TWISTED QUANTUM DRINFELD HECKE ALGEBRAS

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We generalize quantum Drinfeld Hecke algebras by incorporating a 2-cocycle on the associated finite group. We identify these algebras as specializations of deformations of twisted skew group algebras, giving an explicit connection to Hochschild cohomology. We classify these algebras for diagonal actions, as well as for the symmetric groups with their natural representations. Our results show that the parameter spaces for the symmetric groups in the twisted setting is smaller than in the untwisted setting.

1. Introduction

Drinfeld Hecke algebras were defined by V. Drinfeld [1986]. They arise as symplectic reflection algebras in the work of P. Etingof and V. Ginzburg [2002], as braided Cherednik algebras in the work of Y. Bazlov and A. Berenstein [2009], and as graded versions of affine Hecke algebras in the work of G. Lusztig [1989]. They arise in diverse areas, such as representation theory, combinatorics, and orbifold theory, and they were used by I. Gordon [2003] to prove a version of the $n!$ conjecture for Weyl groups.

In this paper, we consider quantum and twisted analogs of Drinfeld Hecke algebras by incorporating quantum parameters as well as a 2-cocycle on the associated finite group. We simultaneously generalize twisted Drinfeld Hecke algebras and quantum Drinfeld Hecke algebras. The former was studied by S. Witherspoon [2007], and the latter was studied in [Levandovskyy and Shepler 2011] and [Naidu and Witherspoon 2011]. T. Chmutova [2005] generalized symplectic reflection algebras by incorporating a 2-cocycle on the associated finite group, and showed that such a 2-cocycle arises naturally for nonfaithful representations. Such a 2-cocycle also arises in orbifold theory, where they are known as discrete torsion [Adem and Ruan 2003; Căldăraru et al. 2004; Vafa and Witten 1995].

Let V be a complex vector space with basis $v_1, v_2, \ldots, v_n$, and $\boldsymbol{q} := (q_{ij})_{1 \le i, j \le n}$, a tuple of nonzero scalars for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. Let $S_{\boldsymbol{q}}(V)$

denote the *quantum symmetric algebra*

$$S_{\boldsymbol{q}}(V) := \mathbb{C}\langle v_1, \ldots, v_n \mid v_i v_j = q_{ij} v_j v_i \text{ for all } 1 \le i, j \le n\rangle.$$

Let G be a finite group acting linearly on V, and $\alpha : G \times G \to \mathbb{C}^\times$, a normalized 2-cocycle on G. Let $\kappa : V \times V \to \mathbb{C}^\alpha G$ be a bilinear map for which $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i)$ for all $1 \le i, j \le n$. Let $T(V)$ be the tensor algebra on V, and define

$$\mathcal{H}_{\boldsymbol{q},\kappa,\alpha} := T(V)\#_\alpha G/(v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j) \mid 1 \le i, j \le n),$$

the quotient of the twisted skew group algebra $T(V)\#_\alpha G$ by the ideal generated by all elements of the form specified. Suppose that the action of G on V induces an action of G on $S_{\boldsymbol{q}}(V)$ by automorphisms, so we may form the twisted skew group algebra $S_{\boldsymbol{q}}(V)\#_\alpha G$. Assigning each v_i degree one and each group element degree zero makes $\mathcal{H}_{\boldsymbol{q},\kappa,\alpha}$ a filtered algebra, and makes $S_{\boldsymbol{q}}(V)\#_\alpha G$ a graded algebra. We call $\mathcal{H}_{\boldsymbol{q},\kappa,\alpha}$ a *twisted quantum Drinfeld Hecke algebra* (over $\mathbb{C}$) if it satisfies the Poincaré–Birkhoff–Witt condition: the associated graded algebra gr $\mathcal{H}_{\boldsymbol{q},\kappa,\alpha}$ is isomorphic, as a graded algebra, to $S_{\boldsymbol{q}}(V)\#_\alpha G$. The space of all maps $\kappa : V \times V \to \mathbb{C}^\alpha G$ for which $\mathcal{H}_{\boldsymbol{q},\kappa,\alpha}$ is a twisted quantum Drinfeld Hecke algebra will be referred to as the *parameter space*.

Main results and organization. In Section 2, we use G. Bergman's Diamond Lemma [1978] to give necessary and sufficient conditions for the algebra $\mathcal{H}_{\boldsymbol{q},\kappa,\alpha}$ to be a twisted quantum Drinfeld Hecke algebra.

In Section 3, we identify the twisted quantum Drinfeld Hecke algebras $\mathcal{H}_{\boldsymbol{q},\kappa,\alpha}$ as specializations of particular types of deformations of the twisted skew group algebras $S_{\boldsymbol{q}}(V)\#_\alpha G$.

Section 4 develops the homological algebra needed for the sections that follow. Specifically, this section is concerned with the computation of the degree two Hochschild cohomology of $S_{\boldsymbol{q}}(V)\#_\alpha G$.

In Section 5, we establish a one-to-one correspondence between the subspace of *constant* Hochschild 2-cocycles (see Section 3 for definition) contained in $\mathrm{HH}^2(S_{\boldsymbol{q}}(V)\#_\alpha G)$ and twisted quantum Drinfeld Hecke algebras associated to the quadruple $(G, V, \boldsymbol{q}, \alpha)$. As a consequence, we show that every constant Hochschild 2-cocycle on $S_{\boldsymbol{q}}(V)\#_\alpha G$ lifts to a deformation of $S_{\boldsymbol{q}}(V)\#_\alpha G$.

In Section 6, we consider diagonal actions of G on a chosen basis for V, and, using results from [Naidu et al. 2011], we classify the corresponding twisted quantum Drinfeld Hecke algebras.

In Section 7, we consider the symmetric groups S_n, $n \ge 5$, with their natural representations, with the unique nontrivial quantum parameters $q_{ij} = -1$, $i \ne j$, and with a cohomologically nontrivial 2-cocycle on S_n, which is unique up to coboundary. We classify the corresponding twisted quantum Drinfeld Hecke algebras. Our results

show that the parameter space in the twisted setting is smaller than in the untwisted setting.

Throughout the paper, let G denote a finite group acting linearly on a complex vector space V with basis $v_1, v_2, \ldots, v_n$. Let $\boldsymbol{q} := (q_{ij})_{1 \leq i, j \leq n}$ denote a tuple of nonzero scalars for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. We work over the complex numbers $\mathbb{C}$, and all tensor products are taken over $\mathbb{C}$ unless otherwise indicated.

2. Necessary and sufficient conditions

In this section, we use Bergman's Diamond Lemma [1978] to give necessary and sufficient conditions for the algebra $\mathcal{H}_{\boldsymbol{q}, \kappa, \alpha}$ (defined in the introduction and recalled below) to be a twisted quantum Drinfeld Hecke algebra. First, we recall the notion of a twisted skew group algebra. Let G be a finite group, and let $\alpha : G \times G \to \mathbb{C}^\times$ be a normalized 2-cocycle on G, that is,

$$\alpha(g_1, g_2)\alpha(g_1 g_2, g_3) = \alpha(g_2, g_3)\alpha(g_1, g_2 g_3) \quad \text{and} \quad \alpha(g, 1) = 1 = \alpha(1, g)$$

for all $g, g_1, g_2, g_3 \in G$. Let A be an algebra on which G acts by automorphisms. The *twisted skew group algebra* $A \#_\alpha G$ is defined as follows. As a vector space, $A \#_\alpha G$ is $A \otimes \mathbb{C}G$. Multiplication on $A \#_\alpha G$ is determined by

$$(a \otimes g)(b \otimes h) := \alpha(g, h)a(^g b) \otimes gh$$

for all $a, b \in A$ and all $g, h \in G$, where a left superscript denotes the action of the group element. The 2-cocycle condition on α ensures that $A \#_\alpha G$ is an associative algebra. Note that A is a subalgebra of $A \#_\alpha G$ via the isomorphism $A \xrightarrow{\sim} A \otimes 1$, and the twisted group algebra $\mathbb{C}^\alpha G$ is a subalgebra of $A \#_\alpha G$ via the isomorphism $\mathbb{C}^\alpha G \xrightarrow{\sim} 1 \otimes \mathbb{C}^\alpha G$. The image of a group element g in the twisted group algebra $\mathbb{C}^\alpha G$ is denoted by t_g. To shorten notation, we write the element $a \otimes g$ of $A \#_\alpha G$ by $a t_g$. Since α is assumed to be normalized, t_1 is the multiplicative identity for $A \#_\alpha G$. For all $g \in G$, we have

$$(t_g)^{-1} = \alpha^{-1}(g, g^{-1})t_{g^{-1}} = \alpha^{-1}(g^{-1}, g)t_{g^{-1}}.$$

Suppose G acts linearly on a complex vector space V with basis $v_1, v_2, \ldots, v_n$, and let $\boldsymbol{q} := (q_{ij})_{1 \leq i, j \leq n}$ denote a tuple of nonzero scalars for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. For each group element $g \in G$, let g_k^i denote the scalar determined by the equation

$$^g v_i = \sum_{k=1}^n g_k^i v_k,$$

and define the *quantum (i, j, k, l)-minor determinant* of g as

$$\det{}_{ijkl}(g) := g_l^j g_k^i - q_{ji} g_l^i g_k^j.$$

The following lemma is used in the proof of Theorem 2.2.

Lemma 2.1. *Suppose that the action of G on V extends to an action on $S_q(V)$ by automorphisms, and let $g \in G$. We have*:

(i) $q_{lk} \det_{ijkl}(g) = -\det_{ijlk}(g)$ *for all i, j, k, l.*

(ii) *For each i, j, if $q_{ij} \neq 1$, then $g_k^i g_k^j = 0$ for all k.*

Proof. For a proof of (i), see [Levandovskyy and Shepler 2011, Lemma 3.2]. Part (ii) follows from the assumption that G acts on $S_q(V)$ by automorphisms and that $q_{ij} \neq 1$: we have ${}^g v_i {}^g v_j = q_{ij} {}^g v_j {}^g v_i$, and so $(\sum_{k=1}^n g_k^i v_k)(\sum_{l=1}^n g_l^j v_l) = q_{ij}(\sum_{k=1}^n g_k^j v_k)(\sum_{l=1}^n g_l^i v_l)$. Equating coefficients of v_k^2 yields $g_k^i g_k^j = q_{ij} g_k^i g_k^j$, and since $q_{ij} \neq 1$, we get $g_k^i g_k^j = 0$. $\square$

Let $\kappa : V \times V \to \mathbb{C}^\alpha G$ be a bilinear map for which $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i)$ for all $1 \leq i, j \leq n$. For each $g \in G$, let $\kappa_g : V \times V \to \mathbb{C}$ be the function determined by the condition

$$\kappa(v, w) = \sum_{g \in G} \kappa_g(v, w) t_g \quad \text{for all } v, w \in V.$$

The condition $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i)$ implies that $\kappa_g(v_i, v_j) = -q_{ij}\kappa_g(v_j, v_i)$ for all $g \in G$.

Recall that the algebra

$$\mathcal{H}_{q,\kappa,\alpha} := T(V)\#_\alpha G/(v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j) \mid 1 \leq i, j \leq n)$$

is called a twisted quantum Drinfeld Hecke algebra if it satisfies the Poincaré–Birkhoff–Witt condition: $\operatorname{gr} \mathcal{H}_{q,\kappa,\alpha} \cong S_q(V)\#_\alpha G$, as graded algebras. This is equivalent to the condition that the set $\{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} t_g \mid m_i \geq 0, g \in G\}$ is a $\mathbb{C}$-basis for $\mathcal{H}_{q,\kappa,\alpha}$.

In the proof of the next theorem, we assume familiarity with, and will freely use, terminology from [Bergman 1978] (for example, "reduction system").

Theorem 2.2. *The algebra $\mathcal{H}_{q,\kappa,\alpha}$ is a twisted quantum Drinfeld Hecke algebra if and only if the following conditions hold.*

(1) *For all $g, h \in G$ and $1 \leq i < j \leq n$,*

$$\frac{\alpha(h, g)}{\alpha(hgh^{-1}, h)}\kappa_g(v_j, v_i) = \sum_{k<l} \det_{ijkl}(h)\kappa_{hgh^{-1}}(v_l, v_k).$$

(2) *For all $g \in G$ and $1 \leq i < j < k \leq n$,*

$$\kappa_g(v_k, v_j)({}^g v_i - q_{ji} q_{ki} v_i) + \kappa_g(v_k, v_i)(q_{kj} v_j - q_{ji} {}^g v_j) + \kappa_g(v_j, v_i)(q_{kj} q_{ki} {}^g v_k - v_k) = 0.$$

Proof. We begin by expressing the algebra $\mathcal{H}_{q,\kappa,\alpha}$ as a quotient of a free associative $\mathbb{C}$-algebra. Let $X = \{v_1, v_2, \ldots, v_n\} \cup \{t_g \mid g \in G\}$, and let $\mathbb{C}\langle X \rangle$ be the free associative $\mathbb{C}$-algebra generated by X. Consider the reduction system

$$S = \{(t_g v_i, {}^g v_i t_g), (t_g t_h, \alpha(g,h) t_{gh}), (v_j v_i, q_{ji} v_i v_j + \kappa(v_j, v_i)) \mid g, h \in G, 1 \le i < j \le n\}$$

for $\mathbb{C}\langle X \rangle$. Let I be the ideal of $\mathbb{C}\langle X \rangle$ generated by the elements

$$t_g v_i - {}^g v_i t_g, \quad t_g t_h - \alpha(g,h) t_{gh}, \quad v_j v_i - q_{ji} v_i v_j - \kappa(v_j, v_i), \quad g, h \in G, 1 \le i < j \le n.$$

In what follows, we use the Diamond Lemma [Bergman 1978] to show that the set

$$\{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} t_g \mid m_i \ge 0, g \in G\}$$

is a $\mathbb{C}$-basis for $\mathbb{C}\langle X \rangle / I$ if and only if the two conditions in the statement of the theorem hold.

Define a partial order $\le$ on the free semigroup $\langle X \rangle$ as follow: First, we declare that $v_1 < v_2 < \cdots < v_n < g$ for all $g \in G$, and then we set $A < B$ if

(i) A is of smaller length than B, or

(ii) A and B have the same length but A is less than B relative to the lexicographical order.

Then $\le$ is a semigroup partial order on $\langle X \rangle$, compatible with the reduction system S, and having the descending chain condition. Thus the hypothesis of the Diamond Lemma holds.

Observe that the set $\langle X \rangle_{\mathrm{irr}}$ of irreducible elements of $\langle X \rangle$ is precisely the alleged $\mathbb{C}$-basis for $\mathbb{C}\langle X \rangle / I$. That is,

$$\langle X \rangle_{\mathrm{irr}} = \{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} t_g \mid m_i \ge 0, g \in G\}.$$

In what follows, we show that all ambiguities of S are resolvable if and only if the two conditions in the statement of the theorem hold. The theorem will then follow by the Diamond Lemma. There are no inclusion ambiguities, but there exist overlap ambiguities, and these correspond to the monomials

$$t_g t_h t_k, \quad t_g t_h v_i, \quad t_h v_j v_i, \quad v_k v_j v_i, \quad \text{where } 1 \le i < j < k \le n, \ g, h \in G.$$

Associativity of the multiplication in the twisted group algebra $\mathbb{C}^\alpha G$ implies that the ambiguities corresponding to the monomials $t_g t_h t_k$ are resolvable. The equality ${}^{gh} v_i = {}^g({}^h v_i)$ implies that the ambiguities corresponding to the monomials $t_g t_h v_i$ are resolvable. Next, we show that the ambiguities corresponding to the monomials $t_h v_j v_i$ are resolvable if and only if condition (1) in the statement of the theorem holds. Applying a reduction to the factor $v_j v_i$ in $t_h v_j v_i$, we get

$$q_{ji} t_h v_i v_j + t_h \kappa(v_j, v_i).$$

Applying a reduction to the factor $t_h v_i$ and then to the resulting factor $t_h v_j$ gives

$$q_{ji}{}^h v_i{}^h v_j t_h + t_h \kappa(v_j, v_i)$$

$$= q_{ji}\left(\sum_{l=1}^{n} h_l^i v_l\right)\left(\sum_{k=1}^{n} h_k^j v_k\right)t_h + t_h \kappa(v_j, v_i)$$

$$= q_{ji}\sum_{l<k} h_l^i h_k^j v_l v_k t_h + q_{ji}\sum_{k<l} h_l^i h_k^j v_l v_k t_h + q_{ji}\sum_{k=1}^{n} h_k^i h_k^j v_k^2 t_h + t_h \kappa(v_j, v_i).$$

Applying a reduction to the factor $v_l v_k$ in the second summation above yields

$$q_{ji}\sum_{l<k} h_l^i h_k^j v_l v_k t_h + q_{ji}\sum_{k<l} h_l^i h_k^j q_{lk} v_k v_l t_h$$

$$+ q_{ji}\sum_{k<l} h_l^i h_k^j \kappa(v_l, v_k) t_h + q_{ji}\sum_{k=1}^{n} h_k^i h_k^j v_k^2 t_h + t_h \kappa(v_j, v_i).$$

Combining the first two summations, expanding $\kappa(v_l, v_k)$ and $\kappa(v_j, v_i)$, and then applying reductions to each term in $\kappa(v_l, v_k)t_h$ and to each term in $t_h \kappa(v_j, v_i)$ gives

$$q_{ji}\sum_{k<l}(h_k^i h_l^j + q_{lk} h_l^i h_k^j)v_k v_l t_h + q_{ji}\sum_{k=1}^{n} h_k^i h_k^j v_k^2 t_h$$

$$+ q_{ji}\sum_{g\in G}\left(\alpha(g, h)\sum_{k<l} h_l^i h_k^j \kappa_g(v_l, v_k)\right)t_{gh} + \sum_{g\in G}\alpha(h, g)\kappa_g(v_j, v_i)t_{hg}$$

$$= q_{ji}\sum_{k<l}(h_k^i h_l^j + q_{lk} h_l^i h_k^j)v_k v_l t_h + q_{ji}\sum_{k=1}^{n} h_k^i h_k^j v_k^2 t_h$$

$$+ \sum_{g\in G}\left(\alpha(hgh^{-1}, h)q_{ji}\sum_{k<l} h_l^i h_k^j \kappa_{hgh^{-1}}(v_l, v_k) + \alpha(h, g)\kappa_g(v_j, v_i)\right)t_{hg}.$$

Next, we apply to $t_h v_j v_i$ a reduction different from the one in the computation above: Applying a reduction to the factor $t_h v_j$ in $t_h v_j v_i$, and then to the resulting factor $t_h v_i$, we get

$$^h v_j{}^h v_i t_h = \left(\sum_{l=1}^{n} h_l^j v_l\right)\left(\sum_{k=1}^{n} h_k^i v_k\right)t_h$$

$$= \sum_{l<k} h_l^j h_k^i v_l v_k t_h + \sum_{k<l} h_l^j h_k^i v_l v_k t_h + \sum_{k=1}^{n} h_k^j h_k^i v_k^2 t_h.$$

Applying a reduction to the factor $v_l v_k$ in the second summation above yields

$$\sum_{l<k} h_l^j h_k^i v_l v_k t_h + \sum_{k<l} q_{lk} h_l^j h_k^i v_k v_l t_h + \sum_{k<l} h_l^j h_k^i \kappa(v_l, v_k) t_h + \sum_{k=1}^{n} h_k^j h_k^i v_k^2 t_h.$$

Combining the first two summations, expanding $\kappa(v_l, v_k)$, and then applying a reduction to each term in $\kappa(v_l, v_k)t_h$ gives

$$\sum_{k<l}(h_k^j h_l^i + q_{lk}h_l^j h_k^i)v_k v_l t_h + \sum_{k=1}^{n} h_k^j h_k^i v_k^2 t_h + \sum_{g \in G}\left(\alpha(g, h)\sum_{k<l}h_l^j h_k^i \kappa_g(v_l, v_k)\right)t_{gh}$$

$$= \sum_{k<l}(h_k^j h_l^i + q_{lk}h_l^j h_k^i)v_k v_l t_h + \sum_{k=1}^{n} h_k^j h_k^i v_k^2 t_h$$

$$+ \sum_{g \in G}\left(\alpha(hgh^{-1}, h)\sum_{k<l}h_l^j h_k^i \kappa_{hgh^{-1}}(v_l, v_k)\right)t_{hg}.$$

By equating coefficients, we see that the final expressions in the previous two computations are equal if and only if

(a) $q_{ji}h_k^i h_l^j + q_{ji}q_{lk}h_l^i h_k^j = h_k^j h_l^i + q_{lk}h_l^j h_k^i$ for all $k < l$,

(b) $q_{ji}h_k^i h_k^j = h_k^j h_k^i$ for all k, and

(c) for all $g \in G$, we have

$$\alpha(hgh^{-1}, h)q_{ji}\sum_{k<l}h_l^i h_k^j \kappa_{hgh^{-1}}(v_l, v_k) + \alpha(h, g)\kappa_g(v_j, v_i)$$
$$= \alpha(hgh^{-1}, h)\sum_{k<l}h_l^j h_k^i \kappa_{hgh^{-1}}(v_l, v_k).$$

Conditions (a) and (b) follow from Lemma 2.1(i) and (ii), respectively. The equation in (c) is equivalent to condition (1) in the statement of the theorem.

Lastly, we show that the ambiguities corresponding to the monomials $v_k v_j v_i$ are resolvable if and only if condition (2) in the statement of the theorem holds. Applying a reduction to the factor $v_k v_j$ in $v_k v_j v_i$, we get

$$q_{kj}v_j v_k v_i + \kappa(v_k, v_j)v_i.$$

Applying a reduction to the factor $v_k v_i$ gives

$$q_{kj}q_{ki}v_j v_i v_k + q_{kj}v_j\kappa(v_k, v_i) + \kappa(v_k, v_j)v_i.$$

Applying a reduction to the factor $v_j v_i$ yields

$$q_{kj}q_{ki}q_{ji}v_i v_j v_k + q_{kj}q_{ki}\kappa(v_j, v_i)v_k + q_{kj}v_j\kappa(v_k, v_i) + \kappa(v_k, v_j)v_i.$$

Expanding $\kappa(v_j, v_i)$, $\kappa(v_k, v_i)$, and $\kappa(v_k, v_j)$, applying reductions to each term in $\kappa(v_j, v_i)v_k$ and to each term in $\kappa(v_k, v_j)v_i$, and then rearranging gives

$$q_{kj}q_{ki}q_{ji}v_i v_j v_k + \sum_{g \in G}(\kappa_g(v_k, v_j)^g v_i + q_{kj}\kappa_g(v_k, v_i)v_j + q_{kj}q_{ki}\kappa_g(v_j, v_i)^g v_k)t_g.$$

Next, we apply to $v_k v_j v_i$ a reduction different from the one in the computation above: Applying a reduction to the factor $v_j v_i$ in $v_k v_j v_i$, we get

$$q_{ji}v_k v_i v_j + v_k\kappa(v_j, v_i).$$

Applying a reduction to the factor $v_k v_i$ gives

$$q_{ji} q_{ki} v_i v_k v_j + q_{ji} \kappa(v_k, v_i) v_j + v_k \kappa(v_j, v_i).$$

Applying a reduction to the factor $v_k v_j$ yields

$$q_{ji} q_{ki} q_{kj} v_i v_j v_k + q_{ji} q_{ki} v_i \kappa(v_k, v_j) + q_{ji} \kappa(v_k, v_i) v_j + v_k \kappa(v_j, v_i).$$

Expanding $\kappa(v_k, v_j)$, $\kappa(v_k, v_i)$, and $\kappa(v_j, v_i)$, and then applying reductions to each term in $\kappa(v_k, v_i) v_j$ gives

$$q_{ji} q_{ki} q_{kj} v_i v_j v_k + \sum_{g \in G} (q_{ji} q_{ki} \kappa_g(v_k, v_j) v_i + q_{ji} \kappa_g(v_k, v_i)^g v_j + \kappa_g(v_j, v_i) v_k) t_g.$$

The final expressions in the two computations above are equal if and only if condition (2) in the statement of the theorem holds. This finishes the proof. $\square$

3. Deformations

The primary goal of this section is to show that the twisted quantum Drinfeld Hecke algebras $\mathcal{H}_{q, \kappa, \alpha}$ are isomorphic to specializations of particular types of deformations of the twisted skew group algebras $S_q(V) \#_\alpha G$.

Let $\hbar$ denote an indeterminate. Recall that, for a $\mathbb{C}$-algebra A, a *deformation of A over $\mathbb{C}[\hbar]$* is an associative $\mathbb{C}[\hbar]$-algebra whose underlying vector space is $A[\hbar] = \mathbb{C}[\hbar] \otimes A$, and which reduces modulo $\hbar$ to the original algebra A. Thus the multiplication μ on $A[\hbar]$ is determined by

$$\mu(a, b) = \mu_0(a, b) + \mu_1(a, b)\hbar + \mu_2(a, b)\hbar^2 + \cdots$$

for all $a, b \in A$, where $\mu_0(a, b)$ is the product in A, the $\mu_i : A \times A \to A$ are $\mathbb{C}$-bilinear maps extended to be bilinear over $\mathbb{C}[\hbar]$, and for each pair (a, b) the sum above is finite. A consequence of associativity of μ is that μ_1 is a *Hochschild 2-cocycle*, that is,

$$(3.1) \qquad a\mu_1(b, c) + \mu_1(a, bc) = \mu_1(ab, c) + \mu_1(a, b)c$$

for all $a, b, c \in A$.

In order to see that the twisted quantum Drinfeld Hecke algebras $\mathcal{H}_{q, \kappa, \alpha}$ may be realized as specializations of deformations of $S_q(V) \#_\alpha G$, we define the algebra

$$\mathcal{H}_{q, \kappa, \alpha, \hbar} := (T(V) \#_\alpha G)[\hbar] / (v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j)\hbar \mid 1 \le i, j \le n).$$

Assigning $\hbar$ degree zero, each v_i degree one, and each t_g ($g \in G$) degree zero, we see that $\mathcal{H}_{q, \kappa, \alpha, \hbar}$ is a filtered algebra, and that $(S_q(V) \#_\alpha G)[\hbar]$ is a graded algebra. We call the algebra $\mathcal{H}_{q, \kappa, \alpha, \hbar}$ a *twisted quantum Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$* if gr $\mathcal{H}_{q, \kappa, \alpha, \hbar} \cong (S_q(V) \#_\alpha G)[\hbar]$, as graded algebras. Specializing a twisted quantum

Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$ to $\hbar = 1$ yields the twisted quantum Drinfeld Hecke algebra over $\mathbb{C}$, as defined earlier.

In the following theorem, by the *degree* of μ_i we mean its degree as a function between graded algebras.

Theorem 3.2. *Every twisted quantum Drinfeld Hecke algebra $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ over $\mathbb{C}[\hbar]$ is isomorphic to some deformation $\mu = \mu_0 + \mu_1\hbar + \mu_2\hbar^2 + \cdots$ of $S_q(V)\#_\alpha G$ over $\mathbb{C}[\hbar]$ with $\deg \mu_i = -2i$ for all $i \geq 1$.*

Proof. Suppose that $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ is a twisted quantum Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$. Consider the natural projection $T(V)\#_\alpha G \to S_q(V)\#_\alpha G$, and let $s : S_q(V)\#_\alpha G \to T(V)\#_\alpha G$ be the $\mathbb{C}$-linear section determined by the ordering $v_1, v_2, \ldots, v_n$ of the basis of V. For example, $s(v_2 v_1^2 t_g) = q_{21}^2 v_1^2 v_2 t_g$.

Extend s to a $\mathbb{C}[\hbar]$-linear map $\tilde{s} : (S_q(V)\#_\alpha G)[\hbar] \to (T(V)\#_\alpha G)[\hbar]$, and let p denote the natural projection from $(T(V)\#_\alpha G)[\hbar]$ to $\mathcal{H}_{q,\kappa,\alpha,\hbar}$. Since $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ is a twisted quantum Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$, the composition $f := p \circ \tilde{s}$ is an isomorphism of $\mathbb{C}[\hbar]$-modules.

Next, define a $\mathbb{C}[\hbar]$-bilinear multiplication μ on $(S_q(V)\#_\alpha G)[\hbar]$ by

$$\mu := f^{-1} \circ \text{mult} \circ (f \times f),$$

where mult is the multiplication map in $\mathcal{H}_{q,\kappa,\alpha,\hbar}$. Since μ is $\mathbb{C}[\hbar]$-bilinear, it must necessarily be a power series

$$\mu = \mu_0 + \mu_1\hbar + \mu_2\hbar + \cdots,$$

where the μ_i are $\mathbb{C}$-bilinear maps from $(S_q(V)\#_\alpha G) \times (S_q(V)\#_\alpha G)$ to $S_q(V)\#_\alpha G$. Note that, by definition of f, the map μ_0 is precisely the multiplication map in $S_q(V)\#_\alpha G$, and so μ is a deformation $S_q(V)\#_\alpha G$ over $\mathbb{C}[\hbar]$. By definition, the map f is an isomorphism between the $\mathbb{C}[\hbar]$-algebras $(S_q(V)\#_\alpha G[\hbar], \mu)$ and $\mathcal{H}_{q,\kappa,\alpha,\hbar}$, proving that $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ is isomorphic to a deformation of $S_q(V)\#_\alpha G$ over $\mathbb{C}[\hbar]$.

Finally we prove the degree condition on the μ_i. Given elements

$$a = v_1^{\beta_1} v_2^{\beta_2} \cdots v_n^{\beta_n} t_g \quad \text{and} \quad b = v_1^{\gamma_1} v_2^{\gamma_2} \cdots v_n^{\gamma_n} t_h$$

in $S_q(V)\#_\alpha G$, to find $\mu_1(a, b), \mu_2(a, b), \ldots$, we must put the product $f(a)f(b) \in \mathcal{H}_{q,\kappa,\alpha,\hbar}$ in the normal form by repeatedly applying the relations defining $\mathcal{H}_{q,\kappa,\alpha,\hbar}$. Induction on the degree $\sum_{k=1}^n \beta_k + \gamma_k$ of ab implies that $\deg \mu_i = -2i$ for all $i \geq 1$, as claimed. $\qquad\square$

Lemma 3.3. *The algebra $\mathcal{H}_{q,\kappa,\alpha}$ is a twisted quantum Drinfeld Hecke algebra over $\mathbb{C}$ if and only if $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ is a twisted quantum Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$.*

Proof. The proof given for $\mathcal{H}_{q,\kappa,\alpha}$ in Theorem 2.2 generalizes for $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ by extending scalars to $\mathbb{C}[\hbar]$. That is, $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ is a twisted quantum Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$ if and only if the two conditions in Theorem 2.2 hold. $\qquad\square$

Corollary 3.4. *Every twisted quantum Drinfeld Hecke algebra $\mathcal{H}_{q,\kappa,\alpha}$ is isomorphic to a specialization of a deformation $\mu = \mu_0 + \mu_1\hbar + \mu_2\hbar^2 + \cdots$ of $S_q(V)\#_\alpha G$ over $\mathbb{C}[\hbar]$ with $\deg \mu_i = -2i$ for all $i \geq 1$.*

A Hochschild 2-cocycle on $S_q(V)\#_\alpha G$ is said to be *constant* if it is of degree -2 as a function between graded algebras. In the next section, it is shown that such 2-cocycles correspond to certain *constant* polynomials, justifying the choice of terminology.

Proposition 3.5. *Let $\mathcal{H}_{q,\kappa,\alpha}$ be a twisted quantum Drinfeld Hecke algebra. The map $\kappa : V \times V \to \mathbb{C}^\alpha G$ is equal to the* quantum skew-symmetrization *of some constant Hochschild 2-cocycle μ_1 on $S_q(V)\#_\alpha G$, that is,*

$$\kappa(v_i, v_j) = \mu_1(v_i, v_j) - q_{ij}\mu_1(v_j, v_i)$$

for all i, j.

Proof. By Lemma 3.3, $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ is a twisted quantum Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$. By Theorem 3.2, associated to $\mathcal{H}_{q,\kappa,\alpha,\hbar}$ is a deformation $\mu = \mu_0 + \mu_1\hbar + \mu_2\hbar^2 + \cdots$ of $S_q(V)\#_\alpha G$ over $\mathbb{C}[\hbar]$ with $\deg \mu_i = -2i$ for all $i \geq 1$. Note that μ_1 is a constant Hochschild 2-cocycle on $S_q(V)\#_\alpha G$. We claim that κ is equal to the quantum skew-symmetrization of μ_1.

Let f be the map defined in the proof of Theorem 3.2. For any two monomials $a, b \in S_q(V)\#_\alpha G$, the value of $\mu_1(a, b)$ is determined by writing the product $f(a)f(b) \in \mathcal{H}_{q,\kappa,\alpha,\hbar}$ in the normal form by repeatedly applying the relations defining $\mathcal{H}_{q,\kappa,\alpha,\hbar}$. If $i \leq j$, the product $f(v_i)f(v_j) = v_iv_j$ is already in the desired form, so $\mu_1(v_i, v_j) = 0$. If $i > j$, we write $v_iv_j \longrightarrow q_{ij}v_jv_i + \kappa(v_i, v_j)\hbar$, and so $\kappa(v_i, v_j) = \mu_1(v_i, v_j)$. If $i \leq j$, we have $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i) = -q_{ij}\mu_1(v_j, v_i)$. Thus $\kappa(v_i, v_j) = \mu_1(v_i, v_j) - q_{ij}\mu_1(v_j, v_i)$ for all i, j. $\square$

The proof of the following theorem is a generalization of [Naidu and Witherspoon 2011, Theorem 2.2]; see also [Witherspoon 2007, Theorem 3.2].

Theorem 3.6. *Every deformation $\mu = \mu_0 + \mu_1\hbar + \mu_2\hbar^2 + \cdots$ of $S_q(V)\#_\alpha G$ over $\mathbb{C}[\hbar]$ with $\deg \mu_i = -2i$ for all $i \geq 1$ is isomorphic to some twisted quantum Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$.*

Proof. Suppose that $\mu = \mu_0 + \mu_1\hbar + \mu_2\hbar^2 + \cdots$ is a deformation of $S_q(V)\#_\alpha G$ over $\mathbb{C}[\hbar]$ with $\deg \mu_i = -2i$ for all $i \geq 1$. In what follows, we identity $T(V)\#_\alpha G$ with the free associative $\mathbb{C}$-algebra generated by the set $\{v_1, v_2, \ldots, v_n\} \cup \{t_g \mid g \in G\}$ subject to the relations $t_g v_i = {}^g v_i t_g$ and $t_g t_h = \alpha(g, h)t_{gh}$ for all $i \in \{1, 2, \ldots, n\}$ and all $g, h \in G$. Define a map $\phi : (T(V)\#_\alpha G)[\hbar] \to (S_q(V)\#_\alpha G)[\hbar]$ as follows. First, set $\phi(v_i) = v_i$ and $\phi(t_g) = t_g$ for all $i \in \{1, 2, \ldots, n\}$ and all $g \in G$. Since $\deg \mu_k = -2k$ for all $k \geq 1$, we have

$$\mu_k(\mathbb{C}^\alpha G, \mathbb{C}^\alpha G) = \mu_k(\mathbb{C}^\alpha G, V) = \mu_k(V, \mathbb{C}^\alpha G) = 0$$

for all $k \geq 1$. This implies that the relations $t_g v_i = {}^g v_i t_g$ and $t_g t_h = \alpha(g, h) t_{gh}$ hold in the algebra $((S_q(V) \#_\alpha G)[\hbar], \mu)$, and so we obtain a $\mathbb{C}$-algebra homomorphism on $T(V) \#_\alpha G$, which extends to a $\mathbb{C}[\hbar]$-algebra homomorphism ϕ from $(T(V) \#_\alpha G)[\hbar]$ to $(S_q(V) \#_\alpha G)[\hbar]$, where the algebra structure on the latter is given by μ.

Next, we show that ϕ is surjective. It is enough to show that each monomial $v_{i_1} \cdots v_{i_m} t_g$ is in the image of ϕ. The proof is by induction on the degree of the monomial. Suppose that all monomials of degree less than m are in the image of ϕ. In particular, $\phi(X) = v_{i_2} \cdots v_{i_m} g$ for some $X \in (T(V) \#_\alpha G)[\hbar]$. Then

$$
\begin{aligned}
\phi(v_{i_1} X) &= \mu(v_{i_1}, \phi(X)) \\
&= \mu(v_{i_1}, v_{i_2} \cdots v_{i_m} t_g) \\
&= v_{i_1} \cdots v_{i_m} t_g + \mu_1(v_{i_1}, v_{i_2} \cdots v_{i_m} t_g) \hbar + \mu_2(v_{i_1}, v_{i_2} \cdots v_{i_m} t_g) \hbar^2 + \cdots .
\end{aligned}
$$

Since $\deg(\mu_k) = -2k$, by the induction hypothesis, each $\mu_k(v_{i_1}, v_{i_2} \cdots v_{i_m} t_g)$ is in the image of ϕ. Therefore, $v_{i_1} \cdots v_{i_m} t_g$ is in the image of ϕ, and it follows that ϕ is surjective.

Finally, we determine the kernel of ϕ. Since $\deg(\mu_1) = -2$, we can define a bilinear map $\kappa : V \times V \to \mathbb{C}^\alpha G$ by setting $\kappa(v_i, v_j) := \mu_1(v_i, v_j) - q_{ij} \mu_1(v_j, v_i)$ for all i, j. Let I denote the ideal in $(T(V) \#_\alpha G)[\hbar]$ generated by the elements

$$
v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j) \hbar.
$$

Since $\mu_k(v_i, v_j) = 0$ for all $k \geq 2$, we have

$$
\phi(v_i v_j) = \mu(v_i, v_j) = v_i v_j + \mu_1(v_i, v_j) \hbar,
$$
$$
\phi(v_j v_i) = \mu(v_j, v_i) = v_j v_i + \mu_1(v_j, v_i) \hbar,
$$

and so I is contained in the kernel of ϕ. The form of the relations and surjectivity of ϕ imply that the kernel of ϕ is precisely I, and it follows that the deformation $((S_q(V) \#_\alpha G)[\hbar], \mu)$ is isomorphic to the twisted quantum Drinfeld Hecke algebra $\mathcal{H}_{q, \kappa, \alpha, \hbar}$ over $\mathbb{C}[\hbar]$. $\qquad\square$

4. Computing $HH^2(S_q(V) \#_\alpha G)$

Let A be an algebra on which the finite group G acts by automorphisms, and let α be a 2-cocycle on G. This section is concerned with the computation of the Hochschild cohomology $HH^\bullet(A \#_\alpha G)$ of the twisted skew group algebra $A \#_\alpha G$. We are particularly interested in degree two cohomology in the case when A is the quantum symmetric algebra $S_q(V)$. The results of this section are used in the sections that follow.

Recall that the Hochschild cohomology of an algebra R is

$$
HH^\bullet(R) := \text{Ext}^\bullet_{R^e}(R, R),
$$

where the enveloping algebra $R^e := R \otimes R^{\mathrm{op}}$ acts on R by left and right multiplication. When R is a twisted skew group algebra $A\#_\alpha G$ in a characteristic not dividing the order of the finite group G, by [Ştefan 1995, Corollary 3.4], there is an action of G on $\mathrm{HH}^\bullet(A, A\#_\alpha G) = \mathrm{Ext}^\bullet_{A^e}(A, A\#_\alpha G)$ for which $\mathrm{HH}^\bullet(A\#_\alpha G)$ is isomorphic to $\mathrm{HH}^\bullet(A, A\#_\alpha G)^G$, the space of elements of $\mathrm{HH}^\bullet(A, A\#_\alpha G)$ that are invariant under the action of G. Thus, one can compute $\mathrm{HH}^\bullet(A\#_\alpha G)$ by first computing $\mathrm{HH}^\bullet(A, A\#_\alpha G)$ and then determining the space of G-invariant elements. When A is the quantum symmetric algebra $S_q(V)$, we compute $\mathrm{HH}^\bullet(S_q(V), S_q(V)\#_\alpha G)$ using the quantum Koszul resolution, recalled below.

The *quantum exterior algebra* $\bigwedge_q(V)$ associated to the tuple $q = (q_{ij})$ is

$$\bigwedge\nolimits_q(V) := \mathbb{C}\langle v_1, \ldots, v_n \mid v_i v_j = -q_{ij} v_j v_i \text{ for all } 1 \le i, j \le n \rangle.$$

Since we are working in characteristic 0, the defining relations imply in particular that $v_i^2 = 0$ for each v_i in $\bigwedge_q(V)$. This algebra has a basis given by all $v_{i_1} \cdots v_{i_m}$ $(0 \le m \le n, 1 \le i_1 < \cdots < i_m \le n)$; we write such a basis element as $v_{i_1} \wedge \cdots \wedge v_{i_m}$ by analogy with the ordinary exterior algebra.

By [Wambst 1993, Proposition 4.1(c)], the following is a free $S_q(V)^e$-resolution of $S_q(V)$:

$$(4.1) \quad \cdots \to S_q(V)^e \otimes \bigwedge\nolimits_q^2(V) \xrightarrow{d_2} S_q(V)^e \otimes \bigwedge\nolimits_q^1(V) \xrightarrow{d_1} S_q(V)^e \xrightarrow{\mathrm{mult}} S_q(V) \to 0,$$

that is, for $1 \le m \le n$, the degree m term is $S_q(V)^e \otimes \bigwedge_q^m(V)$; the differential d_m is defined by

$$d_m(1^{\otimes 2} \otimes v_{j_1} \wedge \cdots \wedge v_{j_m})$$
$$= \sum_{i=1}^m (-1)^{i+1} \left[\left(\prod_{s=1}^i q_{j_s, j_i} \right) v_{j_i} \otimes 1 - \left(\prod_{s=i}^m q_{j_i, j_s} \right) \otimes v_{j_i} \right] \otimes v_{j_1} \wedge \cdots \wedge \hat{v}_{j_i} \wedge \cdots \wedge v_{j_m}$$

whenever $1 \le j_1 < \cdots < j_m \le n$, and mult denotes the multiplication map. The complex (4.1) is a quantum version of the usual Koszul resolution for a polynomial ring.

Suppose that the action of G on V induces an action on $\bigwedge_q(V)$. Thus, there is an action of G on the quantum Koszul complex (4.1), that is, an action of G on each $S_q(V)^e \otimes \bigwedge_q^i(V)$ that commutes with the differentials.

Assume that $\mathrm{HH}^\bullet(S_q(V)\#_\alpha G)$ has been computed using the quantum Koszul resolution. So, elements of $\mathrm{HH}^\bullet(S_q(V)\#_\alpha G)$ are given as G-invariant elements of $\mathrm{HH}^\bullet(S_q(V), S_q(V)\#_\alpha G)$. For our purposes, we need to find representatives for elements in $\mathrm{HH}^2(S_q(V)\#_\alpha G)$ that are given as maps from $(S_q(V)\#_\alpha G) \otimes (S_q(V)\#_\alpha G)$ to $S_q(V)\#_\alpha G$ satisfying the 2-cocycle condition (3.1). To this end, we consider

chain maps between the quantum Koszul resolution (4.1) and the bar resolution of A:

$$\cdots \longrightarrow S_{\boldsymbol{q}}(V)^{\otimes 4} \xrightarrow{\delta_2} S_{\boldsymbol{q}}(V)^{\otimes 3} \xrightarrow{\delta_1} S_{\boldsymbol{q}}(V)^e \xrightarrow{\text{mult}} S_{\boldsymbol{q}}(V) \longrightarrow 0$$

$$\Psi_2 \Big\updownarrow \Phi_2 \qquad \qquad \Psi_1 \Big\updownarrow \Phi_1 \qquad = \Big\uparrow \qquad = \Big\uparrow$$

$$\cdots \longrightarrow S_{\boldsymbol{q}}(V)^e \otimes \textstyle\bigwedge_{\boldsymbol{q}}^2 V \xrightarrow{d_2} S_{\boldsymbol{q}}(V)^e \otimes \textstyle\bigwedge_{\boldsymbol{q}}^1 V \xrightarrow{d_1} S_{\boldsymbol{q}}(V)^e \xrightarrow{\text{mult}} S_{\boldsymbol{q}}(V) \longrightarrow 0.$$

Here the differentials δ_i in the bar resolution are defined as

$$\delta_i(a_0 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^{i} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}$$

for all $a_0, \ldots, a_{i+1} \in A$. We will only need to know the values of Ψ_2 on elements of the form $1 \otimes v_i \otimes v_j \otimes 1$, and these can be chosen to be

$$(4.2) \qquad \Psi_2(1 \otimes v_i \otimes v_j \otimes 1) = \begin{cases} 1 \otimes 1 \otimes v_i \wedge v_j & \text{if } i < j, \\ 0 & \text{if } i \geq j. \end{cases}$$

Chain maps Φ_i are defined in [Naidu et al. 2011], and more generally in [Wambst 1993], that embed the quantum Koszul resolution as a subcomplex of the bar resolution. We will only need Φ_2, and this is defined by

$$(4.3) \qquad \Phi_m(1 \otimes 1 \otimes v_i \wedge \wedge v_j) = 1 \otimes v_i \otimes v_j \otimes 1 - q_{ij} \otimes v_j \otimes v_i \otimes 1$$

for all $1 \leq i, j \leq n$.

We define the Reynold's operator, or averaging map, which ensures G-invariance of the image, compensating for the possibility that Ψ_2 may not preserve the action of G:

$$\mathscr{R}_2 : \text{Hom}_{\mathbb{C}}(S_{\boldsymbol{q}}(V)^{\otimes 2}, S_{\boldsymbol{q}}(V)\#_\alpha G) \to \text{Hom}_{\mathbb{C}}(S_{\boldsymbol{q}}(V)^{\otimes 2}, S_{\boldsymbol{q}}(V)\#_\alpha G)^G,$$

$$\mathscr{R}_2(\gamma) := \frac{1}{|G|} \sum_{g \in G} {}^g\gamma.$$

The following map tells how to extend a function defined on $S_{\boldsymbol{q}}(V)^{\otimes 2}$ to a function defined on $(S_{\boldsymbol{q}}(V)\#_\alpha G)^{\otimes 2}$ [Căldăraru et al. 2004]:

$$\Theta_2^* : \text{Hom}_{\mathbb{C}}(S_{\boldsymbol{q}}(V)^{\otimes 2}, S_{\boldsymbol{q}}(V)\#_\alpha G)^G \to \text{Hom}_{\mathbb{C}}((S_{\boldsymbol{q}}(V)\#_\alpha G)^{\otimes 2}, S_{\boldsymbol{q}}(V)\#_\alpha G),$$

$$\Theta_2^*(\kappa)(a_1 t_{g_1} \otimes a_2 t_{g_2}) := \alpha(g_1, g_2)\kappa(a_1 \otimes {}^{g_1}a_2)t_{g_1 g_2}.$$

The theorem below is from [Căldăraru et al. 2004]; see also [Shepler and Witherspoon 2012].

Theorem 4.4. *Suppose that the action of G on V extends to an action on $\bigwedge_{\boldsymbol{q}}(V)$ by automorphisms. The map*

$$\Theta_2^* \mathscr{R}_2 \Psi_2^* : \text{Hom}_{\mathbb{C}}\big(\textstyle\bigwedge_{\boldsymbol{q}}^2(V), S_{\boldsymbol{q}}(V)\#_\alpha G\big) \to \text{Hom}_{\mathbb{C}}(S_{\boldsymbol{q}}(V)^{\otimes 2}, S_{\boldsymbol{q}}(V)\#_\alpha G)$$

induces an isomorphism

$$\mathrm{HH}^2(S_q(V), S_q(V)\#_\alpha G)^G \xrightarrow{\sim} \mathrm{HH}^2(S_q(V)\#_\alpha G).$$

Moreover, $\Theta_2^* \mathscr{R}_2 \Psi_2^*$ *maps* $\mathrm{HH}^2(S_q(V), S_q(V)\#_\alpha G)$ *onto* $\mathrm{HH}^2(S_q(V)\#_\alpha G)$.

Next, we will introduce some notation and give some formulas that are useful in the sections that follow. For each $g \in G$, the space $S_q(V)t_g \subseteq S_q(V)\#_\alpha G$ is a (left) $S_q(V)^e$-module via the action

$$(a \otimes b) \cdot (ct_g) := act_g b = ac(^g b)t_g$$

for all $a, b, c \in S_q(V)$ and all $g \in G$. Note that $\mathrm{HH}^2(S_q(V), S_q(V)\#_\alpha G)$ is isomorphic to the direct sum $\bigoplus_{g \in G} \mathrm{HH}^2(S_q(V), S_q(V)t_g)$.

We wish to express the formula for the differentials d_m in the quantum Koszul resolution (4.1) in a more convenient form. To this end, let $\mathbb{N}^n$ denote the set of all n-tuples of elements from $\mathbb{N}$. The *length* of $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, denoted $|\gamma|$, is the sum $\sum_{i=1}^n \gamma_i$. For each $\gamma \in \mathbb{N}^n$, define $v^\gamma := v_1^{\gamma_1} v_2^{\gamma_2} \cdots v_n^{\gamma_n}$. For each $i \in \{1, \ldots, n\}$, define $[i] \in \mathbb{N}^n$ by $[i]_j = \delta_{i,j}$, for all $j \in \{1, \ldots, n\}$. For each $\beta = (\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$, let $v^{\wedge \beta}$ denote the vector $v_{j_1} \wedge \cdots \wedge v_{j_m} \in \bigwedge_q^m(V)$ determined by the conditions $m = |\beta|$, $\beta_{j_k} = 1$ for all $k \in \{1, \ldots, m\}$, and $j_1 < \ldots < j_m$. For each $\beta \in \{0, 1\}^n$ with $|\beta| = m$, we have

$$d_m(1^{\otimes 2} \otimes v^{\wedge \beta}) = \sum_{i=1}^n \delta_{\beta_i, 1}(-1)^{\sum_{s=1}^{i-1} \beta_s}\left[\left(\prod_{s=1}^i q_{s,i}^{\beta_s}\right)v_i \otimes 1 - \left(\prod_{s=i}^n q_{i,s}^{\beta_s}\right) \otimes v_i\right] \otimes v^{\wedge(\beta - [i])}.$$

Removing the term $S_q(V)$ from the quantum Koszul resolution (4.1), applying the functor $\mathrm{Hom}_{S_q(V)^e}(\,\cdot\,, S_q(V)t_g)$, and then identifying

$$\mathrm{Hom}_{S_q(V)^e}\left(S_q(V)^e \otimes \textstyle\bigwedge_q^\bullet(V), S_q(V)t_g\right) \cong \mathrm{Hom}_{\mathbb{C}}\left(\textstyle\bigwedge_q^\bullet(V), S_q(V)t_g\right)$$

with $S_q(V)t_g \otimes \bigwedge_{q^{-1}}^\bullet(V^*)$, we obtain the complex

$$(4.5) \quad 0 \to S_q(V)t_g \xrightarrow{d_1^*} S_q(V)t_g \otimes \textstyle\bigwedge_{q^{-1}}^1(V^*) \xrightarrow{d_2^*} S_q(V)t_g \otimes \textstyle\bigwedge_{q^{-1}}^2(V^*) \to \cdots.$$

For all $a \in S_q(V)$ and all $\beta \in \{0, 1\}^n$ with $|\beta| = m - 1$, the differential d_m^* sends the element $at_g \otimes (v^*)^{\wedge \beta}$ to

$$(4.6) \quad \sum_{i=1}^n \delta_{\beta_i, 0}(-1)^{\sum_{s=1}^i \beta_s}\left[\left(\left(\prod_{s=1}^i q_{s,i}^{\beta_s}\right)v_i a - \left(\prod_{s=i}^n q_{i,s}^{\beta_s}\right)a(^g v_i)\right)t_g\right] \otimes (v^*)^{\wedge(\beta + [i])}.$$

For later use, we record the following formula. Let $\eta \in (S_q(V)\#_\alpha G) \otimes \bigwedge_{q^{-1}}^2(V^*)$. Then

$$(4.7) \quad [\Theta_2^* \mathscr{R}_2 \Psi_2^*(\eta)](v_i \otimes v_j) = \frac{1}{|G|} \sum_{g \in G}{}^g\big(\eta(\Psi_2(1 \otimes {}^{g^{-1}}v_i \otimes {}^{g^{-1}}v_j \otimes 1))\big).$$

The elements of $\left((S_q(V)\#_\alpha G)\otimes\bigwedge^2_{q^{-1}}(V^*)\right)^G$ that correspond to *constant* Hochschild 2-cocycles, i.e., those of degree -2 as maps from $(S_q(V)\#_\alpha G)\otimes(S_q(V)\#_\alpha G)$ to $S_q(V)\#_\alpha G$, are precisely those in $\left(\mathbb{C}^\alpha G\otimes\bigwedge^2_{q^{-1}}(V^*)\right)^G$, due to the form of the chain map Ψ_2. Note that the intersection of the image of d_2^* with $\mathbb{C}^\alpha G\otimes\bigwedge^2_{q^{-1}}(V^*)$ is 0. Applying our earlier formula, letting $\beta=[j]+[k]$,

$$(4.8)\quad d_3^*(t_g\otimes v_j^*\wedge v_k^*)$$
$$=\sum_{i\notin\{j,k\}}(-1)^{\sum_{s=1}^i\beta_s}\left[\left(\left(\prod_{s=1}^i q_{s,i}^{\beta_s}\right)v_i-\left(\prod_{s=i}^n q_{i,s}^{\beta_s}\right)^g v_i\right)t_g\right]\otimes(v^*)^{\wedge(\beta+[i])}.$$

5. Constant Hochschild 2-cocycles

In this section, we establish the following bijection:

$$\left\{\begin{array}{c}\text{constant Hochschild}\\\text{2-cocycles on }S_q(V)\#_\alpha G\end{array}\right\}\longleftrightarrow\left\{\begin{array}{c}\text{twisted quantum Drinfeld}\\\text{Hecke algebras }\mathscr{H}_{q,\kappa,\alpha}\end{array}\right\}.$$

We also show that every constant Hochschild 2-cocycles on $S_q(V)\#_\alpha G$ lifts to a deformation of $S_q(V)\#_\alpha G$.

We use the following two lemmas shortly.

Lemma 5.1. *The action of G on V extends to an action on $\bigwedge_q(V)$ by automorphisms if, and only if, for all $g\in G$, $i\neq j$, and $k<l$,*

$$(1-q_{ij}q_{lk})g_k^i g_l^j+(q_{ij}-q_{lk})g_l^i g_k^j=0.$$

Proof. See [Naidu and Witherspoon 2011, Lemma 4.2]. ☐

Lemma 5.2. *Suppose that the action of G on V extends to an action, by automorphisms, on $S_q(V)$ and on $\bigwedge_q(V)$. Then, for all $g\in G$ and all i, j, k, l ($i<j$, $k<l$), if $g_l^i g_k^j\neq 0$, then $q_{lk}=q_{ij}$, and if $g_k^i g_l^j\neq 0$, then $q_{lk}=q_{ij}^{-1}$.*

Proof. See [Naidu and Witherspoon 2011, Lemma 4.3]. ☐

Proposition 3.5 showed that every twisted quantum Drinfeld Hecke algebra arises from the quantum skew-symmetrization of a constant Hochschild 2-cocycle. The following theorem shows that the converse is also true. The proof of the following theorem involves the maps Θ_2^*, $\mathscr{R}_2$, Ψ_2^*, and d_3^* defined in Section 4.

Theorem 5.3. *Suppose that the action of G on V extends to an action, by automorphisms, on $S_q(V)$ and on $\bigwedge_q(V)$. Let α be a normalized 2-cocycle on G, let μ_1 be a constant Hochschild 2-cocycle on $S_q(V)\#_\alpha G$, and let $\kappa:V\times V\to\mathbb{C}^\alpha G$ be the quantum skew-symmetrization of μ_1. Then $\mathscr{H}_{q,\kappa,\alpha}$ is a twisted quantum Drinfeld Hecke algebra.*

Proof. We show that the map κ satisfies the conditions of [Theorem 2.2](). Let η be a G-invariant element of

$$\mathrm{Hom}_{\mathbb{C}}\big(\textstyle\bigwedge^2_{\boldsymbol{q}}(V),\, S_{\boldsymbol{q}}(V)\#_{\alpha}G\big) \cong (S_{\boldsymbol{q}}(V)\#_{\alpha}G)\otimes \textstyle\bigwedge^2_{\boldsymbol{q}^{-1}}(V^*)$$

such that $[\Theta_2^*\mathscr{R}_2\Psi_2^*](\eta)=\mu_1$. Since μ_1 is a *constant* Hochschild 2-cocycle, the image of η as a map from $\bigwedge^2_{\boldsymbol{q}}(V)$ to $S_{\boldsymbol{q}}(V)\#_{\alpha}G$ is contained in $\mathbb{C}^{\alpha}G$, or, equivalently, η belongs to $(\mathbb{C}^{\alpha}G)\otimes\bigwedge^2_{\boldsymbol{q}^{-1}}(V^*)$.

For all $1\le k,l\le n$, we have $[\Psi_2^*(\eta)](v_k\otimes v_l-q_{kl}v_l\otimes v_k)=\eta(v_k\wedge v_l)$. This equality and the G-invariance of η imply that $\kappa(v_i,v_j)=\eta(v_i\wedge v_j)$ for all $1\le i,j\le n$. Indeed, we have

$$
\begin{aligned}
\kappa(v_i,v_j) &= [\Theta_2^*\mathscr{R}_2\Psi_2^*(\eta)](v_i\otimes v_j-q_{ij}v_j\otimes v_i)\\
&= \frac{1}{|G|}\sum_{g\in G}\Theta_2^*({}^g(\Psi_2^*(\eta)))(v_i\otimes v_j-q_{ij}v_j\otimes v_i)\\
&= \frac{1}{|G|}\sum_{g\in G}{}^g((\Psi_2^*(\eta))^{g^{-1}}(v_i\otimes v_j-q_{ij}v_j\otimes v_i))\\
&= \frac{1}{|G|}\sum_{g\in G}{}^g\bigg((\Psi_2^*(\eta))\bigg(\sum_{k,l}(g^{-1})^i_k(g^{-1})^j_l(v_k\otimes v_l-q_{ij}v_l\otimes v_k)\bigg)\bigg)\\
&= \frac{1}{|G|}\sum_{g\in G}{}^g\bigg(\sum_{k,l}(g^{-1})^i_k(g^{-1})^j_l(\Psi_2^*(\eta))(v_k\otimes v_l-q_{ij}v_l\otimes v_k)\bigg)\\
&= \frac{1}{|G|}\sum_{g\in G}{}^g\bigg(\sum_{k,l}(g^{-1})^i_k(g^{-1})^j_l\,\eta(v_k\wedge v_l)\bigg)\\
&= \frac{1}{|G|}\sum_{g\in G}{}^g(\eta(^{g^{-1}}(v_i\wedge v_j)))\\
&= \frac{1}{|G|}\sum_{g\in G}({}^g\eta)(v_i\wedge v_j)\\
&= \eta(v_i\wedge v_j).
\end{aligned}
$$

Next, write

$$\eta=\sum_{g\in G}\sum_{1\le r<s\le n}\eta^g_{rs}t_g\otimes v_r^*\wedge v_s^*\in\mathbb{C}^{\alpha}G\otimes\textstyle\bigwedge^2_{\boldsymbol{q}^{-1}}(V^*)\subseteq(S_{\boldsymbol{q}}(V)\#_{\alpha}G)\otimes\textstyle\bigwedge^2_{\boldsymbol{q}^{-1}}(V^*).$$

The calculation above implies that $\kappa_g(v_i,v_j)=\eta^g_{ij}$ for all $i<j$ and all $g\in G$. Since η is a Hochschild 2-cocycle, we have $d_3^*(\eta)=0$. Using [(4.8)](), we see that, for all $1\le i<j<k\le n$, we must have

$$\sum_{g\in G}(\eta^g_{jk}v_i t_g-\eta^g_{jk}q_{ij}q_{ik}{}^g v_i t_g-\eta^g_{ik}q_{ij}v_j t_g+\eta^g_{ik}q_{jk}{}^g v_j t_g+\eta^g_{ij}q_{ik}q_{jk}v_k t_g-\eta^g_{ij}{}^g v_k t_g)=0.$$

Equivalently,

$$-\eta_{jk}^g(q_{ij}q_{ik}{}^g v_i - v_i) - \eta_{ik}^g(q_{ij}v_j - q_{jk}{}^g v_j) - \eta_{ij}^g({}^g v_k - q_{ik}q_{jk}v_k) = 0$$

for all $1 \leq i < j < k \leq n$ and all $g \in G$.

Multiplying both sides by $q_{ji}q_{ki}q_{kj}$ yields

$$-q_{jk}\eta_{jk}^g({}^g v_i - q_{ji}q_{ki}v_i) - q_{ki}\eta_{ik}^g(q_{kj}v_j - q_{ji}{}^g v_j) - q_{ji}\eta_{ij}^g(q_{kj}q_{ki}{}^g v_k - v_k) = 0.$$

Now substituting $\kappa_g(v_k, v_j)$, $\kappa_g(v_k, v_i)$, and $\kappa_g(v_j, v_i)$ for $-q_{jk}\eta_{jk}^g$, $-q_{ki}\eta_{ik}^g$, and $-q_{ji}\eta_{ij}^g$, respectively, we obtain

$$\kappa_g(v_k, v_j)({}^g v_i - q_{ji}q_{ki}v_i) + \kappa_g(v_k, v_i)(q_{kj}v_j - q_{ji}{}^g v_j) + \kappa_g(v_j, v_i)(q_{kj}q_{ki}{}^g v_k - v_k) = 0,$$

which is condition (2) of Theorem 2.2.

Next, we show that κ also satisfies condition (1) of Theorem 2.2. Since η is G-invariant, we have $\eta({}^h v_i \wedge {}^h v_j) = {}^h(\eta(v_i \wedge v_j))$ for all i, j and all $h \in G$. We have

$$\begin{aligned}
\eta({}^h v_i \wedge {}^h v_j) &= \sum_{k,l} h_k^i h_l^j \eta(v_k \wedge v_l) \\
&= \sum_{k<l} h_k^i h_l^j \eta(v_k \wedge v_l) - \sum_{k<l} q_{lk} h_l^i h_k^j \eta(v_k \wedge v_l) \\
&= \sum_{k<l,\ g\in G} (h_k^i h_l^j - q_{lk} h_l^i h_k^j)\eta_{kl}^g t_g,
\end{aligned}$$

and for all $i < j$, we have

$$\begin{aligned}
{}^h(\eta(v_i \wedge v_j)) &= {}^h\left(\sum_{g\in G} \eta_{ij}^g t_g\right) \\
&= \sum_{g\in G} \eta_{ij}^g t_h t_g (t_h)^{-1} \\
&= \sum_{g\in G} \frac{\alpha(h, g)\alpha(hg, h^{-1})}{\alpha(h^{-1}, h)} \eta_{ij}^g t_{hgh^{-1}} \\
&= \sum_{g\in G} \frac{\alpha(h, g)}{\alpha(hgh^{-1}, h)} \eta_{ij}^g t_{hgh^{-1}}.
\end{aligned}$$

Equating the coefficients of $t_{hgh^{-1}}$, we find that, for all $i < j$ and all $h, g \in G$, we have

$$\frac{\alpha(h, g)}{\alpha(hgh^{-1}, h)} \eta_{ij}^g = \sum_{k<l} (h_k^i h_l^j - q_{lk} h_l^i h_k^j)\eta_{kl}^{hgh^{-1}}.$$

Substituting $\kappa_g(v_i, v_j)$ and $\kappa_{hgh^{-1}}(v_k, v_l)$ for η_{ij}^g and $\eta_{kl}^{hgh^{-1}}$, respectively, and then multiplying both sides by $-q_{ji}$ yields

$$\frac{\alpha(h, g)}{\alpha(hgh^{-1}, h)}\kappa_g(v_j, v_i) = \sum_{k<l}(q_{ji}q_{lk}h_l^i h_k^j - q_{ji}h_k^i h_l^j)\kappa_{hgh^{-1}}(v_k, v_l).$$

Substituting $-q_{kl}\kappa_{hgh^{-1}}(v_l, v_k)$ for $\kappa_{hgh^{-1}}(v_k, v_l)$, and then using Lemma 5.2, we obtain

$$\frac{\alpha(h, g)}{\alpha(hgh^{-1}, h)}\kappa_g(v_j, v_i) = \sum_{k<l}\det_{ijkl}(h)\kappa_{hgh^{-1}}(v_l, v_k),$$

which is condition (1) of Theorem 2.2. □

The proof of the following theorem involves the map Φ_2^* defined in Section 4.

Theorem 5.4. *Let α be a normalized 2-cocycle on G. Suppose that the action of G on V extends to an action, by automorphisms, on $S_{\boldsymbol{q}}(V)$ and on $\bigwedge_{\boldsymbol{q}}(V)$. The assignment*

$$\mu_1 \mapsto \mathscr{H}_{\boldsymbol{q},\kappa,\alpha}$$

where κ is the quantum skew-symmetrization of μ_1 is a bijection from the space of equivalence classes of constant Hochschild 2-cocycles on $S_{\boldsymbol{q}}(V)\#_\alpha G$ to the space of twisted quantum Drinfeld Hecke algebras associated to the quadruple $(G, V, \boldsymbol{q}, \alpha)$.

Proof. Proposition 3.5 showed that the assignment specified in the statement of the theorem is surjective. To see that the assignment is also injective, let μ_1 and μ_1' be constant Hochschild 2-cocycles on $S_{\boldsymbol{q}}(V)\#_\alpha G$ such that their quantum skew-symmetrizations are equal. We have

$$\begin{aligned}
[\Phi_2^*(\mu_1)](1 \otimes 1 \otimes v_i \otimes v_j) &= \mu_1(v_i, v_j) - q_{ij}\mu_1(v_j, v_i)\\
&= \mu_1'(v_i, v_j) - q_{ij}\mu_1'(v_j, v_i)\\
&= [\Phi_2^*(\mu_1')](1 \otimes 1 \otimes v_i \otimes v_j),
\end{aligned}$$

so $\Phi_2^*(\mu_1) = \Phi_2^*(\mu_1')$, and it follows that μ_1 and μ_1' are cohomologous. □

Theorem 5.5. *Let α be a normalized 2-cocycle on G. Suppose that the action of G on V extends to an action, by automorphisms, on $S_{\boldsymbol{q}}(V)$ and on $\bigwedge_{\boldsymbol{q}}(V)$. Each constant Hochschild 2-cocycle on $S_{\boldsymbol{q}}(V)\#_\alpha G$ lifts to a deformation of $S_{\boldsymbol{q}}(V)\#_\alpha G$ over $\mathbb{C}[\hbar]$.*

Proof. Let μ_1' be a constant Hochschild 2-cocycle on $S_{\boldsymbol{q}}(V)\#_\alpha G$. By Theorem 5.3, μ_1' gives rise to a twisted quantum Drinfeld Hecke algebra $\mathscr{H}_{\boldsymbol{q},\kappa,\alpha}$, where κ is the quantum skew-symmetrization of μ_1'. By Lemma 3.3, $\mathscr{H}_{\boldsymbol{q},\kappa,\alpha,\hbar}$ is a twisted quantum Drinfeld Hecke algebra over $\mathbb{C}[\hbar]$. By Theorem 3.2, associated to $\mathscr{H}_{\boldsymbol{q},\kappa,\alpha,\hbar}$ is a deformation $\mu = \mu_0 + \mu_1\hbar + \mu_2\hbar^2 + \cdots$ of $S_{\boldsymbol{q}}(V)\#_\alpha G$. The proof of Proposition 3.5 shows that κ is the quantum skew-symmetrization of μ_1, and it follows from Theorem 5.4 that μ_1' is cohomologous to μ_1. □

6. Diagonal actions

As before, let G be a finite group acting linearly on a vector space V with basis $v_1, \ldots, v_n$. Assume that $v_1, \ldots, v_n$ are common eigenvectors for G. In this case, the Hochschild cohomology $\mathrm{HH}^{\bullet}(S_q(V), S_q(V)\#G)$ was computed in [Naidu et al. 2011]. Let α be a normalized 2-cocycle on G. In this section, we use results from [Naidu et al. 2011] to give an explicit description of the subspace of $\mathrm{HH}^2(S_q(V)\#_\alpha G)$ consisting of constant Hochschild 2-cocycles. As a consequence, we obtain a classification of twisted quantum Drinfeld Hecke algebras associated to the quadruple (G, V, q, α).

Let $\lambda_{g,i} \in \mathbb{C}$ be the scalars for which $^g v_i = \lambda_{g,i} v_i$ for all $g \in G$ and all $i \in \{1, \ldots, n\}$. For each $g \in G$, define

$$(6.1) \quad C_g := \left\{ \gamma \in (\mathbb{N} \cup \{-1\})^n \mid \text{ for each } i \in \{1, \ldots, n\}, \prod_{s=1}^n q_{is}^{\gamma_s} = \lambda_{g,i} \text{ or } \gamma_i = -1 \right\}.$$

Theorem 6.2 [Naidu et al. 2011]. *If G acts diagonally on V, then*

$$\mathrm{HH}^{\bullet}(S_q(V), S_q(V)\#G)$$

is isomorphic to the graded vector subspace of $(S_q(V)\#_G) \otimes \bigwedge_{q^{-1}}(V^*)$ given by

$$\mathrm{HH}^m(S_q(V), S_q(V)\#G) \cong \bigoplus_{g \in G} \bigoplus_{\substack{\beta \in \{0,1\}^n \\ |\beta|=m}} \bigoplus_{\substack{\tau \in \mathbb{N}^n \\ \tau - \beta \in C_g}} \mathrm{span}_{\mathbb{C}}\{(v^\tau t_g) \otimes (v^*)^{\wedge\beta}\},$$

for all $m \in \mathbb{N}$.

Corollary 6.3. *The constant Hochschild 2-cocycles representing elements in the cohomology $\mathrm{HH}^2(S_q(V), S_q(V)\#G)$ form a vector space having as a basis the set of all*

$$t_g \otimes v_r^* \wedge v_s^*,$$

where $r < s$ and $g \in G$ satisfy $q_{rr'} q_{sr'} = \lambda_{g,r'}$ for all $r' \notin \{r, s\}$.

Note that the $S_q(V)$-bimodule structure of $S_q(V)\#_\alpha G$ does not depend on the 2-cocycle α, and so $\mathrm{HH}^2(S_q(V), S_q(V)\#_\alpha G) = \mathrm{HH}^2(S_q(V), S_q(V)\#G)$.

Let $\mathcal{R}$ denote a complete set of representatives of conjugacy classes in G, let $C_G(a)$ denote the centralizer of a in G, and let $[G/C_G(a)]$ denote a complete set of representatives of left cosets of $C_G(a)$ in G. In the following theorem, the notation $\delta_{i,j}$ is the Kronecker delta.

Theorem 6.4. *The constant Hochschild 2-cocycles representing elements in the cohomology $\mathrm{HH}^2(S_q(V)\#_\alpha G)$ form a vector space having as a basis the set of all*

$$\sum_{g \in [G/C_G(a)]} \frac{\alpha(g, a)}{\alpha(gag^{-1}, g)} \lambda_{g,r}^{-1} \lambda_{g,s}^{-1} t_{gag^{-1}} \otimes v_r^* \wedge v_s^*,$$

where $r < s$ and $a \in \mathcal{R}$ satisfy $q_{rr'}q_{sr'} = \lambda_{a,r'}$ for all $r' \notin \{r, s\}$, and $\lambda_{h,r}\lambda_{h,s} = \alpha(h, a)/\alpha(a, h)$ for all $h \in C_G(a)$.

Proof. We show that the space of G-invariant elements of the vector space given in Corollary 6.3 is precisely the vector space stated in the theorem. The stated result then follows from Theorem 4.4.

First, we show that the scalar $(\alpha(g, a)/\alpha(gag^{-1}, g))\lambda_{g,r}^{-1}\lambda_{g,s}^{-1}$ is independent of choice of representative g of a coset of $C_G(a)$ under the assumption that $\lambda_{h,r}\lambda_{h,s} = \alpha(h, a)/\alpha(a, h)$ for all $h \in C_G(a)$. Suppose that $gag^{-1} = g'ag'^{-1}$. Then $g' = gh$ for some $h \in C_G(a)$, and we have

$$\frac{\alpha(g', a)}{\alpha(g'ag'^{-1}, g')}\lambda_{g',r}^{-1}\lambda_{g',s}^{-1} = \frac{\alpha(gh, a)}{\alpha(gag^{-1}, gh)}\lambda_{g,r}^{-1}\lambda_{g,s}^{-1}\lambda_{h,r}^{-1}\lambda_{h,s}^{-1}.$$

Substituting $\lambda_{h,r}\lambda_{h,s} = \alpha(h, a)/\alpha(a, h)$ yields

$$\frac{\alpha(gh, a)\alpha(a, h)}{\alpha(gag^{-1}, gh)\alpha(h, a)}\lambda_{g,r}^{-1}\lambda_{g,s}^{-1}.$$

Applying the 2-cocycle condition of α to the triple (g, h, a) gives

$$\alpha(gh, a)/\alpha(h, a) = \alpha(g, ha)/\alpha(g, h).$$

Making this substitution in the expression above yields

$$\frac{\alpha(g, ha)\alpha(a, h)}{\alpha(gag^{-1}, gh)\alpha(g, h)}\lambda_{g,r}^{-1}\lambda_{g,s}^{-1}.$$

Applying the 2-cocycle condition of α to the triple (g, a, h) gives $\alpha(g, ha)\alpha(a, h) = \alpha(ga, h)\alpha(g, a)$. Making this substitution in the expression above yields

$$\frac{\alpha(ga, h)\alpha(g, a)}{\alpha(gag^{-1}, gh)\alpha(g, h)}\lambda_{g,r}^{-1}\lambda_{g,s}^{-1}.$$

Finally, applying the 2-cocycle condition of α to the triple (gag^{-1}, g, h) gives

$$\frac{\alpha(ga, h)}{\alpha(gag^{-1}, gh)\alpha(g, h)} = \frac{1}{\alpha(gag^{-1}, g)}.$$

Making this substitution in the expression above yields

$$\frac{\alpha(g, a)}{\alpha(gag^{-1}, g)}\lambda_{g,r}^{-1}\lambda_{g,s}^{-1},$$

proving that the scalar above is independent of choice of representative g of a coset of $C_G(a)$ under the assumption that $\lambda_{h,r}\lambda_{h,s} = \alpha(h, a)/\alpha(a, h)$ for all $h \in C_G(a)$. Thus, each of the alleged basis element is well defined, and is evidently G-invariant.

Conversely, let $\eta = \sum_{a \in G} \sum \eta_{rs}^a t_a \otimes v_r^* \wedge v_s^*$, where η_{rs}^a are scalars and the second sum runs over all $r < s$ that satisfy $q_{rr'} q_{sr'} = \lambda_{a,r'}$ for all $r' \notin \{r, s\}$. We have

$$^g\eta = \sum_{a \in G} \eta_{rs}^a t_g t_a (t_g)^{-1} \otimes{}^g(v_r^*) \wedge{}^g(v_s^*) = \sum_{a \in G} \frac{\alpha(g, a)}{\alpha(gag^{-1}, g)} \lambda_{g,r}^{-1} \lambda_{g,s}^{-1} \eta_{rs}^a t_{gag^{-1}} \otimes v_r^* \wedge v_s^*.$$

Assume that η is G-invariant. Then

$$\eta_{rs}^{gag^{-1}} = \frac{\alpha(g, a)}{\alpha(gag^{-1}, g)} \lambda_{g,r}^{-1} \lambda_{g,s}^{-1} \eta_{rs}^a,$$

for all $g \in G$. Letting $g = h \in C_G(a)$ yields

$$\lambda_{h,r} \lambda_{h,s} = \frac{\alpha(h, a)}{\alpha(a, h)},$$

showing that η is in the span of the alleged basis elements. The stated result now follows from Theorem 4.4. $\qquad\square$

The proof of the following theorem involves the maps Θ_2^*, $\mathcal{R}_2$, and Ψ_2^* defined in Section 4.

Theorem 6.5. *The maps $\kappa : V \times V \to \mathbb{C}^\alpha G$ for which $\mathcal{H}_{q,\kappa,\alpha}$ is a twisted quantum Drinfeld Hecke algebra form a vector space with basis consisting of maps*

$$f_{r,s,a} : V \times V \to \mathbb{C}^\alpha G,$$
$$(v_i, v_j) \mapsto (\delta_{i,r} \delta_{j,s} - q_{sr} \delta_{i,s} \delta_{j,r}) \sum_{g \in [G/C_G(a)]} \frac{\alpha(g, a)}{\alpha(gag^{-1}, g)} \lambda_{g,r}^{-1} \lambda_{g,s}^{-1} t_{gag^{-1}},$$

where $r < s$ and $a \in \mathcal{R}$ satisfy $q_{rr'} q_{sr'} = \lambda_{a,r'}$ for all $r' \notin \{r, s\}$ and $\lambda_{h,r} \lambda_{h,s} = \alpha(h, a)/\alpha(a, h)$ for all $h \in C_G(a)$.

Proof. Let $\eta = \sum_{g \in [G/C_G(a)]} (\alpha(g, a)/\alpha(gag^{-1}, g)) \lambda_{g,r}^{-1} \lambda_{g,s}^{-1} t_{gag^{-1}} \otimes v_r^* \wedge v_s^*$, where $r < s$ and $a \in \mathcal{R}$ satisfy the conditions specified in Theorem 6.4. In the proof of Theorem 5.3 we saw that $[\Theta_2^* \mathcal{R}_2 \Psi_2^*(\eta)](v_i \otimes v_j - q_{ij} v_j \otimes v_i) = \eta(v_i \wedge v_j)$, and the latter is equal to

$$(\delta_{i,r} \delta_{j,s} - q_{sr} \delta_{i,s} \delta_{j,r}) \sum_{g \in [G/C_G(a)]} \frac{\alpha(g, a)}{\alpha(gag^{-1}, g)} \lambda_{g,r}^{-1} \lambda_{g,s}^{-1} t_{gag^{-1}}.$$

The stated result now follows from Theorems 4.4 and 5.4. $\qquad\square$

7. Symmetric groups: natural representations

In this section, we classify twisted quantum Drinfeld Hecke algebras for the symmetric groups S_n, $n \geq 4$, acting naturally on a vector space of dimension n.

Consider the natural action of S_n on a vector space V with ordered basis $v_1, \dots, v_n$. Let $\boldsymbol{q} := (q_{ij})_{1 \leq i, j \leq n}$ denote a tuple of nonzero scalars for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. The action of S_n extends to an action on the

quantum symmetric algebra $S_q(V)$ by automorphisms if and only if either $q_{ij} = 1$ for all i, j or $q_{ij} = -1$ for all $i \neq j$. The tuple corresponding to the former will be denoted by $\mathbf{1}$, and the tuple corresponding to the latter by $-\mathbf{1}$. The action of S_n on V extends to an action on the quantum exterior algebra $\bigwedge_{-\mathbf{1}}$ by automorphisms. Note that the algebra $\bigwedge_{-\mathbf{1}}$ is commutative.

The Schur multiplier $\mathrm{H}^2(S_n, \mathbb{C}^\times)$ of the symmetric group S_n is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for all $n \geq 4$ [Schur 2001]. Let α be a 2-cocycle on S_n, and let $[\alpha]$ denote the image of α in $\mathrm{H}^2(S_n, \mathbb{C}^\times)$. A classification of twisted quantum Drinfeld Hecke algebras for S_n, acting naturally on a vector space of dimension n, is given in [Ram and Shepler 2003] for $[\alpha] = 1$ and $\mathbf{q} = \mathbf{1}$, in [Wambst 1993] for $[\alpha] \neq 1$ and $\mathbf{q} = \mathbf{1}$, and in [Naidu and Witherspoon 2011] for $[\alpha] = 1$ and $\mathbf{q} = -\mathbf{1}$. The goal of this section is to address the remaining case: $[\alpha] \neq 1$ and $\mathbf{q} = -\mathbf{1}$.

Next, we recall a Schur covering group of S_n, which we use to obtain a cohomologically nontrivial 2-cocycle on S_n. Let T_n be the group with generators $t_1, \ldots, t_{n-1}, z$ and relations

$$z^2 = 1,$$
$$t_r^2 = 1 \qquad \text{for } 1 \leq r \leq n-1,$$
$$t_r t_s = t_s t_r z \qquad \text{for } |r - s| > 1 \text{ and } 1 \leq r, s \leq n-1,$$
$$t_r t_{r+1} t_r = t_{r+1} t_r t_{r+1} \quad \text{for } 1 \leq r \leq n-2,$$
$$z t_r = t_r z \qquad \text{for } 1 \leq r \leq n-1.$$

The group T_n is a central extension of S_n by $\langle z \rangle$:

$$1 \to \langle z \rangle \to T_n \xrightarrow{p} S_n \to 1,$$

where the surjection p sends z to 1 and sends t_r to the transposition $(r\,r+1)$. The group T_n is a Schur covering group of S_n [Schur 2001].

We define certain distinguished elements of T_n: For every $r, s \in \{1, \ldots, n\}$, $r \neq s$, denote by $[rs]$ the element of T_n defined recursively as follows:

$$[r\,r+1] := t_r,$$
$$[rs] := t_r[r+1\,s]t_r z \quad \text{if } r < s-1,$$
$$[rs] := [sr]z \qquad \text{if } r > s.$$

Note that $p([rs]) = (rs)$.

Next, we define a section $u : S_n \to T_n$ of the surjection $p : T_n \to S_n$ by $u(\sigma) = u_\sigma$. If $\sigma \in S_n$ is the k-cycle $(a_1, \ldots, a_k)$, where $a_1, \ldots, a_k \in \{1, \ldots, n\}$ and a_1 is the smallest element of the set $\{a_1, \ldots, a_k\}$, define

$$u_\sigma := [a_1 a_k][a_1 a_{k-1}] \cdots [a_1 a_2].$$

If $\sigma \in S_n$ is the product $(a_1, \ldots, a_k)(b_1, \ldots, b_l) \cdots$ of disjoint cycles, where a_1 is the smallest element of the set $\{a_1, \ldots, a_k\}$, b_1 is the smallest element of the set $\{b_1, \ldots, b_l\}$, and so on, and $a_1 < b_1 < \cdots$, define

$$u_\sigma := u_{(a_1, \ldots, a_k)} u_{(b_1, \ldots, b_l)} \cdots .$$

It is evident that $u : S_n \to T_n$ is a section, that is, $pu = \mathrm{id}_{S_n}$.

Consider any irreducible representation of the group T_n. Since the element z is central and has order two, it must necessarily act on this representation as multiplication by either 1 or -1. Assume the latter. In this case, we obtain a cohomologically nontrivial (normalized) 2-cocycle $\alpha : S_n \times S_n \to \mathbb{C}^\times$ defined by

(7.1)
$$\alpha(\sigma, \tau) := \begin{cases} 1 & \text{if } u_\sigma u_\tau u_{\sigma\tau}^{-1} = 1, \\ -1 & \text{if } u_\sigma u_\tau u_{\sigma\tau}^{-1} = z, \end{cases}$$

for all $\sigma, \tau \in S_n$.

Our goal is to classify twisted quantum Drinfeld Hecke algebras associated to the quadruple $(S_n, V, -\mathbf{1}, \alpha)$, where V is the natural representation of S_n and $-\mathbf{1}$ is the tuple defined earlier in this section. To this end, in what follows, we establish several lemmas that aid in accomplishing our goal.

Since the subgroup $\langle z \rangle$ of T_n is central, there is an action of S_n on T_n induced by conjugation. If σ belongs to S_n and v belongs to T_n, we denote by $\sigma \triangleright v$ the result of σ acting upon v. We have $\sigma \triangleright v = \hat{\sigma} v (\hat{\sigma})^{-1}$, where $\hat{\sigma}$ is any element in the set $p^{-1}(\sigma)$.

For each $\sigma \in S_n$, let $\epsilon(\sigma)$ denote the signature of σ:

$$\epsilon(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is an even permutation,} \\ 1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

The following result from [Vendramin 2012] will be put to use shortly.

Lemma 7.2. *For all distinct $r, s \in \{1, \ldots, n\}$ and all $\sigma \in S_n$, we have*

$$\sigma \triangleright [rs] = [\sigma(r)\sigma(s)] z^{\epsilon(\sigma)}.$$

For later use, we record two lemmas.

Lemma 7.3. *For all distinct $r, r', s, s' \in \{1, \ldots, n\}$, we have*

$$[rs][sr']z = [rr'][rs] = [r's][rr']z.$$

Proof. We have $[rs]^{-1}[rr'][rs] = (rs)^{-1} \triangleright [rr'] = (rs) \triangleright [rr']$, and, by Lemma 7.2, the last expression equals $[sr']z$, proving the first equality. The second equality is proved similarly. $\square$

Lemma 7.4. *For all distinct $r, r', s, s' \in \{1, \ldots, n\}$, we have*

$$[rs][r's'] = [r's'][rs]z.$$

Proof. We have $[rs][r's'][rs]^{-1} = (rs) \triangleright [r's']$, and, by Lemma 7.2, the last expression equals $[r's']z$.   □

For all distinct $r, s, r', s' \in \{1, \ldots, n\}$, let $d(r, s, r', s')$ denote the number of inequalities

$$\min\{r, s\} > \min\{r', s'\}, \quad r > s, \quad r' > s'$$

that hold. For all distinct $r, s, r', s' \in \{1, \ldots, n\}$ and all $\sigma \in S_n$, define

$$d_\sigma(r, s, r', s') := d(\sigma(r), \sigma(s), \sigma(r'), \sigma(s')).$$

For later use, we record the following obvious result.

Lemma 7.5. *For all distinct $r, s, r', s' \in \{1, \ldots, n\}$, we have*

$$|d(r, s, r', s') - d(r, s, s', r')| = 1 = |d(r, s, r', s') - d(s, r, r', s')|.$$

We need the following lemma, which is a generalization of [Vendramin 2012, Lemma 3.7].

Lemma 7.6. *Let σ be any element of S_n.*

(a) *For all $r, s \in \{1, \ldots, n\}$ with $r < s$, we have*

$$\sigma \triangleright u_{(rs)} = \begin{cases} u_{\sigma(rs)\sigma^{-1}} z^{\epsilon(\sigma)} & \text{if } \sigma(r) < \sigma(s), \\ u_{\sigma(rs)\sigma^{-1}} z^{\epsilon(\sigma)+1} & \text{if } \sigma(r) > \sigma(s). \end{cases}$$

(b) *For all distinct $r, s, r', s' \in \{1, \ldots, n\}$ with $r < s, r' < s'$, and $r < r'$, we have*

$$\sigma \triangleright u_{(rs)(r's')} = u_{\sigma(rs)(r's')\sigma^{-1}} z^{d_\sigma(r,s,r',s')}.$$

(c) *For all distinct $r, s, r' \in \{1, \ldots, n\}$ with $r < s$ and $r < r'$, we have*

$$\sigma \triangleright u_{(rsr')} = u_{\sigma(rsr')\sigma^{-1}}.$$

Proof. (a) By Lemma 7.2, $\sigma \triangleright u_{(rs)} = \sigma \triangleright [rs] = [\sigma(r)\sigma(s)]z^{\epsilon(\sigma)}$. If $\sigma(r) < \sigma(s)$, then

$$[\sigma(r)\sigma(s)]z^{\epsilon(\sigma)} = u_{(\sigma(r)\sigma(s))} z^{\epsilon(\sigma)} = u_{\sigma(rs)\sigma^{-1}} z^{\epsilon(\sigma)}.$$

If $\sigma(r) > \sigma(s)$, then

$$[\sigma(r)\sigma(s)]z^{\epsilon(\sigma)} = [\sigma(s)\sigma(r)]z^{\epsilon(\sigma)+1} = u_{(\sigma(s)\sigma(r))} z^{\epsilon(\sigma)+1} = u_{\sigma(rs)\sigma^{-1}} z^{\epsilon(\sigma)+1}.$$

(b) Again, by Lemma 7.2,

$$\begin{aligned} \sigma \triangleright u_{(rs)(r's')} = \sigma \triangleright [rs][r's'] &= (\sigma \triangleright [rs])(\sigma \triangleright [r's']) \\ &= [\sigma(r)\sigma(s)]z^{\epsilon(\sigma)}[\sigma(r')\sigma(s')]z^{\epsilon(\sigma)} \\ &= [\sigma(r)\sigma(s)][\sigma(r')\sigma(s')]. \end{aligned}$$

If $\min\{\sigma(r), \sigma(s)\} > \min\{\sigma(r'), \sigma(s')\}$, then, using Lemma 7.4, we rewrite the product above as $[\sigma(r')\sigma(s')][\sigma(r)\sigma(s)]z$. If $\sigma(r) > \sigma(s)$, we replace $[\sigma(r)\sigma(s)]$ by

$[\sigma(s)\sigma(r)]z$. Similarly, if $\sigma(r') > \sigma(s')$, we replace $[\sigma(r')\sigma(s')]$ by $[\sigma(s')\sigma(r')]z$. Since the element z has order two, the stated result follows. For example, suppose that $d_\sigma(r, s, r', s') = 3$. Then $\sigma(r) > \sigma(s)$, $\sigma(r') > \sigma(s')$, and $\sigma(s) > \sigma(s')$, and in this case we write

$$
\begin{aligned}
[\sigma(r)\sigma(s)][\sigma(r')\sigma(s')] &= [\sigma(r')\sigma(s')][\sigma(r)\sigma(s)]z = [\sigma(s')\sigma(r')]z[\sigma(s)\sigma(r)]zz \\
&= [\sigma(s')\sigma(r')][\sigma(s)\sigma(r)]z \\
&= u_{(\sigma(s')\sigma(r'))(\sigma(s)\sigma(r))}z \\
&= u_{\sigma(rs)(r's')\sigma^{-1}}z.
\end{aligned}
$$

(c) Again, by Lemma 7.2,

$$
\begin{aligned}
\sigma \triangleright u_{(rsr')} = \sigma \triangleright [rr'][rs] &= (\sigma \triangleright [rr'])(\sigma \triangleright [rs]) \\
&= [\sigma(r)\sigma(r')]z^{\epsilon(\sigma)}[\sigma(r)\sigma(s)]z^{\epsilon(\sigma)} \\
&= [\sigma(r)\sigma(r')][\sigma(r)\sigma(s)].
\end{aligned}
$$

Case (c$_1$). $\sigma(r) < \sigma(r')$ and $\sigma(r) < \sigma(s)$. In this case,

$$
[\sigma(r)\sigma(r')][\sigma(r)\sigma(s)] = u_{(\sigma(r)\sigma(s)\sigma(r'))} = u_{\sigma(rsr')\sigma^{-1}}.
$$

Case (c$_2$). Either $\sigma(s) < \sigma(r) < \sigma(r')$ or $\sigma(s) < \sigma(r') < \sigma(r)$. Using the first equality of Lemma 7.3,

$$
\begin{aligned}
[\sigma(r)\sigma(r')][\sigma(r)\sigma(s)] &= [\sigma(r)\sigma(s)][\sigma(s)\sigma(r')]z = [\sigma(s)\sigma(r)]z[\sigma(s)\sigma(r')]z \\
&= u_{(\sigma(s)\sigma(r')\sigma(r))} \\
&= u_{\sigma(rsr')\sigma^{-1}}.
\end{aligned}
$$

Case (c$_3$). Either $\sigma(r') < \sigma(r) < \sigma(s)$ or $\sigma(r') < \sigma(s) < \sigma(r)$. Using the second equality of Lemma 7.3,

$$
\begin{aligned}
[\sigma(r)\sigma(r')][\sigma(r)\sigma(s)] &= [\sigma(r')\sigma(s)][\sigma(r)\sigma(r')]z \\
&= [\sigma(r')\sigma(s)][\sigma(r')\sigma(r)]zz \\
&= u_{(\sigma(r')\sigma(r)\sigma(s))} \\
&= u_{\sigma(rsr')\sigma^{-1}}. \qquad \square
\end{aligned}
$$

We now turn our attention to the Hochschild cohomology of $S_{-1}(V)\#_\alpha S_n$.

Theorem 7.7 [Naidu and Witherspoon 2011, Theorem 6.8]. *Assume that $n \geq 4$. The constant Hochschild 2-cocycles representing elements in $\mathrm{HH}^2(S_{-1}(V), S_{-1}(V)\#S_n)$ form a vector subspace of $(S_{-1}(V)\#G) \otimes \bigwedge_{-1}(V^*)$ having as a basis the set of all*

$$\eta_1 = t_1 \otimes v_r^* \wedge v_s^* \qquad\qquad\qquad\qquad\qquad\qquad (r < s),$$

$$\eta_2 = t_{(rs)} \otimes v_r^* \wedge v_s^* \qquad\qquad\qquad\qquad\qquad\quad (r < s),$$

$$\eta_3 = t_{(rs)} \otimes (v_r^* \wedge v_{r'}^* + v_s^* \wedge v_{r'}^*) \qquad\qquad\quad (r < s),$$

$$\eta_4 = t_{(rs)(r's')} \otimes (v_r^* \wedge v_{r'}^* + v_r^* \wedge v_{s'}^* + v_s^* \wedge v_{r'}^* + v_s^* \wedge v_{s'}^*) \quad (r < s, r' < s', r < r'),$$

$$\eta_5 = t_{(rsr')} \otimes (v_r^* \wedge v_s^* + v_s^* \wedge v_{r'}^* + v_r^* \wedge v_{r'}^*) \qquad (r < s, r < r').$$

Note that the $S_{-1}(V)$-bimodule structure of $S_{-1}(V)\#_\alpha G$ does not depend on the 2-cocycle α, and so $\mathrm{HH}^2(S_{-1}(V), S_{-1}(V)\#_\alpha G) = \mathrm{HH}^2(S_{-1}(V), S_{-1}(V)\#G)$.

The lemma below involves the maps Θ_2^*, $\mathscr{R}_2$, and Ψ_2^* defined in Section 4. Recall that the image of an element $\sigma \in S_n$ in the twisted group algebra $\mathbb{C}^\alpha S_n$ is denoted by t_σ. Also, recall the definition of the 2-cocycle α given in (7.1).

Lemma 7.8. *For all $i \neq j$,*

$$[(\Theta_2^*\mathscr{R}_2\Psi_2^*)(\eta_a)](v_i \otimes v_j)$$

$$= \begin{cases} \dfrac{1}{n(n-1)}t_1 & \text{if } a = 1, \\[2mm] 0 & \text{if } a = 2, \\[2mm] 0 & \text{if } a = 3 \text{ and } n \geq 5, \\[2mm] 0 & \text{if } a = 4, \\[2mm] \dfrac{1}{n(n-1)(n-2)} \displaystyle\sum_{k \neq i, j} (2t_{(ijk)} + t_{(ikj)}) & \text{if } a = 5, \end{cases}$$

Proof. Using (4.7),

$$[(\Theta_2^*\mathscr{R}_2\Psi_2^*)(\eta_1)](v_i \otimes v_j) = \frac{1}{n!} \sum_{\sigma \in S_n} {}^\sigma(\eta_1(\Psi_2(1 \otimes v_{\sigma^{-1}(i)} \otimes v_{\sigma^{-1}(j)} \otimes 1)))$$

$$= \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma^{-1}(i) < \sigma^{-1}(j)}} {}^\sigma(\eta_1(1 \otimes 1 \otimes v_{\sigma^{-1}(i)} \otimes v_{\sigma^{-1}(j)}))$$

$$= \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i, \sigma(s)=j}} {}^\sigma(t_1)$$

$$= \frac{1}{n(n-1)}t_1.$$

Similarly,

$$[(\Theta_2^*\mathscr{R}_2\Psi_2^*)(\eta_2)](v_i \otimes v_j) = \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i, \sigma(s)=j}} {}^\sigma(t_{(rs)}).$$

Applying the conjugation action in $\mathbb{C}^\alpha G$, we get

$$\frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(s)=j}} \frac{\alpha(\sigma, (rs))}{\alpha(\sigma(rs)\sigma^{-1}, \sigma)} t_{\sigma(rs)\sigma^{-1}} = \left(\frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(s)=j}} \frac{\alpha(\sigma, (rs))}{\alpha((ij), \sigma)} \right) t_{(ij)}.$$

The scalar $\alpha(\sigma, (rs))/\alpha((ij), \sigma)$ in the summation above is determined by the following element of T_n:

$$u_\sigma u_{(rs)} u_{\sigma(rs)}^{-1} u_{(ij)\sigma} u_\sigma^{-1} u_{(ij)}^{-1} = u_\sigma u_{(rs)} u_\sigma^{-1} u_{\sigma(rs)\sigma^{-1}}^{-1}.$$

By Lemma 7.6(a),

$$u_\sigma u_{(rs)} u_\sigma^{-1} u_{\sigma(rs)\sigma^{-1}}^{-1} = \begin{cases} z^{\epsilon(\sigma)} & \text{if } i < j, \\ z^{\epsilon(\sigma)+1} & \text{if } i > j. \end{cases}$$

Since we assume n is greater than or equal to 4, the set $\{\sigma \in S_n \mid \sigma(r)=i,\ \sigma(s)=j\}$ contains an equal number of odd and even permutations, and so

$$\sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(s)=j}} \frac{\alpha(\sigma, (rs))}{\alpha((ij), \sigma)} = 0,$$

proving that $[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\eta_2)](v_i \otimes v_j) = 0$.

Next, we consider the $a = 3$ case. In addition to the stated assumption $r < s$, assume further that $r < r'$ and $s < r'$. The other cases can be handled similarly. We have

$$[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\eta_3)](v_i \otimes v_j) = \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(r')=j}} {}^\sigma(t_{(rs)}) + \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\, \sigma(r')=j}} {}^\sigma(t_{(rs)}).$$

Applying the conjugation action in $\mathbb{C}^\alpha G$, we get

$$\frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(r')=j}} \frac{\alpha(\sigma, (rs))}{\alpha((i\sigma(s)), \sigma)} t_{(i\sigma(s))} + \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\, \sigma(r')=j}} \frac{\alpha(\sigma, (rs))}{\alpha((\sigma(r)i), \sigma)} t_{(\sigma(r)i)}$$

$$= \frac{1}{n!} \sum_{k \neq i, j} \left(\sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(r')=j,\, \sigma(s)=k}} \frac{\alpha(\sigma, (rs))}{\alpha((ik), \sigma)} + \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\, \sigma(r')=j,\, \sigma(r)=k}} \frac{\alpha(\sigma, (rs))}{\alpha((ik), \sigma)} \right) t_{(ik)}.$$

The scalar $\alpha(\sigma, (rs))/\alpha((ik), \sigma)$ in the first of the two inner summations above is determined by the element $u_\sigma u_{(rs)} u_\sigma^{-1} u_{\sigma(rs)\sigma^{-1}}^{-1}$ of T_n. Again, by Lemma 7.6(a),

$$u_\sigma u_{(rs)} u_\sigma^{-1} u_{\sigma(rs)\sigma^{-1}}^{-1} = \begin{cases} z^{\epsilon(\sigma)} & \text{if } i < k, \\ z^{\epsilon(\sigma)+1} & \text{if } i > k. \end{cases}$$

Since n is assumed to be greater than or equal to 5, the set $\{\sigma \in S_n \mid \sigma(r)=i,\ \sigma(r')=j,\ \sigma(s)=k\}$ contains an equal number of odd and even permutations, and so

$$\sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(r')=j,\, \sigma(s)=k}} \frac{\alpha(\sigma, (rs))}{\alpha((ik), \sigma)} = 0.$$

Similarly,

$$\sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\,\sigma(r')=j,\,\sigma(r)=k}} \frac{\alpha(\sigma,(rs))}{\alpha((ik),\sigma)} = 0,$$

and it follows that $[(\Theta_2^* \mathscr{R}_2 \Psi_2^*)(\eta_3)](v_i \otimes v_j) = 0$.

For the $a = 4$ case, in addition to the stated assumptions $r < s$, $r' < s'$, $r < r'$, assume further that $r < s'$, $s < r'$, and $s < s'$. The other cases can be handled similarly. We have

$$[(\Theta_2^* \mathscr{R}_2 \Psi_2^*)(\eta_4)](v_i \otimes v_j)$$

$$= \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\,\sigma(r')=j}} {}^\sigma(t_{(rs)(r's')}) + \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\,\sigma(s')=j}} {}^\sigma(t_{(rs)(r's')})$$

$$+ \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\,\sigma(r')=j}} {}^\sigma(t_{(rs)(r's')}) + \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\,\sigma(s')=j}} {}^\sigma(t_{(rs)(r's')}).$$

Applying the conjugation action in $\mathbb{C}^\alpha G$, we get

$$\frac{1}{n!} \Bigg(\sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\,\sigma(r')=j}} \frac{\alpha(\sigma,(rs)(r's'))}{\alpha((i\sigma(s))(j\sigma(s')),\sigma)} t_{(i\sigma(s))(j\sigma(s'))}$$

$$+ \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\,\sigma(s')=j}} \frac{\alpha(\sigma,(rs)(r's'))}{\alpha((i\sigma(s))(\sigma(r')j),\sigma)} t_{(i\sigma(s))(\sigma(r')j)}$$

$$+ \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\,\sigma(r')=j}} \frac{\alpha(\sigma,(rs)(r's'))}{\alpha((\sigma(r)i)(j\sigma(s')),\sigma)} t_{(\sigma(r)i)(j\sigma(s'))}$$

$$+ \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\,\sigma(s')=j}} \frac{\alpha(\sigma,(rs)(r's'))}{\alpha((\sigma(r)i)(\sigma(r')j),\sigma)} t_{(\sigma(r)i)(\sigma(r')j)} \Bigg)$$

$$= \frac{1}{n!} \sum_{k,l \notin \{i,j\}} \Bigg(\sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\,\sigma(r')=j \\ \sigma(s)=k,\,\sigma(s')=l}} \frac{\alpha(\sigma,(rs)(r's'))}{\alpha((ik)(jl),\sigma)} + \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\,\sigma(s')=j \\ \sigma(s)=k,\,\sigma(r')=l}} \frac{\alpha(\sigma,(rs)(r's'))}{\alpha((ik)(jl),\sigma)}$$

$$+ \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\,\sigma(r')=j \\ \sigma(r)=k,\,\sigma(s')=l}} \frac{\alpha(\sigma,(rs)(r's'))}{\alpha((ik)(jl),\sigma)} + \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\,\sigma(s')=j \\ \sigma(r)=k,\,\sigma(r')=l}} \frac{\alpha(\sigma,(rs)(r's'))}{\alpha((ik)(jl),\sigma)} \Bigg) t_{(ik)(jl)}.$$

The scalar $\alpha(\sigma,(rs)(r's'))/\alpha((ik)(jl),\sigma)$ in the first of the four inner summations above is determined by the element $u_\sigma u_{(rs)(r's')} u_\sigma^{-1} u_{\sigma(rs)(r's')\sigma^{-1}}^{-1}$ of T_n. By Lemma 7.6(b),

$$u_\sigma u_{(rs)(r's)} u_\sigma^{-1} u_{\sigma(rs)(r's')\sigma^{-1}}^{-1} = z^{d_\sigma(r,s,r's')} = z^{d(i,k,j,l)}.$$

Thus,

$$\sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(r')=j \\ \sigma(s)=k,\, \sigma(s')=l}} \frac{\alpha(\sigma, (rs)(r's'))}{\alpha((ik)(jl), \sigma)} = (n-4)!(-1)^{d(i,k,j,l)}.$$

Similarly, the second, third, and fourth summations are equal to $(n-1)!$ times $(-1)^{d(i,k,l,j)}$, $(-1)^{d(k,i,j,l)}$, and $(-1)^{d(k,i,l,j)}$, respectively. From Lemma 7.5 it follows that the sum of the four summations above is equal to zero, and so $[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\eta_4)](v_i \otimes v_j) = 0$.

Finally, for the $a = 5$ case, in addition to the stated assumptions $r < s$, $r < r'$, assume further that $s < r'$. Again, the other case can be handled similarly. We have

$$[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\eta_5)](v_i \otimes v_j)$$

$$= \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(s)=j}} {}^\sigma(t_{(rsr')}) + \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\, \sigma(r')=j}} {}^\sigma(t_{(rsr')}) + \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(r')=j}} {}^\sigma(t_{(rsr')})$$

$$= \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(s)=j}} \frac{\alpha(\sigma, (rsr'))}{\alpha((ij\sigma(r')), \sigma)} t_{(ij\sigma(r'))} + \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\, \sigma(r')=j}} \frac{\alpha(\sigma, (rsr'))}{\alpha((\sigma(r)ij), \sigma)} t_{(\sigma(r)ij)}$$

$$+ \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(r')=j}} \frac{\alpha(\sigma, (rsr'))}{\alpha((i\sigma(s)j), \sigma)} t_{(i\sigma(s)j)}$$

$$= \frac{1}{n!} \sum_{k,l \notin \{i,j\}} \left[\left(\sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(s)=j,\, \sigma(r')=k}} \frac{\alpha(\sigma, (rsr'))}{\alpha((ijk), \sigma)} + \sum_{\substack{\sigma \in S_n \\ \sigma(s)=i,\, \sigma(r')=j,\, \sigma(r)=k}} \frac{\alpha(\sigma, (rsr'))}{\alpha((ijk), \sigma)} \right) t_{(ijk)} \right.$$

$$\left. + \sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(r')=j,\, \sigma(s)=k}} \frac{\alpha(\sigma, (rsr'))}{\alpha((ikj), \sigma)} t_{(ikj)} \right].$$

The scalar $\alpha(\sigma, (rsr'))/\alpha((ijk), \sigma)$ in the first of the three inner summations above is determined by the element $u_\sigma u_{(rsr')} u_\sigma^{-1} u_{\sigma(rsr')\sigma^{-1}}^{-1}$ of T_n. By Lemma 7.6(c), $u_\sigma u_{(rsr')} u_\sigma^{-1} u_{\sigma(rsr')\sigma^{-1}}^{-1} = 1$. Thus,

$$\sum_{\substack{\sigma \in S_n \\ \sigma(r)=i,\, \sigma(s)=j\sigma(r')=k}} \frac{\alpha(\sigma, (rsr')}{\alpha((ijk), \sigma)} = (n-3)!.$$

Similarly, the second and third summations are also equal to $(n-3)!$. It follows that

$$[(\Theta_2^* \mathcal{R}_2 \Psi_2^*)(\eta_5)](v_i \otimes v_j) = \frac{1}{n(n-1)(n-2)} \sum_{k \neq i,j} (2t_{(ijk)} + t_{(ikj)}). \qquad \square$$

Combining Theorems [7.7], [4.4], [5.4], and [Lemma 7.8] establishes the following.

Theorem 7.9. *Assume that $n \geq 5$. The maps $\kappa : V \times V \to \mathbb{C}^\alpha S_n$ for which $\mathcal{H}_{-1,\kappa,\alpha}$ is a twisted quantum Drinfeld Hecke algebra form a two-dimensional vector space with basis consisting of bilinear maps $\kappa_1 : V \times V \to \mathbb{C}^\alpha S_n$ and $\kappa_2 : V \times V \to \mathbb{C}^\alpha S_n$ determined by*

$$\kappa_1(v_i, v_j) = t_1 \quad \text{and} \quad \kappa_2(v_i, v_j) = \sum_{k \neq i, j} (t_{(ijk)} + t_{(ikj)}),$$

for all $i \neq j$.

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