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Kawasaki's formula is a tool to compute holomorphic Euler characteristics of vector bundles on a compact orbifold $\mathscr{X}$. Let $\mathscr{X}$ be an orbispace with perfect obstruction theory which admits an embedding in a smooth orbifold. One can then construct the virtual structure sheaf and the virtual fundamental class of $\mathscr{X}$. In this paper we prove that Kawasaki's formula "behaves well" with working "virtually" on $\mathscr{X}$ in the following sense: if we replace the structure sheaves, tangent and normal bundles in the formula by their virtual counterparts then Kawasaki's formula stays true. Our motivation comes from studying the quantum K-theory of a complex manifold X (Givental and Tonita, 2014), with the formula applied to Kontsevich moduli spaces of genus-0 stable maps to X.

1. Introduction

Given a manifold $\mathscr{X}$ and a vector bundle V on $\mathscr{X}$, then the Hirzebruch–Riemann–Roch formula states that

$$\chi(\mathscr{X}, V) = \int_{\mathscr{X}} \mathrm{ch}(V) T\, d(T_{\mathscr{X}}).$$

Kawasaki [1979] generalized this formula to the case when $\mathscr{X}$ is an orbifold. He reduces the computation of Euler characteristics on $\mathscr{X}$ to the computation of certain cohomological integrals on *the inertia orbifold $I\mathscr{X}$*:

$$\chi(\mathscr{X}, V) = \sum_{\mu} \frac{1}{m_\mu} \int_{\mathscr{X}_\mu} T\, d(T_{\mathscr{X}_\mu})\, \mathrm{ch}\Big(\frac{\mathrm{Tr}(V)}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\Big). \tag{1}$$

We explain below the ingredients in the formula:

$I\mathscr{X}$ is defined as follows: around any point $p \in \mathscr{X}$ there is a local chart $(\widetilde{U}_p, G_p)$ such that locally $\mathscr{X}$ is represented as the quotient of $\widetilde{U}_p$ by G_p. Consider the set of conjugacy classes $(1) = (h_p^1), (h_p^2), \dots, (h_p^{n_p})$ in G_p. Define

$$I\mathscr{X} := \{(p, (h_p^i)) \mid i = 1, 2, \dots, n_p\}.$$

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Pick an element h_p^i in each conjugacy class. Then a local chart on $I\mathscr{X}$ is given by

$$\coprod_{i=1}^{n_p} \widetilde{U}_p^{(h_p^i)}/Z_{G_p}(h_p^i),$$

where $Z_{G_p}(h_p^i)$ is the centralizer of h_p^i in G_p. Denote by $\mathscr{X}_\mu$ the connected components of the inertia orbifold (we'll often refer to them as Kawasaki strata). The multiplicity m_μ associated to each $\mathscr{X}_\mu$ is given by

$$m_\mu := \left|\ker(Z_{G_p}(g) \to \mathrm{Aut}(\widetilde{U}_p^g))\right|.$$

For a vector bundle V we will denote by V^* the dual bundle to V. The restriction of V to $\mathscr{X}_\mu$ decomposes in characters of the g action. Let $E_r^{(l)}$ be the subbundle of the restriction of E to $\mathscr{X}_\mu$ on which g acts with eigenvalue $e^{2\pi i l/r}$. Then the trace $\mathrm{Tr}(V)$ is defined to be the orbibundle whose fiber over the point $(p, (g))$ of $\mathscr{X}_\mu$ is

$$\mathrm{Tr}(V) := \sum_l e^{\frac{2\pi i l}{r}} E_r^{(l)}.$$

Finally, $\Lambda^\bullet N_\mu^*$ is the K-theoretic Euler class of the normal bundle N_μ of $\mathscr{X}_\mu$ in $\mathscr{X}$. $\mathrm{Tr}(\Lambda^\bullet N_\mu^*)$ is invertible because the symmetry g acts with eigenvalues different from 1 on the normal bundle to the fixed point locus. We call the terms corresponding to the identity component in the formula *fake Euler characteristics*:

$$\chi^f(\mathscr{X}, V) = \int_{\mathscr{X}} \mathrm{ch}(V) T\,d(T_{\mathscr{X}}).$$

In the case where $\mathscr{X}$ is a global quotient, formula (1) is the Lefschetz fixed point formula.

Now let $\mathscr{X}$ be a compact, complex orbispace (Deligne–Mumford stack) with a perfect obstruction theory $E^{-1} \to E^0$. This is used to define the intrinsic normal cone, which is embedded in E_1 — the dual bundle to E^{-1} (see [Li and Tian 1998; Behrend and Fantechi 1997]). The virtual structure sheaf $\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}$ was defined in [Lee 2004] as the K-theoretic pullback by the zero section of the structure sheaf of this cone. Let $I\mathscr{X} = \coprod_\mu \mathscr{X}_\mu$ be the inertia orbifold of $\mathscr{X}$. We denote by i_μ the inclusion of a stratum $\mathscr{X}_\mu$ in $\mathscr{X}$. For a bundle V on $\mathscr{X}$, we write $i_\mu^* V = V_\mu^f \oplus V_\mu^m$ for its decomposition as the direct sum of the fixed part and the moving part under the action of the symmetry associated to $\mathscr{X}_\mu$. To avoid ugly notation we will often simply write V^m, V^f. The virtual normal bundle to $\mathscr{X}_\mu$ in $\mathscr{X}$ is defined as $[E_0^m] - [E_1^m]$. We will in addition assume that $\mathscr{X}$ admits an embedding j in a smooth compact orbifold $\mathscr{Y}$. This is always true for the moduli spaces of genus-0 stable maps $X_{0,n,d}$ because an embedding $X \hookrightarrow \mathbb{P}^N$ induces an embedding $X_{0,n,d} \hookrightarrow (\mathbb{P}^N)_{0,n,d}$.

Theorem 1.1. *Denote by N_μ^{vir} the virtual normal bundle of $\mathscr{X}_\mu$ in $\mathscr{X}$. Then*

$$\chi(\mathscr{X}, j^*(V)\otimes \mathcal{O}_{\mathscr{X}}^{\mathrm{vir}}) = \sum_\mu \frac{1}{m_\mu}\chi^f\left(\mathscr{X}_\mu, \frac{\mathrm{Tr}(V_\mu\otimes \mathcal{O}_{\mathscr{X}_\mu}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet(N_\mu^{\mathrm{vir}})^*)}\right). \tag{2}$$

Remark 1.2. A perfect obstruction theory $E^{-1}\to E^0$ on $\mathscr{X}$ induces canonically a perfect obstruction theory on $\mathscr{X}_\mu$ by taking the fixed part of the complex $E_\mu^{-1,f}\to E_\mu^{0,f}$. The proof is the same as that of Proposition 1 in [Graber and Pandharipande 1999]. This is then used to define the sheaf $\mathcal{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}$.

Remark 1.3. It is proved in [Fantechi and Göttsche 2010] that if $\mathscr{X}$ is a scheme, the Grothendieck–Riemann–Roch theorem is compatible with virtual fundamental classes and virtual fundamental sheaves, that is,

$$\chi^f(\mathscr{X}, V\otimes\mathcal{O}_{\mathscr{X}}^{\mathrm{vir}}) = \int_{[\mathscr{X}]} \mathrm{ch}(V\otimes \mathcal{O}_{\mathscr{X}}^{\mathrm{vir}})\cdot T\,d(T^{\mathrm{vir}}),$$

where $[\mathscr{X}]$ is the virtual fundamental class of $\mathscr{X}$ and T^{vir} is its virtual tangent bundle. Their arguments carry over to the case when $\mathscr{X}$ is a stack.

Remark 1.4. The bundles V to which we apply Theorem 1.1 in [Givental and Tonita 2014] are (sums and products of) cotangent line bundles L_i and evaluation classes $\mathrm{ev}_i^*(a_i)$ (where a_i are K-theoretic classes on the target). They are pullbacks of the corresponding bundles on $(\mathbb{P}^N)_{0,n,d}$.

2. Proof of Theorem 1.1

Before proving Theorem 1.1 we recall a couple of background facts and lemmata on K-theory which we will use.

Let $K_0(X)$ be the Grothendieck group of coherent sheaves on X. Given a map $f : X\to Y$, the K-theoretic pullback $f^*(\mathscr{F}) : K_0(Y)\to K_0(X)$ is defined as the alternating sum of derived functors $\mathrm{Tor}^i_{\mathcal{O}_Y}(\mathscr{F}, \mathcal{O}_X)$, provided that the sum is finite. This is always true for instance if f is flat or if it is a regular embedding.

For any fiber square

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ B' & \xrightarrow{\ i\ } & B \end{array}$$

with i a regular embedding one can define K-theoretic refined Gysin homomorphisms $i^! : K_0(V)\to K_0(V')$ (see [Lee 2004]). One way to define the map $i^!$ is the following: The class $i_*(\mathcal{O}_{B'})\in K^0(B)$ has a finite resolution of vector bundles, which is exact off B'. We pull it back to V and then cap (i.e., tensor product) with classes in $K_0(V)$, to get a class on $K_0(V)$ with homology supported on V', which we can regard as an element of $K_0(V')$, because there is a canonical isomorphism between complexes on V with homology supported on V' and $K_0(V')$.

In the following two lemmata, X, Y, Y' are assumed DM stacks. We will use the following result:

Lemma 2.1. *Consider the diagram*:

$$\begin{array}{ccc} \iota^* C_{X/Y} & \longrightarrow & C_{X/Y} \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow{\scriptstyle j} \\ Y' & \xrightarrow{i} & Y \end{array}$$

with i a regular embedding and j an embedding, $C_{X/Y}$ is the normal cone of X in Y and both squares are fiber diagrams. Then

$$i^![\mathbb{O}_{C_{X/Y}}] = [\mathbb{O}_{C_{X'/Y'}}] \in K_0(\iota^* C_{X/Y}). \tag{3}$$

This is stated and proved in [Lee 2004, Lemma 2]. The proof is based on a more general statement (Lemma 1 of [Lee 2004]), which has been worked out in [Kresch 1999] on the level of Chow rings. Since K-theoretic statements are stronger, we give below the key ingredient which allows one to carry over Kresch's proof to K-theory:

Lemma 2.2. *Let $f : X \to Y$ be a closed embedding and let $g : Y \to \mathbb{P}^1$ be a surjection such that $g \circ f$ is flat. Denote by X_0 and Y_0 the fibers over 0 of $g \circ f$ and g, respectively. Moreover, assume that the restriction of f to $X \setminus X_0$ is an isomorphism. Then if i is the inclusion of $\{0\}$ in $\mathbb{P}^1$, we have $i^!(\mathbb{O}_Y) = \mathbb{O}_{X_0} \in K_0(Y_0)$.*

Proof. The skyscraper sheaves at all points of $\mathbb{P}^1$ represent the same element in $K_0(\mathbb{P}^1)$, hence if we pull back a resolution of any point $P \in \mathbb{P}^1$ by g we get the same elements of $K_0(Y)$. On the other hand since f is an isomorphism above $\mathbb{P}^1 \setminus \{0\}$, pulling back by g of the structure sheaf of a point $P \neq 0$ is the same as pulling back by $g \circ f$ followed by f_*. By what we said above we can replace P with 0. Now from the flatness of $g \circ f$ above 0 the pullback of the structure sheaf of 0 by $g \circ f$ is the structure sheaf of the fiber X_0. The result then follows from the definition of $i^!$. □

Remark 2.3. Lemma 2.2 allows one to show Lemma 2.1: intermediately one shows, following [Kresch 1999] (notation is as in Lemma 2.1), that $[\mathbb{O}_{C_1}] = [\mathbb{O}_{C_2}]$ in $K_0(C_{X'}Y \times_Y C_X Y)$, where $C_1 := C_{i^* C_X Y}(C_X Y)$ and $C_2 := C_{j^* C_{Y'} Y}(C_{Y'} Y)$.

We now go on to prove Theorem 1.1. We have

$$\chi(\mathscr{X}, j^*V \otimes \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}) = \chi(\mathscr{Y}, V \otimes j_*\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}).$$

Kawasaki's formula applied to the sheaf $V \otimes j_*\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}$ on $\mathscr{Y}$ gives

$$\chi(\mathscr{Y}, V \otimes j_*\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}) = \sum_{\mu} \frac{1}{m_\mu} \chi^f\left(\mathscr{Y}_\mu, \frac{\mathrm{Tr}(V_\mu \otimes i_\mu^* j_*\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\right). \tag{4}$$

From the fiber diagram

$$\begin{array}{ccc} \mathscr{X}_\mu & \xrightarrow{i'_\mu} & \mathscr{X} \\ {\scriptstyle j'}\big\downarrow & & {\scriptstyle j}\big\downarrow \\ \mathscr{Y}_\mu & \xrightarrow{i_\mu} & \mathscr{Y} \end{array}$$

and Theorem 6.2 in [Fulton 1998] (where this is proved for Chow rings) we have $i_\mu^* j_*\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}} = j'_* i_\mu^! \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}$. Plugging this in (4) gives

$$\chi^f\left(\mathscr{Y}_\mu, \frac{\mathrm{Tr}(V_\mu \otimes i_\mu^* j_*\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\right) = \chi^f\left(\mathscr{Y}_\mu, \frac{\mathrm{Tr}(V_\mu \otimes j'_* i_\mu^! \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\right). \tag{5}$$

Let G_μ be the cyclic group generated by one element of the conjugacy class associated to $\mathscr{X}_\mu$. Then we will show that

$$\mathrm{Tr}\left(\frac{i_\mu^! \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}}{\Lambda^\bullet(N_\mu^*)}\right) = \mathrm{Tr}\left(\frac{\mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}}{\Lambda^\bullet(N_\mu^{\mathrm{vir}})^*}\right) \tag{6}$$

in the G_μ-equivariant K-ring of $\mathscr{X}_\mu$. This is essentially the computation of Section 3 in [Graber and Pandharipande 1999] carried out in $\mathbb{C}^*$-equivariant K-theory. Relation (6) then follows by embedding the group G_μ in the torus and specializing the value of the variable t in the ground ring of $\mathbb{C}^*$-equivariant K-theory to a $|G_\mu|$-root of unity.

If we define a cone $D := C_{\mathscr{X}/\mathscr{Y}} \times_{\mathscr{X}} E_0$, then this is a $T_{\mathscr{Y}}$ cone (see [Behrend and Fantechi 1997]). The virtual normal cone D^{vir} is defined as $D/T_{\mathscr{Y}}$ and $\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}$ is the pullback by the zero section of the structure sheaf of D^{vir}. Alternatively there is a fiber diagram

$$\begin{array}{ccc} T_{\mathscr{Y}} & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ \mathscr{X} & \xrightarrow{0_{E_1}} & E_1 \end{array}$$

where the bottom map is the zero section of E_1. Then one can define $\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}$ as $0_{T_{\mathscr{Y}}}^* 0_{E_1}^! [\mathbb{O}_D]$. We'll prove formula (6) following closely the calculation in [Graber and Pandharipande 1999]. First, by definition of $\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}$ and by commutativity of Gysin maps, we have

$$i_\mu^! \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}} = i_\mu^! 0_{T_{\mathscr{Y}}}^* 0_{E_1}^! [\mathbb{O}_D] = 0_{T_{\mathscr{Y}}}^* 0_{E_1}^! i_\mu^! [\mathbb{O}_D]. \tag{7}$$

We pull back relation (3) to $(i'_\mu)^* D = (i'_\mu)^*(C_{\mathscr{X}/\mathscr{Y}} \times E_0)$ to get

$$i_\mu^! [\mathbb{O}_D] = [\mathbb{O}_{D_\mu} \times (E_0^m)^*]. \tag{8}$$

In the equality above we have used the fact that $D_\mu = C_{\mathscr{X}_\mu/\mathscr{Y}_\mu} \times E_0^f$ and we identified the sheaf of sections of the bundle E_0^m with the dual bundle $(E_0^m)^*$. Plugging (8) in (7) we get

$$i_\mu^! \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}} = 0_{T_{\mathscr{Y}}}^* 0_{E_1}^! [\mathbb{O}_{D_\mu} \times (E_0^m)^*]. \tag{9}$$

Notice that the action of $T_{\mathscr{Y}_\mu}$ leaves $D_\mu \times (E_0^m)^*$ invariant (it acts trivially on $(E_0^m)^*$). Now we can write $0_{T_{\mathscr{Y}}}^* = 0_{T_{\mathscr{Y}_\mu^f}}^* \times 0_{T_{\mathscr{Y}_\mu^m}}^*$ and since $D_\mu^{\mathrm{vir}} = D_\mu / T_{\mathscr{Y}_\mu}$ we rewrite (9) as

$$i_\mu^! \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}} = 0_{T_{\mathscr{Y}_\mu^m}}^* 0_{E_1}^! [\mathbb{O}_{D_\mu^{\mathrm{vir}}} \times (E_0^m)^*]. \tag{10}$$

The proof of Lemma 1 in [Graber and Pandharipande 1999] works in our set-up as well: it uses excess intersection formula which holds in K-theory. It shows that the following relation holds in the $\mathbb{C}^*$-equivariant K-ring of $\mathscr{X}_\mu$:

$$0_{T\mathscr{Y}_\mu^m}^* 0_{E_1}^! [\mathbb{O}_{D_\mu^{\mathrm{vir}}} \times (E_0^m)^*] = 0_{E_0^m}^* \big(0_{E_1}^! [\mathbb{O}_{D_\mu^{\mathrm{vir}}} \times (E_0^m)^*]\big) \cdot \frac{\Lambda^\bullet (T_{\mathscr{Y}^m})^*}{\Lambda^\bullet (E_0^m)^*}. \tag{11}$$

The class $0_{E_1}^! [\mathbb{O}_{D_\mu^{\mathrm{vir}}} \times E_0^m]$ lives in the $\mathbb{C}^*$-equivariant K-ring of E_0^m. The class doesn't depend on the bundle map $E_0^m \to E_1^m$ so we can assume this map to be 0. Then by excess intersection formula and the definition of $\mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}$ we get

$$0_{E_0^m}^* \big(0_{E_1}^! [\mathbb{O}_{D_\mu^{\mathrm{vir}}} \times (E_0^m)^*]\big) = \mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}} \cdot \Lambda^\bullet (E_1^m)^*. \tag{12}$$

Formula (12) holds because $D_\mu^{\mathrm{vir}} \times (E_0^m) \subset E_1^f \times E_0^m$ and $0_{E_1}^!$ acts as $0_{E_1^f}^! \times 0_{E_1^m}^!$ on factors. $0_{E_1^f}^! [\mathbb{O}_{D_\mu^{\mathrm{vir}}}] = \mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}$ by definition of $\mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}$. By excess intersection formula applied to the fiber square

$$\begin{array}{ccc} E_0^m & \longrightarrow & E_0^m \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow \\ \mathscr{X}_\mu & \xrightarrow{0_{E_1^m}} & E_1^m \end{array}$$

we have $0_{E_0^m}^* 0_{E_1^m}^! [(E_0^m)^*] = 0_{E_0^m}^* \pi^* \Lambda^\bullet (E_1^m)^* = \Lambda^\bullet (E_1^m)^*$. Plugging formula (12) in (11) (note that $N_\mu = T_{\mathscr{Y}_\mu^m}$ and $N_\mu^{\mathrm{vir}} = [E_0^m] - [E_1^m]$) and taking traces proves (6).

We now plug (6) in (5) and then pull back to $\mathscr{X}_\mu$ to get

$$\chi^f\left(\mathscr{Y}_\mu, \frac{\mathrm{Tr}(V_\mu \otimes j_* i_\mu^* \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\right) = \chi^f\left(\mathscr{Y}_\mu, \mathrm{Tr}(V_\mu) \otimes j'_* \frac{\mathrm{Tr}(\mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet (N_\mu^{\mathrm{vir}})^*)}\right)$$
$$= \chi^f\left(\mathscr{X}_\mu, \frac{\mathrm{Tr}(V_\mu \otimes \mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet (N_\mu^{\mathrm{vir}})^*)}\right). \qquad \square$$

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