examples $\phi(z_1, z_2) = (\psi(z_1, z_2), 0)$ for ψ inner were given by MacCluer [1984] and Cima, Stanton, and Wogen [Cima et al. 1984]. Other than the Carleson measure characterization there is no satisfactory criteria known for general symbols up to present time. Since a holomorphic linear map ϕ can not guarantee C_ϕ is bounded on Hardy and Bergman spaces when $n > 1$, one may concentrate on finding a good criteria for smooth holomorphic $\phi \in C^\infty(\bar{B}_n)$ so that C_ϕ is bounded on Hardy spaces, $H^2(B_n)$, and Bergman spaces, $A^2(B_n)$.

When ϕ is smooth up to the boundary, Warren Wogen [1988] found a necessary and sufficient condition for C_ϕ to be bounded on $H^p(B_n)$. This was generalized to $A^p_\alpha(B_n)$ in [Koo and Smith 2007], where the authors also showed what is called the jump phenomenon: if ϕ is smooth up to the boundary and C_ϕ is not bounded on $A^p_\alpha(B_n)$, then $C_\phi : A^p_\alpha(B_n) \not\to A^p_{\alpha-\epsilon}(B_n)$ for all $0 \leq \epsilon < \frac{1}{4}$. It was also proved [Koo and Park 2010] that the boundedness of $C_\phi : A^p_\alpha(B_n) \to A^p_\alpha(B_n)$ is equivalent to the compactness of $C_\phi : A^p_\alpha(B_n) \to A^p_{\alpha+1/4}(B_n)$ when ϕ is smooth up to the boundary. Wogen's original proof [1988] is quite long and involves various local analyses of the inducing map. Koo and Wang [2010] gave a much simpler proof of Wogen's result using certain compactness argument.

In this paper, we generalize the boundedness criteria and the jump phenomenon of composition operators with smooth symbols to bounded strictly pseudoconvex domains in $\mathbb{C}^n$. We adapt the compactness argument of [Koo and Wang 2010] in our proof. Our main theorem is the following, with $Q_\phi(\zeta)$ defined as in (3-1).

Theorem 1.1. *Let $0 < p < \infty$ and $\alpha \geq -1$. Let $\phi : D \to D$ be a holomorphic map with $\phi \in C^4(\bar{D})$. Then the following are equivalent.*

(1) $C_\phi : A^p_\alpha(D) \to A^p_\alpha(D)$ *is bounded.*

(2) $C_\phi : A^p_\alpha(D) \to A^p_{\alpha+1/4}(D)$ *is compact.*

(3) $Q_\phi(\zeta) < 1$ *on* $\phi^{-1}(\partial D)$.

Moreover, if $C_\phi : A^p_\alpha(D) \not\to A^p_\alpha(D)$, *then* $C_\phi : A^p_\alpha(D) \not\to A^p_{\alpha+\epsilon}(D)$ *for all* $0 < \epsilon < \frac{1}{4}$.

Remark. For $\phi(z) = (z_1 + z_2^2/2, 0) : B_2 \to B_2$, we know $C_\phi : A^p_\alpha(B_2) \to A^p_{\alpha+1/4}(B_2)$ is bounded [Koo and Smith 2007] but not compact [Koo and Park 2010].

In Section 2, we review well-known facts on strictly pseudoconvex domains D and Wogen's result on the unit ball. In Section 3, we study local behavior of maps on D which are smooth on $\bar{D}$, especially holomorphic self-maps of D. We prove our main theorem in Section 4.

Throughout the paper we use the same letter C to denote various positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants C will be often specified in parentheses. For nonnegative quantities X and Y the notation $X \lesssim Y$ or $Y \gtrsim X$ means $X \le CY$ for some inessential constant C. Similarly, we write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Background

Strictly pseudoconvex domain. A C^2-domain $D \subset \mathbb{C}^n$ is strictly pseudoconvex if there is a defining function $r \in C^2(\mathbb{C}^n)$ such that

$$D = \{z \in \mathbb{C}^n : r(z) > 0\}$$

and there exists $C > 0$ such that

$$C|w|^2 \le -\sum_{j=1}^{n} \frac{\partial^2 r(\zeta)}{\partial \zeta_i \partial \bar{\zeta}_j} w_i \overline{w}_j \tag{2-1}$$

for all $\zeta \in \partial D$ and for all $w \in \mathbb{C}^n$. For $\epsilon > 0$, let

$$D_\epsilon = \{z \in D : r(z) > \epsilon\}.$$

For $z, w \in \bar{D}$, define a quasimetric $d(z, w)$ by

$$d(z, w) = r(z) + r(w) + \left| \sum_{j=1}^{n} \frac{\partial r(w)}{\partial w_j}(z_j - w_j) \right| + |z - w|^2. \tag{2-2}$$

For $z, w \in \bar{D}$, let

$$X(z, w) = r(w) + \sum_{j=1}^{n} \frac{\partial r(w)}{\partial w_j}(z_j - w_j) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 r(w)}{\partial w_i \partial w_j}(z_j - w_j)(z_k - w_k).$$

Note that, by Taylor expansion of r near w, we get

$$r(z) = -r(w) + 2\,\mathrm{Re}\, X(z, w) + \sum_{i,j=1}^{n} \frac{\partial^2 r(w)}{\partial w_i \partial \overline{w}_j}(z_i - w_i)(\bar{z}_j - \overline{w}_j) + O(|z-w|^3).$$

Thus, when D is strictly pseudoconvex and $z \in \bar{D}$ is near $\eta \in \partial D$,

$$\mathrm{Re}\, X(z, \eta) \geq 0 \tag{2-3}$$

by (2-1). Moreover, it is well known from work of C. Fefferman [1974] that there exists $\delta_D > 0$ such that

$$|X(z, w)| \approx d(z, w) \tag{2-4}$$

for all $(z, w) \in R_{\delta_D}$, where

$$R_\delta = \{(z, w) \in \bar{D} \times \bar{D} : r(z) + r(w) + |z - w| < \delta\}.$$

Carleson measures. For any $\zeta \in \partial D$, we can define a Carleson region centered at ζ with radius δ by

$$\mathscr{C}(\zeta, \delta) = \{z \in D : d(z, \zeta) < \delta\}.$$

A positive Borel measure μ on $\bar{D}$ is said to be a *Carleson measure* if there is a constant $M > 0$ such that, for all $\zeta \in \partial D$ and $\delta > 0$,

$$\mu(\overline{\mathscr{C}(\zeta, \delta)}) \leq M\sigma(\overline{\mathscr{C}(\zeta, \delta)} \cap \partial D),$$

and such a measure μ is said to be a *vanishing Carleson measure* if

$$\lim_{\delta \to 0} \sup_{\zeta \in \partial D} \frac{\mu(\overline{\mathscr{C}(\zeta, \delta)})}{\sigma(\overline{\mathscr{C}(\zeta, \delta)} \cap \partial D)} = 0.$$

Also, for $\alpha > -1$, a positive Borel measure μ on D is said to be an α*-Carleson measure* if there is a constant $M > 0$ such that, for all $\zeta \in \partial D$ and $\delta > 0$,

$$\mu(\mathscr{C}(\zeta, \delta)) \leq M V_\alpha(\mathscr{C}(\zeta, \delta)),$$

and such a measure μ is said to be a *vanishing α-Carleson measure* if

$$\lim_{\delta \to 0} \sup_{\zeta \in \partial D} \frac{\mu(\mathscr{C}(\zeta, \delta))}{V_\alpha(\mathscr{C}(\zeta, \delta))} = 0.$$

By [Krantz and Li 1994] the V_α-volume of $\mathscr{C}(\zeta, \delta)$ and the surface area of the intersection $\overline{\mathscr{C}(\zeta, \delta)} \cap \partial D$ are

$$V_\alpha(\mathscr{C}(\zeta, \delta)) \approx \delta^{n+1+\alpha} \quad \text{and} \quad \sigma(\overline{\mathscr{C}(\zeta, \delta)} \cap \partial D) \approx \delta^n, \tag{2-5}$$

respectively.

The next theorem follows from Hörmander's work [1967] on Carleson measures, the work on Bergman and Szegő kernels by Fefferman [1974] and Phong and Stein [1977], together with Krantz and Li's [1994; 1995a; 1995b] work on Hardy spaces and Bergman spaces.

Theorem 2.1. *Let D be a smooth bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $0 < p < \infty$ and $\alpha > -1$. Let μ be a positive Borel measure on $\bar{D}$ and ν a positive Borel measure on D.*

(1) *The inclusion $H^p(D) \hookrightarrow L^p(\mu)$ is continuous if and only if μ is a Carleson measure, and compact if and only if μ is a vanishing Carleson measure.*

(2) *The inclusion $A^p_\alpha(D) \hookrightarrow L^p(\nu)$ is continuous if and only if ν is an α-Carleson measure, and compact if and only if μ is a vanishing α-Carleson measure.*

Let $\phi : D \to D$ be a holomorphic mapping and, for a holomorphic function f on D, let

$$C_\phi(f)(z) = f \circ \phi(z).$$

Since D is bounded, ϕ has admissible limit $\phi^*(\zeta)$ almost everywhere in ∂D. So, when $\xi \in \partial D$, we define $\phi(\xi) =: \phi^*(\xi)$. Let $\sigma \circ \phi^{-1}$ and $V_\alpha \circ \phi^{-1}$ be the measures on $\bar{D}$ and D defined by

$$\sigma \circ \phi^{-1}(E) = \int_{\phi^{*-1}(E)} d\sigma(\zeta)$$

for all $E \subset \bar{D}$ and

$$V_\alpha \circ \phi^{-1}(E) = \int_{\phi^{-1}(E)} dV_\alpha(z)$$

for all $E \subset D$, respectively. Then, by a change of variables, we have

$$\int_{\partial D} |C_\phi f(\zeta)|^p \, d\sigma(\zeta) = \int_{\bar{D}} |f(z)|^p \, d\sigma \circ \phi^{-1}(z)$$

and

$$\int_D |C_\phi f(z)|^p \, dV_\alpha(z) = \int_D |f(z)|^p \, dV_\alpha \circ \phi^{-1}(z).$$

Therefore, as a corollary of Theorem 2.1 we have the following characterization.

Corollary 2.2. *Let $0 < p < \infty$, $\alpha, \beta > -1$, and $\phi : D \to D$ be a holomorphic mapping.*

(1) *$C_\phi : H^p(D) \to H^p(D)$ is bounded if and only if $\sigma \circ \phi^{-1}$ is a Carleson measure, and compact if and only if $\sigma \circ \phi^{-1}$ is a vanishing Carleson measure.*

(2) *$C_\phi : H^p(D) \to A^p_\alpha(D)$ is bounded if and only if $V_\alpha \circ \phi^{-1}$ is a Carleson measure, and compact if and only if $V_\alpha \circ \phi^{-1}$ is a vanishing Carleson measure.*

(3) *$C_\phi : A^p_\alpha(D) \to A^p_\beta(D)$ bounded if and only if $V_\beta \circ \phi^{-1}$ is an α-Carleson measure, and compact if and only if $V_\beta \circ \phi^{-1}$ is a vanishing α-Carleson measure.*

Wogen's theorem. Let $\phi : B_n \to B_n$ be holomorphic and $\phi \in C^4(\bar{B}_n)$. Then Wogen proved [1988] the following characterization for C_ϕ to be bounded in $H^2(B_n)$, which was generalized by Koo and Smith to $A^p_\alpha(B_n)$ [2007], and by Koo and Park to holomorphic Sobolev spaces [2010]. For $z, \zeta \in \mathbb{C}^n$ and a smooth function g, let

$$\mathfrak{D}_\zeta g(z) = \sum_{j=1}^{n} \zeta_j \frac{\partial g}{\partial z_j}(z) \quad \text{and} \quad \mathfrak{D}_{\bar{\zeta}} g(z) = \sum_{j=1}^{n} \bar{\zeta}_j \frac{\partial g}{\partial \bar{z}_j}(z). \tag{2-6}$$

For $z, w \in \mathscr{C}^n$, let $\langle z, w\rangle$ be the Hermitian inner product defined by

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j.$$

Theorem 2.3. *Let $\phi : B_n \to B_n$ be holomorphic and $\phi \in C^4(\bar{B}_n)$. Let $0 < p < \infty$, $\alpha \geq -1$. For $\eta \in \partial B_n$, let $H_\eta(z) = \langle \phi(z), \eta\rangle$. Then $C_\phi : A^p_\alpha(B_n) \to A^p_\alpha(B_n)$ is bounded if and only if*

$$|\mathfrak{D}_{\tau\tau} H_\eta(\zeta)| < \mathfrak{D}_\zeta H_\eta(\zeta)$$

for all $\zeta, \eta, \tau \in \partial B_n$ such that

$$\zeta \in \phi^{-1}(\partial B_n), \quad \eta = \phi(\zeta), \quad \langle \zeta, \tau\rangle = 0.$$

Koo and Smith [2007] proved that the following jump phenomenon occurs when C_ϕ is not bounded.

Theorem 2.4. *Let $\phi : B_n \to B_n$ be holomorphic and $\phi \in C^4(\bar{B}_n)$. Let $0 < p < \infty$, $\alpha \geq -1$. If C_ϕ is not bounded on $A^p_\alpha(B_n)$, then $C_\phi : A^p_\alpha(B_n) \not\to A^p_{\alpha+\epsilon}(B_n)$ for all $0 \leq \epsilon < \frac{1}{4}$.*

The following was proved for the critical index $\epsilon = \frac{1}{4}$ [Koo and Park 2010].

Theorem 2.5. *Let $\phi : B_n \to B_n$ be holomorphic and $\phi \in C^4(\bar{B}_n)$. Let $0 < p < \infty$ and $\alpha \geq -1$. Then $C_\phi : A^p_\alpha(B_n) \to A^p_\alpha(B_n)$ is bounded if and only if $C_\phi : A^p_\alpha(B_n) \to A^p_{\alpha+1/4}(B_n)$ is compact.*

3. Local estimates of smooth holomorphic maps on D

Throughout this section we assume that $\phi : D \to D$ is a holomorphic mapping with $\phi \in C^4(\bar{D})$ where D is a bounded strictly pseudoconvex domain with a smooth boundary. For $z \in \mathbb{C}^n$, we use the following notation:

$$z = (z_1, z_2, \ldots, z_n) = (z_1, z') = (z_1, z_2, z''), \quad z_j = x_j + iy_j \ (1 \leq j \leq n).$$

For w near ∂D, let

$$\nu(w) = |\partial r(w)|^{-1} \partial r(w),$$

where

$$\partial r(z) = \left(\frac{\partial r(z)}{\partial z_1}, \dots, \frac{\partial r(z)}{\partial z_n}\right).$$

For $\eta \in \partial D$, let

$$\phi_\eta(z) = X(\phi(z), \eta)$$

and let

$$Q_\phi(\zeta, \eta) = \sup_\tau \left\{ \left| \frac{\mathcal{D}^2_{\tau\tau}\phi_\eta(\zeta)}{\mathcal{D}_{\nu(\zeta)}\phi_\eta(\zeta)} - \frac{\mathcal{D}^2_{\tau\tau} r(\zeta)}{|\partial r(\zeta)|} \right| \cdot \frac{|\partial r(\zeta)|}{|\mathcal{D}^2_{\tau\bar{\tau}} r(\zeta)|} : \langle \tau, \nu(\zeta) \rangle = 0 \right\}.$$

If $\eta = \phi(\zeta)$, we let

$$Q_\phi(\zeta) = Q_\phi(\zeta, \phi(\zeta)). \tag{3-1}$$

For $D = B_n$, it is easy to check that $\phi_\eta = 2H_\eta - 2$ and the condition on Theorem 2.3 is equivalent to $Q_\phi(\zeta) < 1$ for all $\zeta \in \phi^{-1}(\partial D)$.

Proposition 3.1. *Let $\zeta \in \partial D$ and $\eta = \phi(\zeta) \in \partial D$. Then*

(1) $\mathcal{D}_{\nu(\zeta)}\phi_\eta(\zeta) > 0$,

(2) $\mathcal{D}_\tau \phi_\eta(\zeta) = 0$ *for all τ with $\langle \nu(\zeta), \tau \rangle = 0$,*

(3) $Q_\phi(\zeta) \le 1$.

Proof. Let $\zeta, \eta \in \partial D$, and $\langle \nu(\zeta), \tau \rangle = 0$. Without loss of generality, we may choose local coordinates near $(\zeta, \eta) \in \partial D \times \partial D \subset \mathbb{C}^{2n}$ such that

$$\zeta = \eta = (0, \dots, 0), \quad \nu(\zeta) = \nu(\eta) = (1, 0, \dots, 0), \quad \tau = (0, 1, 0, \dots, 0).$$

For $1 \le i, j \le n$, let

$$r_i = \frac{\partial r(\zeta)}{\partial z_i}, \quad r_{ij} = \frac{\partial^2 r(\zeta)}{\partial z_i \partial z_j}, \quad r_{i\bar{j}} = \frac{\partial^2 r(\zeta)}{\partial z_i \partial \bar{z}_j},$$

and let

$$a_i = \frac{\partial r(\eta)}{\partial z_i}, \quad a_{ij} = \frac{\partial^2 r(\eta)}{\partial z_i \partial z_j}.$$

Also, for $1 \le i, j, \ell \le n$, let

$$b_i^\ell = \frac{\partial \phi_\ell(\zeta)}{\partial z_i}, \quad b_{ij}^\ell = \frac{\partial^2 \phi_\ell(\zeta)}{\partial z_i \partial z_j}.$$

From the definition of X, we have

$$\begin{aligned}\phi_\eta(z) &=: X(\phi(z), \eta) \\ &= \sum_{j=1}^n \frac{\partial r(\eta)}{\partial \eta_j}(\phi_j(z) - \eta_j) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 r(\eta)}{\partial \eta_i \partial \eta_j}(\phi_i(z) - \eta_i)(\phi_j(z) - \eta_j),\end{aligned}$$

and thus

$$\phi_\eta(z) = a_1\phi_1(z) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\phi_i(z)\phi_j(z). \tag{3-2}$$

Since the harmonic function $\operatorname{Re}\phi_1$ takes a minimum at ζ and $\nu(\zeta)$ is the inward normal vector at $\zeta \in \partial D$, by Hopf's lemma, we have

$$b_1^1 = \frac{\partial\phi_1(\zeta)}{\partial\zeta_1} = \frac{\partial \operatorname{Re}\phi_1}{\partial x_1}(\zeta) > 0. \tag{3-3}$$

Since $\nu(\zeta) = (1, 0, \dots, 0)$, for z near ζ

$$r(z) = 2r_1x_1 + O(|z|^2) \quad (r_1 > 0).$$

Therefore, there are $\epsilon, \delta > 0$ such that

$$z = (x_1, z') \in D \quad \text{if } 0 < x_1 \le \delta \quad \text{and} \quad |z'|^2 = \epsilon|x_1|.$$

Then, for all (x_1, z') with $0 < x_1 \le \delta$ and $|z'|^2 = \epsilon|z_1|$, we have

$$0 \le \operatorname{Re}\phi_1(x_1, z') = \operatorname{Re}\left(b_1^1x_1 + \sum_{j=2}^{n} b_j^1 z_j\right) + O(|z|^2).$$

From this, we can easily deduce that

$$b_j^1 = \frac{\partial\phi_1(\zeta)}{\partial\zeta_j} = 0 \quad (2 \le j \le n). \tag{3-4}$$

Then, from (3-2), (3-3), and (3-4), we have

$$\begin{aligned}\phi_\eta(z) &= a_1\left(b_1^1z_1 + \frac{1}{2}\sum_{i,j=1}^{n} b_{ij}^1 z_iz_j\right) + \frac{1}{2}\sum_{k,\ell=1}^{n}\left(\sum_{i,j=1}^{n} a_{ij}b_k^ib_\ell^j\right)z_kz_\ell + O(|z|^3)\\ &= a_1b_1^1\left[z_1 + \frac{1}{2a_1b_1^1}\sum_{i,j=1}^{n}\left[a_1b_{ij}^1 + \sum_{k,\ell=1}^{n} a_{k\ell}b_i^kb_j^\ell\right]z_iz_j\right] + O(|z|^3).\end{aligned}$$

From this we easily conclude (1) and (2).

For (3), let

$$c_{ij} = \frac{r_1}{2a_1b_1^1}\left[a_1b_{ij}^1 + \sum_{k,\ell=1}^{n} a_{k\ell}b_i^kb_j^\ell\right] - \frac{r_{ij}}{2}. \tag{3-5}$$

Then we get

$$\begin{aligned}\text{(3-6)}\quad \phi_\eta(z) = \frac{a_1 b_1^1}{r_1}\Big[r_1 z_1 + \frac{1}{2}\sum_{i,j=1}^n r_{ij} z_i z_j + \frac{1}{2}\sum_{i,j=1}^n r_{i\bar{j}} z_i \bar{z}_j\Big] \\ + \frac{a_1 b_1^1}{r_1}\Big[\sum_{i,j=1}^n c_{ij} z_i z_j - \frac{1}{2}\sum_{i,j=1}^n r_{i\bar{j}} z_i \bar{z}_j\Big] + O(|z|^3).\end{aligned}$$

Note that, for z near ζ,

$$r(z) = 2\,\mathrm{Re}\Big(r_1 z_1 + \frac{1}{2}\sum_{i,j=1}^n r_{ij} z_i z_j + \frac{1}{2}\sum_{i,j=1}^n r_{i\bar{j}} z_i \bar{z}_j\Big) + O(|z|^3).$$

Now consider a point $(s, te^{i\theta}, 0'')$ near ζ, with $s, t \geq 0$. (Here and below, $0''$ stands for the origin in $\mathscr{C}^{n-2}$; see start of Section 3.) We have

$$r(s, te^{i\theta}, 0'') = 2r_1 s + (\mathrm{Re}(r_{22}e^{2i\theta}) + r_{2\bar{2}})t^2 + O(s^2 + st + t^3),$$

and thus

$$\text{(3-7)}\qquad r(s, te^{i\theta}, 0'') \approx t^{5/2} \quad \text{if } s = t^{5/2} - \frac{1}{2r_1}\big(\mathrm{Re}(r_{22}e^{2i\theta}) + r_{2\bar{2}}\big)t^2.$$

Then, with $z := (s, te^{i\theta}, 0'')$, by (2-3) and (3-6), we have

$$\begin{aligned} 0 \leq \mathrm{Re}\,\phi_\eta(z) &= \frac{a_1 b_1^1}{2r_1} r(z) + \frac{a_1 b_1^1}{r_1}\mathrm{Re}\big(c_{22}t^2 e^{2i\theta} - \tfrac{1}{2} r_{2\bar{2}} t^2\big) + O(t^3) \\ &= \frac{a_1 b_1^1}{r_1}\mathrm{Re}\big(c_{22}e^{2i\theta} - \tfrac{1}{2} r_{2\bar{2}}\big)t^2 + O(t^{5/2})\end{aligned}$$

for all θ. Thus

$$\mathrm{Re}\big(c_{22}e^{2i\theta} - \tfrac{1}{2} r_{2\bar{2}}\big) \geq 0, \quad \theta \in [0, 2\pi].$$

This implies

$$|c_{22}| \leq -\frac{r_{2\bar{2}}}{2}.$$

Since $\nu(\zeta) = (1, 0, \ldots, 0)$ and $\tau = (0, 1, 0, \ldots, 0)$, by (3-6) we have

$$c_{22} = r_1 \frac{1}{2}\frac{\partial^2 \phi_\eta(\zeta)}{\partial\zeta_2\partial\zeta_2}\Big(\frac{\partial\phi_\eta(\zeta)}{\partial\zeta_1}\Big)^{-1} - \frac{r_{22}}{2} = \frac{|\partial r(\zeta)|}{2}\Big(\frac{\mathscr{D}^2_{\tau\tau}\phi_\eta(\zeta)}{\mathscr{D}_{\nu(\zeta)}\phi_\eta(\zeta)} - \frac{\mathscr{D}^2_{\tau\tau} r(\zeta)}{|\partial r(\zeta)|}\Big).$$

Therefore, we have

$$\frac{|\partial r(\zeta)|}{2}\Big|\frac{\mathscr{D}^2_{\tau\tau}\phi_\eta(\zeta)}{\mathscr{D}_{\nu(\zeta)}\phi_\eta(\zeta)} - \frac{\mathscr{D}^2_{\tau\tau} r(\zeta)}{|\partial r(\zeta)|}\Big| = |c_{22}| \leq -\frac{1}{2}\frac{\partial^2 r(\zeta)}{\partial z_2 \partial\bar{z}_2} = -\tfrac{1}{2}\mathscr{D}^2_{\tau\bar{\tau}} r(\zeta). \qquad \square$$

The following lemma is the key local estimate for the proof of (3) $\Longrightarrow$ (1) of Theorem 1.1. First we introduce some notation. For $\delta > 0$, let

$$\begin{aligned} V_\delta &= \{\xi \in \partial D : |X(\xi, \zeta)| < \delta \text{ for some } \zeta \in \phi^{-1}(\partial D)\}, \\ W_\delta &= \{\eta \in \partial D : |X(\eta, \phi(\zeta))| < \delta \text{ for some } \zeta \in \phi^{-1}(\partial D)\}, \\ K &= \{(\zeta, \phi(\zeta)) \in \partial D \times \partial D : \zeta \in \phi^{-1}(\partial D)\}, \\ K_\delta &= \{(z, \eta) \in \bar{D} \times \partial D : |X(z, \zeta)| + |X(\phi(\zeta), \eta)| < \delta, \zeta \in \phi^{-1}(\partial D)\}. \end{aligned}$$

Lemma 3.2. *Suppose $Q_\phi(\xi) < 1$ on $\phi^{-1}(\partial D)$. Then there are $\delta > 0$ and $C > 1$ such that, for all $(z, \eta) \in K_\delta$,*

$$\frac{1}{C}\big(|X(\phi(\zeta), \eta)| + |X(z, \zeta)|\big) \le |X(\phi(z), \eta)| \le C\big(|X(\phi(\zeta), \eta)| + |X(z, \zeta)|\big), \tag{3-8}$$

where the point $\zeta \in \partial D$ is defined by the relation

$$\min\{|X(\phi(w), \eta)| : w \in \overline{O_z}\} = |X(\phi(\zeta), \eta)|$$

and O_z is the connected component of $\phi^{-1}(\mathscr{C}(\eta, \delta))$ containing z.

Proof. Since $\phi \in C^2(\bar{D})$, there are $\epsilon, \delta > 0$ such that $Q_\phi(z, \eta) \le 1 - \epsilon$ for all $(z, \eta) \in K_\delta$. Fix $(z, \eta) \in K_\delta$ and let ζ be any point such that

$$\min\{|X(\phi(w), \eta)| : w \in\} = |X(\phi(\zeta), \eta)|.$$

Note that $\zeta \in \partial D$, since $\phi_\eta(w) = X(\phi(w), \eta)$ is an open map as a holomorphic function on D. Without loss of generality, we may choose local coordinates near $(\zeta, \eta) \in \partial D \times \partial D \subset \mathbb{C}^{2n}$ as in the proof of Proposition 3.1 so that

$$\zeta = \eta = (0, \dots, 0), \quad \nu(\zeta) = \nu(\eta) = (1, 0, \dots, 0).$$

Then, by Taylor expansion of ϕ_η at ζ, we have

$$\phi_\eta(z) = \phi_\eta(\zeta) + \sum_{j=1}^{n} a_j z_j + \frac{1}{2} \sum_{i,j=2}^{n} a_{ij}\, z_i z_j + O(|z_1|^2 + |z_1|\,|z'| + |z'|^3).$$

By Proposition 3.1(1), we have $\mathscr{D}_{\nu(\zeta)}\phi_\eta(\zeta) > 0$ when $\eta = \phi(\zeta)$. Therefore, by shrinking δ if necessary, we may assume that $\mathscr{D}_{\nu(\zeta)}\phi_\eta(\zeta) \neq 0$ for all $(\zeta, \eta) \in K_\delta$, and thus

$$a_1 = \frac{\partial \phi_\eta}{\partial z_1}(\zeta) = \mathscr{D}_{\nu(\zeta)}\phi_\eta(\zeta) \neq 0.$$

Since ζ is the local minimum point of $|\phi_\eta|$, by Taylor expansion of $\phi_\eta(z)$ at ζ with $z = (s, te^{i\theta}, 0'')$ as in (3-7), we see that

$$a_j = \frac{\partial \phi_\eta}{\partial z_j}(\zeta) = 0 \quad \text{ if } j \ge 2.$$

Thus we have

$$\phi_\eta(z) = \phi_\eta(\zeta) + a_1 z_1 + \frac{1}{2}\sum_{i,j=2}^{n} a_{ij}\; z_i z_j + O(|z_1|^2 + |z_1|\,|z'| + |z'|^3). \tag{3-9}$$

Note that by assumption we have $Q_\phi(\zeta,\eta) \le 1-\epsilon$, since $(\zeta,\eta)\in K_\delta$. Define F and G on $\mathbb{C}^{n-1}$ by

$$F(z') = \frac{1}{2}\sum_{i,j=2}^{n}\left(\frac{a_{ij}}{a_1} - \frac{r_{ij}}{r_1}\right) z_i z_j, \qquad G(z') = -(1-\epsilon)\sum_{i,j=2}^{n}\frac{r_{i\bar{j}}}{r_1}\; z_i \bar{z}_j.$$

Then the condition $Q_\phi(\zeta,\eta)\le 1-\epsilon$ implies $|\mathcal{D}_{\tau'\tau'}F| \le \mathcal{D}_{\tau'\bar{\tau}'}G$ for all $\tau'\in\mathbb{C}^{n-1}$. But straightforward calculations show that

$$\mathcal{D}_{\tau'\tau'}F(z') = 2F(\tau'), \quad \mathcal{D}_{\tau'\bar{\tau}'}G(z') = G(\tau').$$

Therefore, we have

$$\left|\sum_{i,j=2}^{n}\left(\frac{a_{ij}}{a_1} - \frac{r_{ij}}{r_1}\right) z_i z_j\right| \le -(1-\epsilon)\sum_{i,j=2}^{n}\frac{r_{i\bar{j}}}{r_1}\; z_i \bar{z}_j.$$

Since D is strictly pseudoconvex, from this inequality together with (2-1), we have

$$-\sum_{i,j=2}^{n}\frac{r_{i\bar{j}}}{r_1}\; z_i\bar{z}_j - \left|\sum_{i,j=2}^{n}\left(\frac{a_{ij}}{a_1} - \frac{r_{ij}}{r_1}\right) z_i z_j\right| \ge \epsilon C|z'|^2.$$

Therefore, by (3-9) we have

$$\begin{aligned}
&|\mathrm{Re}(\phi_\eta(z) - \phi_\eta(\zeta))| \\
&\quad\ge |a_1|\,\mathrm{Re}\left(z_1 + \frac{1}{2}\sum_{i,j=2}^{n}\frac{r_{ij}}{r_1} z_i z_j + \frac{1}{2}\sum_{i,j=2}^{n}\frac{r_{i\bar{j}}}{r_1}\; z_i\bar{z}_j\right) \\
&\qquad - |a_1|\left(\frac{1}{2}\sum_{i,j=2}^{n}\frac{r_{i\bar{j}}}{r_1} z_i\bar{z}_j + \frac{1}{2}\left|\sum_{i,j=2}^{n}\left(\frac{a_{ij}}{a_1} - \frac{r_{ij}}{r_1}\right) z_i z_j\right|\right) + O(|z_1|^2 + |z_1|\,|z'| + |z'|^3) \\
&\quad\ge \frac{|a_1|}{2r_1} r(z) + |a_1|\frac{\epsilon C|z'|^2}{2} + O(|z_1|^2 + |z_1|\,|z'| + |z'|^3).
\end{aligned}$$

Since $|\phi_\eta(z) - \phi_\eta(\zeta)| \lesssim |\phi_\eta(z) - \phi_\eta(\zeta)| + |\mathrm{Re}(\phi_\eta(z) - \phi_\eta(\zeta))|$, by (3-9) we then have

$$|\phi_\eta(z) - \phi_\eta(\zeta)| \gtrsim \left|a_1 z_1 + \frac{1}{2}\sum_{i,j=2}^{n} a_{ij} z_i z_j\right| + |z'|^2 + O(|z_1|^2 + |z_1|\,|z'| + |z'|^3).$$

Since $|a+b|+c > |a|/M + (Mc-|b|)/M$ for any $M \geq 1$, we see that there is $C > 0$ such that

$$|\phi_\eta(z) - \phi_\eta(\zeta)| \geq C(|z_1| + |z'|^2) + O(|z_1|^2 + |z_1||z'| + |z'|^3). \tag{3-10}$$

Note that by (2-4) we have

$$\begin{aligned} |X(z,\zeta)| &\approx d(z,\zeta) \\ &= r(z) + r_1|z_1| + |z'|^2 \\ &\approx |z_1| + |z'|^2 + O(|z_1|^2 + |z_1||z'| + |z'|^3). \end{aligned}$$

Therefore, from (3-10), there exist $C > 1$ (by shrinking $\delta > 0$ if necessary) such that

$$|X(\phi(z),\eta) - X(\phi(\zeta),\eta)| \geq \frac{1}{C}|X(z,\zeta)|, \quad |z| < \delta.$$

Note that if $|X(\phi(\zeta),\eta)| < \dfrac{1}{2C}|X(z,\zeta)|$, the triangular inequality yields

$$|X(\phi(z),\eta)| \gtrsim [|X(\phi(\zeta),\eta)| + |X(z,\zeta)|], \quad |z| < \delta.$$

This inequality also holds when

$$|X(\phi(\zeta),\eta)| \geq \frac{1}{2C}|X(z,\zeta)|,$$

since $|X(\phi(z),\eta)|$ has a minimum at ζ. The constants involved depend continuously on η throughout the calculations, and thus, by shrinking $\delta > 0$ again if necessary, there are $C > 0$ and $\delta > 0$ such that

$$|X(\phi(z),\eta)| \geq C[|X(\phi(\zeta),\eta)| + |X(z,\zeta)|] \tag{3-11}$$

for all $(z,\eta) \in K_\delta$.

Since

$$|X(z,\zeta)| \approx |z_1| + |z'|^2 + O(|z_1|^2 + |z_1||z'| + |z'|^3),$$

the converse inequality follows from (3-9). □

We use the same notation as in the proof of Proposition 3.1, and let

$$r_{222} = \frac{\partial^3 r(\zeta)}{\partial z_2^3}, \quad r_{22\bar{2}} = \frac{\partial^3 r(\zeta)}{\partial z_2^2 \partial \bar{z}_2}.$$

We use the following lemma to prove the jump phenomenon when C_ϕ is not bounded on $A^p_\alpha(D)$.

Lemma 3.3. *Let $\zeta = (0,\dots,0) \in \partial D$ with*

$$\nu(\zeta) = (1,0,\dots,0),$$

and let R be a holomorphic polynomial

$$R(z_1,z_2)=r_1z_1+(r_{12}+r_{1\bar 2})z_1z_2+\frac{(r_{22}+r_{2\bar 2})}{2}z_2^2+\frac{(r_{222}+3r_{22\bar 2})}{6}z_2^3. \tag{3-12}$$

Let $a\in\mathbb{C}$, $b\in\mathbb{R}$, and

$$g(z)=(1+az_2)R(z_1,z_2)+ibz_2^3+O(|z_1|^2+|z_2|^4+|z''|^2).$$

Then, for $\alpha\ge -1$, there is $C>0$ such that, for all $\delta>0$,

$$V_{\alpha+1/4}(\{z\in D:|g(z)|\le\delta\})\ge C\delta^{n+\alpha+1}.$$

Proof. It suffices to prove for $\delta>0$ small, and hence we assume $\delta>0$ is sufficiently small. For the rest of proof we assume

$$z'=(z_2,z'')\in A_\delta:=\{(z_2,z'')\in\mathbb{C}^{n-1}:x_2^4+y_2^2+|z''|^2\le\delta\}. \tag{3-13}$$

From the fact that $\nu(\zeta)=(1,0,\dots,0)$, there are constants $p_j\in\mathbb{R}$ for $1\le j\le 5$ such that

$$\begin{aligned} r(z_1,z_2,z'')=r_1x_1+p_1x_1x_2+p_2y_1x_2+p_3x_2^2+p_4x_2^3+p_5x_2y_2&\\ +O(x_1^2+y_1^2+y_2^2+x_2^4+|z''|^2).&\end{aligned} \tag{3-14}$$

Also, there are $q_j\in\mathbb{R}$ for $1\le j\le 5$ such that

$$\begin{aligned}\operatorname{Im}[R(z_1+iy_1,z_2)+ibz_2^3]=r_1y_1+q_1y_1x_2+q_1x_1x_2+q_3x_2y_2+q_4x_2^2+q_5x_2^3&\\ +O(x_1^2+y_1^2+y_2^2+x_2^4),&\end{aligned} \tag{3-15}$$

since $|z_1||y_2|+|x_2^2y_2|=O(x_1^2+y_1^2+y_2^2+x_2^4)$.

Taking $\delta>0$ sufficiently small if necessary, we may assume $r_1+p_1x_2\ge r_1/2$ and $r_1+q_1x_2\ge r_1/2$. Let $(u,v)=(u(z_2),v(z_2))\in\mathbb{R}^2$ be the solution of the equations

$$\begin{aligned}0&=(r_1+p_1x_2)u+p_2x_2v+p_3x_2^2+p_4x_2^3+p_5x_2y_2,\\ 0&=(r_1+q_1x_2)v+q_2x_2u+q_3x_2y_2+q_4x_2^2+q_5x_2^3.\end{aligned}$$

Since $z'\in A_\delta$, the solution (u,v) always exists and satisfies

$$|u|+|v|\lesssim\delta^{1/2}.$$

Hence, by (3-14) and (3-15), we have

$$r(u+iv,z_2,z'')=O(\delta),\quad \operatorname{Im}[R(u+iv,z_2)+ibz_2^3]=O(\delta). \tag{3-16}$$

By (2-1) we have $r_{2\bar 2}\in\mathbb{R}$, and thus

$$\operatorname{Re}[r_{2\bar 2}z_2(z_2-\bar z_2)]=-2r_{2\bar 2}y_2^2.$$

Therefore,

$$
\begin{aligned}
2\operatorname{Re}[R(z_1,z_2)]& \\
&= r(z_1,z_2,0'')+2\operatorname{Re}[r_{1\bar 2}z_1(z_2-\bar z_2)]+\operatorname{Re}[r_{2\bar 2}z_2(z_2-\bar z_2)] \\
&\qquad\qquad +\operatorname{Re}[r_{22\bar 2}z_2^2(z_2-\bar z_2)]+O(|z_1|^2+|z_2|^4) \\
&= r(z_1,z_2,0'')-4y_2\operatorname{Im}[r_{1\bar 2}z_1]-2r_{2\bar 2}y_2^2-2y_2\operatorname{Re}[r_{22\bar 2}z_2^2]+O(|z_1|^2+|z_2|^4) \\
&= r(z_1,z_2,0'')+O(|z_1|^2+|z_1y_2|+y_2^2+|y_2|\,|z_2|^2+|z_2|^4) \\
&= r(z_1,z_2,0'')+O(x_1^2+y_1^2+y_2^2+x_2^4).
\end{aligned}
$$

Therefore, from (3-16) we have

$$2\operatorname{Re}[R(u+iv,z_2)]=O(\delta),$$

and thus, from the second equation of (3-16), we have

$$|R(u+iv,z_2)|\approx|\operatorname{Re}[R(u+iv,z_2)]|+|\operatorname{Im}[R(u+iv,z_2)]|=O(\delta).$$

From these estimates we then have

$$
\begin{aligned}
|g(u+iv,z')|&\lesssim|\operatorname{Re}[R(u+iv,z_2)]|+|z_2|\,|R(u+iv,z_2)| \\
&\qquad +|\operatorname{Im}[R(u+iv,z_2)+ibz_2^3]|+O(|u+iv|^2+x_2^4+y_2^2+|z''|^2) \\
&= O(\delta).
\end{aligned}
$$

Since $\partial g(\zeta)/\partial z_1=r_1$, by taking δ sufficiently small if necessary, we have

$$z_1=u(z_2)+iv(z_2)+O(\delta)\implies|g(z)|\lesssim\delta. \tag{3-17}$$

Let

$$B_\delta^C(z_2):=\{z_1:u(z_2)+C\delta\le x_1\le u(z_2)+2C\delta,\ v(z_2)\le y_1\le v(z_2)+\delta\}$$

and

$$\Lambda_\delta^C=\{z:z'\in A_\delta,\ z_1\in B_\delta^C(z_2)\}.$$

Then, by (3-14), there is $C>0$ such that, for all $z\in\Lambda_\delta^C$, we have

$$r(z)\approx\delta,$$

and from (3-17), for all $z\in\Lambda_\delta^C$, we have

$$|g(z_1,z_2,z'')|\lesssim\delta.$$

Therefore, there are constants $c,C>0$ such that

$$V_{\alpha+1/4}(\{z\in D:|g(z)|\le\delta\})\ge V_{\alpha+1/4}(\Lambda_{c\delta}^C)\gtrsim\delta^{\alpha+1/4}V(\Lambda_{c\delta}^C).$$

Since $B^C_\delta(z_2)$ is a rectangle with area $C\delta^2$ for a fixed z_2, from the definition of A_δ in (3-13) we have

$$V_{\alpha+1/4}(\{z \in D : |g(z)| \le \delta\}) \gtrsim \delta^{\alpha+1/4} V(\Lambda^C_{c\delta}) \approx \delta^{\alpha+n+1}.$$

The proof is complete, since the constants suppressed in the inequalities throughout our calculations are independent of δ. □

4. Proof of Theorem 1.1

First, we prove the last statement, the jump phenomenon, assuming the equivalence of (1), (2), and (3).

Let $0 < \epsilon < \frac{1}{4}$ and suppose

$$C_\phi : A^p_\alpha(D) \to A^p_{\alpha+\epsilon}(D)$$

is bounded. Then

$$C_\phi : A^p_\alpha(D) \to A^p_{\alpha+1/4}(D)$$

is compact, since the inclusion the map $I : A^p_{\alpha+\epsilon}(D) \hookrightarrow A^p_{\alpha+1/4}(D)$ is compact. Thus, from the equivalence of (1) and (2) we conclude the boundedness of

$$C_\phi : A^p_\alpha(D) \to A^p_\alpha(D).$$

To prove the equivalence of (1), (2), and (3), note that (1) $\Longrightarrow$ (2) is trivial since the inclusion map $I : A^p_\alpha(D) \hookrightarrow A^p_{\alpha+1/4}(D)$ is compact. Thus, it suffices to show that (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1). First (3) $\Longrightarrow$ (1) follows from the following theorem.

Theorem 4.1. *Let $0 < p < \infty$ and $\alpha \ge -1$. Let $\phi : D \to D$ be a holomorphic map with $\phi \in C^4(\bar{D})$. If $Q_\phi(\zeta) < 1$ on $\phi^{-1}(\partial D)$, then C_ϕ is bounded on $A^p_\alpha(D)$.*

Proof. Let $\mu = \sigma \circ \phi^{-1}$ and $\mu_\alpha = V_\alpha \circ \phi^{-1}$ for $\alpha > -1$. By Corollary 2.2, it suffices to show that there exist $\delta_0 > 0$ and $M > 0$ such that, for all $\eta \in \partial D$ and $0 < \delta < \delta_0$,

$$\mu(\overline{C(\eta, \delta)}) \le M\delta^n \tag{4-1}$$

and

$$\mu_\alpha(C(\eta, \delta)) \le M\delta^{n+1+\alpha}. \tag{4-2}$$

We may assume $\delta > 0$ is sufficiently small, since, otherwise, (4-1) and (4-2) hold trivially. Note that $\phi(D) \cap \partial D = \varnothing$ since ϕ is a holomorphic self-map of D. Thus $\phi(\bar{D}) \cap [\partial D \setminus V] = \varnothing$ for any neighborhood $V \subset \partial D$ of $\partial D \cap \phi(\partial D)$. By (2-4), with W_δ as defined right before Lemma 3.2, it suffices to show that there are constants $\delta_1 > 0$ and $\delta_2 > 0$ such that (4-1) and (4-2) hold for all $\delta < \delta_1$ and $\eta \in W_{\delta_2}$. Choose δ_1 and δ_2 small so that Lemma 3.2 holds with $\delta = \delta_0 := (\delta_1 + \delta_2)$, and let $C > 1$ be the corresponding constant in Lemma 3.2.

For $\eta \in W_{\delta_2}$, let O_j be any component of $\phi^{-1}(\mathscr{C}(\eta, \delta_0))$ which also intersects with $\phi^{-1}(\mathscr{C}(\eta, \delta_0/2C))$. Let $\zeta_j \in \overline{O_j}$ be a point such that

$$\min\{|X(\phi(w), \eta)| : w \in \overline{O_j}\} = |X(\phi(\zeta_j), \eta)|.$$

Since $|X(\phi(\zeta_j), \eta)| \le \delta_0/2C$, by (3-8) we have

$$\phi(\mathscr{C}(\zeta_j, \delta_0/2C)) \subset \mathscr{C}(\eta, \delta_0).$$

Therefore, $\mathscr{C}(\zeta_j, \delta_0/2C) \subset O_j$, since O_j is a component which contains ζ_j. This implies that the number of components O_j has an upper bound $M < \infty$ independent of η, since

$$M\delta_0^{n+1+\alpha} \approx \sum_{j=1}^{M} V_\alpha(\mathscr{C}(\zeta_j, \delta_0/2C)) \le V_\alpha(\phi^{-1}(\mathscr{C}(\eta, \delta_0))) \lesssim 1.$$

Now fix such a component O_j as above. Then, by Lemma 3.2,

$$O_j \cap \phi^{-1}(\mathscr{C}(\eta, \delta)) \subset \mathscr{C}(\zeta_j, C\delta)$$

for all $\delta < \delta_0$.

Then, (4-1) and (4-2) follows immediately since the number of components has a uniform upper bound M. □

Next, (2) $\Longrightarrow$ (3) follows from the following theorem together with the Carleson measure criteria, Corollary 2.2.

Theorem 4.2. *Let $\phi : D \to D$ be a holomorphic map with $\phi \in C^4(\bar{D})$. Suppose $\zeta, \eta = \phi(\zeta) \in \partial D$ and $Q_\phi(\zeta) = 1$. Then there is C > 0 such that, for all $\delta > 0$,*

$$V_{\alpha+1/4}(\phi^{-1}(\mathscr{C}(\eta, \delta))) \ge C V_\alpha(\mathscr{C}(\eta, \delta))$$

and

$$V_{-3/4} \circ \phi^{-1}(\overline{\mathscr{C}(\eta, \delta)}) \ge C\sigma(\overline{\mathscr{C}(\eta, \delta)} \cap \partial D).$$

Proof. For $z \in \mathbb{C}^n$, let $z = (z_1, \dots, z_n) = (z_1, z') = (z_1, z_2, z'')$. Near $(\zeta, \eta) \in \partial D \times \partial D$, we choose the same coordinates as in the proof of Proposition 3.1 so that

$$\zeta = \eta = (0, \dots, 0), \quad \nu(\zeta) = \nu(\eta) = (1, 0, \dots, 0).$$

By change of coordinates in z' variables if necessary, we may assume $Q_\phi(\zeta) = 1$ for $\tau = (0, 1, 0, \dots, 0)$, that is,

$$\left| \frac{\mathscr{D}^2_{\tau\tau}\phi_\eta(\zeta)}{\mathscr{D}_{\nu(\zeta)}\phi_\eta(\zeta)} - \frac{\mathscr{D}^2_{\tau\tau} r(\zeta)}{|\partial r(\zeta)|} \right| \cdot \frac{|\partial r(\zeta)|}{|\mathscr{D}^2_{\tau\bar{\tau}} r(\zeta)|} = 1 \quad (\tau = (0, 1, 0, \dots, 0)).$$

Since this relation is invariant under rotation in the z_2 variable, we may assume

$$\frac{\mathscr{D}^2_{\tau\tau}\phi_\eta(\zeta)}{\mathscr{D}_{\nu(\zeta)}\phi_\eta(\zeta)} - \frac{r_{22}}{r_1} = \frac{r_{2\bar{2}}}{r_1}$$

By (1) and (2) of Proposition 3.1, we have

$$\phi_\eta(z) = a_1 z_1 + \sum_{j=2}^{n} a_{2j} z_2 z_j + a_{32} z_2^3 + O(|z_1|^2 + |z_2|^4 + |z''|^2) \tag{4-3}$$

with $a_1 > 0$. Therefore, the condition $Q_\phi(\zeta) = 1$ is equivalent to

$$\frac{2a_{22}}{a_1} - \frac{r_{22}}{r_1} = \frac{r_{2\bar{2}}}{r_1}. \tag{4-4}$$

Let $R(z_1, z_2)$ be as in (3-12). Then, by (4-3) and (4-4), we get

$$\begin{aligned}\phi_\eta(z) = \frac{a_1}{r_1}(1 + Az_2)R(z_1, z_2) + Bz_2^3 \\ + \sum_{j=3}^{n} a_{2j} z_2 z_j + O\Bigl(|z_1|^2 + |z_2|^4 + |z_1||z_3|^2 + \sum_{j=4}^{n} |z_j|^2\Bigr),\end{aligned}$$

where

$$A = \frac{a_{12}}{a_1} - \frac{(r_{22} + r_{2\bar{2}})a_{12}}{2r_1}, \quad B = a_{32} - \frac{(r_{222} + 3r_{22\bar{2}})a_1}{6r_1} - A\frac{(r_{22} + r_{2\bar{2}})a_1}{2r_1}.$$

Then, by Lemma 3.3, to complete the proof it suffices to show that

$$\operatorname{Re} B = 0, \quad a_{2j} = 0 \quad (j = 3, \dots, n).$$

Since $\nu(\zeta) = (1, 0, \dots, 0)$, for $(s, t) \in \mathbb{R}^2$ we have

$$r(s, t, te^{i\theta}, 0, \dots, 0) = 2r_1 s + O(s^2 + t^2).$$

Thus, for each $\theta, t \in \mathbb{R}$, there is $s \in \mathbb{R}$ with $|s| \lesssim t^2$ such that $\operatorname{Re}[R(s, t)] = r(s, t, te^{i\theta}, 0, \dots, 0) = 0$.

Since $\operatorname{Re} \phi_\eta(s, t, te^{i\theta}, 0, \dots, 0) \geq 0$ by (2-3), we get

$$\begin{aligned} 0 &\leq \operatorname{Re} \phi_\eta(s, t, te^{i\theta}, 0, \dots, 0) \\ &= \operatorname{Re}\Bigl[\frac{a_1}{r_1}(1 + At)R(s, t) + Bt^3 + a_{23} t^2 e^{i\theta}\Bigr] + O(s^2 + t^4) \\ &= \operatorname{Re}\Bigl[\frac{a_1}{r_1} At R(s, t) + Bt^3 + a_{23} t^2 e^{i\theta}\Bigr] + O(s^2 + t^4) \\ &= \operatorname{Re}[Bt^3 + a_{23} t^2 e^{i\theta}] + O(s^2 + t^4) \end{aligned}$$

for all θ. This implies $a_{23} = 0$, and, with the same argument, we get

$$a_{2j} = 0 \quad (j = 3, \dots, n).$$

Also, note that $r(s, \pm t, 0'') = 2r_1 s + O(s^2 + t^2)$ which implies that for each $\pm t$

there is $s = s(\pm t)$ such that $r(s, \pm t, 0'') = 0$ with $|s(\pm t)| \lesssim t^2$. Then, by (2-3), with $s = s(\pm t)$ we have

$$\begin{aligned} 0 \le \operatorname{Re} \phi_\eta(s, \pm t, 0'') &= \frac{a_1}{r_1} \operatorname{Re}[R(s, \pm t)] \pm t^3 \operatorname{Re} B + O(t\,|\operatorname{Im}[R(s, \pm t)]| + t^4) \\ &= \pm t^3 \operatorname{Re} B + O(t^4). \end{aligned}$$

Therefore, we get $\operatorname{Re} B = 0$ and the proof is complete. □

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