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ON REPRESENTATIONS OF $\mathrm{GL}_{2n}(F)$ WITH A SYMPLECTIC PERIOD

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Given a nonarchimedean local field F, we classify the irreducible admissible representations of $\mathrm{GL}_4(F)$ and $\mathrm{GL}_6(F)$ that bear a nontrivial linear form invariant under the groups $\mathrm{Sp}_2(F)$ and $\mathrm{Sp}_3(F)$, respectively. We propose a few conjectures for the case of $\mathrm{GL}_{2n}(F)$, $n > 3$.

1. Introduction

Let $G = \mathrm{GL}_{2n}(F)$ for F a nonarchimedean local field of characteristic 0 and let H be a symplectic subgroup of G of rank n. A representation π of G is said to have a symplectic period (or to be H-distinguished) if $\mathrm{Hom}_H(\pi|_H, \mathbb{C}) \neq 0$. give a complete list of irreducible admissible representations of $\mathrm{GL}_4(F)$ and $\mathrm{GL}_6(F)$ having a symplectic period. We also make a few conjectural statements for $\mathrm{GL}_{2n}(F)$ at the end.

The motivation for this problem comes from the work of Klyachko [1983] in the case of finite fields. He found a set of representations generalizing the Gelfand–Graev model, after which Heumos and Rallis [1990] studied the analogous notion in the p-adic case. They also proved multiplicity-one theorems in the symplectic case.

Continuing this line of investigation, Offen and Sayag [2007a; 2007b; 2008] proved the uniqueness property of the Klyachko models and multiplicity-one results for irreducible admissible representations. They also showed the existence of the Klyachko model for unitary representations. To state the results precisely we need to introduce notation.

Let δ be a square integrable representation of $\mathrm{GL}_r(F)$. Denote by $U(\delta, m)$ the unique irreducible quotient of the representation,

$$\nu^{(m-1)/2}\delta \times \nu^{(m-3)/2}\delta \times \cdots \times \nu^{-(m-1)/2}\delta.$$

Proposition 1.1 [Offen and Sayag 2007a]. *For $i = 1, \ldots, t$, let δ_i be square-integrable representations of $\mathrm{GL}_{r_i}(F)$ and m_i be positive integers. Let χ_i be a*

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character of $\mathrm{GL}_{2m_i r_i}(F)$. *Then the representation*

$$\chi_1 U(\delta_1, 2m_1) \times \cdots \times \chi_t U(\delta_t, 2m_t)$$

has a symplectic period.

Further define

$$\mathcal{B} = \{U(\delta, 2m),\ \nu^{\alpha} U(\delta, 2m) \times \nu^{-\alpha} U(\delta, 2m)\},$$

where δ varies over the discrete series representations and $\alpha \in \mathbb{R}$ such that $|\alpha| < \frac{1}{2}$.

Theorem 1.2 [Offen and Sayag 2007b]. *Let* $\pi = \tau_1 \times \cdots \times \tau_r$ *such that* $\tau_i \in \mathcal{B}$. *Then* π *has a symplectic period. Conversely, if* π *is an irreducible unitary representation with a symplectic period, there exist* $\tau_1, \ldots, \tau_r \in \mathcal{B}$ *such that* $\pi = \tau_1 \times \cdots \times \tau_r$.

A natural question now is to classify all irreducible admissible representations that admit a symplectic model. For $\mathrm{GL}_4(F)$ and $\mathrm{GL}_6(F)$ we have:

Theorem 1.3. *Using the notation introduced for Proposition 1.1, an irreducible admissible representation of* $\mathrm{GL}_4(F)$ *with a symplectic period is a product of factors* $\chi_i U(\delta_i, 2n_i)$, *where the* χ_i *are (not necessarily unitary) characters of* $F^{\times}$.

Theorem 1.4. *Using the notation introduced for Proposition 1.1, an irreducible admissible representation of* $\mathrm{GL}_6(F)$ *with a symplectic period is either a product of* $\chi_i U(\delta_i, 2n_i)$ *(the* χ_i *are not necessarily unitary), or is a twist of* $Z([1, \nu], [\nu, \nu^4])$ *or its dual.*

A few words about the proofs. It is a consequence of the uniqueness of the Klyachko models that irreducible cuspidal representations (which are generic) cannot have a symplectic period. Since any nonsupercuspidal irreducible representation is a quotient of a representation of the form $\mathrm{Ind}_{P_{k,2n-k}}^{\mathrm{GL}_{2n}}(\rho \otimes \tau)$, $\rho \in \mathrm{Irr}(\mathrm{GL}_k(F))$, $\tau \in \mathrm{Irr}(\mathrm{GL}_{2n-k}(F))$ it is enough to study the problem for representations of these types. For $\mathrm{GL}_4(F)$ and $\mathrm{GL}_6(F)$, this reduces the problem to the analysis of representations of the type $\pi_1 \times \pi_2$ and $\pi_1 \times \pi_2 \times \pi_3$, where each π_i is an irreducible representation of $\mathrm{GL}_2(F)$. For the $\mathrm{GL}_4(F)$ case, using Mackey theory we obtain an exhaustive list of (not necessarily irreducible) representations. Then we study every possible quotient to obtain a complete list of irreducible $\mathrm{Sp}_2(F)$-distinguished representations of $\mathrm{GL}_4(F)$. In the $\mathrm{GL}_6(F)$ case, we first reduce the problem to the case when none of the π_i are cuspidal. Next we reduce it to the case when at most one of the π_i is an irreducible principal series. Then we do a case-by-case analysis (for each π_i to be one of the three types of irreducible representations of $\mathrm{GL}_2(F)$ — a character, an irreducible principal series or a twist of the Steinberg representation, with at most one being an irreducible principal series), analyzing all possible subquotients for symplectic periods. A common way of showing that an irreducible subquotient is not H-distinguished, especially in the $\mathrm{GL}_6(F)$ case, is to express it as a quotient of a representation, which is then shown not to have a symplectic period using Mackey theory.

A word on the organization of the paper. Section 2 notation and preliminary notions used in the paper. Orbit structures and Mackey theory are covered in detail in Section 3. We analyze the representations of the form $\pi_1 \times \pi_2$ and obtain the theorem for $\mathrm{GL}_4(F)$ in Section 4. In Section 5, we analyze the representations of the form $\pi_1 \times \pi_2 \times \pi_3$, collecting all the irreducible $\mathrm{Sp}_3(F)$-distinguished subquotients. Using this analysis we obtain the theorem for $\mathrm{GL}_6(F)$. In Section 6 we make a few conjectures for the general case based on the available examples.

2. Notation and preliminaries

Notation. Throughout the paper, F will denote a nonarchimedean local field of characteristic 0.

Following the notation of [Bernstein and Zelevinsky 1976], we denote the set of all smooth representations of an l-group G by $\mathrm{Alg}(G)$ and the subset of all irreducible admissible representations by $\mathrm{Irr}(G)$. If $\pi \in \mathrm{Alg}(G)$, we denote by $\tilde{\pi}$, its contragredient.

Any character of $\mathrm{GL}_n(F)$ can be thought of as a character of $F^\times$ via the determinant map. Given a character χ of $F^\times$ and a smooth representation π of $\mathrm{GL}_n(F)$ we will denote the twist of π by χ simply by $\chi\pi$, $\chi\pi(g) := \chi(\det(g))\pi(g)$. Unless otherwise mentioned, St_n and 1_n will be used to denote the Steinberg and the trivial character of $\mathrm{GL}_n(F)$. The norm character $\nu(g) := |\det g|$ will be denoted by ν.

Let $P_{n_1,\dots,n_r}$ be the group of block upper triangular matrices corresponding to the tuple $(n_1, \dots, n_r)$. Let $N_{n_1,\dots,n_r}$ denote its unipotent radical. Let $\delta_{P_{n_1,\dots,n_r}}$ denote the modular function of the group $P_{n_1,\dots,n_r}$. Since a parabolic normalizes its unipotent radical, this defines a character of $P_{n_1,\dots,n_r}$ (the module of the automorphism $n \to pnp^{-1}$ of $N_{n_1,\dots,n_r}$ for $p \in P_{n_1,\dots,n_r}$). Call this character $\delta_{N_{n_1,\dots,n_r}}$. Then we have $\delta_{N_{n_1,\dots,n_r}} = \delta_{P_{n_1,\dots,n_r}}$. For an element $p \in P_{n_1,\dots,n_r}$, with its Levi part equal to $\mathrm{diag}(g_1, \dots, g_r)$, we have

$$\delta_{P_{n_1,\dots,n_r}}(p) = |\det g_1|^{n_2+\cdots+n_r} |\det g_2|^{-n_1+n_3+\cdots+n_r} \cdots |\det g_r|^{-n_1\cdots-n_{r-1}}. \tag{2-1}$$

The induced representation of $(\sigma, H, W) \in \mathrm{Alg}(H)$ to G is the following space of locally constant functions

$$\mathrm{Ind}_H^G \sigma = \{f : G \to W \mid f(hg) = \delta_H^{1/2} \delta_G^{-1/2} \sigma(h) f(g) \text{ for all } h \in H,\ g \in G\},$$

where δ_G and δ_H are the modular functions of G and H respectively. G acts on the space by right action. Compact induction from H to G is denoted by $\mathrm{ind}_H^G \sigma$ and is the subspace of $\mathrm{Ind}_H^G \sigma$ consisting of functions compactly supported mod H. Occasionally we will use nonnormalized induction (see Remark 2.22

of [Bernstein and Zelevinsky 1976] for the definition), although unless otherwise mentioned induction is always normalized. Given representations $\rho_i \in \mathrm{Irr}(\mathrm{GL}_{n_i}(F))$ $(i = 1, \ldots, r)$, extend $\rho_1 \otimes \cdots \otimes \rho_r$ to $P_{n_1,\ldots,n_r}$ so that it is trivial on $N_{n_1,\ldots,n_r}$. We denote by $\rho_1 \times \cdots \times \rho_r$ the representation $\mathrm{Ind}_{P_{n_1,\ldots,n_r}}^{\mathrm{GL}_n}(\rho_1 \otimes \cdots \otimes \rho_r)$.

The Jacquet functor with respect to a unipotent subgroup N is denoted by r_N and is always normalized.

If $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F))$, then there exists a partition of n and a multiset of cuspidal representations $\{\rho_1, \ldots, \rho_r\}$ corresponding to it such that π can be embedded in $\rho_1 \times \cdots \times \rho_r$. This multiset is uniquely determined by π and called its *cuspidal support*. For the purposes of this paper, for a smooth representation of finite length define it to be the union (as a set) of all the supports of its irreducible subquotients.

Preliminaries on segments. We briefly recall the notation and the basic definition of segments as introduced in [Zelevinsky 1980]. Given a cuspidal representation ρ of $\mathrm{GL}_m(F)$, a *segment* is a set of the form $\{\rho, \rho\nu, \ldots, \rho\nu^{k-1}\}$, with $k > 0$; we also write it as $[\rho, \rho\nu^{k-1}]$. Given a segment $\Delta = [\rho, \rho\nu^{k-1}]$, the unique irreducible submodule and the unique irreducible quotient of $\rho \times \cdots \times \rho\nu^{k-1}$ are denoted by $Z(\Delta)$ and $Q(\Delta)$ respectively.

For $\Delta_1 = [\rho_1, \nu^{k_1-1}\rho_1]$ and $\Delta_2 = [\rho_2, \nu^{k_2-1}\rho_2]$, we say that Δ_1 and Δ_2 are *linked* if $\Delta_1 \nsubseteq \Delta_2$, $\Delta_2 \nsubseteq \Delta_1$ and $\Delta_1 \cup \Delta_2$ is also a segment. If Δ_1 and Δ_2 are linked and $\Delta_1 \cap \Delta_2 = \phi$, then we say that Δ_1 and Δ_2 are juxtaposed. If Δ_1 and Δ_2 are linked and $\rho_2 = \nu^k\rho_1$, where $k > 0$, we say that Δ_1 *precedes* Δ_2. Given a multiset $\mathfrak{a} = \{\Delta_1, \ldots, \Delta_r\}$ of segments, let

$$\pi(\mathfrak{a}) := Z(\Delta_1) \times \cdots \times Z(\Delta_r).$$

If Δ_i does not precede Δ_j for any $i < j$, $\pi(\mathfrak{a})$ is known to have a unique irreducible submodule, which will be denoted by $Z(\Delta_1, \ldots, \Delta_r)$. By Theorem 6.1 of [Zelevinsky 1980], this submodule is independent of the ordering of the segments as long as the "does not precede" condition is satisfied. Hence we simply denote it by $Z(\mathfrak{a})$. In this situation, a similar statement holds for quotients as well and the unique irreducible quotient of $Q(\Delta_1) \times \cdots \times Q(\Delta_r)$ is denoted by $Q(\mathfrak{a})$. For example, the trivial character 1_n of $\mathrm{GL}_n(F)$ is $Z([\nu^{-(n-1)/2}, \nu^{(n-1)/2}])$, while St_n is $Q([\nu^{-(n-1)/2}, \nu^{(n-1)/2}])$.

We say a multiset $\mathfrak{a} = \{\Delta_1, \ldots, \Delta_r\}$ is on the cuspidal line of ρ, where ρ is a cuspidal representation of some $\mathrm{GL}_n(F)$, if $\Delta_i \subset \{\nu^k\rho\}_{k\in\mathbb{Z}}$ for all i.

Preliminaries on $\mathbf{GL}_n(F)$ and symplectic periods. We now collect a few basic results on $\mathrm{GL}_n(F)$ and symplectic periods needed in the sequel. The following result is used to calculate explicitly the quotients and the submodules in quite a few cases in the proofs of the main theorems.

Theorem 2.1 [Zelevinsky 1980]. *Let Δ_1 and Δ_2 be segments. If Δ_1 and Δ_2 are linked, put $\Delta_3 = \Delta_1 \cup \Delta_2$ and $\Delta_4 = \Delta_1 \cap \Delta_2$. The representation $\pi = Z(\Delta_1) \times Z(\Delta_2)$ is irreducible if and only if Δ_1 and Δ_2 are not linked. If Δ_1 and Δ_2 are linked then π has length 2. If Δ_2 precedes Δ_1 then π has a unique irreducible submodule $Z(\Delta_1, \Delta_2)$ and a unique irreducible quotient $Z(\Delta_3) \times Z(\Delta_4)$. If Δ_1 precedes Δ_2 then π has a unique irreducible submodule $Z(\Delta_3) \times Z(\Delta_4)$ and a unique irreducible quotient $Z(\Delta_1, \Delta_2)$.*

Using the Zelevinsky involution and Rodier's theorem that $Q(\Delta_1, \Delta_2)$ is taken to $Z(\Delta_1, \Delta_2)$ we have a quotient version of this lemma.

Theorem 2.2. *Let Δ_1 and Δ_2 be segments. If Δ_1 and Δ_2 are linked, put $\Delta_3 = \Delta_1 \cup \Delta_2$ and $\Delta_4 = \Delta_1 \cap \Delta_2$. The representation $\pi = Q(\Delta_1) \times Q(\Delta_2)$ is irreducible if and only if Δ_1 and Δ_2 are not linked. If Δ_1 and Δ_2 are linked then π has length 2. If Δ_2 precedes Δ_1 then π has the unique irreducible submodule $Q(\Delta_3) \times Q(\Delta_4)$. If Δ_1 precedes Δ_2 then π has the unique irreducible quotient $Q(\Delta_3) \times Q(\Delta_4)$.*

Lemma 2.3 [Casselman 1995]. *Let $\eta = \rho_1 \times \cdots \times \rho_m$, where $\rho_i \in \mathrm{Irr}(\mathrm{GL}_i(F))$. Define $\breve{\eta} = \tilde{\rho}_m \times \cdots \times \tilde{\rho}_1$. Then π is an irreducible quotient of η if and only if $\tilde{\pi}$ is an irreducible quotient of $\breve{\eta}$.*

Let $\mathrm{Ext}^1_G(\cdot, \mathbb{C})$ be the derived group of the $\mathrm{Hom}_G(\cdot, \mathbb{C})$ functor (for details, see [Prasad 1990; 1993]).

Lemma 2.4. *Let $H = \mathrm{Sp}_n(F)$. Then $\mathrm{Ext}^1_H(\mathbb{C}, \mathbb{C})$ is trivial.*

Proof. An element of $\mathrm{Ext}^1_H(\mathbb{C}, \mathbb{C})$ corresponds to an exact sequence

$$0 \to \mathbb{C} \xrightarrow{i} V \xrightarrow{j} \mathbb{C} \to 0$$

of H-modules, or equivalently a homomorphism from H to the group of upper triangular unipotent subgroup of $\mathrm{GL}_2(\mathbb{C})$. Since H has no abelian quotients, there are no such nontrivial maps and we have the lemma. □

Theorem 2.5 [Offen and Sayag 2008]. *Let $\pi \in \mathrm{Irr}(\mathrm{GL}_n(F))$. If π embeds in a Klyachko model, it does so in a unique Klyachko model and with multiplicity at most one.*

3. Orbit structures and Mackey theory

Let X be a subspace of a symplectic space $(V, \langle\,,\rangle)$ of dimension $2n$. Let

$$X^{\perp} = \{y \in V \mid \langle y, x\rangle = 0 \text{ for all } x \in X\}.$$

Define $\mathrm{Rad}\, X = X \cap X^{\perp}$. Note that $X/\mathrm{Rad}\, X$ inherits the symplectic structure of V, becomes a nondegenerate symplectic space and hence has even dimension.

The next lemma is a variant of the classical theorem of Witt for quadratic forms.

Lemma 3.1 (Witt). (a) *Let X_1, X_2 be subspaces of V of same dimension. Then there exists a symplectic automorphism ϕ of V, taking X_1 to X_2 if and only if* $\dim \operatorname{Rad} X_1 = \dim \operatorname{Rad} X_2$.

(b) *Let X_1, X_2 be subspaces of V and $\phi : X_1 \to X_2$ be a symplectic isomorphism. Then ϕ extends to a symplectic automorphism of V.*

It follows from this lemma that if X is a k-dimensional subspace of V, and P_X is the parabolic subgroup of $\mathrm{GL}(V)$ consisting of automorphisms of V leaving X invariant, then $\mathrm{Sp}(V)\backslash\mathrm{GL}(V)/P_X$ is in bijective correspondence with integers i, $0 \le i \le \dim X$, such that $\dim X - i$ is even. To get a set of representatives for these double cosets, let

$$\{e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_n\}$$

be the standard symplectic basis of V; i.e., $\langle e_i, f_j\rangle = \delta_{ij}$. Define

$$\begin{aligned}
Y_r &:= \langle e_1, \dots, e_r\rangle,\\
Y_r^\vee &:= \langle f_1, \dots, f_r\rangle,\\
S_{k,r} &:= \langle e_{r+1}, \dots, e_{(k+r)/2}, f_{r+1}, \dots, f_{(k+r)/2}\rangle,\\
T_{k,r} &:= \langle e_{\frac{k+r}{2}+1}, \dots, e_n, f_{\frac{k+r}{2}+1}, \dots, f_n\rangle,\\
X_{k,r} &:= Y_r + S_{k,r}.
\end{aligned}$$

Note that $\mathrm{GL}(V)/P_X$ is the set of all k-dimensional subspaces of V on which $\mathrm{Sp}(V)$ acts in a natural way. Therefore $\mathrm{Sp}(V)\backslash\mathrm{GL}(V)/P_X$ is represented by a certain set of k-dimensional subspaces of V, which can be taken to be the spaces $X_{k,r}$ with $0 \le r \le k$ such that $k - r$ is even.

Since $\dim X = \dim X_{k,r}$, there exists an automorphism $g \in \mathrm{GL}(V)$ taking X to $X_{k,r}$. This automorphism gives an isomorphism from P_X to $P_{X_{k,r}}$. Using this isomorphism a representation of P_X can be considered to be a representation of $P_{X_{k,r}}$. By Mackey theory, the restriction of the representation $\mathrm{Ind}_{P_X}^{\mathrm{GL}(V)}(\sigma)$ to $\mathrm{Sp}(V)$ is obtained by gluing the representations:

$$\mathrm{ind}_{(\mathrm{Sp}(V)\cap P_{X_{k,r}})}^{\mathrm{Sp}(V)}(\delta_{P_X}^{1/2}\sigma|_{(\mathrm{Sp}(V)\cap P_{X_{k,r}})}),$$

where the induction is nonnormalized. The isomorphism of P_X with $P_{X_{k,r}}$ takes the unipotent radical of P_X to the unipotent radical of $P_{X_{k,r}}$ and hence the representation of $P_{X_{k,r}}$ so obtained is of the same kind that appears in parabolic induction. This is a special case for maximal parabolics of Proposition 3 of [Offen 2006].

For an isotropic subspace Y of V, the subgroup Q_Y of $\mathrm{Sp}(V)$ stabilizing Y is a parabolic subgroup of $\mathrm{Sp}(V)$, with Levi decomposition

$$Q_Y = \big(\mathrm{GL}(Y) \times \mathrm{Sp}(Y^\perp/Y)\big) \ltimes U,$$

where U is the subgroup of $\mathrm{Sp}(V)$ preserving $Y \subset Y^{\perp}$ and acting trivially on Y, $Y^{\perp}/Y$ and $V/Y^{\perp}$.

We fix a symplectic basis of V and identify the group of linear transformations with the corresponding group of matrices, although we emphasize that the following proposition and its corollary are independent of the choice of the basis.

Proposition 3.2. *The subgroup $H_{k,r}$ of $\mathrm{Sp}(V)$ stabilizing the subspace $X_{k,r}$ of V is*

$$H_{k,r} = \big(\mathrm{GL}(Y_r) \times \mathrm{Sp}(S_{k,r}) \times \mathrm{Sp}(T_{k,r})\big).U_{k,r},$$

where $U_{k,r}$ is the unipotent group inside $\mathrm{Sp}(V)$ consisting of automorphisms of V of the form

$$\begin{pmatrix} I_r & A & B & C \\ 0 & I_{k-r} & 0 & A' \\ 0 & 0 & I_{2n-(k+r)} & B' \\ 0 & 0 & 0 & I_r \end{pmatrix},$$

where $A \in \mathrm{Hom}(S_{k,r}, Y_r)$, $B \in \mathrm{Hom}(T_{k,r}, Y_r)$, the matrix $C \in \mathrm{Hom}(Y_r^{\vee}, Y_r)$ is symmetric, and $A' \in \mathrm{Hom}(Y_r^{\vee}, S_{k,r})$, $B' \in \mathrm{Hom}(Y_r^{\vee}, T_{k,r})$ are adjoint to A, B.

Proof. Note that $H_{k,r}$ is nothing but the symplectic automorphisms of V preserving the flag $0 \subset Y_r = X_{k,r} \cap X_{k,r}^{\perp} \subset X_{k,r} \subset X_{k,r} + X_{k,r}^{\perp} = X_{k,r} + T_{k,r} = Y_r^{\perp} \subset V$. Hence $H_{k,r}$ acts on the successive quotients of this filtration, giving rise to a surjective homomorphism to $\mathrm{GL}(Y_r) \times \mathrm{Sp}(S_{k,r}) \times \mathrm{Sp}(T_{k,r})$ with kernel $U_{k,r}$ consisting of the subgroup of $\mathrm{Sp}(V)$ preserving the flag and acting trivially on successive quotients. Clearly $U_{k,r}$ acts trivially on the isotropic subspace Y_r, on $Y_r^{\perp}$ and on $Y_r^{\perp}/Y_r = S_{k,r} + T_{k,r}$. The well-known knowledge of the structure of the parabolic in $\mathrm{Sp}(V)$ defined by Y_r proves the assertion of the proposition. □

Corollary 3.3. (1) *The modular character $\delta_{k,r}$ of the group $H_{k,r}$ is*

$$\delta_{k,r}(\mathrm{diag}(g, h_1, h_2, {}^t g^{-1})) = |\det g|^{r+a+b+1},$$

where $r = \dim Y_r$, $a = \dim S_{k,r} = k - r$, $b = \dim T_{k,r} = 2n - (k + r)$, and $g \in \mathrm{GL}(Y_r)$.

(2) *By* (2-1) *we have $\delta_P(\mathrm{diag}(g, h_1, h_2, {}^t g^{-1})) = |\det g|^{2r+a+b}$, where we set $P = P_{(r+a,b+r)}$. Thus*

$$\frac{\delta_P^{1/2}}{\delta_{k,r}}\big(\mathrm{diag}(g, h_1, h_2, {}^t g^{-1})\big) = |\det g|^{-1-(a+b)/2} = |\det g|^{-(n-r+1)}. \qquad \square$$

Define M to be the group $\mathrm{GL}(Y_r) \times \mathrm{Sp}(S_{k,r}) \times \mathrm{Sp}(T_{k,r})$ and identify it with

$$\mathrm{GL}_r(F) \times \mathrm{Sp}_{(k-r)/2}(F) \times \mathrm{Sp}_{(2n-k-r)/2}(F)$$

via the fixed basis. Call H the group $\mathrm{Sp}_n(F)$ defined with respect to this symplectic basis. Further let $N = N_1 \times N_2$, where N_1 and N_2 are the unipotent subgroups of $\mathrm{GL}_k(F)$ and $\mathrm{GL}_{2n-k}(F)$ corresponding to the partitions $(r,\ k-r)$ and $(2n-k-r,\ r)$, respectively. Let $\sigma_1 \in \mathrm{Irr}(\mathrm{GL}_k(F))$ and $\sigma_2 \in \mathrm{Irr}(\mathrm{GL}_{2n-k}(F))$. Call σ the representation of $P = P_{(k,2n-k)}$ obtained by extending $\sigma_1 \otimes \sigma_2$ to P in the usual way.

By Frobenius reciprocity and Corollary 3.3, we get

$$\mathrm{Hom}_H\big(\mathrm{ind}_{H_{k,r}}^H(\delta_P^{1/2}\sigma|_{H_{k,r}}), \mathbb{C}\big) = \mathrm{Hom}_{MU_{k,r}}\big(\nu^{-(n-r+1)}\sigma_1 \otimes \sigma_2, \mathbb{C}\big).$$

Clearly,

$$\mathrm{Hom}_{M.U_{k,r}}\big(\nu^{-(n-r+1)}\sigma_1 \otimes \sigma_2, \mathbb{C}\big) = \mathrm{Hom}_{MN}\big(\nu^{-(n-r+1)}\sigma_1 \otimes \sigma_2, \mathbb{C}\big).$$

Since the normalized Jacquet functor is left adjoint to normalized induction by Proposition 1.9(b) of [Bernstein and Zelevinsky 1977], we obtain

$$\begin{aligned}\mathrm{Hom}_{MN}&\big(\nu^{-(n-r+1)}\sigma_1 \otimes \sigma_2, \mathbb{C}\big)\\ &= \mathrm{Hom}_M\big(r_N(\nu^{-(n-r+1)}\sigma_1 \otimes \sigma_2), \delta_N^{-1/2}\big)\\ &= \mathrm{Hom}_M\big(\nu^{-(n-r+1)}\delta_{N_1}^{1/2} r_{N_1}(\sigma_1) \otimes \delta_{N_2}^{1/2} r_{N_2}(\sigma_2), \mathbb{C}\big).\end{aligned}$$

Now let A and B have determinant 1. By (2-1), we have

$$\delta_{N_1}\begin{pmatrix} g & * \\ 0 & A \end{pmatrix} = |\det g|^{(k-r)}, \delta_{N_2}\begin{pmatrix} B & * \\ 0 & {}^t g^{-1} \end{pmatrix} = |\det g|^{2n-(k+r)}.$$

Define α to be the character of M such that $\alpha(\mathrm{diag}(g, h_1, h_2, {}^t g^{-1})) = \nu^{-1}(g)$. Plugging in the value of the delta functions we get

$$\mathrm{Hom}_H\big(\mathrm{ind}_{H_{k,r}}^H(\delta_P^{1/2}\sigma|_{H_{k,r}}), \mathbb{C}\big) = \mathrm{Hom}_M\big(\alpha(r_{N_1}(\sigma_1) \otimes r_{N_2}(\sigma_2)), \mathbb{C}\big). \tag{3-1}$$

From this we have the following lemma for $\mathrm{GL}_{2n}(F)$.

Lemma 3.4. *Let $\pi_i = Z(\Delta_1^i, \dots, \Delta_{k_i}^i) \in \mathrm{Irr}(\mathrm{GL}_{n_i}(F))$ for $i = 1, \dots, s$ be such that the following conditions are satisfied*:

(1) *For $i \neq j$, the segments $\Delta_{m_i}^i$ and $\Delta_{m_j}^j$ are disjoint and not linked, for all $m_i = 1, \dots, k_i$ and all $m_j = 1, \dots, k_j$.*

(2) *$\sum_{i=1}^{s} n_i$ is even and $\pi := \pi_1 \times \cdots \times \pi_s$ has a symplectic period.*

Then each n_i is even and every π_i has a symplectic period.

Proof. Condition (1) forces π to be irreducible (by Proposition 8.5 of [Zelevinsky 1980]). Thus it is enough to prove the lemma for $s = 2$.

Let $\pi_1 \in \mathrm{Irr}(\mathrm{GL}_{n_1}(F))$ and $\pi_2 \in \mathrm{Irr}(\mathrm{GL}_{n_2}(F))$. Now, since $r_{N_1}(\pi_1)$ lies in $\mathrm{Alg}(\mathrm{GL}_r(F) \times \mathrm{GL}_{n_1-r}(F))$ and the functor r_{N_1} takes finite length representations into ones of finite length ([ibid.], Proposition 1.4), up to semisimplification it is of the form $\sum_{i=1}^{t_1} \pi_{1i} \otimes \tau_{1i}$ for some $t_1 > 0$, where $\pi_{1i} \in \mathrm{Irr}(\mathrm{GL}_r(F))$ and $\tau_{1i} \in \mathrm{Irr}(\mathrm{GL}_{n_1-r}(F))$ for all $i = 1, \dots, t_1$. Similarly, up to semisimplification, $r_{N_2}(\pi_2)$ is equal to $\sum_{j=1}^{t_2} \tau_{2j} \otimes \pi_{2j}$, where $\tau_{2j} \in \mathrm{Irr}(\mathrm{GL}_{n_2-r}(F))$ and $\pi_{2j} \in \mathrm{Irr}(\mathrm{GL}_r(F))$.

We claim that for any $\theta \in \mathrm{Irr}(\mathrm{GL}_m(F))$, the cuspidal support (page 438) of $r_N(\theta)$ is always a subset (as a set) of the cuspidal support of θ. Assume $\theta = Z(\Delta_1, \dots, \Delta_l)$. The claim follows from the geometrical lemma (Lemma 2.12 of [Bernstein and Zelevinsky 1977]) applied to $r_N(Z(\Delta_1) \times \cdots \times Z(\Delta_l))$, along with the observation that $r_N(\theta)$ is a submodule of it.

Together with condition (1) of the lemma, this claim implies the vanishing of $\mathrm{Hom}_{\mathrm{GL}_r}(\nu^{-1}\pi_{1i} \otimes \tilde{\pi_{2j}}, \mathbb{C})$ for every pair i, j. By (3-1) and the realization of contragredient representations due to Gelfand and Kazhdan (cf. Theorem 7.3 of [Bernstein and Zelevinsky 1976]), this implies

$$\mathrm{Hom}_H\big(\mathrm{ind}_{H_{n_1,r}}^H(\delta_{P_{n_1,n_2}}^{1/2}(\pi_1 \otimes \pi_2)|_{H_{n_1,r}}), \mathbb{C}\big) = 0$$

unless $r = 0$. This along with condition (2) forces n_1, n_2 to be even and π_1, π_2 both to have symplectic periods. □

Lemma 3.5. *Let Δ_1 and Δ_2 be segments of even lengths such that their intersection is of odd length. Then the representation $\theta = Z(\Delta_1, \Delta_2)$ has a symplectic period.*

Proof. If possible, let $\mathrm{Hom}_H(\theta, \mathbb{C}) = 0$. Define the segments $\Delta_3 = \Delta_1 \cup \Delta_2$ and $\Delta_4 = \Delta_1 \cap \Delta_2$. Without loss of generality assume Δ_1 precedes Δ_2. By Theorem 2.1, θ sits inside the following exact sequence of $\mathrm{GL}_{2n}(F)$ modules:

$$0 \to \theta \to Z(\Delta_2) \times Z(\Delta_1) \to Z(\Delta_3) \times Z(\Delta_4) \to 0.$$

Observe that Δ_3 and Δ_4 are segments of odd length. So, $Z(\Delta_3) \times Z(\Delta_4)$ has a mixed Klyachko model by Theorem 3.7 of [Offen and Sayag 2007b] and hence by Theorem 2.5, it is not H-distinguished. Since $\mathrm{Hom}_H\big(Z(\Delta_2) \times Z(\Delta_1), \mathbb{C}\big) = 0$ if $\mathrm{Hom}_H\big(Z(\Delta_3) \times Z(\Delta_4), \mathbb{C}\big) = 0$ and $\mathrm{Hom}_H(\theta, \mathbb{C}) = 0$, we obtain a contradiction with Proposition 1.1. □

Lemma 3.6. *If Δ_1 and Δ_2 are juxtaposed segments of even lengths in the cuspidal line of 1_1 (the trivial representation of* $\mathrm{GL}_1(F)$)*, the representation $\theta = Z(\Delta_1, \Delta_2)$ does not have a symplectic period.*

Proof. Define $\Delta_3 = \Delta_1 \cup \Delta_2$ and let $2n$ be its length. In fact, twisting it by an appropriate power of ν, without loss of generality we can take Δ_3 to be $[\nu^{-\frac{2n-1}{2}}, \nu^{\frac{2n-1}{2}}]$ and hence $Z(\Delta_3) = 1$. Let

$$\Delta_1 = [\nu^{-\frac{2n-1}{2}}, \nu^{\frac{a}{2}}] \quad \text{and} \quad \Delta_2 = [\nu^{\frac{b}{2}}, \nu^{\frac{2n-1}{2}}].$$

Let $k = \frac{2n-1}{2} - \frac{b}{2} + 1$, the length of Δ_2. Now assume $k \leq n$.

Let $\mu_1 = Z(\Delta_2)$ and $\mu_2 = Z(\Delta_1)$. Let us first calculate $\mathrm{Hom}_H(\mu_1 \times \mu_2, \mathbb{C})$. By (3-1), for $r \neq 0$, $\mathrm{Hom}_H\big(\mathrm{ind}_{H_{k,r}}^H(\delta_P^{1/2}\mu_1 \otimes \mu_2|_{H_{k,r}}), \mathbb{C}\big)$ is isomorphic to

$$\mathrm{Hom}_{\mathrm{GL}_r(F)\times \mathrm{Sp}_{\frac{k-r}{2}}(F)\times \mathrm{Sp}_{n-\frac{k+r}{2}}(F)}\big(\nu^{n-\frac{2k-r}{2}-1} \otimes \nu^{n-1-\frac{k-r}{2}} \otimes \nu^{-\frac{k+r}{2}} \otimes \nu^{n-k-\frac{r}{2}}, \mathbb{C}\big),$$

where $\mathrm{GL}_r(F)$ acts on the last term via the contragredient. Now, consider

$$\mathrm{Hom}_{\mathrm{GL}_r(F)}\big(\nu^{n-\frac{2k-r}{2}-1} \otimes \nu^{-(n-k-\frac{r}{2})}, \mathbb{C}\big).$$

This is nonzero only if $n - \frac{2k-r}{2} - 1 = n - k - \frac{r}{2}$, which is impossible since k is even by the hypothesis of the lemma. Thus

$$\mathrm{Hom}_H\big(\mathrm{ind}_{H_{k,r}}^H(\delta_P^{1/2}\mu_1 \otimes \mu_2|_{H_{k,r}}), \mathbb{C}\big) = 0 \quad \text{if } r \neq 0.$$

On the other hand, if $r = 0$ we have

$$\mathrm{Hom}_H\big(\mathrm{ind}_{H_{k,0}}^H(\delta_P^{1/2}\mu_1 \otimes \mu_2|_{H_{k,0}}), \mathbb{C}\big) = \mathrm{Hom}_{\mathrm{Sp}_{\frac{k}{2}}(F)}\big(\mu_1, \mathbb{C}\big) \otimes \mathrm{Hom}_{\mathrm{Sp}_{n-\frac{k}{2}}(F)}(\mu_2, \mathbb{C}) = \mathbb{C}.$$

Hence $\mathrm{Hom}_H(\mu_1 \times \mu_2, \mathbb{C})$ is at most one-dimensional. Now, we have the following exact sequence of $\mathrm{GL}_{2n}(F)$ modules (and hence of $\mathrm{Sp}_n(F)$ modules):

$$0 \to Z(\Delta_1, \Delta_2) \xrightarrow{i} Z(\Delta_2) \times Z(\Delta_1) \xrightarrow{j} \mathbb{C} \to 0.$$

Applying the functor $\mathrm{Hom}_{\mathrm{Sp}_n(F)}\big(.., \mathbb{C}\big)$ to it we obtain the long exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{Sp}_n(F)}\big(\mathbb{C}, \mathbb{C}\big) \xrightarrow{j^*} \mathrm{Hom}_{\mathrm{Sp}_n(F)}\big(Z(\Delta_2) \times Z(\Delta_1), \mathbb{C}\big) \xrightarrow{i^*} \mathrm{Hom}_{\mathrm{Sp}_n(F)}\big(Z(\Delta_1, \Delta_2), \mathbb{C}\big) \xrightarrow{r^*} \mathrm{Ext}^1_{\mathrm{Sp}_n(F)}\big(\mathbb{C}, \mathbb{C}\big) \to \cdots$$

Observing that $\mathrm{Ext}^1_{\mathrm{Sp}_n(F)}\big(\mathbb{C}, \mathbb{C}\big) = 0$ (see Lemma 2.4) we get the following short exact sequence:

$$0 \to \mathbb{C} \xrightarrow{j^*} \mathrm{Hom}_{\mathrm{Sp}_n(F)}\big(Z(\Delta_2) \times Z(\Delta_1), \mathbb{C}\big) \xrightarrow{i^*} \mathrm{Hom}_{\mathrm{Sp}_n(F)}\big(Z(\Delta_1, \Delta_2), \mathbb{C}\big) \to 0.$$

Since j^* is injective, $\mathrm{Im}(j^*) = \mathbb{C}$. By exactness, $\mathrm{Ker}(i^*) = \mathbb{C}$ as well. Since $\mathrm{Hom}_{\mathrm{Sp}_n(F)}\big(Z(\Delta_2) \times Z(\Delta_1), \mathbb{C}\big)$ was shown to be at most one-dimensional, it is equal to $\mathrm{Ker}(i^*)$. Thus $\mathrm{Im}(i^*) = 0$. But again by exactness, i^* is surjective, thus implying that

$$\mathrm{Hom}_{\mathrm{Sp}_n(F)}\big(Z(\Delta_1, \Delta_2), \mathbb{C}\big) = 0.$$

Thus we have the lemma if $k \leq n$. Since an irreducible representation has a symplectic period if and only if its contragredient has so, we have the lemma in the case $k > n$. □

4. Analysis in the $\mathrm{GL}_4(F)$ case: proof of Theorem 1.3

In this section we prove Theorem 1.3. We begin with the following lemma.

Lemma 4.1. *Let θ be an irreducible representation of $\mathrm{GL}_4(F)$ with a symplectic period. Then there exists $\pi_i \in \mathrm{Irr}(\mathrm{GL}_2(F))$, $i = 1, 2$ such that θ appears as a quotient of $\pi_1 \times \pi_2$.*

Proof. If θ is a supercuspidal representation of $\mathrm{GL}_4(F)$, it is generic and hence by Theorem 2.5 it doesn't have a symplectic period. Thus θ appears as a quotient of either $\chi_1 \times \theta_3$, $\theta_3 \times \chi_1$ or $\pi_1 \times \pi_2$ (where $\chi_1 \in \mathrm{Irr}(\mathrm{GL}_1(F))$, $\theta_3 \in \mathrm{Irr}(\mathrm{GL}_3(F))$ and $\pi_1, \pi_2 \in \mathrm{Irr}(\mathrm{GL}_2(F))$). In the last case we have nothing left to prove. If θ is a quotient of $\theta_3 \times \chi_1$, by Lemma 2.3, $\tilde{\theta}$ is a quotient of $\tilde{\chi_1} \times \tilde{\theta_3}$. Since an irreducible representation has a symplectic period if and only if its contragredient does, by applying Lemma 2.3 again we are reduced to the first case. So assume θ is a quotient of $\chi_1 \times \theta_3$. Now if θ_3 is cuspidal, $\chi_1 \times \theta_3$ is irreducible and generic. Hence by the disjointness of the symplectic and Whittaker models it cannot have a symplectic period. Thus assume θ_3 isn't cuspidal.

Then θ_3 is a quotient of one of the representations of the form $\chi_1' \times \delta_2$, $\delta_2 \times \chi_1'$ or $\chi_1' \times \chi_1'' \times \chi_1'''$, where $\chi_1', \chi_1'', \chi_1'''$ are characters of $\mathrm{GL}_1(F)$ and δ_2 is a supercuspidal of $\mathrm{GL}_2(F)$.

In the first case, $\chi_1 \times \theta_3$ is a quotient of $\chi_1 \times \chi_1' \times \delta_2$. If $\chi_1 \times \chi_1'$ is irreducible, the lemma is proved. If not, $\chi_1 \times \chi_1' \times \delta_2$ is glued from $Z(\chi_1 \times \chi_1') \times \delta_2$ and $Q(\chi_1 \times \chi_1') \times \delta_2$, where $Z(\chi_1 \times \chi_1')$ and $Q(\chi_1 \times \chi_1')$ are respectively the unique irreducible submodule and unique irreducible quotient of $\chi_1 \times \chi_1'$. Thus any irreducible quotient of $\chi_1 \times \theta_3$ has to be a quotient of one of the two.

In the second case, since $\delta_2 \times \chi_1'$ is irreducible, $\chi_1 \times \delta_2 \times \chi_1' \cong \chi_1 \times \chi_1' \times \delta_2$. Thus we are back to the first case.

In the third case, if both $\chi_1 \times \chi_1'$ and $\chi_1'' \times \chi_1'''$ are irreducible we are done. In case at least one of them is reducible, we get the lemma by breaking $\chi_1 \times \chi_1' \times \chi_1'' \times \chi_1'''$, as in the first case, into subquotients of the required form. □

By this lemma, it is enough to consider representations of the form $\pi_1 \times \pi_2$, where π_1 and π_2 are irreducible representations of $\mathrm{GL}_2(F)$. If $\pi = \pi_1 \times \pi_2$ has an H-distinguished quotient, then π itself is H-distinguished. By Mackey theory we get that $(\pi_1 \times \pi_2)|_{\mathrm{Sp}_2(F)}$ is glued from the two subquotients

$$\mathrm{ind}_{H_{2,0}}^{H}(\delta_{P_{2,2}}^{1/2}\pi_1 \otimes \pi_2|_{H_{2,0}}) \quad \text{and} \quad \mathrm{ind}_{H_{2,2}}^{H}(\delta_{P_{2,2}}^{1/2}\pi_1 \otimes \pi_2|_{H_{2,2}}).$$

Analyzing the two subquotients (using (3-1)), it is easy to see that the necessary conditions for π to have a symplectic period are that either π_1, π_2 are characters of $\mathrm{GL}_2(F)$ or $\pi_2 \cong \nu^{-1}\pi_1$. Any irreducible representation of $\mathrm{GL}_2(F)$ is either a supercuspidal, a character, an irreducible principal series or a twist of the Steinberg representation. Thus any irreducible $\mathrm{Sp}_4(F)$-distinguished representation occurs as a quotient of one of the representations listed in the next proposition.

Proposition 4.2. *Let θ be an irreducible admissible representation of* $\mathrm{GL}_4(F)$ *with a symplectic period. Then θ occurs as a quotient of one of the following representations π of* $\mathrm{GL}_4(F)$:

(1) $\pi = \chi_2 \times \chi_2'$, *where χ_2, χ_2' are characters of* $\mathrm{GL}_2(F)$.

(2) $\pi = \sigma_2 \times \nu^{-1}\sigma_2$, *where σ_2 is a supercuspidal of* $\mathrm{GL}_2(F)$.

(3) $\pi = \chi_1 \times \chi_1' \times \nu^{-1}\chi_1 \times \nu^{-1}\chi_1'$, *where χ_1, χ_1' are characters of $F^\times$ and $\chi_1 \times \chi_1'$ is an irreducible principal series.*

(4) $\pi = Q([\chi_1\nu^{-1/2}, \chi_1\nu^{1/2}]) \times Q([\chi_1\nu^{-3/2}, \chi_1\nu^{-1/2}])$, *where χ_1 is a character of $F^\times$.* □

Now we come to the theorem in the $\mathrm{GL}_4(F)$ case. We state and prove an equivalent version of Theorem 1.3 in terms of the Zelevinsky classification.

Theorem 4.3. *This is the complete list of irreducible admissible representations θ of* $\mathrm{GL}_4(F)$ *with a symplectic period*:

(1) $\theta = Z([\sigma_2, \nu\sigma_2])$, *where σ_2 is a cuspidal representation of* $\mathrm{GL}_2(F)$.

(2) $\theta = Z(\Delta_1, \Delta_2)$, *where $\Delta_1 = [\chi_1\nu^{-1/2}, \chi_1\nu^{1/2}]$ and $\Delta_2 = [\chi_1\nu^{-3/2}, \chi_1\nu^{-1/2}]$ (χ_1 is a character of $F^\times$).*

(3) $\theta =$ *a character of* $\mathrm{GL}_4(F)$.

(4) $\theta = \chi_2 \times \chi_2'$, *where χ_2, χ_2' are characters of* $\mathrm{GL}_2(F)$.

Proof. The strategy of the proof is to consider each representation in the list of Proposition 4.2 and to check, for all irreducible quotients of each one, whether they have a symplectic period.

<u>Case I</u>: $\pi = \chi_2 \times \chi_2'$. If $\chi_2 \times \chi_2'$ is irreducible, $\theta = \pi$ has a symplectic period by Proposition 1.1. So assume otherwise. Let $\chi_2 = Z([\chi_1\nu^{-1/2}, \chi_1\nu^{1/2}])$ and $\chi_2' = Z([\chi_1'\nu^{-1/2}, \chi_1'\nu^{1/2}])$. There are four subcases:

(1) $\chi_1 = \chi_1'\nu$. In this case, $\pi = Z([\chi_1'\nu^{1/2}, \chi_1'\nu^{3/2}]) \times Z([\chi_1'\nu^{-1/2}, \chi_1'\nu^{1/2}])$. By Theorem 2.1, it has a unique irreducible quotient, which is

$$\theta = Z([\chi_1'\nu^{-1/2}, \chi_1'\nu^{3/2}]) \times \chi_1'\nu^{1/2}.$$

By Theorem 3.7 of [Offen and Sayag 2007b], it has a mixed Klyachko model. Hence, by Theorem 2.5, it doesn't have a symplectic period.

(2) $\chi_1 = \chi_1'\nu^{-1}$. Here, $\pi = Z([\chi_1'\nu^{-3/2}, \chi_1'\nu^{-1/2}]) \times Z([\chi_1'\nu^{-1/2}, \chi_1'\nu^{1/2}])$. By Theorem 2.1, this has a unique irreducible quotient

$$\theta = \chi_1' Z([\nu^{-3/2}, \nu^{-1/2}], [\nu^{-1/2}, \nu^{1/2}]),$$

which has a symplectic period by Lemma 3.5. Note that θ is a twist of $U(\mathrm{St}_2, 2)$ and the fact that it has a symplectic period also follows from Proposition 1.1.

(3) $\chi_1 = \chi_1'\nu^2$. In this case, $\pi = Z([\chi_1'\nu^{3/2}, \chi_1'\nu^{5/2}]) \times Z([\chi_1'\nu^{-1/2}, \chi_1'\nu^{1/2}])$. This has a unique irreducible quotient $\theta = Z([\chi_1'\nu^{-1/2}, \chi_1'\nu^{5/2}])$. Thus θ is the character $\chi_1\nu$ of $\mathrm{GL}_4(F)$ and has a symplectic period.

(4) $\chi_1 = \chi_1'\nu^{-2}$. Here, $\pi = Z([\chi_1'\nu^{-5/2}, \chi_1'\nu^{-3/2}]) \times Z([\chi_1'\nu^{-1/2}, \chi_1'\nu^{1/2}])$, which, by Theorem 2.1, has a unique irreducible quotient

$$\theta = Z([\chi_1'\nu^{-5/2}, \chi_1'\nu^{-3/2}], [\chi_1'\nu^{-1/2}, \chi_1'\nu^{1/2}]).$$

By Lemma 3.6, it doesn't have a symplectic period.

Case II: $\pi = \sigma_2 \times \nu^{-1}\sigma_2$. In this case, π has a unique irreducible quotient $U(\nu^{-1/2}\sigma_2, 2) \cong Z([\nu^{-1}\sigma_2, \sigma_2])$. By Proposition 1.1 it has a symplectic period.

Case III: $\pi = \chi_1 \times \chi_1' \times \nu^{-1}\chi_1 \times \nu^{-1}\chi_1'$, where $\chi_1 \times \chi_1'$ is irreducible. There are two further subcases:

(1) $\chi_1' \times \chi_1\nu^{-1}$ is irreducible. This again can be broken down into two subcases.

(1a) $\chi_1' \neq \chi_1\nu^2$. In this case, $\pi \cong \chi_1 \times \nu^{-1}\chi_1 \times \chi_1' \times \nu^{-1}\chi_1'$. The "does not precede" condition (page 438) is satisfied and so π has a unique irreducible quotient. Clearly, π has $Z([\nu^{-1}\chi_1, \chi_1]) \times Z([\nu^{-1}\chi_1', \chi_1'])$ as a quotient. If it's irreducible, it has a symplectic period by Proposition 1.1 and has already been accounted for in case I. So assume the contrary. In that case the segments are linked. But the assumption that $\chi_1' \times \chi_1\nu^{-1}$ is irreducible, together with $\chi_1' \neq \chi_1\nu^2$, forces a contradiction. Hence irreducibility of $Z([\nu^{-1}\chi_1, \chi_1]) \times Z([\nu^{-1}\chi_1', \chi_1'])$ is the only possibility.

(1b) $\chi_1' = \chi_1\nu^2$. In this case,

$$\pi \cong \chi_1'\nu^{-2} \times \chi_1' \times \chi_1'\nu^{-3} \times \chi_1'\nu^{-1} \cong \chi_1'\nu^{-2} \times \chi_1'\nu^{-3} \times \chi_1' \times \chi_1'\nu^{-1}.$$

This representation has $\tau = Z([\chi_1'\nu^{-3}, \chi_1'\nu^{-2}]) \times Z([\chi_1'\nu^{-1}, \chi_1'])$ as a quotient. Since the cuspidal support of π is multiplicity free it has a unique irreducible quotient (by Proposition 2.10 of [Zelevinsky 1980]) and so any H-distinguished irreducible quotient of π is a quotient of τ. Thus they have already been accounted for in case I.

(2) $\chi_1' \times \chi_1 \nu^{-1}$ is reducible. This happens if and only if $\chi_1 = \chi_1'$ or $\chi_1 = \chi_1' \nu^2$. Again, we will deal with the cases separately.

(2a) $\chi_1 = \chi_1'$. The representation is of the form $\chi_1 \times \chi_1 \times \chi_1 \nu^{-1} \times \chi_1 \nu^{-1}$. Since it satisfies the "does not precede" condition (page 438) it has a unique irreducible quotient. It can be easily seen that $\theta = Z([\chi_1 \nu^{-1}, \chi_1]) \times Z([\chi_1 \nu^{-1}, \chi_1])$ is an irreducible quotient of this representation (and so is the unique one). θ has a symplectic period and has already been accounted for in case I.

(2b) $\chi_1 = \chi_1' \nu^2$. In this case, $\pi \cong \chi_1' \nu^2 \times \chi_1' \times \chi_1' \nu \times \chi_1' \nu^{-1} \cong \chi_1' \times \chi_1' \nu^{-1} \times \chi_1' \nu^2 \times \chi_1' \nu$. By an argument similar to the one used in (1b) above, we conclude that this representation has already been accounted for in case I.

Case IV: $\pi = Q([\chi_1 \nu^{-1/2}, \chi_1 \nu^{1/2}]) \times Q([\chi_1 \nu^{-3/2}, \chi_1 \nu^{-1/2}])$. By Theorem 2.2, π has a unique irreducible quotient $\chi_1 Q([\nu^{-3/2}, \nu^{-1/2}], [\nu^{-1/2}, \nu^{1/2}])$. As seen in case I(2), it is a twist of $U(\mathrm{St}_2, 2)$ and has a symplectic period (by Proposition 1.1). □

5. Analysis in the $\mathrm{GL}_6(F)$ case

In this section we obtain the theorem for $\mathrm{GL}_6(F)$. The following lemma reduces the analysis to representations of the form $\pi_1 \times \pi_2 \times \pi_3$, where the π_i are irreducible representations of $\mathrm{GL}_2(F)$.

Lemma 5.1. *Let θ be an irreducible representation of $\mathrm{GL}_6(F)$ with a symplectic period. Then either θ is of the form $Z([\sigma_3, \nu\sigma_3])$, where σ_3 is a supercuspidal representation of $\mathrm{GL}_3(F)$ or it occurs as a subquotient of a representation of the form $\pi_1 \times \pi_2 \times \pi_3$, where $\pi_1 \in \mathrm{Irr}(\mathrm{GL}_2(F))$ for $i = 1, 2, 3$.*

Proof. Since supercuspidal representations are generic, they don't have a symplectic period. Thus θ appears as a subquotient of $\tau_1 \times \tau_2$, where τ_1 and τ_2 are irreducible representations of $\mathrm{GL}_k(F)$ and $\mathrm{GL}_{6-k}(F)$ respectively. By interchanging τ_1 and τ_2 if necessary, we can assume $k \le 3$.

Case 1: $k = 1$. If τ_2 is a cuspidal representation of $\mathrm{GL}_5(F)$, since τ_1 is a character, $\tau_1 \times \tau_2$ is irreducible and generic. Thus τ_2 occurs as a subquotient of a representation induced from a maximal parabolic of $\mathrm{GL}_5(F)$. So θ is either a subquotient of $\tau_1 \times \chi \times \tau$ ($\chi \in \mathrm{Irr}(\mathrm{GL}_1(F))$, $\tau \in \mathrm{Irr}(\mathrm{GL}_4(F))$) or $\tau_1 \times \tau' \times \tau''$ ($\tau' \in \mathrm{Irr}(\mathrm{GL}_2(F))$, $\tau'' \in \mathrm{Irr}(\mathrm{GL}_3(F))$). Thus θ is either a subquotient of $\theta_1 \times \tau$ (where $\theta_1 \in \mathrm{Irr}(\mathrm{GL}_2(F))$)

or of $\theta_2 \times \tau''$ (where $\theta_2 \in \mathrm{Irr}(\mathrm{GL}_3(F))$) thus reducing the lemma to the next two cases.

Case 2: $k = 2$. If τ_2 is a cuspidal representation of $\mathrm{GL}_4(F)$, $\tau_1 \times \tau_2$ is irreducible and doesn't have a symplectic period, by Lemma 3.4. Thus, as earlier, τ_2 occurs as a subquotient of a representation induced from a maximal parabolic of $\mathrm{GL}_4(F)$. So θ is either a subquotient of $\tau_1 \times \chi \times \tau$ ($\chi \in \mathrm{Irr}(\mathrm{GL}_1(F))$, $\tau \in \mathrm{Irr}(\mathrm{GL}_3(F))$) or $\tau_1 \times \tau' \times \tau''$ ($\tau' \in \mathrm{Irr}(\mathrm{GL}_2(F))$, $\tau'' \in \mathrm{Irr}(\mathrm{GL}_2(F))$). In the first scenario θ occurs as a subquotient of $\theta_1 \times \theta_2$ (where $\theta_1, \theta_2 \in \mathrm{Irr}(\mathrm{GL}_3(F))$), reducing the lemma to the next case, while in the second we have the lemma.

Case 3: $k = 3$. We will first show that if either of τ_1, τ_2 (say τ_1) is cuspidal then θ is of the form $Z([\sigma_3, \nu\sigma_3])$. Choose $\tau_2' \in \mathrm{Irr}(\mathrm{GL}_3(F))$ such that θ is a quotient of either $\tau_1 \times \tau_2'$ or $\tau_2' \times \tau_1$.

Assume the former. Then $\tau_1 \times \tau_2'$ also has a nontrivial $\mathrm{Sp}_3(F)$-invariant linear form. Now, $\tau_1 \times \tau_2'|_{\mathrm{Sp}_3(F)}$ is glued from

$$\mathrm{ind}_{H_{3,3}}^{H}(\delta_{P_{3,3}}^{1/2}\tau_1 \otimes \tau_2'|_{H_{3,3}}) \quad \text{and} \quad \mathrm{ind}_{H_{3,1}}^{H}(\delta_{P_{3,3}}^{1/2}\tau_1 \otimes \tau_2'|_{H_{3,1}}).$$

Since τ_1 is cuspidal, by (3-1), $\mathrm{Hom}_H(\mathrm{ind}_{H_{3,1}}^{H}(\delta_{P_{3,3}}^{1/2}\tau_1 \otimes \tau_2'|_{H_{3,1}}), \mathbb{C}) = 0$. Hence,

$$\mathrm{Hom}_H(\mathrm{ind}_{H_{3,3}}^{H}(\delta_{P_{3,3}}^{1/2}\tau_1 \otimes \tau_2'|_{H_{3,3}}), \mathbb{C}) \neq 0,$$

which is true if and only if $\tau_2' = \nu^{-1}\tau_1$, again by (3-1) and a theorem of Gelfand and Kazhdan (see [Bernstein and Zelevinsky 1976, Theorem 7.3]). Thus θ equals $Z([\nu^{-1}\tau_1, \tau_1])$.

If instead θ is a quotient of $\tau_2' \times \tau_1$, replacing θ by $\tilde{\theta}$ gives us the desired result.

Thus assume now that none of the two are cuspidal. Then $\exists\ \chi_i, \theta_i'$ ($i = 1, 2$) such that τ_i is a subquotient of $\chi_i \times \theta_i'$ (where $\chi_i \in \mathrm{Irr}(\mathrm{GL}_1(F))$, $\theta_i' \in \mathrm{Irr}(\mathrm{GL}_2(F))$). Thus θ is a subquotient of $\chi_1 \times \chi_2 \times \theta_1' \times \theta_2'$ and hence the lemma is proved. □

Next we prove a hereditary property for $\mathrm{GL}_6(F)$ using the classification theorem for $\mathrm{GL}_4(F)$.

Proposition 5.2. *Let $\pi_1 \in \mathrm{Irr}(\mathrm{GL}_2(F))$ and $\pi_2 \in \mathrm{Irr}(\mathrm{GL}_4(F))$ be two irreducible representations with symplectic periods. Then $\pi_1 \times \pi_2$ has a symplectic period. Similarly, if π_1, π_2, π_3 are irreducible representations of $\mathrm{GL}_2(F)$, with a symplectic period, then $\pi_1 \times \pi_2 \times \pi_3$ has a symplectic period.*

Proof. Any irreducible representation π of $\mathrm{GL}_2(F)$ having a symplectic period is a character, while by Theorem 1.3 any such representation of $\mathrm{GL}_4(F)$ is either a character, an irreducible product of two characters of $\mathrm{GL}_2(F)$ or a representation of the form $U(\delta, 2)$. The proposition now follows from Proposition 1.1. □

The following lemma is a consequence of Lemma 3.4 and the fact that cuspidal representations are generic (and hence not symplectic).

Lemma 5.3. *Let π_1, π_2, π_3 be irreducible admissible representations of* $\mathrm{GL}_2(F)$. *If one or more of the π_i are cuspidal and θ is an* $\mathrm{Sp}_3(F)$*-distinguished subquotient of $\pi = \pi_1 \times \pi_2 \times \pi_3$ then it is of the form $\chi_2 \times Z([\sigma_2, \nu\sigma_2])$, where χ_2 and σ_2 are a character and a supercuspidal of* $\mathrm{GL}_2(F)$ *respectively.*

Proof. Without loss of generality let π_3 be a supercuspidal. Call it σ_2. Now there can be three cases depending on π_1 and π_2.

Case 1: None of π_1 and π_2 are cuspidal. In this case σ_2 is not in the cuspidal support of $\pi_1 \times \pi_2$ and hence any irreducible subquotient of π is of the form $\sigma_2 \times J$, where J is an irreducible subquotient of $\pi_1 \times \pi_2$. By Lemma 3.4, it doesn't have a symplectic period.

Case 2: Both π_1 and π_2 are cuspidal. In this case π is of the form $\sigma_2 \times \sigma_2' \times \sigma_2''$. If none of the pairs are linked or there is exactly one linked pair among the 3, then again by Lemma 3.4, π doesn't have an $\mathrm{Sp}_3(F)$-distinguished irreducible subquotient. So π has to be either of the form $\sigma_2 \times \nu\sigma_2 \times \nu\sigma_2$, $\sigma_2 \times \sigma_2 \times \nu\sigma_2$ or $\sigma_2 \times \nu\sigma_2 \times \nu^2\sigma_2$ (up to a permutation of the π_i).

If $\pi = \sigma_2 \times \nu\sigma_2 \times \nu^2\sigma_2$ (or a permutation), then it has 4 irreducible subquotients. Of these, $Q([\sigma_2, \nu^2\sigma_2])$ is generic and $Z([\sigma_2, \nu^2\sigma_2])$ doesn't have a symplectic period (by Theorem 1.2). Now consider the subquotient $Z([\sigma_2], [\nu\sigma_2, \nu^2\sigma_2])$. It is the unique irreducible quotient of the representation $\mu_1 \times \mu_2$, where $\mu_1 = \sigma_2$ and $\mu_2 = Z([\nu\sigma_2, \nu^2\sigma_2])$. Now, using (3-1), it can be easily checked that $\mathrm{Hom}_H\big(\mathrm{ind}_{H_{2,0}}^H(\delta_{P_{2,4}}^{1/2}\mu_1 \otimes \mu_2|_{H_{2,0}}), \mathbb{C}\big)$ and $\mathrm{Hom}_H\big(\mathrm{ind}_{H_{2,2}}^H(\delta_{P_{2,4}}^{1/2}\mu_1 \otimes \mu_2|_{H_{2,2}}), \mathbb{C}\big)$ are both 0, thus implying $\mathrm{Hom}_H(\mu_1 \times \mu_2, \mathbb{C}) = 0$. So, $Z([\sigma_2], [\nu\sigma_2, \nu^2\sigma_2])$ doesn't have a symplectic period and by taking contragredients we conclude that neither does $Z([\nu^2\sigma_2], [\sigma_2, \nu\sigma_2])$. Thus π doesn't have any irreducible subquotient carrying a symplectic period.

If $\pi = \sigma_2 \times \nu\sigma_2 \times \nu\sigma_2$ (or a permutation), it is glued from the irreducible representations $\nu\sigma_2 \times Z([\sigma_2, \nu\sigma_2])$ and $\nu\sigma_2 \times Q([\sigma_2, \nu\sigma_2])$. As in the above paragraph, taking $\mu_1 = \nu\sigma_2$ and $\mu_2 = Z([\sigma_2, \nu\sigma_2])$ and using (3-1), it can be easily checked that $\nu\sigma_2 \times Z([\sigma_2, \nu\sigma_2])$ doesn't have a symplectic period. The representation $\nu\sigma_2 \times Q([\sigma_2, \nu\sigma_2])$ is generic and hence doesn't have a symplectic period, by Theorem 2.5. Similarly $\sigma_2 \times \sigma_2 \times \nu\sigma_2$ (or any of its permutations) cannot have an $\mathrm{Sp}_3(F)$-distinguished subquotient either.

Case 3: Exactly one of π_1 and π_2 is cuspidal. Up to a permutation, π then is a representation of the form $\sigma_2 \times \sigma_2' \times \theta'$, where θ' is an irreducible representation of $\mathrm{GL}_2(F)$, which isn't supercuspidal. If σ_2' and σ_2 are linked and θ' is a character then π has an $\mathrm{Sp}_3(F)$-distinguished subquotient of the required form. Otherwise, again by Lemma 3.4, it doesn't have one. □

Thus it reduces the analysis to the cases where each π_i is either a character, an irreducible principal series or a twist of the Steinberg. Note that (up to a permutation of the π_i) there are 10 possible cases. Next we show that if at least two of the π_i are irreducible principal series representations, we need not consider those cases. This reduces the analysis to the remaining 7 cases.

Lemma 5.4. *Let π_1, π_2, π_3 be irreducible admissible representations of $\mathrm{GL}_2(F)$ such that none of them are cuspidal. If two or more of the π_i are irreducible principal series representations and θ is an $\mathrm{Sp}_3(F)$-distinguished subquotient of $\pi = \pi_1 \times \pi_2 \times \pi_3$ then it also appears as a subquotient of $\pi' = \pi_1' \times \pi_2' \times \pi_3'$, where at most one of the π_i' is a principal series representation.*

Proof. If θ is as above, it is a subquotient of a representation of the form $\tau = \chi_1 \times \cdots \times \chi_6$, where each χ_i is a character of $\mathrm{GL}_1(F)$. It is easy to see that Lemma 3.4 implies that unless all the χ_i are in the same cuspidal line, θ is an irreducible product of a character of $\mathrm{GL}_2(F)$ and an irreducible $\mathrm{Sp}_2(F)$-distinguished representation. We count them in the case when all the three π_i are characters. So without loss of generality we can assume the χ_i to be integral powers of the character ν of $\mathrm{GL}_1(F)$. Say a character is linked to another if they are linked as one-element segments (page 438): explicitly, ν^a and ν^b are linked if and only if $a - b = \pm 1$. If no two of the characters appearing in τ are linked, τ is irreducible and generic and so θ cannot be its subquotient. So we can assume $\tau \cong 1 \times \nu \times \nu^a \times \nu^b \times \nu^c \times \nu^d$.

Now, assume that there is a character among $\nu^a, \ldots, \nu^d$ (say ν^a) that is not linked to any of the other characters. Collecting all the ν^a together, we see that $\theta = \nu^a \times \cdots \times \nu^a \times J$ for some irreducible representation J such that $\nu^a \times \cdots \times \nu^a$ and J satisfy the hypothesis of Lemma 3.4. So τ cannot have an $\mathrm{Sp}_3(F)$-distinguished subquotient. Thus we further assume that all the characters among $\nu^a, \ldots, \nu^d$ are linked to some other character.

Note that if there exists a partition of the characters of τ such that at least two different blocks of the partition consist of linked pairs, τ is glued from subquotients of the form $\tau_1 \times \tau_2 \times \nu^{n_1} \times \nu^{n_2}$, where τ_i is either a character or a twist of the Steinberg. Thus θ can also be obtained in the cases when two of the π_i are characters, two of them are twists of the Steinberg or one of the π_i is a character and another one is a twist of the Steinberg.

Thus if we show that, under the hypothesis that any two of the characters of τ are linked and such a partition of them doesn't exist, τ cannot have an irreducible H-distinguished subquotient we are done. Lemma 5.5 precisely does that. □

Lemma 5.5. *Call the characters ν^a and ν^b linked if and only if $a - b = \pm 1$. Let $\tau \cong 1 \times \nu \times \nu^a \times \nu^b \times \nu^c \times \nu^d$ be such that every character of it is linked to some* *other character. Assume that there doesn't exist any partition of the characters with at least two blocks consisting of linked pairs. Then τ cannot have an irreducible* $\mathrm{Sp}_3(F)$*-distinguished subquotient.*

Proof. If possible, let θ be an $\mathrm{Sp}_3(F)$-distinguished subquotient of τ. The hypothesis of the lemma implies that the cuspidal support of τ can have at most ν^{-1} or ν^2 along with 1 and ν. Moreover, ν^{-1} and ν^2 cannot both be there simultaneously and in case the support only consists of 1 and ν, τ is one of the representations $1\times 1\times 1\times 1\times 1\times \nu$ or $1\times\nu\times\nu\times\nu\times\nu\times\nu$ (up to a permutation of the characters).

If τ has ν^{-1} in the cuspidal support, 1 can be there only with multiplicity one and so the only possible forms for τ, up to a permutation of the characters, are these:

$$\nu^{-1}\times 1\times\nu\times\nu\times\nu\times\nu, \qquad \nu^{-1}\times\nu^{-1}\times 1\times\nu\times\nu\times\nu,$$
$$\nu^{-1}\times\nu^{-1}\times\nu^{-1}\times 1\times\nu\times\nu, \quad \nu^{-1}\times\nu^{-1}\times\nu^{-1}\times\nu^{-1}\times 1\times\nu.$$

Consider the last representation first. There exists a permutation of factors such that θ is a quotient of the representation obtained by taking the product in that order. An easy calculation, using arguments similar to those of case 1(a) below (where all three π_i are characters), shows that no permutation gives a product which is H-distinguished. Thus θ cannot have a symplectic period which is a contradiction. So τ cannot be $\nu^{-1}\times\nu^{-1}\times\nu^{-1}\times\nu^{-1}\times 1\times\nu$ (or any permutation of the characters). Similarly one checks that τ cannot be $\nu^{-1}\times\nu^{-1}\times\nu^{-1}\times 1\times\nu\times\nu$ (or any permutation). Since the other two representations are contragredients of the above two representations, they cannot have any H-distinguished subquotients either. Thus τ cannot be any permutation of one of them either and we conclude that τ cannot have ν^{-1} in its cuspidal support.

Observe that the possible values of τ if its cuspidal support has ν^2 instead of ν^{-1} can all be obtained by appropriately twisting the contragredients of the ones obtained in the ν^{-1} case. So τ cannot be one of them either and hence cannot have ν^2 in its cuspidal support.

Thus τ can only have 1s and νs in its cuspidal support. If $\tau = 1\times 1\times 1\times 1\times 1\times\nu$ (or any permutation of the characters), it is glued from $Z([1,\nu])\times 1\times 1\times 1\times 1$ and $Q([1,\nu])\times 1\times 1\times 1\times 1$. For the first one take $\mu_1 = Z([1,\nu])$, $\mu_2 = 1\times 1\times 1\times 1$ and use (3-1), as in case 1(a) below, to conclude that it doesn't have a symplectic period. The second one is generic and hence also cannot have symplectic period. Thus again τ cannot have θ as a subquotient and so it cannot be a permutation of $1\times 1\times 1\times 1\times 1\times\nu$. Taking contragredients we conclude that it cannot be a permutation of $1\times\nu\times\nu\times\nu\times\nu\times\nu$ either. This shows that even 1 and ν cannot be in the cuspidal support of τ. This is a contradiction to our initial assumption that τ has an $\mathrm{Sp}_3(F)$-distinguished subquotient. $\square$

Thus we need to analyze only the remaining seven cases.

Case 1: π_1, π_2, π_3 are all characters. The representations in this case are of the form

$$\pi = Z([\chi_1, \chi_1\nu]) \times Z([\chi_1', \chi_1'\nu]) \times Z([\chi_1'', \chi_1''\nu]).$$

If there are no links among the three segments, the representation is an irreducible product of characters of $\mathrm{GL}_2(F)$ and is symplectic by Proposition 1.1. Assume now that there is exactly one link. Without loss of generality we can assume that $[\chi_1', \chi_1'\nu]$ and $[\chi_1'', \chi_1''\nu]$ are linked, and that $[\chi_1, \chi_1\nu]$ is not linked to either. Clearly then, $\chi_1 \neq \chi_1', \chi_1''$. So, $[\chi_1, \chi_1\nu]$ is disjoint and not linked to either $[\chi_1', \chi_1'\nu]$ or $[\chi_1'', \chi_1''\nu]$. Observe that if a segment Δ_1 is not linked to Δ_2 and Δ_3 (where Δ_2 and Δ_3 are linked), it is not linked to $\Delta_2 \cup \Delta_3$ or $\Delta_2 \cap \Delta_3$ either. So by Theorem 7.1 and Proposition 8.5 of [Zelevinsky 1980], each irreducible subquotient of π is of the form $Z([\chi_1, \chi_1\nu]) \times \theta'$, where θ' is an irreducible subquotient of $Z([\chi_1', \chi_1'\nu]) \times Z([\chi_1'', \chi_1''\nu])$. Moreover if the subquotient is H-distinguished, observe then that $Z([\chi_1, \chi_1\nu])$ and θ' satisfy the hypothesis of Lemma 3.4. Thus by Lemma 3.4 any irreducible $\mathrm{Sp}_3(F)$-distinguished subquotient is an irreducible product of H-distinguished representations of $\mathrm{GL}_2(F)$ and $\mathrm{GL}_4(F)$.

Hence we look at the cases where there are at least two links among the segments. Without loss of generality we can assume χ_1 to be trivial. Following are the eight possible cases:

(a) $Z([1, \nu]) \times Z([\nu, \nu^2]) \times Z([\nu^3, \nu^4])$
(b) $Z([1, \nu]) \times Z([\nu, \nu^2]) \times Z([\nu^2, \nu^3])$
(c) $Z([1, \nu]) \times Z([\nu^2, \nu^3]) \times Z([\nu^4, \nu^5])$
(d) $Z([1, \nu]) \times Z([\nu, \nu^2]) \times Z([\nu, \nu^2])$
(e) $Z([1, \nu]) \times Z([\nu^2, \nu^3]) \times Z([\nu^2, \nu^3])$
(f) $Z([1, \nu]) \times Z([\nu^2, \nu^3]) \times Z([\nu^3, \nu^4])$
(g) $Z([1, \nu]) \times Z([1, \nu]) \times Z([\nu, \nu^2])$
(h) $Z([1, \nu]) \times Z([1, \nu]) \times Z([\nu^2, \nu^3])$

In each case we will evaluate all possible irreducible subquotients using [Zelevinsky 1980, Theorem 7.1], to determine whether it has a symplectic period.

(a) $\pi = Z([1, \nu]) \times Z([\nu, \nu^2]) \times Z([\nu^3, \nu^4])$. In this case, all irreducible subquotients of π are $Z([1, \nu], [\nu, \nu^2], [\nu^3, \nu^4])$, $Z([1, \nu^2], [\nu], [\nu^3, \nu^4])$, $Z([1, \nu^4], [\nu])$ and $Z([1, \nu], [\nu, \nu^4])$. We now analyze each of these representations.

- $\theta = Z([1, \nu], [\nu, \nu^2], [\nu^3, \nu^4])$ is the only irreducible submodule of $Z([\nu^3, \nu^4]) \times Z([\nu, \nu^2]) \times Z([1, \nu])$. Using Lemma 2.3 and taking contragredients we get that θ is the unique irreducible quotient of $\pi = Z([1, \nu]) \times Z([\nu, \nu^2]) \times Z([\nu^3, \nu^4])$. Since $Z([\nu, \nu^2], [\nu^3, \nu^4])$ is a quotient of $Z([\nu, \nu^2]) \times Z([\nu^3, \nu^4])$, θ is also the unique irreducible quotient of $Z([1,\nu]) \times Z([\nu,\nu^2],[\nu^3,\nu^4])$.

Let $\mu_1 = Z([1,\nu])$ and $\mu_2 = Z([\nu,\nu^2],[\nu^3,\nu^4])$. Now, $\mu_1 \times \mu_2$ is glued from

$$\mathrm{ind}_{H_{2,0}}^{H}(\delta_{P_{2,4}}^{1/2}\mu_1 \otimes \mu_2|_{H_{2,0}}) \quad \text{and} \quad \mathrm{ind}_{H_{2,2}}^{H}(\delta_{P_{2,4}}^{1/2}\mu_1 \otimes \mu_2|_{H_{2,2}})$$

(see Section 3). Since μ_2 doesn't have a symplectic period (by Lemma 3.6),

$$\mathrm{Hom}_H\bigl(\mathrm{ind}_{H_{2,0}}^{H}(\delta_{P_{2,4}}^{1/2}\mu_1 \otimes \mu_2|_{H_{2,0}}),\mathbb{C}\bigr) = \mathrm{Hom}_{\mathrm{Sp}_1}(\mu_1,\mathbb{C}) \otimes \mathrm{Hom}_{\mathrm{Sp}_2}(\mu_2,\mathbb{C})$$

is zero, by (3-1). On the other hand, $\mathrm{Hom}_H\bigl(\mathrm{ind}_{H_{2,2}}^{H}(\delta_{P_{2,4}}^{1/2}\mu_1 \otimes \mu_2|_{H_{2,2}}),\mathbb{C}\bigr)$ equals

$$\mathrm{Hom}_{\mathrm{GL}_2\times\mathrm{Sp}_1}\bigl(\nu^{-1}\mu_1 \otimes r_{(2,2),(4)}(\mu_2),\mathbb{C}\bigr). \tag{5-1}$$

An application of the geometrical lemma ([Bernstein and Zelevinsky 1977, Lemma 2.12]) shows that $r_{(2,2),(4)}(Z([\nu,\nu^2]) \times Z([\nu^3,\nu^4]))$ is glued from the irreducible representations $Z([\nu,\nu^2]) \otimes Z([\nu^3,\nu^4])$, $Z([\nu^3,\nu^4]) \otimes Z([\nu,\nu^2])$ and $(\nu \times \nu^3) \otimes (\nu^2 \times \nu^4)$. Jacquet functor being an exact functor, $r_{(2,2),(4)}(\mu_2)$ is glued from one or more of these terms. It can be checked that replacing $r_{(2,2),(4)}(\mu_2)$ by each of these three representations makes the group (5-1) trivial.

Thus, we get that $Z([1,\nu]) \times Z([\nu,\nu^2],[\nu^3,\nu^4])$ doesn't have a symplectic period. If θ had a symplectic period, this would have given a nontrivial $\mathrm{Sp}_3(F)$-invariant linear functional of $Z([1,\nu]) \times Z([\nu,\nu^2],[\nu^3,\nu^4])$ (by composing the one for θ with the quotient map), a contradiction. Hence θ doesn't have a symplectic period.

- If $\theta = Z([1,\nu^2],[\nu],[\nu^3,\nu^4])$, then θ is the unique irreducible submodule of $Z([\nu^3,\nu^4]) \times \nu \times Z([1,\nu^2])$. Using Lemma 2.3 and taking contragredients, θ is the unique irreducible quotient of $Z([1,\nu^2]) \times \nu \times Z([\nu^3,\nu^4])$. Applying Lemma 2.3 again, $\tilde{\theta}$ is the unique irreducible quotient of $Z([\nu^{-4},\nu^{-3}]) \times \nu^{-1} \times Z([\nu^{-2},1])$. By taking $\mu_1 = Z([\nu^{-4},\nu^{-3}])$ and $\mu_2 = \nu^{-1} \times Z([\nu^{-2},1])$ and a similar calculation as above shows that $Z([\nu^{-4},\nu^{-3}]) \times \nu^{-1} \times Z([\nu^{-2},1])$, and hence $\tilde{\theta}$, doesn't have a symplectic period. Thus even θ is not $\mathrm{Sp}_3(F)$-distinguished.
- If $\theta = Z([1,\nu^4],[\nu]) \cong Z([1,\nu^4]) \times \nu$ (by Proposition 8.5 of [Zelevinsky 1980]), by Theorem 2.1, it is an irreducible quotient of $Z([\nu^3,\nu^4]) \times Z([1,\nu^2]) \times \nu$. Now doing a similar calculation as in the first case by taking $\mu_1 = Z([\nu^3,\nu^4])$ and $\mu_2 = Z([1,\nu^2]) \times \nu$, we get that $Z([\nu^3,\nu^4]) \times Z([1,\nu^2]) \times \nu$, and hence θ, doesn't have a symplectic period.
- If $\theta = Z([1,\nu],[\nu,\nu^4])$, it has a symplectic period by Lemma 3.5.

This concludes case (a).

(b) $\pi = Z([1,\nu]) \times Z([\nu,\nu^2]) \times Z([\nu^2,\nu^3])$. Here the irreducible subquotients of π are $Z([1,\nu],[\nu,\nu^2],[\nu^2,\nu^3])$, $Z([1,\nu^2],[\nu],[\nu^2,\nu^3])$, $Z([1,\nu],[\nu,\nu^3],[\nu^2])$, $Z([1,\nu^3],[\nu],[\nu^2])$, $Z([1,\nu^2],[\nu,\nu^3])$, and $Z([1,\nu^3],[\nu,\nu^2])$. We now analyze each of these representations.

- If $\theta = Z([1,\nu],[\nu,\nu^2],[\nu^2,\nu^3])$, we have $\theta = Q([1,\nu^2],[\nu,\nu^3])$ by Theorem A.10(iii) of [Tadić 1986]. Twisting θ by an appropriate power of ν makes it a unitary representation, which then turns out to be $\mathrm{Sp}_3(F)$-distinguished by Theorem 1.2.
- If $\theta = Z([1,\nu^2],[\nu],[\nu^2,\nu^3])$, then θ is the unique irreducible submodule of $Z([\nu^2,\nu^3]) \times \nu \times Z([1,\nu^2])$. Using Lemma 2.3 and taking contragredients we get that θ is the unique irreducible quotient of $\pi = Z([1,\nu^2]) \times \nu \times Z([\nu^2,\nu^3])$. Using Lemma 2.3 again, we get that $\tilde{\theta}$ is the unique irreducible quotient of $Z([\nu^{-3},\nu^{-2}]) \times \nu^{-1} \times Z([\nu^{-2},1])$. By taking $\mu_1 = Z([\nu^{-3},\nu^{-2}])$ and $\mu_2 = \nu^{-1} \times Z([\nu^{-2},1])$, and doing a similar calculation as in (a), we get that $Z([\nu^{-3},\nu^{-2}]) \times \nu^{-1} \times Z([\nu^{-2},1])$ and hence $\tilde{\theta}$, doesn't have a symplectic period. Thus even θ is not $\mathrm{Sp}_3(F)$-distinguished.
- If $\theta = Z([1,\nu],[\nu,\nu^3],[\nu^2])$, it can be obtained by twisting the contragredient of $Z([1,\nu^2],[\nu],[\nu^2,\nu^3])$, which, as showed in the last paragraph, doesn't have a symplectic period.
- If $\theta = Z([1,\nu^3],[\nu],[\nu^2])$, it is the unique irreducible submodule of
$$\nu^2 \times \nu \times Z([1,\nu^3]) \cong Z([1,\nu^3]) \times \nu^2 \times \nu.$$
Thus it is the unique irreducible submodule of $Z([1,\nu^3]) \times Q([\nu,\nu^2])$. Using Lemma 2.3 and taking contragredients we get that θ is the unique irreducible quotient of $Q([\nu,\nu^2]) \times Z([1,\nu^3])$. Now doing a similar calculation as in (a) by taking $\mu_1 = Q([\nu,\nu^2])$ and $\mu_2 = Z([1,\nu^3])$, we get that $Q([\nu,\nu^2]) \times Z([1,\nu^3])$, and hence θ, doesn't have a symplectic period.
- If $\theta = Z([1,\nu^2],[\nu,\nu^3])$, by Theorem A.10(iii) of [Tadić 1986],
$$\theta = Q([1,\nu],[\nu,\nu^2],[\nu^2,\nu^3]).$$
Twisting θ by an appropriate power of ν makes it a unitary representation. By Theorem 1.2 it doesn't have a symplectic period.
- If $\theta = Z([1,\nu^3],[\nu,\nu^2]) \cong Z([1,\nu^3]) \times Z([\nu,\nu^2])$, it has a symplectic period by Proposition 5.2.

This concludes case (b).

(c) $\pi = Z([1,\nu]) \times Z([\nu^2,\nu^3]) \times Z([\nu^4,\nu^5])$. Here the irreducible subquotients of π are $Z([1,\nu],[\nu^2,\nu^3],[\nu^4,\nu^5])$, $Z([1,\nu^3],[\nu^4,\nu^5])$, $Z([1,\nu],[\nu^2,\nu^5])$ and $Z([1,\nu^5])$. Of these, by Lemma 3.6, $Z([1,\nu^3],[\nu^4,\nu^5])$ and $Z([1,\nu],[\nu^2,\nu^5])$

do not have a symplectic period. $Z([1,\nu^5])$ being a character clearly has a symplectic period. We analyze the remaining representation.

- If $\theta = Z([1,\nu],[\nu^2,\nu^3],[\nu^4,\nu^5])$, by definition, it is the unique irreducible submodule of $Z([\nu^4,\nu^5]) \times Z([\nu^2,\nu^3]) \times Z([1,\nu])$. Thus we get that it is the unique irreducible quotient of $\pi = Z([1,\nu]) \times Z([\nu^2,\nu^3]) \times Z([\nu^4,\nu^5])$ (using Lemma 2.3 and taking contragredients). Since $Z([\nu^2,\nu^3],[\nu^4,\nu^5])$ is a quotient of $Z([\nu^2,\nu^3]) \times Z([\nu^4,\nu^5])$, we get θ is a quotient of $Z([1,\nu]) \times Z([\nu^2,\nu^3],[\nu^4,\nu^5])$. By taking $\mu_1 = Z([1,\nu])$ and $\mu_2 = Z([\nu^2,\nu^3],[\nu^4,\nu^5])$, and doing a calculation as in (a), we get that $Z([1,\nu]) \times Z([\nu^2,\nu^3],[\nu^4,\nu^5])$ and hence θ, doesn't have a symplectic period.

This concludes case (c).

(d) $\pi = Z([1,\nu]) \times Z([\nu,\nu^2]) \times Z([\nu,\nu^2])$. In this case, all irreducible subquotients of π are $Z([1,\nu],[\nu,\nu^2],[\nu,\nu^2])$ and $Z([1,\nu^2],[\nu],[\nu,\nu^2])$. We now analyze both of these representations.

- If $\theta = Z([1,\nu^2],[\nu],[\nu,\nu^2]) \cong Z([1,\nu^2]) \times Z([\nu,\nu^2]) \times \nu$, by Theorem 3.7 of [Offen and Sayag 2007b], it has a mixed Klyachko model. Hence by Theorem 2.5, it is not $\mathrm{Sp}_3(F)$-distinguished.
- If $\theta = Z([1,\nu],[\nu,\nu^2],[\nu,\nu^2])$, it is the unique irreducible submodule of $Z([\nu,\nu^2]) \times Z([\nu,\nu^2]) \times Z([1,\nu])$. Thus it is the unique irreducible submodule of $Z([\nu,\nu^2]) \times Z([1,\nu],[\nu,\nu^2]) \cong Z([\nu,\nu^2]) \times Q([1,\nu],[\nu,\nu^2])$ (by Example 11.4 in [Zelevinsky 1980]). By Theorem 1 of [Badulescu et al. 2012], this representation is irreducible and so $\theta \cong Z([\nu,\nu^2]) \times Z([1,\nu],[\nu,\nu^2])$ and has a symplectic period by Proposition 5.2.

This concludes case (d).

(e) $\pi = Z([1,\nu]) \times Z([\nu^2,\nu^3]) \times Z([\nu^2,\nu^3])$. In this case, all irreducible subquotients of π are $Z([1,\nu],[\nu^2,\nu^3],[\nu^2,\nu^3])$ and $Z([1,\nu^3],[\nu^2,\nu^3])$. We now analyze both of these representations.

- If $\theta = Z([1,\nu^3],[\nu^2,\nu^3]) \cong Z([1,\nu^3]) \times Z([\nu^2,\nu^3])$, it has a symplectic period by Proposition 5.2.
- If $\theta = Z([1,\nu],[\nu^2,\nu^3],[\nu^2,\nu^3])$, it is the unique irreducible submodule of $Z([\nu^2,\nu^3]) \times Z([\nu^2,\nu^3]) \times Z([1,\nu])$. Now, $Z([\nu^2,\nu^3]) \times Z([\nu^2,\nu^3]) \times Z([1,\nu])$ is glued from $Z([\nu^2,\nu^3]) \times Z([1,\nu],[\nu^2,\nu^3])$ and $Z([\nu^2,\nu^3]) \times Z([1,\nu^3])$. Using Theorem 1 of [Badulescu et al. 2012], we get the irreducibility of

$$Z([\nu^2,\nu^3]) \times Z([1,\nu],[\nu^2,\nu^3]) \cong Z([\nu^2,\nu^3]) \times Q([1],[\nu,\nu^2],[\nu^3]).$$

Thus $\theta \cong Z([\nu^2,\nu^3]) \times Z([1,\nu],[\nu^2,\nu^3])$, implying that $\tilde{\theta} \cong Z([\nu^{-3},\nu^{-2}]) \times Z([\nu^{-1},1],[\nu^{-3},\nu^{-2}])$. A calculation as in case (a), taking $\mu_1 = Z([\nu^{-3},\nu^{-2}])$

and $\mu_2 = Z([\nu^{-1}, 1], [\nu^{-3}, \nu^{-2}])$, shows that $\tilde{\theta} = \mu_1 \times \mu_2$ doesn't have a symplectic period. Thus even θ is not $\mathrm{Sp}_3(F)$-distinguished.

This concludes case (e).

The remaining cases, (f), (g), and (h), are dealt with by duality: all irreducible subquotients of π are twists of the contragredients of those obtained in cases (a), (d), and (e), respectively. Hence the only subquotients with a symplectic period are up to a twist, duals of the ones already obtained previously.

Case 2: π_1, π_2, π_3 are all twists of Steinberg. The representations that we are looking at in this case are of the form

$$\pi = Q([\chi_1, \chi_1\nu]) \times Q([\chi_1', \chi_1'\nu]) \times Q([\chi_1'', \chi_1''\nu]).$$

The following result will be used repeatedly in the analysis of this case.

Lemma 5.6. *Let $\pi = Q([\nu^a, \nu^{a+1}]) \times Q([\nu^b, \nu^{b+1}]) \times Q([\nu^c, \nu^{c+1}])$. Then π has a symplectic period only if $a = b = c + 1$.*

Proof. Let $\mu_1 = Q([\nu^a, \nu^{a+1}])$ and $\mu_2 = Q([\nu^b, \nu^{b+1}]) \times Q([\nu^c, \nu^{c+1}])$. Since μ_1 doesn't have a symplectic period (by Theorem 2.5), the group

$$\mathrm{Hom}_H\big(\mathrm{ind}_{H_{2,0}}^H(\delta_{P_{2,4}}^{1/2}\mu_1 \otimes \mu_2|_{H_{2,0}}), \mathbb{C}\big) = \mathrm{Hom}_{\mathrm{Sp}_1(F)}\big(\mu_1, 1\big) \otimes \mathrm{Hom}_{\mathrm{Sp}_2(F)}(\mu_2, \mathbb{C})$$

is zero, by (3-1). Thus the other term, $\mathrm{Hom}_H\big(\mathrm{ind}_{H_{2,2}}^H(\delta_{P_{2,4}}^{1/2}\mu_1 \otimes \mu_2|_{H_{2,2}}), \mathbb{C}\big)$, has to be nonzero. Now, $r_{(2,2),(4)}(Q([\nu^b, \nu^{b+1}]) \times Q([\nu^c, \nu^{c+1}])$ is glued from $Q([\nu^b, \nu^{b+1}]) \otimes Q([\nu^c, \nu^{c+1}])$, $Q([\nu^c, \nu^{c+1}]) \otimes Q([\nu^b, \nu^{b+1}])$ and

$$(\nu^{b+1} \times \nu^{c+1}) \otimes (\nu^b \times \nu^c),$$

by Lemma 2.12 of [Bernstein and Zelevinsky 1977]. It can be checked easily that replacing $r_{(2,2),(4)}(\mu_2)$ by the first two of the three representations makes this Hom space 0. Thus,

$$\mathrm{Hom}_{\mathrm{GL}_2(F)\times \mathrm{Sp}_1(F)}\big(\nu^{-1}Q([\nu^a, \nu^{a+1}]) \otimes (\nu^{b+1} \times \nu^{c+1}) \otimes (\nu^b \times \nu^c), \mathbb{C}\big) \neq 0.$$

Solving the equations for this to be nonzero gives the lemma. □

By similar arguments using Lemma 3.4 as in case 1 it can be easily concluded that if there is at most one link among the three segments then π doesn't have an H-distinguished subquotient. Thus we look at the case where there are at least two links among the segments. Since twisting by a character doesn't matter to us, without loss of generality we can assume χ_1 to be trivial. As before we have eight possible cases:

(a) $Q([1, \nu]) \times Q([\nu, \nu^2]) \times Q([\nu^3, \nu^4])$

(b) $Q([1, \nu]) \times Q([\nu, \nu^2]) \times Q([\nu^2, \nu^3])$

(c) $Q([1,\nu])\times Q([\nu^2,\nu^3])\times Q([\nu^4,\nu^5])$

(d) $Q([1,\nu])\times Q([\nu,\nu^2])\times Q([\nu,\nu^2])$

(e) $Q([1,\nu])\times Q([\nu^2,\nu^3])\times Q([\nu^2,\nu^3])$

(f) $Q([1,\nu])\times Q([\nu^2,\nu^3])\times Q([\nu^3,\nu^4])$

(g) $Q([1,\nu])\times Q([1,\nu])\times Q([\nu,\nu^2])$

(h) $Q([1,\nu])\times Q([1,\nu])\times Q([\nu^2,\nu^3])$

(a) $\pi = Q([1,\nu])\times Q([\nu,\nu^2])\times Q([\nu^3,\nu^4])$. Here, all irreducible subquotients of π are $Q([1,\nu],[\nu,\nu^2],[\nu^3,\nu^4])$, $Q([1,\nu^2],[\nu],[\nu^3,\nu^4])$, $Q([1,\nu^4],[\nu])$ and $Q([1,\nu],[\nu,\nu^4])$. We now analyze each of these representations.

- If $\theta = Q([1,\nu],[\nu,\nu^2],[\nu^3,\nu^4])$, it is the unique irreducible quotient of
$$Q([\nu^3,\nu^4])\times Q([\nu,\nu^2])\times Q([1,\nu]),$$
which doesn't have a symplectic period by Lemma 5.6. Hence θ doesn't have one.

- If $\theta = Q([1,\nu^2],[\nu],[\nu^3,\nu^4])$, it is the unique irreducible quotient of
$$Q([\nu^3,\nu^4])\times \nu\times Q([1,\nu^2]).$$
Thus it is an irreducible quotient of $Q([\nu^3,\nu^4])\times Q([1,\nu])\times Q([\nu,\nu^2])$ (by Theorem 2.2), which doesn't have a symplectic period by Lemma 5.6. Hence θ doesn't have one.

- If $\theta = Q([1,\nu^4],[\nu])\cong Q([1,\nu^4])\times\nu$ (by Proposition 8.5 of [Zelevinsky 1980]), it is generic and hence doesn't have a symplectic period (by Theorem 2.5).

- If $\theta = Q([1,\nu],[\nu,\nu^4])$, it is a quotient of $Q([\nu,\nu^2])\times Q([\nu^3,\nu^4])\times Q([1,\nu])$, which doesn't have a symplectic period by Lemma 5.6. Hence it doesn't have one too.

This concludes case (a).

(b) $\pi = Q([1,\nu])\times Q([\nu,\nu^2])\times Q([\nu^2,\nu^3])$. Here the irreducible subquotients of π are $Q([1,\nu],[\nu,\nu^2],[\nu^2,\nu^3])$, $Q([1,\nu^2],[\nu],[\nu^2,\nu^3])$, $Q([1,\nu],[\nu,\nu^3],[\nu^2])$, $Q([1,\nu^3],[\nu],[\nu^2])$, $Q([1,\nu^2],[\nu,\nu^3])$ and $Q([1,\nu^3],[\nu,\nu^2])$.

- If $\theta = Q([1,\nu],[\nu,\nu^2],[\nu^2,\nu^3])$, twisting θ by an appropriate power of ν makes it a unitary representation. By Theorem 1.2 it doesn't have a symplectic period.

- If $\theta = Q([1,\nu^2],[\nu],[\nu^2,\nu^3])$, it is the unique irreducible quotient of
$$Q([\nu^2,\nu^3])\times \nu\times Q([1,\nu^2]).$$
This itself is a quotient of $Q([\nu^2,\nu^3])\times Q([1,\nu])\times Q([\nu,\nu^2])$, which doesn't have a symplectic period by Lemma 5.6. Hence θ doesn't have one.

- If $\theta = Q([1,\nu],[\nu,\nu^3],[\nu^2])$, it can be obtained by twisting the contragredient of $Q([1,\nu^2],[\nu],[\nu^2,\nu^3])$, which as showed in the last paragraph, doesn't have a symplectic period.
- If $\theta = Q([1,\nu^3],[\nu],[\nu^2])$, it is the unique irreducible quotient of $\nu^2 \times \nu \times Q([1,\nu^3])$ and hence of $Z([\nu,\nu^2]) \times Q([1,\nu^3])$. Now doing a similar calculation as in case 1(a) by taking $\mu_1 = Z([\nu,\nu^2])$ and $\mu_2 = Q([1,\nu^3])$ we get that $Z([\nu,\nu^2]) \times Q([1,\nu^3])$, and hence θ, doesn't have a symplectic period.
- If $\theta = Q([1,\nu^2],[\nu,\nu^3])$, it has a symplectic period by Theorem 1.2.
- If $\theta = Q([1,\nu^3],[\nu,\nu^2]) \cong Q([1,\nu^3]) \times Q([\nu,\nu^2])$, (by Proposition 8.5 of [Zelevinsky 1980]), it is generic and hence doesn't have a symplectic period (by Theorem 2.5).

This concludes case (b).

(c) $\pi = Q([1,\nu]) \times Q([\nu^2,\nu^3]) \times Q([\nu^4,\nu^5])$. Here the irreducible subquotients of π are $Q([1,\nu],[\nu^2,\nu^3],[\nu^4,\nu^5])$, $Q([1,\nu^3],[\nu^4,\nu^5])$, $Q([1,\nu],[\nu^2,\nu^5])$ and $Q([1,\nu^5])$.

- If $\theta = Q([1,\nu^5])$, it is generic and hence doesn't have a symplectic period.
- If $\theta = Q([1,\nu],[\nu^2,\nu^3],[\nu^4,\nu^5])$, it is the unique irreducible quotient of $Q([\nu^4,\nu^5]) \times Q([\nu^2,\nu^3]) \times Q([1,\nu])$, which doesn't have a symplectic period by Lemma 5.6. Hence it doesn't have one. The other two cases are dealt similarly.
- If $\theta = Q([1,\nu^3],[\nu^4,\nu^5])$, it is the unique irreducible quotient of $Q([\nu^4,\nu^5]) \times Q([1,\nu^3])$. Now this itself is a quotient of $Q([\nu^4,\nu^5]) \times Q([1,\nu]) \times Q([\nu^2,\nu^3])$, which doesn't have a symplectic period by Lemma 5.6. Hence θ doesn't have one.
- If $\theta = Q([1,\nu],[\nu^2,\nu^5])$, it can be obtained by twisting the contragredient of $Q([1,\nu^3],[\nu^4,\nu^5])$, which as shown in the last paragraph, doesn't have a symplectic period.

This concludes case (c).

(d) $\pi = Q([1,\nu]) \times Q([\nu,\nu^2]) \times Q([\nu,\nu^2])$. In this case, all irreducible subquotients of π are $Q([1,\nu],[\nu,\nu^2],[\nu,\nu^2])$ and $Q([1,\nu^2],[\nu],[\nu,\nu^2])$.

- If $\theta = Q([1,\nu^2],[\nu],[\nu,\nu^2]) \cong Q([1,\nu^2]) \times \nu \times Q([\nu,\nu^2])$, it is generic and hence doesn't have a symplectic period.
- If $\theta = Q([1,\nu],[\nu,\nu^2],[\nu,\nu^2])$, it is the sole irreducible quotient of $Q([\nu,\nu^2]) \times Q([\nu,\nu^2]) \times Q([1,\nu])$. Thus it is the unique irreducible quotient of $Q([\nu,\nu^2]) \times Q([1,\nu],[\nu,\nu^2]) \cong Q([\nu,\nu^2]) \times Z([1,\nu],[\nu,\nu^2])$ (by Example 11.4 in [Zelevinsky 1980]). By Theorem 1 of [Badulescu et al. 2012], this representation is

irreducible and so $\theta \cong Q([\nu,\nu^2]) \times Z([1,\nu],[\nu,\nu^2])$. So it is a quotient of $Q([\nu,\nu^2]) \times Z([1,\nu]) \times Z([\nu,\nu^2])$. Now a calculation as in case 1(a), taking $\mu_1 = Q([\nu,\nu^2])$ and $\mu_2 = Z([1,\nu]) \times Z([\nu,\nu^2])$, yields that $Q([\nu,\nu^2]) \times Z([1,\nu]) \times Z([\nu,\nu^2])$, and hence θ, doesn't have a symplectic period.

This concludes case (d).

(e) $\pi = Q([1,\nu]) \times Q([\nu^2,\nu^3]) \times Q([\nu^2,\nu^3])$. In this case, all irreducible subquotients of π are $Q([1,\nu],[\nu^2,\nu^3],[\nu^2,\nu^3])$ and $Q([1,\nu^3],[\nu^2,\nu^3])$.

- If $\theta = Q([1,\nu^3],[\nu^2,\nu^3]) \cong Q([1,\nu^3]) \times Q([\nu^2,\nu^3])$, (by Proposition 8.5 of [Zelevinsky 1980]), it is generic and hence doesn't have a symplectic period (by Theorem 2.5).
- If $\theta = Q([1,\nu],[\nu^2,\nu^3],[\nu^2,\nu^3])$, it is the unique irreducible quotient of $Q([\nu^2,\nu^3]) \times Q([\nu^2,\nu^3]) \times Q([1,\nu])$. This doesn't have a symplectic period by Lemma 5.6 and so θ doesn't have one too.

This concludes case (e).

As before, in cases (f), (g), and (h) all the irreducible subquotients of π are twists of the contragredients of the ones obtained in cases (a), (d), and (e) respectively. Hence the only subquotients with a symplectic period are up to a twist, duals of the ones already obtained previously.

Cases 3–7: The remaining five cases of $\pi_1 \times \pi_2 \times \pi_3$ are dealt similarly, proving Theorem 1.4. We just mention that no new H-distinguished subquotients are obtained from the other cases.

6. Conjectures for the general case

Theorem 1.3 and Theorem 1.4 prompt us to make certain conjectures for the general $2n$ case. In order to do so we need to set up notation.

Define $\mathfrak{S}'$ as the set of all representations of $\mathrm{GL}_{2n}(F)$ of the form $Z(\Delta_1,\dots,\Delta_r)$ that satisfy the following properties:

(1) All the segments are in the same cuspidal line.

(2) Each segment is of even length.

(3) No two segments have the beginning element in common.

(4) Conditions (1) and (3) imply that there is a natural ordering of the segments (with respect to the beginning element). Arrange $\Delta_1,\dots,\Delta_r$ accordingly. We require that the intersection of each segment with its neighbors is odd in length, in particular is nonempty.

The set $\mathfrak{S}'$ is contained in the set of ladder representations as defined in [Badulescu et al. 2012].

Further define $\mathfrak{S} \subset \cup_{i\geq 1} \mathrm{Irr}(\mathrm{GL}_{2i}(F))$ to be the set of all irreducible products of elements in $\mathfrak{S}'$; i.e.,

$$\mathfrak{S} = \{\pi_1 \times \cdots \times \pi_t \mid \pi_1, \ldots, \pi_t \in \mathfrak{S}' \text{ and the product is irreducible}\}.$$

Let us now state the conjecture in the general case using the above notation.

Conjecture 6.1. *Let θ be an irreducible representation of* $\mathrm{GL}_{2n}(F)$ *carrying a symplectic period. Then there exists $\pi_1, \ldots, \pi_t \in \mathfrak{S}'$ such that*

$$\theta \cong \pi_1 \times \cdots \times \pi_t.$$

In other words, $\theta \in \mathfrak{S}$.

The following proposition verifies the conjecture for unitary representations.

Proposition 6.2. *Let θ be an irreducible unitary representation having a symplectic period. Then $\theta \in \mathfrak{S}$.*

Proof. Let $\delta = Q([\rho\nu^{\frac{1-d}{2}}, \rho\nu^{\frac{d-1}{2}}])$. By Theorem A.10(iii) of [Tadić 1986], $U(\delta, t) = Z(\Delta_1, \ldots, \Delta_d)$, where

$$\begin{aligned}
\Delta_1 &= [(\rho\nu^{\frac{1-d}{2}})\nu^{\frac{1-t}{2}}, (\rho\nu^{\frac{1-d}{2}})\nu^{\frac{t-1}{2}}],\\
\Delta_2 &= [(\rho\nu^{\frac{3-d}{2}})\nu^{\frac{1-t}{2}}, (\rho\nu^{\frac{3-d}{2}})\nu^{\frac{t-1}{2}}], \ldots,\\
\Delta_d &= [(\rho\nu^{\frac{d-1}{2}})\nu^{\frac{1-t}{2}}, (\rho\nu^{\frac{d-1}{2}})\nu^{\frac{t-1}{2}}].
\end{aligned}$$

The intersection of each segment with both its neighbors, if they are arranged in the order of precedence, is of length $t-1$. So if t is even, $U(\delta, t) \in \mathfrak{S}'$. The proposition then follows from Theorem 1.2. □

That $U(\delta, 2m) \in \mathfrak{S}'$ leads to an obvious question generalizing Proposition 1.1, which we state as the next conjecture.

Conjecture 6.3 (hereditary property). *Let $\theta \in \mathfrak{S}'$. Then θ has a symplectic period. Moreover, if $\theta_1, \ldots, \theta_d \in \mathfrak{S}'$ then $\theta_1 \times \cdots \times \theta_d$ has a symplectic period.*

Conjecture 6.1 and Conjecture 6.3 together imply that $\mathfrak{S}$ is precisely the set of H-distinguished representations of the linear groups. Thus Theorem 1.3 and Theorem 1.4 prove the conjectures for $\mathrm{GL}_4(F)$ and $\mathrm{GL}_6(F)$. Note that the above conjectures together imply that the property of having a symplectic period is dependent only on the combinatorial structure of the segments involved and not on the building blocks, i.e., the cuspidal representations. More precisely:

Conjecture 6.4. *Let $\pi \in$* $\mathrm{Irr}(\mathrm{GL}_{2n}(F))$ *be of the form $Z(\Delta_1, \ldots, \Delta_r)$ such that all the segments are in the same cuspidal line. Let $\rho \in$* $\mathrm{Irr}(\mathrm{GL}_m(F))$ *be an element of the line. Let Δ_i' be the segment obtained from Δ_i by replacing ρ with the trivial representation of $F^\times$ and π' be the representation $Z(\Delta_1', \ldots, \Delta_r')$ of* $\mathrm{GL}_{2n/m}(F)$.

(1) *If* $2n/m$ *is even*, π *has a symplectic period if and only if* π' *has a symplectic period.*

(2) *If* $2n/m$ *is odd*, π *doesn't have a symplectic period.*

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