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KEOMKYO SEO

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We estimate the bottom of the spectrum of the Laplace operator on a stable minimal hypersurface in a negatively curved manifold. We also derive various vanishing theorems for L^p harmonic 1-forms on minimal hypersurfaces in terms of the bottom of the spectrum of the Laplace operator. As consequences, the corresponding Liouville type theorems for harmonic functions with finite L^p energy on minimal hypersurfaces in a Riemannian manifold are obtained.

1. Introduction

Hodge theory plays an important role in the topology of compact Riemannian manifolds. Unfortunately, the Hodge theory does not work anymore in noncompact manifolds. However, the L^2-Hodge theory works well in noncompact cases [Anderson 1988; Dodziuk 1982]. In this direction, there are various results for L^2 harmonic 1-forms on stable minimal hypersurfaces. Recall that a minimal hypersurface in a Riemannian manifold is called *stable* if the second variation of its volume is always nonnegative for any normal variation with compact support. More precisely, an n-dimensional minimal hypersurface M in a Riemannian manifold N is called *stable* if it holds that, for any compactly supported Lipschitz function f on M,

$$\int_M |\nabla f|^2 - (|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu)) f^2 \, dv \geq 0,$$

where ν is the unit normal vector of M, $\overline{\mathrm{Ric}}(\nu, \nu)$ denotes the Ricci curvature of N in the ν direction, $|A|^2$ is the square length of the second fundamental form A, and dv is the volume form for the induced metric on M.

Using the nonexistence of L^2 harmonic 1-forms, Palmer [1991] proved that if there exists a codimension-one cycle on a complete minimal hypersurface M in Euclidean space, which does not separate M, M is unstable. Using Bochner's

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vanishing technique, Miyaoka [1993] showed that a complete noncompact stable minimal hypersurface in a nonnegatively curved manifold has no nontrivial L^2 harmonic 1-forms. Pigola, Rigoli, and Setti [Pigola et al. 2005] gave general Liouville type results and the corresponding vanishing theorems on the L^2 cohomology of stable minimal hypersurfaces. Refer to [Carron 2002; Pigola et al. 2008] for a survey in this area. While the L^2 theory is quite well understood, in the case $p \neq 2$, the L^p theory is less developed. See [Scott 1995] for general L^p theory of differential forms on a manifold.

The purpose of this paper is twofold. Firstly, we estimate the smallest spectral value of the Laplace operator on a complete noncompact stable minimal hypersurface in a Riemannian manifold under the assumption on L^p norm of the second fundamental form. Secondly, we obtain various vanishing theorems for L^p harmonic 1-forms on minimal hypersurfaces.

Let M be a complete noncompact Riemannian manifold and let Ω be a compact domain in M. Let $\lambda_1(\Omega) > 0$ denote the first eigenvalue of the Dirichlet boundary value problem

$$\begin{cases} \Delta f + \lambda f = 0 & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ denotes the Laplace operator on M. Then the first eigenvalue $\lambda_1(M)$ is defined by

$$\lambda_1(M) = \inf_{\Omega} \lambda_1(\Omega),$$

where the infimum is taken over all compact domains in M. Cheung and Leung [2001] gave the first eigenvalue estimate for an n-dimensional complete noncompact submanifold M with the norm of its mean curvature vector bounded in the hyperbolic space. In particular, they proved that if M is minimal, the first eigenvalue $\lambda_1(M)$ satisfies

$$\tfrac{1}{4}(n-1)^2 \leq \lambda_1(M).$$

Note that this inequality is sharp because equality holds if M is totally geodesic [McKean 1970]. This result was extended to an n-dimensional complete noncompact submanifold with the norm of its mean curvature vector bounded in a complete simply connected Riemannian manifold with sectional curvature bounded above by a negative constant. More precisely, we have the following theorem.

Theorem [Bessa and Montenegro 2003; Seo 2012]. *Let N be an n-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying $K_N \leq -a^2 < 0$ for a positive constant $a > 0$. Let M be a an m-dimensional complete noncompact submanifold with bounded mean curvature vector H in N satisfying $|H| \leq b < (m-1)a$. Then*

$$\tfrac{1}{4}[(m-1)a - b]^2 \leq \lambda_1(M). \tag{1}$$

On the other hand, Candel [2007] obtained an upper bound for the bottom of the spectrum of a complete simply connected stable minimal surface in 3-dimensional hyperbolic space. With finite L^2 norm of the second fundamental form, one may estimate an upper bound for the bottom of the spectrum of a stable minimal hypersurface in a Riemannian manifold with pinched negative sectional curvature [Dung and Seo 2012; Seo 2011]. In Section 2, we estimate the bottom of the spectrum of the Laplace operator on stable minimal hypersurfaces under the assumption on the L^p norm of the second fundamental form. Indeed, we prove the following.

Theorem. *Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \le K_N \le K_2$, where K_1, K_2 are constants and $K_1 \le K_2 < 0$. Let M be a complete stable non-totally geodesic minimal hypersurface in N. Assume that, for $1-\sqrt{2/n} < p < 1+\sqrt{2/n}$,*

$$\lim_{R\to\infty} R^{-2} \int_{B(R)} |A|^{2p} = 0,$$

where $B(R)$ is a geodesic ball of radius R on M. If $|\nabla K|^2 = \sum_{i,j,k,l,m} K^2_{ijkl;m} \le K_3^2|A|^2$ for some constant $K_3 \ge 0$, we have

$$-K_2\frac{(n-1)^2}{4} \le \lambda_1(M) \le \frac{np^2(2K_3 - n(K_1+K_2))}{2-n(p-1)^2}.$$

The author [2010] proved that if M is an n-dimensional complete stable minimal hypersurface in hyperbolic space with $\lambda_1(M) > (2n-1)(n-1)$, there is no nontrivial L^2 harmonic 1-form on M. This result was generalized [Dung and Seo 2012] to a complete stable minimal hypersurface in a Riemannian manifold with sectional curvature bounded below by a nonpositive constant. In Section 3, we prove an extended result for L^p harmonic 1-forms on a complete noncompact stable minimal hypersurface as follows.

Theorem. *Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying that $K \le K_N$ where $K \le 0$ is a constant. Let M be a complete noncompact stable minimal hypersurface in N. Assume that, for $0 < p < n/(n-1)+\sqrt{2n}$,*

$$\lambda_1(M) > \frac{-2n(n-1)^2p^2K}{2n-[(n-1)p-n]^2}.$$

Then there is no nontrivial L^{2p} harmonic 1-form on M.

Yau [1976] proved that there are no nonconstant L^p harmonic functions on a complete Riemannian manifold for $1 < p < \infty$. Li and Schoen [1984] proved that Yau's result is still true for L^p harmonic functions on a complete manifold of nonnegative Ricci curvature when $0 < p < \infty$. In the case of harmonic forms, Greene and Wu [1974; 1981] announced nonexistence of nontrivial L^p harmonic forms ($1 \le p < \infty$) on complete Riemannian and Kählerian manifolds of nonnegative curvature. See also [Colding and Minicozzi 1996; 1997; 1998; Li and Tam 1987; 1992] for Liouville type theorems for harmonic functions on a complete Riemannian manifold. The Liouville property holds also for harmonic functions on minimal hypersurfaces in a Riemannian manifold. For instance, Schoen and Yau proved the Liouville type theorem on minimal hypersurfaces as follows.

Theorem [Schoen and Yau 1976]. *Let M be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature. If f is a harmonic function on M with finite L^2 energy, f is constant.*

Recall that a function f on a Riemannian manifold M has *finite L^p energy* if $|\nabla f| \in L^p(M)$. As an application of our theorem, we immediately obtain the following, which is a generalization of Schoen and Yau's result (see Corollary 3.10).

Theorem. *Let M be a complete noncompact stable minimal hypersurface in a Riemannian manifold with nonnegative sectional curvature with $\lambda_1(M) > 0$. Then there is no nontrivial harmonic function on M with finite L^p energy for $0 < p < n/(n-1) + \sqrt{2n}$.*

For $n \ge 3$, it is well known [Cao et al. 1997] that an n-dimensional complete stable minimal hypersurface M in Euclidean space cannot have more than one end. This topological result was generalized to minimal hypersurfaces with finite index in Euclidean space and stable minimal hypersurfaces in a nonnegatively curved manifold by Li and Wang [2002; 2004]. If we assume that M has sufficiently small total scalar curvature instead of assuming that M is stable, we can also have the same conclusion [Ni 2001; Seo 2008]. See also [Pigola and Veronelli 2012] for more general results related with L^p norm of the second fundamental form. In the same spirit, Yun [2002] proved that if $M \subset \mathbb{R}^{n+1}$ is a complete minimal hypersurface with sufficiently small total scalar curvature, there is no nontrivial L^2 harmonic 1-form on M. Yun's result was generalized [Dung and Seo 2012] to a complete noncompact stable minimal hypersurface in a complete Riemannian manifold with sectional curvature bounded below by a nonpositive constant. The corresponding vanishing theorems for L^p harmonic 1-forms are obtained in Section 4.

One crucial step in the proofs of our theorems is to obtain an inequality of Simons' type for $|\phi|^p$ rather than $|\phi|$, where ϕ is a geometric quantity which we want to analyze. This kind of inequalities has been used in [Deng 2008; Fu 2012; Shen and Zhu 2005]. Equipped with this Simons' type inequality, we extend the original Bochner technique to our cases.

2. An estimate for the bottom of the spectrum of the Laplace operator

Let M be an n-dimensional manifold immersed in an $(n+1)$-dimensional Riemannian manifold N. We choose a local vector field of orthonormal frames $e_1, \ldots, e_{n+1}$ in N such that the vectors $e_1, \ldots, e_n$ are tangent to M and the vector e_{n+1} is normal to M. With respect to this frame field of N, let K_{ijkl} be a curvature tensor of N. We denote by $K_{ijkl;m}$ the covariant derivative of K_{ijkl}. In this section, we follow the notation of [Schoen et al. 1975].

Theorem 2.1. *Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \le K_N \le K_2$, where K_1, K_2 are constants and $K_1 \le K_2 < 0$. Let M be a complete stable non-totally geodesic minimal hypersurface in N. Assume that, for $1-\sqrt{2/n} < p < 1+\sqrt{2/n}$,*

$$\lim_{R\to\infty} R^{-2} \int_{B(R)} |A|^{2p} = 0,$$

where $B(R)$ is a geodesic ball of radius R on M. If $|\nabla K|^2 = \sum_{i,j,k,l,m} K^2_{ijkl;m} \le K_3^2 |A|^2$ for some constant $K_3 \ge 0$, we have

$$-K_2 \frac{(n-1)^2}{4} \le \lambda_1(M) \le \frac{np^2(2K_3 - n(K_1+K_2))}{2-n(p-1)^2}.$$

Proof. As mentioned in the introduction, one sees that the lower bound of $\lambda_1(M)$ is given as $-K_2(n-1)^2/4$ from inequality (1) [Bessa and Montenegro 2003; Seo 2012]. Namely, the first eigenvalue of an n-dimensional minimal hypersurface in a complete simply connected Riemannian manifold with sectional curvature bounded above by a negative constant K_2 is bounded below by $-K_2(n-1)^2/4$. Therefore, in the rest of the proof, we shall find the upper bound of the first eigenvalue $\lambda_1(M)$.

By [Schoen et al. 1975, (1.22), (1.27)], we have

$$|A|\Delta|A| + 2K_3|A|^2 - n(2K_2 - K_1)|A|^2 + |A|^4 \ge \sum h^2_{ijk} - \bigl|\nabla|A|\bigr|^2$$

at all points where $|A| \ne 0$. Because $K_2 - K_1 \ge 0$, this inequality implies

$$\begin{aligned} |A|\Delta|A| + 2K_3|A|^2 - nK_2|A|^2 + |A|^4 &\ge \sum h^2_{ijk} - \bigl|\nabla|A|\bigr|^2 \\ &= |\nabla A|^2 - \bigl|\nabla|A|\bigr|^2. \end{aligned}$$

Applying the Kato-type inequality

$$|\nabla A|^2 - \bigl|\nabla|A|\bigr|^2 \ge \frac{2}{n}\bigl|\nabla|A|\bigr|^2,$$

due to Y. L. Xin [2005], we get

$$|A|\Delta|A| + (2K_3 - nK_2)|A|^2 + |A|^4 \ge \frac{2}{n}\bigl|\nabla|A|\bigr|^2. \tag{2}$$

For a positive number $p > 0$, we have

$$\begin{aligned} |A|^p \Delta |A|^p &= |A|^p \mathrm{div}(\nabla |A|^p) \\ &= |A|^p \mathrm{div}(p\, |A|^{p-1} \nabla |A|) \\ &= p(p-1)|A|^{2p-2} \big|\nabla |A|\big|^2 + p|A|^{2p-1} \Delta |A| \\ &= \frac{p-1}{p} \big|\nabla |A|^p\big|^2 + p|A|^{2p-2} |A| \Delta |A|. \end{aligned}$$

It follows from inequality (2) that

$$\begin{aligned} &|A|^p \Delta |A|^p \\ &\quad \geq \frac{p-1}{p} \big|\nabla |A|^p\big|^2 + \frac{2p}{n} |A|^{2p-2} \big|\nabla |A|\big|^2 - p|A|^{2p+2} - p(2K_3 - nK_2)|A|^{2p} \\ &\quad = \frac{p-1}{p} \big|\nabla |A|^p\big|^2 + \frac{2}{np} \big|\nabla |A|^p\big|^2 - p|A|^{2p+2} - p(2K_3 - nK_2)|A|^{2p}. \end{aligned}$$

Thus

$$|A|^p \Delta |A|^p + p(2K_3 - nK_2)|A|^{2p} + p|A|^{2p+2} \geq \left(1 - \frac{n-2}{np}\right) \big|\nabla |A|^p\big|^2.$$

Choose a Lipschitz function f with compact support in a geodesic ball $B(R)$ of radius R centered at a point $x \in M$. Multiplying both sides by f^2 and integrating over $B(R)$, we obtain

$$\begin{aligned} \int_{B(R)} f^2 |A|^p \Delta |A|^p + p(2K_3 - nK_2) \int_{B(R)} f^2 |A|^{2p} + p \int_{B(R)} f^2 |A|^{2p+2} \\ \geq \left(1 - \frac{n-2}{np}\right) \int_{B(R)} f^2 \big|\nabla |A|^p\big|^2. \end{aligned}$$

The divergence theorem yields

$$\begin{aligned} &\int_{B(R)} f^2 |A|^p \Delta |A|^p \\ &\quad = \int_{B(R)} \mathrm{div}(f^2 |A|^p \nabla |A|^p) - \int_{B(R)} f^2 \big|\nabla |A|^p\big|^2 - 2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle \\ &\quad = - \int_{B(R)} f^2 \big|\nabla |A|^p\big|^2 - 2 \int_{B(R)} f |A|^p \langle \nabla f, \nabla |A|^p \rangle. \end{aligned}$$

Therefore

$$(3)\quad \left(1-\frac{n-2}{np}\right)\int_{B(R)} f^2\big|\nabla|A|^p\big|^2 \le p(2K_3-nK_2)\int_{B(R)} f^2|A|^{2p}+p\int_{B(R)} f^2|A|^{2p+2} - \int_{B(R)} f^2\big|\nabla|A|^p\big|^2-2\int_{B(R)} f|A|^p\langle\nabla f,\nabla|A|^p\rangle.$$

The stability of M implies that

$$(4)\qquad \int_M |\nabla f|^2-(|A|^2+\overline{\mathrm{Ric}}(e_{n+1}))f^2\ge 0$$

for any compactly supported Lipschitz function f on M. From our assumption on the sectional curvature of N, we see that

$$nK_1\le \overline{\mathrm{Ric}}(e_{n+1})=R_{n+1,1,n+1,1}+\cdots+R_{n+1,n,n+1,n}\le nK_2.$$

Hence the stability inequality (4) gives

$$(5)\qquad \int_M |\nabla f|^2-(|A|^2+nK_1)f^2\ge 0$$

for any compactly supported Lipschitz function f on M. Choose a Lipschitz function f with compact support in a geodesic ball $B(R)\subset M$, as before. Replacing f by $|A|^p f$ in inequality (5), we have

$$\int_M \big|\nabla(|A|^p f)\big|^2-(|A|^{2p+2}f^2+nK_1|A|^{2p}f^2)\ge 0.$$

Thus

$$(6)\quad \int_{B(R)}\big|\nabla|A|^p\big|^2 f^2+\int_{B(R)}|\nabla f|^2|A|^{2p}+2\int_{B(R)} f|A|^p\langle\nabla f,\nabla|A|^p\rangle \ge \int_{B(R)}|A|^{2p+2}f^2+nK_1\int_{B(R)}|A|^{2p}f^2.$$

Combining the inequalities (3) and (6), we get

$$(7)\quad \left(1-\frac{n-2}{np}\right)\int_{B(R)} f^2\big|\nabla|A|^p\big|^2 \le p(2K_3-nK_1-nK_2)\int_{B(R)} f^2|A|^{2p}+(p-1)\int_{B(R)} f^2\big|\nabla|A|^p\big|^2 + p\int_{B(R)}|\nabla f|^2|A|^{2p}+2(p-1)\int_{B(R)} f|A|^p\langle\nabla f,\nabla|A|^p\rangle.$$

On the other hand, from the definition of $\lambda_1(M)$ and the domain monotonicity of eigenvalues, it follows that

$$\lambda_1(M) \le \lambda_1(B(R)) \le \frac{\int_{B(R)} |\nabla f|^2}{\int_{B(R)} f^2} \tag{8}$$

for any compactly supported nonconstant Lipschitz function f on M. Substituting $|A|^p f$ for f in inequality (8), we see that

$$\begin{aligned} \lambda_1(M) \int_{B(R)} |A|^{2p} f^2 &\le \int_{B(R)} \big|\nabla(|A|^p f)\big|^2 \\ &= \int_{B(R)} f^2 \big|\nabla |A|^p\big|^2 + \int_{B(R)} |A|^{2p} |\nabla f|^2 + 2\int_{B(R)} f|A|^p \langle \nabla f, \nabla |A|^p\rangle. \end{aligned} \tag{9}$$

Plugging inequality (9) into (7), we have

$$\begin{aligned} &\Big(1 - \frac{n-2}{np}\Big) \int_{B(R)} f^2 \big|\nabla |A|^p\big|^2 \\ &\le \frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) \bigg(\int_{B(R)} f^2 \big|\nabla|A|^p\big|^2 \\ &\qquad\qquad + |\nabla f|^2 |A|^{2p} + 2\int_{B(R)} f|A|^p \langle \nabla f, \nabla |A|^p\rangle \bigg) \\ &+ (p-1) \int_{B(R)} f^2 \big|\nabla |A|^p\big|^2 + p\int_{B(R)} |\nabla f|^2 |A|^{2p} + 2(p-1) \int_{B(R)} f|A|^p \langle \nabla f, \nabla |A|^p\rangle. \end{aligned}$$

Thus

$$\begin{aligned} \Big(1 - \frac{n-2}{np}\Big) \int_{B(R)} f^2 \big|\nabla |A|^p\big|^2 &\le \Big(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1 \Big) \int_{B(R)} f^2 \big|\nabla |A|^p\big|^2 \\ &\quad + \Big(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p \Big) \int_{B(R)} |\nabla f|^2 |A|^{2p} \\ &\quad + 2\Big(\frac{p}{\lambda_1(M)} (2K_3 - nK_1 - nK_2) + p - 1 \Big) \int_{B(R)} f|A|^p \langle \nabla f, \nabla |A|^p\rangle. \end{aligned} \tag{10}$$

Note that Young's inequality yields

$$2\int_{B(R)} f|A|^p \langle \nabla f, \nabla |A|^p\rangle \le \varepsilon \int_{B(R)} |\nabla f|^2 |A|^{2p} + \frac{1}{\varepsilon} \int_{B(R)} f^2 |\nabla |A|^p|^2 \tag{11}$$

for any $\varepsilon > 0$. From inequalities (10) and (11), it follows that

$$\left(1-\frac{n-2}{np}\right)\int_{B(R)} f^2\big|\nabla|A|^p\big|^2 \\ \le \left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p-1\right)\int_{B(R)} f^2\big|\nabla|A|^p\big|^2 \\ +\left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p\right)\int_{B(R)} |\nabla f|^2|A|^{2p} \\ +\left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p-1\right)\left(\varepsilon\int_{B(R)}|\nabla f|^2|A|^{2p}+\frac{1}{\varepsilon}\int_{B(R)} f^2|\nabla|A|^p|^2\right),$$

which yields that

$$\left[1-\frac{n-2}{np}-\left(1+\frac{1}{\varepsilon}\right)\left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p-1\right)\right]\int_{B(R)} f^2\big|\nabla|A|^p\big|^2 \\ \le \left[(1+\varepsilon)\left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p\right)-\varepsilon\right]\int_{B(R)}|\nabla f|^2|A|^{2p}.$$

For a contradiction, we suppose that

$$\lambda_1(M) > \frac{p(2K_3-nK_1-nK_2)}{1-(n-2)/np-(p-1)} = \frac{np^2(2K_3-n(K_1+K_2))}{2-n(p-1)^2}.$$

Note the assumption that $1-\sqrt{2/n} < p < 1+\sqrt{2/n}$ is equivalent to

$$2-n(p-1)^2 > 0.$$

Choose a sufficiently large $\varepsilon > 0$ satisfying

$$\left[1-\frac{n-2}{np}-\left(1+\frac{1}{\varepsilon}\right)\left(\frac{p}{\lambda_1(M)}(2K_3-nK_1-nK_2)+p-1\right)\right] > 0.$$

Since $|\nabla f| \le 1/R$ by our choice of f, one can conclude that, by letting $R \to \infty$,

$$\int_M |\nabla|A|^p|^2 = 0,$$

where we used the growth condition on $\int_{B(R)} |A|^{2p}$. Thus we see that $|A|$ is constant. Since the volume of M is infinite [Wei 2003], we get $|A| \equiv 0$. This implies that M is totally geodesic, which is impossible by our assumption. Therefore we obtain the upper bound of $\lambda_1(M)$:

$$\lambda_1(M) \le \frac{np^2(2K_3-n(K_1+K_2))}{2-n(p-1)^2}. \qquad \square$$

Dung and the author [2012] gave an estimate of the bottom of the spectrum for the Laplace operator on a complete noncompact stable minimal hypersurface M in a complete simply connected Riemannian manifold with pinched negative sectional curvature under the assumption on L^2-norm of the second fundamental form A of M. In Theorem 2.1, if we take $p = 1$, we get the following.

Corollary 2.2 [Dung and Seo 2012]. *Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \le K_N \le K_2$, where K_1, K_2 are constants and $K_1 \le K_2 < 0$. Let M be a complete stable non-totally geodesic minimal hypersurface in N. Assume that*

$$\lim_{R\to\infty} R^{-2} \int_{B(R)} |A|^2 = 0,$$

where $B(R)$ is a geodesic ball of radius R on M. If $|\nabla K|^2 = \sum_{i,j,k,l,m} K^2_{ijkl;m} \le K_3^2 |A|^2$ for some constant $K_3 > 0$, we have

$$-K_2 \frac{(n-1)^2}{4} \le \lambda_1(M) \le \frac{(2K_3 - n(K_1+K_2))n}{2}.$$

In particular, if N is the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, one sees that $K_1 = K_2 = -1$, and hence $|\nabla K|^2 = 0$, that is, $K_3 = 0$. Moreover, it follows from McKean's result [1970] that the first eigenvalue $\lambda_1(M)$ of any complete totally geodesic hypersurface $M \subset \mathbb{H}^{n+1}$ satisfies $\lambda_1(M) = (n-1)^2/4$. Therefore we have the following consequence which is an extension of the result in [Seo 2011].

Corollary 2.3. *Let M be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |A|^{2p}\, dv < \infty$ for $1 - \sqrt{2/n} < p < 1 + \sqrt{2/n}$. Then we have*

$$-K_2 \frac{(n-1)^2}{4} \le \lambda_1(M) \le \frac{2n^2p^2}{2 - n(p-1)^2}.$$

As another application of Theorem 2.1, we have the following when $n < 8$.

Corollary 2.4. *Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature satisfying $K_1 \le K_N \le K_2$, where K_1, K_2 are constants and $K_1 \le K_2 < 0$ for $n < 8$. Let M be a complete stable non-totally geodesic minimal hypersurface in N. For $p = 1, 2, 3$, if $\int_M |A|^p < \infty$, we have*

$$-K_2 \frac{(n-1)^2}{4} \le \lambda_1(M) \le \frac{np^2(2K_3 - n(K_1+K_2))}{2 - n(p-1)^2}.$$

Proof. Since $\sqrt{2/n} > 1/2$ when $n < 8$, the conclusion can be derived from Theorem 2.1. □

3. Vanishing theorems on minimal hypersurfaces with $\lambda_1(M)$ bounded below

Before we prove the vanishing theorems for L^p harmonic 1-forms on complete minimal hypersurface, we begin with some useful facts.

Lemma 3.1 [Leung 1992]. *Let M be an n-dimensional complete immersed minimal hypersurface in a Riemannian manifold N. If all the sectional curvatures of N are bounded below by a constant K,*

$$\mathrm{Ric} \geq (n-1)K - \frac{n-1}{n}|A|^2.$$

Lemma 3.2 [Wang 2001]. *Let ω be a harmonic 1-form on an n-dimensional Riemannian manifold M. Then*

$$|\nabla\omega|^2 - \big|\nabla|\omega|\big|^2 \geq \frac{1}{n-1}\big|\nabla|\omega|\big|^2. \tag{12}$$

We also need the following well-known Sobolev inequality on a Riemannian manifold.

Lemma 3.3 [Hoffman and Spruck 1974]. *Let M^n be a complete immersed minimal submanifold in a nonpositively curved manifold N^{n+p}, $n \geq 3$. Then, for any $\phi \in W_0^{1,2}(M)$, we have*

$$\left(\int_M |\phi|^{2n/(n-2)}\, dv\right)^{(n-2)/n} \leq C_s \int_M |\nabla\phi|^2\, dv, \tag{13}$$

where C_s is the Sobolev constant which depends only on $n \geq 3$.

A complete Riemannian manifold M is called *nonparabolic* if it admits a nonconstant positive superharmonic function. Otherwise, M is said to be *parabolic*. The following sufficient condition for parabolicity is well known.

Theorem [Grigoryan 1983; 1985; Karp 1982; Varopoulos 1983]. *Let M be a complete Riemannian manifold. If, for any point $p \in M$ and a geodesic ball $B_p(r)$,*

$$\int_1^\infty \frac{r}{\mathrm{Vol}(B_p(r))}\, dr = \infty,$$

M is parabolic.

It immediately follows from this result that if M is nonparabolic,

$$\int_1^\infty \frac{r}{\mathrm{Vol}(B_p(r))}\, dr < \infty,$$

and hence M has infinite volume. Moreover, if $\lambda_1(M) > 0$, M is nonparabolic [Grigoryan 1999]. Therefore one can conclude the following.

Proposition 3.4. *Let M be an n-dimensional complete noncompact Riemannian manifold with $\lambda_1(M) > 0$. Then* $\mathrm{Vol}(M) = \infty$.

Note that, in the case of submanifolds, Cheung and Leung [1998] proved that the volume $\mathrm{Vol}(B_p(r))$ of every complete noncompact submanifold M in the Euclidean or hyperbolic space grows at least as a linear function of r under the assumption that the mean curvature vector H of M is bounded in absolute value.

We are now ready to state and prove vanishing theorems for L^p harmonic 1-forms on a complete noncompact stable minimal hypersurface.

Theorem 3.5. *Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \le K_N$ where $K \le 0$ is a constant. Let M be a complete noncompact stable minimal hypersurface in N. Assume that, for $0 < p < n/(n-1) + \sqrt{2n}$,*

$$\lambda_1(M) > \frac{-2n(n-1)^2 p^2 K}{2n - [(n-1)p - n]^2}.$$

Then there is no nontrivial L^{2p} harmonic 1-form on M.

Proof. We consider two cases: $K < 0$ and $K = 0$.

Case 1: $K < 0$. Let ω be an L^{2p} harmonic 1-form on M, that is,

$$\Delta\omega = 0 \quad \text{and} \quad \int_M |\omega|^{2p}\, dv < \infty.$$

In an abuse of notation, we refer to both a harmonic 1-form and its dual harmonic vector field by ω. Bochner's formula yields

$$\Delta|\omega|^2 = 2(|\nabla\omega|^2 + \mathrm{Ric}(\omega, \omega)).$$

Moreover,

$$\Delta|\omega|^2 = 2(|\omega|\Delta|\omega| + \big|\nabla|\omega|\big|^2).$$

Applying Lemma 3.1 and the Kato-type inequality (12), we see that

$$|\omega|\Delta|\omega| + \frac{n-1}{n}|A|^2|\omega|^2 - (n-1)K|\omega|^2 \ge \frac{1}{n-1}\big|\nabla|\omega|\big|^2. \tag{14}$$

For any positive number p, we have

$$\begin{aligned} |\omega|^p \Delta|\omega|^p &= |\omega|^p \mathrm{div}(\nabla|\omega|^p) \\ &= |\omega|^p \mathrm{div}(p\,|\omega|^{p-1}\nabla|\omega|) \\ &= p(p-1)|\omega|^{2p-2}\big|\nabla|\omega|\big|^2 + p|\omega|^{2p-1}\Delta|\omega| \\ &= \frac{p-1}{p}\big|\nabla|\omega|^p\big|^2 + p|\omega|^{2p-2}|\omega|\Delta|\omega|. \end{aligned}$$

Plugging inequality (14) into the above equality, we have

$$|\omega|^p\Delta|\omega|^p + p(n-1)\left(\frac{|A|^2}{n} - K\right)|\omega|^{2p} \geq \left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right)\big|\nabla|\omega|^p\big|^2.$$

Choose a Lipschitz function f with compact support in a geodesic ball $B(R)$ of radius R centered at $p \in M$. Multiplying both side by f^2 and integrating over $B(R)$, we obtain

$$\left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2$$
$$\leq \int_{B(R)} f^2|\omega|^p\Delta|\omega|^p + \frac{p(n-1)}{n}\int_{B(R)} f^2|A|^2|\omega|^{2p} - p(n-1)K\int_{B(R)} f^2|\omega|^{2p}.$$

The divergence theorem gives

$$\int_{B(R)} f^2|\omega|^p\Delta|\omega|^p = -\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 - 2\int_{B(R)} f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle.$$

Thus

$$(15)\quad \left(1 - \frac{1}{p} + \frac{1}{p(n-1)}\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2$$
$$\leq \frac{p(n-1)}{n}\int_{B(R)} f^2|A|^2|\omega|^{2p} - p(n-1)K\int_{B(R)} f^2|\omega|^{2p}$$
$$- \int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 - 2\int_{B(R)} f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle.$$

Since M is stable,

$$\int_M |\nabla f|^2 - (|A|^2 + \overline{\mathrm{Ric}}(e_{n+1}))f^2 \geq 0$$

for any compactly supported Lipschitz function f on M. From the assumption on the sectional curvature of N, it follows that

$$\int_M |\nabla f|^2 - (|A|^2 + nK)f^2 \geq 0$$

for any compactly supported Lipschitz function f on M. Replacing f by $|\omega|^p f$, we have

$$(16)\quad \int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 + \int_{B(R)} |\nabla f|^2|\omega|^{2p} + 2\int_{B(R)} f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle$$
$$\geq \int_{B(R)} f^2|A|^2|\omega|^{2p} + nK\int_{B(R)} f^2|\omega|^{2p}.$$

Combining the inequalities (15) and (16) gives

$$
\begin{aligned}
\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)&\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 \\
&\le \frac{p(n-1)}{n}\bigg[\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2+\int_{B(R)}|\nabla f|^2|\omega|^{2p} \\
&\qquad +2\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle-nK\int_{B(R)} f^2|\omega|^{2p}\bigg] \\
-p(n-1)K\int_{B(R)} f^2|\omega|^{2p}&-\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2-2\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(17)\quad \left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)&\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 \\
&\le\left(\frac{p(n-1)}{n}-1\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2+\frac{p(n-1)}{n}\int_{B(R)}|\nabla f|^2|\omega|^{2p} \\
-2p(n-1)K\int_{B(R)} f^2|\omega|^{2p}&+2\left(\frac{p(n-1)}{n}-1\right)\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle.
\end{aligned}
$$

Moreover, using the definition of the bottom of the spectrum, we see that

$$
\begin{aligned}
(18)\quad \lambda_1(M)\int_{B(R)}|\omega|^{2p}f^2
&\le\int_{B(R)}\big|\nabla(|\omega|^p f)\big|^2 \\
&=\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2+\int_{B(R)}|\omega|^{2p}|\nabla f|^2+2\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle.
\end{aligned}
$$

From inequalities (17) and (18), it follows that

$$
\begin{aligned}
\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)&\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 \\
&\le\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 \\
&\qquad+\left(\frac{p(n-1)}{n}-\frac{2p(n-1)K}{\lambda_1(M)}\right)\int_{B(R)}|\nabla f|^2|\omega|^{2p} \\
&\qquad+2\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle.
\end{aligned}
$$

Applying Young's inequality, we have

$$2\int_{B(R)} f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle \le \varepsilon\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 + \frac{1}{\varepsilon}\int_{B(R)} |\nabla f|^2|\omega|^{2p}$$

for any $\varepsilon > 0$. Thus

$$\left[2-\frac{1}{p}+\frac{1}{p(n-1)}+\frac{2p(n-1)K}{\lambda_1(M)}-\frac{p(n-1)}{n}-\varepsilon\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\right] \times\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2$$
$$\le\left[\frac{p(n-1)}{n}-\frac{2p(n-1)K}{\lambda_1(M)}+\frac{1}{\varepsilon}\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\right]\int_{B(R)} |\nabla f|^2|\omega|^{2p}.$$

Since

$$\lambda_1(M) > \frac{-2p(n-1)K}{2-1/p+1/(p(n-1))-p(n-1)/n} = \frac{-2n(n-1)^2p^2K}{2n-[(n-1)p-n]^2}$$

by the hypothesis, one can choose a sufficiently small $\varepsilon > 0$ satisfying that

$$\left[2-\frac{1}{p}+\frac{1}{p(n-1)}+\frac{2p(n-1)K}{\lambda_1(M)}-\frac{p(n-1)}{n}-\varepsilon\left(\frac{p(n-1)}{n}-1-\frac{2p(n-1)K}{\lambda_1(M)}\right)\right] > 0.$$

Note that $\int_M |\omega|^{2p} < \infty$, since ω is an L^{2p} harmonic 1-form on M. Letting R tend to infinity, we obtain

$$\int_M \big|\nabla|\omega|^p\big|^2 = 0,$$

which implies that $\big|\nabla|\omega|\big| \equiv 0$. Hence $|\omega| \equiv$ constant. From Proposition 3.4, it follows that $|\omega| \equiv 0$.

Case 2: $K = 0$. Using the inequality (17) and Young's inequality, we obtain

$$\left[2-\frac{1}{p}+\frac{1}{p(n-1)}-\frac{p(n-1)}{n}-\varepsilon\left(\frac{p(n-1)}{n}-1\right)\right]\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2$$
$$\le\left[\frac{p(n-1)}{n}+\frac{1}{\varepsilon}\left(\frac{p(n-1)}{n}-1\right)\right]\int_{B(R)} |\nabla f|^2|\omega|^{2p}.$$

Since $0 < p < n/(n-1)+\sqrt{2n}$, one may choose a sufficiently small $\varepsilon > 0$ satisfying

$$2-\frac{1}{p}+\frac{1}{p(n-1)}-\frac{p(n-1)}{n}-\varepsilon\left(\frac{p(n-1)}{n}-1\right) > 0.$$

Letting R tend to infinity gives

$$\int_{B(R)} \big|\nabla|\omega|^p\big|^2 = 0,$$

which implies that $|\omega| \equiv$ constant. From the assumption that $\lambda_1(M) > 0$ and Proposition 3.4, it follows that $|\omega| \equiv 0$. □

As a consequence of Theorem 3.5, given a complete noncompact stable minimal hypersurface in a nonnegatively curved Riemannian manifold, one has the following result.

Corollary 3.6. *Let N be an $(n+1)$-dimensional complete nonnegatively curved Riemannian manifold. Let M be a complete noncompact stable minimal hypersurface in N with $\lambda_1(M) > 0$. If $n \le 11$, there is no nontrivial L^p harmonic 1-form on M for any $0 < p \le n$.*

Proof. For $n \le 11$, the inequality $2(n/(n-1) + \sqrt{2n}) \ge n$ holds. □

Corollary 3.7. *Let N be an $(n+1)$-dimensional complete nonnegatively curved Riemannian manifold. Let M be a complete noncompact stable minimal hypersurface in N with $\lambda_1(M) > 0$. If $n \le 11$, there is no nontrivial L^2 harmonic 1-form on M.*

In the case of L^2 harmonic 1-forms, Theorem 3.5 gives a generalization of [Dung and Seo 2012] as follows.

Corollary 3.8. *Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \le K_N$ where $K < 0$ is a constant. Let M be a complete noncompact stable minimal hypersurface in N. Assume that*

$$\lambda_1(M) > \frac{-2n(n-1)^2 K}{2n-1}.$$

Then there are no nontrivial L^2 harmonic 1-forms on M.

In particular, if N is $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, Corollary 3.8 improves the previous result of [Seo 2010]. Related to this result, Cavalcante, Mirandola, and Vitório [Cavalcante et al. 2012] obtained the vanishing theorem for L^2 harmonic 1-forms on complete noncompact submanifolds in a Cartan–Hadamard manifold.

Palmer [1991] showed that if there exists a codimension-one cycle in a complete minimal hypersurface M in $\mathbb{R}^{n+1}$ which does not separate M, M is unstable. We obtain a generalization of Palmer's result as follows.

Corollary 3.9. *Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \le K_N$ where $K \le 0$ is a constant. Let M be a complete noncompact minimal hypersurface in N. Assume that*

$$\lambda_1(M) > \frac{-2n(n-1)^2 K}{2n-1}.$$

Suppose that there exists a codimension-one cycle in M which does not separate M. Then M cannot be stable.

Proof. Suppose that M is stable in N. From [Dodziuk 1982], there exists a nontrivial L^2 harmonic 1-form on M, which is a contradiction to Corollary 3.8. □

Let M be a complete Riemannian manifold and let f be a harmonic function on M with finite L^p energy. Then the total differential df is obviously an L^p harmonic 1-form on M. As another application of Theorem 3.5, we prove the following Liouville type theorem for harmonic functions with finite L^p energy on a complete noncompact stable minimal hypersurface, which is a generalization of Schoen and Yau's result [1976], as mentioned in the introduction.

Corollary 3.10. *Let N be an $(n+1)$-dimensional complete Riemannian manifold with sectional curvature satisfying $K \le K_N$ where $K \le 0$ is a constant. Let M be a complete noncompact stable minimal hypersurface in N. Assume that, for $0 < p < n/(n-1) + \sqrt{2n}$,*

$$\lambda_1(M) > \frac{-2n(n-1)^2 p^2 K}{2n - [(n-1)p - n]^2}.$$

Then there is no nontrivial harmonic function on M with finite L^p energy.

So far, we have assumed that $\lambda_1(M) > 0$ for a complete noncompact stable minimal hypersurface M in a nonnegatively curved Riemannian manifold. However, we do not know whether the assumption that $\lambda_1(M) > 0$ is necessary or not. It would be interesting to remove the condition in these results.

4. Vanishing theorems on minimal hypersurfaces with small L^n or L^∞ norm of the second fundamental form

In the following, we prove a vanishing theorem for L^p harmonic 1-forms on a complete stable minimal hypersurface M, assuming that M has sufficiently small total scalar curvature instead of assuming that M is stable.

Theorem 4.1. *Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying that $K_1 \le K_N \le K_2 < 0$, where K_1, K_2 are constants and $n \ge 3$. Let M be a complete minimal hypersurface in N. Assume that $K := K_2/K_1$ satisfies*

$$K > \frac{4(n-2)}{(n-1)^2}.$$

For

$$\frac{(n-1)K}{4}-\frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4}-(n-2)K} < p < \frac{(n-1)K}{4}+\frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4}-(n-2)K},$$

assume that

$$\left(\int_M |A|^n\right)^{2/n} < \frac{n(2p(n-1)-n+2-4p^2K)}{p^2(n-1)^2C_s},$$

where C_s *is the Sobolev constant in [Hoffman and Spruck 1974]. Then there are no nontrivial* L^{2p} *harmonic* 1*-forms on* M.

Proof. A similar argument as in the proof of Theorem 3.5 shows

$$|\omega|^p\Delta|\omega|^p + p(n-1)\left(\frac{|A|^2}{n}-K_1\right)|\omega|^{2p} \geq \left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\big|\nabla|\omega|^p\big|^2$$

for any Lipschitz function f with compact support in a geodesic ball $B(R)$ of radius R centered at a point $p\in M$. Multiplying both sides by f^2, integrating over $B(R)$, and applying the divergence theorem, we see that

$$\begin{aligned}(19)\quad \left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 &\leq \frac{p(n-1)}{n}\int_{B(R)} f^2|A|^2|\omega|^{2p} - p(n-1)K_1\int_{B(R)} f^2|\omega|^{2p}\\ &\quad -\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 - 2\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle.\end{aligned}$$

On the other hand, the Sobolev inequality (13) implies that

$$\begin{aligned}\int_{B(R)} f^2|A|^2|\omega|^{2p} &\leq \left(\int_M |A|^n\right)^{2/n}\left(\int_M (|\omega|^p f)^{(2n)/n-2}\right)^{(n-2)/n}\\ &\leq C_s\left(\int_M |A|^n\right)^{2/n}\int_M \big|\nabla(|\omega|^p f)\big|^2\\ &\leq C_s\left(\int_M |A|^n\right)^{2/n}\left(\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 + \int_{B(R)}|\nabla f|^2|\omega|^{2p}\right.\\ &\qquad\left. + 2\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle\right).\end{aligned}$$

Plugging this inequality into (19) gives

$$(20)\quad \left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 \le \frac{p(n-1)C_s}{n}\left(\int_M |A|^n\right)^{2/n}\int_{B(R)}|\nabla f|^2|\omega|^{2p} + \left(\frac{p(n-1)C_s}{n}\left(\int_M |A|^n\right)^{2/n}-1\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 + 2\left(\frac{p(n-1)C_s}{n}\left(\int_M |A|^n\right)^{2/n}-1\right)\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle - p(n-1)K_1\int_{B(R)} f^2|\omega|^{2p}.$$

An estimate (1) for the bottom of the spectrum yields

$$-\frac{K_2(n-1)^2}{4}\le\lambda_1(M)\le\frac{\int_{B(R)}\big|\nabla(|\omega|^p f)\big|^2}{\int_{B(R)}(|\omega|^p f)^2},$$

which gives

$$(21)\quad \int_{B(R)}(|\omega|^p f)^2 \le -\frac{4}{K_2(n-1)^2}\left(\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2+\int_{B(R)} u|\nabla f|^2|\omega|^{2p} + 2\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle\right).$$

Thus, from inequalities (20) and (21), it follows that

$$\left(1-\frac{1}{p}+\frac{1}{p(n-1)}\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 \le B\int_{B(R)}|\nabla f|^2|\omega|^{2p}+(B-1)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2+2(B-1)\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle,$$

where

$$B=\frac{p(n-1)C_s}{n}\left(\int_M |A|^n\right)^{2/n}+\frac{4p}{(n-1)}\frac{1}{K}.$$

Applying Young's inequality

$$2\int_{B(R)} f|\omega|^p\langle\nabla f,\nabla|\omega|^p\rangle\le\varepsilon\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2+\frac{1}{\varepsilon}\int_{B(R)}|\nabla f|^2|\omega|^{2p}$$

for any $\varepsilon > 0$, we see that

$$\left(2-\frac{1}{p}+\frac{1}{p(n-1)}-B-\varepsilon(B-1)\right)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 \le \left(B+\frac{1}{\varepsilon}(B-1)\right)\int_{B(R)}|\nabla f|^2|\omega|^{2p}.$$

From the assumption on the total curvature of M, one can make

$$\left(2-\frac{1}{p}+\frac{1}{p(n-1)}-B-\varepsilon(B-1)\right) > 0$$

by choosing a sufficiently small $\varepsilon > 0$. Letting $R \to \infty$ and using that ω is an L^{2p} harmonic 1-form, we conclude that

$$\int_M \big|\nabla|\omega|^p\big|^2 = 0.$$

The same argument as before shows that $|\omega| \equiv 0$. □

Corollary 4.2. *Let M be a complete minimal hypersurface in $\mathbb{H}^{n+1}$ satisfying*

$$\left(\int_M |A|^n\right)^{2/n} < \frac{n(-4p^2+2p(n-1)-n+2)}{p^2(n-1)^2C_s}$$

for $1/2 < p < n/2-1$. Then there are no nontrivial L^{2p} harmonic 1-forms on M.

Corollary 4.3. *Under the same conditions as in Theorem 4.1, there is no nontrivial harmonic function on M with finite L^p energy.*

When the L^∞ norm of the second fundamental form of a complete minimal hypersurface is bounded, the following vanishing theorem holds.

Theorem 4.4. *Let N be an $(n+1)$-dimensional complete simply connected Riemannian manifold with sectional curvature K_N satisfying $K_1 \le K_N \le K_2 < 0$, where K_1, K_2 are constants and $n \ge 3$. Let M be a complete noncompact minimal hypersurface in N. Assume that $K := K_2/K_1 > 4(n-2)/(n-1)^2$ and the second fundamental form A satisfies*

$$|A|^2 \le C < \frac{4p^2K_1-(2p(n-1)-n+2)K_2}{4p^2}$$

for

$$\frac{(n-1)K}{4}-\frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4}-(n-2)K} < p < \frac{(n-1)K}{4}+\frac{1}{2}\sqrt{\frac{(n-1)^2K^2}{4}-(n-2)K}.$$

Then there are no nontrivial L^{2p} harmonic 1-forms on M.

Proof. A similar argument as before shows

$$\begin{aligned}\Big(1-\frac{1}{p}+\frac{1}{p(n-1)}\Big)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 &\le \frac{p(n-1)}{n}\int_{B(R)} f^2|A|^2|\omega|^{2p} - p(n-1)K_1\int_{B(R)} f^2|\omega|^{2p}\\ &\quad - \int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 - 2\int_{B(R)} f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle.\end{aligned}$$

Since $|A|^2 \le C$,

$$\begin{aligned}\Big(2-\frac{1}{p}+\frac{1}{p(n-1)}\Big)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 &\le \Big(\frac{p(n-1)C}{n} - p(n-1)K_1\Big)\int_{B(R)} f^2|\omega|^{2p} - 2\int_{B(R)} f|\omega|^p\langle\nabla f, \nabla|\omega|^p\rangle.\end{aligned}$$

Using an estimate for the bottom of the spectrum and Young's inequality again, we have

$$\begin{aligned}\Big(2-\frac{1}{p}+\frac{1}{p(n-1)} - D - \varepsilon(D-1)\Big)\int_{B(R)} f^2\big|\nabla|\omega|^p\big|^2 &\le \Big(D+\frac{1}{\varepsilon}(D-1)\Big)\int_{B(R)} |\nabla f|^2|\omega|^{2p},\end{aligned}$$

where

$$D = \frac{-4}{(n-1)^2K_2}\Big(\frac{p(n-1)C}{n} - p(n-1)K_1\Big).$$

Since

$$C < \frac{4p^2K_1 - (2p(n-1)-n+2)K_2}{4p^2},$$

by our assumption, we may choose a sufficiently small $\varepsilon > 0$ satisfying

$$\Big(2-\frac{1}{p}+\frac{1}{p(n-1)} - D - \varepsilon(D-1)\Big) > 0.$$

Thus we get

$$\int_{B(R)} \big|\nabla|\omega|^p\big|^2 = 0$$

by letting R tend to infinity. Hence $\omega \equiv 0$. □

Corollary 4.5. *Let M be a complete minimal hypersurface in $\mathbb{H}^{n+1}$ with the second fundamental form A satisfying*

$$|A|^2 \le C < \frac{-4p^2 + 2p(n-1) - n + 2}{4p^2}$$

for $1/2 < p < n/2 - 1$. Then there are no nontrivial L^{2p} harmonic 1-forms on M.

Corollary 4.6. *Under the same conditions as in Theorem 4.4, there is no nontrivial harmonic function on M with finite L^p energy.*

We remark that there are lots of examples of minimal hypersurfaces with finite L^n or L^∞ norm of the second fundamental form in $\mathbb{H}^{n+1}$ [do Carmo and Dajczer 1983; Mori 1981; Ripoll 1989; Seo 2011].

References

[Anderson 1988] M. T. Anderson, "L^2 harmonic forms on complete Riemannian manifolds", pp. 1–19 in *Geometry and analysis on manifolds* (Katata/Kyoto, 1987), edited by T. Sunada, Lecture Notes in Math. **1339**, Springer, Berlin, 1988. MR 89j:58004 Zbl 0652.53030

[Bessa and Montenegro 2003] G. P. Bessa and J. F. Montenegro, "Eigenvalue estimates for submanifolds with locally bounded mean curvature", *Ann. Global Anal. Geom.* **24**:3 (2003), 279–290. MR 2004f:53068 Zbl 1060.53063

[Candel 2007] A. Candel, "Eigenvalue estimates for minimal surfaces in hyperbolic space", *Trans. Amer. Math. Soc.* **359**:8 (2007), 3567–3575. MR 2007m:53076 Zbl 1115.53005

[Cao et al. 1997] H.-D. Cao, Y. Shen, and S. Zhu, "The structure of stable minimal hypersurfaces in $\mathbf{R}^{n+1}$", *Math. Res. Lett.* **4**:5 (1997), 637–644. MR 99a:53037 Zbl 0906.53004

[do Carmo and Dajczer 1983] M. do Carmo and M. Dajczer, "Rotation hypersurfaces in spaces of constant curvature", *Trans. Amer. Math. Soc.* **277**:2 (1983), 685–709. MR 85b:53055 Zbl 0518.53059

[Carron 2002] G. Carron, "L^2 harmonic forms on non-compact Riemannian manifolds", pp. 49–59 in *Surveys in analysis and operator theory* (Canberra, 2001), edited by A. Hassell, Proc. Centre Math. Appl. Austral. Nat. Univ. **40**, Austral. Nat. Univ., Canberra, 2002. MR 2003j:58001 Zbl 1038.58023

[Cavalcante et al. 2012] M. Cavalcante, H. Mirandola, and F. Vitorio, "L^2 harmonic 1-forms on submanifolds with finite total curvature", preprint, 2012. arXiv 1201.5392

[Cheung and Leung 1998] L.-F. Cheung and P.-F. Leung, "The mean curvature and volume growth of complete noncompact submanifolds", *Differential Geom. Appl.* **8**:3 (1998), 251–256. MR 99k:53111 Zbl 0942.53037

[Cheung and Leung 2001] L.-F. Cheung and P.-F. Leung, "Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space", *Math. Z.* **236**:3 (2001), 525–530. MR 2002c:53094 Zbl 0990.53029

[Colding and Minicozzi 1996] T. H. Colding and W. P. Minicozzi, II, "Generalized Liouville properties of manifolds", *Math. Res. Lett.* **3**:6 (1996), 723–729. MR 97h:53039 Zbl 0884.58032

[Colding and Minicozzi 1997] T. H. Colding and W. P. Minicozzi, II, "Harmonic functions on manifolds", *Ann. of Math.* (2) **146**:3 (1997), 725–747. MR 98m:53052 Zbl 0928.53030

[Colding and Minicozzi 1998] T. H. Colding and W. P. Minicozzi, II, "Liouville theorems for harmonic sections and applications", *Comm. Pure Appl. Math.* **51**:2 (1998), 113–138. MR 98m:53053 Zbl 0928.58022

[Deng 2008] Q. Deng, "Complete hypersurfaces with constant mean curvature and finite L^p norm curvature in Euclidean spaces", *Arch. Math.* (*Basel*) **90**:4 (2008), 360–373. MR 2009i:53054 Zbl 1137.53017

[Dodziuk 1982] J. Dodziuk, "L^2−harmonic forms on complete manifolds.", in *Semin. differential geometry*, edited by S.-T. Yao, Ann. Math. Stud. **102**, 1982. Zbl 0484.53033

[Dung and Seo 2012] N. T. Dung and K. Seo, "Stable minimal hypersurfaces in a Riemannian manifold with pinched negative sectional curvature", *Ann. Global Anal. Geom.* **41**:4 (2012), 447–460. MR 2891296 Zbl 1242.53073

[Fu 2012] H.-P. Fu, "Bernstein type theorems for complete submanifolds in space forms", *Math. Nachr.* **285**:2-3 (2012), 236–244. MR 2012m:53135 Zbl 06012479

[Greene and Wu 1974] R. E. Greene and H. Wu, "Integrals of subharmonic functions on manifolds of nonnegative curvature", *Invent. Math.* **27** (1974), 265–298. MR 52 #3605 Zbl 0342.31003

[Greene and Wu 1981] R. E. Greene and H. Wu, "Harmonic forms on noncompact Riemannian and Kähler manifolds", *Michigan Math. J.* **28**:1 (1981), 63–81. MR 82e:58005

[Grigoryan 1983] A. A. Grigoryan, "Existence of the Green function on a manifold", *Uspekhi Mat. Nauk* **38**:1(229) (1983), 161–162. In Russian; translated in *Russian Math. Surveys* **38**:1 (1983), 190–191. MR 84i:58128

[Grigoryan 1985] A. A. Grigoryan, "The existence of positive fundamental solutions of the Laplace equation on Riemannian manifolds", *Mat. Sb.* (*N.S.*) **128(170)**:3 (1985), 354–363, 446. MR 87d:58140

[Grigoryan 1999] A. Grigoryan, "Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds", *Bull. Amer. Math. Soc.* (*N.S.*) **36**:2 (1999), 135–249. MR 99k:58195

[Hoffman and Spruck 1974] D. Hoffman and J. Spruck, "Sobolev and isoperimetric inequalities for Riemannian submanifolds", *Comm. Pure Appl. Math.* **27** (1974), 715–727. MR 51 #1676 Zbl 0295.53025

[Karp 1982] L. Karp, "Subharmonic functions, harmonic mappings and isometric immersions.", in *Semin. differential geometry*, edited by S.-T. Yau, Ann. Math. Stud. **102**, 1982. Zbl 0487.53046

[Leung 1992] P. F. Leung, "An estimate on the Ricci curvature of a submanifold and some applications", *Proc. Amer. Math. Soc.* **114**:4 (1992), 1051–1061. MR 92g:53052 Zbl 0753.53003

[Li and Schoen 1984] P. Li and R. Schoen, "L^p and mean value properties of subharmonic functions on Riemannian manifolds", *Acta Math.* **153**:3-4 (1984), 279–301. MR 86j:58147 Zbl 0556.31005

[Li and Tam 1987] P. Li and L.-F. Tam, "Positive harmonic functions on complete manifolds with nonnegative curvature outside a compact set", *Ann. of Math.* (2) **125**:1 (1987), 171–207. MR 88m:58039

[Li and Tam 1992] P. Li and L.-F. Tam, "Harmonic functions and the structure of complete manifolds", *J. Differential Geom.* **35**:2 (1992), 359–383. MR 93b:53033 Zbl 0768.53018

[Li and Wang 2002] P. Li and J. Wang, "Minimal hypersurfaces with finite index", *Math. Res. Lett.* **9**:1 (2002), 95–103. MR 2003b:53066 Zbl 1019.53025

[Li and Wang 2004] P. Li and J. Wang, "Stable minimal hypersurfaces in a nonnegatively curved manifold", *J. Reine Angew. Math.* **566** (2004), 215–230. MR 2005e:53093 Zbl 1050.53049

[McKean 1970] H. P. McKean, "An upper bound to the spectrum of Δ on a manifold of negative curvature", *J. Differential Geometry* **4** (1970), 359–366. MR 42 #1009 Zbl 0197.18003

[Miyaoka 1993] R. Miyaoka, "L^2 harmonic 1-forms on a complete stable minimal hypersurface", pp. 289–293 in *Geometry and global analysis* (Sendai, 1993), edited by T. Kotake et al., Tohoku Univ., Sendai, 1993. MR 96g:53102 Zbl 0912.53042

[Mori 1981] H. Mori, "Minimal surfaces of revolution in H^3 and their global stability", *Indiana Univ. Math. J.* **30**:5 (1981), 787–794. MR 82k:53082 Zbl 0589.53007

[Ni 2001] L. Ni, "Gap theorems for minimal submanifolds in $\mathbf{R}^{n+1}$", *Comm. Anal. Geom.* **9**:3 (2001), 641–656. MR 2002m:53097 Zbl 1020.53041

[Palmer 1991] B. Palmer, "Stability of minimal hypersurfaces", *Comment. Math. Helv.* **66**:2 (1991), 185–188. MR 92m:58023 Zbl 0736.53054

[Pigola and Veronelli 2012] S. Pigola and G. Veronelli, "Remarks on L^p-vanishing results in geometric analysis", *Internat. J. Math.* **23**:1 (2012), 1250008, 18. MR 2888937 Zbl 1252.53072

[Pigola et al. 2005] S. Pigola, M. Rigoli, and A. G. Setti, "Vanishing theorems on Riemannian manifolds, and geometric applications", *J. Funct. Anal.* **229**:2 (2005), 424–461. MR 2006k:53055 Zbl 1087.58022

[Pigola et al. 2008] S. Pigola, M. Rigoli, and A. G. Setti, *Vanishing and finiteness results in geometric analysis: a generalization of the Bochner technique*, Progress in Mathematics **266**, Birkhäuser, Basel, 2008. MR 2009m:58001 Zbl 1150.53001

[Ripoll 1989] J. B. Ripoll, "Helicoidal minimal surfaces in hyperbolic space", *Nagoya Math. J.* **114** (1989), 65–75. MR 91a:53015 Zbl 0699.53067

[Schoen and Yau 1976] R. Schoen and S. T. Yau, "Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature", *Comment. Math. Helv.* **51**:3 (1976), 333–341. MR 55 #11302 Zbl 0361.53040

[Schoen et al. 1975] R. Schoen, L. Simon, and S. T. Yau, "Curvature estimates for minimal hypersurfaces", *Acta Math.* **134**:3-4 (1975), 275–288. MR 54 #11243 Zbl 0323.53039

[Scott 1995] C. Scott, "L^p theory of differential forms on manifolds", *Trans. Amer. Math. Soc.* **347**:6 (1995), 2075–2096. MR 95i:58009 Zbl 0849.58002

[Seo 2008] K. Seo, "Minimal submanifolds with small total scalar curvature in Euclidean space", *Kodai Math. J.* **31**:1 (2008), 113–119. MR 2009f:53095 Zbl 1147.53313

[Seo 2010] K. Seo, "L^2 harmonic 1-forms on minimal submanifolds in hyperbolic space", *J. Math. Anal. Appl.* **371**:2 (2010), 546–551. MR 2011j:58054 Zbl 1195.53087

[Seo 2011] K. Seo, "Stable minimal hypersurfaces in the hyperbolic space", *J. Korean Math. Soc.* **48**:2 (2011), 253–266. MR 2012c:53096 Zbl 1211.53080

[Seo 2012] K. Seo, "Isoperimetric inequalities for submanifolds with bounded mean curvature", *Monatsh. Math.* **166**:3-4 (2012), 525–542. MR 2925153 Zbl 1245.58009

[Shen and Zhu 2005] Y. B. Shen and X. H. Zhu, "On complete hypersurfaces with constant mean curvature and finite L^p-norm curvature in $\mathbf{R}^{n+1}$", *Acta Math. Sin.* (*Engl. Ser.*) **21**:3 (2005), 631–642. MR 2006f:53088 Zbl 1087.53054

[Varopoulos 1983] N. T. Varopoulos, "Potential theory and diffusion on Riemannian manifolds", pp. 821–837 in *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II* (Chicago, 1981), edited by W. Beckner et al., Wadsworth, Belmont, CA, 1983. MR 85a:58103 Zbl 0558.31009

[Wang 2001] X. Wang, "On conformally compact Einstein manifolds", *Math. Res. Lett.* **8**:5-6 (2001), 671–688. MR 2003d:53075 Zbl 1053.53030

[Wei 2003] S. W. Wei, "The structure of complete minimal submanifolds in complete manifolds of nonpositive curvature", *Houston J. Math.* **29**:3 (2003), 675–689. MR 2004f:53075 Zbl 1130.53307

[Xin 2005] Y. L. Xin, "Berstein type theorems without graphic condition", *Asian J. Math.* **9**:1 (2005), 31–44. MR 2006b:53010

[Yau 1976] S. T. Yau, "Some function-theoretic properties of complete Riemannian manifold and their applications to geometry", *Indiana Univ. Math. J.* **25**:7 (1976), 659–670. MR 54 #5502 Zbl 0335.53041

[Yun 2002] G. Yun, "Total scalar curvature and L^2 harmonic 1-forms on a minimal hypersurface in Euclidean space", *Geom. Dedicata* **89** (2002), 135–141. MR 2003a:53091

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KEOMKYO SEO
DEPARTMENT OF MATHEMATICS
SOOKMYUNG WOMEN'S UNIVERSITY
HYOCHANGWONGIL 52
SEOUL 140-742
SOUTH KOREA
kseo@sookmyung.ac.kr

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