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Λ-ADIC BARSOTTI–TATE GROUPS

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To my dear friend Jon Rogawski, who created the UCLA number theory group.

We define Λ-BT groups as well-controlled ind-Barsotti–Tate groups under the action of the Iwasawa algebra and construct a prototypical example of such groups out of modular Jacobians. We then discuss the relation of these groups to Weil numbers of weight 1 and to the nonvanishing problem of the adjoint $\mathscr{L}$-invariant.

1. Introduction

Fix a prime $p \geq 5$ (throughout the paper). For a given valuation ring R, a Λ-BT group $\mathscr{G} = \mathscr{G}_R$ is by definition an inductive limit of (p-divisible) Barsotti–Tate groups $\mathscr{G}_n = \mathscr{G}_{n,R}$ ($0 < n \in \mathbb{Z}$) defined over R with an action of the Iwasawa algebra $\Lambda = \Lambda_W := W[\![x]\!]$ as endomorphisms. Here the limit is taken as an object of the ind-category of commutative group schemes over R or in the (bigger) abelian category of abelian fppf sheaves over R (see [Hida 2012, §1.12.1] for abelian fppf sheaves), W is a discrete valuation ring finite flat over $\mathbb{Z}_p$, and $W[\![x]\!]$ is the ring of power series in one variable. We write K for the quotient field of R and $\mathbb{F}$ for the residue field; $\overline{K}$, $\overline{\mathbb{F}}$ denote algebraic closures thereof. We assume R to be of mixed characteristic $(0, p)$ (so K has characteristic 0 and $\mathbb{F}$ has characteristic p),

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though the definition is obviously valid over any valuation ring (and it is interesting to know if there is some good theory over a general R). We impose the following two conditions:

(CT′) Writing $\gamma = 1+x$, we have $\mathcal{G}_{n,R} = \mathcal{G}_R[\gamma^{p^{n-1}}-1] := \mathrm{Ker}(\gamma^{p^{n-1}}-1 : \mathcal{G}_R \to \mathcal{G}_R)$ (in particular, $\mathcal{G}_{n,R} \hookrightarrow \mathcal{G}_R$ is a closed immersion).

(DV) The geometric generic fiber $\mathcal{G}(\bar{K})$ is isomorphic to $(\Lambda^*)^n$ for the Pontryagin dual $\Lambda^* := \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$, so $T\mathcal{G} = \mathrm{Hom}_\Lambda(\Lambda^*, \mathcal{G}(\bar{K}))$ is Λ-free of rank n.

In the text (see (CT) in Section 3), we often impose a slightly stronger condition than (CT′) here (though we have an example of $\mathcal{G}$ satisfying only (CT′) not (CT); see Remark 5.5). Such group schemes have been studied in [Hida 1986; Mazur and Wiles 1986; Tilouine 1987; Ohta 1995] and more recently in [Cais 2012, §5.4], primarily through the deformation theory of modular forms and Galois representations. In this note, we would like to give some basic facts of Λ-BT groups and to point out their relation to the $\mathcal{L}$-invariant of adjoint square L-functions of modular forms. In some sense, this note is a revisiting of the topics presented in [Hida 1986] through a new formulation via arithmetic geometry, and in near future, we hope to delve into deeper in this direction. We write $\Gamma = \gamma^{\mathbb{Z}_p}$ for the subgroup of $\Lambda^\times$ topologically generated by γ, which is isomorphic to $1 + p\mathbb{Z}_p$ (we fix the isomorphism $1 + p\mathbb{Z}_p \cong \Gamma$ sending $1 + p$ to $1 + x$).

Starting with a brief explanation of $U(p)$-isomorphisms in Section 2, in Section 3, we list expected properties (other than (CT′) and (DV)) of Λ-BT groups, and we discuss seemingly naive questions on Λ-BT groups. Some of the implications of the questions will be discussed in later sections. In Sections 4 and 5, we construct modular Λ-BT groups, and verify most of the properties listed in Section 3 for the modular Λ-BT groups. In Section 6, we prove the remaining properties in Section 3 related to reduction modulo p of the modular Λ-BT groups. In the last two Sections 7 and 8, we relate the theory to the problem of nonvanishing of the adjoint $\mathcal{L}$-invariants. We note that p-adic Hodge theory of the modular Λ-BT group, as well as other aspects, has been studied by Cais [2012], who may have been influenced by my lecture notes [Hida 2005] at CRM, which were the origin of the present paper.

2. $U(p)$-isomorphisms

For $\mathbb{Z}[U]$-modules X and Y, we call a $\mathbb{Z}[U]$-linear map $f : X \to Y$ a U-injection (resp. a U-surjection) if $\mathrm{Ker}(f)$ is killed by a power of U (resp. $\mathrm{Coker}(f)$ is killed by a power of U). If f is a U-injection and a U-surjection, we call f a U-isomorphism. In other words, f is a U-injection, a U-surjection, or a U-isomorphism if, after tensoring with $\mathbb{Z}[U, U^{-1}]$, it becomes an injection, a surjection, or an isomorphism.

In terms of U-isomorphisms, we describe briefly the facts we study in this article (and in later sections, we fill in more details in terms of the ordinary projector e).[1]

Let N be a positive integer prime to p. We consider the (open) modular curve $Y_r := Y_1(Np^r)_{/\mathbb{Q}}$ which classifies elliptic curves E with an embedding $\phi : \mu_{Np^r} \hookrightarrow E[Np^r] = \mathrm{Ker}(Np^r : E \to E)$. Let $R_i = \mathbb{Z}_{(p)}[\mu_{p^i}]$ and $K_i = \mathbb{Q}[\mu_{p^i}]$ ($i = 1, 2, \dots, \infty$). We fix an isomorphism $\mathbb{Z}_p(1) = \varprojlim_r \mu_{p^r}(R_\infty)$ choosing a coherent sequence of primitive roots of unity $\zeta_{p^r} \in \mu_{p^r}(R_r)$ such that $\zeta_{p^{r+1}}^p = \zeta_{p^r}$ for all r, and therefore R_i has a specific primitive root of unity denoted by ζ_{p^i}. Similarly, we fix an isomorphism $\mathbb{Z}/N\mathbb{Z} \cong \mu_N$ over $\mathbb{Q}(\mu_N)$ (given by $m \mapsto \zeta_N^m$) choosing a primitive root of unity ζ_N. We write $\zeta_{Np^r} = \zeta_N\zeta_{p^r}$. Let R be either a valuation ring or a field (over $\mathbb{Z}_{(p)}$) inside $\mathbb{Q}[\mu_{p^\infty}]$ with quotient field K. We write $X_{r,R}$ for the normalization of the j-line $\boldsymbol{P}(j)_{/R}$ in the function field of $Y_{r/K}$. The group $z \in (\mathbb{Z}/p^r\mathbb{Z})^\times$ acts on $Y_{r,K}$ by $\phi \mapsto \phi \circ z$ (and hence on its normalization $X_{r,R}$), as $\mathrm{Aut}(\mu_{Np^r}) \cong (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Thus $\Gamma = 1 + p\mathbb{Z}_p = \gamma^{\mathbb{Z}_p}$ acts on X_r (and its Jacobian) through its image in $(\mathbb{Z}/Np^r\mathbb{Z})^\times$. For $s > r \geq 0$, we define another modular curve $Y^r_{s,K}$ by the geometric quotient of Y_s by $(1 + p^r\mathbb{Z}_p)/(1 + p^s\mathbb{Z}_p) \subset (\mathbb{Z}/Np^s\mathbb{Z})^\times$ and define $X^r_{s,R}$ to be the normalization of $\boldsymbol{P}(j)_{/R}$ in the function field $K(Y^r_{s,K})$. Then $X^r_{s,K}(\mathbb{C})$ is given by $\Gamma^r_s \backslash (\mathfrak{H} \sqcup \boldsymbol{P}^1(\mathbb{Q}))$ for $\Gamma^r_s = \Gamma_1(Np^r) \cap \Gamma_0(p^s)$ ($s > r \geq 0$). Hereafter we take $U = U(p)$ for the Hecke operator $U(p)$ (as defined in [Hida 2012, §3.2.3 and §4.2.1]).

As before, take a valuation ring $R \subset \mathbb{Q}[\mu_{p^\infty}]$ over $\mathbb{Z}_{(p)}$. Let $J_{r,R} = \mathrm{Pic}^0_{X_{r,R}/R}$ be the connected component of the Picard scheme. Then $J_{r,R}$ is the identity connected component of the Néron model of the Jacobian $J_{r/K}$ of $X_{r/K}$. Indeed, by the table of geometric multiplicities of irreducible components of $X_{r/\mathbb{F}_p} = X \times_R \mathbb{F}_p$ in [Katz and Mazur 1985, 13.5.6], the greatest common divisor D of the geometric multiplicities of irreducible components of $X_{r/\mathbb{F}_p}$ is equal to 1. Then by a result of Raynaud [Bosch et al. 1990, Theorem 9.5.4(b)], the identity $D = 1$ implies that $J_{r,R}$ is isomorphic to the identity connected component of the Néron model of $J_{r/K}$ over R. Similarly, $J^r_s := \mathrm{Pic}^0_{X^r_s/R}$ is isomorphic to the identity connected component of the Néron model over R of the Jacobian variety $J^r_{s/K}$ of the modular curve $X^r_{s/K}$. Note that

$$\text{(2-1)}\qquad \Gamma^r_s\backslash\Gamma^r_s\begin{pmatrix}1 & 0\\ 0 & p^{s-r}\end{pmatrix}\Gamma_1(Np^r) = \left\{\begin{pmatrix}1 & a\\ 0 & p^{s-r}\end{pmatrix} \,\middle|\, a \bmod p^{s-r}\right\}$$
$$= \Gamma_1(Np^r)\backslash\Gamma_1(Np^r)\begin{pmatrix}1 & 0\\ 0 & p^{s-r}\end{pmatrix}\Gamma_1(Np^r).$$

Now, write $U^s_r(p^{s-r}) : J^s_r \to J_r$ for the Hecke operator of $\Gamma^s_r\alpha_{s-r}\Gamma_1(Np^r)$ for

[1]This section about U-isomorphisms is from a conference talk at CRM in September in 2005 (see http://www.crm.umontreal.ca/Representations05/indexen.html).

$\alpha_m = \left(\begin{smallmatrix}1&0\\0&p^m\end{smallmatrix}\right)$. As described in [Shimura 1971, Chapter 7], for a modular curve $X(\Gamma) := \Gamma\backslash(\mathfrak{H} \sqcup \boldsymbol{P}^1(\mathbb{Q}))$, each double coset $\Gamma\alpha\Gamma'$ gives rise to a correspondence $X(\Gamma \cap \alpha\Gamma'\alpha^{-1})$ embedded in $X(\Gamma) \times X(\Gamma')$ by $z \mapsto (z, \alpha(z))$. In our cases of $\Gamma = \Gamma_s^r, \Gamma_1(Np^r)$ and $\Gamma' = \Gamma_1(Np^r)$, the modular curve $X(\Gamma\cap\alpha\Gamma'\alpha^{-1})$ is known to be defined over $\mathbb{Q}$ [loc. cit.], and hence the correspondences are also defined over $\mathbb{Q}$. These correspondences defined over $\mathbb{Q}$ act on the Jacobians by morphisms defined over $\mathbb{Q}$ (by Picard and Albanese functoriality, respectively) and its composition relation verified over $\mathbb{C}$ remains valid over $\mathbb{Q}$ (and over any subfield in $\mathbb{C}$). Then we have the following commutative diagram from the above identity, first over $\mathbb{C}$, then over K (via the correspondences defined over $\mathbb{Q}$ and hence over K) and by functoriality (of Picard schemes or Néron models) over R:

$$\begin{tikzcd} J_{r,R} \arrow[r, "\pi^*"] \arrow[d, "u"'] & J^r_{s,R} \arrow[d, "u''"] \arrow[dl, "u'"'] \\ J_{r,R} \arrow[r, "\pi^*"] & J^r_{s,R} \end{tikzcd} \tag{2-2}$$

where the middle u' is given by $U_r^s(p^{s-r})$ and u and u'' are $U(p^{s-r})$. Thus

(u1) $\pi^* : J_{r,R} \to J^r_{s,R}$ is a $U(p)$-isomorphism (for the projection $\pi : X_s^r \to X_r$).

Taking the dual $U^*(p)$ of $U(p)$ with respect to the Rosati involution induced by the canonical polarization on the Jacobians, we have a dual version of the above diagram for $s > r > 0$:

$$\begin{tikzcd} J_{r,R} & J^r_{s,R} \arrow[l, "\pi_*"'] \\ J_{r,R} \arrow[u, "u^*"] \arrow[ur, "u'^*"] & J^r_{s,R} \arrow[l, "\pi_*"'] \arrow[u, "u''^*"'] \end{tikzcd} \tag{2-3}$$

Here the superscript "$*$" indicates the Rosati involution corresponding to the canonical divisor on the Jacobians, and $u^* = U^*(p)^{s-r}$ for the level $\Gamma_1(Np^r)$ and $u''^* = U^*(p)^{s-r}$ for Γ_s^r. Without applying the duality, these morphisms come directly from Hecke correspondences associated to the following coset decomposition:

$$\begin{aligned} \Gamma_s^r\Big\backslash\Gamma_s^r\begin{pmatrix}p^{s-r}s & 0\\ 0 & 1\end{pmatrix}\Gamma_1(Np^r) &= \Gamma_1(Np^r)\Big\backslash\Gamma_1(Np^r)\begin{pmatrix}p^{s-r} & 0\\ 0 & 1\end{pmatrix}\Gamma_s^r \\ &= \left\{\begin{pmatrix}p^{s-r} & a\\ 0 & 1\end{pmatrix} \,\middle|\, a \bmod p^{s-r}\right\} \\ &= \Gamma_1(Np^r)\Big\backslash\Gamma_1(Np^r)\begin{pmatrix}p^{s-r} & 0\\ 0 & 1\end{pmatrix}\Gamma_1(Np^r). \end{aligned} \tag{2-4}$$

Alternatively, the diagram (2-3) follows from Albanese functoriality of Jacobians applied to the Hecke correspondence $U(p)$. In any case, we get

(u1*) $\pi_* : J_{r,R} \to J^r_{s,R}$ is a $U^*(p)$-isomorphism, where π_* is the dual of π^*.

In particular, if we take the ordinary and the coordinary projectors

$$e = \lim_{n\to\infty} U(p)^{n!} \quad \text{and} \quad e^* = \lim_{n\to\infty} U^*(p)^{n!}$$

on $J[p^\infty]$ for $J = J_r, J_s, J^r_s$, noting $U(p^m) = U(p)^m$, we have

$$\pi^* : J_r[p^\infty]^{\mathrm{ord}} \cong J^r_s[p^\infty]^{\mathrm{ord}} \quad \text{and} \quad \pi_* : J^r_s[p^\infty]^{\mathrm{coord}} \cong J_r[p^\infty]^{\mathrm{coord}},$$

where "ord" (resp. "coord") indicates the image of the projector e (resp. e^*). For simplicity, we write $\mathcal{G}_{r,R} := J_r[p^\infty]^{\mathrm{ord}}_{/R}$. The group scheme $J_r[p^n]$ is often neither flat nor finite (for example if $J_{r/\mathbb{F}_p}$ has additive part). Thus for the moment, as explained in the introduction, we take $J_r[p^\infty]^{\mathrm{ord}}_{/R}$ as defined either in the ind-category of group schemes over R or in the abelian category of fppf abelian sheaves. Our point is that we will show that the ordinary part behaves well under Picard functoriality and is represented by a Λ-BT group if $R \supset R_\infty$.

Pick a congruence subgroup Γ defining the modular curve $X = X(\Gamma)$, and write its Jacobian as J. We now identify $J(\mathbb{C})$ with a subgroup of $H^1(\Gamma, \boldsymbol{T})$ (for the Γ-module $\boldsymbol{T} := \mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C} \mid |z| = 1\}$ with trivial Γ-action). Since $\Gamma^r_s \rhd \Gamma_1(Np^s)$, we may consider the finite cyclic quotient group $C := \Gamma^r_s/\Gamma_1(Np^s)$. By the inflation restriction sequence, we have the following commutative diagram with exact rows:

$$\begin{tikzcd} H^1(C,\boldsymbol{T}) \arrow[r, hook] & H^1(\Gamma^r_s,\boldsymbol{T}) \arrow[r] & H^1(\Gamma_1(Np^s),\boldsymbol{T})^{\gamma^{p^{r-1}}=1} \arrow[r] & H^2(C,\boldsymbol{T}) \\ ? \arrow[u] \arrow[r] & J^r_s(\mathbb{C}) \arrow[u, "\cup"'] \arrow[r] & J_s(\mathbb{C})[\gamma^{p^{r-1}}-1] \arrow[u, "\cup"'] \arrow[r] & ? \arrow[u] \end{tikzcd} \tag{2-5}$$

Since C is a finite cyclic group of order p^{s-r} (with generator g) acting trivially on $\boldsymbol{T}$, we have $H^1(C, \boldsymbol{T}) = \mathrm{Hom}(C, \boldsymbol{T}) \cong C$ and

$$H^2(C, \boldsymbol{T}) = \boldsymbol{T}/(1 + g + \cdots + g^{p^{s-r}-1})\boldsymbol{T} = \boldsymbol{T}/p^{s-r}\boldsymbol{T} = 0.$$

By the same token, replacing $\boldsymbol{T}$ by $\mathbb{T}_p := \mathbb{Q}_p/\mathbb{Z}_p$, we get $H^2(C, \mathbb{T}_p) = 0$. By computing explicitly the double coset action of $U(p)$ (see [Hida 1986, Lemma 6.1]), we confirm that $U(p)$ acts on $H^1(C, \boldsymbol{T})$ and $H^1(C, \mathbb{T}_p)$ via multiplication by its degree p, and hence $U(p)^{s-r}$ kill $H^1(C, \boldsymbol{T})$ and $H^1(C, \mathbb{T}_p)$. Hence $J^r_s \to J_s$ is a U-isomorphism over $\mathbb{C}$ and hence over K. We record what we have proved:

$$U(p)^{s-r}(H^1(C, \mathbb{T}_p)) = H^2(C, \boldsymbol{T}) = H^2(C, \mathbb{T}_p) = 0. \tag{2-6}$$

Thus $J_s[\gamma^{p^{r-1}}-1](\mathbb{C})$ is connected, and hence $J_s[\gamma^{p^{r-1}}-1]$ is an abelian variety over $\mathbb{Q}$ (as it is the kernel of the $\mathbb{Q}$-rational endomorphism $\gamma^{p^{r-1}}-1$). By the diagram (2-5), we get an isogeny $i_r^s : J^r_{s,\mathbb{C}} \to J_{s,\mathbb{C}}[\gamma^{p^{r-1}}-1]$ whose kernel is a cyclic group of order p^{s-r}. Since this isogeny i_r^s is induced by the Picard functoriality from the covering map $X_s \twoheadrightarrow X_s^r$ defined over $\mathbb{Q}$, i_r^s is defined over $\mathbb{Q}$. Then it extends to $i^s_{r/R} : J^r_{s,R} \to J_{s,R}$ by the functoriality of Picard schemes (which factors through $J_{s,R}[\gamma^{p^{r-1}}-1] = \mathrm{Ker}(\gamma^{p^{r-1}}-1 : J_{s,R} \to J_{s,R})$). There is a more arithmetic proof of these facts valid for any k-points (for a general field k) in place of $\mathbb{C}$-points of the Jacobians which we hope to discuss in our future article.

Though we do not use this fact in this paper, there is a dual homology version of (2-5) coming from group homology [Brown 1982, VII.6.4]:

$$
\begin{tikzcd}
H_2(C,\boldsymbol{T}) \arrow[r, hook] \arrow[d] & \dfrac{H_1(\Gamma_1(Np^s),\boldsymbol{T})}{(\gamma^{p^{r-1}}-1)H_1(\Gamma_1(Np^s),\boldsymbol{T})} \arrow[r] \arrow[d, "\text{onto}"'] & H_1(\Gamma_s^r,\boldsymbol{T}) \arrow[r] \arrow[d, "\text{onto}"'] & H_1(C,\boldsymbol{T}) \arrow[d] \\
? \arrow[r] & (J_s/(\gamma^{p^{r-1}}-1)J_s)(\mathbb{C}) \arrow[r] & J_s^r(\mathbb{C}) \arrow[r] & ?
\end{tikzcd}
\tag{2-7}
$$

Since $H_j(C,\boldsymbol{T})$ is the Pontryagin dual of $H^j(C,\mathbb{Z})$, we have

$$H_1(C,\boldsymbol{T}) = \mathrm{Hom}(C,\mathbb{Z}) = 0 \quad \text{and} \quad H_2(C,\boldsymbol{T}) = H^2(C,\mathbb{Z}) = \mathbb{Z}/N_s^r\mathbb{Z} \cong C$$

for $N_s^r = 1+g+\cdots+g^{p^{s-r}-1}$ with $g=\gamma^{p^{r-1}}$. This shows that $J^r_{s,\mathbb{Q}}$ is a quotient of $(J_s/(\gamma^{p^{r-1}}-1)J_s)_{/\mathbb{Q}}$ by a finite cyclic group of order p^{s-r} killed by $U(p)^{s-r}$ and also $U^*(p)^{s-r}$.

As we have seen from (2-5), we have a morphism $i^s_{r/R} : J^r_{s,R} \to J_{s,R}[\gamma^{p^{r-1}}-1]$. This morphism composed with $J_{r,R} \to J^r_{s,R}$, induced by Picard functoriality from the covering map $X^r_{s,R} \to X_{r,R}$, gives rise to the morphism $I^s_{r/R} : J_{r,R}[p^\infty] \to J_{s,R}[p^\infty]$, which gives rise to a inductive system $\{\mathcal{G}_{r,R}, I_r^s \mid \mathcal{G}_{r,R} \to \mathcal{G}_{s,R}\}_{s>r}$ of ind-group schemes. Then $\mathcal{G}_R = \varinjlim_r \mathcal{G}_r$ is again a well-defined ind-group scheme. We want to study the control property as in (CT′) for $\mathcal{G}_R$ if either $R=K$ or $R=R_\infty$. Suppose $R=R_\infty=\mathbb{Z}_{(p)}[\mu_{p^\infty}]$. Through the diamond operators, the multiplicative group $\mathbb{Z}_p^\times$ acts on $\mathcal{G}_R$. For $a\in\mathbb{Z}/(p-1)\mathbb{Z}$, write $\mathcal{G}(a)_R$ for the eigenspace (that is, the maximal ind-Barsotti–Tate subgroup) on which $\zeta\in\mu_{p-1}(\mathbb{Z}_p)$ acts via the multiplication by ζ^a. In particular, $\mathcal{G}(0)$ is the μ-fixed part $\mathcal{G}^\mu$ for $\mu=\mu_{p-1}\subset\mathbb{Z}_p^\times$. Regarding $\mathcal{G}_R$ as an fppf p-abelian sheaf (meaning it has values in the category of p-abelian groups), we will show that the projector $x\mapsto 1/(p-1)\sum_{\zeta\in\mu_{p-1}\subset\mathbb{Z}_p^\times}\zeta^{-a}\langle\zeta\rangle(x)$ projects the sheaf $\mathcal{G}_R$ onto $\mathcal{G}(a)_R$. We put $\mathcal{G}_R^{(0)} = \bigoplus_{0<a<p-1}\mathcal{G}(a)_R$; thus $\mathcal{G}_R = \mathcal{G}_R^\mu \oplus \mathcal{G}_R^{(0)}$. Using the good reduction theorem of Langlands and Carayol combined with an analysis of the relation between $\mathcal{G}_R^{(0)}$ and the good abelian quotients from [Mazur and Wiles

1984, Section 3], we will show $\mathcal{G}^{(0)}_{R_\infty}$ is a Λ-BT group (see Sections 4 and 5 for the proof, and Remark 5.5 and Proposition 6.3 for the structure of the complement $\mathcal{G}^{\mu}$).

Roughly, X_{r,R_r} classifies degree p^r cyclic isogenies $\pi : E \to E'$ with some additional data (here "cyclicity" means the kernel of the isogeny is "cyclic" in the sense of Drinfeld as explained in [Katz and Mazur 1985, Chapter 6]). After a base change (tensoring $\mathbb{F}_p$ over R_r), π factors as

$$E \xrightarrow{F^a} E^{(p^a)} \cong E'^{(p^b)} \xrightarrow{V^b} E'$$

for the p-power relative Frobenius F and its dual V (the Verschiebung) for some nonnegative integers a, b with $a+b=r$. Thus $X_{r,\mathbb{F}_p} := X_{r,R_r} \otimes_{R_r} \mathbb{F}_p$ is a union $\bigcup_{a+b=r} X_{(a,b)}$ for $X_{(a,b)}$ classifying cyclic isogenies of type (a, b) as above (with additional data). We define $Y_r = X_{(0,r)} \cup X_{(r,0)}$ inside $X_{r,R_r} \otimes_{R_r} \mathbb{F}_p$ (good components in the sense of [Mazur and Wiles 1984, Section 3]). We will see in Corollary 6.1 that

(u) the projection $J_{r,R_r}[p^\infty] \otimes_{R_r} \mathbb{F}_p \to \mathrm{Pic}^0_{Y_r/\mathbb{F}_p}[p^\infty]$ is a $U(p)$-isomorphism,

where as a correspondence, $U(p) \cap Y_r := U(p) \times_{X_{r,R_r}} Y_r$ induces a correspondence on $Y_r \times Y_r$ and hence it acts on $\mathrm{Pic}^0_{Y_r/\mathbb{F}_p}$.

3. More structures on the modular Λ-BT groups

We list here some more good properties satisfied by the modular Λ-BT group $\mathcal{G}$ (to be constructed in the following two sections) whose proofs will be given in the later sections. We would like to know to what extent a general Λ-BT group $\mathcal{G}_R$ satisfies the following properties, though the only example we know is made out of modular Jacobians. In this section only, we denote by $R = R_\infty$ a general valuation ring with residue field $\mathbb{F}$ of mixed characteristic $(0, p)$ not necessarily in $\mathbb{Q}[\mu_{p^\infty}]$, and suppose $R_\infty = \bigcup_j R_j$ for an increasing sequence of discrete valuation subrings R_j (i.e., $R_j \subset R_{j+1} \subset R$ for all j). Again R_n is not necessarily in $\mathbb{Q}[\mu_{p^n}]$. We write K_n for the quotient field of R_n.

Recall the quotient field K of R, and fix an algebraic closure $\overline{K}$ of K. We have the geometric generic fiber $\mathcal{G}_r[p^n](\overline{K})$ of the (quasi)finite group scheme $\mathcal{G}_r[p^n]$ and put $\mathcal{G}(\overline{K}) = \varinjlim_r \varinjlim_n \mathcal{G}_r[p^n](\overline{K})$ (which we will call the geometric generic fiber of $\mathcal{G}$). We may regard $\mathcal{G}(\overline{K})$ as a discrete Λ-module. Similarly taking the special fiber $\mathcal{G}_r[p^n]_{/\mathbb{F}} := \mathcal{G}_r[p^n] \otimes_R \mathbb{F}$, we define $\mathcal{G}_{\mathbb{F}} = \varinjlim_r \varinjlim_n \mathcal{G}_r[p^n]_{/\mathbb{F}}$ as an ind-group scheme (we call $\mathcal{G}_{\mathbb{F}}$ the special fiber of $\mathcal{G}_R$). We will verify in the next section the following condition for the modular Λ-BT group:

(DV) $\mathcal{G}(\overline{K}) \cong \Lambda^{*n}$ (as Λ-modules) for $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$.

If R has a finite residue field $\mathbb{F} = \mathbb{F}_q$ of characteristic p, we further consider the following properties for $\mathcal{G}$:

(CT) Writing $\gamma = 1+x$, we have $\mathcal{G}_n = \mathrm{Ker}(\gamma^{p^{n-1}} - 1 : \mathcal{G}_R \to \mathcal{G}_R)$ (closed immersion) and $\mathcal{G}_{n,R}$ descends to a Barsotti–Tate group over the discrete valuation ring R_n (for each $0 < n \in \mathbb{Z}$). We have $\mathcal{G}_m \times_{R_m} R_n \cong \mathcal{G}_{m,R_n}$ if $n > m$ (compatibility).

(D) We have a Cartier self-duality $\mathcal{G}_n[p^m] \times \mathcal{G}_n[p^m] \to \mu_{p^m}$ over R_n which, after taking the limit, gives (Galois equivariant) Pontryagin duality $T\mathcal{G} \times \mathcal{G}(\bar{K}) \to \mu_{p^\infty}(\bar{K})$ for $T\mathcal{G} = \varprojlim_n T\mathcal{G}_n$ (for $T\mathcal{G}_n = \varprojlim_m \mathcal{G}_n[p^m](\bar{K})$) with respect to the map $T\mathcal{G}_{n+1} \twoheadrightarrow T\mathcal{G}_n$ dual to $\mathcal{G}_n \hookrightarrow \mathcal{G}_{n+1}$.

(Od) The connected component of $\mathcal{G}_r^\circ[p^n]$ for all $n > 0$ and $r > 0$ is a multiplicative locally free group over the strict henselization of R.

(U) On the special fiber, we have the Frobenius map F and its dual V with $FV = VF = q$. Thus we have a splitting $\mathcal{G}_{\mathbb{F}} = \mathcal{G}^\circ \times \mathcal{G}^{et}$ so that $\mathcal{G}^\circ = \mathrm{Ker}(e_F) = \mathrm{Im}(e_V)$ and $\mathcal{G}^{et} = \mathrm{Ker}(e_V) = \mathrm{Im}(e_F)$ for $e_F = \lim_{n\to\infty} F^{n!}$ and $e_V = \lim_{n\to\infty} V^{n!}$. Then we have a Λ-linear automorphism U of $\mathcal{G}$ such that $U|_{\mathcal{G}_r}$ is defined over R_r for all $r > 0$, U commutes with F and V, and U on $\mathcal{G}^{et}$ lifts $F|_{\mathcal{G}^{et}}$. Moreover, $e = \lim_{n\to\infty} U^{n!} = e_F|_{\mathcal{G}^{et}} + e_V|_{\mathcal{G}^\circ}$ on $\mathcal{G}_{\mathbb{F}}$.

A Λ-BT group satisfying the above properties will be called an ordinary Λ-BT group over R. We prove these properties for the modular Λ-BT groups in the following two sections for $R = \mathbb{Z}_{(p)}[\mu_{p^\infty}]$. We may replace the base ring R in the above conditions by a field of characteristic p (for example $\mathbb{F}_q$); so, the definition of Λ-BT group makes sense over a finite field and $\bar{\mathbb{F}}_p$ (see (Q1) below).

Pick a linear operator $L \in \mathrm{End}_{\Lambda[\mathrm{Gal}(\bar{K}/K)]}(T\mathcal{G})$ whose restriction to $T\mathcal{G}_r$ (for each $r > 0$) commutes with the action of $\mathrm{Gal}(\bar{K}/K_r)$ (the bigger Galois group). Since the Barsotti–Tate group $\mathcal{H}$ over a field k of characteristic 0 is an étale group and therefore is determined by its Galois module, we have $\mathrm{End}_{\mathbb{Z}_p[\mathrm{Gal}(\bar{k}/k)]}(T\mathcal{H}) \cong \mathrm{End}_{BT}(\mathcal{H})$. Then the restriction $L_r \in \mathrm{End}_{\Lambda[\mathrm{Gal}(\bar{K}/K_r)]}(T\mathcal{G}_r)$ gives rise to an endomorphism of $\mathcal{G}_{r,K_r}$ and extends uniquely to an endomorphism of the Barsotti–Tate group $\mathcal{G}_{r,R_r}$ defined over R_r [Tate 1967, Theorem 4]. We write this restriction as $L_r^{BT} \in \mathrm{End}_{BT}(\mathcal{G}_{r,R_r})$. Then we have $L^{BT} = \varprojlim_r L_r^{BT} \in \mathrm{End}_\Lambda(\mathcal{G}_R)$. If confusion is unlikely, we simply write L for L^{BT}.

Suppose that $\det(L) \neq 0$ in Λ as an endomorphism of $T\mathcal{G} \cong \Lambda^n$. Define

$$\mathcal{G}[L]_R = \mathrm{Ker}(L : \mathcal{G}_R \to \mathcal{G}_R),$$

which is a well-defined fppf abelian sheaf over R (as the category of fppf abelian sheaves is abelian; see [Milne 1980, Chapter 2]). We regard $\mathcal{G}[L](\bar{K})$ as an abelian group. Since $\det(L) \neq 0$, by the classification of Λ-modules, the maximal p-divisible subgroup $\mathcal{G}[L](\bar{K})^{\mathrm{div}}$ of $\mathcal{G}[L](\bar{K})$ has finite corank; that is, $\mathcal{G}[L](\bar{K})^{\mathrm{div}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^m$ as an abstract group for a finite $m > 0$, which is called the corank of $\mathcal{G}[L](\bar{K})$. We call $\mathcal{G}[L](\bar{K})^{\mathrm{div}}$ the p-divisible part of $\mathcal{G}[L](\bar{K})$, which is canonically determined inside $\mathcal{G}[L](\overline{K})$. Thus its p^n-torsion subgroup $\mathcal{G}[L](\overline{K})^{\mathrm{div}}[p^n]$ is finite, and we can find a finite $r = r(n) > 0$ such that $\mathcal{G}[L](\overline{K})^{\mathrm{div}}[p^n] \subset \mathcal{G}_r[p^n](\overline{K})$.

Since L_r^{BT} commutes with the action of $\mathrm{Gal}(\overline{K}/K_r)$ and the p-divisible part is unique in $\mathcal{G}[L](\overline{K})$, it follows that $\mathcal{G}[L](\overline{K})^{\mathrm{div}}[p^n]$ is stable under the action of $\mathrm{Gal}(\overline{K}/K_r)$. Thus $\mathcal{G}[L](\overline{K})^{\mathrm{div}}[p^n]$ is the group of geometric points of a finite flat subgroup $\mathcal{G}[L]^{\mathrm{div}}[p^n]_{/K_r}$ of $\mathcal{G}_{r,K_r}[p^n]$ for sufficiently large r. Take the schematic closure G_{n/R_r} of $\mathcal{G}[L]^{\mathrm{div}}[p^n]_{/K_r}$ in $\mathcal{G}_{r,R_r}[p^n]$ defined over R_r. Writing $\mathcal{G}_r[p^n] = \mathrm{Spec}(A)$ for a Hopf R_r-algebra $A = A_n$ with $\mathcal{G}[L]^{\mathrm{div}}[p^n]_{/K_r} = \mathrm{Spec}(A \otimes_{R_r} K_r/I)$ for an ideal I of $A \otimes_{R_r} K_r$, we have $G_{n/R_r} = \mathrm{Spec}(A/A \cap I)$. Note that $A/(A \cap I) \cong (A+I)/I \hookrightarrow A \otimes_{R_r} K_r/I$ is p-torsion free (and hence flat over R_r). Thus G_{n/R_r} is a finite flat subgroup scheme of $\mathcal{G}_{r,R_r}[p^n]$.

Since $\mathcal{G}_{r,R_s} = \mathcal{G}_{r,R_r} \otimes_{R_r} R_s \hookrightarrow \mathcal{G}_{s,R_s}$ $(s > r)$ is a closed immersion, the schematic closure of $\mathcal{G}[L]^{\mathrm{div}}[p^n]_{/K_s} = \mathcal{G}[L]^{\mathrm{div}}[p^n]_{/K_r} \otimes_{K_r} K_s$ coincides with $G_{n/R_r} \otimes_{R_r} R_s$; thus the formation of the base change $G_{n/R} := G_{n/R_r} \otimes_{R_r} R$ is independent of the chosen r. Put $\mathcal{G}[L]^{BT}_{/R} = \varinjlim_n G_{n/R}$, which should be a Barsotti–Tate group over R with the identity $\mathcal{G}[L]^{BT}[p^n]_{/R} = G_{n/R}$. To discuss our naive questions, we just take $\mathcal{G}[L]^{BT}_{/R}$ to be a Barsotti–Tate group as a working hypothesis.

Of course, starting with a self-dual Λ-BT group H with a lift U, $TH \otimes_{\mathbb{Z}_p} \Lambda^*$ gives a constant Λ-BT group. Here we put $TH = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{T}_p, H(\overline{K}))$ (the Tate module of H). We hereafter suppose that all Λ-BT groups we consider are nonconstant. Thus it could be said that the representation of $\mathrm{Gal}(\overline{K}/K)$ on $T\mathcal{G}$ is a nonconstant deformation of $T\mathcal{G}_1$ in the sense of Mazur (see [Mazur 1989] and [Hida 2000a]).

A p-ordinary Barsotti–Tate group H over R is called of $\mathrm{GL}(2g)$*-type* if it is self-dual and there exists a local ring $E \subset \mathrm{End}_{\mathbb{Z}_p}(H_{/R})$ such that we have an isomorphism of E-modules for its Tate module: $TH \cong E^{2g}$. Here we say that H is *p-ordinary* if H satisfies (Od). We call H *minimal* if E is generated by $\mathrm{Tr}(\sigma) \in E$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$, where $\mathrm{Tr}(\sigma) \in E$ is the trace of the action of σ on $TH \cong E^{2g}$. If we have a local $\Lambda[U]$-algebra $\mathbb{T}$ inside $\mathrm{End}_{\Lambda[U]}(\mathcal{G})$ for an ordinary Λ-BT group $\mathcal{G}_R$ such that $T\mathcal{G} \cong \mathbb{T}^{2g}$ and $\mathbb{T}$ is self-adjoint under the duality, we call $\mathcal{G}$ *of* $\mathrm{GL}(2g)$*-type over* $\mathbb{T}$. In this Λ-adic case, we call $\mathcal{G}$ *minimal* if $\mathbb{T}$ is topologically generated by $\mathrm{Tr}(\sigma)$ and U. Suppose that there exists a nonconstant Λ-BT group $\mathcal{G}$ over a valuation ring R inside $\overline{\mathbb{Q}}$. Then we can ask a lot of simple questions:

(Q0) Does there exist R discretely valued?

(Q1) If we are given an ordinary Λ-BT group $\mathcal{G}$ over a finite field $\mathbb{F}$ of characteristic p, can one lift it to a Λ-BT group over R for a suitable R? (Deformation question to characteristic 0.)

(Q2) Is there any systematic way of constructing such an ordinary Λ-BT group $\mathcal{G}$ over a given R? If it exists, does it create all such Λ-BT groups over R of $\mathrm{GL}(2g)$-type? (Construction.)

(Q3) If an ordinary Λ-BT group $\mathcal{G}$ is nonconstant, can $\det(U) \in \mathbb{T}^\times$ be algebraic over W? (Transcendency.)

(Q4) Let us give ourselves a Weil number $\alpha \in \overline{\mathbb{Q}} \cap W$ with $|\alpha| = \sqrt{p}$ of degree $2g$ over $\mathbb{Q}$. Supposing α *ordinary* (in the sense that the minimal polynomial in X of α modulo p is divisible by X^g but not by higher powers), is there a Barsotti–Tate subgroup $\mathcal{G}[U-\alpha]^{BT}_{/R} \subset \mathcal{G}_R$ whose geometric generic fiber is given by $\mathcal{G}[U-\alpha](\overline{K})^{\mathrm{div}}$? Suppose this is the case. Does $\mathcal{G}[U-\alpha]^{BT}_{/R}$ descend to a discrete valuation ring? (Descent.) Here, $\mathcal{G}[U-\alpha]^{BT}$ should be the maximal Barsotti–Tate subgroup in $\mathcal{G}[U-\alpha]_{/R} = \mathrm{Ker}(U-\alpha)$.

(Q5) Under the notation in (Q4), is it possible to embed the Barsotti–Tate part $\mathcal{G}[U-\alpha]^{BT}_{/R}$ of $\mathcal{G}[U-\alpha] = \mathrm{Ker}(U-\alpha)$ into an abelian scheme defined over a finite extension of R? (Relation to abelian varieties.)

(Q6) For a given minimal $\mathcal{G}_{1,R}$ of $\mathrm{GL}(2g)$-type whose Tate module $T\mathcal{G}_1(\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$ is simple as a Galois module, is there a universal $\mathcal{G}$? (Universality.) Here the universality is defined as follows. If we have a minimal Barsotti–Tate group H of $\mathrm{GL}(2g)$-type with a morphism $i : \mathcal{G}_1 \to H$ having kernel represented by a finite group scheme (so, $i \circ \mathrm{Tr}(\sigma|_{\mathcal{G}_1}) = \mathrm{Tr}(\sigma|_H) \circ i$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$), there exists a unique morphism $i_H : H_{/R} \hookrightarrow \mathcal{G}_{/R}$ of Barsotti–Tate groups with finite (group scheme) kernel making the following diagram commute:

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "i"] \arrow[d] & H \arrow[d, "i_H"] \\ \mathcal{G} \arrow[r, equal] & \mathcal{G} \end{tikzcd}$$

Question (Q0) probably has a negative answer. Here is one reason why: Suppose that $\mathcal{G}$ is minimal of $\mathrm{GL}(2)$-type and suppose that $\mathcal{G}$ extends as a Λ-BT group to the integral closure of $\mathbb{Z}[1/N]$ in R. If $\mathcal{G}$ is defined over a discrete valuation ring $R = \mathbb{Z}_p$ or $\mathbb{Z}_{(p)}$, then, by the classification of p-ordinary divisible groups [Raynaud 1974, 4.2], the determinant of the Galois representation on $T\mathcal{G}$ has to be the p-adic cyclotomic character χ. Thus $T\mathcal{G}$ is a deformation of $T\mathcal{G}_1$ which is p-ordinary and of determinant χ. If $T\mathcal{G}_1$ is modular whose residual representation is irreducible over $\mathbb{Q}[\sqrt{p^*}]$ ($p^* = (-1)^{(p-1)/2}p$), by Wiles [1995], the universal Galois deformation ring for p-ordinary deformations unramified outside Np with fixed determinant χ is of finite rank over $\mathbb{Z}_p$. Thus $T\mathcal{G}$ has to be constant; therefore, $\mathcal{G}$ has to be constant. Thus if such a $\mathcal{G}$ exists, at least R contains the p-adic valuation ring of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$.

Suppose $g = 1$. Questions related to the ones given above have been studied in [Hida 1986; Mazur and Wiles 1986; Tilouine 1987; Ohta 1995] for this case.

In this paper, I will give an automorphic way of constructing such $\mathcal{G}$ over $R_\infty = \mathbb{Z}_{(p)}[\mu_{p^\infty}]$. By the solution of Galois deformation problems of ordinary type (Mazur, Wiles–Taylor) and by the solution of Serre's modulo p modularity conjecture (Khare–Wintenberger, Kisin), this gives almost all such Λ-BT groups of GL(2)-type, basically solving (Q2) and (Q6) for GL(2)-type groups. After giving the construction of this modular example in terms of the ordinary projector e (in place of $U(p)$-isomorphisms), we will make some comments on the other questions listed above for the modular Λ-BT group.

4. Construction over $\mathbb{Q}$ via the ordinary projector

Fix a prime $p \geq 5$ and a positive integer N prime to p. Here, we give a down-to-earth construction of the modular Λ-BT group $\mathcal{G}_{\mathbb{Q}}$ over $\mathbb{Q}$ via the ordinary projector e, though we follow the line explained in Section 2. Here we mean by a Λ-BT group over $\mathbb{Q}$ an ind-étale group defined over $\mathbb{Q}$ satisfying conditions (CT) and (DV) from the previous section (as modified by replacing the valuation ring R and R_n by the field $\mathbb{Q}$). Since the category of Barsotti–Tate groups over $\mathbb{Q}$ is equivalent to the category of p-divisible modules of finite corank with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (as any finite flat group scheme over $\mathbb{Q}$ is étale), we are just dealing with such Galois modules. We also prove the corresponding properties (CT), (DV) and (D) over $\mathbb{Q}$ for this $\mathcal{G}$. Note that the conditions (DV) and (CT) concern only the Λ-module structure of the group not the Galois action (under the condition $R = \mathbb{Q}$).

As before, let $J_r = \mathrm{Pic}^0_{X_1(Np^r)/\mathbb{Q}}$ be the Jacobian variety. Similarly we take J^r_s to be the Jacobian variety associated to the modular curve with the congruence subgroup $\Gamma^r_s = \Gamma_1(Np^r) \cap \Gamma_0(p^s)$ $(0 \leq r \leq s)$. From (2-2) with $R = \mathbb{Q}$, since $e = \lim_{n\to\infty} U(p)^{n!}$ acts on $J[p^\infty]$ for $J = J_r,\ J_s,\ J^r_s$ (noting $U(p^m) = U(p)^m$), we have

$$\mathcal{G}_{r,\mathbb{Q}} := J_r[p^\infty]^{\mathrm{ord}} \cong J^r_s[p^\infty]^{\mathrm{ord}},$$

where "ord" indicates the image of e.

For the Jacobian J of $X = X(\Gamma)$ with $\Gamma = \Gamma^r_s$, we identify $J[p^\infty](\mathbb{C})$ with a subgroup of $H^1(\Gamma, \mathbb{T}_p)$ (here $\mathbb{T}_p := \mathbb{Q}_p/\mathbb{Z}_p$, on which Γ acts trivially). Applying the ordinary projector $e = \lim_{n\to\infty} U(p)^{n!}$ to the diagram (2-5) (replacing $\boldsymbol{T}$ there by $\mathbb{T}_p$), we get from (2-6) the controllability

$$\begin{aligned}\mathcal{G}_{s,\mathbb{Q}}[\gamma^{p^{r-1}} - 1] &= \mathrm{Ker}(\gamma^{p^{r-1}} - 1 : J_s[p^\infty]^{\mathrm{ord}} \to J_s[p^\infty]^{\mathrm{ord}}) \\ &= J_r[p^\infty]^{\mathrm{ord}} = \mathcal{G}_{r,\mathbb{Q}}.\end{aligned}$$

Define, as an ind-group scheme over $\mathbb{Q}$ (or, as a p-abelian fppf sheaf),

$$\mathcal{G}_{\mathbb{Q}} := J_\infty[p^\infty]^{\mathrm{ord}} = \varinjlim_r J_r[p^\infty]^{\mathrm{ord}}.$$

For each character $\varepsilon : \Gamma/\Gamma^{p^{r-1}} \to \mu_{p^\infty}$, by the inflation and restriction sequence, we get

$$\mathcal{G}_{\mathbb{Q}}[p^n](\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon][\gamma - \varepsilon(\gamma)] \cong J_r[p^n](\overline{\mathbb{Q}})^{\mathrm{ord}} \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon][\gamma - \varepsilon(\gamma)]$$
$$\cong H^1(X_r^1, \mathbb{T}_p(\varepsilon))^{\mathrm{ord}},$$

where $\mathbb{T}_p(\varepsilon)$ is a Γ_r^1-module isomorphic to $\mathbb{T}_p$ on which Γ_r^1 acts by ε. Thus the group $\mathcal{G}_{\mathbb{Q}}(\overline{\mathbb{Q}}) \otimes \mathbb{Z}[\varepsilon][\gamma - \varepsilon(\gamma)]$ is a nontrivial p-divisible group. Taking the Pontryagin dual $T := \mathcal{G}_{\mathbb{Q}}^*$, by Nakayama's lemma applied to $T/\mathfrak{m}T \cong J_1[p]^{\mathrm{ord}}$ (for the maximal ideal $\mathfrak{m}$ of Λ), we find a surjection $\pi : \Lambda^{2j} \twoheadrightarrow T$ for $2j = \dim_{\mathbb{F}_p} J_1[p]^{\mathrm{ord}}$. Then for a prime $P = P_\varepsilon := (\gamma - \varepsilon(\gamma)) \cap \Lambda$, T/PT is the dual of $\mathcal{G}_{\mathbb{Q}}[P] \otimes \mathbb{Z}[\varepsilon][\gamma - \varepsilon(\gamma)]$, which is $\mathbb{Z}_p$-free of rank $2j$. Thus $\mathrm{Ker}(\pi) \subset P_\varepsilon \Lambda^{2j}$. Moving ε around, from $\bigcap_\varepsilon P_\varepsilon \Lambda^{2j} = \{0\}$, we find that $T \cong \Lambda^{2j}$; therefore, $\mathcal{G}_{\mathbb{Q}}$ is a Λ-BT group satisfying (CT) and (DV) (over $\mathbb{C}$ and hence over $\mathbb{Q}$).

As for (D), the canonical polarization of $J_{r/\mathbb{Q}}$ gives rise to the self-duality pairing $[\cdot,\cdot]$ of $J_r[p^r]$ and $J_r \cong {}^tJ_r$. Let $U^*(p)$ (resp. $T^*(n)$) be the image of $U(p)$ (resp. $T(n)$) under the canonical Rosati involution of J_r in $\mathrm{End}(J_r)$. The Weil involution τ over $\mathbb{Q}(\mu_{Np^r})$ associated to $\left(\begin{smallmatrix} 0 & -1 \\ Np^r & 0 \end{smallmatrix}\right)$ satisfies $\tau U(p) \tau^{-1} = U^*(p)$ and $\tau T(n) \tau^{-1} = T^*(n)$ inside $\mathrm{End}(J_{r/\mathbb{Q}[\mu_{Np^r}]})$ because τ is only defined over $\mathbb{Q}[\mu_{Np^r}]$. See [Hida 1986, Theorem 9.3] for more details. Thus, twisting the pairing by τ and $U(p)^{-r}$, we get the self-duality pairing $\langle \cdot,\cdot \rangle_r = [\cdot, \tau \circ U(p)^{-r}(\cdot)]$ of $\mathcal{G}_r[p^m]$. Writing $R_s^r : \mathcal{G}_r \hookrightarrow \mathcal{G}_s$ for the inclusion, and $N_r^s = \sum_{j=1}^{p^{s-r}} \gamma_r^j : \mathcal{G}_s \to \mathcal{G}_r$ with $\gamma_r = \gamma^{p^{r-1}}$, we can verify by computation that $\langle R_s^r(x), y \rangle_s = \langle x, N_r^s(y) \rangle_r$ (see [Ohta 1995, §4.1], for instance). From this we get (D) over K_∞.

5. Construction over $\mathbb{Z}_{(p)}[\mu_{p^\infty}]$

Hereafter, for simplicity, we assume that N is cube-free, and we make the construction of $\mathcal{G}$ over $R_\infty := \mathbb{Z}_{(p)}[\mu_{p^\infty}]$. Under this assumption, the ordinary Hecke algebra $h_2^{\mathrm{ord}}(\Gamma_r^s; \mathbb{Z}_p) \subset \mathrm{End}_{\mathbb{Z}_p}(J_s^r[p^\infty]^{\mathrm{ord}})$ generated by Hecke operators $T(n)$ and $U(q)$ is known to be reduced (it has no nontrivial nilradical; see [Hida 2013, Corollary 1.2]). From this fact, $S_2(\Gamma_s^r)$ has a basis of Hecke eigenforms for all Hecke operators $T(n)$ and $U(q)$. If N is not cube-free, we need to consider old-forms $f(dz) \in S_2(\Gamma_1(Np^r))$ for Hecke eigenforms f and for suitable $d \mid N$ (and abelian varieties associated to such forms), which complicates the arguments, though all arguments we give actually go through if we consider cusp forms which are eigenforms for $T(n)$ with n prime to N. Hereafter, if we say f is a Hecke eigenform, we mean that f is an eigenvector of $T(n)$ for all n prime to Np and $U(q)$ for all primes $q \mid Np$. The Hecke eigenforms we consider may not be new-forms of exact level Np^r.

The Tate module $T_r = TJ_r[p^\infty]^{\mathrm{ord}}(\overline{\mathbb{Q}})$ carries Galois representations (constructed by Eichler and Shimura) of Hecke eigenforms satisfying the following properties (see [Hida 2012, §4.2]):

(1) Cusp forms in $S_2(\Gamma_1^0)$ (with $\Gamma_1^0 = \Gamma_1(N) \cap \Gamma_0(p)$).

(2) All cusp forms in $S_2(\Gamma_1(Np^m))$ whose Neben character has p-conductor equal to p^m for $m = 1, 2, \dots, r$.

By a theorem of Langlands and Carayol (see [Carayol 1986]), the ℓ-adic Galois representation ($\ell \neq p$) associated to such a Hecke eigenform f does not ramify at p over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^r}])$ except for in case (1). In case (1), it is semistable at p. Thus the abelian subvariety A_f attached to f extends to a semiabelian scheme over $\mathbb{Z}_{(p)}[\mu_{p^r}]$ (see [Serre and Tate 1968, §1] and [Bosch et al. 1990, §7.4]). Let ${}^tG_r = \sum_{f \text{ as above}} A_f \subset J_r$. Thus we have an inclusion ${}^tG_r \hookrightarrow J_r$ defined over $\mathbb{Q}$. Let $J_r \cong {}^tJ_r \twoheadrightarrow G_r$ be the dual quotient. Thus, by definition, we have a commutative diagram defined over $\mathbb{Q}$ for all n prime to Np and all primes $q \mid Np$:

$$\begin{tikzcd} J_r \arrow[r] \arrow[d, "{T^*(n),U^*(q)}"'] & G_r \arrow[d, "{T^*(n),U^*(q)}"] \\ J_r \arrow[r] & G_r \end{tikzcd}$$

where the superscript "$*$" indicates the Rosati involution induced from the polarization of G_r (coming from the canonical polarization of J_r). Since the space spanned by the Hecke eigenforms described in (1) and (2) above is stable under $T(n)$, $T^*(n)$ and $U(q)$, $U^*(q)$, actually we also have the following commutative diagram defined over $\mathbb{Q}$:

$$\begin{tikzcd} J_r \arrow[r] \arrow[d, "{T(n),U(q)}"'] & G_r \arrow[d, "{T(n),U(q)}"] \\ J_r \arrow[r] & G_r \end{tikzcd}$$

We can also justify this by noting that the Rosati involution induced by the polarization on G_r from the canonical polarization on J_r sends $T^*(n)|_{G_r}$ to $T(n)|_{G_r}$ and $U^*(q)|_{G_r}$ to $U(q)|_{G_r}$ for all n prime to Np and all primes $q \mid Np$.

A Hecke eigenform $f \in S_2(\Gamma_1(Np^m))$, new at p, satisfies (1) or (2) above if and only if $f|U(p) = f|U^*(p) \neq 0$ (see [Miyake 1989, Theorem 4.6.17]). Thus $U(p) \in \mathrm{End}({}^tG_{r/\mathbb{Q}})$ is an isogeny. Recall the quotient field K of R. By [Hida 2013, Proposition 1.1, Corollary 1.2],

(5-1) a sufficiently large power $U(p)^M$ projects $J_{r,K}$ onto ${}^tG_{r,K}$.

In other words, we have the following commutative diagram for general R:

$$\begin{tikzcd} {}^tG_{r,R} \arrow[r, "i"] \arrow[d] & J_{r,R} \arrow[d] \arrow[dl] \\ {}^tG_{r,R} \arrow[r, "i"'] & J_{r,R} \end{tikzcd} \tag{5-2}$$

where the vertical and diagonal arrows are given by $U(p)^M$, and ${}^tG_{r,R} \xrightarrow{i} J_{r,R}$ is the Néron extension of the inclusion $i : {}^tG_{r,K} \hookrightarrow J_{r,K}$ (note that the extended i might not be an immersion). We also have the dual $i^* : J_{r,R} \to G_{r,R}$ which is the Néron extension of the projection $i^* : J_{r,K} \to G_{r,K}$.

For any abelian subvariety A of J_r stable under $U(q)$ for all q dividing Np and under $T(n)$ for all n prime to Np, if there exists an abelian subvariety B stable under the same Hecke operators such that $A + B = J_r$ and $A \cap B$ is finite, the abelian subvariety B is uniquely determined by A (the multiplicity-one theorem; see [Gelbart 1975] and [Miyake 1989, §4.6]). The abelian subvariety B is called the *complement* of A in J_r.

By construction, G_r and ${}^tG_r$ extend to semiabelian schemes over $R_r := \mathbb{Z}_{(p)}[\mu_{p^r}]$. The group $\mu = \mu_{p-1} \subset \mathbb{Z}_p^\times$ acts on J_r, ${}^tG_r$ and G_r by the diamond operators. If we define ${}^tG_r^{(0)}$ in ${}^tG_r$ to be the complement of abelian subvariety fixed by μ, then ${}^tG_r^{(0)}$ and its dual quotient $G_r^{(0)}$ extend to abelian schemes over R_r. Anyway, we take the Néron models G_{r,R_r}, ${}^tG_{r,R_r}$, $G^{(0)}_{r,R_r}$ and ${}^tG^{(0)}_{r,R_r}$ over R_r of the abelian varieties G_{r,K_r}, ${}^tG_{r,K_r}$, $G^{(0)}_{r,K_r}$ and ${}^tG^{(0)}_{r,K_r}$, and we take their p-divisible groups. Here the p-divisible group $G^{(0)}_{r,R_r}$ is a Barsotti–Tate group over R_r. The μ-fixed parts $G_{r,R_r}[p^n]^\mu$ and ${}^tG_{r,R_r}[p^n]^\mu$ are at worst quasifinite flat groups schemes.

Theorem 5.1. *Recall $\mathcal{G}_{r,R_r} = J_{r,R_r}[p^\infty]^{\mathrm{ord}}$. We have the two isomorphisms*

$${}^tG_{r,R_r}[p^\infty]^{\mathrm{ord}} \xrightarrow[\sim]{i} \mathcal{G}_{r,R_r} \xrightarrow[\sim]{i*} G_{r,R_r}[p^\infty]^{\mathrm{ord}}$$

canonically over $R_r := \mathbb{Z}_{(p)}[\mu_{p^r}]$.

This theorem might appear tautological. However note that a priori $\mathcal{G}_{r,R_r}[p^n]$ is not known even to be a flat group scheme, but we know that ${}^tG_r[p^\infty]^{(0),\mathrm{ord}}$ and $G_r[p^\infty]^{(0),\mathrm{ord}}$ are Barsotti–Tate groups, where the superscript "(0)" indicates the complement of the μ-fixed part. Thus to show that $\mathcal{G}^{(0)}_{r,R_r} = \bigoplus_{0<a<p-1} \mathcal{G}_{r,R_r}(a)$ is a Barsotti–Tate group, it appears that we need to make a difficult analysis of the inclusion $\mathcal{G}_{r,R_r} \subset J_{r,R_r}$ (in the category of fppf abelian sheaves over R_r) to claim that $\mathcal{G}^{(0)}[p^n]$ is represented by a finite-flat group scheme as in the theorem, since $\mathcal{G}^{(0)}_{r,R_r}[p^n]$ is a priori not even known to be represented by a flat group over R_r. However, suppose that we find two ($U(p)$-equivariant) morphisms of group schemes

$${}^tG_{r,R_r}[p^n]^{\mathrm{ord}} \xrightarrow{\mathcal{L}} \mathcal{G}_{r,R_r}[p^n] \quad \text{and} \quad \mathcal{G}_{r,R_r}[p^n] \xrightarrow{\mathcal{R}} G_{r,R_r}[p^n]^{\mathrm{ord}}$$

(for each n) such that the composite $\mathcal{R} \circ \mathcal{L}$ is an isomorphism, which implies that $\mathcal{R}$ (resp. $\mathcal{L}$) is an epimorphism (resp. a monomorphism) of fppf abelian sheaves. Note that the category of fppf abelian sheaves over R_r is an abelian category. By (5-1) and (5-2), $\mathrm{Ker}(\mathcal{R})$ is projected down into $\mathrm{Im}(\mathcal{L})$ by $U(p)^M$. Since $U(p)^M$ are automorphisms of the three fppf abelian sheaves, $\mathcal{L}$ and $\mathcal{R}$ must be isomorphisms, showing $\mathcal{G}_r^{(0)}[p^n]$ is a finite flat group over R_r for all n (and hence $\mathcal{G}_r^{(0)}$ is a Barsotti–Tate group over R_r). This point is the nontriviality of the theorem. A geometric analysis in depth of $\mathcal{G}_{r,R_r}$ has been done in [Cais 2012, §5.4], and $\mathcal{G}^{(0)} = \bigoplus_{0<a<p-1} \mathcal{G}(a)$ is shown directly by Cais to be a Λ-BT group, but our short-cut might be worth recording.

To prove the theorem, without making a difficult analysis of the group scheme $\mathcal{G}_{r,R_r}[p^n]$, the following lemma is quite useful:

Lemma 5.2. *Let R be a discrete valuation ring with fraction field K. Let G_K and G'_K be either both Barsotti–Tate groups or both abelian varieties over K. If G_K and G'_K are abelian varieties, let G_R and G'_R be the identity connected component of the Néron models over R of G_K and G'_K. If G_K and G'_K are Barsotti–Tate groups, we assume we have Barsotti–Tate groups G_R and G'_R over R whose generic fibers are isomorphic to G_K and G'_K, respectively.*

1. *Suppose that we have a surjective morphism $f_K : G_K \to G'_K$ and an endomorphism $g_K : G_K \to G_K$ such that the map* $\mathrm{Ker}(f_K : G_K \to G'_K) \hookrightarrow G_K$ *factors through* $\mathrm{Ker}(g_K : G_K \to G_K) \hookrightarrow G_K$. *Then for the extensions $f : G_R \to G'_R$ and $g : G_R \to G_R$ over R,* $\mathrm{Ker}(f)$ *is a closed subscheme of* $\mathrm{Ker}(g)$ *in the abelian case and is a closed ind-subgroup scheme in the Barsotti–Tate case.*
2. *Suppose we have an injective morphism $f_K : G'_K \to G_K$ and an endomorphism $g_K : G_K \to G_K$ such that* $\mathrm{Coker}(f_K : G'_K \to G_K)$ *is the surjective image of* $\mathrm{Coker}(g_K : G_K \to G_K)$. *Then, for the extensions $f : G'_R \to G_R$ and $g : G_R \to G_R$ over R,* $\mathrm{Coker}(f)$ *is a quotient group of* $\mathrm{Coker}(g)$.

Here, strictly speaking, a "surjective" morphism between fppf abelian sheaves means an epimorphism in the abelian category of fppf abelian sheaves over K.

Proof. We first prove assertion (1). We note that the category of groups schemes fppf over a base S is a full subcategory of the category of abelian fppf sheaves. Thus we may regard G_K and G'_K as abelian fppf sheaves over K in this proof. Since the category of fppf abelian sheaves is an abelian category (because of the existence of the sheafification functor from presheaves to sheaves under fppf topology described in [Milne 1980, §II.2]), the assumption that the map $\mathrm{Ker}(f_K : G_K \to G'_K) \hookrightarrow G_K$ factors through $\mathrm{Ker}(g_K : G_K \to G_K) \hookrightarrow G_K$ (that is, $\mathrm{Ker}(f_K) \subset \mathrm{Ker}(g_K)$) implies that there exists a morphism $f'_K : G'_K \to G_K$ of fppf abelian sheaves over K such that $f'_K \circ f_K = g_K$. If we have unique extensions $f : G_R \to G'_R$, $g : G_R \to G_R$,

$f' : G'_R \to G_R$ of these morphisms, we have $f' \circ f = g$ by uniqueness. This implies $\mathrm{Ker}(f)$ is a closed subscheme of $\mathrm{Ker}(g)$ in the abelian case and is a closed ind-group scheme in the Barsotti–Tate case.

If G_K and G'_K are abelian schemes, since G_R and G'_R are the connected components of the Néron models of G_K and G'_K, any generic morphism ϕ_K of these schemes extends to a unique morphism over R (see [Bosch et al. 1990, Proposition 7.4.3]).

If G_R and G'_R are Barsotti–Tate groups, the extensions f and f' exist and are unique by [Tate 1967, Theorem 4]. This finishes the proof of (1).

The second assertion is heuristically the dual of the first, with respect to taking dual abelian schemes or Cartier dual Barsotti–Tate groups of G_R and G'_R. We give a direct proof supplied by the referee as the duality of cokernels with kernels may not be valid in this generality. By hypothesis, G'_K is an abelian subvariety (or p-divisible subgroup) of G_K and $g_K : G_K \to G_K$ factors through G'_K. By the Néron functoriality (or Tate's theorem) $g : G_R \to G_R$ factors through $f : G'_R \to G_R$, so $g(G_R) \subset f(G'_R)$ as fppf abelian subsheaves of G_R. This latter inclusion is precisely the meaning of the conclusion of (2). $\square$

Proof of Theorem 5.1. Over $\mathbb{Q}$ we have ${}^tG_{r,\mathbb{Q}}[p^\infty]^{\mathrm{ord}} \subset \mathscr{G}_{r,\mathbb{Q}}$, from the definition. Let $B_{\mathbb{Q}}$ be the identity connected component of $\mathrm{Ker}(J_{r,\mathbb{Q}} \to G_{r,\mathbb{Q}})$, which is the complement of ${}^tG_{r,\mathbb{Q}}$. By definition, e kills $B_{\mathbb{Q}}[p^n]$ for all n; thus, it kills the p-primary part $H[p^\infty]$ of the finite group scheme $H = B_{\mathbb{Q}} \cap {}^tG_{r,\mathbb{Q}}$ over $\mathbb{Q}$. Thus, over $\mathbb{Q}$, we have the identity in the theorem. Since H is finite, H is killed by $M \cdot U(p)^{M'}$ for an integer M prime to p and another integer M' sufficiently large. We apply the first statement of the lemma to the projection $f_K : {}^tG_{r,K_r} \to G_{r,K_r}$ and $g_K = M \cdot U(p)^{M'}$ for $R = R_r$ and $K = K_r$. Thus, by the lemma, we have $\mathrm{Ker}(f) \subset \mathrm{Ker}(M \cdot U(p)^{M'})$; thus, we get a monomorphism ${}^tG_{r,R_r}[p^\infty]^{\mathrm{ord}} \xrightarrow{\iota} G_{r,R_r}[p^\infty]^{\mathrm{ord}}$ of p-abelian fppf sheaves over $R_r = \mathbb{Z}_{(p)}[\mu_{p^r}]$. Thus,

$$ {}^tG_{r,R_r}[p^\infty]^{(0),\mathrm{ord}} \xrightarrow{\iota^{(0)}} G_{r,R_r}[p^\infty]^{(0),\mathrm{ord}} $$

is a monomorphism of Barsotti–Tate groups of equal corank (here, the corank is the $\mathbb{Z}_p$-corank of the geometric generic fiber). Thus, generically $\iota^{(0)}$ is an isomorphism, which is enough to conclude that $\iota^{(0)}$ is an isomorphism by a result of Tate (see [Tate 1967, Corollary 2 on p. 181]). Since $\iota^{(0)} = \mathscr{R} \circ \mathscr{L}$ for the Néron extension $\mathscr{L} : {}^tG_{r,R_r}[p^\infty]^{(0),\mathrm{ord}} \to \mathscr{G}^{(0)}_{r,R_r}$ of the inclusion ${}^tG_{r,K_r} \to J_{r,K_r}$ and the Néron extension $\mathscr{R} : \mathscr{G}^{(0)}_{r,R_r} \to G_{r,R_r}[p^\infty]^{(0),\mathrm{ord}}$ of the projection $J_{r,K_r} \to G_{r,K_r}$, we conclude that $\mathscr{L}$ is a monomorphism of fppf p-abelian sheaves. By (5-1) and (5-2) combined with the injectivity of $\mathscr{L}$, a high power $U(p)^M$ projects $\mathscr{G}_{r,R_r}[p^n]$ into ${}^tG_{r,R_r}[p^n]^{\mathrm{ord}}$ for all $n > 0$, where we regard $U(p)^M$ as the Néron extension of a projection $U(p)^M : J_{r,K_r} \to {}^tG_{r,K_r}$; compare (5-1). Thus $\mathscr{L}$ is an epimorphism of abelian fppf sheaves, and so also an isomorphism of the Barsotti–Tate group ${}^tG_{r,R_r}[p^\infty]^{(0),\mathrm{ord}}$ onto the fppf abelian sheaf $\mathcal{G}^{(0)}_{r,R_r}$. In other words, the sheaf $\mathcal{G}^{(0)}_{r,R_r}$ is represented by a Barsotti–Tate group ${}^tG_{r,R_r}[p^\infty]^{(0),\mathrm{ord}}$. Now $J_{r,R_r}[p^n]^{(0),\mathrm{ord}}$ is proven to be a finite flat group scheme (without any analysis of the complicated group scheme $J_{r,R_r}[p^n]$), and $\mathcal{G}^{(0),\mathrm{ord}}_{r,R_r}$ is a Barsotti–Tate group over R_r.

In general, for any flat quasifinite group scheme $A_{/R}$, we have a functorial exact sequence

$$0 \longrightarrow FA \longrightarrow A \longrightarrow EA \longrightarrow 0, \tag{5-3}$$

where FA is a finite flat group scheme and EA is étale quasifinite with trivial closed fiber (see [Mazur 1978, Lemma 1.1]). Since ${}^tG_{r,R_r}$ and G_{r,R_r} are semiabelian, their finite p-power torsion points form quasifinite flat group schemes over R_r (see [Bosch et al. 1990, Lemma 7.3.2], for instance). Applying the above exact sequence to ${}^tG_r[p^n]^{\mathrm{ord}}$ and $G_r[p^n]^{\mathrm{ord}}$ for each $n > 0$ and defining

$${}^tG[p^\infty]^{\mathrm{ord,\,BT}} = \varinjlim_n F({}^tG[p^n]^{\mathrm{ord}}) \quad \text{and} \quad G[p^\infty]^{\mathrm{ord,\,BT}} = \varinjlim_n F(G[p^n]^{\mathrm{ord}}),$$

we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & {}^tG_{r,R_r}[p^\infty]^{\mathrm{ord,\,BT}} \arrow[r, hook] \arrow[d, "\iota_{BT}"'] & {}^tG_{r,R_r}[p^\infty]^{\mathrm{ord}} \arrow[r, two heads] \arrow[d, "\iota"'] & {}^tE_r^{\mathrm{ord}} \arrow[r] \arrow[d, "\iota_\eta"] & 0 \\
0 \arrow[r] & G_{r,R_r}[p^\infty]^{\mathrm{ord,\,BT}} \arrow[r, hook] & G_{r,R_r}[p^\infty]^{\mathrm{ord}} \arrow[r, two heads] & E_r^{\mathrm{ord}} \arrow[r] & 0
\end{tikzcd}$$

where the subscript “BT” indicates the maximal Barsotti–Tate subgroups. Here ${}^tE_r^{\mathrm{ord}}[p^n]$ and $E_r^{\mathrm{ord}}[p^n]$ have empty closed fiber, and ${}^tE_r^{\mathrm{ord}}(\overline{\mathbb{Q}})$ and $E_r^{\mathrm{ord}}(\overline{\mathbb{Q}})$ are each isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^m$ for some $m \geq 0$. The morphism ι_η (regarded as ${}^tE_r^{\mathrm{ord}}(\overline{\mathbb{Q}}) \to E_r^{\mathrm{ord}}(\overline{\mathbb{Q}})$) is an isomorphism by the construction over $\mathbb{Q}$ done in Section 4. Since ι_{BT} is a monomorphism, by the same argument as above, ι_{BT} is an isomorphism. This implies that ι is also an isomorphism. Then again $\mathcal{L} : {}^tG_{r,R_r}[p^\infty]^{\mathrm{ord}} \to \mathcal{G}_{r,R_r}$ is an isomorphism by (5-1) and (5-2), which implies $\mathcal{R} : \mathcal{G}_{r,R_r} \cong G_{r,R_r}[p^\infty]^{\mathrm{ord}}$. This finishes the proof. □

Lemma 5.3. *The natural morphism*: ${}^tG_{r,R_s}[p^\infty]^{\mathrm{ord}} \to {}^tG_{s,R_s}[p^\infty]^{\mathrm{ord}}$ *is a closed immersion for* $s > r$.

Proof. We have a morphism of semiabelian schemes ${}^tG_{r,R_s} \xrightarrow{i} {}^tG_{s,R_s}$, whose kernel is killed by $U(p)^M$ for sufficiently large M (by Lemma 5.2 applied to $f = i$ and $g = U(p)^M$). Thus i induces a closed immersion of the Barsotti–Tate part i_{BT} : ${}^tG_{r,R_s}[p^\infty]^{\mathrm{ord,\,BT}} \to {}^tG_{s,R_s}[p^\infty]^{\mathrm{ord,\,BT}}$. Since $E({}^tG_r[p^\infty]^{\mathrm{ord}}) \to E({}^tG_s[p^\infty]^{\mathrm{ord}})$ is a closed immersion, we get the desired result. □

Theorem 5.4. *Over $R_s := \mathbb{Z}_{(p)}[\mu_{p^s}]$, the natural inclusion $\mathcal{G}_{r,R_s}$ into $\mathcal{G}_{s,R_s}$ is a closed immersion whose image is equal to the kernel $\mathrm{Ker}(\gamma^{p^{r-1}}-1)$ on $\mathcal{G}_s$ for all $s > r$. In particular, the complement $\mathcal{G}^{(0)}$ of the fixed part of $\mathcal{G}$ by the action of μ is a Λ-BT group over R_∞.*

Proof. The first assertion proves the condition (CT) for the modular Λ-BT group, and hence $\mathcal{G}^{(0)}_{R_\infty}$ is a Λ-BT group over R_∞, as the condition (DV) was already proven in Section 4.

Thus we prove the first assertion. By Lemma 5.2, we have a sequence

$$0 \longrightarrow \mathcal{G}_{r,R_s} \xrightarrow{\ i\ } \mathcal{G}_{s,R_s} \xrightarrow{\gamma^{p^{r-1}}-1} \mathcal{G}_{s,R_s}$$

in which i is a closed immersion by Lemma 5.3. Look at $N^s_r : \mathcal{G}_{s,R_s} \to \mathcal{G}_{s,R_s}$ with $N^s_r = \sum_{\sigma\in\Gamma^{p^{r-1}}/\Gamma^{p^{s-1}}} \sigma$ and the inclusion $i : \mathcal{G}_{r,R_s} \to \mathcal{G}_{s,R_s}$. By applying (2) of Lemma 5.2 to $g_K = N^s_r$ and $f_K = i$ for $K = K_s$ and $R = R_s$, we see $\mathrm{Coker}(i)$ is a surjective image of $\mathrm{Coker}(N^s_r)$. Thus, we have the sequence

$$0 \longrightarrow \mathcal{G}_{r,R_s} \xrightarrow{\ i\ } \mathcal{G}_{s,R_s} \xrightarrow{\ \pi\ } \mathcal{G}_{s-r,R_s} \longrightarrow 0, \tag{5-4}$$

where i is a closed immersion and π is an epimorphism (of abelian fppf sheaves). The generic fiber of the sequence is exact by the result in the previous section. Thus we need to prove the exactness of the sequence (5-4) in the category of abelian fppf sheaves over R_s.

Applying the functor F in (5-3), we get the sequence of the Barsotti–Tate parts:

$$0 \longrightarrow \mathcal{G}^{BT}_{r,R_s} \xrightarrow{\ i^{BT}\ } \mathcal{G}^{BT}_{s,R_s} \xrightarrow{\ \pi^{BT}\ } \mathcal{G}^{BT}_{s-r,R_s} \longrightarrow 0. \tag{5-5}$$

Again π^{BT} is an epimorphism and i^{BT} is a closed immersion. Truncating the sequence to its finite layers, we get the third sequence

$$0 \longrightarrow \mathcal{G}^{BT}_{r,R_s}[p^n] \xrightarrow{\ i^{BT}_n\ } \mathcal{G}^{BT}_{s,R_s}[p^n] \xrightarrow{\ \pi^{BT}_n\ } \mathcal{G}^{BT}_{s-r,R_s}[p^n] \longrightarrow 0 \tag{5-6}$$

with epimorphism π^{BT}_n and closed immersion i^{BT}_n for each $n > 0$. Then $\mathrm{Ker}(\pi^{BT}_n)$ is represented by a finite flat group scheme (see [Hida 2012, §1.12.1], for instance). Writing $\mathrm{Ker}(\pi^{BT}_n) = \mathrm{Spec}(A)$, we have $\mathrm{Im}(i^{BT}_n) = \mathrm{Spec}(A/I)$ for an ideal I. Since $\mathrm{rank}_{R_s} A = \mathrm{rank}_{R_s} A/I$ as they have the same generic geometric fiber, we have $I = 0$ and $\mathrm{Ker}(\pi^{BT}_n) = \mathrm{Im}(i^{BT}_n)$ for all $n > 0$. In other words, the sequence (5-5) is exact.

By the result in Section 4, we have the exact sequence

$$0 \longrightarrow E(\mathcal{G}_{r,R_s}) \xrightarrow{\ i^{et}\ } E(\mathcal{G}_{s,R_s}) \xrightarrow{\ \pi^{et}\ } E(\mathcal{G}_{s-r,R_s}) \longrightarrow 0.$$

This combined with exactness of (5-5) implies exactness of the sequence (5-4) as desired. □

Remark 5.5. Over a nonnoetherian nondiscrete valuation ring such as R_∞, the distinction between Barsotti–Tate (or crystalline) Galois representations and semistable multiplicative Galois representations is murky. To give an example of this, start with a modular rational elliptic curve E with multiplicative reduction at p of conductor Np (so $p \nmid N$). Then the associated modular form f satisfies $f|U(p) = \epsilon f$ and $f|U^*(p) = \epsilon^{-1} f$ for a root of unity ϵ. Thus f has p-slope 0.

Consider the ordinary universal Galois deformation space S of the Galois representation ρ_E on the p-adic Tate module $T_p E_{/\mathbb{Q}}$. By definition, S is a formal scheme over $\mathbb{Z}_p$. Write $\boldsymbol{h} \subset \mathrm{End}_\Lambda(\mathcal{G})$ for the subalgebra generated over $\Lambda = \mathbb{Z}_p[\![x]\!]$ by all Hecke operators $T(n)$ and $U(q)$. The formal scheme S is often identified with $\mathrm{Spf}(\mathbb{T})$ for a local ring $\mathbb{T}$ of $\boldsymbol{h}$ by an "$R = T$" theorem (see [Wiles 1995] or [Hida 2000b, Theorem 5.29]), so it is flat over $\mathbb{Z}_p$ in such a good case, or even smooth over $\mathbb{Z}_p$ if $\mathbb{T} = \Lambda = \mathbb{Z}_p[\![\Gamma]\!]$. To make our argument easy, suppose that $S = \mathrm{Spf}(\Lambda)$ (so $\mathrm{Spec}(\Lambda)(\mathbb{Z}_p) \cong \Gamma \cong \mathbb{Z}_p$, for which we write $S(\mathbb{Z}_p)$ by abuse of notation). Then the subset $S^{\mathrm{crys}}(\mathbb{Z}_p) \subset S(\mathbb{Z}_p)$ corresponding to crystalline representations is p-adically dense (i.e., f is a p-adic limit of Hecke eigenforms f_k of weight $k > 2$ of level N prime to p). Thus $E[p^n](\overline{\mathbb{Q}})$ is, over $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, the reduction modulo p^n of a crystalline Galois representation and a multiplicative Galois representation at the same time. Instead of S^{crys}, we can take the subset S^{pBT} of potentially Barsotti–Tate Galois representations in S. Then S^{pBT} is Zariski dense in the scheme $S = \mathrm{Spec}(\Lambda)_{/\mathbb{Z}_p}$. Indeed, identifying $\mathrm{Spec}(\Lambda)$ with

$$\widehat{\mathbb{G}}_{m/\mathbb{Z}_p} = \mathrm{Spec}(\widehat{\mathbb{Z}_p[t, t^{-1}]})$$

(for $\widehat{\mathbb{Z}_p[t, t^{-1}]} = \varprojlim_r \mathbb{Z}_p[t, t^{-1}]/(t^{p^r} - 1)$) and making the identification $\gamma = t$, we have an identification $S^{pBT} = \mu_{p^\infty}(\overline{\mathbb{Q}}_p) - \{1\}$ inside $S(\overline{\mathbb{Q}}_p)$. Thus the Galois module $E[p^n](\overline{\mathbb{Q}}_p)$ can be realized as a generic geometric fiber of a finite flat group scheme G_n defined over a highly wild p-ramified subring R of R_∞. Since the generic fiber does not determine G_n over highly p-ramified ring (see [Raynaud 1974] and [Bosch et al. 1990, §7.5]), we have ambiguity. However, if we can pick G_n inside $\mathcal{G}_{R_\infty}$, it is expected to be unique. Thus $E[p^\infty]$ would be given as a generic fiber of a Barsotti–Tate group over R_∞. In particular, $\mathcal{G}$ is close to a Λ-BT group satisfying the following condition in place of (CT):

(ct) $\mathcal{G}_{R_\infty}[\gamma^{p^{r-1}} - 1]$ (for each $r > 0$) is a Barsotti–Tate group over R_∞.

Thus, decomposing $\mathcal{G}$ as $\mathcal{G}^\mu \oplus \mathcal{G}^{(0)}$, where $\mathcal{G}^\mu$ is the fixed part of $\mathcal{G}$ under $\mu := \mu_{p-1} \subset \mathbb{Z}_p^\times$, control of $\mathcal{G}^\mu$ is not equivalent to having nontrivial (nonflat) cokernel $\mathcal{G}^\mu[\gamma^{p^{r-1}} - 1]/\mathcal{G}^\mu_r$, since $\mathcal{G}^\mu_r(\overline{\mathbb{F}}_p)$ may even be finite. The Barsotti–Tate group

$\mathcal{G}^\mu[\gamma^{p^{r-1}}-1]$ over R_∞ does not descend to R_r. We give here two of our three arguments proving (ct) and will give the third in the next section.

Here is our first argument. Take the abelian subvariety $A_{r,\mathbb{Q}}=\sum_f A_f\subset {}^tG_{r,\mathbb{Q}}$, with f running over Hecke eigenforms satisfying condition (2) at the beginning of this section. Thus A_f and A_r have good reduction over R_∞. Then, writing ${}^tG_r=A_r+B_r$ for the complementary abelian subvariety $B_{r,\mathbb{Q}}$, the subvariety B_r is unique and is the image of the Jacobian of X_1^0 in J_r over $\mathbb{Q}$. Then by (DV), one can show that

$$\varinjlim_r A_r^{\mathrm{ord}}[p^\infty](\overline{\mathbb{Q}})=\varinjlim_r J_r^{\mathrm{ord}}[p^\infty](\overline{\mathbb{Q}})=\mathcal{G}(\overline{\mathbb{Q}}).$$

Indeed, it is easy to see that $\bigcup_{r>0}\Lambda^*[(\gamma^{p^r}-1)/(\gamma-1)]=\Lambda^*$ for the Pontryagin dual Λ^* of Λ, and hence, identifying $\mathcal{G}(\overline{\mathbb{Q}})$ with $(\Lambda^*)^{2j}$ as Λ-modules, we have

$$\mathcal{G}(\overline{\mathbb{Q}})\supset\varinjlim_r A_r^{\mathrm{ord}}[p^\infty](\overline{\mathbb{Q}})\supset\bigcup_{r>0}(\Lambda^*)^{2j}\left[\frac{\gamma^{p^r}-1}{\gamma-1}\right]=(\Lambda^{2j})^*=\mathcal{G}(\overline{\mathbb{Q}}).$$

Then we can go through the argument proving Theorem 5.1, replacing ${}^tG_r$ by A_r to show that $\mathcal{G}_{R_\infty}=\varinjlim_r A_r^{\mathrm{ord}}[p^\infty]_{/R_\infty}$ and get the desired result.

Here is a more direct argument without using A_r. Let

$$\widehat{R}=\widehat{R}_\infty=\bigcup_r\widehat{R}_r=R_\infty\otimes_{\mathbb{Z}}\mathbb{Z}_p\subset\overline{\mathbb{Q}}_p,$$

where $\widehat{R}_r=\varprojlim_n R_r/p^nR_r\cong R_r\otimes_{\mathbb{Z}}\mathbb{Z}_p$ for finite r is the p-adic completion. Define $F(\mathcal{G}_{\widehat{R}})=\varinjlim_{r,n}F(\mathcal{G}_{r,\widehat{R}}[p^n])$ and $E(\mathcal{G}_{\widehat{R}})=\varinjlim_{r,n}E(\mathcal{G}_{r,\widehat{R}}[p^n])$ for the functors F, E in (5-3). Since injective limits (in the category of fppf abelian sheaves) are exact, we get an exact sequence of ind-group schemes over $\widehat{R}$:

$$0\longrightarrow F(\mathcal{G}_{\widehat{R}})\longrightarrow\mathcal{G}_{\widehat{R}}\longrightarrow E(\mathcal{G}_{\widehat{R}})\longrightarrow 0.$$

Note that $E(\mathcal{G}_{r,\widehat{R}_r})(\overline{\mathbb{Q}}_p)\cong(\mathbb{Q}_p/\mathbb{Z}_p)^m$ (for m the dimension of multiplicative part of the reduction modulo p of $J^0_{1/\mathbb{Z}_p}$) and that $E(\mathcal{G}_{r,\widehat{R}_r})(\overline{\mathbb{Q}}_p)$ is killed by $x=\gamma-1$. Thus $E(\mathcal{G}_{\widehat{R}})(\overline{\mathbb{Q}}_p)$ is killed by x (and is still embedded in $(\mathbb{Q}_p/\mathbb{Z}_p)^m$). Note that $\mathcal{G}_{\widehat{R}}(\overline{\mathbb{Q}}_p)\cong(\Lambda^*)^{2j}$ for $2j=\dim_{\mathbb{F}_p}J_1[p](\overline{\mathbb{Q}})$. Since Λ^* is Λ-divisible, it does not have any quotient killed by x (except for $\{0\}$). Thus we get $F(\mathcal{G}_{\widehat{R}})=\mathcal{G}_{\widehat{R}}$ as we claimed.

There is a third more geometric argument (Proposition 6.3) showing the identity $\mathcal{G}(\overline{\mathbb{F}}_p)[\gamma^{p^{r-1}}-1]/\mathcal{G}_r(\overline{\mathbb{F}}_p)\cong(\mathbb{Q}_p/\mathbb{Z}_p)^m$ (for finite $r>0$) of the geometric special fibers. Therefore $\mathcal{G}(\overline{\mathbb{F}}_p)$ actually covers the multiplicative part. We give the details of this argument in the following section, after preparing some notation regarding the special fibers of modular curves (over discrete valuation rings).

6. Mod p modular curves

We keep the simplifying assumption that N is cube-free. Hereafter, we write $\mathcal{G}$ and $\mathcal{G}_r$ for the modular Λ-BT group and its r-th layer made out of the Jacobian of X_r. We prove the properties (Od) and (U) for $\mathcal{G}$ over R_∞ now. We consider the following Drinfeld-style moduli problem classifying $(E, \phi'_p, \phi_N)_{/A}$ over $\mathbb{Z}_{(p)}$, where $\phi_N : \mu_N \hookrightarrow E[N]$ is a closed immersion of group schemes over A and ϕ'_p is a pair of isogenies $\pi : E \to E'$ and ${}^t\pi : E' \to E$ of degree p^r together with points $P \in E(A)$ and $P' \in E'(A)$ such that $\mathrm{Ker}(\pi)$ is equal to the relative Cartier divisor $\sum_{j=0}^{p^r-1}[jP] \subset E$ and $\mathrm{Ker}({}^t\pi)$ is equal to the relative Cartier divisor $\sum_{j=0}^{p^r-1}[jP'] \subset E'$. The canonical Cartier duality pairing $\mathrm{Ker}(\pi) \times \mathrm{Ker}({}^t\pi) \to \mu_{p^r}$ gives a point $\zeta_{p^r} = \langle P, P' \rangle$. Thus, this moduli problem is defined over R_r. This problem at p is called the balanced $\Gamma_1(p^r)$ moduli problem in [Katz and Mazur 1985, §3.3]. As shown in Theorem 13.11.4 of the same book, this problem is represented by a regular affine scheme over $\mathbb{Z}_{(p)}[\mu_{p^r}]$ with regular projective compactification X'_r whose generic fiber is X_{r,K_r}. Recall the normalization X_{r,R_r} of $\boldsymbol{P}^1(j)_{/R_r}$ in X_{r,K_r}. Every regular scheme is normal (see [Matsumura 1986, Theorem 19.4], for instance), so $X'_{r,R_r} = X_{r,R_r}$. The special fiber $X_{r,\mathbb{F}_p}$ of X_{r,R_r} has the following description:

$$X_{r,\mathbb{F}_p} = X'_{r,\mathbb{F}_p} = X_{(r,0)} \cup X_{(0,r)} \cup \bigcup_{\substack{a+b=r,\ a>0,\ b>0 \\ u \in (\mathbb{Z}/p^{\min(a,b)}\mathbb{Z})^\times}} X_{(a,b,u)},$$

for smooth irreducible projective curves $X_{(a,b,u)}$ intersecting only at supersingular points (see [Katz and Mazur 1985, Theorem 13.11.4]). The curves $X_{(r,0)}$ and $X_{(0,r)}$ are smooth geometrically irreducible (by a theorem of Igusa).

The open curve obtained from $X_{(r,0)}$ by removing supersingular points and cusps represents the moduli problem classifying triples $(E, \mu_{p^r} \hookrightarrow E, \phi_N)$, and the corresponding open curve obtained from $X_{(0,r)}$ classifies $(E, \mathbb{Z}/p^r\mathbb{Z} \hookrightarrow E, \phi_N)$. This curve is called the Igusa curve, and hence we write $I_r = I_{r,\mathbb{F}_p} = X_{(0,r),\mathbb{F}_p}$. We have $I_{r,\mathbb{F}_p} \cong X^{(p^r)}_{(r,0),\mathbb{F}_p}$ (the base change by the p^r-th power Frobenius map) canonically. Since $X_{(r,0)}$ is defined over $\mathbb{F}_p$, we have actually $X_{(r,0),\mathbb{F}_p} \cong I_{r,\mathbb{F}_p}$. All this follows from [Katz and Mazur 1985, Theorem 13.11.4].

We put $Y_r = I_r \cup X_{(r,0)}$ which is the Zariski closure of the image of the disjoint union $I_r \sqcup X_{(r,0)}$ in X_r. This curve Y_r is introduced just above (u) in Section 2. Fix an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Over $\overline{\mathbb{F}}_p$, the two components of Y_r intersect only at supersingular points (and the crossing is an ordinary double point). On the middle components $X_{(a,b)} = \bigcup_u X_{(a,b,u)}$ with $ab \neq 0$, $\pi : E \to E'$ factors as

$$E \xrightarrow{F^a} E^{(p^a)} \overset{(*)}{\cong} E'^{(p^b)} \xrightarrow{V^b} E'$$

(and the middle isomorphism $(*)$ is determined by the datum $u \in (\mathbb{Z}/p^{\min(a,b)}\mathbb{Z})^\times$ outside the crossing).

As before, let J_r (resp. G_r and ${}^tG_r$) be the identity connected component of the Néron model of $J_{r/\mathbb{Q}}$ (resp. G_r and ${}^tG_r$) over $R_r := \mathbb{Z}_{(p)}[\mu_{p^r}]$. Mazur and Wiles [1984, Chapter 3] have shown the existence of a canonical isogeny $\mathrm{av}(\mathrm{Pic}^0_{Y_r/\mathbb{F}_p}) \to av(G_{r/\mathbb{F}_p})$, where av denotes the abelian variety part. By a theorem of Raynaud [Bosch et al. 1990, Theorem 9.4.5], we have $J_{r,R_r} = \mathrm{Pic}^0_{X_r/R_r}$. Thus, taking the special fiber, we have a surjection $J_{r,\mathbb{F}_p} = \mathrm{Pic}^0_{X_r/\mathbb{F}_p} \twoheadrightarrow \mathrm{Pic}^0_{Y_r/\mathbb{F}_p}$ corresponding to the inclusion $Y_{r,\mathbb{F}_p} = Y'_{r,\mathbb{F}_p} \hookrightarrow X'_{r,\mathbb{F}_p} = X_{r,\mathbb{F}_p}$. Then by Theorem 5.1 combined with [Mazur and Wiles 1984, Proposition on p. 267], we find:

Corollary 6.1. *We have* $\mathcal{G}_{r,\mathbb{F}_p} \cong \mathrm{Pic}^0_{Y_r/\mathbb{F}_p}[p^\infty]^{\mathrm{ord}} \cong G_{r,\mathbb{F}_p}[p^\infty]^{\mathrm{ord}}$.

Proof. Adding the toric part to the isogeny in [Mazur and Wiles 1984], we have an isogeny

$$\mathrm{Pic}^0_{Y_r/\mathbb{F}_p}[p^\infty]^{\mathrm{ord}} \to G_r[p^\infty]^{\mathrm{ord}}_{/\mathbb{F}_p},$$

but the projection: $J_r[p^\infty]^{\mathrm{ord}}_{/\mathbb{F}_p} \cong \mathrm{Pic}^0_{Y_r/\mathbb{F}_p}[p^\infty]^{\mathrm{ord}}$ composed with this isogeny is the special fiber of the isomorphism in Theorem 3.1. □

In [Mazur and Wiles 1984, Section 3.3], it is shown that the $U(p)$ operator on the abelian quotient

$$\mathrm{Pic}^0_{X_{(r,0)}/\mathbb{F}_p} \times \mathrm{Pic}^0_{I_r/\mathbb{F}_p}$$

of $\mathrm{Pic}^0_{Y_r/\mathbb{F}_p}$ has the following matrix shape:

$$\begin{pmatrix} F & * \\ 0 & V\langle p^{(p)}\rangle \end{pmatrix} \text{ on } \mathrm{Pic}^0_{I_r/\mathbb{F}_p} \times \mathrm{Pic}^0_{X_{(r,0)}/\mathbb{F}_p} \tag{6-1}$$

for the p-power relative Frobenius F and its dual V. If $N = 1$, then $U(p) = \left(\begin{smallmatrix} F & 0 \\ 0 & V \end{smallmatrix}\right)$ is semisimple on $\mathrm{Pic}^0_{I_r/\mathbb{F}_p} \times \mathrm{Pic}^0_{X_{(r,0)}/\mathbb{F}_p}$. Here, $\langle p^{(p)}\rangle$ is the diamond operator for $p \in (\mathbb{Z}/N\mathbb{Z})^\times$. This proves the conditions (Od) and (U) for the modular Λ-BT group $\mathcal{G}$. Moreover, writing $j_{r,\mathbb{F}_p} = \mathrm{Pic}^0_{I_r/\mathbb{F}_p}$ (the Jacobian of the r-th layer of the Igusa tower), we confirm that the generic geometric fiber $\mathcal{G}_r(\overline{\mathbb{F}}_p)$ coincides with $j_r[p^\infty](\overline{\mathbb{F}}_p)$ as the Frobenius map F (which equals $U(p)$ on j_r) is an automorphism on the geometric points of j_r and V is topologically nilpotent on $\mathrm{Pic}^0_{X_{(r,0)}/\mathbb{F}_p}[p^\infty](\overline{\mathbb{F}}_p)$.

We prepare some results to show that $\mathcal{G}(\overline{\mathbb{F}}_p)$ is Λ-injective (the third proof of (ct) in Remark 5.5).

Lemma 6.2. *Let $f : X \to Y$ be a finite flat Galois covering with Galois group G of projective smooth connected curves over $\overline{\mathbb{F}}_p$ unramified outside a finite set $S \subset Y(\overline{\mathbb{F}}_p)$. Assume that $G \cong \mathbb{Z}/p^m\mathbb{Z}$ and that every point in S fully ramifies in X, so we have a bijection $f^{-1}(S) \cong S$ induced by f. Then, writing $J_?$ for the Jacobian* *variety for* $? = X, Y$, *the pullback map* $f^* : J_Y(\overline{\mathbb{F}}_p) \hookrightarrow J_X(\overline{\mathbb{F}}_p)$ *is injective, and we have an isomorphism*

$$J_X(\overline{\mathbb{F}}_p)^G/f^*J_Y(\overline{\mathbb{F}}_p) \cong \left\{ D \in \bigoplus_{s\in f^{-1}(S)} \mathbb{Z}/p^m\mathbb{Z}[s] \,\middle|\, \deg(D) = 0 \right\}$$

of finite groups, where $[s]$ *is the divisor on* X *corresponding to the point* s, $J_X(\overline{\mathbb{F}}_p)^G$ *denotes the* G*-invariant subgroup* $H^0(G, J_X(\overline{\mathbb{F}}_p))$, *and* $\deg(\sum_s a_s[s]) = \sum_s a_s$ *for* $a_s \in \mathbb{Z}/p^m\mathbb{Z}$.

We call the quotient group $J_X(\overline{\mathbb{F}}_p)^G/f^*J_Y(\overline{\mathbb{F}}_p)$ the *ambiguous class group* and write it as $\mathrm{Amb}_{X/Y}$.

Proof. Write $\overline{\mathbb{F}}_p(?)$ for the function field of $? = X, Y$. Since $J_Y[n](\overline{\mathbb{F}}_p)$ for a positive integer n is canonically isomorphic to the Galois group of an abelian extension K_n of $\overline{\mathbb{F}}_p(Y)$ unramified everywhere, while X/Y ramifies fully at S, $\overline{\mathbb{F}}_p(X)$ is linearly disjoint from K_n over $\overline{\mathbb{F}}_p(Y)$; so, $f^* : J_Y[n] \to J_X[n]$ is injective. Since $J_Y(\overline{\mathbb{F}}_p) = \bigcup_n J_Y[n]$, we get the injectivity of $f^* : J_Y \to J_X$.

Let $\mathrm{Div}_?$ for $? = X_{\overline{\mathbb{F}}_p}, Y_{\overline{\mathbb{F}}_p}$ be the divisor group of ? and $\mathrm{Div}^0_?$ be the subgroup of degree 0 divisors. Then for the subgroup $P_? = \{\mathrm{div}(g) \mid g \in \overline{\mathbb{F}}_p(?)^\times\}$ of principal divisors, we have $\mathrm{Pic}_{?/\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p) = \mathrm{Div}_?/P_?$. Consider the subgroup $R_S = \bigoplus_{s\in f^{-1}(S)} \mathbb{Z}[s] \subset \mathrm{Div}_X$.

Write $D \sim D'$ if the two divisors are linearly equivalent. If $D^\sigma \sim D$ for $D \in \mathrm{Div}_X$ ($\sigma \in G$), writing $D^\sigma - D = \mathrm{div}(g_\sigma)$, we find $g_\sigma^\tau g_\tau/g_{\sigma\tau} \in \overline{\mathbb{F}}_p^\times$. Thus $g \mapsto g_\sigma$ is a 1-cocycle of G having values in $\overline{\mathbb{F}}_p(X)^\times/\overline{\mathbb{F}}_p^\times$. By the long exact sequence attached to the short exact sequence $\overline{\mathbb{F}}_p^\times \hookrightarrow \overline{\mathbb{F}}_p(X)^\times \twoheadrightarrow \overline{\mathbb{F}}_p(X)^\times/\overline{\mathbb{F}}_p^\times$, combined with the fact that $H^2(G, \overline{\mathbb{F}}_p^\times) = 0$ (since $\overline{\mathbb{F}}_p^\times$ is a prime-to-p-torsion module), we conclude from $H^1(G, \overline{\mathbb{F}}_p(X)^\times) = 0$ (Hilbert's theorem 90) that $H^1(G, \overline{\mathbb{F}}_p(X)^\times/\overline{\mathbb{F}}_p^\times) = 0$. Thus $g_\sigma = h - h^\sigma$ for $h \in \overline{\mathbb{F}}_p(X)^\times$, and $D + \mathrm{div}(h) \in \mathrm{Div}_X^G$. This shows that $\mathrm{Pic}_X(\overline{\mathbb{F}}_p)^G$ is the surjective image of Div_X^G.

We have $\mathrm{Div}_X^G = f^*\mathrm{Div}_Y + R_S$ with $R_S \cap f^*\mathrm{Div}_Y = p^m R_S$ as $s \in f^{-1}(S)$ ramifies fully in X/Y. Thus

$$\mathrm{Div}_X^G/f^*\mathrm{Div}_Y \cong R_S/p^m R_S.$$

Suppose $D \in \mathrm{Div}_X^G$ is principal, so $D = \mathrm{div}(g)$ for $g \in \overline{\mathbb{F}}_p(X)$. Then, for $\sigma \in G$, $g^{\sigma-1} = g^\sigma/g$ is a constant in $\overline{\mathbb{F}}_p^\times$. Thus $\sigma \mapsto g^{\sigma-1}$ is a homomorphism of G into $\overline{\mathbb{F}}_p^\times$, which must be trivial as $\overline{\mathbb{F}}_p^\times$ does not have any nontrivial p-subgroup. Thus $g \in f^*(\overline{\mathbb{F}}_p(Y))$. This shows $R_S/p^m R_S$ injects into $\mathrm{Pic}_X(\overline{\mathbb{F}}_p)^G/f^*\mathrm{Pic}_Y(\overline{\mathbb{F}}_p)$, and in fact $R_S/p^m R_S \cong \mathrm{Pic}_X(\overline{\mathbb{F}}_p)^G/f^*\mathrm{Pic}_Y(\overline{\mathbb{F}}_p)$ as $\mathrm{Pic}_X(\overline{\mathbb{F}}_p)^G$ is the surjective image of $\mathrm{Div}_X^G = f^*\mathrm{Div}_Y + R_S$. Since J_X is the degree 0 component of Pic_X, we confirm that

$$J_X(\overline{\mathbb{F}}_p)^G/J_Y(\overline{\mathbb{F}}_p) \cong \{D \in R_S/p^m R_S \mid \deg(D) = 0\}. \qquad \square$$

Proposition 6.3. *The geometric special fiber $\mathcal{G}(\overline{\mathbb{F}}_p)$ is Λ-injective (isomorphic to $(\Lambda^*)^j$), $\mathcal{G}(\overline{\mathbb{F}}_p) \cong (\mathcal{G}/\mathcal{G}^\circ)(\overline{\mathbb{Q}}_p)$ by the reduction map as Λ-modules, and $\mathcal{G}_{\mathbb{F}_p}[\gamma^{p^{r-1}}-1]$ is a Barsotti–Tate group over $\mathbb{F}_p$. Here, $\mathcal{G}^\circ$ is the connected component of $\mathcal{G}$ over $\widehat{R}_\infty = R_\infty \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ (whose group of generic geometric points $\mathcal{G}^\circ(\overline{\mathbb{Q}}_p)$ is the kernel of the reduction map).*

Here, j is as in Section 4 and is given by $2j = \dim_{\mathbb{F}_p} J_1[p](\overline{\mathbb{Q}})$. In the following proof, we consider the Igusa tower unramified outside supersingular points:

$$\cdots \longrightarrow I_{r+1,\mathbb{F}_p} \longrightarrow I_{r,\mathbb{F}_p} \longrightarrow \cdots \longrightarrow I_{0,\mathbb{F}_p} := X^0_{1/\mathbb{F}_p}$$

over $\mathbb{F}_p$. Write $j_{r,\mathbb{F}_p}$ for the Jacobian variety of $I_{r,\mathbb{F}_p}$. Mazur and Wiles [1983] studied $\mathcal{G}^{(0)}(\overline{\mathbb{F}}_p) = \varinjlim_r j_r[p^\infty](\overline{\mathbb{F}}_p)^{(0)}$, where the superscript "(0)" indicates the complement of the fixed part of $\mu := \mu_{p-1} \subset \mathbb{Z}_p^\times$. In particular, they showed the control

$$\mathcal{G}^{(0)}(\overline{\mathbb{F}}_p)[\gamma^{p^{r-1}}-1] = j_r[p^\infty](\overline{\mathbb{F}}_p)^{(0)}$$

for all $r > 0$ (as [ibid., (1) in §2]). The control fails between $\mathcal{G}(\overline{\mathbb{F}}_p)^\mu$ and $j_r[p^\infty](\overline{\mathbb{F}}_p)^\mu$, and the idea for the proof is to compute the failure using Lemma 6.2.

Proof. We only prove the first two assertions, as the last one follows from the second argument in Remark 5.5 after making the base change from R_∞ to $\mathbb{F}_p$. Let S_r be the finite set of supersingular points of I_r. The diamond operator action is equal to the action of $\mathrm{Gal}(I_\infty/X_1^0) = \mathbb{Z}_p^\times$. Since $I_s \to I_r$ $(s > r > 0)$ fully ramifies over each point of S_r having the Galois group isomorphic to $(1+p^r\mathbb{Z}_p)/(1+p^s\mathbb{Z}_p) \cong \mathbb{Z}/p^{s-r}\mathbb{Z}_p$, the failure of the control of $j_s[p^\infty](\overline{\mathbb{F}}_p)$ can be computed by Lemma 6.2, and we get

$$j_s[p^\infty][\gamma^{p^{r-1}}-1]/j_r[p^\infty] \cong \mathrm{Amb}_{I_s/I_r} = \{D \in R_{S_s}/p^{s-r}R_{S_s} \mid \deg(D) = 0\}$$

under the notation in the proof of Lemma 6.2. Passing to the injective limit with respect to s, we get, for $m = |S_0| - 1$,

$$\mathcal{G}(\overline{\mathbb{F}}_p)[\gamma^{p^{r-1}}-1]/\mathcal{G}_r(\overline{\mathbb{F}}_p) = \varinjlim_s j_s[p^\infty][\gamma^{p^{r-1}}-1]/j_r[p^\infty] = \varinjlim_s \mathrm{Amb}_{I_s/I_r} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^m.$$

As is well known (see [Hida 2012, (4.14)], for instance), the dimension of the multiplicative part of J_1^0 is given by $m = |S_0| - 1$. This shows that $\mathcal{G}(\overline{\mathbb{F}}_p)[\gamma^{p^{r-1}}-1]$ is a p-divisible module of finite $\mathbb{Z}_p$-corank for all r, and hence $\mathcal{G}(\overline{\mathbb{F}}_p) \cong (\Lambda^*)^j$ (a Λ-injective module of Λ-corank j) by the same argument proving $\mathcal{G}(\overline{\mathbb{Q}}) \cong (\Lambda^*)^{2j}$ in Section 4. The reduction map (over $\widehat{R}_\infty$) induces an injection $(\mathcal{G}/\mathcal{G}^\circ)(\overline{\mathbb{Q}}_p) = (\mathcal{G}^{BT}/\mathcal{G}^\circ)(\overline{\mathbb{Q}}_p) \to \mathcal{G}(\overline{\mathbb{F}}_p)$ (as we have proven $\mathcal{G}^{BT} = F(\mathcal{G}_{\widehat{R}_\infty}) = \mathcal{G}$ in Remark 5.5). Then, comparing the corank, we get an isomorphism $(\mathcal{G}/\mathcal{G}^\circ)(\overline{\mathbb{Q}}_p) \cong \mathcal{G}(\overline{\mathbb{F}}_p)$. □

7. The α-eigenspace in $\mathcal{G}_{R_\infty}$

The modular Λ-BT group $\mathcal{G}$ has coefficients $\Lambda = \mathbb{Z}_p[\![x]\!] = \mathbb{Z}_p[\![\Gamma]\!]$ (for $\Gamma = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$). To get a Λ_W-BT group for $\Lambda_W = W[\![x]\!] = W[\![\Gamma]\!]$, we can extend endomorphisms to a valuation ring W bigger than $\mathbb{Z}_p$; that is, we consider an fppf abelian sheaf $\mathcal{G}_{R_\infty} \otimes W$ defined by $A \mapsto \mathcal{G}_{R_\infty} \otimes W(A) := \mathcal{G}_{R_\infty}(A) \otimes_{\mathbb{Z}_p} W$ for A running over fppf extensions of R_∞. Suppose that W is finite flat over $\mathbb{Z}_p$, then forgetting about the action of W, $\mathcal{G}_{R_\infty} \otimes W$ is isomorphic to $\mathcal{G}_{R_\infty}^{\mathrm{rank}\,W}$ and hence is represented by a Λ_W-BT group. Note that $\mathcal{G}_{R_\infty} \otimes W$ is not a base change of $\mathcal{G}$ to W (the operation is just extending endomorphisms from Λ to Λ_W formally). For the Jacobian J_r and a free $\mathbb{Z}$-module L of finite rank, we can form the endomorphism extension $J_r \otimes L$ which send A to $J_r(A) \otimes_{\mathbb{Z}} L$ as an fppf abelian sheaf over R_r (or $\mathbb{Q}$). This fppf abelian sheaf is represented by an abelian variety, again written $J_r \otimes L$, defined over R_r (or $\mathbb{Q}$). We can take L to be the subalgebra $\mathbb{Z}[\mu_{p^r}, \alpha]$ in $\overline{\mathbb{Q}}$ generated by an algebraic integer α and p^r-th roots of unity.

We fix an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and often identify $\alpha \in \overline{\mathbb{Q}}$ with $i_p(\alpha) \in \overline{\mathbb{Q}}_p$ without attaching "i_p" (if confusion is unlikely). For an eigenvalue $\alpha \in \overline{\mathbb{Q}}$ of $U(p)$ on $S_2(\Gamma_1(p^r))$, we consider the subfield $\mathbb{Q}_p(\mu_{p^r}, \alpha)$ in $\overline{\mathbb{Q}}_p$ generated by $i_p(\alpha)$ over $\mathbb{Q}_p(\mu_{p^r})$. Note here $\mathbb{Q}(\alpha)$ may not contain μ_{p^r} even if α is realized as a Hecke eigenvalue of a new form in $S_2(\Gamma_0(Np^{r+1}), \chi)$ for χ having p-conductor p^{r+1}. Let W be the p-adic integer ring of $\mathbb{Q}_p(\mu_{p^r}, \alpha)$. We put $\mathfrak{G}_{R_\infty} := \mathcal{G}_{R_\infty} \otimes W$ (the endomorphism extension). We want to know when $\mathfrak{G}_{R_\infty}[U(p) - \alpha]$ is contained in $\mathfrak{G}_{r,R_\infty} = \mathfrak{G}_{R_\infty}[\gamma^{p^{r-1}} - 1] = \mathcal{G}_{r,R_\infty} \otimes W$.

Look into the Hecke algebra $\boldsymbol{h}_W$ over Λ_W defined by

$$\boldsymbol{h}_W = \Lambda_W[\{T(n) \otimes 1\}_{p \nmid n}, U(p) \otimes 1] \subset \mathrm{End}_\Lambda(\mathfrak{G}_{R_\infty}),$$

where $T(n) \otimes 1$ sends $g \otimes w$ to $T(n)(g) \otimes w$ for $g \otimes w \in (\mathcal{G}_{R_\infty} \otimes W)(A)$ with $w \in W$ and $g \in \mathcal{G}_{R_\infty}(A)$. Hereafter, we just write simply $T(n)$ (resp. $U(p)$) for $T(n) \otimes 1$ (resp. $U(p) \otimes 1$). This is the big p-ordinary Hecke algebra over Λ_W, which is free of finite rank over Λ_W. Take a local ring $\mathbb{T}$ of $\boldsymbol{h}_W$ with maximal ideal $\mathfrak{m}$. We give ourselves a Hecke eigenvalue α given by $f|U(p) = \alpha f$ for $f \in S_2(\Gamma_0(p^r), \varepsilon)$ with $\mathbb{T} \cdot f \neq 0$. Regard ε as a character of $\mathbb{Z}_p^\times \supset \Gamma = 1 + p\mathbb{Z}_p$. Since $W \supset \mu_{p^r}(\overline{\mathbb{Q}}_p)$, W contains $\varepsilon(\gamma)$.

For a module or an fppf abelian sheaf M over R_∞ on which $\boldsymbol{h}_W$ acts via endomorphisms, adding the subscript $\mathbb{T}$, we indicate the $\mathbb{T}$-eigenspace. Therefore if M is an fppf abelian sheaf,

$$M_{\mathbb{T}}(A) = \{1_{\mathbb{T}}(x) \in M(A) \mid x \in M(A)\} = \{h(x) \mid x \in M(A), h \in \mathbb{T}\} = \mathbb{T}(M)$$

for the idempotent $1_{\mathbb{T}}$ of $\mathbb{T}$ in $\boldsymbol{h}_W$. Since $\mathbb{T}$ is a direct ring summand of $\boldsymbol{h}_W$, $M_{\mathbb{T}}$ is a direct summand of M as an fppf sheaf. In particular,

$$\mathfrak{G}_{\mathbb{T}}(A) = \{1_{\mathbb{T}}(x) \in \mathfrak{G}(A) \mid x \in \mathfrak{G}(A)\} = \{h(x) \mid x \in \mathfrak{G}(A), h \in \mathbb{T}\}. \tag{7-1}$$

Since $\mathfrak{G}_{\mathbb{T}}$ is a direct summand of the Λ_W-BT group $\mathfrak{G}_{R_\infty}$, $\mathfrak{G}_{\mathbb{T}}$ is a Λ_W-BT group over R_∞, and more generally $\mathfrak{G}_{\mathbb{T},r} := \mathfrak{G}_{\mathbb{T}}[\gamma^{p^{r-1}} - 1]$ is a Barsotti–Tate group over R_∞. We write $t = \gamma - \varepsilon(\gamma) \in \Lambda_W$, where $\gamma = 1 + x$ as before is the fixed generator of Γ. Then we expect:

Conjecture 7.1. *Let $f|U(p) = \alpha f$ for a Hecke eigenform $f \in S_2(\Gamma_0(p^r), \varepsilon)$ with $\mathbb{T}(f) \neq 0$, and put $t = \gamma - \varepsilon(\gamma)$. Let*

$$\mathfrak{G}_{\mathbb{T}}[U(p) - \alpha, t^n] = \{x \in \mathfrak{G}_{\mathbb{T}}(\overline{\mathbb{Q}}) : x \mid U(p) = \alpha x \text{ and } t^n x = 0\}.$$

Then there exists a positive integer $s \geq r$ independent of $n < \infty$ such that the p-divisible part $\mathfrak{G}_{\mathbb{T}}[U(p) - \alpha, t^n]^{\mathrm{div}}$ is contained in $\mathfrak{G}_{\mathbb{T},s}(\overline{\mathbb{Q}})$.

The scalar $\alpha \in W$ is regarded as an operator $g \otimes w \mapsto g \otimes \alpha w$ acting on $\mathfrak{G}_{\mathbb{T}}$ under the notation introduced above. Since the Tate module $T\mathfrak{G}_{\mathbb{T},s} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (after extending scalars to $\mathbb{Q}_p$) is a multiplicity-free semisimple $\mathbb{T}$-module, this conjecture implies that $T(\mathfrak{G}_{\mathbb{T}}[U(p) - \alpha, t^n]^{\mathrm{div}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a multiplicity-free semisimple $\mathbb{T}$-module, and moreover, by an isogeny, the Barsotti–Tate group $\mathfrak{G}_{\mathbb{T}}[U(p) - \alpha, t^n]^{BT}$ can be brought into the abelian variety $J_s \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_{p^r}, \alpha]$ (the endomorphism extension). Thus this conjecture is a semisimplicity conjecture for the α-eigenspace of $U(p)$ and conjecturally answers (to some extent) the question (Q5).

In the following section, we relate a weaker version of this conjecture to the nonvanishing problem of a certain $\mathscr{L}$-invariant which was conjectured earlier.

8. The adjoint $\mathscr{L}$-invariant

We use the notation introduced in the previous section. Let $f \in S_2(\Gamma_1(p^r))$ be a Hecke eigenform with $f|U = \alpha f$ for an algebraic integer α with either $|\alpha^\sigma| = \sqrt{p}$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ or $\alpha = \pm 1$. Take a prime $\mathfrak{p}$ of the integer ring $\overline{\mathbb{Z}}$ of $\overline{\mathbb{Q}}$. We assume that $\alpha \not\equiv 0 \bmod \mathfrak{p}$. Such an eigenform is called a $\mathfrak{p}$-ordinary eigenform. We now relate Conjecture 7.1 to a conjecture of Greenberg on the nonvanishing of an $\mathscr{L}$-invariant. This may be the only heuristic reason supporting the validity of Conjecture 7.1 at this moment. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(W)$ be the $\mathfrak{p}$-adic Galois representation of f.

Suppose $W \supset \mathbb{Z}_p[\mu_{p^r}, \alpha]$ as before. Recall that $\boldsymbol{h}_W := \boldsymbol{h} \otimes_{\mathbb{Z}_p} W$ acts on the endomorphism extension (not the base change) $\mathfrak{G} = \mathscr{G} \otimes_{\mathbb{Z}_p} W$ by $h \otimes w(x \otimes w') = h(x) \otimes ww'$ for $x \in \mathfrak{G}(A) = \mathscr{G}(A) \otimes_{\mathbb{Z}_p} W$. Then $\gamma = 1 + x$ acts on f as $f|\gamma = \varepsilon(\gamma) f$ for a finite order character $\varepsilon : \Gamma \to W^\times$. Let $\mathbb{T}$ be the local ring of $\boldsymbol{h}_W$ acting nontrivially on f, and consider $\mathfrak{G}_{\mathbb{T}}$ defined in (7-1). Thus $\mathfrak{G}_{\mathbb{T}}$ is a Λ-adic Barsotti–Tate group over R_∞. Let $t = \gamma - \varepsilon(\gamma)$. Here is a weaker version of Conjecture 7.1 directly related to the adjoint $\mathscr{L}$-invariant:

Conjecture 8.1. *The p-divisible part $\mathfrak{G}_\alpha$ of*

$$\{x \in \mathfrak{G}_{\mathbb{T}}(\overline{\mathbb{Q}}) \mid x|U = \alpha \cdot x \text{ and } t^2 x = 0\}$$

is contained in $\mathfrak{G}_{\mathbb{T}}[\gamma^s - 1](\overline{\mathbb{Q}})$ for sufficiently large s. In particular, the action of Γ on the Tate module $T\mathfrak{G}_\alpha \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semisimple.

Let P be the kernel of $\lambda : \mathbb{T} \to W$ given by $f|h = \lambda(h)f$. Then the P-adically completed localization $\mathbb{T}_P$ is canonically isomorphic to $K[\![t]\!]$ for the quotient field K of W. Let $\boldsymbol{a}(p)$ be the image of $U = U(p)$ in $\mathbb{T}_P$. Consider $\boldsymbol{a}'(p) = d\boldsymbol{a}(p)/dt = d\boldsymbol{a}(p)/dx$. Numerically, $\boldsymbol{a}'(p)$ is almost always a unit in Λ, but there are exceptions; for example, if we take $p = 53$ and $\alpha = -1$, $\boldsymbol{a}'(p)$ is a nonunit. Suppose $\boldsymbol{a}'(p) \in \Lambda^\times$. Then $\boldsymbol{a}(p) = \alpha$ can happen on the Λ-adic Tate module $T = T\mathfrak{G}_{\mathbb{T}}$ with multiplicity 1. Thus $T/(\boldsymbol{a}(p) - \alpha)T \cong W^2$, and

$$\mathfrak{G}_\alpha = \mathrm{Ker}(U - \alpha : \mathfrak{G}_{\mathbb{T}} \to \mathfrak{G}_{\mathbb{T}}) \subset J_r[p^\infty](\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}_p} W.$$

By the control theorem of $\boldsymbol{h}$ (see [Hida 2012, Sections 3.1–3.2], for instance), $\boldsymbol{a}(p)(\gamma^k - 1)$ is a $U(p)$-eigenvalue for a weight $k+2$ modular form; so, by the solution of the Ramanujan–Petersson conjecture due to Deligne, $|\boldsymbol{a}(p)(\gamma^k - 1)| = p^{(k+1)/2}$ Thus as a function of x, $\boldsymbol{a}(p)$ assumes infinitely many distinct values. Thus $\boldsymbol{a}(p)$ is transcendental over W. In particular, $\boldsymbol{a}'(p) \neq 0$. Thus, for almost all f, $t^2 \nmid (\boldsymbol{a}(p) - \alpha)$, and the conjecture holds for almost all f. Here, for simplicity, we used Deligne's result to conclude $t^2 \nmid (\boldsymbol{a}(p) - \alpha)$ for most f; there is a more elementary proof of this in [Hida 2011].

We let ρ act by conjugation on the trace 0 subspace of $M_2(W)$, which is called the adjoint square representation $\mathrm{Ad}(\rho)$ of ρ. Since $\mathrm{Ad}(\rho)([p, \mathbb{Q}_p])$ has an eigenvalue 1, the p-adic L-function $L_p^{\mathrm{an}}(s, \mathrm{Ad}(\rho)) = L_p^{\mathrm{an}}(s, \mathrm{Ad}(f))$ has an exceptional zero at $s = 1$ (see [Hida 2011, Section 2]). Following [Mazur et al. 1986], we give an analytic definition of the $\mathscr{L}$-invariant of $L_p^{\mathrm{an}}(s, \mathrm{Ad}(f))$ as

$$L_p^{\mathrm{an}}(1, \mathrm{Ad}(f)) = \mathscr{L}^{\mathrm{an}}(\mathrm{Ad}(f)) \frac{L(1, \mathrm{Ad}(f))}{c_+(\mathrm{Ad}(f))}$$

for a Shimura period $c_+(\mathrm{Ad}(f)(1))$, appropriately normalized. Note here that $L(1, \mathrm{Ad}(f))$ is nonzero. We have a power series $\Phi^{\mathrm{an}}(x) \in W[\![x]\!]$ such that $L_p^{\mathrm{an}}(s, \mathrm{Ad}(f)) = \Phi^{\mathrm{an}}(\gamma^{s-1} - 1)$ regarding $\gamma \in 1 + p\mathbb{Z}_p$. We have $\Phi^{\mathrm{an}}(x) = x\Psi^{\mathrm{an}}(x)$ with $\Psi^{\mathrm{an}}(x) \in W[\![x]\!]$, and $\mathscr{L}^{\mathrm{an}}(\mathrm{Ad}(f))$ is a nonzero constant multiple of $\Psi^{\mathrm{an}}(0)$.

Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension. The arithmetic p-adic L-function is defined as $L_p(s, \mathrm{Ad}(\rho)) = \Phi(\gamma^{s-1} - 1)$ for the characteristic power series $\Phi(x) \in W[\![x]\!]$ of the adjoint square Selmer group $\mathrm{Sel}_{\mathbb{Q}_\infty}(\mathrm{Ad}(\rho))$ defined by Greenberg (see [Greenberg 1994] and [Hida 2011]), where we identify $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ with $\Gamma = 1 + p\mathbb{Z}_p$ by the cyclotomic character, and the Iwasawa algebra $W[\![\Gamma]\!]$ with $W[\![x]\!]$ by $\Gamma \ni 1 + p \mapsto 1 + x$. It is known that $\Phi(x) = x\Psi(x)$ with $\Psi(x) \in W[\![x]\!]$ (see

[Greenberg 1994]). Thus, the arithmetic L-function $L_p(s, \mathrm{Ad}(\rho))$ has a zero at $s=1$. Greenberg has defined his $\mathscr{L}$-invariant $\mathscr{L}(\mathrm{Ad}(\rho))$ by purely Galois cohomological means and proved that $\Psi(0)$ is a multiple of $\mathscr{L}(\mathrm{Ad}(\rho))$ by a simple constant (up to units in W; see [ibid.]).

The main conjecture in this setting predicts the equality $\Phi(x)=\Phi^{\mathrm{an}}(x)$ up to units in $W[\![x]\!]$ (assuming that ρ is residually absolutely irreducible). The conjecture has been proven in many cases by [Urban 2006]. We assume the following condition:

(H) $\bar{\rho}=(\rho \bmod \mathfrak{m}_W)$ is absolutely irreducible over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{p^*}])$ for $p^*=(-1)^{(p-1)/2}p$, or the semisimplification of ρ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the sum of two distinct characters, or $\alpha=\pm 1$.

Under this circumstance, regarding $\gamma=1+p\in 1+p\mathbb{Z}_p$, it is known that

$$\mathscr{L}(\mathrm{Ad}(\rho))=-2\log_p(\gamma)\alpha^{-1}\frac{d\boldsymbol{a}(p)}{dx}\Big|_{t=0}. \tag{8-1}$$

This follows from [Greenberg and Stevens 1993] if $\alpha=1$, because in this case $\mathscr{L}(\mathrm{Ad}(\rho))=\mathscr{L}(\rho)$. Otherwise, it is proven in [Hida 2004] and [2011]. Though Greenberg made the following conjecture in a more general setting, if we limit ourselves to $\mathrm{Ad}(\rho)$ for the ordinary modular Galois representation ρ, Conjecture 8.1 is equivalent to the following conjecture under (H):

Conjecture 8.2 (R. Greenberg). $\mathscr{L}(\mathrm{Ad}(\rho))\neq 0$.

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