Pacific
Journal of
Mathematics

A VIRTUAL KAWASAKI–RIEMANN–ROCH FORMULA

VALENTIN TONITA

Volume 268 No. 1 March 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 268, No. 1, 2014
dx.doi.org/10.2140/pjm.2014.268.249

A VIRTUAL KAWASAKI–RIEMANN–ROCH FORMULA

VALENTIN TONITA

Kawasaki's formula is a tool to compute holomorphic Euler characteristics of vector bundles on a compact orbifold $\mathscr{X}$. Let $\mathscr{X}$ be an orbispace with perfect obstruction theory which admits an embedding in a smooth orbifold. One can then construct the virtual structure sheaf and the virtual fundamental class of $\mathscr{X}$. In this paper we prove that Kawasaki's formula "behaves well" with working "virtually" on $\mathscr{X}$ in the following sense: if we replace the structure sheaves, tangent and normal bundles in the formula by their virtual counterparts then Kawasaki's formula stays true. Our motivation comes from studying the quantum K-theory of a complex manifold X (Givental and Tonita, 2014), with the formula applied to Kontsevich moduli spaces of genus-0 stable maps to X.

1. Introduction

Given a manifold $\mathscr{X}$ and a vector bundle V on $\mathscr{X}$, then the Hirzebruch–Riemann–Roch formula states that

$$\chi(\mathscr{X}, V) = \int_{\mathscr{X}} \mathrm{ch}(V) Td(T_{\mathscr{X}}).$$

Kawasaki [1979] generalized this formula to the case when $\mathscr{X}$ is an orbifold. He reduces the computation of Euler characteristics on $\mathscr{X}$ to the computation of certain cohomological integrals on *the inertia orbifold* $I\mathscr{X}$:

$$\chi(\mathscr{X}, V) = \sum_{\mu} \frac{1}{m_\mu} \int_{\mathscr{X}_\mu} Td(T_{\mathscr{X}_\mu})\, \mathrm{ch}\Big(\frac{\mathrm{Tr}(V)}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\Big). \tag{1}$$

We explain below the ingredients in the formula:

$I\mathscr{X}$ is defined as follows: around any point $p \in \mathscr{X}$ there is a local chart $(\widetilde{U}_p, G_p)$ such that locally $\mathscr{X}$ is represented as the quotient of $\widetilde{U}_p$ by G_p. Consider the set of conjugacy classes $(1) = (h_p^1), (h_p^2), \ldots, (h_p^{n_p})$ in G_p. Define

$$I\mathscr{X} := \{(p, (h_p^i)) \mid i = 1, 2, \ldots, n_p\}.$$

MSC2010: 19L10.
Keywords: Gromov–Witten theory, Riemann–Roch type formulae.

Pick an element h_p^i in each conjugacy class. Then a local chart on $I\mathscr{X}$ is given by

$$\coprod_{i=1}^{n_p} \widetilde{U}_p^{(h_p^i)} / Z_{G_p}(h_p^i),$$

where $Z_{G_p}(h_p^i)$ is the centralizer of h_p^i in G_p. Denote by $\mathscr{X}_\mu$ the connected components of the inertia orbifold (we'll often refer to them as Kawasaki strata). The multiplicity m_μ associated to each $\mathscr{X}_\mu$ is given by

$$m_\mu := \left|\ker(Z_{G_p}(g) \to \operatorname{Aut}(\widetilde{U}_p^g))\right|.$$

For a vector bundle V we will denote by V^* the dual bundle to V. The restriction of V to $\mathscr{X}_\mu$ decomposes in characters of the g action. Let $E_r^{(l)}$ be the subbundle of the restriction of E to $\mathscr{X}_\mu$ on which g acts with eigenvalue $e^{2\pi i l/r}$. Then the trace $\operatorname{Tr}(V)$ is defined to be the orbibundle whose fiber over the point $(p, (g))$ of $\mathscr{X}_\mu$ is

$$\operatorname{Tr}(V) := \sum_l e^{\frac{2\pi i l}{r}} E_r^{(l)}.$$

Finally, $\Lambda^\bullet N_\mu^*$ is the K-theoretic Euler class of the normal bundle N_μ of $\mathscr{X}_\mu$ in $\mathscr{X}$. $\operatorname{Tr}(\Lambda^\bullet N_\mu^*)$ is invertible because the symmetry g acts with eigenvalues different from 1 on the normal bundle to the fixed point locus. We call the terms corresponding to the identity component in the formula *fake Euler characteristics*:

$$\chi^f(\mathscr{X}, V) = \int_{\mathscr{X}} \operatorname{ch}(V) T\,d(T_{\mathscr{X}}).$$

In the case where $\mathscr{X}$ is a global quotient, formula (1) is the Lefschetz fixed point formula.

Now let $\mathscr{X}$ be a compact, complex orbispace (Deligne–Mumford stack) with a perfect obstruction theory $E^{-1} \to E^0$. This is used to define the intrinsic normal cone, which is embedded in E_1 — the dual bundle to E^{-1} (see [Li and Tian 1998; Behrend and Fantechi 1997]). The virtual structure sheaf $\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}$ was defined in [Lee 2004] as the K-theoretic pullback by the zero section of the structure sheaf of this cone. Let $I\mathscr{X} = \coprod_\mu \mathscr{X}_\mu$ be the inertia orbifold of $\mathscr{X}$. We denote by i_μ the inclusion of a stratum $\mathscr{X}_\mu$ in $\mathscr{X}$. For a bundle V on $\mathscr{X}$, we write $i_\mu^* V = V_\mu^f \oplus V_\mu^m$ for its decomposition as the direct sum of the fixed part and the moving part under the action of the symmetry associated to $\mathscr{X}_\mu$. To avoid ugly notation we will often simply write V^m, V^f. The virtual normal bundle to $\mathscr{X}_\mu$ in $\mathscr{X}$ is defined as $[E_0^m] - [E_1^m]$. We will in addition assume that $\mathscr{X}$ admits an embedding j in a smooth compact orbifold $\mathscr{Y}$. This is always true for the moduli spaces of genus-0 stable maps $X_{0,n,d}$ because an embedding $X \hookrightarrow \mathbb{P}^N$ induces an embedding $X_{0,n,d} \hookrightarrow (\mathbb{P}^N)_{0,n,d}$.

Theorem 1.1. *Denote by N_μ^{vir} the virtual normal bundle of $\mathscr{X}_\mu$ in $\mathscr{X}$. Then*

$$\chi(\mathscr{X}, j^*(V)\otimes\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}) = \sum_\mu \frac{1}{m_\mu}\chi^f\left(\mathscr{X}_\mu, \frac{\mathrm{Tr}(V_\mu\otimes\mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet(N_\mu^{\mathrm{vir}})^*)}\right). \tag{2}$$

Remark 1.2. A perfect obstruction theory $E^{-1}\to E^0$ on $\mathscr{X}$ induces canonically a perfect obstruction theory on $\mathscr{X}_\mu$ by taking the fixed part of the complex $E_\mu^{-1,f}\to E_\mu^{0,f}$. The proof is the same as that of Proposition 1 in [Graber and Pandharipande 1999]. This is then used to define the sheaf $\mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}$.

Remark 1.3. It is proved in [Fantechi and Göttsche 2010] that if $\mathscr{X}$ is a scheme, the Grothendieck–Riemann–Roch theorem is compatible with virtual fundamental classes and virtual fundamental sheaves, that is,

$$\chi^f(\mathscr{X}, V\otimes\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}}) = \int_{[\mathscr{X}]}\mathrm{ch}(V\otimes\mathbb{O}_{\mathscr{X}}^{\mathrm{vir}})\cdot Td(T^{\mathrm{vir}}),$$

where $[\mathscr{X}]$ is the virtual fundamental class of $\mathscr{X}$ and T^{vir} is its virtual tangent bundle. Their arguments carry over to the case when $\mathscr{X}$ is a stack.

Remark 1.4. The bundles V to which we apply Theorem 1.1 in [Givental and Tonita 2014] are (sums and products of) cotangent line bundles L_i and evaluation classes $\mathrm{ev}_i^*(a_i)$ (where a_i are K-theoretic classes on the target). They are pullbacks of the corresponding bundles on $(\mathbb{P}^N)_{0,n,d}$.

2. Proof of Theorem 1.1

Before proving Theorem 1.1 we recall a couple of background facts and lemmata on K-theory which we will use.

Let $K_0(X)$ be the Grothendieck group of coherent sheaves on X. Given a map $f: X\to Y$, the K-theoretic pullback $f^*(\mathscr{F}): K_0(Y)\to K_0(X)$ is defined as the alternating sum of derived functors $\mathrm{Tor}^i_{\mathbb{O}_Y}(\mathscr{F}, \mathbb{O}_X)$, provided that the sum is finite. This is always true for instance if f is flat or if it is a regular embedding.

For any fiber square

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\ i\ } & B \end{array}$$

with i a regular embedding one can define K-theoretic refined Gysin homomorphisms $i^!: K_0(V)\to K_0(V')$ (see [Lee 2004]). One way to define the map $i^!$ is the following: The class $i_*(\mathbb{O}_{B'})\in K^0(B)$ has a finite resolution of vector bundles, which is exact off B'. We pull it back to V and then cap (i.e., tensor product) with classes in $K_0(V)$, to get a class on $K_0(V)$ with homology supported on V', which we can regard as an element of $K_0(V')$, because there is a canonical isomorphism between complexes on V with homology supported on V' and $K_0(V')$.

In the following two lemmata, X, Y, Y' are assumed DM stacks. We will use the following result:

Lemma 2.1. *Consider the diagram*:

$$\begin{array}{ccc} \iota^* C_{X/Y} & \longrightarrow & C_{X/Y} \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\ \iota\ } & X \\ \downarrow & & \downarrow{\scriptstyle j} \\ Y' & \xrightarrow{\ i\ } & Y \end{array}$$

with i a regular embedding and j an embedding, $C_{X/Y}$ is the normal cone of X in Y and both squares are fiber diagrams. Then

$$i^![\mathcal{O}_{C_{X/Y}}] = [\mathcal{O}_{C_{X'/Y'}}] \in K_0(\iota^* C_{X/Y}). \tag{3}$$

This is stated and proved in [Lee 2004, Lemma 2]. The proof is based on a more general statement (Lemma 1 of [Lee 2004]), which has been worked out in [Kresch 1999] on the level of Chow rings. Since K-theoretic statements are stronger, we give below the key ingredient which allows one to carry over Kresch's proof to K-theory:

Lemma 2.2. *Let $f : X \to Y$ be a closed embedding and let $g : Y \to \mathbb{P}^1$ be a surjection such that $g \circ f$ is flat. Denote by X_0 and Y_0 the fibers over* 0 *of $g \circ f$ and g, respectively. Moreover, assume that the restriction of f to $X \setminus X_0$ is an isomorphism. Then if i is the inclusion of* $\{0\}$ *in $\mathbb{P}^1$, we have $i^!(\mathcal{O}_Y) = \mathcal{O}_{X_0} \in K_0(Y_0)$.*

Proof. The skyscraper sheaves at all points of $\mathbb{P}^1$ represent the same element in $K_0(\mathbb{P}^1)$, hence if we pull back a resolution of any point $P \in \mathbb{P}^1$ by g we get the same elements of $K_0(Y)$. On the other hand since f is an isomorphism above $\mathbb{P}^1 \setminus \{0\}$, pulling back by g of the structure sheaf of a point $P \neq 0$ is the same as pulling back by $g \circ f$ followed by f_*. By what we said above we can replace P with 0. Now from the flatness of $g \circ f$ above 0 the pullback of the structure sheaf of 0 by $g \circ f$ is the structure sheaf of the fiber X_0. The result then follows from the definition of $i^!$. □

Remark 2.3. Lemma 2.2 allows one to show Lemma 2.1: intermediately one shows, following [Kresch 1999] (notation is as in Lemma 2.1), that $[\mathcal{O}_{C_1}] = [\mathcal{O}_{C_2}]$ in $K_0(C_{X'}Y \times_Y C_X Y)$, where $C_1 := C_{i^* C_X Y}(C_X Y)$ and $C_2 := C_{j^* C_{Y'} Y}(C_{Y'} Y)$.

We now go on to prove Theorem 1.1. We have

$$\chi(\mathcal{X}, j^*V \otimes \mathcal{O}_{\mathcal{X}}^{\mathrm{vir}}) = \chi(\mathcal{Y}, V \otimes j_*\mathcal{O}_{\mathcal{X}}^{\mathrm{vir}}).$$

Kawasaki's formula applied to the sheaf $V \otimes j_*\mathcal{O}_{\mathcal{X}}^{\mathrm{vir}}$ on $\mathcal{Y}$ gives

$$\chi(\mathcal{Y}, V \otimes j_*\mathcal{O}_{\mathcal{X}}^{\mathrm{vir}}) = \sum_{\mu} \frac{1}{m_\mu} \chi^f\left(\mathcal{Y}_\mu, \frac{\mathrm{Tr}(V_\mu \otimes i_\mu^* j_*\mathcal{O}_{\mathcal{X}}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\right). \tag{4}$$

From the fiber diagram

$$\begin{array}{ccc} \mathcal{X}_\mu & \xrightarrow{i'_\mu} & \mathcal{X} \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{Y}_\mu & \xrightarrow{i_\mu} & \mathcal{Y} \end{array}$$

and Theorem 6.2 in [Fulton 1998] (where this is proved for Chow rings) we have $i_\mu^* j_*\mathcal{O}_{\mathcal{X}}^{\mathrm{vir}} = j'_* i_\mu^! \mathcal{O}_{\mathcal{X}}^{\mathrm{vir}}$. Plugging this in (4) gives

$$\chi^f\left(\mathcal{Y}_\mu, \frac{\mathrm{Tr}(V_\mu \otimes i_\mu^* j_*\mathcal{O}_{\mathcal{X}}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\right) = \chi^f\left(\mathcal{Y}_\mu, \frac{\mathrm{Tr}(V_\mu \otimes j'_* i_\mu^! \mathcal{O}_{\mathcal{X}}^{\mathrm{vir}})}{\mathrm{Tr}(\Lambda^\bullet N_\mu^*)}\right). \tag{5}$$

Let G_μ be the cyclic group generated by one element of the conjugacy class associated to $\mathcal{X}_\mu$. Then we will show that

$$\mathrm{Tr}\left(\frac{i_\mu^! \mathcal{O}_{\mathcal{X}}^{\mathrm{vir}}}{\Lambda^\bullet(N_\mu^*)}\right) = \mathrm{Tr}\left(\frac{\mathcal{O}_{\mathcal{X}_\mu}^{\mathrm{vir}}}{\Lambda^\bullet(N_\mu^{\mathrm{vir}})^*}\right) \tag{6}$$

in the G_μ-equivariant K-ring of $\mathcal{X}_\mu$. This is essentially the computation of Section 3 in [Graber and Pandharipande 1999] carried out in $\mathbb{C}^*$-equivariant K-theory. Relation (6) then follows by embedding the group G_μ in the torus and specializing the value of the variable t in the ground ring of $\mathbb{C}^*$-equivariant K-theory to a $|G_\mu|$-root of unity.

If we define a cone $D := C_{\mathcal{X}/\mathcal{Y}} \times_{\mathcal{X}} E_0$, then this is a $T_{\mathcal{Y}}$ cone (see [Behrend and Fantechi 1997]). The virtual normal cone D^{vir} is defined as $D/T_{\mathcal{Y}}$ and $\mathcal{O}_{\mathcal{X}}^{\mathrm{vir}}$ is the pullback by the zero section of the structure sheaf of D^{vir}. Alternatively there is a fiber diagram

$$\begin{array}{ccc} T_{\mathcal{Y}} & \longrightarrow & D \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{0_{E_1}} & E_1 \end{array}$$

where the bottom map is the zero section of E_1. Then one can define $\mathcal{O}_{\mathcal{X}}^{\mathrm{vir}}$ as $0_{T_{\mathcal{Y}}}^* 0_{E_1}^! [\mathcal{O}_D]$. We'll prove formula (6) following closely the calculation in [Graber and Pandharipande 1999]. First, by definition of $\mathcal{O}_{\mathscr{X}}^{\mathrm{vir}}$ and by commutativity of Gysin maps, we have

$$i_\mu^! \mathcal{O}_{\mathscr{X}}^{\mathrm{vir}} = i_\mu^! 0_{T_{\mathscr{Y}}}^* 0_{E_1}^! [\mathcal{O}_D] = 0_{T_{\mathscr{Y}}}^* 0_{E_1}^! i_\mu^! [\mathcal{O}_D]. \tag{7}$$

We pull back relation (3) to $(i'_\mu)^* D = (i'_\mu)^*(C_{\mathscr{X}/\mathscr{Y}} \times E_0)$ to get

$$i_\mu^! [\mathcal{O}_D] = [\mathcal{O}_{D_\mu} \times (E_0^m)^*]. \tag{8}$$

In the equality above we have used the fact that $D_\mu = C_{\mathscr{X}_\mu/\mathscr{Y}_\mu} \times E_0^f$ and we identified the sheaf of sections of the bundle E_0^m with the dual bundle $(E_0^m)^*$. Plugging (8) in (7) we get

$$i_\mu^! \mathcal{O}_{\mathscr{X}}^{\mathrm{vir}} = 0_{T_{\mathscr{Y}}}^* 0_{E_1}^! [\mathcal{O}_{D_\mu} \times (E_0^m)^*]. \tag{9}$$

Notice that the action of $T_{\mathscr{Y}_\mu}$ leaves $D_\mu \times (E_0^m)^*$ invariant (it acts trivially on $(E_0^m)^*$). Now we can write $0_{T_{\mathscr{Y}}}^* = 0_{T_{\mathscr{Y}_\mu^f}}^* \times 0_{T_{\mathscr{Y}_\mu^m}}^*$ and since $D_\mu^{\mathrm{vir}} = D_\mu / T_{\mathscr{Y}_\mu}$ we rewrite (9) as

$$i_\mu^! \mathcal{O}_{\mathscr{X}}^{\mathrm{vir}} = 0_{T_{\mathscr{Y}_\mu^m}}^* 0_{E_1}^! [\mathcal{O}_{D_\mu^{\mathrm{vir}}} \times (E_0^m)^*]. \tag{10}$$

The proof of Lemma 1 in [Graber and Pandharipande 1999] works in our set-up as well: it uses excess intersection formula which holds in K-theory. It shows that the following relation holds in the $\mathbb{C}^*$-equivariant K-ring of $\mathscr{X}_\mu$:

$$0_{T\mathscr{Y}_\mu^m}^* 0_{E_1}^! [\mathcal{O}_{D_\mu^{\mathrm{vir}}} \times (E_0^m)^*] = 0_{E_0^m}^* \big(0_{E_1}^! [\mathcal{O}_{D_\mu^{\mathrm{vir}}} \times (E_0^m)^*]\big) \cdot \frac{\Lambda^\bullet (T_{\mathscr{Y}^m})^*}{\Lambda^\bullet (E_0^m)^*}. \tag{11}$$

The class $0_{E_1}^! [\mathcal{O}_{D_\mu^{\mathrm{vir}}} \times E_0^m]$ lives in the $\mathbb{C}^*$-equivariant K-ring of E_0^m. The class doesn't depend on the bundle map $E_0^m \to E_1^m$ so we can assume this map to be 0. Then by excess intersection formula and the definition of $\mathcal{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}$ we get

$$0_{E_0^m}^* \big(0_{E_1}^! [\mathcal{O}_{D_\mu^{\mathrm{vir}}} \times (E_0^m)^*]\big) = \mathcal{O}_{\mathscr{X}_\mu}^{\mathrm{vir}} \cdot \Lambda^\bullet (E_1^m)^*. \tag{12}$$

Formula (12) holds because $D_\mu^{\mathrm{vir}} \times (E_0^m) \subset E_1^f \times E_0^m$ and $0_{E_1}^!$ acts as $0_{E_1^f}^! \times 0_{E_1^m}^!$ on factors. $0_{E_1^f}^! [\mathcal{O}_{D_\mu^{\mathrm{vir}}}] = \mathcal{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}$ by definition of $\mathcal{O}_{\mathscr{X}_\mu}^{\mathrm{vir}}$. By excess intersection formula applied to the fiber square

$$\begin{array}{ccc} E_0^m & \longrightarrow & E_0^m \\ \pi \big\downarrow & & \big\downarrow \\ \mathscr{X}_\mu & \xrightarrow{\;0_{E_1^m}\;} & E_1^m \end{array}$$

we have $0_{E_0^m}^* 0_{E_1^m}^! [(E_0^m)^*] = 0_{E_0^m}^* \pi^* \Lambda^\bullet (E_1^m)^* = \Lambda^\bullet (E_1^m)^*$. Plugging formula (12) in (11) (note that $N_\mu = T_{\mathscr{Y}_\mu^m}$ and $N_\mu^{\mathrm{vir}} = [E_0^m] - [E_1^m]$) and taking traces proves (6).

We now plug (6) in (5) and then pull back to $\mathscr{X}_\mu$ to get

$$\chi^f\left(\mathscr{Y}_\mu, \frac{\operatorname{Tr}(V_\mu \otimes j_* i_\mu^* \mathbb{O}_{\mathscr{X}}^{\mathrm{vir}})}{\operatorname{Tr}(\Lambda^\bullet N_\mu^*)}\right) = \chi^f\left(\mathscr{Y}_\mu, \operatorname{Tr}(V_\mu) \otimes j'_* \frac{\operatorname{Tr}(\mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}})}{\operatorname{Tr}(\Lambda^\bullet (N_\mu^{\mathrm{vir}})^*)}\right)$$
$$= \chi^f\left(\mathscr{X}_\mu, \frac{\operatorname{Tr}(V_\mu \otimes \mathbb{O}_{\mathscr{X}_\mu}^{\mathrm{vir}})}{\operatorname{Tr}(\Lambda^\bullet (N_\mu^{\mathrm{vir}})^*)}\right). \qquad \square$$

Acknowledgements

I would like to thank Alexander Givental for suggesting the problem and for useful discussions. Thanks are also due to Yuan-Pin Lee who patiently answered my questions on the material in his work [Lee 2004] and to Hsian-Hua Tseng who read a preliminary draft of the paper.

References

[Behrend and Fantechi 1997] K. Behrend and B. Fantechi, "The intrinsic normal cone", *Invent. Math.* **128**:1 (1997), 45–88. MR 98e:14022 Zbl 0909.14006

[Fantechi and Göttsche 2010] B. Fantechi and L. Göttsche, "Riemann–Roch theorems and elliptic genus for virtually smooth schemes", *Geom. Topol.* **14**:1 (2010), 83–115. MR 2011a:14016 Zbl 1194.14017

[Fulton 1998] W. Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **2**, Springer, Berlin, 1998. MR 99d:14003 Zbl 0885.14002

[Givental and Tonita 2014] A. Givental and V. Tonita, "The Hirzebruch–Riemann–Roch theorem in true genus-0 quantum K-theory", pp. 43–90 in *Symplectic, Poisson and noncommutative geometry*, Mathematical Sciences Research Publications **62**, Cambridge Univ. Press, New York, 2014. arXiv 1106.3136

[Graber and Pandharipande 1999] T. Graber and R. Pandharipande, "Localization of virtual classes", *Invent. Math.* **135**:2 (1999), 487–518. MR 2000h:14005 Zbl 0953.14035

[Kawasaki 1979] T. Kawasaki, "The Riemann–Roch theorem for complex V-manifolds", *Osaka J. Math.* **16**:1 (1979), 151–159. MR 80f:58042 Zbl 0405.32010

[Kresch 1999] A. Kresch, "Canonical rational equivalence of intersections of divisors", *Invent. Math.* **136**:3 (1999), 483–496. MR 2000d:14005 Zbl 0923.14003

[Lee 2004] Y.-P. Lee, "Quantum K-theory, I: Foundations", *Duke Math. J.* **121**:3 (2004), 389–424. MR 2005f:14107 Zbl 1051.14064

[Li and Tian 1998] J. Li and G. Tian, "Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties", *J. Amer. Math. Soc.* **11**:1 (1998), 119–174. MR 99d:14011 Zbl 0912.14004

Received October 19, 2012. Revised September 23, 2013.

VALENTIN TONITA
KAVLI INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE (KAVLI IPMU)
5-1-5 KASHIWANOHA
KASHIWA 277-8583
JAPAN
valentin.tonita@ipmu.jp

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US $410/year for the electronic version, and $535/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 268 No. 1 March 2014
