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LINKED TRIPLES OF QUATERNION ALGEBRAS

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Let F be a field of characteristic different from 2, and let Q_1, Q_2, Q_3 be quaternion algebras over F such that any element in $\mathrm{Br}(F)$ generated by Q_1, Q_2, Q_3 has index at most 2. For the triple $\{Q_1, Q_2, Q_3\}$ we construct a certain invariant, lying in $I^3(F)$, which is a 4-fold Pfister form, provided the algebras Q_1, Q_2, Q_3 have a common slot. Among other results we prove that the algebras Q_1, Q_2, Q_3 have a common slot if and only if the torsion of the group $\mathrm{CH}^2(X_1 \times X_2 \times X_3)$ is zero, where X_i is the projective conic associated with the algebra Q_i.

Let F be a field with char $F \neq 2$, let $Q_1, \ldots, Q_n$ be quaternion algebras over F, and $G \subset {}_2\mathrm{Br}(F)$ the group generated by these quaternions. We call the collection $\{Q_1, \ldots, Q_n\}$ linked if $\mathrm{ind}(\alpha) \leq 2$ for any $\alpha \in G$. We say that the collection $\{Q_1, \ldots, Q_n\}$ has a common slot if there are $a, b_1, \ldots, b_n \in F^*$ such that $Q_i = (a, b_i)$ for each $1 \leq i \leq n$. Obviously, if the collection $\{Q_1, \ldots, Q_n\}$ has a common slot, then it is linked. The opposite statement is true for $n = 2$ [Scharlau 1985], but is not true for $n \geq 3$. Indeed, it was shown in [Peyre 1995] that if $\sqrt{-1} \in F$, $a, b, c \in F^*$, and $a \cup b \cup c \neq 0$ in $H^3(F, \mathbb{Z}/2\mathbb{Z})$, then the triple $\{(a, b), (a, c), (b, c)\}$ is linked, but has no common slot. However, this example does not make clear whether there exists some obstruction that does not permit a given linked triple of quaternion algebras to have a common slot. In this note we construct such an obstruction, which lies in the Witt ring of the field F; more precisely, it lies in $I^3(F)$.

We use the notation usual in the theory of quadratic forms, which can be found, for instance, in the books [Lam 2005; Scharlau 1985; Elman et al. 2008]. The word "form" always means a quadratic form over a field of characteristic different from 2. For a form φ defined on a linear space V over a field F, by $D(\varphi)$ we denote the set of nonzero values $\varphi(v)$, where $v \in V$. The n-fold Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ is the form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$. (Take notice of signs!) Sometimes, slightly abusing notation, we consider regular forms over F as their images in the Witt ring $W(F)$, and, vice versa, elements of $W(F)$ as the associated *anisotropic* forms. In particular, we call the corresponding elements in $W(F)$ n-fold Pfister forms. The signs $+$

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and $-$ are used for operations in $W(F)$. The dimension of the anisotropic form associated with $\varphi \in W(F)$ is denoted by $\dim \varphi$. By $F(\varphi)$ we denote the function field of the projective quadric associated with the *form* φ over the field F. By $i_1(\varphi)$ we denote the first Witt index of the form φ, which is defined as follows: $i_1(\varphi) = n$, where $\varphi_{F(\varphi)} \simeq \psi \perp n\mathbb{H}$ and ψ is the anisotropic part of $\varphi_{F(\varphi)}$.

Let $\mathfrak{T} = \{Q_1, Q_2, Q_3\}$ be a linked triple. For any quaternion algebra α denote by $c(\alpha)$ the 2-fold Pfister form corresponding to α. Put

$$c(\mathfrak{T}) = c(Q_1) + c(Q_2) + c(Q_3) - c(Q_1 + Q_2) - c(Q_1 + Q_3) \\ - c(Q_2 + Q_3) + c(Q_1 + Q_2 + Q_3) \in W(F).$$

The invariant $c(\mathfrak{T})$ has the following properties:

Proposition 1. (1) $c(\mathfrak{T}) \in I^3(F)$, $c(\mathfrak{T})$ *is divisible by each* $c(Q_i)$, *and* $\dim c(\mathfrak{T})$ *equals either* 0, 8, *or* 16. *Moreover*, *if* $\dim c(\mathfrak{T}) = 16$, *then* $1 \in D(c(\mathfrak{T}))$.

(2) $c(\mathfrak{T})$ *is a sum of two elements*, *similar to a* 4*-fold and a* 3*-fold Pfister form*, *respectively. In particular*, $c(\mathfrak{T}) \pmod{I^4(F)}$ *is a symbol in* $I^3(F)/I^4(F)$.

(3) *If* $\mathfrak{T}$ *has a common slot*, *then* $c(\mathfrak{T})$ *is a* 4*-fold Pfister form.*

Proof. (1) It is obvious that the elements $c(Q_1) + c(Q_2) - c(Q_1 + Q_2)$ and $c(Q_3) + c(Q_1 + Q_2 + Q_3) - c(Q_2 + Q_3) - c(Q_1 + Q_3)$ both are from $I^3(F)$ and split by the extension $F(SB(Q_1))/F$. Hence they are divisible by $c(Q_1)$ [Scharlau 1985]. Since the dimensions of the elements $c(Q_1) - c(Q_1 + Q_2)$, $c(Q_2) - c(Q_2 + Q_3)$ and $c(Q_3) - c(Q_1 + Q_3)$ are at most 4, we get

$$\dim\bigl(c(Q_1) + c(Q_2) + c(Q_3) - c(Q_1 + Q_2) - c(Q_1 + Q_3) - c(Q_2 + Q_3)\bigr) \le 12.$$

Hence $\dim c(\mathfrak{T}) \le 16$, and if $\dim c(\mathfrak{T}) = 16$, then the Pfister form $c(Q_1 + Q_2 + Q_3)$ is a direct summand of $c(\mathfrak{T})$, and so $1 \in D(c(\mathfrak{T}))$ (here we consider $c(\mathfrak{T})$ as the corresponding quadratic form). Furthermore, since $c(\mathfrak{T}) \in I^3(F)$ and is divisible by $c(Q_1)$, it follows that $\dim c(\mathfrak{T})$ is divisible by 8. By symmetry $c(\mathfrak{T})$ is divisible by each $c(Q_i)$.

(2) It follows at once from part (1) that

$$\begin{aligned} c(\mathfrak{T}) &= c(Q_1) \otimes \langle a_1, a_2, a_3, a_4 \rangle \\ &= c(Q_1) \otimes \langle a_1, a_2, a_3, a_1 a_2 a_3 \rangle + c(Q_1) \otimes \langle -a_1 a_2 a_3, a_4 \rangle \end{aligned}$$

for some $a_i \in F^*$. The first summand is similar to a 4-fold Pfister form, and the second one to a 3-fold Pfister form.

(3) Assume that $Q_i = (a, b_i)$ for some $a, b_1, b_2, b_3 \in F^*$. Then a straightforward computation shows that

$$c(\mathfrak{T}) = \langle\langle a \rangle\rangle \otimes \langle 1, -b_1, -b_2, -b_3, b_1 b_2, b_1 b_3, b_2 b_3, -b_1 b_2 b_3 \rangle = \langle\langle a, b_1, b_2, b_3 \rangle\rangle. \qquad \square$$

As another example of computation of $c(\mathfrak{T})$, which will be used in the sequel, we have the following:

Proposition 2. *Let F be a field, $a, b_1, b_2 \in F^*$. Let Q be a quaternion algebra over F such that $Q_{F(\sqrt{b_I})} = 0$ for any nonempty $I \subset \{1, 2\}$, where $b_I = \prod_{i \in I} b_i$. Then the triple $\{(a, b_1), (a, b_2), Q\}$ is linked, and $c(\mathfrak{T}) = \langle\langle a \rangle\rangle c(Q)$. Moreover, if $\{(a, b_1), (a, b_2), Q\}$ has a common slot, then $c(\mathfrak{T}) = 0$.*

Proof. For any nonempty $I \subset \{1, 2\}$ there is some element $c_I \in F^*$ such that $Q = (b_I, c_I)$. Obviously,

$$Q + \sum_{i \in I} (a, b_i) = (ac_I, b_I),$$

hence the triple $\mathfrak{T}$ is linked. Furthermore, in view of Proposition 1(3) and the equality $\langle\langle xy, z \rangle\rangle = \langle\langle x, z \rangle\rangle + \langle\langle y, z \rangle\rangle - \langle\langle x, y, z \rangle\rangle$ we get

$$\begin{aligned} c(\mathfrak{T}) &= \langle\langle a, b_1 \rangle\rangle + \langle\langle a, b_2 \rangle\rangle - \langle\langle a, b_1 b_2 \rangle\rangle + Q + \sum_{I \subset \{1,2\}} (-1)^{|I|} \langle\langle ac_I, b_I \rangle\rangle \\ &= \langle\langle a, b_1, b_2 \rangle\rangle + Q + \sum_{I \subset \{1,2\}} (-1)^{|I|} \big(\langle\langle a, b_I \rangle\rangle + Q - \langle\langle a, b_I, c_I \rangle\rangle \big) \\ &= \langle\langle a, b_1, b_2 \rangle\rangle + \sum_{I \subset \{1,2\}} (-1)^{|I|} \langle\langle a, b_I \rangle\rangle + \sum_{I \subset \{1,2\}} (-1)^{|I|+1} \langle\langle a \rangle\rangle c(Q) \\ &= \langle\langle a, b_1, b_2 \rangle\rangle - \langle\langle a, b_1, b_2 \rangle\rangle + \langle\langle a \rangle\rangle c(Q) = \langle\langle a \rangle\rangle c(Q). \end{aligned}$$

By Proposition 1(3), the assumption that the triple $\{(a, b_1), (a, b_2), Q\}$ has a common slot implies $c(\mathfrak{T}) = 0$. □

As a consequence we obtain another proof of the example from [Peyre 1995].

Corollary 3. *Let F be a field, $\sqrt{-1} \in F$, $a, b, c \in F^*$. If $a \cup b \cup c \neq 0$ in $H^3(F, \mathbb{Z}/2\mathbb{Z})$, then the triple $\mathfrak{T} = \{(a, b), (a, c), (b, c)\}$ is linked, but does not have a common slot.*

Proof. Since $\langle\langle b, c \rangle\rangle = \langle\langle b, -bc \rangle\rangle = \langle\langle b, bc \rangle\rangle$, we have

$$(a, b) + (a, c) + (b, c) = (a, bc) + (b, bc) = (ab, bc),$$

so the triple is linked. Since

$$(b, c)_{F(\sqrt{b})} = (b, c)_{F(\sqrt{c})} = (b, c)_{F(\sqrt{bc})} = 0,$$

we get by Proposition 2 that $c(\mathfrak{T}) = \langle\langle a, b, c \rangle\rangle \neq 0$. Also by Proposition 2 we get that the triple $\mathfrak{T}$ has no common slot. □

Recall that a field L is called linked if any two quaternion algebras over L have a common slot [Elman et al. 2008]. A natural question arises whether there exists a linked field L and a triple of quaternion algebras $\{Q_1, Q_2, Q_3\}$ over L without a common slot. The answer is in affirmative, and one can construct such a field L even with a few additional properties. To do this we need a couple of lemmas.

Lemma 4. *Let φ be an anisotropic Albert form (i.e., an anisotropic 6-dimensional form with trivial discriminant) over a field F, π the 2-fold Pfister form over $F(\varphi)$ similar to $\varphi_{F(\varphi)}$. Then $\pi \notin h + I^4(F(\varphi))$ for any $h \in I^2(F)$.*

Proof. We will induct on $\dim h$. Assume the converse, i.e., that there is $h \in I^2(F)$ such that $\pi \in h + I^4(F(\varphi))$. Consider a few cases.

(a) $h = 0$. Then $\pi = 0$, hence $\varphi_{F(\varphi)} = 0$. Choose any $a \in F^*$ such that the form $\varphi_{F(\sqrt{a})}$ is isotropic. In particular, $a \notin F^{*2}$. Obviously, $\varphi_{F(\sqrt{a})} = 0$, i.e., $\varphi \simeq \langle\langle a\rangle\rangle\psi$ for some 3-dimensional form ψ. Comparing discriminants, we get $a \in F^{*2}$, a contradiction.

(b) The form $\varphi_{F(h)}$ is isotropic. Then there exists $c \in F^*$ such that either $\varphi = ch$, or $\varphi = c(h - \tilde{h})$ for some form $\tilde{h} \neq 0$ with $\dim h = \dim \tilde{h} = 4$ [Merkurjev 1991]. In both cases by dimension count we have $\pi = h_{F(\varphi)}$. If $\varphi \simeq ch$, then we have $\varphi_{F(\varphi)} = c\pi$; therefore, $\dim(\varphi \perp \langle -c\rangle)_{F(\varphi)} = 3$. If $c \in D(\varphi)$, then the field $F(\varphi)$ splits a proper subform of φ, which is impossible, since $i_1(\varphi) = 1$ [Karpenko and Merkurjev 2003]. (Another, more elementary way to come to a contradiction is to make the form $(\varphi \perp \langle -c\rangle)_{\mathrm{an}}$ a Pfister neighbor, not splitting the form φ [Hoffmann 1995].) If $c \notin D(\varphi)$, then $\dim(\varphi \perp \langle -c\rangle) = 7$. Moreover, $i_1(\varphi \perp \langle -c\rangle) = 1$, since $\varphi_{F(\varphi\perp\langle -c\rangle)}$ is anisotropic [Merkurjev 1991]. Since $c \in D(\varphi_{F(\varphi\perp\langle -c\rangle)})$, changing F for $F(\varphi \perp \langle -c\rangle)$, we get a contradiction again.

If $\varphi = c(h - \tilde{h})$ and $\varphi_{F(\varphi)} = u\pi = uh$ for some $u \in F(\varphi)^*$, then $c\tilde{h} = ch - uh \in I^3(F(\varphi))$, hence $\tilde{h}_{F(\varphi)} = 0$, which is also impossible.

(c) $\varphi_{F(h)}$ is anisotropic, $h \neq 0$. Then, passing to the field $F(h)$, the forms $(h_{F(h)})_{\mathrm{an}}$, $\pi_{F(h)(\varphi)}$ and the Albert form $\varphi_{F(h)}$, we can conclude by induction on $\dim h$ that this case is impossible as well, which finishes the proof. $\square$

Lemma 5. *Let F be a field, Q_1, Q_2, Q_3 pairwise distinct nontrivial quaternion algebras, φ an anisotropic Albert form. Suppose $\{Q_1, Q_2, Q_3\}$ is not a linked triple. Then either the triple $\mathfrak{T} = \{Q_{1F(\varphi)}, Q_{2F(\varphi)}, Q_{3F(\varphi)}\}$ is not linked, or it is linked and $c(\mathfrak{T}_{F(\varphi)}) \notin I^4(F(\varphi))$. In particular, the triple $\mathfrak{T}_{F(\varphi)}$ has no common slot.*

Proof. Notice first that all $\sum_{i\in I} Q_i$ ($I \subset \{1, 2, 3\}$) are pairwise distinct. Suppose the triple $\mathfrak{T}$ is linked. Then there is a unique $\alpha \in G = \langle Q_1, Q_2, Q_3\rangle \in {}_2\mathrm{Br}(F)$ such that $\mathrm{ind}(\alpha) = 4$, and, moreover, φ is an Albert form for α. Assume that $c(\mathfrak{T}) \in I^4(F(\varphi))$. Then we get $c(\alpha)_{F(\varphi)} \in h + I^4(F(\varphi))$ for some form $h \in I^2(F)$, which contradicts Lemma 4. $\square$

Theorem 6. *Suppose that the triple $\{Q_1, Q_2, Q_3\}$ over a field F is not linked. Then there exists a field extension L/F with the following properties*:

(1) *The field L is linked. In particular, the triple $\{Q_1, Q_2, Q_3\}_L$ is linked.*

(2) *The triple $\{Q_1, Q_2, Q_3\}_L$ has no common slot.*

(3) *The field L has no proper odd degree extension.*

(4) *Any 9-dimensional form over L is isotropic.*

(5) *Up to isomorphism there is a unique nontrivial 3-fold Pfister form over L, namely the one similar to $c(\{Q_1, Q_2, Q_3\}_L)$.*

Proof. We will apply a procedure similar to the one used in the construction of fields with prescribed even U-invariant [Merkurjev 1991]. First, applying Lemma 5 a few times, we can construct a field extension K/F such that the triple $\{Q_1, Q_2, Q_3\}_K$ is linked, but has no common slot. Further, splitting if needed some 4-fold Pfister form over K, we pass to the field K_1/K such that $c(\{Q_1, Q_2, Q_3\}_{K_1})$ is similar to some 3-fold Pfister form τ over K_1. Next, splitting all Albert forms, all forms of dimension at least 9, and all 3-fold Pfister forms differing from τ, then passing to a maximal odd degree extension, we construct a tower of fields $K_1 \subset K_2 \subset \cdots$ such that $L = \bigcup_i K_i$ satisfies all the required properties. The point is that for each i the form τ_{K_i} remains nontrivial; this proves the absence of a common slot of the triple $\{Q_1, Q_2, Q_3\}_{K_i}$. □

Theorem 6 has an application for Chow groups of codimension 2 of the product of three projective conics. We give numerous examples of increasing the torsion of these groups when passing to some field extension. We need the following statement, which is an immediate consequence of Theorem 4.1, Proposition 6.1 and Remark 4.1 from [Peyre 1995].

Peyre's Theorem. *Let F be a field, Q_1, Q_2, Q_3 quaternion algebras over F, and X_1, X_2, X_3 the corresponding projective conics. Denote by G the subgroup of ${}_2\operatorname{Br}(F)$ generated by all Q_i. Then*:

(1) *The torsion of the group $\operatorname{CH}^2(X_1 \times X_2 \times X_3)$ is either zero, or $\mathbb{Z}/2\mathbb{Z}$.*

(2) *Denote by d the least common multiple of all the numbers $\operatorname{ind}(\alpha)$, where α runs over all the elements of G. The following two assertions are equivalent*:

 (i) *The algebras Q_i have a common splitting field of degree dm, where m is an odd integer.*

 (ii) *The torsion of the group $\operatorname{CH}^2(X_1 \times X_2 \times X_3)$ equals 0.*

(3) *If the algebras Q_i have a common slot, the sequence*

$$F^*/F^{*2} \otimes G \xrightarrow{\text{cup product}} H^3(F, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\text{res}} H^3\big(F(X_1 \times X_2 \times X_3), \mathbb{Z}/2\mathbb{Z}\big)$$

is exact.

Corollary 7. *Let F be a field, $\mathfrak{T} = \{Q_1, Q_2, Q_3\}$ a nonlinked triple of quaternion algebras. Suppose that either at least one of the elements $Q_1 + Q_2$, $Q_1 + Q_3$, $Q_2 + Q_3$, $Q_1 + Q_2 + Q_3$ has index 2, or $\operatorname{ind}(Q_1 + Q_2 + Q_3) = 8$. Denote by X_i the projective conic corresponding to Q_i. Further, let L/F be the field extension constructed in Theorem 6 for the triple $\mathfrak{T}$. Then the torsion of $\mathrm{CH}^2(X_1 \times X_2 \times X_3)$ is zero, but the torsion of $\mathrm{CH}^2(X_{1L} \times X_{2L} \times X_{3L})$ is $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Let us compute the number d for the algebras Q_1, Q_2, Q_3. Since the triple $\{Q_1, Q_2, Q_3\}$ is not linked, $d \geq 4$. In the first case, since the sum of a few Q_i has index 2, we get $d \neq 8$, i.e., $d = 4$. Assume that either $\operatorname{ind}(Q_1 + Q_2) = 2$ or $\operatorname{ind}(Q_1 + Q_2 + Q_3) = 2$. Choose $a \in F^*$ such that $Q_{3F(\sqrt{a})} = 0$. Then $Q_{1F(\sqrt{a})}$ and $Q_{2F(\sqrt{a})}$ have a common slot, hence for the algebras Q_1, Q_2, Q_3 there exists a common splitting field extension K/F of degree 4. By Peyre's theorem we conclude that the torsion of $\mathrm{CH}^2(X_1 \times X_2 \times X_3)$ is zero. In the second case it is obvious that $d = 8$, and there is a triquadratic common splitting field extension for the Q_i. Hence in this case the torsion of $\mathrm{CH}^2(X_1 \times X_2 \times X_3)$ is zero as well. On the other hand, for the algebras Q_{1L}, Q_{2L}, Q_{3L} we have $d = 2$ in both cases, but there is no common splitting extension E/L of degree $2m$ where m is odd. Indeed, existence of such an extension implies $m = 1$, since L has no proper odd degree extension. This means that the triple $\mathfrak{T}_L$ has a common slot, which is not the case. Applying Peyre's theorem again, we get that the torsion of $\mathrm{CH}^2(X_{1L} \times X_{2L} \times X_{3L})$ is $\mathbb{Z}/2\mathbb{Z}$. □

In the examples above the invariant $c(\mathfrak{T})$ was a Pfister form, either 3-fold or 4-fold. In general this is not the case, as shown by a generic example of a linked triple.

Proposition 8. *Let k be a field, a, b_1, b_2, c, d indeterminates. Further, let φ_1, φ_2, φ_3 be the Albert forms corresponding to the biquaternion algebras*

$$(a, b_1) + (c, d), \quad (a, b_2) + (c, d), \quad (a, b_1 b_2) + (c, d),$$

respectively. Put $F = k(\varphi_1, \varphi_2, \varphi_3)$. Then:

(1) *The triple $\mathfrak{T} = \{(a, b_1)_F, (a, b_2)_F, (c, d)_F\}$ is linked.*
(2) $\dim c(\mathfrak{T}) = 16$, *and* $c(\mathfrak{T}) \notin I^4(F)$.
(3) *The form $c(\mathfrak{T}_{F(c(\mathfrak{T}))})$ is similar, but not equal, to a 3-fold Pfister form over $F(c(\mathfrak{T}))$.*

Proof. (1) This is obvious by the definition of linked triple.

(2) We have $(c, d)_{k(\sqrt{ac})} = (a, d)_{k(\sqrt{ac})}$. Put

$$\mathfrak{P} = \left\{(a, b_1)_{k(\sqrt{ac})}, (a, b_2)_{k(\sqrt{ac})}, (c, d)_{k(\sqrt{ac})}\right\}.$$

By Proposition 1 we get $c(\mathfrak{P}) = \langle\langle a, b_1, b_2, d\rangle\rangle_{k(\sqrt{ac})} \neq 0$. Notice that $\mathfrak{T}_{F(\sqrt{ac})} = \mathfrak{P}_{k(\sqrt{ac})(\varphi_1, \varphi_2, \varphi_3)}$, and the forms $\varphi_1, \varphi_2, \varphi_3$ are isotropic over $k(\sqrt{ac})$. Hence the field $k(\sqrt{ac})(\varphi_1, \varphi_2, \varphi_3)$ is purely transcendental over $k(\sqrt{ac})$, which implies that

$$\dim c(\mathfrak{T}_{F(\sqrt{ac})}) = \dim c\big(\mathfrak{P}_{k(\sqrt{ac})(\varphi_1,\varphi_2,\varphi_3)}\big) = 16.$$

By Proposition 1 we get $\dim c(\mathfrak{T}) = 16$. Assume now that $c(\mathfrak{T}) \in I^4(F)$. To come to a contradiction it suffices to construct a field extension L/F such that $\dim c(\mathfrak{T}_L) = 8$. Let

$$\tau_1 \simeq \langle b_1, -c, -d, cd\rangle, \quad \tau_2 \simeq \langle b_2, -c, -d, cd\rangle, \quad \tau_3 \simeq \langle b_1 b_2, -c, -d, cd\rangle.$$

Obviously, the algebra $Q = (c, d)_{k(\tau_1,\tau_2,\tau_3)}$ satisfies the hypothesis of Proposition 2 with respect to the elements $b_1, b_2, b_1 b_2$. Moreover, $Q \neq 0$ [Scharlau 1985] and a is an indeterminate over $k(\tau_1, \tau_2, \tau_3)$. Therefore,

$$c\big(\{(a, b_1), (a, b_2), (c, d)_{k(\tau_1,\tau_2,\tau_3)}\}\big) = \langle\langle a\rangle\rangle Q \neq 0.$$

On the other hand, since the Albert forms $\varphi_{i\,k(\tau_i)}$ are isotropic, the field extension $k(\tau_1, \tau_2, \tau_3, \varphi_1, \varphi_2, \varphi_3)/k(\tau_1, \tau_2, \tau_3)$ is purely transcendental, so

$$\dim c(\mathfrak{T}_L) = 8, \quad \text{where } L = k(\tau_1, \tau_2, \tau_3, \varphi_1, \varphi_2, \varphi_3) = F(\tau_1, \tau_2, \tau_3).$$

(3) Here we consider $c(\mathfrak{T})$ as the corresponding anisotropic form. In view of part (2) we have $\dim c(\mathfrak{T}_{F(c(\mathfrak{T}))}) = 8$, hence the form $\mathfrak{T}_{F(c(\mathfrak{T}))}$ is similar to a 3-fold Pfister form over $F(c(\mathfrak{T}))$. By Proposition 1 we have $1 \in D(c(\mathfrak{T}))$, i.e., $c(\mathfrak{T}) = \langle 1\rangle \perp \psi$ for some 15-dimensional form δ over F. Suppose $1 \in D(c(\mathfrak{T}_{F(c(\mathfrak{T}))}))$. Then $\dim \delta_{F(c(\mathfrak{T}))} = \dim \delta_{F(\delta)} = 7$, i.e., $i_1(\delta) = 4$. Since the first Witt index of an odd dimensional form is odd [Karpenko 2003], we get a contradiction. □

It turns out that existence of a common slot for the triple $\{Q_1, Q_2, Q_3\}$ over the field F is equivalent to isotropicity of a certain 9-dimensional form over $F(t)$.

Proposition 9. *The following conditions are equivalent*:

(1) *The triple* $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ *has a common slot.*

(2) *The system of quadratic forms*

$$\begin{cases} \varphi_1 = a_1 x_1^2 + b_1 x_2^2 - a_1 b_1 x_3^2 - a_2 x_4^2 - b_2 x_5^2 + a_2 b_2 x_6^2 = 0 \\ \varphi_2 = -a_2 x_4^2 - b_2 x_5^2 + a_2 b_2 x_6^2 + a_3 x_7^2 + b_3 x_8^2 - a_3 b_3 x_9^2 = 0 \end{cases}$$

has a nontrivial zero.

(3) *The form*

$$\Phi \simeq \varphi_1 + t\varphi_2 \simeq \langle a_1, b_1, -a_1 b_1\rangle \perp (t+1)\langle -a_2, -b_2, a_2 b_2\rangle \perp t\langle a_3, b_3, -a_3 b_3\rangle$$

is isotropic over $F(t)$.

Proof. By [Brumer 1978] the conditions (2) and (3) are equivalent. If at least one of the algebras (a_i, b_i) is trivial, then it is easy to see that both conditions (1) and (2) hold. Assume that each (a_i, b_i) is nontrivial. Then (2) is equivalent to existence of a nonzero row $(x_1, \dots, x_9)$ such that

$$0 \neq a_1x_1^2 + b_1x_2^2 - a_1b_1x_3^2 = a_2x_4^2 + b_2x_5^2 - a_2b_2x_6^2 = a_3x_7^2 + b_3x_8^2 - a_3b_3x_9^2,$$

which in turn is equivalent to (1). □

Proposition 10. *Let F be a field, $\{Q_1, Q_2, Q_3\}$ a triple over F without a common slot, and L/F a field extension. Then the triple $\{Q_1, Q_2, Q_3\}_L$ has no common slot as well, if L/F is either purely transcendental, or of odd degree, or $L = F(\tau)$, where τ is a form over F, $\dim \tau \geq 9$.*

Proof. We keep the notation of Proposition 9. If L/F is purely transcendental, then the claim is obvious. If L/F is an odd degree extension, then the assertion follows from Proposition 9 and Springer's theorem [Scharlau 1985]. Assume now that τ is a form over F, $\dim \tau \geq 9$, $L = F(\tau)$, and that $\{Q_1, Q_2, Q_3\}_L$ has a common slot. We may assume that τ is anisotropic, for otherwise the extension $F(\tau)/F$ is purely transcendental. By Proposition 9 the form $\Phi_{F(t)(\tau)}$ is isotropic. Since $\dim \Phi = \dim \tau = 9$, it follows that the form $\tau_{F(t)(\Phi)}$ is isotropic as well [Izhboldin 2000]. On the other hand, the form $\Phi_{F(t)(\sqrt{-a_1a_3t})}$ is isotropic, hence the form $\tau_{F(t)(\sqrt{-a_1a_3t})}$ is isotropic, which is impossible, since the extension $F(t)(\sqrt{-a_1a_3t})/F$ is purely transcendental. □

Corollary 11. *Let K/F be a field extension of degree $2m$, where m is odd. Suppose that Q_1, Q_2, Q_3 are quaternion algebras such that $Q_{1K} = Q_{2K} = Q_{3K} = 0$. Then the triple $\{Q_1, Q_2, Q_3\}$ has a common slot.*

Proof. We will induct on m, the case $m = 1$ being trivial. Suppose first that $K = F(\alpha)$, and denote by $p(t)$ the irreducible monic polynomial for α. As earlier, let X_i be the projective conic corresponding to Q_i. By the hypothesis there is a morphism $\operatorname{Spec} K \to X_1 \times X_2 \times X_3$. This map gives rise to the morphism

$$\operatorname{Spec} L[t]/p(t) = \operatorname{Spec} L \otimes_F K \to X_{1L} \times X_{2L} \times X_{3L},$$

where L is a maximal odd-degree field extension of F. Since $\deg p$ is not divisible by 4, and any finite extension of L is 2-primary, there is an irreducible $f \in L[t]$ such that $f | p$ and $\deg f \leq 2$. Therefore, we get a morphism $\operatorname{Spec} L[t]/f(t) \to X_{1L} \times X_{2L} \times X_{3L}$. If $\deg f = 1$, then $Q_{1L} = Q_{2L} = Q_{3L} = 0$, hence, since L/F is an odd degree extension, $Q_1 = Q_2 = Q_3 = 0$. If $\deg f = 2$, then the triple $\{Q_1, Q_2, Q_3\}_L$ has a common slot, hence by Proposition 10, the triple $\{Q_1, Q_2, Q_3\}$ has a common slot as well, and the corollary is proved in the case $K = F(\alpha)$.

In the general case we have a tower of field extensions $F \subset F(\alpha) \subset K$, $F(\alpha) \neq F$, and $[F(\alpha) : F]$ is not divisible by 4. If $[F(\alpha) : F] = 2n$, where n is odd, then

$[K : F(\alpha)]$ is odd, hence $Q_{1F(\alpha)} = Q_{2F(\alpha)} = Q_{3F(\alpha)} = 0$, and by induction $\{Q_1, Q_2, Q_3\}$ has a common slot. If $[F(\alpha) : F]$ is odd, then again by induction with respect to the extension $K/F(\alpha)$ the triple $\{Q_1, Q_2, Q_3\}_{F(\alpha)}$ has a common slot. By Proposition 10, $\{Q_1, Q_2, Q_3\}$ has a common slot as well. □

Corollary 12. *The linked triple* $\{Q_1, Q_2, Q_3\}$ *has a common slot if and only if the torsion of the group* $\mathrm{CH}^2(X_1 \times X_2 \times X_3)$ *is zero.*

Proof. By Peyre's theorem, the group $\mathrm{CH}^2(X_1 \times X_2 \times X_3)$ is zero if and only if there is an extension K/F of degree $2m$, with m odd, such that $Q_{1K} = Q_{2K} = Q_{3K} = 0$. By Corollary 11 this is equivalent to the triple $\{Q_1, Q_2, Q_3\}$ having a common slot. □

Peyre's theorem implies the following curious result on *four* quaternion algebras.

Proposition 13. *Let* k *be a field,* $a_1, a_2, a_3, a_4 \in k^*$, $D \in {}_2\mathrm{Br}(k)$, $F = k(x)$ *the rational function field,* X_i *the projective conics corresponding to the algebras* (a_i, x) *over* F. *Then the following conditions are equivalent*:

(1) *There exist* $b_1, b_2, b_3, b_4 \in k^*$ *such that* $D = \sum_{i=1}^{4}(a_i, b_i)$.

(2) $D \cup (x)_{F(X)} = 0$ *in* $H^3(F(X), \mathbb{Z}/2\mathbb{Z})$, *where* $X = X_1 \times X_2 \times X_3 \times X_4$.

Proof. The implication (1) $\Longrightarrow$ (2) is trivial, since if $D = \sum_{i=1}^{4}(a_i, b_i)$, then

$$D \cup (x) = \sum_{i=1}^{4}(a_i, b_i, x),$$

hence $D\cup(x)_{F(X)} = 0$. Suppose now that $D\cup(x)_{F(X)} = 0$. We have $F(X_4) = k(u, v)$, where $u^2 - av^2 = x$. Therefore, $D \cup (u^2 - av^2)_{K(Y_1\times Y_2\times Y_3)} = 0$, where $K = k(u, v)$, and Y_i is the conic corresponding to the algebra $(a_i, u^2 - av^2)$. Put $t = u/v$, $a = a_4$. By applying Peyre's theorem to the algebras $(a_i, t^2 - a)$ $(i = 1, 2, 3)$, we get

$$\begin{aligned} D \cup (t^2 - a) = (a_1, t^2 - a, p_1(t)) + (a_2, t^2 - a, p_2(t)) \\ +(a_3, t^2 - a, p_3(t)) \in H^3(k(t), \mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

for some $p_i(t) \in k(t)$. Obviously, we can rewrite the last equality as

$$D \cup (t^2 - a) = \sum_{I\subset\{1,2,3\}} (a_I, t^2 - a, f_I),$$

where $a_I = \prod_{i\in I} a_i$, and $f_I \in k[t]$ are some pairwise coprime polynomials. Moreover, we may suppose that each f_I is coprime with t^2-a. Now apply the well-known exact sequence

$$\begin{aligned} 0 \to H^3(k, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\text{res}} H^3(k(t), \mathbb{Z}/2\mathbb{Z}) \\ \xrightarrow{\coprod \partial_p} \coprod_{p\in\mathbb{P}^1_k} H^2(k_p, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{N} H^2(k, \mathbb{Z}/2\mathbb{Z}) \to 0 \end{aligned}$$

to the symbols (a_I, t^2-a, f_I). Since the polynomials f_I are pairwise coprime and $\partial_p(D\cup(t^2-a))=0$ for any prime polynomial $p\neq t^2-a$, we have

$$\partial_p(a_I, t^2-a, f_I)=0$$

for such p. Also $\partial_\infty(a_I, t^2-a, f_I)=0$. By the exact sequence above we have $N_{k(\sqrt{a})/k}(\partial_{t^2-a}(a_I, t^2-a, f_I))=0$. Hence $\partial_{t^2-a}(a_I, t^2-a, f_I))=(a_I, b_I)$ for some $b_I\in k^*$. Applying the above exact sequence again, we get

$$(a_I, t^2-a, f_I)=(a_I, b_I, t^2-a).$$

Therefore,

$$\begin{aligned}D_{k(\sqrt{a})}&=\partial_{t^2-a}(D\cup(t^2-a))\\&=\partial_{t^2-a}\Bigl(\sum_{I\subset\{1,2,3\}}(a_I, t^2-a, f_I)\Bigr)=\sum_{I\subset\{1,2,3\}}(a_I, b_I)_{k(\sqrt{a})}.\end{aligned}$$

It follows that $D=(a,b)+\sum_{I\subset\{1,2,3\}}(a_I, b_I)$ for some $b\in k^*$, which implies (1). $\square$

Corollary 14. *Assume F is a C_2-field, which means that for each $d\geq 1$ any homogeneous polynomial in d^2+1 variables of degree d over F has a nontrivial zero* [Scharlau 1985]. *Then any triple of quaternion algebras $\{Q_1, Q_2, Q_3\}$ over F has a common slot.*

Proof. Any two 9-dimensional quadratic forms over F have a common nontrivial zero [Scharlau 1985], hence we are done by Proposition 9. $\square$

In the case of global fields we can say more.

Proposition 15. *Let F be a global field. Any finite collection $Q_1,\dots,Q_n$ of quaternion algebras over F has a common slot.*

Proof. It is well known from the class field theory that the natural restriction map $\partial:\mathrm{Br}(F)\to\coprod_v \mathrm{Br}(F_v)$ is injective, where v runs over all valuations of F, and F_v is the completion of F with respect to v. For any finite extension L/F, the diagram

$$\begin{array}{ccc}\mathrm{Br}(F) & \xrightarrow{\partial=\coprod\partial_v} & \coprod_v \mathrm{Br}\,F_v\\ {\scriptstyle\mathrm{res}}\downarrow & & \downarrow{\scriptstyle\mathrm{res}}\\ \mathrm{Br}(L) & \xrightarrow{\partial=\coprod\partial_w} & \coprod_w \mathrm{Br}\,L_w\end{array}$$

is commutative. Let $\{v_1,\dots,v_m\}$ be all valuations over F such that $\partial_{v_i}(Q_j)\neq 0$ for some j. Choose $d\in F^*$ such that $d\notin {F_{v_i}}^{*2}$ and $v_i(d)=0$ for each v_i. Then

$$\partial\circ\mathrm{res}_{F(\sqrt{d})/F}(Q_j)=\mathrm{res}_{F(\sqrt{d})/F}\circ\partial(Q_j)=0,$$

hence $\mathrm{res}_{F(\sqrt{d})/F}(Q_j)=0$, so $Q_j=(d,e_j)$ for some $e_j\in F^*$. $\square$

The above results prompt the following:

Open questions.

(1) Does there exist a linked triple $\mathfrak{T}$ over a field F without common slot such that $c(\mathfrak{T}) \in I^4(F)$? (In view of Proposition 9, an equivalent version of this question is the one where $c(\mathfrak{T}) \in I^4(F)$ is changed for $c(\mathfrak{T}) = 0$.)

(2) Assume that φ is an anisotropic form, $\dim \varphi \geq 5$, $\mathfrak{T}$ is a linked triple without common slot. Is it true that $\mathfrak{T}_{F(\varphi)}$ is without common slot as well? (Notice that if $\dim \varphi = 4$, then the answer is negative in general. For instance, under the notation of Corollary 3 the triple $\mathfrak{T}_{F(\langle a,b,ab,c\rangle)}$ obviously has a common slot, namely c).

(3) Let F be a field such that any form of dimension at least 5 over F is isotropic. Does any triple of quaternion algebras $\{Q_1, Q_2, Q_3\}$ over F have a common slot? (By Corollary 14 and Proposition 15 the answer is positive if F is either a C_2-field or a nonreal global field.)

(4) Suppose $\mathrm{cd}_2(F) = 2$. Is it true that any linked triple has a common slot?

Certainly, these questions are not independent of one another. For instance, if the answer to question (2) is positive, then the answer to question (1) is positive as well, and the answers to questions (3) and (4) are negative.

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