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We give a counterexample to a proposition claimed to be proven in an earlier paper of ours and used in the proof of its main theorem. We also show how to salvage the main result of that paper under additional hypotheses.

Let $X \subset \mathbb{P}^N$ be a nondegenerate smooth projective variety such that X^* is a hypersurface. Let $L \subset \mathbb{P}^N$ be a linear subspace such that for general $x \in X$ we have $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$. We say that $L^\perp$ is an *unexpected equisingular space* in X^* (see Definition 3.2.1 of [Abuaf 2011], hereafter cited as [A]) if the general hyperplane containing $\langle L, T_{X,x} \rangle$ has the same multiplicity in X^* as a general hyperplane containing L. In [A], the following side-result, whose aim was to discuss a necessary hypothesis in our main theorem, was stated in Section 3 ("Open question and corollaries"):

Theorem 3.2.2 of [A]. *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate projective variety such that X^* is a hypersurface. Let $L \subset X$ be a linear space with* $\dim(L) = \dim(X) - 1$. *Assume that $L^\perp$ is an unexpected equisingular linear space in X^* such that* $\operatorname{mult}_{L^\perp} X^* = 2$. *Then X is the cubic scroll surface in $\mathbb{P}^4$.*

Its proof was based on this proposition:

Proposition 3.2.3 of [A]. *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that* $\operatorname{mult}_{[h]} X^* = 2$. *The scheme-theoretic tangency locus of H with X is one of the following*:

- *An irreducible hyperquadric and in this case* $|\mathscr{C}_{[h]}(X^*)|^* = \operatorname{Tan}(H, X)$.
- *The union of two (not necessarily distinct) linear spaces.*
- *A linear space with at least one embedded component.*

This proposition is false as shown by the following example.

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Example 1. Let V be a vector space of dimension 6 and let $W = \mathbb{G}(3, V) \subset \mathbb{P}(\bigwedge^3 V)$ be the Grassmannian of $\mathbb{C}^3 \subset V$ in its Plücker embedding. The dual of X is a quartic hypersurface in $\mathbb{P}(\bigwedge^3 V^*)$. We can decompose $\bigwedge^3 V^*$ as

$$\mathbb{C} \oplus U \oplus U^* \oplus \mathbb{C},$$

where U is identified with the space of 3×3 matrices (see [Landsberg and Manivel 2001, Section 5] for more details). We denote by C the determinant on U, which can be seen as a map $S^3U \to \mathbb{C}$ or as a map $S^2U \to U^*$. We also denote by C^* the determinant on U^*.

It is shown in (ibid.) that an equation (up to an automorphism of $\mathbb{P}(\bigwedge^3 V^*)$) of W^* is

$$\begin{aligned} &Q(x, X, Y, y) \\ &\quad = \bigl(3xy - \tfrac{1}{2}\langle X, Y\rangle\bigr)^2 + \tfrac{1}{3}\bigl(yC(X^{\otimes 3}) + xC^*(Y^{\otimes 3})\bigr) - \tfrac{1}{6}\bigl\langle C^*(Y^{\otimes 2}), C(X^{\otimes 2})\bigr\rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the standard pairing between U and U^*. The partial derivatives of Q give the equations of the variety of "stationary secants" to $W^\perp := \mathbb{G}(3, V^*) \subset \mathbb{P}(\bigwedge^3 V^*)$, which we denote by $\sigma_+(W^\perp)$. The Jacobian criterion shows that the variety $\sigma_+(W^\perp)$ is singular precisely along $W^\perp$. However, a simple Taylor expansion of Q around the point $[1, 0, \dots, 0] \in W^\perp$ shows that, contrary to what is claimed in Proposition 5.10 of (ibid.), $W^\perp$ is not defined by all the second derivatives of Q. The orbit structure of the action of SL_6 on $\mathbb{P}(\bigwedge^3 V^*)$ is

$$W^\perp \subset \sigma_+(W^\perp) \subset W^* \subset \mathbb{P}\bigl(\textstyle\bigwedge^3 V^*\bigr).$$

Since $W^\perp$ is the deepest strata in W^* and all the second derivatives of the equation of W^* do not vanish on $W^\perp$, we conclude that there are no point of multiplicity bigger than 2 in W^*. However one can prove (see (ibid.) for instance) that a point in $W^\perp$ is tangent to W along a cone over $\mathbb{P}^2 \times \mathbb{P}^2$. This gives a counterexample to the above proposition. Note that an easy computation shows that if $p = (p_0, P_0, P_1, p_1) \in \mathbb{P}(\bigwedge^3 V^*)$ is a generic point then the cubic hypersurface (which we denote by $\mathcal{P}(Q, p)$) defined by the equation $p_0 \frac{\partial Q}{\partial x} + P_0 \frac{\partial Q}{\partial X} + P_1 \frac{\partial Q}{\partial Y} + p_1 \frac{\partial Q}{\partial y}$ is smooth. Moreover the polar $\mathcal{P}(W^*, p) := W^* \cap \mathcal{P}(Q, p)$ has multiplicity 3 along $W^\perp$.

In [A] I claim that I "prove" Proposition 3.2.3 in the appendix. This proof relies on the following statement:

Lemma A.3 of [A]. *Let $Z \subset \mathbb{P}^N$ be an irreducible and reduced hypersurface, whose defining equation is denoted by f_Z. Let $z \in Z$ and let $k \in \{-1, \dots, N-2\}$. Then one of the following holds for general $D \in \mathbb{G}(k, N)$:*

• $z \notin P(Z, D)$.

• $$\operatorname{mult}_z P(Z, D) = \operatorname{mult}_z Z . \operatorname{mult}_z P(f_Z, D)$$

if $\dim(Z_{sing}^{(z)}) < \dim P(Z, D)$, *where* $Z_{sing}^{(z)}$ *is an irreducible component of* Z_{sing} *of maximal dimension passing through* z.

• $$\operatorname{mult}_z P(Z, D) < \operatorname{mult}_z Z . \operatorname{mult}_z P(f_Z, D)$$

if $\dim(Z_{sing}^{(z)}) \geq \dim P(Z, D)$, *where* $Z_{sing}^{(z)}$ *is an irreducible component of* Z_{sing} *of maximal dimension passing through* z.

This lemma is also false as shown by Example 1. Indeed the hypersurface $\mathcal{P}(Q, p)$ is smooth for generic p, the hypersurface W^* has multiplicity 2 along $W^\perp$, but the polar $\mathcal{P}(W^*, p) := W^* \cap \mathcal{P}(Q, p)$ has multiplicity 3 along $W^\perp$. The mistake in the proof of the lemma can be easily found. On line 5, page 14 of [A], I write “Let $(Z_i)_{i \in I}$ be a stratification of Z such that Z_i is smooth and Z is normally flat along Z_i for all $i \in I$. Such a stratification exists, due to the open nature of normal flatness [...]. Consider the Gauss map $G : Z \to (\mathbb{P}^N)^*$. It restricts to a map $G_i : Z_i \to (\mathbb{P}^N)^*$ ... .” This last sentence is nonsense since the Gauss map is not defined on the singular locus of Z.

I used Lemma A.3 of [A] in the form of the following corollary:

Corollary A.4 of [A]. *Let* $Z \subset \mathbb{P}^N$ *be an hypersurface and let* $z \in Z$ *such that* $\operatorname{mult}_z Z = 2$ *and let* $k \in \{-1, \dots N-2\}$. *Then, for generic* $D \in \mathbb{G}(k, N)$, *we have* $\operatorname{mult}_z \mathcal{P}(Z, D) \leq 2$.

This corollary is again false as shown in Example 1, but it seems natural to use its conclusion as an hypothesis. Indeed the rest of the proof of Proposition 3.2.3 of [A] is correct, and thus we get the following result:

Proposition 2 (replacement for Proposition 3.2.3 of [A]). *Let* $X \subset \mathbb{P}^N$ *be a smooth, irreducible, nondegenerate projective variety such that* X^* *is a hypersurface. Let* $[h] \in X^*$ *be such that* $\operatorname{mult}_{[h]} X^* = 2$ *and that for all* $k \in \{-1, \dots, N-2\}$ *and generic* $D \in \mathbb{G}(k, N)$, *we have* $\operatorname{mult}_{[h]} \mathcal{P}(X^*, D) \leq 2$. *The scheme theoretic tangency locus of* H *with* X *is one of the following*:

- *An irreducible hyperquadric and in this case* $|\mathcal{C}_{[h]}(X^*)|^* = \operatorname{Tan}(H, X)$.
- *The union of two (not necessarily distinct) linear spaces.*
- *A linear space with at least one embedded component.*

Finally, we can formulate a version of Theorem 3.2.2 of [A], whose proof relies on the above proposition:

Theorem 3 (replacement for Theorem 3.2.2 of [A]). *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate projective variety such that X^* is a hypersurface. Let $L \subset X$ be a linear space with* $\dim(L) = \dim(X) - 1$. *Assume that $L^\perp$ is an unexpected equisingular linear space in X^* such that* $\mathrm{mult}_{L^\perp}(X^*) = 2$. *Assume moreover that for all $[h] \in L^\perp$, for all $k \in \{-1, \dots, N-2\}$ and generic $D \in \mathbb{G}(k, N)$, we have* $\mathrm{mult}_{[h]} \mathcal{P}(X^*, D) \leq 2$. *Then X is the cubic scroll surface in $\mathbb{P}^4$.*

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