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# DISCRETE SEMICLASSICAL ORTHOGONAL POLYNOMIALS OF CLASS ONE

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**We study discrete semiclassical orthogonal polynomials of class $s = 1$. By considering particular solutions of the Pearson equation, we obtain five canonical families of such polynomials. We also consider limit relations between these and other families of orthogonal polynomials.**

## 1. Introduction

Discrete orthogonal polynomials with respect to uniform lattices have attracted the interest of researchers from many points of view [Nikiforov et al. 1985]. A first approach comes from the discretization of hypergeometric second-order linear differential equations and thus the classical discrete orthogonal polynomials (Charlier, Krawtchouk, Meixner, Hahn) appear in a natural way. As a consequence of the symmetrization problem for the above second-order difference equations, one can deduce that such polynomials are orthogonal with respect to (discrete) measures. This yields the so-called Pearson equation that the measure satisfies.

In the last twenty years, new families of discrete orthogonal polynomials have been considered in the literature, taking into account the so-called canonical spectral transformations of the orthogonality measure. Under a Uvarov transformation, mass points are added to the discrete measure; sequences of orthogonal polynomials with respect to the new measure have been studied extensively in this case (see [Chihara 1985; Álvarez and Marcellán 1995a; Álvarez et al. 1995], among others). Under a Christoffel transformation, the discrete measure is multiplied by a polynomial; a few results are available in this case [Ronveaux and Salto 2001].

From a structural point of view, some effort has been made to translate to the discrete case the well-known theory of semiclassical orthogonal polynomials (see [Maroni 1991]). In particular, characterizations of such polynomials in terms of structure relations of the first and second kind, as well as discrete holonomic equations (second-order linear difference equations with polynomial coefficients of fixed degree and where the degree of the polynomial appears as a parameter) were

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given in [Marcellán and Salto 1998]. Linear spectral perturbations of semiclassical linear functionals have been studied in the Uvarov case [Godoy et al. 1997].

On the other hand, we must point out that the linear canonical spectral transformations (Christoffel, Uvarov, Geronimus) of classical discrete orthogonal polynomials yield discrete semiclassical orthogonal polynomials. But, as a first step, the problem of classification of discrete semiclassical linear functional of class one remains open. Symmetric discrete semiclassical linear functionals of class one have been described in [Maroni and Mejri 2008]. A classification of $D$-semiclassical linear functional of class one was given in [Belmehdi 1992] and of those of class two in [Marcellán et al. 2012].

This article provides a constructive method for finding $D_w$-semiclassical orthogonal polynomials, based on the Pearson equation satisfied by the corresponding linear functional. We will focus our attention on the classification of $D_1$-semiclassical linear functionals of class $s = 1$. In such a way, new families of linear functionals appear. Notice that an alternative method is based on the Laguerre–Freud equations satisfied by the coefficients of the three-term recurrence relations associated with these orthogonal polynomials. Their complexity increases with the class of the linear functional and the solution is cumbersome. Basic references concerning this approach are [Foupouagnigni et al. 1998] as well as [Maroni and Mejri 2008].

The structure of the article is as follows: Section 2 deals with the basic definitions and the theoretical background we will need in the sequel. In Section 3 we describe the $D_1$-classical linear functionals as $D_1$-semiclassical of class $s = 0$. The fact that most of the semiclassical linear functionals of class $s = 1$ are related to the class $s = 0$ will prove to be very useful later on. Indeed, in Section 4, a classification of such semiclassical linear functionals is given. Some of them are not known in the literature, as far as we know. Finally, Section 5 studies limit relations for semiclassical orthogonal polynomials of class $s = 1$.

## 2. Preliminaries and basic background

**Definition 1.** Let $\{\mu_n\}_{n\geq 0}$ be a sequence of complex numbers and let $\mathscr{L}$ be a linear complex-valued function defined on the linear space $\mathbb{P}$ of polynomials with complex coefficients by

$$\langle \mathscr{L}, x^n \rangle = \mu_n.$$

Then $\mathscr{L}$ is called the moment functional determined by the moment sequence $\{\mu_n\}_{n\geq 0}$, and $\mu_n$ is called the moment of order $n$.

Given a moment functional $\mathscr{L}$, the formal Stieltjes function of $\mathscr{L}$ is defined by

$$S_{\mathscr{L}}(z) = -\sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}.$$

For any moment functional $\mathscr{L}$ and any polynomial $q(x)$, we define the moment functional $q\mathscr{L}$ by

$$\langle q\mathscr{L}, P\rangle = \langle \mathscr{L}, qP\rangle, \quad P \in \mathbb{P}.$$

**Definition 2.** Let $\mathscr{L}$ be the linear functional associated with the moment sequence $\{\mu_n\}_{n\ge 0}$ and

$$\Delta_n = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}.$$

We call $\mathscr{L}$ *regular* if $\Delta_n \neq 0$ for all $n \in \mathbb{N}_0 := \{n \in \mathbb{Z} : n \ge 0\}$. We call it *positive definite* if $\Delta_n > 0$ for all $n \in \mathbb{N}_0$.

**Definition 3.** A sequence of polynomials $\{P_n(x)\}_{n\ge 0}$, with $\deg P_n = n$, is said to be an *orthogonal polynomial sequence* with respect to a regular linear functional $\mathscr{L}$ if there exists a sequence of nonzero real numbers $\{\zeta_n\}_{n\ge 0}$ such that

$$\langle \mathscr{L}, P_k P_n\rangle = \zeta_n \delta_{k,n}, \quad k, n \in \mathbb{N}_0.$$

If $\zeta_n = 1$, then $\{P_n(x)\}_{n\ge 0}$ is said to be an *orthonormal polynomial sequence*. If the linear functional is positive definite, such a sequence is unique under the assumption that each entry has a positive real leading coefficient.

**Theorem 4** [Chihara 1978, Theorem 4.4]. *Let $\{b_n\}_{n\ge 0}$ and $\{\gamma_n\}_{n\ge 0}$, with $\gamma_n \neq 0$ for every $n \in \mathbb{N}_0$, be arbitrary sequences of complex numbers and let $\{P_n(x)\}$ be a sequence of monic polynomials defined by the three-term recurrence relation*

$$P_{n+1}(x) = (x - b_n)P_n(x) - \gamma_n P_{n-1}(x), \tag{1}$$

*with $P_{-1} = 0$ and $P_0 = 1$. Then, there is a unique linear functional $\mathscr{L}$ such that $\mathscr{L}(1) = \gamma_0$ and*

$$\langle \mathscr{L}, P_k(x)P_n(x)\rangle = \gamma_0\gamma_1 \cdots \gamma_n \delta_{k,n}.$$

If the linear functional is positive definite and $\{p_n(x)\}_{n\ge 0}$ is the corresponding orthonormal polynomial sequence, formula (1) becomes

$$a_{n+1}p_{n+1}(x) = (x - b_n)p_n(x) - a_n p_{n-1}(x),$$

where $a_n$ is a real number and $a_n^2 = \gamma_n$.

**Definition 5.** Let $\mathscr{L}$ be a linear functional and $U^* : \mathbb{P} \to \mathbb{P}$ a linear operator. The linear functional $U\mathscr{L}$ is defined by

$$\langle U\mathscr{L}, P\rangle = -\langle \mathscr{L}, U^* P\rangle, \quad P \in \mathbb{P}.$$

**Example 6.** If $U$ is the standard derivative operator $D$, we have $U^* = U = D$.

**Definition 7.** A regular linear functional $\mathscr{L}$ is called *$U$-semiclassical* if it satisfies the Pearson equation $U(\phi\mathscr{L}) + \psi\mathscr{L} = 0$ or, equivalently,

$$\langle U(\phi\mathscr{L}) + \psi\mathscr{L}, P\rangle = 0, \quad P \in \mathbb{P},$$

where $\phi, \psi$ are two polynomials and $\phi$ is monic. The corresponding orthogonal sequence $\{P_n(x)\}_{n\geq 0}$ is called $U$-semiclassical.

Semiclassical linear functionals with respect to several choices of operators have been studied in the literature. For example, when $U = D$ (the standard derivative operator), the theory of $D$-semiclassical linear functionals has been exhaustively studied by P. Maroni and coworkers (see [Maroni 1991] for an excellent survey on this topic).

If $U = D_\omega$, where

$$D_\omega f(x) = \frac{f(x+\omega) - f(x)}{\omega}, \quad \omega \neq 0,$$

a regular linear functional $\mathscr{L}$ is said to be *$D_\omega$-semiclassical* if there exist polynomials $\phi, \psi$, where $\phi$ is monic and $\deg \psi \geq 1$, such that $D_\omega(\phi\mathscr{L}) + \psi\mathscr{L} = 0$.

Notice that

$$D_1 f(x) = f(x+1) - f(x) = \Delta f(x),$$
$$D_{-1} f(x) = f(x) - f(x-1) = \nabla f(x)$$

are the forward and backward difference operators, respectively, and

$$\lim_{\omega\to 0} D_\omega f(x) = Df(x) = f'(x).$$

If $U = D_\omega$, we define $U^* = D_{-\omega}$. With this definition, we have $\Delta^* = \nabla$ and when $\omega \to 0$ we recover the identity $U^* = D = U$.

The concept of the class of a $D_\omega$-semiclassical linear functional plays a central role in giving a constructive theory of such linear functionals.

**Definition 8.** If $\mathscr{L}$ is a $D_\omega$-semiclassical linear functional, the *class* $s$ of $\mathscr{L}$ is defined by

$$s = \min_{\phi,\psi} \max\{\deg \phi - 2, \deg \psi - 1\},$$

among all polynomials $\phi, \psi$ such that the Pearson equation holds. Notice that the class $s$ is always nonnegative.

For any complex number $c$, we introduce the linear map $\theta_c : \mathbb{P} \to \mathbb{P}$, defined by

$$\theta_c(p)(x) = \frac{p(x) - p(c)}{x - c}.$$

**Theorem 9** [Maroni 1991]. *A regular linear functional $\mathscr{L}$ satisfying the Pearson equation*

$$D_\omega(\phi\mathscr{L}) + \psi\mathscr{L} = 0$$

*is of class $s$ if and only if*

$$\prod_{c\in Z(\phi)} \big(|\psi(c-\omega) + (\theta_c\phi)(c-\omega)| + |\langle \mathscr{L}, \theta_{c-\omega}(\psi + \theta_c\phi)\rangle|\big) > 0,$$

*where $Z(\phi)$ denotes the set of zeros of the polynomial $\phi(x)$.*

*When there exists $c \in Z(\phi)$ such that*

$$\psi(c-\omega) + (\theta_c\phi)(c-\omega) = \langle \mathscr{L}, \theta_{c-\omega}(\psi + \theta_c\phi)\rangle = 0,$$

*the Pearson equation becomes*

$$D_\omega[(\theta_c\phi)\mathscr{L}] + [\theta_{c-\omega}(\psi + \theta_c\phi)]\mathscr{L} = 0.$$

**Remark 10.** When $s = 0$, we obtain the $D_\omega$-classical orthogonal polynomials (see [Abdelkarim and Maroni 1997]). For $\omega = 1$, several characterizations of classical orthogonal polynomials were given in [García et al. 1995]. Indeed, we explain in more detail in the next section the main characteristics of these polynomials and their corresponding linear functionals.

The $D_1$-semiclassical linear functionals have been studied by F. Marcellán and L. Salto [1998] and they are characterized following the same ideas as in the $D$ case. P. Maroni and M. Mejri [2008] deduced the Laguerre–Freud equations for the coefficients of the three-term recurrence relation of $D_w$-semiclassical orthogonal polynomials of class $s = 1$. In the symmetric case, when the moments of odd order vanish, they deduced the explicit values of these coefficients, and the integral representations of the corresponding linear functionals are given.

On the other hand, the Pearson equation yields a difference equation for the moments of the linear functional, and, as a consequence, we get a linear difference equation with polynomial coefficients satisfied by the Stieltjes function associated with the linear functional:

**Theorem 11.** *If $\mathscr{L}$ is a $D_\omega$-semiclassical moment functional, the formal Stieltjes function of $\mathscr{L}$ satisfies the nonhomogeneous first-order linear difference equation*

$$\phi(z)D_\omega S_{\mathscr{L}}(z) = a(z)S_{\mathscr{L}}(z) + b(z),$$

*where $a(z)$ and $b(z)$ are polynomials depending on $\phi$ and $\psi$, with* $\deg a \le s+1$ *and* $\deg b \le s$.

## 3. Discrete semiclassical orthogonal polynomials

We consider linear functionals

$$\langle \mathscr{L}, P\rangle = \sum_{x=0}^{\infty} P(x)\rho(x),$$

for some positive weight function $\rho(x)$ supported on a countable subset of the real line. With this choice, the Pearson equation

$$\langle \Delta(\phi\mathscr{L}) + \psi\mathscr{L}, P\rangle = 0, \quad P \in \mathbb{P},$$

yields

$$\Delta(\phi\rho) + \psi\rho = 0. \tag{2}$$

We rewrite this equation as

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\phi(x)-\psi(x)}{\phi(x+1)} = \frac{\lambda(x)}{\phi(x+1)}, \tag{3}$$

with

$$\phi(x) = x(x+\beta_1)(x+\beta_2)\cdots(x+\beta_r),$$

and

$$\lambda(x) = c(x+\alpha_1)(x+\alpha_2)\cdots(x+\alpha_l).$$

Since the Pochhammer symbol $(\alpha)_x$ defined by $(\alpha)_0 = 1$ and

$$(a)_x = a(a+1)\cdots(a+x-1), \quad x \in \mathbb{N}, \tag{4}$$

satisfies the identity

$$\frac{(\alpha)_{x+1}}{(\alpha)_x} = x+\alpha, \quad x \in \mathbb{N}_0,$$

we obtain

$$\rho(x) = \frac{(\alpha_1)_x\cdots(\alpha_l)_x}{(\beta_1+1)_x\cdots(\beta_r+1)_x}\frac{c^x}{x!}. \tag{5}$$

We will denote the orthogonal polynomials associated with $\rho(x)$ by

$$P_n^{(l,r)}(x;\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_r;c).$$

The moments of the weight function (5) are given by

$$\mu_n = \sum_{x=0}^{\infty} x^n \frac{(\alpha_1)_x\cdots(\alpha_l)_x}{(\beta_1+1)_x\cdots(\beta_r+1)_x}\frac{c^x}{x!}, \quad n = 0,1,\dots.$$

By [Olver et al. 2010, 16.2], they exist if one of the following conditions holds:

(6) $l \leq r$ and $c \in \mathbb{C}$.

(7) $l \geq r+1$, $c \in \mathbb{C}$, and one or more of the top parameters $\alpha_i$ is a nonpositive integer.

(8) $l = r+1$, and $|c| < 1$.

(9) $l = r+1$, $|c| = 1$, and $\operatorname{Re}(\beta_1 + \cdots + \beta_r - \alpha_1 - \cdots - \alpha_l) > 0$.

**3.1.** ***Discrete classical polynomials.*** Let $s = 0$. We solve the Pearson equation (3) with $\deg \psi = 1$ and $1 \leq \deg \phi \leq 2$. Three canonical cases appear (see [Nikiforov et al. 1985]), according to the following table, where $\lambda = \psi + \phi$:

| $\deg \lambda$ | $\deg \phi$ | $\deg \psi$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 1 |
| 2 | 2 | 1 |

**Case 1**: If $\deg \lambda = 0$ and $\deg \phi = 1$, we can take

$$\lambda(x) = c, \quad \phi(x) = x, \quad \psi(x) = \phi(x) - \lambda(x) = x - c, \tag{10}$$

and from (5) we obtain

$$\rho(x) = \frac{c^x}{x!}, \quad c > 0, \quad x \in \mathbb{N}_0. \tag{11}$$

The family of orthogonal polynomials associated with the weight function (11) is known as the Charlier polynomials; we denote them by $P_n^{(0,0)}(x;c)$. They have the hypergeometric representation (see [Koekoek et al. 2010, 9.14.1])

$$P_n^{(0,0)}(x;c) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix}; -\frac{1}{c}\right), \tag{12}$$

where the hypergeometric function ${}_pF_q(z)$ is defined by

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{13}$$

The monic Charlier polynomials $\widehat{P}_n^{(0,0)}(x;c)$ are given by

$$\widehat{P}_n^{(0,0)}(x;c) = (-c)^n P_n^{(0,0)}(x;c). \tag{14}$$

It is usual to denote these polynomials by

$$C_n(x;a) = P_n^{(0,0)}(x;a). \tag{15}$$

**Case 2**: If $\deg \lambda = 1$ and $\deg \phi = 1$, we can take

$$\lambda(x) = c(x+\alpha), \quad \phi(x) = x, \quad \psi(x) = (1-c)x - c\alpha, \tag{16}$$

and from (5) we have

(17) $$\rho(x) = (\alpha)_x \frac{c^x}{x!}, \quad \alpha > 0, \quad 0 < c < 1, \quad x \in \mathbb{N}_0.$$

From (8), the condition $0 < c < 1$ is needed for the moments to exist. The first moment $\mu_0$ is given by

(18) $$\mu_0 = \sum_{x=0}^{\infty} (\alpha)_x \frac{c^x}{x!} = (1-c)^{-\alpha}.$$

The family of orthogonal polynomials associated with the weight function (17) is known as the Meixner polynomials; we denote them by $P_n^{(1,0)}(x;\alpha;c)$. They have the hypergeometric representation (see [Koekoek et al. 2010, 9.10.1])

(19) $$P_n^{(1,0)}(x;\alpha;c) = {}_2F_1\left(\begin{matrix}-n, -x \\ \alpha\end{matrix}; 1-\frac{1}{c}\right),$$

and the monic Meixner polynomials $\widehat{P}_n^{(1,0)}(x;\alpha;c)$ are given by

(20) $$\widehat{P}_n^{(1,0)}(x;\alpha;c) = (\alpha)_n \left(\frac{c}{c-1}\right)^n P_n^{(1,0)}(x;\alpha;c).$$

It is usual to denote these polynomials by

$$M_n(x;\beta,c) = P_n^{(1,0)}(x;\beta;c).$$

If we want $c$ to be unbounded, we can use (7) and set $\alpha = -N$, with $N \in \mathbb{N}$. For the weight function to be positive we need $c < 0$, and we obtain the Krawtchouk polynomials $P_n^{(1,0)}(x;-N;c)$, with

(21) $$\rho(x) = (-N)_x \frac{c^x}{x!}, \quad c < 0, \quad N \in \mathbb{N}, \quad x \in [0, N],$$

and

(22) $$\phi(x) = x, \quad \psi(x) = (1-c)x + cN.$$

It is usual to denote these polynomials by

$$K_n(x;p,N) = P_n^{(1,0)}\left(x;-N;\frac{p}{p-1}\right).$$

**Case 3**: If $\deg\lambda = 2$ and $\deg\phi = 2$, we can take

$$\lambda(x) = c(x+\alpha_1)(x+\alpha_2), \quad \phi(x) = x(x+\beta).$$

Thus,

$$\psi(x) = \phi(x) - \lambda(x) = (1-c)x^2 + x(\beta - c\alpha_1 - c\alpha_2) - c\alpha_1\alpha_2,$$

and since $\deg \psi = 1$, we must have $c = 1$. Hence,

$$\phi(x) = x(x+\beta), \quad \psi(x) = x(\beta - \alpha_1 - \alpha_2) - \alpha_1\alpha_2, \tag{23}$$

and

$$\rho(x) = \frac{(\alpha_1)_x(\alpha_2)_x}{(\beta+1)_x}\frac{1}{x!}, \quad x \in \mathbb{N}_0. \tag{24}$$

From (9), we need $\operatorname{Re}(\beta + 1 - \alpha_1 - \alpha_2) > 0$ for the moments to exist. The first moment $\mu_0$ is given by (see [Olver et al. 2010, 15.4.20])

$$\mu_0 = \sum_{x=0}^{\infty} \frac{(\alpha_1)_x(\alpha_2)_x}{(\beta+1)_x}\frac{1}{x!} = \frac{\Gamma(\beta+1)\Gamma(\beta+1-\alpha_1-\alpha_2)}{\Gamma(\beta+1-\alpha_1)\Gamma(\beta+1-\alpha_2)}.$$

Thus, we need $\alpha_1, \alpha_2 > 0$ and $\beta+1 > \alpha_1+\alpha_2$. The family of orthogonal polynomials associated with the weight function (24) is known as the Hahn polynomials; we denote them by $P_n^{(2,1)}(x;\alpha_1,\alpha_2,\beta;1)$. They have the hypergeometric representation [Erdélyi et al. 1953, 10.23.12]

$$P_n^{(2,1)}(x;\alpha_1,\alpha_2,\beta;1) = {}_3F_2\left(\begin{matrix}-n,\ -x,\ n+\alpha_1+\alpha_2-\beta-1 \\ \alpha_1,\ \alpha_2\end{matrix};\ 1\right). \tag{25}$$

In the literature (see [Koekoek et al. 2010, 9.5.1]), the so-called Hahn polynomials $Q_n(x;\alpha,\gamma,N)$ correspond to the choice $\alpha_1 = \alpha+1,\ \alpha_2 = -N,\ \gamma = -N-\beta-1$, with $N \in \mathbb{N}$.

Another family of Hahn polynomials is

$$h_n(x;\alpha,\beta,N) = P_n^{(2,1)}(x;\beta+1, 1-N, -N-\alpha; 1); \tag{26}$$

see page 34 in [Nikiforov et al. 1985]. In fact two different families of Hahn polynomials are considered in that reference; the polynomials involved in the corresponding Pearson equations are related by negating the variable $x$. Indeed, for the second family we also have a relation

$$\tilde{h}_n(x;\mu,\nu,N) = P_n^{(2,1)}(x;1-N-\nu, 1-N, \mu; 1), \tag{27}$$

as well as

$$h_n(x;\alpha,\beta,N) = \tilde{h}_n(x;-N-\alpha,-N-\beta,N). \tag{28}$$

## 4. Discrete semiclassical polynomials of class one

When $s = 1$, we solve the Pearson equation (2) with $\deg \psi = 2$ and $1 \le \deg \phi \le 3$, and obtain five canonical cases:

| $\deg\lambda$ | $\deg\phi$ | $\deg\psi$ |
|---|---|---|
| 0 | 2 | 2 |
| 1 | 2 | 2 |
| 2 | 1 | 2 |
| 2 | 2 | 2 |
| 3 | 3 | 2 |

**Case 4**: If $\deg\lambda = 0$ and $\deg\phi = 2$, we can take

$$\lambda(x) = c, \quad \phi(x) = x(x+\beta), \quad \psi(x) = x^2 + \beta x - c,$$

and from (5) we have

$$\rho(x) = \frac{1}{(\beta+1)_x}\frac{c^x}{x!}, \quad x \in \mathbb{N}_0, \tag{29}$$

where $\beta > -1$ and $c > 0$. The family of orthogonal polynomials associated with the weight function (29) is known as the generalized Charlier polynomials; we denote them by $P_n^{(0,1)}(x;\beta;c)$ and study them in Section 4.1 below.

**Case 5**: If $\deg\lambda = 1$ and $\deg\phi = 2$, we can take

$$\lambda(x) = c(x+\alpha), \quad \phi(x) = x(x+\beta), \quad \psi(x) = x^2 + (\beta - c)x - c\alpha.$$

From (5), we have

$$\rho(x) = \frac{(\alpha)_x}{(\beta+1)_x}\frac{c^x}{x!}, \quad x \in \mathbb{N}_0, \tag{30}$$

where $\alpha(\beta+1) > 0$ and $c > 0$. The family of orthogonal polynomials associated with the weight function (30) is known as the generalized Meixner polynomials; we denote them by $P_n^{(1,1)}(x;\alpha,\beta;c)$ and study them in Section 4.2.

**Case 6**: If $\deg\lambda = 2$ and $\deg\phi = 1$, we can take

$$\lambda(x) = c(x+\alpha_1)(x+\alpha_2), \quad \phi(x) = x.$$

From (5), we have

$$\rho(x) = (\alpha_1)_x(\alpha_2)_x\frac{c^x}{x!},$$

and from (7) we need $\alpha_2 = -N$, with $N \in \mathbb{N}$, for the moments to exist. Setting $\alpha_1 = \alpha$, we get

$$\lambda(x) = c(x+\alpha)(x-N), \quad \phi(x) = x, \quad \psi(x) = -cx^2 + x(Nc - c\alpha + 1) + Nc\alpha.$$

The family of orthogonal polynomials associated with the weight function

$$\rho(x) = (\alpha)_x(-N)_x\frac{c^x}{x!}, \quad x \in [0,N], \tag{31}$$

with $c<0$ and $\alpha>0$, will be referred to as the *generalized Krawtchouk polynomials*; we will denote them by $P_n^{(2,0)}(x;\alpha,-N;c)$ and study them in Section 4.3.

**Case 7**: If $\deg\lambda=2$ and $\deg\phi=2$, we can take

$$\lambda(x)=c(x+\alpha_1)(x+\alpha_2),\quad \phi(x)=x(x+\beta),$$
$$\psi(x)=(1-c)x^2+(\beta-c\alpha_1-c\alpha_2)x-c\alpha_1\alpha_2.$$

From (5), we have

$$\rho(x)=\frac{(\alpha_1)_x(\alpha_2)_x}{(\beta+1)_x}\frac{c^x}{x!},\quad x\in\mathbb{N}_0, \tag{32}$$

and from (8) we need $0<c<1$, with $\alpha_1\alpha_2(\beta+1)>0$. The family of orthogonal polynomials associated with the weight function (32) will be referred to as the *generalized Hahn polynomials of type I*; we will denote them by $P_n^{(2,1)}(x;\alpha_1,\alpha_2;\beta;c)$ and study them in Section 4.4.

**Case 8**: If $\deg\lambda=3$ and $\deg\phi=3$, we can take

$$\lambda(x)=c(x+\alpha_1)(x+\alpha_2)(x+\alpha_3),\quad \phi(x)=x(x+\beta_1)(x+\beta_2),$$
$$\psi(x)=x(x+\beta_1)(x+\beta_2)-c(x+\alpha_1)(x+\alpha_2)(x+\alpha_3).$$

For $\psi(x)$ to be of second degree we need $c=1$. Thus,

$$\lambda(x)=(x+\alpha_1)(x+\alpha_2)(x+\alpha_3),\quad \phi(x)=x(x+\beta_1)(x+\beta_2),$$
$$\psi(x)=-x^2(\alpha_1+\alpha_2-\beta_1+\alpha_3-\beta_2)-x(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3-\beta_1\beta_2)-\alpha_1\alpha_2\alpha_3,$$

and from (5) we obtain

$$\rho(x)=\frac{(\alpha_1)_x(\alpha_2)_x(\alpha_3)_x}{(\beta_1+1)_x(\beta_2+1)_x}\frac{1}{x!},\quad x\in\mathbb{N}_0. \tag{33}$$

For the moments to exist, (9) gives $\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3>0$, while positivity demands that $\alpha_1\alpha_2\alpha_3(\beta_1+1)(\beta_2+1)>0$. The family of orthogonal polynomials associated with the weight function (33) will be referred to as *generalized Hahn polynomials of type II*; we will denote them by $P_n^{(3,2)}(x;\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;1)$ and study them in Section 4.5.

**4.1.** ***Generalized Charlier polynomials.*** This is Case 4 above, with weight function given by (29) (with $\beta>-1$ and $c>0$). The first moments are

$$\mu_0=c^{-\frac{\beta}{2}}I_\beta(2\sqrt{c})\Gamma(\beta+1),\quad \mu_1=c^{\frac{1-\beta}{2}}I_{\beta+1}(2\sqrt{c})\Gamma(\beta+1),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind [Olver et al. 2010, 10.25.2].

Hounkonnou, Hounga and Ronveaux studied the semiclassical polynomials associated with the weight function

$$\rho_r(x) = \frac{c^x}{(x!)^r}, \quad r = 0, 1, \dots \tag{34}$$

(see [Hounkonnou et al. 2000]). For $r = 2$, they derived the Laguerre–Freud equations for the recurrence coefficients and a second-order difference equation. Note that from (34) we have

$$\frac{\rho_r(x+1)}{\rho_r(x)} = \frac{c}{(x+1)^r},$$

and from (3) we conclude that

$$\lambda_r(x) = c, \quad \phi_r(x) = x^r, \quad \psi_r(x) = x^r - c,$$

and therefore the orthogonal polynomials associated with $\rho_r(x)$ are of class $r-1$. The case $r = 2$ is a particular example of (29) with $\beta = 0$.

Van Assche and Foupouagnigni [2003] also considered (34) with $r = 2$. Simplifying the Laguerre–Freud equations from [Hounkonnou et al. 2000], they got

$$u_{n+1} + u_{n-1} = \frac{1}{\sqrt{c}} \frac{n u_n}{1 - u_n^2} \quad \text{and} \quad v_n = \sqrt{c}\, u_{n+1} u_n,$$

with $\gamma_n = c(1-u_n^2)$ and $\beta_n = v_n + n$. They showed that these equations are related to the discrete Painlevé II equation $\mathrm{dP_{II}}$ [Van Assche 2007]

$$x_{n+1} + x_{n-1} = \frac{(an+b)x_n + c}{1 - x_n^2}.$$

They also obtained the asymptotic behavior

$$\lim_{n\to\infty} \gamma_n = c, \quad \lim_{n\to\infty} v_n = 0,$$

and concluded that the asymptotic zero distribution is given by the uniform distribution on [0, 1], as is the case for the usual Charlier polynomials [Kuijlaars and Van Assche 1999].

Smet and Van Assche [2012] studied the orthogonal polynomials associated with the weight function (29). They obtained the Laguerre–Freud equations

$$(a_{n+1}^2 - c)(a_n^2 - c) = c(b_n - n)(b_n - n + \beta), \quad b_n + b_{n-1} = n - 1 - \beta + \frac{cn}{a_n^2}, \tag{35}$$

for the orthonormal polynomials. They showed that these equations are a limiting case of the discrete Painlevé IV equation $dP_{IV}$ [Van Assche 2007]

$$x_{n+1}x_n = \frac{(y_n - \delta n - E)^2 - A}{y_n^2 - B},$$
$$y_n + y_{n-1} = \frac{\delta n + E - \delta/2 - C}{1 + Dx_n} + \frac{\delta n + E - \delta/2 + C}{1 + x_n/D}.$$

Finally, Filipuk and Van Assche [2013] related the system (35) to the (continuous) fifth Painlevé equation $P_V$.

**4.2. *Generalized Meixner polynomials.*** This is Case 5 above, and the weight function is given by (30), with $\alpha(\beta+1) > 0$ and $c > 0$. The first moments are

$$\mu_0 = M(\alpha, \beta+1; c), \quad \mu_1 = \frac{\alpha c}{\beta+1} M(\alpha+1, \beta+2; c),$$

where $M(a, b; z)$ is the confluent hypergeometric function [Olver et al. 2010, 13.2.2].

Ronveaux [1986] considered the semiclassical polynomials associated with the weight function

$$\rho_r(x) = \prod_{j=1}^{r} (\alpha_j)_x \frac{c^x}{(x!)^r}, \quad r = 1, 2, \dots,$$

and in [Ronveaux 2001] he made some conjectures on the asymptotic behavior of the recurrence coefficients.

Smet and Van Assche [2012] studied the orthogonal polynomials associated with the weight function (30). They obtained the Laguerre–Freud equations

$$\begin{aligned}(u_n + v_n)(u_{n+1} + v_n) &= \frac{\alpha-1}{c^2} v_n (v_n - c)\left(v_n - c\frac{\alpha-1-\beta}{\alpha-1}\right),\\ (u_n + v_n)(u_{n+1} + v_{n-1}) &= \frac{u_n}{u_n - \frac{cn}{\alpha-1}} (u_n + c)\left(u_n + c\frac{\alpha-1-\beta}{\alpha-1}\right),\end{aligned} \tag{36}$$

for the orthonormal polynomials, with

$$a_n^2 = cn - (\alpha-1)u_n, \quad b_n = n + \alpha + c - \beta - 1 - \frac{\alpha-1}{c} v_n.$$

They also proved that the system (36) is a limiting case of the asymmetric discrete Painlevé IV equation $\alpha$-$dP_{IV}$ [Van Assche 2007].

Filipuk and Van Assche [2011] showed that the system (36) can be obtained from the Bäcklund transformation of the fifth Painlevé equation $P_V$. The particular case of (30) when $\beta = 0$ was considered by Boelen, Filipuk, and Van Assche [Boelen et al. 2011].

If we set $\alpha = -N$, $N \in \mathbb{N}$, in (30), we obtain

$$\rho(x) = \frac{(-N)_x}{(\beta+1)_x} \frac{c^x}{x!},$$

where we now have $\beta > -1$ and $c < 0$. This case was analyzed in [Boelen et al. 2013].

*Singular limits.* If we let $\alpha \to 0$ and $\beta \to -1$ in (30), we have $\rho(x) \to \tilde{\rho}(x)$, where $\tilde{\rho}(x)$ is a new weight function satisfying the Pearson equation

$$\Delta[(x-1)x\tilde{\rho}] + [x-(c+1)]x\tilde{\rho} = 0. \tag{37}$$

Assuming $\tilde{\rho}$ satisfies $x\tilde{\rho}(x) = xu(x)$ for some weight function $u(x)$ we get

$$\Delta[(x-1)xu] + [x-(c+1)]xu = 0. \tag{38}$$

Using the product rule

$$\Delta(fg) = f\Delta g + g\Delta f + \Delta f \Delta g \tag{39}$$

in (38), we have

$$xu + (x-1)\Delta(xu) + \Delta(xu) + [x-(c+1)]xu = 0,$$

or

$$x\Delta(xu) + [x-(c+1)+1]xu = 0.$$

Dividing by $x$, we obtain

$$\Delta(xu) + (x-c)u = 0.$$

Comparing with (10), we see that $u(x)$ is the weight function corresponding to the Charlier polynomials (11), and therefore (37) implies that

$$\tilde{\rho}(x) = \frac{c^x}{x!} + M\delta(x), \tag{40}$$

where $\delta(x)$ is the Dirac delta function.

The orthogonal polynomials $P_n^{(1,1)}(x; 0, -1; c)$ associated with the weight function (40) were first studied by Chihara [1985]. He showed that they satisfy the three-term recurrence relation (1) with

$$b_n = c\frac{n}{n+1}\frac{D_n}{D_{n+1}} + (n+1)\frac{D_{n+1}}{D_n}, \qquad \gamma_n = c\frac{n^2}{n+1}\frac{D_n^2}{D_{n-1}D_{n+1}},$$

where

$$D_n = \frac{c^n}{n!}\frac{M}{e^c + MK_{n-1}}, \qquad K_n = \sum_{j=0}^{n} \frac{c^j}{j!}, \qquad K_{-1} = 0.$$

Note that for $D_n$ to be well-defined for all $n$, we need $M > -1$, since $K_n \nearrow e^c$.

Bavinck and Koekoek [1995] obtained a difference equation satisfied by these polynomials and Álvarez-Nodarse, García, and Marcellán [Álvarez et al. 1995] found the hypergeometric representation

$$P_n^{(1,1)}(x;\,0;\,-1;\,c) = (-c)^n\,{}_3F_1\left(\begin{matrix}-n,\,-x,\,1+x/D_n\\ x/D_n\end{matrix};\,-\frac{1}{c}\right).$$

Since $\lim_{z\to\infty}\dfrac{(1+z)_x}{(z)_x}=1$, we see that

$$\lim_{M\to 0} P_n^{(1,1)}(x;0,-1;c) = \widehat{C}_n(x;c),$$

where $\widehat{C}_n(x;c)$ is the monic Charlier polynomial (14).

**4.3.** ***Generalized Krawtchouk polynomials.*** This is Case 6 above, and the weight function is given by (31), with $c<0$, $N\in\mathbb{N}$, and $\alpha>0$. The first moments are

$$\mu_0 = C_N\Big(-\alpha;-\frac{1}{c}\Big),\qquad \mu_1 = -c\alpha N C_{N-1}\Big(-\alpha-1;-\frac{1}{c}\Big),$$

where $C_n(x;a)$ is the Charlier polynomial (15).

To our knowledge, these polynomials have not appeared before in the literature.

**4.4.** ***Generalized Hahn polynomials of type I.*** This is Case 7 above, and the weight function is given by (32), with $0<c<1$ and $\alpha_1\alpha_2(\beta+1)>0$. The first moments are

$$\mu_0 = {}_2F_1\left(\begin{matrix}\alpha_1,\,\alpha_2\\ \beta+1\end{matrix};\,c\right),\qquad \mu_1 = c\frac{\alpha_1\alpha_2}{\beta+1}\,{}_2F_1\left(\begin{matrix}\alpha_1+1,\,\alpha_2+1\\ \beta+2\end{matrix};\,c\right),$$

where ${}_2F_1\left(\begin{matrix}a,\,b\\ c\end{matrix};\,z\right)$ is the hypergeometric function.

*Singular limits.* (a) If we let $\alpha_2\to 0,\beta\to -1$ and $\alpha_1=\alpha$ in (32), we have $\rho(x)\to\tilde{\rho}(x)$, where $\tilde{\rho}(x)$ is a new weight function satisfying the Pearson equation

$$\Delta[(x-1)x\tilde{\rho}] + [(1-c)x-(1+c\alpha)]x\tilde{\rho} = 0. \tag{41}$$

Assuming that $\tilde{\rho}(x)$ satisfies $x\tilde{\rho}(x)=xu(x)$ for some weight function $u(x)$, we get

$$\Delta[(x-1)xu] + [(1-c)x-(1+c\alpha)]xu = 0. \tag{42}$$

Using the product rule (39) in (42), we have

$$xu+(x-1)\Delta(xu)+\Delta(xu)+[(1-c)x-(1+c\alpha)]xu=0,$$

or

$$x\Delta(xu)+[(1-c)x-(1+c\alpha)+1]xu=0.$$

Dividing by $x$, we obtain $\Delta(xu)+[(1-c)x-c\alpha]u=0$. Comparing with (16), we see that $u(x)$ is the weight function corresponding to the Meixner polynomials (17), and therefore (37) implies that

$$\tilde{\rho}(x)=(\alpha)_x\frac{c^x}{x!}+M\delta(x). \tag{43}$$

The orthogonal polynomials associated with the weight function (43) were first studied by Chihara [1985]. He showed that they satisfy the three-term recurrence relation (1) with

$$b_n=\frac{c(\alpha+n)}{c-1}\frac{n}{n+1}\frac{B_n}{B_{n+1}}+\frac{n+1}{c-1}\frac{B_{n+1}}{B_n},\qquad \gamma_n=\frac{c}{(c-1)^2}\frac{n^2(\alpha+n)}{n+1}\frac{B_n^2}{B_{n-1}B_{n+1}},$$

where

$$B_n=\frac{c^n(\alpha)_n}{(1-c)n!}\frac{M}{(1-c)^{-\alpha}+MK_{n-1}},\qquad K_n=\sum_{j=0}^{n}(\alpha)_j\frac{c^j}{j!},\qquad K_{-1}=0.$$

For $B_n$ to be well-defined for all $n$, we need $M>-1$, since $K_n\nearrow(1-c)^{-\alpha}$.

In [Brezinski et al. 1991], Richard Askey proposed the problem of finding a second-order difference equation satisfied by these polynomials. The problem was solved in [Bavinck and van Haeringen 1994]; in [Álvarez et al. 1995] the hypergeometric representation

$$P_n^{(2,1)}(x;\alpha,0,-1;c)=(\alpha)_n\left(\frac{c}{c-1}\right)^n{}_3F_2\left(\begin{matrix}-n,\ -x,\ 1+x/B_n\\ \alpha,\ x/B_n\end{matrix};\ 1-\frac{1}{c}\right)$$

was given. In this case,

$$\lim_{M\to 0}P_n^{(2,1)}(x;\alpha,0,-1;c)=\widehat{M}_n(x;\alpha,c),$$

where $\widehat{M}_n(x;\alpha,c)$ is the monic Meixner polynomial (20).

(b) If $\alpha_1=-N$, $N\in\mathbb{N}$, we can remove the restriction that $0<c<1$ and take any $c<0$, with $\alpha_2\notin[-N,0]$, $\beta\notin[-N-1,-1]$, and $\alpha_2(\beta+1)>0$. If we let $\alpha_2\to-(N-1)$ and $\beta\to-N$, we have $\rho(x)\to\tilde{\rho}(x)$, where $\tilde{\rho}(x)$ is a new weight function satisfying the Pearson equation

$$\begin{aligned}&\psi(x)=(1-c)x^2+(\beta-c\alpha_1-c\alpha_2)x-c\alpha_1\alpha_2,\\ &\Delta[x(x-N)\tilde{\rho}]+[(1-c)x+c(N-1)](x-N)\tilde{\rho}=0.\end{aligned}\tag{44}$$

Assuming that $\tilde{\rho}(x)$ satisfies $(x-N)\tilde{\rho}(x)=(x-N)u(x)$ for some weight function $u(x)$, we get

$$\Delta[x(x-N)u]+[(1-c)x+c(N-1)](x-N)u=0. \tag{45}$$

Using the product rule (39) in (45), we have

$$xu + (x - N + 1)\Delta(xu) + [(1-c)x + c(N-1)](x-N)u = 0,$$

or

$$(x - N + 1)\Delta(xu) + (x - N + 1)(x + Nc - cx)u = 0.$$

Dividing by $x - N + 1$, we obtain $\Delta(xu) + [(1-c)x + cN]u = 0$. Comparing with (22), we see that $u(x)$ is the weight function corresponding to the Krawtchouk polynomials (21), and therefore (44) implies that

$$\tilde{\rho}(x) = (-N)_x \frac{c^x}{x!} + M\delta(x - N).$$

**4.5.** ***Generalized Hahn polynomials of type II.*** This is Case 8, and the weight function is (33), with $\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3 > 0$ and $\alpha_1\alpha_2\alpha_3(\beta_1+1)(\beta_2+1) > 0$. The first moments are

$$\mu_0 = {}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3\\ \beta_1+1, \beta_2+1\end{matrix}; 1\right),$$

$$\mu_1 = \frac{\alpha_1\alpha_2\alpha_3}{(\beta_1+1)(\beta_2+1)}\, {}_3F_2\left(\begin{matrix}\alpha_1+1, \alpha_2+1, \alpha_3+1\\ \beta_1+2, \beta_2+2\end{matrix}; 1\right),$$

where ${}_3F_2\left(\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix}; z\right)$ is the hypergeometric function.

To our knowledge, these polynomials have not appeared before in the literature.

*Singular limits.* (a) If we let $\alpha_3 \to 0$, $\beta_2 \to -1$, $\beta_1 = \beta$ in (33), we have $\rho(x) \to \tilde{\rho}(x)$, where $\tilde{\rho}(x)$ is a new weight function satisfying the Pearson equation

$$\Delta[(x-1)(x+\beta)x\tilde{\rho}] + [(\beta - 1 - \alpha_1 - \alpha_2)x - \alpha_1\alpha_2 - \beta]x\tilde{\rho} = 0. \tag{46}$$

Assuming that $\tilde{\rho}(x)$ satisfies $x\tilde{\rho}(x) = xu(x)$ for some weight function $u(x)$, we get

$$\Delta[(x-1)(x+\beta)xu] + [(\beta - 1 - \alpha_1 - \alpha_2)x - \alpha_1\alpha_2 - \beta]xu = 0. \tag{47}$$

Using the product rule (39) in (47), we have

$$(x+\beta)xu + x\Delta[(x+\beta)xu] + [(\beta - 1 - \alpha_1 - \alpha_2)x - \alpha_1\alpha_2 - \beta]xu = 0,$$

or

$$x\Delta[(x+\beta)xu] + [(\beta - \alpha_1 - \alpha_2)x - \alpha_1\alpha_2]xu = 0.$$

Dividing by $x$, we obtain

$$\Delta[(x+\beta)xu] + [(\beta - \alpha_1 - \alpha_2)x - \alpha_1\alpha_2]u = 0.$$

Comparing with (23), we see that $u(x)$ is the weight function corresponding to the Hahn polynomials (24); therefore (46) implies that

$$\tilde{\rho}(x) = \frac{(\alpha_1)_x(\alpha_2)_x}{(\beta+1)_x}\frac{1}{x!} + M\delta(x). \tag{48}$$

(b) Similarly, if we let $\alpha_3 = -N, \alpha_2 \to -(N-1), \beta_2 \to -N$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\alpha(\beta+1) < 0$ in (33), we have $\rho(x) \to \tilde{\rho}(x)$, where $\tilde{\rho}(x)$ is a new weight function satisfying the Pearson equation

$$\Delta[x(x+\beta)(x-N)\tilde{\rho}] + [(\beta-\alpha+N-1)x + \alpha(N-1)](x-N)\tilde{\rho} = 0. \tag{49}$$

Assuming that $\tilde{\rho}(x)$ satisfies $(x-N)\tilde{\rho}(x) = (x-N)u(x)$ for some weight function $u(x)$, we get

$$\Delta[x(x+\beta)(x-N)u] + [(\beta-\alpha+N-1)x + \alpha(N-1)](x-N)u = 0. \tag{50}$$

Using the product rule (39) in (50), we have

$$(x+\beta)xu + (x-N+1)\Delta[(x+\beta)xu] \\ + [(\beta-\alpha+N-1)x + \alpha(N-1)](x-N)u = 0,$$

or

$$(x-N+1)\Delta[(x+\beta)xu] + (x-N+1)[(\beta-\alpha+N)x + \alpha N]u = 0.$$

Dividing by $x-N+1$, we obtain

$$\Delta[(x+\beta)xu] + [(\beta-\alpha+N)x + \alpha N]u = 0.$$

Comparing with (23), we see that $u(x)$ is the weight function corresponding to the truncated Hahn polynomials (26), and therefore (49) implies that

$$\tilde{\rho}(x) = \frac{(\alpha)_x(-N)_x}{(\beta+1)_x}\frac{1}{x!} + M\delta(x-N). \tag{51}$$

The orthogonal polynomials associated with the weight functions (48) and (51) were first studied in [Álvarez and Marcellán 1995b].

## 5. Limit relations between polynomials

From the identities (see [Koekoek et al. 2010])

$$\lim_{\lambda\to\infty}\frac{(\lambda\alpha)_x}{\lambda^x} = \alpha^x \quad \text{and} \quad \lim_{\lambda\to\infty}\frac{(\lambda\alpha)_x}{(\lambda\beta)_x} = \left(\frac{\alpha}{\beta}\right)^x,$$

we have the following limit relations:

1. generalized Hahn polynomials of type II to generalized Hahn polynomials of type I
$$\lim_{\alpha\to\infty} P_n^{(3,2)}(x;\alpha_1,\alpha_2,\alpha,\beta,\alpha/c;1) = P_n^{(2,1)}(x;\alpha_1,\alpha_2,\beta;c),$$
2. generalized Hahn polynomials of type I to generalized Krawtchouk polynomials
$$\lim_{\beta\to\infty} P_n^{(2,1)}(x;\alpha,-N,\beta;c\beta) = P_n^{(2,0)}(x;\alpha,-N;c),$$
3. generalized Hahn polynomials of type I to generalized Meixner polynomials
$$\lim_{\alpha_2\to\infty} P_n^{(2,1)}(x;\alpha,\alpha_2,\beta;c/\alpha_2) = P_n^{(1,1)}(x;\alpha,\beta;c),$$
4. generalized Meixner polynomials to generalized Charlier polynomials
$$\lim_{\alpha\to\infty} P_n^{(1,1)}(x;\alpha,\beta;c/\alpha) = P_n^{(0,1)}(x;\beta;c),$$
5. generalized Meixner polynomials to Meixner polynomials
$$\lim_{\beta\to\infty} P_n^{(1,1)}(x;\alpha,\beta;c\beta) = M_n(x;\alpha;c),$$
6. generalized Charlier polynomials to Charlier polynomials
$$\lim_{\beta\to\infty} P_n^{(0,1)}(x;\beta;c\beta) = C_n(x;c).$$

We also have the following singular limits, where "$\oplus\,\delta(x-x_0)$" denotes the addition of a delta function to the measure of orthogonality at the point $x_0$:

1. generalized Meixner polynomials to Charlier-Dirac polynomials
$$\lim_{\substack{\alpha\to 0\\ \beta\to -1}} P_n^{(1,1)}(x;\alpha,\beta;c) = C_n(x;c)\oplus\delta(x),$$
2. generalized Hahn polynomials of type I to truncated Hahn polynomials
$$\lim_{\substack{\alpha_2\to -N\\ c\to 1}} P_n^{(2,1)}(x;\alpha,\alpha_2,\beta;c) = Q_n(x;\alpha,\beta,N),$$
3. generalized Hahn polynomials of type I to Meixner–Dirac polynomials
$$\lim_{\substack{\alpha_2\to 0\\ \beta\to -1}} P_n^{(2,1)}(x;\alpha,\alpha_2,\beta;c) = M_n(x;\alpha;c)\oplus\delta(x),$$
4. generalized Hahn polynomials of type I to Krawtchouk–Dirac polynomials
$$\lim_{\substack{\alpha_2\to -N+1\\ \beta\to -N}} P_n^{(2,1)}(x;-N,\alpha_2,\beta;c) = K_n(x;-N;c)\oplus\delta(x-N),$$
5. generalized Hahn polynomials of type II to Hahn–Dirac polynomials
$$\lim_{\substack{\alpha_2\to 0\\ \beta_2\to -1}} P_n^{(3,2)}(x;\alpha,\alpha_2,-N,\beta,\beta_2;1) = Q_n(x;\alpha,\beta,N)\oplus\delta(x),$$

6. generalized Hahn polynomials of type II to Hahn–Dirac polynomials

$$\lim_{\substack{\alpha_2 \to -N+1 \\ \beta_2 \to -N}} P_n^{(3,2)}(x;\alpha,\alpha_2,-N,\beta,\beta_2;1) = Q_n(x;\alpha,\beta,N) \oplus \delta(x-N).$$

We can summarize these results in the following scheme:

$$\begin{tikzcd}
P_n^{(3,2)} \arrow[r] \arrow[d] & \text{Hahn} \oplus \delta(x) \arrow[d] & \text{Hahn} \oplus \delta(x-N) \arrow[d] & \\
P_n^{(2,1)} \arrow[r] \arrow[d] \arrow[dr] & \text{Meixner} \oplus \delta(x) \arrow[dr] & \text{Krawtchouk} \oplus \delta(x-N) \arrow[d] & \\
P_n^{(2,0)} & P_n^{(1,1)} \arrow[r] \arrow[d] \arrow[dr] & \text{Charlier} \oplus \delta(x) & \text{Hahn} \arrow[dl] \\
 & P_n^{(0,1)} \arrow[dr] & \text{Meixner–Krawtchouk} \arrow[d] & \\
 & & \text{Charlier} &
\end{tikzcd}$$

## 6. Concluding remarks

We have described the discrete semiclassical orthogonal polynomials of class $s = 1$ using the different choices for the polynomials in the canonical Pearson equation that the corresponding linear functional satisfies. We have focused our attention to the case where the linear functional has a representation in terms of a discrete positive measure supported on a countable subset of the real line. Some new families of orthogonal polynomials appear, as well as some families of orthogonal polynomials (generalized Charlier, generalized Krawtchouk, and generalized Meixner) which have attracted the interest of researchers in the last years, since the coefficients of their three-term recurrence relations are related to discrete and continuous Painlevé equations. We have also studied limit relations between these families of orthogonal polynomials, having in mind an analogue of the Askey tableau for classical orthogonal polynomials. It would be very interesting to find the equations satisfied by the coefficients of the three-term recurrence relations for the above new sequences of semiclassical orthogonal polynomials. Furthermore, an analysis of the class $s = 2$ will also be welcome in order to get a complete classification of such a class as well as to check if new families of orthogonal polynomials appear as in the case of the $D$-semiclassical orthogonal polynomials pointed out in [Marcellán et al. 2012].

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## References

[Abdelkarim and Maroni 1997] F. Abdelkarim and P. Maroni, "The $D_\omega$-classical orthogonal polynomials", *Results Math.* **32**:1-2 (1997), 1–28. MR 99b:33009 Zbl 0889.42020

[Álvarez and Marcellán 1995a] R. Álvarez-Nodarse and F. Marcellán, "Difference equation for modifications of Meixner polynomials", *J. Math. Anal. Appl.* **194**:1 (1995), 250–258. MR 96k:33009 Zbl 0834.39005

[Álvarez and Marcellán 1995b] R. Álvarez-Nodarse and F. Marcellán, "The modification of classical Hahn polynomials of a discrete variable", *Integral Transform. Spec. Funct.* **3**:4 (1995), 243–262. MR 97j:33010 Zbl 0849.33007

[Álvarez et al. 1995] R. Álvarez-Nodarse, A. G. García, and F. Marcellán, "On the properties for modifications of classical orthogonal polynomials of discrete variables", *J. Comput. Appl. Math.* **65**:1-3 (1995), 3–18. MR 98b:33018 Zbl 0865.42023

[Bavinck and Koekoek 1995] H. Bavinck and R. Koekoek, "On a difference equation for generalizations of Charlier polynomials", *J. Approx. Theory* **81**:2 (1995), 195–206. MR 96h:33002 Zbl 0865.33006

[Bavinck and van Haeringen 1994] H. Bavinck and H. van Haeringen, "Difference equations for generalized Meixner polynomials", *J. Math. Anal. Appl.* **184**:3 (1994), 453–463. MR 95h:33004 Zbl 0824.33005

[Belmehdi 1992] S. Belmehdi, "On semi-classical linear functionals of class $s = 1$: classification and integral representations", *Indag. Math.* (*N.S.*) **3**:3 (1992), 253–275. MR 94e:33038 Zbl 0783.33003

[Boelen et al. 2011] L. Boelen, G. Filipuk, and W. Van Assche, "Recurrence coefficients of generalized Meixner polynomials and Painlevé equations", *J. Phys. A* **44**:3 (2011), Article ID #035202. MR 2011j:33036 Zbl 1217.34135

[Boelen et al. 2013] L. Boelen, G. Filipuk, C. Smet, W. Van Assche, and L. Zhang, "The generalized Krawtchouk polynomials and the fifth Painlevé equation", *J. Differ. Equations Appl.* **19**:9 (2013), 1437–1451. MR 3173496 Zbl 1279.33013

[Brezinski et al. 1991] C. Brezinski, L. Gori, and A. Ronveaux (editors), *Orthogonal polynomials and their applications* (Erice, 1990), IMACS Ann. Comput. Appl. Math. **9**, J. C. Baltzer, Basel, 1991. MR 94m:42003 Zbl 0812.00027

[Chihara 1978] T. S. Chihara, *An introduction to orthogonal polynomials*, Mathematics and its Applications **13**, Gordon and Breach, New York, 1978. MR 58 #1979 Zbl 0389.33008

[Chihara 1985] T. S. Chihara, "Orthogonal polynomials and measures with end point masses", *Rocky Mountain J. Math.* **15**:3 (1985), 705–719. MR 87a:42037 Zbl 0586.33007

[Erdélyi et al. 1953] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*, vol. II, McGraw-Hill, New York, 1953. MR 15,419i Zbl 0052.29502

[Filipuk and Van Assche 2011] G. Filipuk and W. Van Assche, "Recurrence coefficients of a new generalization of the Meixner polynomials", *SIGMA Symmetry Integrability Geom. Methods Appl.* **7** (2011), Paper 068. MR 2861208 Zbl 1250.33015

[Filipuk and Van Assche 2013] G. Filipuk and W. Van Assche, "Recurrence coefficients of generalized Charlier polynomials and the fifth Painlevé equation", *Proc. Amer. Math. Soc.* **141**:2 (2013), 551–562. MR 2996960 Zbl 1269.34095

[Foupouagnigni et al. 1998] M. Foupouagnigni, M. N. Hounkonnou, and A. Ronveaux, "Laguerre–Freud equations for the recurrence coefficients of $D_\omega$ semi-classical orthogonal polynomials of class one", *J. Comput. Appl. Math.* **99**:1-2 (1998), 143–154. MR 2000e:33008 Zbl 0931.33009

[García et al. 1995] A. G. García, F. Marcellán, and L. Salto, "A distributional study of discrete classical orthogonal polynomials", *J. Comput. Appl. Math.* **57**:1-2 (1995), 147–162. MR 96m:33004 Zbl 0853.33009

[Godoy et al. 1997] E. Godoy, F. Marcellán, L. Salto, and A. Zarzo, "Perturbations of discrete semiclassical functionals by Dirac masses", *Integral Transform. Spec. Funct.* **5**:1-2 (1997), 19–46. MR 98f:33015 Zbl 0877.33002

[Hounkonnou et al. 2000] M. N. Hounkonnou, C. Hounga, and A. Ronveaux, "Discrete semi-classical orthogonal polynomials: generalized Charlier", *J. Comput. Appl. Math.* **114**:2 (2000), 361–366. MR 2001m:33010 Zbl 0971.42015

[Koekoek et al. 2010] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer, Berlin, 2010. MR 2011e:33029 Zbl 1200.33012

[Kuijlaars and Van Assche 1999] A. B. J. Kuijlaars and W. Van Assche, "Extremal polynomials on discrete sets", *Proc. London Math. Soc.* (3) **79**:1 (1999), 191–221. MR 2000g:42033 Zbl 1033.42024

[Marcellán and Salto 1998] F. Marcellán and L. Salto, "Discrete semi-classical orthogonal polynomials", *J. Differ. Equations Appl.* **4**:5 (1998), 463–496. MR 2000e:33010 Zbl 0916.33006

[Marcellán et al. 2012] F. Marcellán, M. Sghaier, and M. Zaatra, "On semiclassical linear functionals of class $s = 2$: classification and integral representations", *J. Differ. Equations Appl.* **18**:6 (2012), 973–1000. MR 2929632 Zbl 1250.42081

[Maroni 1991] P. Maroni, "Une théorie algébrique des polynômes orthogonaux: application aux polynômes orthogonaux semi-classiques", pp. 95–130 in *Orthogonal polynomials and their applications* (Erice, 1990), edited by C. Brezinski et al., IMACS Ann. Comput. Appl. Math. **9**, Baltzer, Basel, 1991. MR 95i:42018 Zbl 0944.33500

[Maroni and Mejri 2008] P. Maroni and M. Mejri, "The symmetric $D_\omega$-semi-classical orthogonal polynomials of class one", *Numer. Algorithms* **49**:1-4 (2008), 251–282. MR 2010a:42100 Zbl 1181.42028

[Nikiforov et al. 1985] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Классические ортогональные полиномы дискретной переменной*, Nauka, Moscow, 1985. Translated as *Classical orthogonal polynomials of a discrete variable*, Springer, Berlin, 1991. MR 92m:33019 Zbl 0576.33001

[Olver et al. 2010] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (editors), *NIST handbook of mathematical functions*, Cambridge University Press, 2010. MR 2012a:33001 Zbl 1198.00002

[Ronveaux 1986] A. Ronveaux, "Discrete semi-classical orthogonal polynomials: generalized Meixner", *J. Approx. Theory* **46**:4 (1986), 403–407. MR 87f:33030 Zbl 0591.42015

[Ronveaux 2001] A. Ronveaux, "Asymptotics for recurrence coefficients in the generalized Meixner case", *J. Comput. Appl. Math.* **133**:1–2 (2001), 695–696.

[Ronveaux and Salto 2001] A. Ronveaux and L. Salto, "Discrete orthogonal polynomials: polynomial modification of a classical functional", *J. Differ. Equations Appl.* **7**:3 (2001), 323–344. MR 2004d:33010 Zbl 0991.33002

[Smet and Van Assche 2012] C. Smet and W. Van Assche, "Orthogonal polynomials on a bi-lattice", *Constr. Approx.* **36**:2 (2012), 215–242. MR 2957309 Zbl 1262.34103

[Van Assche 2007] W. Van Assche, "Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials", pp. 687–725 in *Difference equations, special functions and orthogonal polynomials* (Munich, 2005), edited by S. Elaydi et al., World Scientific, Hackensack, NJ, 2007. MR 2009k:42055 Zbl 1128.39013

[Van Assche and Foupouagnigni 2003] W. Van Assche and M. Foupouagnigni, "Analysis of non-linear recurrence relations for the recurrence coefficients of generalized Charlier polynomials", *J. Nonlinear Math. Phys.* **10**:Suppl. 2 (2003), 231–237. MR 2005f:33020

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