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Let R be a commutative noetherian ring and M a finitely generated R -module. In this paper, we reconstruct M from its Koszul homology with respect to a suitable sequence of elements of R by taking direct summands, syzygies and extensions, and count the number of those operations. Using this result, we consider generation and classification of certain subcategories of the category of finitely generated R -modules, its bounded derived category and the singularity category of R .

1. Introduction

For the past five decades, a lot of classification theorems of subcategories of abelian categories and triangulated categories have been given in ring theory, representation theory, algebraic geometry and algebraic topology; see, for instance, [Balmer 2002; 2005; Benson et al. 2011; Dao and Takahashi 2014; Friedlander and Pevtsova 2007; Gabriel 1962; Hopkins and Smith 1998; Hovey 2001; Krause 2008; Krause and Stevenson 2013; Neeman 1992; Stevenson 2014; Takahashi 2010; 2013; Thomason 1997]. Reconstruction of an object from its *support* in the spectrum of a suitable commutative ring plays a crucial role in the proofs of those theorems.

The notion of dimension for triangulated categories was introduced by Bondal and Van den Bergh [2003] and by Rouquier [2008]; analogues for abelian categories were introduced by Dao and Takahashi [2011; 2012a]. They essentially indicate the number of *extensions* necessary to build all objects out of a single object. There are many related studies; for example, see [Aihara and Takahashi 2011; Avramov et al. 2010a; Ballard et al. 2012; Bergh et al. 2010; Burke et al. 2012; Christensen 1998; Krause and Kussin 2006; Dao and Takahashi 2012b; Oppermann 2009; Orlov 2009b; Rouquier 2006; Schoutens 2003; Takahashi 2009].

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In this paper, we study reconstructing a given module from its Koszul homology and counting the number of necessary operations. Our main result is the following theorem.

Theorem 1.1. *Let R be a commutative noetherian ring, and let M be a finitely generated R -module. Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of R such that M is locally free on $D(\mathbf{x})$. Then there exists a positive integer k such that the Koszul complex $K(\mathbf{x}^k, M)$ is equivalent to a complex of finitely generated R -modules*

$$(0 \rightarrow N \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow 0),$$

where P_0, \dots, P_{n-1} are projective and M is a direct summand of N . In particular, M can be built out of the Koszul homologies $H_0(\mathbf{x}^k, M), \dots, H_n(\mathbf{x}^k, M)$ by taking n syzygies, n extensions and 1 direct summand.

Note that since the free locus of a finitely generated R -module is an open subset of $\text{Spec } R$ in the Zariski topology, there exist many such sequences \mathbf{x} that satisfy the assumption of the theorem. We shall prove a more general result in [Theorem 3.1](#).

[Theorem 1.1](#) has a lot of applications. To state some of them, we fix notation. Let $\text{mod } R$ be the category of finitely generated R -modules and $\mathbf{D}_b(R)$ the bounded derived category of $\text{mod } R$. We denote by $\mathbf{D}_{\text{sg}}(R)$ the *singularity category* of R . This category has been introduced and studied by Buchweitz [\[1986\]](#) in connection with Cohen–Macaulay modules over Gorenstein rings. In recent years, it has been investigated by Orlov [\[2004; 2006; 2009a; 2011; 2012\]](#) in relation to the homological mirror symmetry conjecture.

Let $S(R)$ be the set of prime ideals \mathfrak{p} of R such that $R_{\mathfrak{p}}$ is not a field, and denote by $\text{Sing } R$ the singular locus of R . Applying [Theorem 1.1](#), we can prove the following result on classification of subcategories.

Corollary 1.2. *Let R be a commutative noetherian ring.*

(1) *There is a one-to-one correspondence between:*

- (a) *the specialization-closed subsets of $S(R)$,*
- (b) *the resolving subcategories of $\text{mod } R$ generated by a Serre subcategory of $\text{mod } R$.*

(2) *There are one-to-one correspondences among:*

- (a) *the specialization-closed subsets of $\text{Sing } R$,*
- (b) *the thick subcategories of $\mathbf{D}_b(R)$ generated by R and a Serre subcategory of $\text{mod } R$,*
- (c) *the thick subcategories of $\mathbf{D}_{\text{sg}}(R)$ generated by a Serre subcategory of $\text{mod } R$.*

When R is local, let $\text{mod}^\circ(R)$ (respectively, $\mathbf{D}_b^\circ(R)$, $\mathbf{D}_{\text{sg}}^\circ(R)$) be the full subcategories of $\text{mod } R$ (respectively, $\mathbf{D}_b(R)$, $\mathbf{D}_{\text{sg}}(R)$) consisting of modules (respectively, complexes) that are locally free (respectively, perfect, zero) on the punctured spectrum of R . Applying [Theorem 1.1](#), we can prove the following result on generation of subcategories.

Corollary 1.3. *Let R be a commutative noetherian local ring of Krull dimension d with residue field k .*

(1) *Every object in $\text{mod}^\circ(R)$ is built out of a module of finite length by taking d extensions in $\text{mod } R$, up to finite direct sums, direct summands and syzygies.*

(2) *Every object in $\mathbf{D}_{\text{sg}}^\circ(R)$ is built out of a module of finite length by taking d extensions in $\mathbf{D}_{\text{sg}}(R)$, up to finite direct sums, direct summands and shifts.*

In particular, one has that $\text{mod}^\circ(R)$ is generated by k as a resolving subcategory of $\text{mod } R$, that $\mathbf{D}_b^\circ(R)$ is generated by R and k as a thick subcategory of $\mathbf{D}_b(R)$, and that $\mathbf{D}_{\text{sg}}^\circ(R)$ is generated by k as a thick subcategory of $\mathbf{D}_{\text{sg}}(R)$.

[Corollary 1.3](#) yields variants of results shown by Schoutens [\[2003\]](#) and Takahashi [\[2009; 2010\]](#). It also recovers a result on isolated singularities given by Keller–Murfet–Van den Bergh [\[2011\]](#). Furthermore, utilizing it, one can show the following result.

Corollary 1.4. *Let R be a commutative noetherian ring. The following are equivalent for a resolving subcategory \mathcal{X} of $\text{mod } R$:*

(1) *\mathcal{X} is generated by a Serre subcategory of $\text{mod } R$.*

(2) *\mathcal{X} is closed under tensor products and transposes.*

Hence there is a one-to-one correspondence between the specialization-closed subsets of $S(R)$ and the resolving subcategories of $\text{mod } R$ closed under tensor products and transposes.

The last assertion of this corollary greatly improves the main result of [\[Takahashi 2013\]](#). Indeed, it removes the superfluous assumptions that R is local and that R is Cohen–Macaulay.

The organization of this paper is as follows. In the next [Section 2](#) we prepare some fundamental notions. In [Section 3](#) we state and prove the most general result in this paper, which includes [Theorem 1.1](#). In the final [Section 4](#) we apply the results shown in the preceding section to find out the structure of certain subcategories, and give several results including [Corollaries 1.2, 1.3 and 1.4](#).

2. Basic definitions

This section is devoted to stating the definitions and basic properties of notions which we will *freely* use in the later sections. We begin with our convention.

Convention 2.1. Throughout the present paper, let R be a commutative noetherian ring with identity. We assume that all R -modules are finitely generated, that all R -complexes are homologically bounded, and that all subcategories of categories are full.

In what follows, \mathcal{T} and \mathcal{A} denote a triangulated category and an abelian category with enough projective objects, respectively.

Definition 2.2. (1) For a subcategory \mathcal{X} of an additive category \mathcal{C} , the *additive closure* $\text{add}_{\mathcal{C}} \mathcal{X}$ of \mathcal{X} is defined to be the smallest subcategory of \mathcal{C} containing \mathcal{X} and closed under finite direct sums and direct summands.

(2) A *Serre subcategory* of \mathcal{A} is defined to be a subcategory of \mathcal{A} closed under subobjects, quotients and extensions.

(3) A *thick subcategory* of \mathcal{T} is by definition a triangulated subcategory of \mathcal{T} closed under direct summands. The *thick closure* of a subcategory \mathcal{X} of \mathcal{T} is defined as the smallest thick subcategory of \mathcal{T} containing \mathcal{X} , and denoted by $\text{thick}_{\mathcal{T}} \mathcal{X}$ or simply by $\text{thick} \mathcal{X}$. When \mathcal{X} consists of a single object M , we denote it by $\text{thick}_{\mathcal{T}} M$ or $\text{thick} M$.

(4) We denote by $\text{proj} \mathcal{A}$ the subcategory of \mathcal{A} consisting of projective objects.

(5) Let $P = (\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow 0)$ be a projective resolution of $M \in \mathcal{A}$. Then for each $n > 0$ we define the *n -th syzygy* $\Omega^n M$ of M (with respect to P) as the image of d_n . This is uniquely determined up to projective summands.

(6) We define a *resolving subcategory* of \mathcal{A} as a subcategory of \mathcal{A} containing $\text{proj} \mathcal{A}$ and closed under direct summands, extensions and syzygies. The *resolving closure* of a subcategory \mathcal{X} of \mathcal{A} is by definition the smallest resolving subcategory of \mathcal{A} containing \mathcal{X} , and denoted by $\text{res}_{\mathcal{A}} \mathcal{X}$ or simply by $\text{res} \mathcal{X}$. When \mathcal{X} consists of a single object M , we denote it by $\text{res}_{\mathcal{A}} M$ or $\text{res} M$.

(7) Let X, Y be complexes of objects of \mathcal{A} .

(a) A homomorphism $f : X \rightarrow Y$ of complexes is called a *quasiisomorphism* if the induced map $H_i(f) : H_i(X) \rightarrow H_i(Y)$ on the i -th homologies is an isomorphism for all integers i .

(b) We say that X is *equivalent* to Y if there exists a sequence X^0, X^1, \dots, X^n of complexes such that $X^0 = X$, $X^n = Y$, and there is a quasiisomorphism between X^i and X^{i+1} for all $0 \leq i \leq n - 1$. Then we write $X \simeq Y$.

Remark 2.3. (1) A Serre subcategory is defined for an arbitrary abelian category.

(2) A resolving subcategory is usually defined as a subcategory containing the projective objects and closed under direct summands, extensions and kernels of epimorphisms. This definition and ours are equivalent.

(3) Let \mathcal{X} be a resolving subcategory of \mathcal{A} . Let M be an object of \mathcal{X} and $n > 0$ an integer. The n -th syzygy of M with respect to *some* projective resolution of M is in \mathcal{X} if and only if the n -th syzygy of M with respect to *every* projective resolution of M is in \mathcal{X} .

We recall the notions of balls in \mathcal{T} and \mathcal{A} introduced in [Bondal and Van den Bergh 2003; Dao and Takahashi 2011; Rouquier 2008].

Definition 2.4. (1a) For a subcategory \mathcal{X} of \mathcal{T} we denote by $\langle \mathcal{X} \rangle$ the smallest subcategory of \mathcal{T} containing \mathcal{X} that is closed under finite direct sums, direct summands and shifts; in symbols, $\langle \mathcal{X} \rangle = \text{add}_{\mathcal{T}}\{X[i] \mid i \in \mathbb{Z}, X \in \mathcal{X}\}$. When \mathcal{X} consists of a single object M , we simply denote it by $\langle M \rangle$.

(1b) For subcategories \mathcal{X}, \mathcal{Y} of \mathcal{T} we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of \mathcal{T} consisting of objects M which fits into an exact triangle $X \rightarrow M \rightarrow Y \rightsquigarrow$ in \mathcal{T} with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We set $\mathcal{X} \diamond \mathcal{Y} = \langle \mathcal{X} \rangle * \langle \mathcal{Y} \rangle$.

(1c) Let \mathcal{C} be a subcategory of \mathcal{T} . We define the *ball of radius r centered at \mathcal{C}* as

$$\langle \mathcal{C} \rangle_r = \begin{cases} \langle \mathcal{C} \rangle & (r = 1), \\ \langle \mathcal{C} \rangle_{r-1} \diamond \mathcal{C} = \langle \langle \mathcal{C} \rangle_{r-1} \rangle * \langle \mathcal{C} \rangle & (r \geq 2). \end{cases}$$

If \mathcal{C} consists of a single object M , then we simply denote it by $\langle M \rangle_r$. We write $\langle \mathcal{C} \rangle_r^{\mathcal{T}}$ when we should specify that \mathcal{T} is the ground category where the ball is defined.

(2a) For a subcategory \mathcal{X} of \mathcal{A} we denote by $[\mathcal{X}]$ the smallest subcategory of \mathcal{A} containing $\text{proj } \mathcal{A}$ and \mathcal{X} that is closed under finite direct sums, direct summands and syzygies, that is, $[\mathcal{X}] = \text{add}_{\mathcal{A}}(\text{proj } \mathcal{A} \cup \{\Omega^i X \mid i \geq 0, X \in \mathcal{X}\})$. When \mathcal{X} consists of a single object M , we simply denote it by $[M]$.

(2b) For subcategories \mathcal{X}, \mathcal{Y} of \mathcal{A} we denote by $\mathcal{X} \circ \mathcal{Y}$ the subcategory of \mathcal{A} consisting of objects M which fits into an exact sequence $0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0$ in \mathcal{A} with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We set $\mathcal{X} \bullet \mathcal{Y} = [[\mathcal{X}] \circ [\mathcal{Y}]]$.

(2c) Let \mathcal{C} be a subcategory of \mathcal{A} . We define the *ball of radius r centered at \mathcal{C}* as

$$[\mathcal{C}]_r = \begin{cases} [\mathcal{C}] & (r = 1), \\ [\mathcal{C}]_{r-1} \bullet \mathcal{C} = [[\mathcal{C}]_{r-1}] \circ [\mathcal{C}] & (r \geq 2). \end{cases}$$

If \mathcal{C} consists of a single object M , then we simply denote it by $[M]_r$. We write $[\mathcal{C}]_r^{\mathcal{A}}$ when we should specify that \mathcal{A} is the ground category where the ball is defined.

Remark 2.5 [Bondal and Van den Bergh 2003; Dao and Takahashi 2011; Rouquier 2008]. (1) Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{C}$ be subcategories of \mathcal{T} .

(a) An object $M \in \mathcal{T}$ belongs to $\mathcal{X} \diamond \mathcal{Y}$ if and only if there is an exact triangle $X \rightarrow Z \rightarrow Y \rightsquigarrow$ with $X \in \langle \mathcal{X} \rangle$, $Y \in \langle \mathcal{Y} \rangle$, and M a direct summand of Z .

(b) One has $(\mathcal{X} \diamond \mathcal{Y}) \diamond \mathcal{Z} = \mathcal{X} \diamond (\mathcal{Y} \diamond \mathcal{Z})$ and $\langle \mathcal{C} \rangle_a \diamond \langle \mathcal{C} \rangle_b = \langle \mathcal{C} \rangle_{a+b}$ for all $a, b > 0$.

(2) Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{C}$ be subcategories of \mathcal{A} .

- (a) An object $M \in \mathcal{A}$ belongs to $\mathcal{X} \bullet \mathcal{Y}$ if and only if there is an exact sequence $0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$ with $X \in [\mathcal{X}]$ and $Y \in [\mathcal{Y}]$ such that M is a direct summand of Z .
- (b) One has $(\mathcal{X} \bullet \mathcal{Y}) \bullet \mathcal{Z} = \mathcal{X} \bullet (\mathcal{Y} \bullet \mathcal{Z})$ and $[\mathcal{C}]_a \bullet [\mathcal{C}]_b = [\mathcal{C}]_{a+b}$ for all $a, b > 0$.

Definition 2.6. An R -complex is called *perfect* if it is a bounded complex of projective R -modules. The *singularity category* $\mathbf{D}_{\text{sg}}(R)$ of R is defined as the Verdier quotient of $\mathbf{D}_b(R)$ by the perfect complexes. For the definition of a Verdier quotient, we refer to [Neeman 2001, Remark 2.1.9]. Whenever we discuss the singularity category $\mathbf{D}_{\text{sg}}(R)$, we identify each object or subcategory of $\text{mod } R$ with its image in $\mathbf{D}_{\text{sg}}(R)$ by the composition of the canonical functors $\text{mod } R \rightarrow \mathbf{D}_b(R) \rightarrow \mathbf{D}_{\text{sg}}(R)$.

Remark 2.7 [Dao and Takahashi 2012b, Lemma 2.4]. (1) For all $X \in \mathbf{D}_b(R)$ there exists an exact triangle $P \rightarrow X \rightarrow M[n] \rightsquigarrow$ in $\mathbf{D}_b(R)$ such that P is a perfect complex, M is a module and n is an integer. In particular, $X \cong M[n]$ in $\mathbf{D}_{\text{sg}}(R)$.
 (2) For every $M \in \text{mod } R$ and every $n \geq 0$ there is an isomorphism $M \cong \Omega^n M[n]$ in $\mathbf{D}_{\text{sg}}(R)$. Hence, for a subcategory \mathcal{C} of $\text{mod } R$ and an integer $k > 0$, each module in $[\mathcal{C}]_k^{\text{mod } R}$ belongs to $(\mathcal{C})_k^{\mathbf{D}_{\text{sg}}(R)}$.

We introduce subcategories which will be investigated in Section 4.

Definition 2.8. Let Φ be a subset of $\text{Spec } R$. Set $\Phi^c = \text{Spec } R \setminus \Phi$. We denote by $e^\Phi(R)$ (respectively, $\text{mod}^\Phi(R)$) the subcategory of $\text{mod } R$ consisting of R -modules M such that $M_{\mathfrak{p}} = 0$ (respectively, $M_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$ -free) for all $\mathfrak{p} \in \Phi^c$. Also, $\mathbf{D}_b^\Phi(R)$ (respectively, $\mathbf{D}_{\text{sg}}^\Phi(R)$) denotes the subcategory of $\mathbf{D}_b(R)$ (respectively, $\mathbf{D}_{\text{sg}}(R)$) consisting of R -complexes X such that $X_{\mathfrak{p}}$ isomorphic to a perfect $R_{\mathfrak{p}}$ -complex in $\mathbf{D}_b(R_{\mathfrak{p}})$ (respectively, $X_{\mathfrak{p}} \cong 0$ in $\mathbf{D}_{\text{sg}}(R_{\mathfrak{p}})$) for all $\mathfrak{p} \in \Phi^c$. We have that $e^\Phi(R)$ is a Serre subcategory of $\text{mod } R$, that $\text{mod}^\Phi(R)$ is a resolving subcategory of $\text{mod } R$, and that $\mathbf{D}_b^\Phi(R), \mathbf{D}_{\text{sg}}^\Phi(R)$ are thick subcategories of $\mathbf{D}_b(R), \mathbf{D}_{\text{sg}}(R)$ respectively.

Definition 2.9. (1) For an R -module M we denote by $\text{NF}(M)$ the *nonfree locus* of M , that is, the set of prime ideals \mathfrak{p} of R such that the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is nonfree. As is well-known, $\text{NF}(M)$ is a closed subset of $\text{Spec } R$ in the Zariski topology.

(2) For an R -complex M we denote by $\text{IPD}(M)$ the *infinite projective dimension locus* of M , that is, the set of prime ideals \mathfrak{p} of R such that the $R_{\mathfrak{p}}$ -complex $M_{\mathfrak{p}}$ has infinite projective dimension.

(3) For a subcategory \mathcal{X} of $\text{mod } R$ we set $\text{Supp } \mathcal{X} = \bigcup_{M \in \mathcal{X}} \text{Supp } M$ and $\text{NF}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \text{NF}(M)$.

(4) For a subcategory \mathcal{X} of $\mathbf{D}_b(R)$ we set $\text{IPD}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \text{IPD}(M)$.

(5) For a subcategory \mathcal{X} of $\mathbf{D}_{\text{sg}}(R)$ we set $\text{Supp}_{\text{sg}}(\mathcal{X}) = \bigcup_{M \in \mathcal{X}} \text{IPD}(M)$.

Definition 2.10. (1) Let M be an R -module.

(a) Let \mathbf{x} be a sequence of elements of R . Then $K(\mathbf{x}, M)$ denotes the *Koszul complex* of M with respect to \mathbf{x} . We call $H_i(\mathbf{x}, M) := H_i(K(\mathbf{x}, M))$ the i -th *Koszul homology* ($i \in \mathbb{Z}$) and $H(\mathbf{x}, M) := \bigoplus_{i \in \mathbb{Z}} H_i(\mathbf{x}, M)$ the *Koszul homology* of M with respect to \mathbf{x} .

(b) Let $P_1 \xrightarrow{d} P_0 \rightarrow M \rightarrow 0$ be a projective presentation of M . Then the cokernel of the R -dual map of d is called the *transpose* of M and denoted by $\text{Tr } M$. This is uniquely determined up to projective summands.

(2) A subset Φ of $\text{Spec } R$ is called *specialization-closed* if $V(\mathfrak{p}) \subseteq \Phi$ for all $\mathfrak{p} \in \Phi$. This is nothing but a union of closed subsets of $\text{Spec } R$ in the Zariski topology.

(3) We denote by $\text{Sing } R$ the *singular locus* of R , namely, the set of prime ideals \mathfrak{p} of R such that $R_{\mathfrak{p}}$ is not a regular local ring.

(4) A local ring R with maximal ideal \mathfrak{m} is called an *isolated singularity* if $\text{Sing } R \subseteq \{\mathfrak{m}\}$.

3. Reconstruction from Koszul homology

In this section, we consider reconstructing a given module from its Koszul homology by taking direct summands, extensions and syzygies. We start by stating and proving the most general result in this paper; actually, almost all of the other results given in this paper are deduced from this.

Theorem 3.1. *Let M be an R -module. Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of R such that $x_p \text{Ext}_R^q(M, \Omega^r M) = 0$ for all $1 \leq p \leq n$ and $1 \leq q, r \leq p$. Let P be a projective resolution of M . Then $K(\mathbf{x}, M)$ is equivalent to a complex*

$$X = (0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0)$$

such that $X_i = \bigoplus_{j=0}^i P_j^{\oplus \binom{n}{i-j}}$ for each $0 \leq i \leq n-1$ and $X_n = \bigoplus_{j=0}^n (\Omega^j M)^{\oplus \binom{n}{j}}$.

Proof. We prove the theorem by induction on n . Let us first consider the case where $n = 1$. Multiplication by x_1 makes a pullback diagram:

$$\begin{array}{ccccccc} \sigma : & 0 & \longrightarrow & \Omega M & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \parallel & & \uparrow & & x_1 \uparrow & & \\ & x_1 \sigma : & 0 & \longrightarrow & \Omega M & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Since $x_1 \text{Ext}_R^1(M, \Omega M) = 0$, we see that the exact sequence $x_1 \sigma$ splits and get an isomorphism $N \cong \Omega M \oplus M$. Thus we obtain a short exact sequence of complexes

$$0 \rightarrow W \rightarrow X \rightarrow K(x_1, M) \rightarrow 0,$$

where $W = (0 \rightarrow \Omega M \xrightarrow{=} \Omega M \rightarrow 0)$ and $X = (0 \rightarrow \Omega M \oplus M \rightarrow P_0 \rightarrow 0)$. As W is acyclic, $K(x_1, M)$ is equivalent to X .

Next we assume $n \geq 2$. The induction hypothesis implies that $K(x_1, \dots, x_{n-1}, M)$ is equivalent to a complex

$$Y = (0 \rightarrow Y_{n-1} \xrightarrow{f} Y_{n-2} \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 \rightarrow 0)$$

with $Y_i = \bigoplus_{j=0}^i P_j \oplus \binom{n-1}{i-j}$ for $0 \leq i \leq n-2$ and $Y_{n-1} = \bigoplus_{j=0}^{n-1} (\Omega^j M) \oplus \binom{n-1}{j}$. In general, taking a tensor product with a perfect complex preserves equivalence of complexes (cf. [Christensen 2000, A.4.1]). Hence we have

$K(\mathbf{x}, M)$

$$\begin{aligned} &= K(x_1, \dots, x_{n-1}, M) \otimes_R K(x_n, R) \simeq Y \otimes_R K(x_n, R) \\ &= (0 \rightarrow Y_{n-1} \xrightarrow{g} Y_{n-1} \oplus Y_{n-2} \xrightarrow{d_{n-1}} Y_{n-2} \oplus Y_{n-3} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} Y_1 \oplus Y_0 \xrightarrow{d_1} Y_0 \rightarrow 0) \\ &=: Z, \end{aligned}$$

where $g = \binom{(-1)^{n-1}x_n}{f}$. Note that there is an exact sequence $0 \rightarrow \Omega Y_{n-1} \rightarrow Q \xrightarrow{\pi} Y_{n-1} \rightarrow 0$ with $Q = \bigoplus_{j=0}^{n-1} P_j \oplus \binom{n-1}{j}$. Consider the pullback diagram

$$\begin{array}{ccccccc} \tau: & 0 & \longrightarrow & \Omega Y_{n-1} & \longrightarrow & Q \oplus Y_{n-2} & \xrightarrow{h} & Y_{n-1} \oplus Y_{n-2} & \longrightarrow & 0 \\ & & & \parallel & & \uparrow & & g \uparrow & & \\ g^*(\tau): & 0 & \longrightarrow & \Omega Y_{n-1} & \longrightarrow & L & \longrightarrow & Y_{n-1} & \longrightarrow & 0, \end{array}$$

where $h = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ and $g^* = \text{Ext}_R^1(g, \Omega Y_{n-1})$. As Y_{n-2} is projective, the map g^* can be identified with the multiplication map

$$\text{Ext}_R^1(Y_{n-1}, \Omega Y_{n-1}) \xrightarrow{(-1)^{n-1}x_n} \text{Ext}_R^1(Y_{n-1}, \Omega Y_{n-1}).$$

There are isomorphisms

$$\begin{aligned} \text{Ext}_R^1(Y_{n-1}, \Omega Y_{n-1}) &\cong \bigoplus_{j,k=0}^{n-1} \text{Ext}_R^1(\Omega^j M, \Omega(\Omega^k M)) \oplus \binom{(n-1)+(n-1)}{j+k} \\ &\cong \bigoplus_{j,k=0}^{n-1} \text{Ext}_R^{j+1}(M, \Omega^{k+1} M) \oplus \binom{(n-1)+(n-1)}{j+k}, \end{aligned}$$

and hence x_n annihilates $\text{Ext}_R^1(Y_{n-1}, \Omega Y_{n-1})$. Therefore $g^*(\tau)$ is a split exact sequence, and we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega Y_{n-1} & \longrightarrow & Q \oplus Y_{n-2} & \xrightarrow{h} & Y_{n-1} \oplus Y_{n-2} & \longrightarrow & 0 \\ & & \parallel & & \uparrow l & & g \uparrow & & \\ 0 & \longrightarrow & \Omega Y_{n-1} & \longrightarrow & \Omega Y_{n-1} \oplus Y_{n-1} & \longrightarrow & Y_{n-1} & \longrightarrow & 0, \end{array}$$

with exact rows. We observe that the complex Z is equivalent to the complex

$$X = (0 \rightarrow \Omega Y_{n-1} \oplus Y_{n-1} \xrightarrow{l} Q \oplus Y_{n-2} \xrightarrow{d_{n-1}h} Y_{n-2} \oplus Y_{n-3} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_2} Y_1 \oplus Y_0 \xrightarrow{d_1} Y_0 \rightarrow 0).$$

There are equalities

$$\begin{aligned} \Omega Y_{n-1} \oplus Y_{n-1} &= \bigoplus_{j=0}^n (\Omega^j M)^{\oplus \binom{n}{j}}, \\ Q \oplus Y_{n-2} &= \bigoplus_{j=0}^{n-1} P_j^{\oplus \binom{n-1}{n-1-j}}, \\ Y_i \oplus Y_{i-1} &= \bigoplus_{j=0}^i P_j^{\oplus \binom{n}{i-j}} \end{aligned}$$

for $1 \leq i \leq n-2$ and $Y_0 = P_0$. Thus we are done. \square

Using [Theorem 3.1](#), we obtain the following corollary.

Corollary 3.2. *Let M and \mathbf{x} be as in [Theorem 3.1](#).*

(1) *If \mathbf{x} is a regular sequence on M , then $\Omega^n(M/\mathbf{x}M) \cong \bigoplus_{k=0}^n (\Omega^k M)^{\oplus \binom{n}{k}}$ in mod R .*

(2) *For each $1 \leq i \leq n$ there exists an exact sequence of R -modules*

$$0 \rightarrow H_i(\mathbf{x}, M) \rightarrow E_i \rightarrow \Omega E_{i-1} \rightarrow 0$$

with $E_0 = H_0(\mathbf{x}, M)$ such that M is a direct summand of E_n . Hence M is built out of $H_0(\mathbf{x}, M), \dots, H_n(\mathbf{x}, M)$ by taking n syzygies, n extensions and 1 direct summand. In particular, M belongs to the ball $[H(\mathbf{x}, M)]_{n+1}^{\text{mod } R}$.

(3) *There is an exact triangle*

$$F \rightarrow K(\mathbf{x}, M) \rightarrow \bigoplus_{k=0}^n (\Omega^k M)^{\oplus \binom{n}{k}}[n] \rightsquigarrow$$

in $\mathbf{D}_b(R)$, where $F = (0 \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow 0)$ is a perfect complex.

(4) *The module M belongs to the ball $\langle R \oplus K(\mathbf{x}, M) \rangle_{n+1}^{\mathbf{D}_b(R)}$.*

(5) *One has $K(\mathbf{x}, M) \cong \bigoplus_{k=0}^n M^{\oplus \binom{n}{k}}[k]$ in $\mathbf{D}_{\text{sg}}(R)$. In particular, M is a direct summand of $K(\mathbf{x}, M)$ in $\mathbf{D}_{\text{sg}}(R)$.*

Proof. We use the notation of [Theorem 3.1](#) and its assertion.

(1) Since \mathbf{x} is regular on M , we have an equivalence $K(\mathbf{x}, M) \simeq M/\mathbf{x}M$. There is an exact sequence

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_0 \rightarrow M/\mathbf{x}M \rightarrow 0$$

of R -modules. As $X_n = \bigoplus_{j=0}^n (\Omega^j M)^{\oplus \binom{n}{j}}$ and X_i is projective for all $0 \leq i \leq n-1$, the module X_n is the n -th syzygy of $M/\mathbf{x}M$ as an R -module.

(2) For each $0 \leq i \leq n$ take a truncation $X^i = (0 \rightarrow X_n \rightarrow \cdots \rightarrow X_{i+1} \rightarrow X_i \rightarrow 0)$ of X with $(X^i)_j = X_{i+j}$ for $0 \leq j \leq n$. Then there is a short exact sequence

$$0 \rightarrow X_{i-1} \rightarrow X^{i-1} \rightarrow X^i[1] \rightarrow 0$$

of complexes for each $1 \leq i \leq n$. The long exact sequence in homology gives an exact sequence $0 \rightarrow H_1(X^{i-1}) \rightarrow H_0(X^i) \rightarrow X_{i-1} \rightarrow H_0(X^{i-1}) \rightarrow 0$ of modules. As X_{i-1} is projective, we have an exact sequence

$$0 \rightarrow H_1(X^{i-1}) \rightarrow H_0(X^i) \rightarrow \Omega H_0(X^{i-1}) \rightarrow 0$$

for all $1 \leq i \leq n$. Notice $H_1(X^{i-1}) = H_i(\mathbf{x}, M)$, $H_0(X^0) = H_0(\mathbf{x}, M)$ and $H_0(X^n) = X_n$. Setting $E_i = H_0(X^i)$ for $0 \leq i \leq n$, we obtain desired exact sequences.

(3) Truncating the complex X provides such an exact triangle.

(4) Decomposing F into short exact sequences of complexes, we observe that F is in $\langle R \rangle_n^{\mathbf{D}_b(R)}$. As M is a direct summand of $\bigoplus_{k=0}^n (\Omega^k M)^{\oplus \binom{n}{k}}$, the assertion follows from (3).

(5) By (3) we have an isomorphism $K(\mathbf{x}, M) \cong \bigoplus_{k=0}^n (\Omega^k M)^{\oplus \binom{n}{k}}[n]$ in $\mathbf{D}_{\text{sg}}(R)$. Since $M \cong \Omega^k M[k]$ in $\mathbf{D}_{\text{sg}}(R)$, we are done. □

Remark 3.3. (1) [Corollary 3.2\(1\)](#) is a refinement of [[Takahashi 2010](#), Proposition 2.2], which shows the same conclusion under the additional assumption that \mathbf{x} is a regular sequence on R annihilating more Ext modules.

(2) [Corollary 3.2\(5\)](#) can also be shown by using the proof of [[Dao and Takahashi 2012b](#), Proposition 2.3]. It also implies that M belongs to $\langle R \oplus K(\mathbf{x}, M) \rangle_m^{\mathbf{D}_b(R)}$ for some integer $m > 0$. However, it cannot determine how big/small m is, while [Corollary 3.2\(4\)](#) can.

We are interested in existence of a sequence \mathbf{x} as in [Theorem 3.1](#). The lemma below guarantees that such a sequence always exists. Moreover, one can make such a sequence as a power of an arbitrary sequence whose defining closed subset covers the nonfree locus.

Lemma 3.4. *Let M be an R -module. Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of R with $\text{NF}(M) \subseteq V(\mathbf{x})$. Then there exists an integer $k > 0$ such that the sequence $\mathbf{x}^k = x_1^k, \dots, x_n^k$ annihilates $\text{Ext}_R^i(M, N)$ for all $i > 0$ and all $N \in \text{mod } R$.*

Proof. Let I be an ideal of R with $\text{NF}(M) = V(I)$. Then by [[Dao and Takahashi 2012a](#), Remark 5.2(1)] there exists an integer $p > 0$ such that $I^p \text{Ext}_R^i(M, N) = 0$ for all $i > 0$ and all $N \in \text{mod } R$. By assumption, we have $(\mathbf{x}^q) \subseteq I$ for some $q > 0$. Setting $k = pq$ completes the proof. □

Combining [Theorem 3.1](#), [Corollary 3.2\(2\)](#) and [Lemma 3.4](#), we immediately obtain the following result, which includes [Theorem 1.1](#).

Corollary 3.5. *Let M be an R -module. Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of R with $\text{NF}(M) \subseteq V(\mathbf{x})$. Then there exists an integer $k > 0$ such that $K(\mathbf{x}^k, M)$ is equivalent to a complex*

$$(0 \rightarrow N \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow 0),$$

where each P_i is projective and M is a direct summand of N . Hence, M is built out of $H_0(\mathbf{x}^k, M), \dots, H_n(\mathbf{x}^k, M)$ by taking n syzygies, n extensions and 1 direct summand. In particular, M is in $[H(\mathbf{x}^k, M)]_{n+1}^{\text{mod } R}$.

4. Generation of subcategories

In this section, we apply our results obtained in the previous section to investigate generation of subcategories. To be precise, for a subset Φ of $\text{Spec } R$ we analyze the structure of the subcategories $\text{mod}^\Phi(R)$, $\mathbf{D}_b^\Phi(R)$ and $\mathbf{D}_{\text{sg}}^\Phi(R)$. We also consider classification of these subcategories.

First of all, we want to make a generator of $\text{mod}^\Phi(R)$ as a resolving subcategory of $\text{mod } R$ and generators of $\mathbf{D}_b^\Phi(R)$, $\mathbf{D}_{\text{sg}}^\Phi(R)$ as thick subcategories of $\mathbf{D}_b(R)$, $\mathbf{D}_{\text{sg}}(R)$. In fact, $e^\Phi(R)$ gives generators of these three subcategories:

Theorem 4.1. *Let Φ be a subset of $\text{Spec } R$. Then one has equalities*

- (1) $\text{mod}^\Phi(R) = \text{res}_{\text{mod } R}(e^\Phi(R)),$
- (2) $\mathbf{D}_b^\Phi(R) = \text{thick}_{\mathbf{D}_b(R)}(\{R\} \cup e^\Phi(R)),$
- (3) $\mathbf{D}_{\text{sg}}^\Phi(R) = \text{thick}_{\mathbf{D}_{\text{sg}}(R)}(e^\Phi(R)).$

Proof. (1) It is obvious that $e^\Phi(R)$ is contained in $\text{mod}^\Phi(R)$, and hence so is its resolving closure. To show the opposite inclusion, let M be an object of $\text{mod}^\Phi(R)$. Then by definition $\text{NF}(M)$ is contained in Φ . It is seen from [Corollary 3.5](#) that there is a sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of R with $\text{NF}(M) = V(\mathbf{x})$ such that M belongs to $\text{res}_{\text{mod } R} H(\mathbf{x}, M)$. Since $H(\mathbf{x}, M)$ is annihilated by \mathbf{x} , we have

$$\text{Supp } H(\mathbf{x}, M) \subseteq V(\mathbf{x}) = \text{NF}(M) \subseteq \Phi,$$

which shows $H(\mathbf{x}, M) \in e^\Phi(R)$. Consequently, M is in $\text{res}_{\text{mod } R}(e^\Phi(R))$.

(2) Clearly, $\mathbf{D}_b^\Phi(R)$ contains R and $e^\Phi(R)$, and the thick closure of $\{R\} \cup e^\Phi(R)$. Let X be an object of $\mathbf{D}_b^\Phi(R)$. Then there is an exact triangle

$$P \rightarrow X \rightarrow M[n] \rightsquigarrow$$

in $\mathbf{D}_b(R)$ such that P is a perfect R -complex, M is an R -module and n is an integer. We use the *large restricted flat dimension* of M , namely

$$\text{Rfd}_R M = \sup_{p \in \text{Spec } R} \{\text{depth } R_p - \text{depth}_{R_p} M_p\}$$

By [Avramov et al. 2010b, Theorem 1.1] this is finite. Put $r = \text{Rfd}_R M$. Let \mathfrak{p} be a prime ideal in Φ^c . Localizing the above exact triangle at \mathfrak{p} , we see that the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ has finite projective dimension. Hence

$$\text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq r.$$

Setting $N = \Omega^r M$, we note that N belongs to $\text{mod}^{\Phi}(R)$, hence to $\text{res}_{\text{mod } R}(e^{\Phi}(R))$ by (1). Therefore N is in $\text{thick}_{\mathbf{D}_b(R)}(\{R\} \cup e^{\Phi}(R))$, and so is M . As $P \in \text{thick}_{\mathbf{D}_b(R)} R$, the object X belongs to $\text{thick}_{\mathbf{D}_b(R)}(\{R\} \cup e^{\Phi}(R))$ by the above exact triangle.

(3) This equality is obtained by using (2). □

One can describe the structure of $e^{\Phi}(R)$ in more detail, which makes more visible representations of $\text{mod}^{\Phi}(R)$, $\mathbf{D}_b^{\Phi}(R)$ and $\mathbf{D}_{\text{sg}}^{\Phi}(R)$.

Corollary 4.2. *Let Φ be a subset of $\text{Spec } R$. Then $e^{\Phi}(R)$ is the smallest subcategory of $\text{mod } R$ containing R/\mathfrak{p} for all $\mathfrak{p} \in \Phi^{\text{sp}}$ and closed under extensions. Here Φ^{sp} denotes the largest specialization-closed subset of $\text{Spec } R$ contained in Φ . Hence*

$$\begin{aligned} \text{mod}^{\Phi}(R) &= \text{res}_{\text{mod } R} \{ R/\mathfrak{p} \mid \mathfrak{p} \in \Phi^{\text{sp}} \}, \\ \mathbf{D}_b^{\Phi}(R) &= \text{thick}_{\mathbf{D}_b(R)} \{ R, R/\mathfrak{p} \mid \mathfrak{p} \in \Phi^{\text{sp}} \}, \\ \mathbf{D}_{\text{sg}}^{\Phi}(R) &= \text{thick}_{\mathbf{D}_{\text{sg}}(R)} \{ R/\mathfrak{p} \mid \mathfrak{p} \in \Phi^{\text{sp}} \}. \end{aligned}$$

Proof. The last assertion follows from Theorem 4.1.

We claim that $\Phi^{\text{sp}} = \text{Supp}(e^{\Phi}(R))$ holds. Indeed, it is evident that $\text{Supp}(e^{\Phi}(R))$ is a specialization-closed subset of $\text{Spec } R$ contained in Φ . Let Ψ be a specialization-closed subset of $\text{Spec } R$ contained in Φ . Then we have $e^{\Psi}(R) \subseteq e^{\Phi}(R)$, and hence $\Psi = \text{Supp}(e^{\Psi}(R)) \subseteq \text{Supp}(e^{\Phi}(R))$. Thus the claim holds.

Let \mathcal{X} be the smallest subcategory of $\text{mod } R$ containing R/\mathfrak{p} for all $\mathfrak{p} \in \Phi^{\text{sp}}$ and closed under extensions. First, let \mathfrak{p} be a prime ideal in Φ^{sp} . As Φ^{sp} is specialization-closed, we have $\text{Supp}(R/\mathfrak{p}) = \mathbf{V}(\mathfrak{p}) \subseteq \Phi^{\text{sp}} \subseteq \Phi$, whence R/\mathfrak{p} belongs to $e^{\Phi}(R)$. Since $e^{\Phi}(R)$ is closed under extensions, $e^{\Phi}(R)$ contains \mathcal{X} . Next, let M be a module in $e^{\Phi}(R)$. Take a filtration

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$$

of submodules of M such that $M_{i-1}/M_i \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \text{Spec } R$ for each $1 \leq i \leq n$. Then \mathfrak{p}_i is in $\text{Supp } M$, and so in $\text{Supp}(e^{\Phi}(R))$. By the claim, we have $\mathfrak{p}_i \in \Phi^{\text{sp}}$ for all $1 \leq i \leq n$. Decomposing the above filtration into short exact sequences, we see that M is in \mathcal{X} . Therefore \mathcal{X} contains $e^{\Phi}(R)$, and the proof is completed. □

The next result, which includes part of Corollary 1.3, follows immediately from Corollary 4.2. Note that the objects of $\text{mod}^{[m]}(R)$ are the R -modules that are locally free on the punctured spectrum of R .

Corollary 4.3. (1) $\mathbf{D}_b(R) = \text{thick}\{ R, R/\mathfrak{p} \mid \mathfrak{p} \in \text{Sing } R \}$.

(2) $\mathbf{D}_{\text{sg}}(R) = \text{thick}\{ R/\mathfrak{p} \mid \mathfrak{p} \in \text{Sing } R \}$.

(3) *If R is a local ring with maximal ideal \mathfrak{m} and residue field k , then* $\text{mod}^{\{\mathfrak{m}\}}(R) = \text{res}(k)$, $\mathbf{D}_b^{\{\mathfrak{m}\}}(R) = \text{thick}(R \oplus k)$ and $\mathbf{D}_{\text{sg}}^{\{\mathfrak{m}\}}(R) = \text{thick}(k)$.

Remark 4.4. The equalities in (1) and (2) can also be shown using Theorem VI.8 of [Schoutens 2003], while similar results to (3) have been obtained in Theorem 2.4 of [Takahashi 2010] as well as by H. Abe and O. Iyama (work in progress).

As a common consequence of the two assertions of Corollary 4.3, one can recover [Keller et al. 2011, Proposition A.2]:

Corollary 4.5. *Let R be an isolated singularity with residue field k . Then $\mathbf{D}_b(R) = \text{thick}(R \oplus k)$ and $\mathbf{D}_{\text{sg}}(R) = \text{thick}(k)$.*

Next, we make a closer investigation on the inner structure of subcategories. In fact, we can refine the assertions as to $\text{mod}^{\{\mathfrak{m}\}}(R)$ and $\mathbf{D}_{\text{sg}}^{\{\mathfrak{m}\}}(R)$ in Corollary 4.3(3) in terms of balls in the abelian category $\text{mod } R$ and the triangulated category $\mathbf{D}_{\text{sg}}(R)$. Denote by $\text{fl}(R)$ the subcategory of $\text{mod } R$ consisting of modules of finite length. The following theorem holds, which is the main part of Corollary 1.3.

Theorem 4.6. *Let R be a d -dimensional local ring with maximal ideal \mathfrak{m} . Then there are equalities*

$$\text{mod}^{\{\mathfrak{m}\}}(R) = [\text{fl}(R)]_{d+1}^{\text{mod } R} \quad \text{and} \quad \mathbf{D}_{\text{sg}}^{\{\mathfrak{m}\}}(R) = \langle \text{fl}(R) \rangle_{d+1}^{\mathbf{D}_{\text{sg}}(R)}.$$

Proof. (1) Let us show the first equality. It clearly holds when $d = 0$, so we assume $d > 0$. Let M be an R -module in $\text{mod}^{\{\mathfrak{m}\}}(R)$. Take any system of parameters $\mathbf{x} = x_1, \dots, x_d$ of R . As M is in $\text{mod}^{\{\mathfrak{m}\}}(R)$, we have $\text{NF}(M) \subseteq \{\mathfrak{m}\} = \mathbf{V}(\mathbf{x})$. Corollary 3.5 implies that M belongs to $[\text{H}(\mathbf{x}^k, M)]_{d+1}$ for some $k > 0$. Since the R -module $\text{H}(\mathbf{x}^k, M)$ is annihilated by the \mathfrak{m} -primary ideal (\mathbf{x}^k) , it has finite length. Thus we obtain $M \in [\text{fl}(R)]_{d+1}$, and the first equality follows.

(2) We prove the second equality. Let X be an R -complex in $\mathbf{D}_{\text{sg}}^{\{\mathfrak{m}\}}(R)$. Note that $X \cong \Omega^d M[n]$ in $\mathbf{D}_{\text{sg}}(R)$ for some R -module M and some integer n . By the Auslander–Buchsbaum formula, we see that $\Omega^d M$ belongs to $\text{mod}^{\{\mathfrak{m}\}}(R) = [\text{fl}(R)]_{d+1}$. Now the second equality follows from the first one. \square

Here is an immediate consequence of Theorem 4.6.

Corollary 4.7. *If R is a d -dimensional isolated singularity, $\mathbf{D}_{\text{sg}}(R) = \langle \text{fl}(R) \rangle_{d+1}$.*

Remark 4.8. (1) Rewording the second equality in Theorem 4.6 by the terminology introduced in [Aihara et al. 2014], one has the following inequality:

$$\text{fl}(R) \text{-tri.dim } \mathbf{D}_{\text{sg}}^{\{\mathfrak{m}\}}(R) \leq \dim R.$$

- (2) Theorem A in [Takahashi 2009] constructs *some* object in $\text{mod}^{(m)}(R)$ from *every* object in $\text{mod } R$ and counts the number of necessary operations (containing syzygies). In contrast to this, Theorem 4.6 constructs *every* object in $\text{mod}^{(m)}(R)$ from *some* object in $\text{fl}(R)$ and counts the number of necessary operations.
- (3) Similar equalities to the first equality in Theorem 4.6 are given for $\text{mod } R$ in [Schoutens 2003, Theorem VI.8] and [Burke et al. 2012, Theorem 2], but these are different from ours in respect of how to count operations. The biggest difference is that neither of those two results counts the number of necessary extensions.
- (4) In the case where R is Cohen–Macaulay, Corollary 4.7 also follows from [Aihara et al. 2014, 4.5.1], because every maximal Cohen–Macaulay R -module is a direct summand of the d -th syzygy of some module of finite length by [Takahashi 2010, Proposition 2.2].

Finally, we are interested in classifying resolving and thick subcategories by using $\text{mod}^\Phi(R)$, $\mathbf{D}_b^\Phi(R)$ and $\mathbf{D}_{\text{sg}}^\Phi(R)$. For this purpose, we prepare a lemma:

Lemma 4.9. (1) *The assignments $\mathcal{X} \mapsto \text{Supp } \mathcal{X}$ and $\Phi \mapsto e^\Phi(R)$ make a one-to-one correspondence between the Serre subcategories of $\text{mod } R$ and the specialization-closed subsets of $\text{Spec } R$.*

(2) *Let Φ be a specialization-closed subset of $\text{Spec } R$. Then $\text{NF}(\text{mod}^\Phi(R)) = \Phi \cap S(R)$ and $\text{IPD}(\mathbf{D}_b^\Phi(R)) = \text{Supp}_{\text{sg}}(\mathbf{D}_{\text{sg}}^\Phi(R)) = \Phi \cap \text{Sing } R$.*

Proof. (1) This is Gabriel’s classification theorem [1962] for Serre subcategories.

(2) Let $\mathfrak{p} \in \Phi$. Then $\text{IPD}(R/\mathfrak{p}) \subseteq \text{NF}(R/\mathfrak{p}) \subseteq V(\mathfrak{p}) \subseteq \Phi$. Hence R/\mathfrak{p} belongs to $\text{mod}^\Phi(R)$, $\mathbf{D}_b^\Phi(R)$ and $\mathbf{D}_{\text{sg}}^\Phi(R)$. If $\mathfrak{p} \in S(R)$ (respectively, $\text{Sing } R$), then $\mathfrak{p} \in \text{NF}(R/\mathfrak{p})$ (respectively, $\text{IPD}(R/\mathfrak{p})$). The assertion now follows. \square

We can obtain the following theorem, which includes Corollary 1.2.

Theorem 4.10. (1) *The assignment $\Phi \mapsto \text{mod}^\Phi(R)$ is a bijection from the set of specialization-closed subsets of $\text{Spec } R$ contained in $S(R)$ to the set of resolving closures $\text{res}_{\text{mod } R} \mathcal{X}$, where \mathcal{X} runs through the Serre subcategories of $\text{mod } R$.*

(2) *The assignment $\Phi \mapsto \mathbf{D}_b^\Phi(R)$ is a bijection from the set of specialization-closed subsets of $\text{Spec } R$ contained in $\text{Sing } R$ to the set of thick closures $\text{thick}_{\mathbf{D}_b(R)}(\{R\} \cup \mathcal{X})$, where \mathcal{X} runs through the Serre subcategories of $\text{mod } R$.*

(3) *The assignment $\Phi \mapsto \mathbf{D}_{\text{sg}}^\Phi(R)$ is a bijection from the set of specialization-closed subsets of $\text{Spec } R$ contained in $\text{Sing } R$ to the set of thick closures $\text{thick}_{\mathbf{D}_{\text{sg}}(R)} \mathcal{X}$, where \mathcal{X} runs through the Serre subcategories of $\text{mod } R$.*

Proof. In view of Theorem 4.1, the three assignments make well-defined maps, and they are injective by Lemma 4.9(2). Thus it only remains to show that they are surjective.

(1) Let \mathcal{X} be a Serre subcategory of $\text{mod } R$. According to [Lemma 4.9\(1\)](#), we have $\mathcal{X} = e^Z(R)$ for some specialization-closed subset Z of $\text{Spec } R$. Putting $\Phi = Z \cap S(R)$, we easily see that Φ is a specialization-closed subset of $\text{Spec } R$ which is contained in $S(R)$ and satisfies $\text{mod}^Z(R) = \text{mod}^\Phi(R)$. [Theorem 4.1](#) implies $\text{res}_{\text{mod } R} \mathcal{X} = \text{mod}^\Phi(R)$.

(2), (3) We use the proof of (1). Set $\Psi = Z \cap \text{Sing } R$. Then Ψ is a specialization-closed subset of $\text{Spec } R$ contained in $\text{Sing } R$ such that the equalities $\mathbf{D}_b^Z(R) = \mathbf{D}_b^\Psi(R)$ and $\mathbf{D}_{\text{sg}}^Z(R) = \mathbf{D}_{\text{sg}}^\Psi(R)$ hold. Hence the surjectivity of the map follows from [Theorem 4.1](#). \square

The next statement subsumes [Corollary 1.4](#) and also some earlier results: namely, (1) and the equivalence of (b)–(d) in (2) are proved in [[Takahashi 2013](#), Theorem 1.1 and Proposition 4.6] under the assumption that R is a Cohen–Macaulay local ring. Our results show that this assumption is superfluous.

Corollary 4.11. (1) *The assignments $\Phi \mapsto \text{mod}^\Phi(R)$ and $\mathcal{X} \mapsto \text{NF}(\mathcal{X})$ gives mutually inverse bijections between*

- (a) *the specialization-closed subsets of $\text{Spec } R$ contained in $S(R)$, and*
 - (b) *the resolving subcategories of $\text{mod } R$ closed under tensor products and transposes.*
- (2) *Let \mathcal{X} be a resolving subcategory of $\text{mod } R$. Then the following are equivalent:*
- (a) *\mathcal{X} is the resolving closure of a Serre subcategory of $\text{mod } R$.*
 - (b) *\mathcal{X} is closed under tensor products and transposes.*
 - (c) *R/\mathfrak{p} belongs to \mathcal{X} for all $\mathfrak{p} \in \text{NF}(\mathcal{X})$.*
 - (d) *For all $\mathfrak{p} \in \text{NF}(\mathcal{X})$ there exists $M \in \mathcal{X}$ such that $\kappa(\mathfrak{p})$ is a direct summand of $M_{\mathfrak{p}}$.*

Proof. Recall that we have proved in [Corollary 4.3\(3\)](#) that if R is a local ring with maximal ideal \mathfrak{m} and residue field k , then the equality $\text{mod}^{(\mathfrak{m})}(R) = \text{res}_{\text{mod } R}(k)$ holds. Hence, in view of [[Dao and Takahashi 2014](#), Lemma 3.2], we see that all the ten assertions in [[Takahashi 2013](#), Lemma 2.5] hold without the assumption that R is Cohen–Macaulay. Therefore, it is observed from [[Dao and Takahashi 2014](#), Proposition 3.3] and the proof of [[Takahashi 2013](#), Proposition 3.1] that one can remove from [[Takahashi 2013](#), Proposition 3.1] the two assumptions that R is local and that R is Cohen–Macaulay. Thus, the proof of [[Takahashi 2013](#), Theorem 3.3] actually proves that the statement [[Takahashi 2013](#), Theorem 3.3] holds without the assumption that R is a Cohen–Macaulay local ring. Since [[Takahashi 2013](#), Lemma 4.5] (respectively, [[Takahashi 2013](#), Lemma 4.4]) is still valid for an arbitrary commutative noetherian ring (respectively, local ring) R , so are [[Takahashi 2013](#), Proposition 4.6 and Theorem 4.7]. Now our [Theorem 4.10](#) completes the proof. \square

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RYO TAKAHASHI
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
FUROCHO, CHIKUSAKU
NAGOYA 464-8602
JAPAN
takahashi@math.nagoya-u.ac.jp

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University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Don Blasius
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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Princeton NJ 08544-1000
yang@math.princeton.edu

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