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THE ASYMPTOTIC BEHAVIOR OF PALAIS–SMALE SEQUENCES ON MANIFOLDS WITH BOUNDARY

SÉRGIO ALMARAZ

We describe the asymptotic behavior of Palais–Smale sequences associated to certain Yamabe-type equations on manifolds with boundary. We prove that each of those sequences converges to a solution of the limit equation plus a finite number of "bubbles" which are obtained by rescaling fundamental solutions of the corresponding Euclidean equations.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold with boundary ∂M and dimension $n \geq 3$. For $u \in H^1(M)$, we consider the following family of equations, indexed by $\nu \in \mathbb{N}$:

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ \dfrac{\partial}{\partial \eta_g} u - h_\nu u + u^{\frac{n}{n-2}} = 0 & \text{on } \partial M, \end{cases} \tag{1-1}$$

and their associated functionals

$$I_g^\nu(u) = \frac{1}{2}\int_M |du|_g^2 \, dv_g + \frac{1}{2}\int_{\partial M} h_\nu u^2 \, d\sigma_g - \frac{n-2}{2(n-1)}\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} \, d\sigma_g. \tag{1-2}$$

Here, $\{h_\nu\}_{\nu \in \mathbb{N}}$ is a sequence of functions in $C^\infty(\partial M)$, Δ_g is the Laplace–Beltrami operator, and η_g is the inward unit normal vector to ∂M. Moreover, dv_g and $d\sigma_g$ are the volume forms of M and ∂M respectively and $H^1(M)$ is the Sobolev space $H^1(M) = \{u \in L^2(M) : du \in L^2(M)\}$.

Definition 1.1. We say that $\{u_\nu\}_{\nu \in \mathbb{N}} \subset H^1(M)$ is a *Palais–Smale* sequence for $\{I_g^\nu\}$ if

(i) $\{I_g^\nu(u_\nu)\}_{\nu \in \mathbb{N}}$ is bounded, and

(ii) $dI_g^\nu(u_\nu) \to 0$ strongly in $H^1(M)'$ as $\nu \to \infty$.

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In this paper we establish a result describing the asymptotic behavior of those Palais–Smale sequences. This work is inspired by Struwe's theorem [1984] for equations $\Delta u + \lambda u + |u|^{4/(n-2)}u = 0$ on Euclidean domains. We refer the reader to [Druet et al. 2004, Chapter 3] for a version of Struwe's theorem on closed Riemannian manifolds, and to [Cao et al. 2001; Chabrowski and Girão 2002; Pierotti and Terracini 1995] for similar equations with boundary conditions.

Roughly speaking, as $\nu \to \infty$ and $h_\nu \to h_\infty$, we prove that each Palais–Smale sequence $\{u_\nu \geq 0\}_{\nu\in\mathbb{N}}$ is $H^1(M)$-asymptotic to a nonnegative solution of the limit equations

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ \dfrac{\partial}{\partial \eta_g} u - h_\infty u + u^{\frac{n}{n-2}} = 0 & \text{on } \partial M, \end{cases} \tag{1-3}$$

plus a finite number of "bubbles" obtained by rescaling fundamental positive solutions of the Euclidean equations

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial}{\partial y_n} u + u^{\frac{n}{n-2}} = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1-4}$$

where $\mathbb{R}^n_+ = \{(y_1, \dots, y_n) \in \mathbb{R}^n \,:\, y_n \geq 0\}$.

Palais–Smale sequences frequently appear in the blow-up analysis of geometric problems. In the particular case when h_∞ is $(n-2)/2$ times the boundary mean curvature, the equations (1-3) are satisfied by a positive smooth function u representing a conformal scalar-flat Riemannian metric $u^{4/(n-2)}g$ with positive constant boundary mean curvature. The existence of those metrics is the Yamabe-type problem for manifolds with boundary introduced in [Escobar 1992].

An application of our result is the blow-up analysis performed in [Almaraz 2012] for the proof of a convergence theorem for a Yamabe-type flow introduced in [Brendle 2002].

We now begin to state our theorem more precisely.

Convention. We assume that there is some $h_\infty \in C^\infty(\partial M)$ and $C > 0$ such that $h_\nu \to h_\infty$ in $L^2(\partial M)$ as $\nu \to \infty$ and $|h_\nu(x)| \leq C$ for all $x \in \partial M$, $\nu \in \mathbb{N}$. This obviously implies that $h_\nu \to h_\infty$ in $L^p(\partial M)$ as $\nu \to \infty$, for any $p \geq 1$.

Notation. If (M, g) is a Riemannian manifold with boundary ∂M, we will denote by $D_r(x)$ the metric ball in ∂M with center at $x \in \partial M$ and radius r.

If $z_0 \in \mathbb{R}^n_+$, we set $B^+_r(z_0) = \{z \in \mathbb{R}^n_+ \,:\, |z - z_0| < r\}$. We define

$$\partial^+ B^+_r(z_0) = \partial B^+_r(z_0) \cap \mathbb{R}^n_+, \quad \text{and} \quad \partial' B^+_r(z_0) = B^+_r(z_0) \cap \partial\mathbb{R}^n_+.$$

Thus, $\partial' B^+_r(z_0) = \varnothing$ if $z_0 = (z_0^1, \dots, z_0^n)$ satisfies $z_0^n > r$.

We define the Sobolev space $D^1(\mathbb{R}^n_+)$ as the completion of $C_0^\infty(\mathbb{R}^n_+)$ with respect to the norm

$$\|u\|_{D^1(\mathbb{R}^n_+)} = \sqrt{\int_{\mathbb{R}^n_+} |du(y)|^2 \, dy}.$$

It follows from a Liouville-type theorem established in [Li and Zhu 1995] (see also [Escobar 1990] and [Chipot et al. 1996]) that any nonnegative solution in $D^1(\mathbb{R}^n_+)$ to the equations (1-4) is of the form

$$U_{\epsilon,a}(y) = \left(\frac{\epsilon}{(y_n + \epsilon/(n-2))^2 + |\bar{y} - a|^2} \right)^{\frac{n-2}{2}}, \quad a \in \mathbb{R}^{n-1},\ \epsilon > 0, \tag{1-5}$$

or is identically zero; see Remark 2.5. By [Escobar 1988] or [Beckner 1993] we have the sharp Euclidean Sobolev inequality

$$\left(\int_{\partial \mathbb{R}^n_+} |u(y)|^{\frac{2(n-1)}{n-2}} \, dy \right)^{\frac{n-2}{n-1}} \le K_n^2 \int_{\mathbb{R}^n_+} |du(y)|^2 \, dy, \tag{1-6}$$

for $u \in D^1(\mathbb{R}^n_+)$, which has the family of functions (1-5) as extremal functions. Here,

$$K_n = \left(\frac{n-2}{2} \right)^{-1/2} \sigma_{n-1}^{-\frac{1}{2(n-1)}},$$

where σ_{n-1} is the area of the unit $(n-1)$-sphere in $\mathbb{R}^n$. Up to a multiplicative constant, the functions defined by (1-5) are the only nontrivial extremal ones for the inequality (1-6).

Definition 1.2. Fix $x_0 \in \partial M$ and geodesic normal coordinates for ∂M centered at x_0. Let $(x_1, \dots, x_{n-1})$ be the coordinates of $x \in \partial M$ and $\eta_g(x)$ be the inward unit vector normal to ∂M at x. For small $x_n \ge 0$, the point $\exp_x(x_n \eta_g(x)) \in M$ is said to have *Fermi coordinates* $(x_1, \dots, x_n)$ (*centered at* x_0).

For small $\rho > 0$ the Fermi coordinates centered at $x_0 \in \partial M$ define a smooth map $\psi_{x_0} : B^+_\rho(0) \subset \mathbb{R}^n_+ \to M$.

We define the functional I_g^∞ by the same expression as I_g^ν, with $h_\nu = h_\infty$ for all ν, and state our main theorem as follows:

Theorem 1.3. *Let (M^n, g) be a compact Riemannian manifold with boundary ∂M and dimension $n \ge 3$. Suppose $\{u_\nu \ge 0\}_{\nu \in \mathbb{N}}$ is a Palais–Smale sequence for $\{I_g^\nu\}$. Then there exist $m \in \{0, 1, 2, \dots\}$, a nonnegative solution $u^0 \in H^1(M)$ of* (1-3), *and m nontrivial nonnegative solutions $U^j = U_{\epsilon_j, a_j} \in D^1(\mathbb{R}^n_+)$ of* (1-4), *sequences $\{R^j_\nu > 0\}_{\nu \in \mathbb{N}}$, and sequences $\{x^j_\nu\}_{\nu \in \mathbb{N}} \subset \partial M$, $1 \le j \le m$, the whole satisfying the following conditions for $1 \le j \le m$, possibly after taking subsequences*:

(i) $R^j_\nu \to \infty$ *as* $\nu \to \infty$.

(ii) x^j_ν *converges as* $\nu \to \infty$.

(iii) $\left\| u_\nu - u^0 - \sum_{j=1}^{m} \eta_\nu^j u_\nu^j \right\|_{H^1(M)} \to 0$ *as* $\nu \to \infty$, *where*

$$u_\nu^j(x) = (R_\nu^j)^{(n-2)/2} U^j (R_\nu^j \psi_{x_\nu^j}^{-1}(x)) \quad \textit{for } x \in \psi_{x_\nu^j}(B_{2r_0}^+(0)).$$

Here, $r_0 > 0$ *is small*, *the*

$$\psi_{x_\nu^j} : B_{2r_0}^+(0) \subset \mathbb{R}_+^n \to M$$

are Fermi coordinates centered at $x_\nu^j \in \partial M$, *and the* η_ν^j *are smooth cutoff functions such that* $\eta_\nu^j \equiv 1$ *in* $\psi_{x_\nu^j}(B_{r_0}^+(0))$ *and* $\eta_\nu^j \equiv 0$ *in* $M \backslash \psi_{x_\nu^j}(B_{2r_0}^+(0))$.

Moreover,

$$I_g^\nu(u_\nu) - I_g^\infty(u^0) - \frac{m}{2(n-1)} K_n^{-2(n-1)} \to 0 \quad \textit{as } \nu \to \infty,$$

and we can assume that for all $i \neq j$

$$\frac{R_\nu^i}{R_\nu^j} + \frac{R_\nu^j}{R_\nu^i} + R_\nu^i R_\nu^j d_g(x_\nu^i, x_\nu^j)^2 \to \infty \quad \textit{as } \nu \to \infty. \tag{1-7}$$

Remark 1.4. Relations of the type (1-7) were previously obtained in [Bahri and Coron 1988; Brezis and Coron 1985].

2. Proof of the main theorem

The rest of this paper is devoted to the proof of Theorem 1.3, which will be carried out in several lemmas. Our presentation will follow the same steps as Chapter 3 of [Druet et al. 2004], with the necessary modifications.

Lemma 2.1. *Let* $\{u_\nu\}$ *be a Palais–Smale sequence for* $\{I_g^\nu\}$. *Then there exists* $C > 0$ *such that* $\|u_\nu\|_{H^1(M)} \le C$ *for all* ν.

Proof. It suffices to prove that $\|du_\nu\|_{L^2(M)}$ and $\|u_\nu\|_{L^2(\partial M)}$ are uniformly bounded. The proof follows the same arguments as [Druet et al. 2004, p. 27]. ☐

Define I_g as the functional in (1-2) when $h_\nu \equiv 0$ for all ν.

Lemma 2.2. *Let* $\{u_\nu \ge 0\}$ *be a Palais–Smale sequence for* $\{I_g^\nu\}$ *such that* $u_\nu \rightharpoonup u^0 \ge 0$ *in* $H^1(M)$, *and set* $\hat{u}_\nu = u_\nu - u^0$. *Then* $\{\hat{u}_\nu\}$ *is a Palais–Smale sequence for* $\{I_g\}$ *and satisfies*

$$I_g(\hat{u}_\nu) - I_g^\nu(u_\nu) + I_g^\infty(u^0) \to 0 \quad \textit{as } \nu \to \infty. \tag{2-1}$$

Moreover, u^0 *is a* (*weak*) *solution of* (1-3).

Proof. First, observe that $u_\nu \rightharpoonup u^0$ in $H^1(M)$ implies that $u_\nu \to u^0$ in $L^{\frac{n}{n-2}}(\partial M)$ and a.e. in ∂M. Using the facts that $dI_g^\nu(u_\nu)\phi \to 0$ for any $\phi \in C^\infty(\overline{M})$ and $h_\nu \to h_\infty$ in $L^p(\partial M)$ for any $p \ge 1$, it is not difficult to see that the last assertion of Lemma 2.2 follows.

In order to prove (2-1), we first observe that

$$I_g^\nu(u_\nu) = I_g(\hat{u}_\nu) + I_g^\infty(u^0) - \frac{(n-2)}{2(n-1)} \int_{\partial M} \Phi_\nu \, d\sigma_g + o(1),$$

where $\Phi_\nu = |\hat{u}_\nu + u^0|^{\frac{2(n-1)}{n-2}} - |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} - |u^0|^{\frac{2(n-1)}{n-2}}$ and $o(1) \to 0$ as $\nu \to \infty$. Then (2-1) follows from the fact that there exists $C > 0$ such that

$$\int_{\partial M} \Phi_\nu \, d\sigma_g \le C \int_{\partial M} |\hat{u}_\nu|^{\frac{n}{n-2}} |u^0| \, d\sigma_g + C \int_{\partial M} |u^0|^{\frac{n}{n-2}} |\hat{u}_\nu| \, d\sigma_g \quad \text{for all } \nu,$$

and, by basic integration theory, the right side of this last inequality goes to 0 as $\nu \to \infty$.

Now we prove that $\{\hat{u}_\nu\}$ is a Palais–Smale sequence for I_g. Let $\phi \in C^\infty(M)$. Observe that

$$\left| \int_{\partial M} h_\nu u_\nu \phi \, d\sigma_g - \int_{\partial M} h_\infty u_\nu \phi \, d\sigma_g \right| \\ \le \|u_\nu\|_{L^2(\partial M)} \|h_\nu - h_\infty\|_{L^{2(n-1)}(\partial M)} \|\phi\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)}$$

by Hölder's inequality. Then, by the Sobolev embedding theorem,

$$\int_{\partial M} h_\nu u_\nu \phi \, d\sigma_g = \int_{\partial M} h_\infty u^0 \phi \, d\sigma_g + o(\|\phi\|_{H^1(M)}),$$

from which follows that

$$dI_g^\nu(u_\nu)\phi = dI_g(\hat{u}_\nu)\phi - \int_{\partial M} \psi_\nu \phi \, d\sigma_g + o(\|\phi\|_{H^1(M)}), \tag{2-2}$$

where $\psi_\nu = |\hat{u}_\nu + u^0|^{\frac{2}{n-2}}(\hat{u}_\nu + u^0) - |\hat{u}_\nu|^{\frac{2}{n-2}} \hat{u}_\nu - |u^0|^{\frac{2}{n-2}} u^0$.

Next we observe that there exists $C > 0$ such that

$$\int_{\partial M} |\psi_\nu \phi| \, d\sigma_g \le C \int_{\partial M} |\hat{u}_\nu|^{\frac{2}{n-2}} |u^0||\phi| \, d\sigma_g + C \int_{\partial M} |u^0|^{\frac{2}{n-2}} |\hat{u}_\nu||\phi| \, d\sigma_g$$

for all ν, and use Hölder's inequality and basic integration theory to obtain

$$\begin{aligned} \int_{\partial M} |\psi_\nu \phi| \, d\sigma_g \\ &\le \Big(\big\| |\hat{u}_\nu|^{\frac{2}{n-2}} u^0 \big\|_{L^{\frac{2(n-1)}{n}}(\partial M)} + \big\| |u^0|^{\frac{2}{n-2}} \hat{u}_\nu \big\|_{L^{\frac{2(n-1)}{n}}(\partial M)} \Big) \|\phi\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)} \\ &= o\big(\|\phi\|_{L^{\frac{2(n-1)}{n-2}}(\partial M)} \big). \end{aligned}$$

We can use this and the Sobolev embedding theorem in (2-2) to conclude that

$$dI_g^\nu(u_\nu)\phi = dI_g(\hat{u}_\nu)\phi + o(\|\phi\|_{H^1(M)}),$$

finishing the proof. $\square$

Lemma 2.3. *Let $\{\hat{u}_\nu\}_{\nu\in\mathbb{N}}$ be a Palais–Smale sequence for I_g such that $\hat{u}_\nu \rightharpoonup 0$ in $H^1(M)$ and $I_g(\hat{u}_\nu) \to \beta$ as $\nu \to \infty$ for some $\beta < K_n^{-2(n-1)}/(2(n-1))$. Then $\hat{u}_\nu \to 0$ in $H^1(M)$ as $\nu \to \infty$.*

Proof. Since

$$\int_M |d\hat{u}_\nu|^2\, dv_g - \int_{\partial M} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_g = dI_g(\hat{u}_\nu)\cdot \hat{u}_\nu = o(\|\hat{u}_\nu\|_{H^1(M)})$$

and $\{\|\hat{u}_\nu\|_{H^1(M)}\}$ is uniformly bounded due to Lemma 2.1, we can see that

$$\begin{aligned} \beta + o(1) = I_g(\hat{u}_\nu) &= \frac{1}{2(n-1)} \int_{\partial M} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_g + o(1) \\ &= \frac{1}{2(n-1)} \int_M |d\hat{u}_\nu|_g^2\, dv_g + o(1), \end{aligned} \tag{2-3}$$

which already implies $\beta \ge 0$. At the same time, as proved in [Li and Zhu 1997], there exists $B = B(M,g) > 0$ such that

$$\left(\int_{\partial M} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_g\right)^{\frac{n-2}{n-1}} \le K_n^2 \int_M |d\hat{u}_\nu|_g^2\, dv_g + B\int_{\partial M} |\hat{u}_\nu|^2\, d\sigma_g.$$

Since $H^1(M)$ is compactly embedded in $L^2(\partial M)$, we have $\|\hat{u}_\nu\|_{L^2(\partial M)} \to 0$. Then

$$(2(n-1)\beta + o(1))^{\frac{n-2}{n-1}} \le 2(n-1)K_n^2\beta + o(1),$$

from which we conclude that either

$$\frac{K_n^{-2(n-1)}}{2(n-1)} \le \beta + o(1)$$

or $\beta = 0$. Hence, our hypotheses imply $\beta = 0$. Using (2-3) finishes the proof. □

Define the functional

$$E(u) = \tfrac{1}{2}\int_{\mathbb{R}^n_+} |du(y)|^2\, dy - \frac{n-2}{2(n-1)} \int_{\partial\mathbb{R}^n_+} |u(y)|^{\frac{2(n-1)}{n-2}}\, dy$$

for $u \in D^1(\mathbb{R}^n_+)$ and observe that $E(U_{\epsilon,a}) = \dfrac{K_n^{-2(n-1)}}{2(n-1)}$ for any $a \in \mathbb{R}^{n-1}$, $\epsilon > 0$.

Lemma 2.4. *Let $\{\hat{u}_\nu\}_{\nu\in\mathbb{N}}$ be a Palais–Smale sequence for I_g. Suppose $\hat{u}_\nu \rightharpoonup 0$ in $H^1(M)$, but not strongly. Then there exist a sequence $\{R_\nu > 0\}_{\nu\in\mathbb{N}}$ with $R_\nu \to \infty$, a convergent sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset \partial M$, and a nontrivial solution $u \in D^1(\mathbb{R}^n_+)$ of*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial}{\partial y_n} u - |u|^{2/(n-2)} u = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{2-4}$$

the whole such that, up to a subsequence, the following holds: If

$$\hat{v}_\nu(x) = \hat{u}_\nu(x) - \eta_\nu(x) R_\nu^{\frac{n-2}{2}} u(R_\nu \psi_{x_\nu}^{-1}(x)),$$

then $\{\hat{v}_\nu\}_{\nu \in \mathbb{N}}$ *is a Palais–Smale sequence for* I_g *satisfying* $\hat{v}_\nu \rightharpoonup 0$ *in* $H^1(M)$ *and*

$$\lim_{\nu \to \infty} \big(I_g(\hat{u}_\nu) - I_g(\hat{v}_\nu) \big) = E(u).$$

Here, the $\psi_{x_\nu} : B^+_{2r_0}(0) \subset \mathbb{R}^n_+ \to M$ *are Fermi coordinates centered at* x_ν *and the* $\eta_\nu(x)$ *are smooth cutoff functions such that* $\eta_\nu \equiv 1$ *in* $\psi_{x_\nu}(B^+_{r_0}(0))$ *and* $\eta_\nu \equiv 0$ *in* $M \backslash \psi_{x_\nu}(B^+_{2r_0}(0))$.

Proof. By the density of $C^\infty(M)$ in $H^1(M)$ we can assume that $\hat{u}_\nu \in C^\infty(M)$. We can also assume that $I_g(\hat{u}_\nu) \to \beta$ as $\nu \to \infty$ and, since $dI_g(\hat{u}_\nu) \to 0$ in $H^1(M)'$, we obtain

$$\lim_{\nu \to \infty} \int_{\partial M} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} \, d\sigma_g = 2(n-1)\beta \geq K_n^{-2(n-1)},$$

as in the proof of Lemma 2.3. Hence, given $t_0 > 0$ small we can choose $x_0 \in \partial M$ and $\lambda_0 > 0$ such that

$$\int_{D_{t_0}(x_0)} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} \, d\sigma_g \geq \lambda_0$$

up to a subsequence. Now we set

$$\mu_\nu(t) = \max_{x \in \partial M} \int_{D_t(x)} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} \, d\sigma_g$$

for $t > 0$, and, for any $\lambda \in (0, \lambda_0)$, choose sequences $\{t_\nu\} \subset (0, t_0)$ and $\{x_\nu\} \subset \partial M$ such that

$$\lambda = \mu_\nu(t_\nu) = \int_{D_{t_\nu}(x_\nu)} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}} \, d\sigma_g. \tag{2-5}$$

We can also assume that x_ν converges. Now, we choose $r_0 > 0$ small such that for any $x_0 \in \partial M$ the Fermi coordinates $\psi_{x_0}(z)$ centered at x_0 are defined for all $z \in B^+_{2r_0}(0) \subset \mathbb{R}^n_+$ and satisfy

$$\tfrac{1}{2}|z - z'| \leq d_g(\psi_{x_0}(z), \psi_{x_0}(z')) \leq 2|z - z'| \quad \text{for any } z, z' \in B^+_{r_0}(0).$$

For each ν we consider Fermi coordinates

$$\psi_\nu = \psi_{x_\nu} : B^+_{2r_0}(0) \to M.$$

For any $R_\nu \geq 1$ and $y \in B^+_{R_\nu r_0}(0)$, we set

$$\tilde{u}_\nu(y) = R_\nu^{-\frac{n-2}{2}} \hat{u}_\nu(\psi_\nu(R_\nu^{-1} y)) \quad \text{and} \quad \tilde{g}_\nu(y) = (\psi_\nu^* g)(R_\nu^{-1} y).$$

Let us consider $z \in \mathbb{R}^n_+$ and $r > 0$ such that $|z| + r < R_\nu r_0$. Then we have

$$\int_{B_r^+(z)} |d\tilde{u}_\nu|^2_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} = \int_{\psi_\nu(R_\nu^{-1}B_r^+(z))} |d\hat{u}_\nu|^2_g\, dv_g,$$

and, if in addition $z \in \partial\mathbb{R}^n_+$,

$$\begin{aligned}(2\text{-}6)\qquad \int_{\partial' B_r^+(z)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_{\tilde{g}_\nu} &= \int_{\psi_\nu(R_\nu^{-1}\partial' B_r^+(z))} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_g \\ &\leq \int_{D_{2R_\nu^{-1}r}(\psi_\nu(R_\nu^{-1}z))} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_g,\end{aligned}$$

where we have used the fact that

$$\psi_\nu\big(R_\nu^{-1}\partial' B_r^+(z)\big) = \psi_\nu\big(\partial' B^+_{R_\nu^{-1}r}(R_\nu^{-1}z)\big) \subset D_{2R_\nu^{-1}r}\big(\psi_\nu(R_\nu^{-1}z)\big).$$

Given $r \in (0, r_0)$ we fix $t_0 \leq 2r$. Then, given a $\lambda \in (0, \lambda_0)$ to be fixed later, we set $R_\nu = 2rt_\nu^{-1} \geq 2rt_0^{-1} \geq 1$. It follows from (2-5) and (2-6) that

$$(2\text{-}7)\qquad \int_{\partial' B_r^+(z)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_{\tilde{g}_\nu} \leq \lambda.$$

Moreover, since $\psi_\nu(\partial' B^+_{2R_\nu^{-1}r}(0)) = D_{t_\nu}(x_\nu)$, we have

$$(2\text{-}8)\qquad \int_{\partial' B_{2r}^+(0)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_{\tilde{g}_\nu} = \int_{D_{t_\nu}(x_\nu)} |\hat{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_g = \lambda.$$

Choosing r_0 smaller if necessary, we can suppose that

$$(2\text{-}9)\qquad \frac{1}{2}\int_{\mathbb{R}^n_+} |du|^2\, dy \leq \int_{\mathbb{R}^n_+} |du|^2_{\tilde{g}_{x_0,R}}\, dv_{\tilde{g}_{x_0,R}} \leq 2\int_{\mathbb{R}^n_+} |du|^2\, dy$$

for any $R \geq 1$ and any $u \in D^1(\mathbb{R}^n_+)$ such that $\operatorname{supp}(u) \subset B^+_{2r_0R}(0)$. Here, $\tilde{g}_{x_0,R}(y) = (\psi_{x_0}^* g)(R^{-1}y)$. We can also assume that

$$(2\text{-}10)\qquad \frac{1}{2}\int_{\partial\mathbb{R}^n_+} |u|\, dy \leq \int_{\partial\mathbb{R}^n_+} |u|\, d\sigma_{\tilde{g}_{x_0,R}} \leq 2\int_{\partial\mathbb{R}^n_+} |u|\, dy$$

for all $u \in L^1(\partial\mathbb{R}^n_+)$ such that $\operatorname{supp}(u) \subset \partial' B^+_{2r_0R}(0)$.

Let $\tilde{\eta}$ be a smooth cutoff function on $\mathbb{R}^n$ such that $0 \leq \tilde{\eta} \leq 1$, $\tilde{\eta}(z) = 1$ for $|z| \leq \frac{1}{4}$, and $\tilde{\eta}(z) = 0$ for $|z| \geq \frac{3}{4}$. We set $\tilde{\eta}_\nu(y) = \tilde{\eta}(r_0^{-1}R_\nu^{-1}y)$.

It is easy to check that

$$\left\{\int_{\mathbb{R}^n_+} |d(\tilde{\eta}_\nu\tilde{u}_\nu)|^2_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu}\right\}$$

is uniformly bounded. Then the inequality (2-9) implies that $\{\tilde{\eta}_\nu\tilde{u}_\nu\}$ is uniformly bounded in $D^1(\mathbb{R}^n_+)$ and we can assume that $\tilde{\eta}_\nu \tilde{u}_\nu \rightharpoonup u$ in $D^1(\mathbb{R}^n_+)$ for some u.

Claim 1. *Let us set $r_1 = r_0/24$. There exists $\lambda_1 = \lambda_1(n)$ such that for any $0 < r < r_1$ and $0 < \lambda < \min\{\lambda_1, \lambda_0\}$ we have*

$$\tilde{\eta}_\nu \tilde{u}_\nu \to u \textit{ in } H^1(B^+_{2Rr}(0)) \quad \textit{as } \nu \to \infty,$$

for any $R \geq 1$ satisfying $R \leq R_\nu$ for all ν large.

Proof. We consider $r \in (0, r_1)$, $\lambda \in (0, \lambda_0)$ and choose $z_0 \in \partial\mathbb{R}^n_+$ such that $|z_0| < 3(2R-1)r_1$. By Fatou's lemma,

$$\int_r^{2r} \liminf_{\nu\to\infty} \left\{ \int_{\partial^+ B^+_\rho(z_0)} \big(|d(\tilde{\eta}_\nu \tilde{u}_\nu)|^2 + |\tilde{\eta}_\nu \tilde{u}_\nu|^2\big)\, d\sigma_\rho \right\} d\rho$$

$$\leq \liminf_{\nu\to\infty} \int_{B^+_{2r}(z_0)} \big(|d(\tilde{\eta}_\nu \tilde{u}_\nu)|^2 + |\tilde{\eta}_\nu \tilde{u}_\nu|^2\big)\, dy \leq C,$$

where $d\sigma_\rho$ is the volume form on $\partial^+ B^+_\rho(z_0)$ induced by the Euclidean metric. Thus there exists $\rho \in [r, 2r]$ such that, up to a subsequence,

$$\int_{\partial^+ B^+_\rho(z_0)} \big(|d(\tilde{\eta}_\nu \tilde{u}_\nu)|^2 + |\tilde{\eta}_\nu \tilde{u}_\nu|^2\big)\, d\sigma_\rho \leq C \quad \text{for all } \nu.$$

Hence, $\big\{\|\tilde{\eta}_\nu \tilde{u}_\nu\|_{H^1(\partial^+ B^+_\rho(z_0))}\big\}$ is uniformly bounded, and, since the embedding

$$H^1(\partial^+ B^+_\rho(z_0)) \subset H^{1/2}(\partial^+ B^+_\rho(z_0))$$

is compact, we can assume that

$$\tilde{\eta}_\nu \tilde{u}_\nu \to u \text{ in } H^{1/2}(\partial^+ B^+_\rho(z_0)) \quad \text{as } \nu \to \infty.$$

We set $\mathscr{A} = B^+_{3r}(z_0) - \overline{B^+_\rho(z_0)}$, and let $\{\phi_\nu\} \subset D^1(\mathbb{R}^n_+)$ be such that

$$\phi_\nu = \begin{cases} \tilde{\eta}_\nu \tilde{u}_\nu - u, & \text{in } B^+_{\rho+\epsilon}(z_0), \\ 0, & \text{in } \mathbb{R}^n_+ \setminus B^+_{3r-\epsilon}(z_0), \end{cases}$$

with $\epsilon > 0$ small. Then

$$\|\tilde{\eta}_\nu \tilde{u}_\nu - u\|_{H^{1/2}(\partial^+ B^+_\rho(z_0))} = \|\phi_\nu\|_{H^{1/2}(\partial^+ B^+_\rho(z_0))} \to 0 \quad \text{as } \nu \to \infty,$$

and there exists $\{\phi^0_\nu\} \subset D^1(\mathscr{A})$ such that

$$\|\phi_\nu + \phi^0_\nu\|_{H^1(\mathscr{A})} \leq C\|\phi_\nu\|_{H^{1/2}(\partial^+ \mathscr{A})} = C\|\phi_\nu\|_{H^{1/2}(\partial^+ B^+_\rho(z_0))}$$

for some $C > 0$ independent of ν. Here, $D^1(\mathscr{A})$ is the closure of $C^\infty_0(\mathscr{A})$ in $H^1(\mathscr{A})$, and we have set $\partial^+\mathscr{A} = \partial\mathscr{A} \cap (\mathbb{R}^n_+ \setminus \partial\mathbb{R}^n_+)$ and $\partial'\mathscr{A} = \partial\mathscr{A} \cap \partial\mathbb{R}^n_+$.

The sequence of functions $\{\zeta_\nu\} = \{\phi_\nu + \phi^0_\nu\} \subset D^1(\mathbb{R}^n_+)$ satisfies

$$\zeta_\nu = \begin{cases} \tilde{\eta}_\nu \tilde{u}_\nu - u & \text{in } \overline{B^+_\rho(z_0)}, \\ \phi_\nu + \phi^0_\nu & \text{in } B^+_{3r}(z_0)\backslash\overline{B^+_\rho(z_0)}, \\ 0 & \text{in } \mathbb{R}^n_+\backslash B^+_{3r}(z_0). \end{cases}$$

In particular, $\zeta_\nu \to 0$ in $H^1(\mathscr{A})$. We set

$$\tilde{\zeta}_\nu(x) = R_\nu^{\frac{n-2}{2}} \zeta_\nu(R_\nu \psi_\nu^{-1}(x)) \quad \text{if } x \in \psi_\nu(B^+_{6r_1}(0)),$$

and $\tilde{\zeta}_\nu(x) = 0$ otherwise. Since we are assuming $|z_0| < 3(2R-1)r_1 \le 3(2R_\nu - 1)r_1$ for all ν large, $B^+_{3r}(z_0) \subset B^+_{6r_1 R_\nu}(0)$. Hence,

$$\tilde{\zeta}_\nu(x) = \begin{cases} R_\nu^{\frac{n-2}{2}}(\tilde{\eta}_\nu \tilde{u}_\nu - u)(R_\nu \psi_\nu^{-1}(x)) & \text{if } x \in \psi_\nu(R_\nu^{-1}\overline{B^+_\rho(z_0)}), \\ R_\nu^{\frac{n-2}{2}}(\phi_\nu + \phi^0_\nu)(R_\nu \psi_\nu^{-1}(x)) & \text{if } x \in \psi_\nu\big(R_\nu^{-1}(\overline{B^+_{3r}(z_0)}\backslash B^+_\rho(z_0))\big), \end{cases}$$

and $\tilde{\zeta}_\nu(x) = 0$ otherwise, and

$$\begin{aligned} \text{(2-11)}\quad & dI_g(\hat{u}_\nu)\cdot\tilde{\zeta}_\nu \\ &= dI_g(\hat{\eta}_\nu \hat{u}_\nu)\cdot\tilde{\zeta}_\nu \\ &= \int_{B^+_{3r}(z_0)} \langle d(\tilde{\eta}_\nu \tilde{u}_\nu), d\zeta_\nu\rangle_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} - \int_{\partial' B^+_{3r}(z_0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2}{n-2}}(\tilde{\eta}_\nu \tilde{u}_\nu)\zeta_\nu\, d\sigma_{\tilde{g}_\nu}, \end{aligned}$$

where $\hat{\eta}_\nu(x) = \tilde{\eta}(r_0^{-1}\psi_\nu^{-1}(x))$.

Since there exists $C > 0$ such that $\|\tilde{\zeta}_\nu\|_{H^1(M)} \le C\|\zeta_\nu\|_{D^1(\mathbb{R}^n_+)}$, the sequence $\{\tilde{\zeta}_\nu\}$ is uniformly bounded in $H^1(M)$. Hence,

$$dI_g(\hat{u}_\nu)\cdot\tilde{\zeta}_\nu \to 0 \quad \text{as } \nu \to \infty. \tag{2-12}$$

Noting that $\zeta_\nu \to 0$ in $H^1(\mathscr{A})$ and $\zeta_\nu \rightharpoonup 0$ in $D^1(\mathbb{R}^n_+)$, we obtain

$$\begin{aligned} \int_{B^+_{3r}(z_0)} \langle d(\tilde{\eta}_\nu \tilde{u}_\nu), d\zeta_\nu\rangle_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} &= \int_{B^+_\rho(z_0)} \langle d(\zeta_\nu + u), d\zeta_\nu\rangle_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} + o(1) \\ &= \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} + o(1). \end{aligned} \tag{2-13}$$

Similarly,

$$\int_{\partial' B^+_{3r}(z_0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2}{n-2}}(\tilde{\eta}_\nu \tilde{u}_\nu)\zeta_\nu\, d\sigma_{\tilde{g}_\nu} = \int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_{\tilde{g}_\nu} + o(1). \tag{2-14}$$

Using (2-11), (2-12), (2-13) and (2-14) we conclude that

$$\int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} = \int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_{\tilde{g}_\nu} + o(1). \tag{2-15}$$

Using again the facts that $\zeta_\nu \to 0$ in $H^1(\mathscr{A})$ and $\zeta_\nu \rightharpoonup 0$ in $D^1(\mathbb{R}^n_+)$, we can apply the inequality

$$\left| |\tilde{\eta}_\nu \tilde{u}_\nu - u|^{\frac{2(n-1)}{n-2}} - |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} + |u|^{\frac{2(n-1)}{n-2}} \right| \\ \leq C|u|^{\frac{n}{n-2}} |\tilde{\eta}_\nu \tilde{u}_\nu - u| + C|\tilde{\eta}_\nu \tilde{u}_\nu - u|^{\frac{n}{n-2}} |u|$$

to see that

$$\int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} = \int_{\partial' B^+_\rho(z_0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} - \int_{\partial' B^+_\rho(z_0)} |u|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1).$$

This implies

$$\int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} \leq \int_{\partial' B^+_\rho(z_0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1) \\ = \int_{\partial' B^+_\rho(z_0)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} + o(1), \tag{2-16}$$

where we have used the fact that $\tilde{\eta}_\nu(z) = 1$ for all $z \in B^+_\rho(z_0)$.

If $N = N(n) \in \mathbb{N}$ is such that $\partial' B^+_2(0)$ is covered by N discs in $\partial\mathbb{R}^n_+$ of radius 1 with center in $\partial' B^+_2(0)$, then we can choose points $z_i \in \partial' B^+_{2r}(z_0)$, $i = 1, \ldots, N$, satisfying

$$\partial' B^+_\rho(z_0) \subset \partial' B^+_{2r}(z_0) \subset \bigcup_{i=1}^N \partial' B^+_r(z_i).$$

Hence, using (2-7), (2-15) and (2-16), we see that

$$\int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu} + o(1) = \int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} \leq N\lambda + o(1). \tag{2-17}$$

It follows from (2-9), (2-10) and the Sobolev inequality (1-6) that

$$\left(\int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} d\sigma_{\tilde{g}_\nu} \right)^{\frac{n-2}{n-1}} \leq 2^{\frac{n-2}{n-1}} \left(\int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}} dx \right)^{\frac{n-2}{n-1}} \\ \leq 2^{\frac{n-2}{n-1}} K_n^2 \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2 dx \\ \leq 2^{1+\frac{n-2}{n-1}} K_n^2 \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu} dv_{\tilde{g}_\nu}.$$

Then using (2-15) and (2-17) we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} &= \int_{\partial\mathbb{R}^n_+} |\zeta_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_{\tilde{g}_\nu} + o(1) \\
&\le \left(2^{1+\frac{n-2}{n-1}} K_n^2\right)^{\frac{n-1}{n-2}} \left(\int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu}\right)^{\frac{n-1}{n-2}} + o(1) \\
&\le 2^{1+\frac{n-1}{n-2}} K_n^{\frac{2(n-1)}{n-2}} (N\lambda + o(1))^{\frac{1}{n-2}} \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} + o(1).
\end{aligned}
$$

Now we set $\lambda_1 = \dfrac{K_n^{-2(n-1)}}{2^{2n-3}N}$ and assume that $\lambda < \lambda_1$. Then

$$
2^{1+\frac{n-1}{n-2}} (N\lambda)^{\frac{1}{n-2}} K_n^{\frac{2(n-1)}{n-2}} < 1,
$$

and we conclude that

$$
\lim_{\nu\to\infty} \int_{\mathbb{R}^n_+} |d\zeta_\nu|^2_{\tilde{g}_\nu}\, dv_{\tilde{g}_\nu} = 0.
$$

Hence, $\zeta_\nu \to 0$ in $D^1(\mathbb{R}^n_+)$. Since $r \le \rho$, we have

$$
\tilde{\eta}_\nu \tilde{u}_\nu \to u \quad \text{in } H^1(B_r^+(z_0)). \tag{2-18}
$$

Now let us choose any $z_0 = ((z_0)^1, \dots, (z_0)^n) \in \mathbb{R}^n_+$ satisfying $(z_0)^n > r/2$ and $|z_0| < 3(2R-1)r_1$. Using this choice of z_0 and $r' = r/6$ replacing r, the process above can be performed with some obvious modifications. In this case, we have $\partial' B^+_{3r'}(z_0) = \varnothing$ and the boundary integrals vanish. Hence, the equality (2-15) already implies that $\tilde{\eta}_\nu \tilde{u}_\nu \to u$ in $H^1(B^+_{r'}(z_0))$.

If $N_1 = N_1(R,n) \in \mathbb{N}$ and $N_2 = N_2(R,n) \in \mathbb{N}$ are such that the half-ball $B^+_{2R}(0)$ is covered by N_1 half-balls of radius 1 with centers in $\partial' B^+_{2R}(0)$, plus N_2 balls of radius $1/6$ with centers in $\{z = (z^1, \dots, z^n) \in B^+_{2R}(0) : z^n > 1/2\}$, then the half-ball $B^+_{2Rr}(0)$ is covered by N_1 half-balls of radius r with centers in $\partial' B^+_{2Rr}(0)$, plus N_2 balls of radius $r/6$ with center in $\{z = (z^1, \dots, z^n) \in B^+_{2Rr}(0) : z^n > r/2\}$.

Hence, $\tilde{\eta}_\nu \tilde{u}_\nu \to u$ in $H^1(B^+_{2Rr}(0))$, finishing the proof of Claim 1. □

Using (2-8), (2-10) and Claim 1 with $R = 1$, we see that

$$
\begin{aligned}
\lambda &= \int_{\partial' B_r^+(0)} |\tilde{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_{\tilde{g}_\nu} \\
&= \int_{\partial' B_r^+(0)} |\tilde{\eta}_\nu \tilde{u}_\nu|^{\frac{2(n-1)}{n-2}}\, d\sigma_{\tilde{g}_\nu} \\
&\le 2 \int_{\partial' B_r^+(0)} |u|^{\frac{2(n-1)}{n-2}}\, dx + o(1).
\end{aligned} \tag{2-19}
$$

It follows that $u \not\equiv 0$, due to (1-6).

Claim 2. *We have* $\lim_{\nu\to\infty} R_\nu = \infty$. *In particular*, *Claim 1 can be stated for any* $R \geq 1$.

Proof. Suppose for a contradiction that, up to a subsequence, $R_\nu \to R'$ as $\nu \to \infty$, for some $1 \leq R' < \infty$. Then, since $\hat{u}_\nu \rightharpoonup 0$ in $H^1(M)$, we have $\tilde{u}_\nu \rightharpoonup 0$ in $H^1(B^+_{2r}(0))$. This contradicts the fact that

$$\tilde{u}_\nu \tilde{\eta}_\nu \to u \not\equiv 0 \quad \text{in } H^1(B^+_{2r}(0)),$$

which is obtained by applying Claim 1 with $R = 1$. This proves Claim 2. □

That u is a (weak) solution of (2-4) follows easily from the fact that $\{\hat{u}_\nu\}$ is a Palais–Smale sequence for I_g and $\tilde{\eta}_\nu \tilde{u}_\nu \to u$ in $D^1(\mathbb{R}^n_+)$.

Now, we set

$$V_\nu(x) = \eta_\nu(x) R_\nu^{\frac{n-2}{2}} u(R_\nu \psi^{-1}_{x_\nu}(x))$$

for $x \in \psi_{x_\nu}(B^+_{2r_0}(0))$, and 0 otherwise. The proof of the following claim is totally analogous to step 3 on p. 37 of [Druet et al. 2004] with some obvious modifications.

Claim 3. *We have* $\hat{u}_\nu - V_\nu \rightharpoonup 0$, *as* $\nu \to \infty$, *in* $H^1(M)$. *Moreover*, *as* $\nu \to \infty$,

$$dI_g(V_\nu) \to 0 \quad \textit{and} \quad dI_g(\hat{u}_\nu - V_\nu) \to 0$$

strongly in $H^1(M)'$, *and*

$$I_g(\hat{u}_\nu) - I_g(\hat{u}_\nu - V_\nu) \to E(u).$$

We finally observe that if $r'_0 > 0$ is also sufficiently small then $|(\eta_\nu - \eta'_\nu)V_\nu| \to 0$ as $\nu \to \infty$, where η'_ν is a smooth cutoff function such that $\eta'_\nu \equiv 1$ in $\psi_{x_\nu}(B^+_{r'_0}(0))$ and $\eta'_\nu \equiv 0$ in $M \backslash \psi_{x_\nu}(B^+_{2r'_0}(0))$. Hence, the statement of Lemma 2.4 holds for any $r_0 > 0$ sufficiently small, finishing the proof. □

Proof of Theorem 1.3. According to Lemma 2.1, the Palais–Smale sequence $\{u_\nu\}$ for I^ν_g is uniformly bounded in $H^1(M)$. Hence, we can assume that $u_\nu \rightharpoonup u^0$ in $H^1(M)$, and $u_\nu \to u^0$ a.e. in M, for some $0 \leq u^0 \in H^1(M)$. By Lemma 2.2, u^0 is a solution to the equations (1-3). Moreover, $\hat{u}_\nu = u_\nu - u^0$ is Palais–Smale for I_g and satisfies

$$I_g(\hat{u}_\nu) = I^\nu_g(u_\nu) - I^\infty_g(u^0) + o(1).$$

If $\hat{u}_\nu \to 0$ in $H^1(M)$, then the theorem is proved. If $\hat{u}_\nu \rightharpoonup 0$ in $H^1(M)$ but not strongly, then we apply Lemma 2.4 to obtain a new Palais–Smale sequence $\{\hat{u}^1_\nu\}$ satisfying

$$I_g(\hat{u}^1_\nu) \leq I_g(\hat{u}_\nu) - \beta^* + o(1) = I^\nu_g(u_\nu) - I^\infty_g(u^0) - \beta^* + o(1),$$

where $\beta^* = \dfrac{K_n^{-2(n-1)}}{2(n-1)}$. The term β^* appears in this inequality because $E(u) \geq \beta^*$

for any nontrivial solution $u \in D^1(\mathbb{R}^n_+)$ to the equations (1-1). This can be seen using the Sobolev inequality (1-6).

Now we again have either $\hat{u}^1_\nu \to 0$ in $H^1(M)$, in which case the theorem is proved, or we apply Lemma 2.4 to obtain a new Palais–Smale sequence $\{\hat{u}^2_\nu\}$. The process follows by induction and stops, by virtue of Lemma 2.3, once we obtain a Palais–Smale sequence $\{\hat{u}^m_\nu\}$ with $I_g(\hat{u}^m_\nu)$ converging to some $\beta < \beta^*$.

We are now left with the proof of (1-7) and the fact that the U^j obtained by the process above are of the form (1-5). To that end, we can follow the proof of Lemma 3.3 in [Druet et al. 2004], with some simple changes, to obtain the relation (1-7) and to prove that the U^j are nonnegative. For the reader's convenience this is outlined below.

Claim. *The functions u^0 and U^j obtained above are nonnegative. Moreover, the identity* (1-7) *holds.*

Proof. That u^0 is nonnegative is straightforward. To prove that the U^j are also nonnegative, set $\hat{u}_\nu = u_\nu - u^0$ and $\mu^j_\nu = 1/R^j_\nu$.

Given integers $N \in [1, m]$ and $s \in [0, N-1]$, we will prove that there exist an integer p and sequences $\{\tilde{x}^k_\nu\}_{\nu\in\mathbb{N}} \subset \partial M$ and $\{\lambda^k_\nu > 0\}_{\nu\in\mathbb{N}}$ for each $k = 1, \dots, p$, such that $d_g(x^N_\nu, \tilde{x}^k_\nu)/\mu^N_\nu$ is bounded and $\lim_{\nu\to\infty} \lambda^k_\nu/\mu^N_\nu = 0$, and such that

$$\int_{\Omega^N_\nu(R)\setminus\bigcup_{k=1}^p \widetilde{\Omega}^k_\nu(R')} \left|\hat{u}_\nu - \sum_{j=1}^{s} u^j_\nu - u^N_\nu\right|^{\frac{2n}{n-2}} dv_g = o(1) + \epsilon(R') \tag{2-20}$$

for any $R, R' > 0$. Here, $\Omega^N_\nu(R) = \psi_{x^N_\nu}(B^+_{R\mu^N_\nu}(0))$, $\widetilde{\Omega}^k_\nu(R') = \psi_{\tilde{x}^k_\nu}(B^+_{R'\lambda^k_\nu}(0))$ and $\epsilon(R') \to 0$ as $R' \to \infty$.

We prove (2-20) by reverse induction on s. It follows from Claim 2 in the proof of Lemma 2.4 that

$$\int_{\Omega^N_\nu(R)} \left|\hat{u}_\nu - \sum_{j=1}^{N-1} u^j_\nu - u^N_\nu\right|^{\frac{2n}{n-2}} dv_g = o(1),$$

so that (2-20) holds for $s = N-1$.

Assuming (2-20) holds for some $s \in [1, N-1]$, let us prove it does for $s-1$.

We first consider the case when $d_g(x^s_\nu, x^N_\nu)$ does not converge to zero as $\nu \to \infty$. In this case, we can assume $\Omega^N_\nu(R) \cap \Omega^s_\nu(\widetilde{R}) = \varnothing$ for any $\widetilde{R} > 0$. Then after rescaling we have

$$\int_{\Omega^N_\nu(R)\setminus\bigcup_{k=1}^p \widetilde{\Omega}^k_\nu(R')} |u^s_\nu|^{\frac{2n}{n-2}} dv_g \le C \int_{\mathbb{R}^n_+\setminus B^+_{\widetilde{R}}(0)} |U^s|^{\frac{2n}{n-2}} dy. \tag{2-21}$$

Since $\widetilde{R} > 0$ is arbitrary and $U^s \in L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)$, the left side of (2-21) converges to zero as $\nu \to \infty$. Hence, (2-20) still holds replacing s by $s-1$.

Now consider the case when $d_g(x_\nu^s, x_\nu^N) \to 0$ as $\nu \to \infty$. According to Claim 2 in the proof of Lemma 2.4, given $\widetilde{R} > 0$, we have

$$\int_{\Omega_\nu^s(\widetilde{R})} \left| \hat{u}_\nu - \sum_{j=1}^{s} u_\nu^j \right|^{\frac{2n}{n-2}} dv_g = o(1).$$

Using the induction hypothesis (2-20), we then conclude that

$$\int_{(\Omega_\nu^N(R)\setminus\bigcup_{k=1}^{p}\widetilde{\Omega}_\nu^k(R'))\cap\Omega_\nu^s(\widetilde{R})} |u_\nu^N|^{\frac{2n}{n-2}} dv_g = o(1) + \epsilon(R').$$

First assume that $d_g(x_\nu^s, x_\nu^N)/\mu_\nu^N \to \infty$. Rescaling by μ_ν^N and using coordinates centered at x_ν^N, it's not difficult to see that $d_g(x_\nu^s, x_\nu^N)/\mu_\nu^s \to \infty$. Hence we can assume that $\Omega_\nu^N(R) \cap \Omega_\nu^s(\widetilde{R}) = \varnothing$ for any $\widetilde{R} > 0$, and we proceed as in the case when $d_g(x_\nu^s, x_\nu^N)$ does not converge to 0 to conclude that (2-20) holds for $s-1$.

If $d_g(x_\nu^s, x_\nu^N)/\mu_\nu^N$ does not go to infinity, we can assume that it converges. One can then check that $\mu_\nu^s/\mu_\nu^N \to 0$. We set $\tilde{x}_\nu^{p+1} = x_\nu^s$ and $\lambda_\nu^{p+1} = \mu_\nu^s$, so that $\lambda_\nu^{p+1}/\mu_\nu^N \to 0$ as $\nu \to \infty$. Observing that

$$\int_{\Omega_\nu^N(R)\setminus\bigcup_{k=1}^{p+1}\widetilde{\Omega}_\nu^k(R')} |u_\nu^s|^{\frac{2n}{n-2}} dv_g \le \int_{M\setminus\Omega_\nu^s(R')} |u_\nu^s|^{\frac{2n}{n-2}} dv_g \le \epsilon(R'),$$

it follows that (2-20) holds when we replace p by $p+1$ and s by $s-1$.

This proves (2-20). The above also proves (1-7).

We fix an integer $N \in [1, m]$ and $s = 0$. Let $\tilde{y}_\nu^k \in \partial\mathbb{R}_+^n$ be such that $\tilde{x}_\nu^k = \psi_{x_\nu^N}^N(\mu_\nu^N \tilde{y}_\nu^k)$, for $k = 1, \dots, p$. For each k, the sequence $\{\tilde{y}_\nu^k\}_{\nu\in\mathbb{N}}$ is bounded, so there exists $\tilde{y}^k \in \partial\mathbb{R}_+^n$ such that $\lim_{\nu\to\infty} \tilde{y}_\nu^k = \tilde{y}^k$, possibly after taking a subsequence. We set $\tilde{X} = \bigcup_{k=1}^{p} \tilde{y}^k$ and

$$\tilde{u}_\nu^N(y) = (\mu_\nu^N)^{\frac{n-2}{2}} \hat{u}_\nu^N(\psi_{x_\nu^N}(\mu_\nu^N y))\,.$$

It follows from (2-20) that

$$\tilde{u}_\nu^N \to U^N \quad \text{in } L_{\text{loc}}^{\frac{2n}{n-2}}(B_R^+(0)\setminus\widetilde{X}) \quad \text{as } \nu \to \infty.$$

Therefore we can assume that $\tilde{u}_\nu \to U^N$ a.e. in $\mathbb{R}_+^n$ as $\nu \to \infty$.

If we set

$$\tilde{u}_\nu^{0,N}(y) = (\mu_\nu^N)^{\frac{n-2}{2}} u^0(\psi_{x_\nu^N}(\mu_\nu^N y)),$$

it's easy to prove that

$$\tilde{u}_\nu^{0,N} \to 0 \quad \text{in } L_{\text{loc}}^{\frac{2n}{n-2}}(B_R^+(0)) \quad \text{as } \nu \to \infty.$$

Hence, $\tilde{u}_{\nu}^{0,N} \to 0$ a.e. in $\mathbb{R}^n_+$ as $\nu \to \infty$. Setting

$$v_{\nu}^{N}(y) = (\mu_{\nu}^{N})^{\frac{n-2}{2}} u_{\nu}^{N}(\psi_{x_{\nu}^{N}}(\mu_{\nu}^{N} y)),$$

we see that $v_{\nu}^{N} \to U^{N}$ a.e. in $\mathbb{R}^n_+$ as $\nu \to \infty$. In particular, U^N is nonnegative. This proves the claim. □

Remark 2.5. For the regularity of the U^j we can use [Cherrier 1984, théorème 1]. Although that theorem was established for compact manifolds, we can use the conformal equivalence between $\mathbb{R}^n_+$ and $B^n \backslash \{\text{point}\}$ and a removable singularities theorem (see Lemma 2.7 on p. 1821 of [Almaraz 2011]) to apply it in B^n.

Thus we are able to use the result in [Li and Zhu 1995] to conclude that the U^j are of the form (1-5), so we can write $U^j = U_{\epsilon_j, a_j}$.

This finishes the proof of Theorem 1.3. □

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THE CUP SUBALGEBRA OF A II_1 FACTOR GIVEN BY A SUBFACTOR PLANAR ALGEBRA IS MAXIMAL AMENABLE

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To every subfactor planar algebra was associated a II_1 factor with a canonical abelian subalgebra generated by the cup tangle. Using Popa's approximative orthogonality property, we show that this cup subalgebra is maximal amenable.

Introduction

The study of maximal abelian subalgebras (MASAs) was initiated by Dixmier [1954], who introduced an invariant coming from the normalizer. Other invariants were provided later, such as the Takesaki equivalence relation [1963], the Tauer length [1965], the Pukánszky invariant [1960] or the δ-invariant [Popa 1983b].

Popa [1983a] exhibited an example of a MASA $A \subset M$ in a II_1 factor that is maximal amenable.

This example answers negatively a question of Kadison asking if every abelian subalgebra of a II_1 factor (with separable predual) is included in a copy of the hyperfinite II_1 factor. We recall that a von Neumann algebra is hyperfinite if and only if it is amenable by the famous theorem of Connes [1976]. Popa introduced the notion of approximative orthogonality property (AOP) and showed that any singular MASA with the AOP is maximal amenable. Then he proved that the generator MASA in a free group factor is singular and has the AOP.

Using the same scheme of proof, Cameron et al. [2010] showed that the radial MASA in the free group factor is maximal amenable. Shen [2006], Jolissaint [2010] and Houdayer [2012] provided other examples of maximal amenable MASAs.

In this paper, we provide maximal amenable MASAs in II_1 factors using subfactor planar algebras. The theory of subfactors has been initiated by Jones [1983]. He introduced the standard invariant that has been formalized as a Popa system by Popa [1995] and as a subfactor planar algebra by Jones [1999; 2011]. Popa [1993; 1995; 2002] proved that any standard invariant comes from a subfactor. Popa and Shlyakhtenko [2003] proved that the subfactor can be realized in the infinite

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free group factor $L(\mathbb{F}_\infty)$. Using planar algebras, random matrix models and free probability, Guionnet et al. [2010; 2011] (see also [Jones et al. 2010]) showed that any finite depth standard invariant can be realized as a subfactor of an interpolated free group factor. Using the same construction, Hartglass [2013] proved that any infinite depth subfactor is realized in $L(\mathbb{F}_\infty)$.

The construction in [Jones et al. 2010] associated a II_1 factor M to a subfactor planar algebra $\mathcal{P}$. This II_1 factor contains a generic MASA $A \subset M$ that we call the *cup subalgebra* (see page 22). We now state our main theorem:

Theorem 0.1. *For any nontrivial subfactor planar algebra $\mathcal{P}$, the cup subalgebra is maximal amenable.*

The construction of Jones et al. has been extended for unshaded planar algebras in [Brothier 2012; Brothier et al. 2012]. In those constructions, we have proven that the cup subalgebra is still a MASA. It seems very plausible that it is also maximal amenable. Note that the cup subalgebra is analogous to the *radial MASA* in a free group factor. We don't know if for a certain subfactor planar algebra those two subalgebras are isomorphic or not.

1. Approximative orthogonality property and maximal amenability

We briefly recall Popa's approximative orthogonality property for an abelian subalgebra $A \subset M$ and how it implies the maximal amenability of A, whenever $A \subset M$ is a singular MASA.

Definition 1.1 [Popa 1983a, Lemma 2.1]. Consider a tracial von Neumann algebra (M, tr) and a subalgebra $A \subset M$. Let ω be a free ultrafilter on $\mathbb{N}$. Then $A \subset M$ has the approximative orthogonality property if for any $x \in M^\omega \ominus A^\omega \cap A'$ and any $b \in M \ominus A$ we have $xb \perp bx$ in $L^2(M^\omega)$, that is, $\lim_{n\to\omega} \mathrm{tr}(x_n b x_n^* b^*) = 0$, where $(x_n)_n$ is a representative of x.

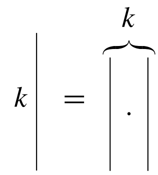
Remark 1.2. By polarization, the definition of AOP is equivalent to asking that for any $x_1, x_2 \in M^\omega \ominus A^\omega \cap A'$ and any $b_1, b_2 \in M \ominus A$ we have $x_1 b_1 \perp b_2 x_2$.

We recall the fundamental theorem of Popa that is contained in the proof of [Popa 1983a, Theorem 3.2]. A more detailed explanation of it has been given in [Cameron et al. 2010, Lemma 2.2 and Corollary 2.3].

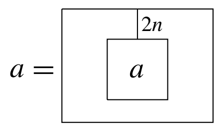
Theorem 1.3 [Popa 1983a]. *Let $A \subset M$ be a singular MASA with the AOP in a* II_1 *factor M. Then $A \subset M$ is maximal amenable.*

2. Construction of the cup subalgebra

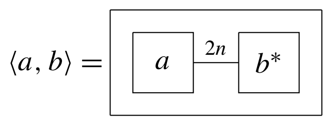
Construction of a* $\mathbf{II_1}$ *factor from a subfactor planar algebra. Consider a subfactor planar algebra $\mathcal{P} = (\mathcal{P}_n)_{n\geqslant 0}$ with modulus $\delta > 1$. Let us recall the construction given in [Jones et al. 2010]. We assume that the reader is familiar with planar algebras. For more details on planar algebras, see [Jones 1999; 2011] or the introduction of [Peters 2010]. Let $\mathrm{Gr}(\mathcal{P})$ be the graded vector space equal to the algebraic direct sum $\bigoplus_{n\geqslant 0}\mathcal{P}_n$. We decorate strands in a planar tangle with natural numbers to represent cabling of that strand. For example:



An element $a\in\mathcal{P}_n$ will be represented as a box:

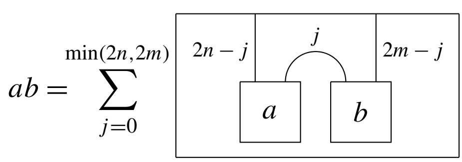


We assume that the distinguished first interval is at the top left of the box. We consider the inner product $\langle\cdot,\cdot\rangle$ on each $\mathcal{P}_n$:



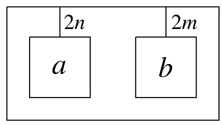
for all $a, b\in\mathcal{P}_n$.

We extend this inner product on $\mathrm{Gr}(\mathcal{P})$ in such a way that the spaces $\mathcal{P}_n$ are pairwise orthogonal. We still write $\mathcal{P}_n$ when it is considered as the n-graded part of $\mathrm{Gr}(\mathcal{P})$. Let $\mathcal{H}$ be the Hilbert space equal to the completion of $\mathrm{Gr}(\mathcal{P})$ for its pre-Hilbert structure. Note that $\mathcal{H}$ is the Hilbert space equal to the orthogonal direct sum of the spaces $\mathcal{P}_n$. We define a multiplication on $\mathrm{Gr}(\mathcal{P})$ given by the tangle



for all $a\in\mathcal{P}_n$, $b\in\mathcal{P}_m$.

For a fixed $a\in\mathrm{Gr}(\mathcal{P})$, the map $b\in\mathrm{Gr}(\mathcal{P})\mapsto ab\in\mathrm{Gr}(\mathcal{P})$ is bounded for the inner product $\langle\cdot,\cdot\rangle$. This gives us a representation of the $*$-algebra $\mathrm{Gr}(\mathcal{P})$ on $\mathcal{H}$. We denote by M the von Neumann algebra equal to the bicommutant of this representation. It is a II_1 factor by [Jones et al. 2010]. We identify the graded algebra $\mathrm{Gr}(\mathcal{P})$ and its image in the von Neumann algebra M. The unique faithful normal trace tr of M is the one coming from the planar algebra structure of $\mathcal{P}$. It is equal to the formula $\mathrm{tr}(a)=\langle a,1\rangle$, where 1 is the unity of $\mathrm{Gr}(\mathcal{P})$. Let $L^2(M)$ be the Hilbert space coming from the Gelfand–Naimark–Segal construction over the trace tr. Note that the standard representation of the von Neumann algebra M on the Hilbert space $L^2(M)$ is conjugate to the action of M on the Hilbert space $\mathcal{H}$. We will identify those two representations. Also, we identify M with its image in $L^2(M)$. The left and right actions of M on the Hilbert space $L^2(M)$ are denoted by π and ρ, so $\pi(x)\rho(y)z = xzy$, for $x, y, z \in M$. The norm of M is denoted by $\|\cdot\|$ and that of $L^2(M)$ by $\|\cdot\|_2$, or by $\|\cdot\|$ if the context is clear. We define a multiplication on $\mathrm{Gr}(\mathcal{P})$ by requiring that if $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$, then $a \bullet b \in \mathcal{P}_{n+m}$ is given by

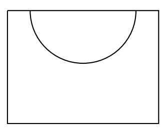
$a \bullet b =$ 

We remark that $\|a \bullet b\|_2 = \|a\|_2\|b\|_2$, for all $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$. By the triangle inequality, the bilinear function

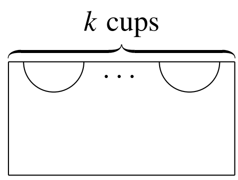
$$\mathrm{Gr}(\mathcal{P}) \times \mathrm{Gr}(\mathcal{P}) \to \mathrm{Gr}(\mathcal{P}), \quad (a, b) \mapsto a \bullet b,$$

is continuous for the norm $\|\cdot\|_2$. We extend this operation to $L^2(M) \times L^2(M)$ and still denote it by $\bullet$.

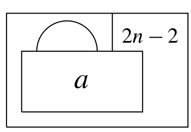
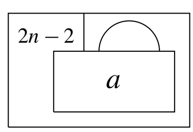
The cup subalgebra. The cup subalgebra $A \subset M$ is the abelian von Neumann algebra generated by the self-adjoint element cup:



We denote cup by the symbol $\cup$. Also we use the following notation:

$\cup^{\bullet k} =$ 

We use the convention that $0 = \cup^{\bullet k}$ for $k < 0$ and $1 = \cup^{\bullet 0}$. Let $n \geqslant 1$ and V_n be the subspace of $\mathcal{P}_n$ of elements which vanish when a cap is placed at the top right and vanish when a cap is placed at the top left, i.e.,

$V_n = \Big\{ a \in \mathcal{P}_n,$  $=$  $= 0 \Big\}.$

We denote by V the orthogonal direct sum of the V_n:

$$V = \bigoplus_{n=1}^{\infty} V_n.$$

Let $\ell^2(\mathbb{N})$ be the separable Hilbert space with orthonormal basis $\{e_n,\ n \geqslant 0\}$ and $S \in \mathbb{B}(\ell^2(\mathbb{N}))$ the unilateral shift operator.

Proposition 2.1 [Jones et al. 2010, Theorem 4.9]. *The map*

$$\Theta : L^2(M) \to \ell^2(\mathbb{N}) \oplus (\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N}))$$

defined by

$$\delta^{-k/2} \cup^{\bullet k} \mapsto e_k \oplus 0, \quad \delta^{-(l+r)/2} \cup^{\bullet l} \bullet v \bullet \cup^{\bullet r} \mapsto 0 \oplus e_l \otimes v \otimes e_r,$$

defines a unitary transformation, where $k, l, r \geqslant 0$, $v \in V$ and δ is the modulus of the planar algebra. We have

$$\Theta\pi\left(\frac{\cup - 1}{\delta^{1/2}}\right)\Theta^* = \begin{pmatrix} S + S^* - q_{e_0} & 0 \\ 0 & (S + S^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} \end{pmatrix}$$

and

$$\Theta\rho\left(\frac{\cup - 1}{\delta^{1/2}}\right)\Theta^* = \begin{pmatrix} S + S^* - q_{e_0} & 0 \\ 0 & 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (S + S^*) \end{pmatrix},$$

where q_{e_0} is the rank-one projection on $\mathbb{C}e_0$ and 1_V, $1_{\ell^2(\mathbb{N})}$ are the identity operators of the Hilbert spaces V and $\ell^2(\mathbb{N})$.

Corollary 2.2. *The cup subalgebra is a singular MASA.*

Proof. The A-bimodule $L^2(M) \ominus L^2(A)$ is isomorphic to an infinite direct sum of the coarse bimodule $L^2(A) \otimes L^2(A)$. This implies that $A \subset M$ is maximal abelian. See [Jones et al. 2010] for more details. Suppose that there exists a unitary u in the normalizer of A inside M which is orthogonal to A. It generates a A-subbimodule

$$\mathcal{K} \subset \bigoplus_{j=0}^{\infty} L^2(A) \otimes L^2(A). \tag{1}$$

We have the inclusion (1) if and only if the automorphism $a \in A \mapsto uau^*$ is trivial. This implies that $u \in A' \cap M$. Hence $u \in A$, a contradiction. Therefore, $A \subset M$ is singular. $\square$

Basic facts on the unilateral shift operator. Consider the semicircular measure

$$d\nu(t) = \frac{\sqrt{4 - t^2}}{2\pi}\, dt$$

defined on the interval $[-2; 2]$. Let $P_i \in \mathbb{R}[X]$ be the family of polynomials such that

$$P_0(X) = 1, \quad P_1(X) = X, \quad P_i(X) = XP_{i-1}(X) - P_{i-2}(X) \quad \text{for } i \geqslant 2. \tag{2}$$

By [Voiculescu et al. 1992, Example 3.4.2], the map

$$\Psi : \ell^2(\mathbb{N}) \to L^2([-2;2],\nu), \quad e_i \mapsto P_i, \tag{3}$$

defines a unitary transformation. Further, for any continuous function $f \in \mathscr{C}([-2;2])$ we have $(\Psi^* f(S+S^*)\Psi)(t) = tf(t)$ for almost every $t \in [-2;2]$.

Lemma 2.3. *For $I \geqslant 0$, let $R_I : [-2;2] \to \mathbb{R}$ be given by $R_I(t) = \sum_{i=0}^{I} P_i(t)^2$. The sequence $(R_I)_{I\geqslant 0}$ converges uniformly to $+\infty$.*

Proof. Let us prove the simple convergence to $+\infty$. Suppose there exists $t_0 \in [-2;2]$ such that the sequence $(R_I(t_0))_k$ does not converge to $+\infty$. The polynomials P_i have real coefficient. Hence, for any $t \in [-2;2]$, $P_i(t)$ is real; thus, $(R_I(t_0))_k$ is an increasing sequence in $\mathbb{R}$. If this sequence does not diverge, then it is bounded. Then, the sequence $(P_i(t_0))_i$ is square summable. In particular we have $\lim_{i\to\infty} P_i(t_0) = 0$. We put $\varepsilon_i = P_i(t_0)$. We have that $\varepsilon_{i+1} = t_0\varepsilon_i - \varepsilon_{i-1}$ and $\lim_{i\to\infty} \varepsilon_i = 0$. There is only one sequence that satisfies those axioms and it is the sequence equal to zero. Since $0 \neq 1 = P_0(t_0) = \varepsilon_0$, we arrive at a contradiction and thus, $\lim_{I\to\infty} S_I(t) = +\infty$ for any $t \in [-2;2]$. To conclude we use the following well known result due to Dini: Let $(f_I)_I$ be a sequence of continuous functions from a compact topological space K to $\mathbb{R}$ such that $f_I \leqslant f_{I+1}$. If for any $t \in K$, $\lim_{I\to\infty} f_I(t) = +\infty$, then the sequence $(f_I)_I$ converges uniformly to $+\infty$. □

Proof of Theorem 0.1. According to Theorem 1.3 and Corollary 2.2, it is sufficient to show that the cup subalgebra has the AOP. Fix $x \in M^\omega \ominus A^\omega \cap A'$ and $b \in M \ominus A$. Let us show that $xb \perp bx$. By the Kaplansky density theorem we can assume that there exists $J \geqslant 1$ such that $b \in \bigoplus_{j=0}^{J} \mathscr{P}_j$. Suppose that $\|x\| \leqslant 1$ and fix a sequence $x_n \in M$ which is a representative of x such that $x_n \in M \ominus A$ and $\|x_n\| \leqslant 1$ for all $n \geqslant 0$.

Consider the closed subspaces of $L^2(M)$ given by

$$\begin{aligned} Y_L &= \overline{\operatorname{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r},\ l, r \leqslant L,\ v \in V\},\\ Z_L &= \overline{\operatorname{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r},\ l \text{ or } r \leqslant L,\ v \in V\}, \end{aligned}$$

for all $L \geqslant 0$. Note that b is in Y_{J-1}.

We claim that for any $z \in M$ which is orthogonal to A and Z_{J-1} we have

$$zb \perp bz. \tag{4}$$

The element z is a weak limit of finite linear combinations of $\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j}$, where $i, j \geqslant J$ and $v \in V$. The element b is a finite linear combination of $\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r}$, where $k, r \leqslant J-1$ and $\tilde{v} \in V$. We have

$$\begin{aligned}(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j})(\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r}) \\ = (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j+k} \bullet \tilde{v} \bullet \cup^{\bullet r}) + (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j+k-1} \bullet \tilde{v} \bullet \cup^{\bullet r}) + \cdots \\ + \delta^k(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j-k} \bullet \tilde{v} \bullet \cup^{\bullet r}) + \delta^k(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j-k-1} \bullet \tilde{v} \bullet \cup^{\bullet r}),\end{aligned}$$

for any $i, j \geqslant J$ and $k, r \leqslant J-1$. It is easy to see that $v \bullet \cup^{\bullet n} \bullet \tilde{v}$ is an element of V for any n. Hence, the product $(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j})(\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r})$ is in the vector space

$$\overline{\operatorname{span}}\{\cup^{\bullet l} \bullet w \bullet \cup^{\bullet r}, l \geqslant J, w \in V, r \leqslant J-1\}$$

and so is zb. A similar computation shows that bz is in the closed vector space

$$\overline{\operatorname{span}}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet r}, l \leqslant J-1, w \in V, r \geqslant J\}.$$

Therefore, we have $zb \perp bz$. This proves (4). Hence, if we show that x is in the orthogonal of Z^{ω}_{J-1} then we would have proven that xb is orthogonal to bx. Consider $Q_J : L^2(M) \to Z_{J-1}$, the orthogonal projection of range Z_{J-1}. We remark that

$$\Theta Q_J \Theta^* = \bigoplus_{j=0}^{J-1} \big((q_{e_j} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})}) \oplus (1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes q_{e_j})\big),$$

where Θ is the unitary transformation defined in Proposition 2.1 and 1_V, $1_{\ell^2(\mathbb{N})}$ are the identity operators of V and $\ell^2(\mathbb{N})$. By symmetry, it is sufficient to show that

$$\lim_{n\to\omega} \|(q_{e_j} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})})\xi_n\| = 0 \quad \text{for any } j \geqslant 0, \tag{5}$$

where $\xi_n := \Theta(x_n)$. We know that $x \in M^{\omega} \cap A'$. Hence by conjugation by Θ we obtain the equation

$$\lim_{n\to\omega} \|((S+S^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} - 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (S+S^*))\xi_n\| = 0. \tag{6}$$

We will show that (6) implies (5).

All the operators involved in our context act trivially on the factor V. For simplicity of the notations we stop writing the extra "$\otimes 1_V \otimes$" in the formula and denote the identity operator $1_{\ell^2(\mathbb{N})}$ by 1. Therefore, we assume that ξ_n is a vector of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. Equations (5) and (6) become

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1)\xi_n\| = 0 \quad \text{for any } i \geqslant 0 \tag{7}$$

and

$$\lim_{n\to\omega} \|((S+S^*) \otimes 1 - 1 \otimes (S+S^*))\xi_n\| = 0. \tag{8}$$

Consider the partial isometry $v_i \in \mathbb{B}(\ell^2(\mathbb{N}))$ such that $v_i^* v_i = q_{e_i}$ and $v_i v_i^* = q_{e_0}$. We claim that for all $i \geqslant 0$ we have

$$\lim_{n\to\omega} \|((v_i \otimes 1) - (q_{e_0} \otimes P_i(S+S^*)))\xi_n\| = 0, \tag{9}$$

where $\{P_i\}_i$ is the family of polynomials defined in (2). For all $k \geqslant 2$ we have

$$\begin{aligned}(S+S^*)^k &\otimes 1 - 1 \otimes (S+S^*)^k \\ &= ((S+S^*) \otimes 1 - 1 \otimes (S+S^*)) \circ \left(\sum_{j=0}^{k-1} (S+S^*)^j \otimes (S+S^*)^{k-1-j}\right).\end{aligned}$$

Therefore, (8) implies that

$$\lim_{n\to\omega} \|(P(S+S^*) \otimes 1 - 1 \otimes P(S+S^*))\xi_n\| = 0 \quad \text{for all polynomials } P.$$

In particular,

$$\lim_{n\to\omega} \|(P_i(S+S^*) \otimes 1 - 1 \otimes P_i(S+S^*))\xi_n\| = 0 \quad \text{for all } i \geqslant 0.$$

Note that $P_i(S+S^*)(e_0) = e_i$ for all $i \geqslant 0$. Furthermore, P_i has real coefficients. Therefore, the operator $P_i(S+S^*)$ is self-adjoint. We have

$$\begin{aligned}\langle q_{e_0} \circ P_i(S+S^*)e_l, e_r\rangle &= \langle P_i(S+S^*)e_l, q_{e_0}e_r\rangle = \delta_{r,0}\langle P_i(S+S^*)e_l, e_0\rangle \\ &= \delta_{r,0}\langle e_l, P_i(S+S^*)e_0\rangle = \delta_{r,0}\delta_{l,i},\end{aligned}$$

where $i, l, r \geqslant 0$ and $\delta_{n,m}$ is the Kronecker symbol. Hence $q_{e_0} \circ P_i(S+S^*) = v_i$, for all $i \geqslant 0$. We have

$$\lim_{n\to\omega} \|(q_{e_0} \otimes 1) \circ (P_i(S+S^*) \otimes 1 - 1 \otimes P_i(S+S^*))\xi_n\| = 0.$$

Therefore, we have

$$\lim_{n\to\omega} \|(v_i \otimes 1 - q_{e_0} \otimes P_i(S+S^*))\xi_n\| = 0.$$

This proves the claim. We have

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1 - v_i^* q_{e_0} \otimes P_i(S+S^*))\xi_n\| = 0.$$

This means that

$$\lim_{n\to\omega} \|(q_{e_i} \otimes 1)\xi_n - (v_i^* \otimes P_i(S+S^*)) \circ (q_{e_0} \otimes 1)\xi_n\| = 0.$$

Hence, we have

$$\begin{aligned}\lim_{n\to\omega} \|(q_{e_i} \otimes 1)\xi_n\| &\leqslant \lim_{n\to\omega} \|(v_i^* \otimes P_i(S+S^*)) \circ (q_{e_0} \otimes 1)\xi_n\| \\ &\leqslant \|v_i^* \otimes P_i(S+S^*)\| \lim_{n\to\omega} \|(q_{e_0} \otimes 1)\xi_n\|.\end{aligned}$$

Therefore, to prove (7) it is sufficient to show that

$$\lim_{n\to\omega} \|(q_{e_0}\otimes 1)\xi_n\| = 0.$$

Let us fix $\varepsilon > 0$; we have to find an element of the ultrafilter $E \in \omega$ such that $\|(q_{e_0}\otimes 1)\xi_n\| < \varepsilon$ for any $n \in E$. By the triangle inequality, we have

$$\|(q_{e_0}\otimes P_i(S+S^*))\xi_n\| \leqslant \|(q_{e_0}\otimes P_i(S+S^*))\xi_n - (v_i\otimes 1)\xi_n\| + \|(v_i\otimes 1)\xi_n\|,$$

for all $i \geqslant 0$. We have $\|(v_i\otimes 1)\xi_n\| \leqslant \|\xi_n\| \leqslant 1$; thus,

$$\begin{aligned}\|(v_i\otimes 1)\xi_n\|^2 \geqslant \|(q_{e_0}\otimes P_i(S+S^*))\xi_n\|^2 - \|(q_{e_0}\otimes P_i(S+S^*))\xi_n-(v_i\otimes 1)\xi_n\|^2 \\ -2\|(q_{e_0}\otimes P_i(S+S^*))\xi_n-(v_i\otimes 1)\xi_n\|.\end{aligned} \tag{10}$$

By Lemma 2.3, there exists an integer $I \in \mathbb{N}$ such that $\inf_{t\in[-2;2]} S_I(t) > \frac{2}{\varepsilon}$. We have

$$\begin{aligned}\sum_{i=0}^{I} \|(q_{e_0}\otimes P_i(S+S^*))\xi_n\|^2 &= \sum_{i=0}^{I} \|(1\otimes P_i(S+S^*))\circ(q_{e_0}\otimes 1)\xi_n\|^2 \\ &= \sum_{i=0}^{I}\int_{[-2;2]} \|P_i(t)((q_{e_0}\otimes\Psi)\xi_n)(t)\|^2\, d\nu(t) \\ &= \int_{[-2;2]} \|((q_{e_0}\otimes\Psi)\xi_n)(t)\|^2 \sum_{i=0}^{I} P_i(t)^2\, d\nu(t) \\ &\geqslant \frac{2}{\varepsilon}\|(q_{e_0}\otimes\Psi)\xi_n\|^2 = \frac{2}{\varepsilon}\|(q_{e_0}\otimes 1)\xi_n\|^2,\end{aligned} \tag{11}$$

where Ψ is the unitary transformation defined in (3).

By (9), there exists an element of the ultrafilter $E \in \omega$ such that for any $n \in E$ and $i \in \{0, \dots, I\}$ we have

$$\|((q_{e_0}\otimes P_i(S+S^*)) - (v_i\otimes 1))\xi_n\| < \tfrac{1}{4}. \tag{12}$$

By Pythagoras' theorem and the inequalities (10), (11) and (12) we have

$$\begin{aligned}1 &\geqslant \|\xi_n\|^2 = \sum_{i\geqslant 0}\|(q_{e_i}\otimes 1)\xi_n\|^2 \geqslant \sum_{i=0}^{I}\|(q_{e_i}\otimes 1)\xi_n\|^2 = \sum_{i=0}^{I}\|(v_i\otimes 1)\xi_n\|^2 \\ &\geqslant \sum_{i=0}^{I}\|(q_{e_0}\otimes P_i(S+S^*))\xi_n\|^2 - (I+1)\Big(\frac{1}{4^2}+2\cdot\frac{1}{4}\Big) \\ &\geqslant \frac{2(I+1)}{\varepsilon}\|(q_{e_0}\otimes 1)\xi_n\| - (I+1).\end{aligned}$$

This implies

$$\|(q_{e_0}\otimes 1)\xi_n\| \leqslant \varepsilon \quad \text{for all } n \in E.$$

We have proved that

$$\lim_{n\to\omega} \|(q_{e_0} \otimes 1)\xi_n\|_2 = 0.$$

Therefore, $\lim_{n\to\omega} \|Q_J(x_n)\| = 0$ which implies that x is orthogonal to Z^{ω}_{J-1}. The equality (4) implies that $xb \perp bx$. Thus, the cup subalgebra $A \subset M$ has the AOP. By Corollary 2.2, $A \subset M$ is a singular MASA. Hence, by Theorem 1.3, the cup subalgebra is maximal amenable.

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REPRESENTATION THEORY OF TYPE B AND C STANDARD LEVI W-ALGEBRAS

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We classify the finite-dimensional irreducible representations with integral central character of finite W-algebras $U(\mathfrak{g}, e)$ associated to standard Levi nilpotent orbits in classical Lie algebras of types B and C. This classification is given explicitly in terms of the highest weight theory for finite W-algebras.

1. Introduction

Let e be a nilpotent element in the Lie algebra $\mathfrak{g}$ of a reductive algebraic group G over $\mathbb{C}$. The finite W-algebra $U(\mathfrak{g}, e)$ associated to the pair $(\mathfrak{g}, e)$ is an associative algebra obtained from $U(\mathfrak{g})$ by a certain quantum Hamiltonian reduction. There has been a great deal of recent interest in finite W-algebras and their representation theory; for an overview, see the survey article [Losev 2011b].

In [Brown and Goodwin 2013a; 2013b], we gave a combinatorial classification of the finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules, where $\mathfrak{g}$ is a classical Lie algebra and e is an even-multiplicity nilpotent element; we recall that e is said to be *even multiplicity* if all parts of the Jordan type of e occur with even multiplicity. This classification is given in terms of the highest weight theory for finite W-algebras from [Brundan et al. 2008].

Now recall that a nilpotent element e of $\mathfrak{g}$ is said to be of *standard Levi type* if e is in the regular nilpotent orbit of a Levi subalgebra of $\mathfrak{g}$. It is easy to check that if $\mathfrak{g}$ is of classical type and e is even multiplicity, then e is standard Levi. In this paper, we extend the results of [Brown and Goodwin 2013a] to classify the finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules with integral central character, where $\mathfrak{g}$ is of type B or C and e is any *standard Levi nilpotent* element; see Theorem 1.2. We plan to deal with the case of any (not necessarily integral) central characters in future work, where different methods will be required. We recall (see, for example, the footnote to [Premet 2007, Question 5.1]) that the centre of $U(\mathfrak{g}, e)$ is canonically identified with the centre of $U(\mathfrak{g})$, which allows one to define integral central characters.

The situation for $\mathfrak{g}$ of type D and e standard Levi, but not even-multiplicity, is more awkward. In this case the combinatorics become more complicated and the

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statement of the classification of finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules cannot be given simply in terms of a row-equivalent to column-strict condition as in Theorem 1.2.

We remark here that finite W-algebras corresponding to nilpotent elements of standard Levi type are a natural class to consider. This is because such finite W-algebras are particularly amenable to the highest weight theory from [Brundan et al. 2008], as explained in Section 2C.

Losev and Ostrik [2013] have achieved a classification of the finite-dimensional $U(\mathfrak{g}, e)$-modules of integral central character for any reductive Lie algebra $\mathfrak{g}$ in the following manner: Losev [2010] gave a surjection from the primitive ideals of finite codimension of $U(\mathfrak{g}, e)$ to the primitive ideals of $U(\mathfrak{g})$ having associated variety equal to the closure $\overline{G \cdot e}$ of the G-orbit of e. There is a natural action of the component group C of the centralizer of e in G on the set of primitive ideals of $U(\mathfrak{g}, e)$, as explained, for example, in the introduction to [Losev 2011a]. This last paper extends the results of [Losev 2010] to show that the fibres of the above surjection are precisely C-orbits. The classification in [Losev and Ostrik 2013] is accomplished by describing the fibres of this map, i.e., determining the stabiliser of the C-orbit for each fibre. The primitive ideals with associated variety equal to $\overline{G \cdot e}$ can be described thanks to the methods of a variety of mathematicians in the 1970s and 1980s; see for example [Jantzen 1983] and the references therein for details.

We go on to explain the results of this paper in more detail, so we take $\mathfrak{g}$ to be of type B or C; that is, $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{g} = \mathfrak{sp}_{2n}$ for some $n \in \mathbb{Z}_{\geq 2}$. We recall that nilpotent orbits in $\mathfrak{g}$ are parametrized by their Jordan type. Thus they are given by partitions of $2n+1$ (respectively $2n$) where all even (respectively odd) parts occur with even multiplicity when $\mathfrak{g} = \mathfrak{so}_{2n+1}$ (respectively $\mathfrak{g} = \mathfrak{sp}_{2n}$). In this paper we consider only nilpotent orbits which are standard Levi but not even-multiplicity, as the latter are dealt with in [Brown and Goodwin 2013a; 2013c]. This means that the Jordan type of e is given by a partition of the form

$$\boldsymbol{p} = (p_1^{2a_1} < p_2^{2a_2} < \cdots < p_{d-1}^{2a_{d-1}} < p_d^{2a_d+1} < p_{d+1}^{2a_{d+1}} < \cdots < p_r^{2a_r});$$

that is, all parts of $\boldsymbol{p}$ occur with even multiplicity except for one part p_d, which occurs with odd multiplicity. This description of partitions corresponding to standard Levi nilpotent orbits follows, for example, from the explicit description of Levi subgroups regular nilpotent elements given in [Jantzen 2004, §4.5, §6.3]. It will be more convenient for us to reindex this partition and write it as

$$\boldsymbol{p} = (p_1^2 \leq p_2^2 \leq \cdots \leq p_{d-1}^2 < p_0 \leq p_d^2 \leq \cdots \leq p_r^2).$$

In this paper, we only consider finite-dimensional irreducible representations for $U(\mathfrak{g}, e)$ with integral central character. As we explain in Section 2C, such representations occur only when e is a special nilpotent element in the sense of [Lusztig 1979]. In terms of the partition $\boldsymbol{p}$, this means that the dual partition of $\boldsymbol{p}$ is the Jordan type of a nilpotent orbit in $\mathfrak{g}$. Explicitly, this means that p_i must be odd for all $i \geq d$ when $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or p_i must be even for all $i \leq d$ when $\mathfrak{g} = \mathfrak{sp}_{2n}$. This can be deduced from the description of special symbols in [Lusztig 1979] and [Barbasch and Vogan 1982, Theorem 18]; see also [Collingwood and McGovern 1993, Proposition 6.3.7]. For the remainder of the paper we assume that $\boldsymbol{p}$ is a partition satisfying these conditions.

We use *symmetric pyramids* to describe much of the combinatorics underlying $U(\mathfrak{g}, e)$-modules. The symmetric pyramid for $\boldsymbol{p}$, denoted by P, is a finite connected collection of boxes in the plane such that

- the boxes are arranged in connected rows;
- the boxes are symmetric with respect to both the y-axis and the x-axis;
- each box is 2 units by 2 units;
- the lengths of the rows from top to bottom are given by

$$p_1 \dots, p_r, p_0, p_r, \dots, p_1.$$

An *s-table* with underlying symmetric pyramid P is a skew-symmetric (with respect to the origin) filling of P with complex numbers. We define $\mathrm{sTab}(P)$ to be a certain set of s-tables depending on whether $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$. For $\mathfrak{g} = \mathfrak{sp}_{2n}$ we let $\mathrm{sTab}(P)$ denote the set of s-tables with underlying symmetric pyramid P such that all entries are integers, whereas for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ we define $\mathrm{sTab}(P)$ to be the s-tables such that either all entries are in $\mathbb{Z}$ or all entries are in $\frac{1}{2}+\mathbb{Z}$. Let $\mathrm{sTab}^{\leq}(P)$ denote the elements of $\mathrm{sTab}(P)$ that have nondecreasing rows. As explained in Section 3C, the elements of $\mathrm{sTab}^{\leq}(P)$ parametrize the irreducible highest weight $U(\mathfrak{g}, e)$-modules; given $A \in \mathrm{sTab}(P)$, we write $L(A)$ for the corresponding irreducible highest weight $U(\mathfrak{g}, e)$-module.

An example of an s-table in $\mathrm{sTab}^{\leq}(P)$, when $\mathfrak{g} = \mathfrak{sp}_{2n}$, $\boldsymbol{p} = (2^2, 4, 5^2)$ and P is the symmetric pyramid for $\boldsymbol{p}$, is this:

$$\begin{array}{c} \boxed{6}\,\boxed{7} \\ \boxed{2}\,\boxed{4}\,\boxed{5}\,\boxed{8}\,\boxed{9} \\ \boxed{-3}\,\boxed{-1}\,\boxed{1}\,\boxed{3} \\ \boxed{-9}\,\boxed{-8}\,\boxed{-5}\,\boxed{-4}\,\boxed{-2} \\ \boxed{-7}\,\boxed{-6} \end{array} \tag{1.1}$$

The *left justification* of an s-table is the diagram created by left-justifying all of the s-table's rows. We say an s-table is *justified row-equivalent to column-strict* if the row equivalence class of its left justification contains a table in which every column is strictly decreasing; we note that there can be a gap in the middle of some columns, and we require entries to be strictly decreasing across this gap. We write $\mathrm{sTab}^{c}(P)$ for the set of all $A \in \mathrm{sTab}(P)$ that are justified row-equivalent to column-strict. It is easy to see that the example of the s-table above is an element of $\mathrm{sTab}^{c}(P)$.

Recall that C denotes the component group of the centralizer of e in G. In Section 7A, we define an action of C on the subset of $\mathrm{sTab}^{\leq}(P)$ corresponding to finite-dimensional $U(\mathfrak{g}, e)$-modules.

Now we can state the main theorem of this paper:

Theorem 1.2. *Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$, let $\boldsymbol{p}$ be a partition corresponding to a standard Levi special nilpotent orbit in $\mathfrak{g}$, let e be an element of this orbit and let P be the symmetric pyramid for $\boldsymbol{p}$. Then*

$$\{L(A) \mid A \in \mathrm{sTab}^{\leq}(P),\ A \text{ is } C\text{-conjugate to some } B \in \mathrm{sTab}^{c}(P)\}$$

is a complete set of isomorphism classes of finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules with integral central character. Moreover, the C-action on s-tables agrees with the C-action on finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules.

Analogous results to [Brown and Goodwin 2013a, Corollaries 5.17 and 5.18] hold in the present situation. So when all parts of $\boldsymbol{p}$ have the same parity, if $L(A)$ is finite-dimensional, then, in fact, A is row-equivalent to column-strict as an s-table. Thus in this case $L(A)$ can be obtained as a subquotient of the restriction of a finite-dimensional $U(\mathfrak{g}(0))$-module via the Miura map. We refer the reader to the discussion before Corollary 5.18 in that reference for more details, and to Section 2A below for the definition of $\mathfrak{g}(0)$.

Theorem 1.2 and the correspondence of finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules and primitive ideals of $U(\mathfrak{g})$ with associated variety $\overline{G \cdot e}$ discussed above allow us to deduce the following corollary. It gives an explicit description of the primitive ideals of $U(\mathfrak{g})$ having associated variety equal to $\overline{G \cdot e}$ and integral central character. A method to classify these primitive ideals was originally given in [Barbasch and Vogan 1982]. In the corollary, $L(\lambda_A)$ denotes the irreducible highest weight $U(\mathfrak{g})$-module defined from an s-table A as explained in Section 3C below.

Corollary 1.3. *The set of primitive ideals with integral central character and associated variety $\overline{G \cdot e}$ is equal to*

$$\{\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_A) \mid A \in \mathrm{sTab}^{c}(P) \cap \mathrm{sTab}^{\leq}(P)\}.$$

Below we give an outline of the proof of Theorem 1.2.

The key step is to deal with the case where $\boldsymbol{p}$ has three parts. We deal with this case using the relationship between finite-dimensional irreducible representations of $U(\mathfrak{g}, e)$ and primitive ideals of $U(\mathfrak{g})$ with associated variety equal to $\overline{G \cdot e}$. Using this and results of Barbasch and Vogan and Garfinkle on primitive ideals, we are able to classify finite-dimensional irreducible modules for $U(\mathfrak{g}, e)$ and explicitly describe the component group action. These results are stated in Theorems 5.4 and 6.17.

In Section 7, we use inductive methods to deduce Theorem 1.2. The important ingredients here are "Levi subalgebras" of $U(\mathfrak{g}, e)$ as defined in [Brown and Goodwin 2013a, §3] and changing highest weight theories. The latter is dealt with in [Brown and Goodwin 2013c] for the case of an even-multiplicity nilpotent orbit, and we observe here that there is an analogous theory in the present situation; see Proposition 4.6.

We note that if we were able to deal with the case where $\boldsymbol{p}$ has three parts by other means, for example from an explicit presentation of the finite W-algebras, then we would be able to remove the dependence on the results of Losev, Barbasch and Vogan, and Garfinkle. It would, therefore, be interesting and useful to have a presentation of such finite W-algebras.

2. Overview of finite W-algebras

2A. ***Definition of the finite W-algebra $U(\mathfrak{g}, e)$.*** Let G be a reductive algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. The finite W-algebra $U(\mathfrak{g}, e)$ is defined in terms of a nilpotent element $e \in \mathfrak{g}$. By the Jacobson–Morozov Theorem, e embeds into an $\mathfrak{sl}_2$-triple (e, h, f). The $\operatorname{ad} h$ eigenspace decomposition gives a grading on $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j), \tag{2.1}$$

where $\mathfrak{g}(j) = \{x \in \mathfrak{g} \mid [h, x] = jx\}$. Define the character $\chi : \mathfrak{g} \to \mathbb{C}$ by $\chi(x) = (x, e)$, where $(\cdot\,,\cdot)$ is a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$. Then we can define a nondegenerate symplectic form $\langle \cdot\,,\cdot \rangle$ on $\mathfrak{g}(-1)$ by $\langle x, y \rangle = \chi([y, x])$. Choose a Lagrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}(-1)$ with respect to $\langle \cdot\,,\cdot \rangle$, and let $\mathfrak{m} = \mathfrak{l} \oplus \bigoplus_{j \le -2} \mathfrak{g}(j)$. Let $\mathfrak{m}_\chi = \{m - \chi(m) \mid m \in \mathfrak{m}\}$. The adjoint action of $\mathfrak{m}$ on $U(\mathfrak{g})$ leaves the left ideal $U(\mathfrak{g})\mathfrak{m}_\chi$ invariant, so there is an induced adjoint action of $\mathfrak{m}$ on $Q_\chi = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$. The space of fixed points $Q_\chi^{\mathfrak{m}}$ inherits a well-defined multiplication from $U(\mathfrak{g})$, making it an associative algebra, and we define the finite W-algebra to be

$$U(\mathfrak{g}, e) = Q_\chi^{\mathfrak{m}} = \{u + U(\mathfrak{g})\mathfrak{m}_\chi \in Q_\chi \mid [x, u] \in U(\mathfrak{g})\mathfrak{m}_\chi \text{ for all } x \in \mathfrak{m}\}.$$

We also recall here that the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ maps into $U(\mathfrak{g}, e)$ via the inclusion $Z(\mathfrak{g}) \subseteq U(\mathfrak{g})$. Moreover, it is known that this defines an isomorphism between

$Z(\mathfrak{g})$ and the centre of $U(\mathfrak{g}, e)$; see the footnote to [Premet 2007, Question 5.1]. We use this isomorphism to identify the centre of $U(\mathfrak{g}, e)$ with $Z(\mathfrak{g})$, which in particular allows us to define integral central characters for $U(\mathfrak{g}, e)$-modules.

Remark 2.2. There are different equivalent definitions of the finite W-algebra in the literature. Above we have given the Whittaker model definition, as it is the shortest and most convenient for our purposes here.

2B. ***Skryabin's equivalence and Losev's map of primitive ideals.*** The left $U(\mathfrak{g})$-module Q_χ is also a right $U(\mathfrak{g}, e)$-module, so there is a functor

$$\mathcal{S} : U(\mathfrak{g}, e)\text{-mod} \to U(\mathfrak{g})\text{-mod}, \quad M \mapsto Q_\chi \otimes_{U(\mathfrak{g},e)} M,$$

where M is a $U(\mathfrak{g}, e)$-module. Skryabin [2002] showed that $\mathcal{S}$ is an equivalence of categories between $U(\mathfrak{g}, e)$-mod and the category of *Whittaker modules for e*, the category of $U(\mathfrak{g})$-modules on which $\mathfrak{m}_\chi$ acts locally nilpotently.

For an algebra A, let $\operatorname{Prim} A$ denote the set of primitive ideals of A. Losev [2011a] showed that there exists a map

$$\cdot^\dagger : \operatorname{Prim} U(\mathfrak{g}, e) \to \operatorname{Prim} U(\mathfrak{g}), \quad I \mapsto I^\dagger$$

with the following properties:

(1) It preserves central characters: $I \cap Z(\mathfrak{g}) = I^\dagger \cap Z(\mathfrak{g})$ for any $I \in \operatorname{Prim}(U(\mathfrak{g}, e))$, under the identification of the centre of $U(\mathfrak{g}, e)$ with $Z(\mathfrak{g})$.

(2) It behaves well with respect to Skryabin's equivalence in the sense that

$$\operatorname{Ann}_{U(\mathfrak{g})} \mathcal{S}(M) = (\operatorname{Ann}_{U(\mathfrak{g},e)} M)^\dagger$$

for every irreducible $U(\mathfrak{g}, e)$-module M.

(3) Its restriction to $\operatorname{Prim}_0 U(\mathfrak{g}, e)$, the set of primitive ideals of $U(\mathfrak{g}, e)$ of finite codimension, is a surjection onto $\operatorname{Prim}_e U(\mathfrak{g})$, the set of primitive ideals of $U(\mathfrak{g})$ with associated variety equal to $\overline{G \cdot e}$.

(4) Its fibres restricted to $\operatorname{Prim}_0 U(\mathfrak{g}, e)$ are C-orbits, where C is the component group of the centralizer of e. See, for example, the introduction to [Losev 2011a] for an explanation of the action of C on $\operatorname{Prim}_0 U(\mathfrak{g}, e)$.

2C. ***Highest weight theory and Losev's map.*** By using the highest weight theory for finite W-algebras developed by Brundan, Kleshchev and Goodwin in [Brundan et al. 2008] (abbreviated [BGK] in this section), the map $\cdot^\dagger$ from the previous subsection can be explicitly calculated in terms of highest weight modules for $U(\mathfrak{g}, e)$ and $U(\mathfrak{g})$.

The key part of this highest weight theory is the use of a minimal Levi subalgebra $\mathfrak{g}_0$ that contains e. In [BGK, Theorem 4.3] it is proved that there is a certain subquotient of $U(\mathfrak{g}, e)$ that is isomorphic to $U(\mathfrak{g}_0, e)$. Then, in [BGK, §4.2], it is explained how a choice of a parabolic subalgebra $\mathfrak{q}$ with Levi factor $\mathfrak{g}_0$ leads to a highest weight theory for $U(\mathfrak{g}, e)$ in which $U(\mathfrak{g}_0, e)$ plays the role of the Cartan subalgebra in the usual highest weight theory for reductive Lie algebras. This leads to a definition of Verma modules for $U(\mathfrak{g}, e)$ by "parabolically inducing" $U(\mathfrak{g}_0, e)$-modules up to $U(\mathfrak{g}, e)$-modules. Then [BGK, Theorem 4.5] says that these Verma modules have irreducible heads, and that any finite-dimensional irreducible $U(\mathfrak{g}, e)$-module is isomorphic to one of these irreducible heads. This gives a method to explicitly parametrize finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules, though a classification of $U(\mathfrak{g}_0, e)$-modules in general is unknown at present.

When e is of standard Levi type, the classification of $U(\mathfrak{g}_0, e)$-modules is known. By a theorem of Kostant [1978] and the Harish-Chandra isomorphism, we have that $U(\mathfrak{g}_0, e) \cong Z(\mathfrak{g}_0) \cong S(\mathfrak{t})^{W_0}$, where $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}$ and W_0 is the Weyl group of $\mathfrak{g}_0$. Hence the finite-dimensional irreducible $U(\mathfrak{g}_0, e)$-modules are all one-dimensional, and they are parametrized by the W_0-orbits on $\mathfrak{t}^*$. We choose $\mathfrak{t}$ as specified in [BGK, §5.1], and let $\Lambda \in \mathfrak{t}^*/W_0$ be a W_0-orbit. In [BGK] an explicit isomorphism $U(\mathfrak{g}_0, e) \to S(\mathfrak{t})^{W_0}$ is given. Using this isomorphism and our choice of $\mathfrak{q}$, we let $M(\Lambda, \mathfrak{q})$ denote the Verma module for $U(\mathfrak{g}, e)$ induced from Λ, and we write $L(\Lambda, \mathfrak{q})$ for the irreducible head of $M(\Lambda, \mathfrak{q})$. We note that there are "shifts" involved in the isomorphisms above and thus in the definition of $M(\Lambda, \mathfrak{q})$ in [BGK, Sections 4 and 5].

Let $\mathfrak{u}$ be the nilradical of $\mathfrak{q}$ and let $\mathfrak{b}_0$ be a Borel subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{t}$, so that $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{u}$ is a Borel subalgebra of $\mathfrak{g}$. For $\lambda \in \mathfrak{t}^*$, let $L(\lambda, \mathfrak{b})$ denote the highest weight irreducible $\mathfrak{g}$-module defined in terms of $\mathfrak{b}$, with highest weight $\lambda - \rho$ (where ρ is the half-sum of the positive roots for $\mathfrak{b}$).

The theorem below allows us to explicitly calculate Losev's map $\cdot^\dagger$ on primitive ideals in terms of highest weight modules. In [BGK, §5.1] it was shown that this theorem follows from [Miličić and Soergel 1997, Theorem 5.1] and [BGK, Conjecture 5.3]. Also this last conjecture was verified in [Losev 2012, Theorem 5.1.1], except for a technical point which was resolved in [Brown and Goodwin 2013a, Proposition 3.10].

Theorem 2.3. *Let $\Lambda \in \mathfrak{t}^*/W_0$ and let $\lambda \in \Lambda$ be antidominant for $\mathfrak{b}_0$. Then*

$$(\operatorname{Ann}_{U(\mathfrak{g},e)} L(\Lambda, \mathfrak{q}))^\dagger = \operatorname{Ann}_{U(\mathfrak{g})} L(\lambda, \mathfrak{b}).$$

One consequence of this theorem is that if e is not a special nilpotent element, then $U(\mathfrak{g}, e)$ has no finite-dimensional irreducible representations of integral central character. This is due to results of Barbasch and Vogan [1982; 1983], which imply that the associated variety of $\operatorname{Ann}_{U(\mathfrak{g})} L(\lambda, \mathfrak{b})$ is a special nilpotent orbit if and only if λ is integral.

The following theorem is Conjecture 5.2 of [BGK], which follows from Conjecture 5.3 of the same paper, as is explained in there.

Theorem 2.4. *Let $\Lambda \in \mathfrak{t}^*/W_0$ and let $\lambda \in \Lambda$ be antidominant for $\mathfrak{b}_0$. Then $L(\Lambda, \mathfrak{q})$ is finite-dimensional if and only if the associated variety of $\operatorname{Ann}_{U(\mathfrak{g})} L(\lambda, \mathfrak{b})$ is equal to $\overline{G \cdot e}$.*

3. Combinatorics of s-tables and finite W-algebras

3A. ***Realizations of $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$.*** In the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$, we realize $\mathfrak{g}$ in the following way: Let $V = \mathbb{C}^{2n+1}$ have basis $\{e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}\}$. Then we take $\mathfrak{gl}_{2n+1} = \operatorname{End}(V)$ as having basis $\{e_{i,j} \mid i, j = 0, \pm 1, \dots, \pm n\}$, where $e_{i,j} \in \operatorname{End}(V)$ is defined via $e_{i,j}(e_k) = \delta_{j,k} e_i$. We define the bilinear form $(\cdot\,,\cdot)$ on V by declaring that $(e_i, e_j) = \delta_{i,-j}$. Then we set

$$\mathfrak{g} = \mathfrak{so}_{2n+1} = \{x \in \mathfrak{gl}_{2n+1} \mid (xv, w) = -(v, xw) \text{ for all } v, w \in V\}.$$

Note that $\mathfrak{g}$ has basis $\{f_{i,j} \mid i, j = 0, \pm 1, \dots, \pm n,\ i + j > 0\}$, where $f_{i,j} = e_{i,j} - e_{-j,-i}$. We choose $\mathfrak{t} = \{f_{i,i} \mid i = 1, \dots, n\}$ as a maximal toral subalgebra, so that $\mathfrak{t}^*$ has basis $\{\epsilon_i \mid i = 1, \dots, n\}$, where $\epsilon_i \in \mathfrak{t}^*$ is defined via $\epsilon_i(f_{j,j}) = \delta_{i,j}$ for $i, j > 0$. We write Φ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Let $\mathfrak{b} = \langle f_{i,j} \mid i \le j \rangle$ be the Borel subalgebra of upper-triangular matrices in $\mathfrak{g}$. Then the corresponding system of positive roots is given by

$$\Phi^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\} \cup \{\epsilon_i \mid i = 1, \dots, n\}.$$

For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we let $V = \mathbb{C}^{2n}$ have basis $\{e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\}$. Then we realize $\mathfrak{gl}_{2n} = \operatorname{End}(V)$ as having basis $\{e_{i,j} \mid i, j = \pm 1, \dots, \pm n\}$, where $e_{i,j} \in \operatorname{End}(V)$ is defined via $e_{i,j}(e_k) = \delta_{j,k} e_i$. We define the bilinear form $(\cdot\,,\cdot)$ on V by declaring that $(e_i, e_j) = \operatorname{sign}(i)\delta_{i,-j}$, and set

$$\mathfrak{g} = \mathfrak{sp}_{2n} = \{x \in \mathfrak{gl}_{2n} \mid (xv, w) = -(v, xw) \text{ for all } v, w \in V\}.$$

Then $\mathfrak{g}$ has basis $\{f_{i,j} \mid i, j = \pm 1, \dots, \pm n,\ i + j \ge 0\}$, where $f_{i,j} = e_{i,j} - \operatorname{sign}(i)\operatorname{sign}(j) e_{-j,-i}$. We choose $\mathfrak{t} = \{f_{i,i} \mid i = 1, \dots, n\}$ as a maximal toral subalgebra, so that $\mathfrak{t}^*$ has basis $\{\epsilon_i \mid i = 1, \dots, n\}$, where $\epsilon_i \in \mathfrak{t}^*$ is defined via $\epsilon_i(f_{j,j}) = \delta_{i,j}$ for $i, j > 0$. We write Φ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. We choose the Borel subalgebra $\mathfrak{b} = \langle f_{i,j} \mid i \le j \rangle$ of upper-triangular matrices in $\mathfrak{g}$. Then the corresponding system of positive roots is given by

$$\Phi^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\} \cup \{2\epsilon_i \mid i = 1, \dots, n\}.$$

3B. ***Standard Levi nilpotent elements and symmetric pyramids.*** Recall from the introduction that we are considering nilpotent orbits in $\mathfrak{g}$ that are special and standard

Levi, but not even-multiplicity. The Jordan type for such a nilpotent orbit is of the form

$$\boldsymbol{p} = (p_1^2 \le \cdots \le p_{d-1}^2 < p_0 \le p_d^2 \le \cdots \le p_r^2). \tag{3.1}$$

Moreover, p_i must be odd for all $i \ge d$ when $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or p_i must be even for all $i < d$ when $\mathfrak{g} = \mathfrak{sp}_{2n}$. As explained in the introduction, the condition that $\boldsymbol{p}$ has only one part of odd multiplicity is due to the standard Levi assumption, and the parity conditions are due to the assumption that the corresponding orbit is special.

Also recall from the introduction the definition of the symmetric pyramid P for $\boldsymbol{p}$. We form a diagram K called the *coordinate pyramid for* $\boldsymbol{p}$ by filling the boxes of P with $1, \dots, n, -n, \dots, -1$ if $\mathfrak{g} = \mathfrak{sp}_{2n}$ or with $1, \dots, n, 0, -n, \dots, -1$ if $\mathfrak{g} = \mathfrak{so}_{2n}$, across rows from top to bottom. For example, for $\mathfrak{g} = \mathfrak{sp}_{18}$ and $\boldsymbol{p} = (2^2, 4, 5^2)$, we have

$$K = \begin{array}{ccccc} & 1 & 2 & & \\ 3 & 4 & 5 & 6 & 7 \\ & 8 & 9 & -9 & -8 \\ -7 & -6 & -5 & -4 & -3 \\ & -2 & -1 & & \end{array}$$

We let $\operatorname{col}(i)$ denote the x-coordinate of the centre of the box of K that contains i. However, we use $\operatorname{row}(i)$ to denote the row of K that contains i when we label the rows of K by $1, \dots, r, 0, -r, \dots, -1$ from top to bottom, so that p_i is the length of row i.

We define $e \in \mathfrak{g}$ by

$$e = \sum_{i,j} f_{i,j}, \tag{3.2}$$

where the sum is over all adjacent pairs $\begin{array}{|c|c|}\hline i & j \\ \hline\end{array}$ in K, so that e is in the nilpotent G-orbit with Jordan type $\boldsymbol{p}$.

We also use K to conveniently define many of the objects required for the definition of $U(\mathfrak{g}, e)$ and the highest weight theory.

Let $h = \sum_{i=1}^{n} -\operatorname{col}(i) f_{i,i}$; then (e, h, f) is an $\mathfrak{sl}_2$-triple for some $f \in \mathfrak{g}$. Furthermore, the grading from (2.1) on $\mathfrak{g}$ is given by $\mathfrak{g}(k) = \langle f_{i,j} \mid \operatorname{col}(j) - \operatorname{col}(i) = k \rangle$. For our Lagrangian subspace of $\mathfrak{g}(-1)$, we let

$$\mathfrak{l} = \langle f_{i,j} \mid \operatorname{col}(i) - \operatorname{col}(j) = 1,\ \operatorname{row}(i) < \operatorname{row}(j) \rangle.$$

Then we have that $\mathfrak{m} = \mathfrak{l} \oplus \langle f_{i,j} \mid \operatorname{col}(i) - \operatorname{col}(j) > 1 \rangle$, and we use these choices of e and $\mathfrak{m}$ to form the finite W-algebra $U(\mathfrak{g}, e)$ as in Section 2A.

We take $\mathfrak{g}_0 = \langle f_{i,j} \mid \mathrm{row}(i) = \mathrm{row}(j)\rangle$. So $\mathfrak{g}_0$ is a minimal Levi subalgebra which contains e, and e is a regular nilpotent element of $\mathfrak{g}_0$. In the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$, we have

$$\mathfrak{g}_0 \cong \mathfrak{so}_{p_0} \oplus \bigoplus_{i=1}^{r} \mathfrak{gl}_{p_i}$$

and in the case $\mathfrak{g} = \mathfrak{sp}_{2n}$ we have

$$\mathfrak{g}_0 \cong \mathfrak{sp}_{p_0} \oplus \bigoplus_{i=1}^{r} \mathfrak{gl}_{p_i}.$$

We choose $\mathfrak{q} = \langle f_{i,j} \mid$ the row containing i is above or equal to the row containing $j\rangle$. Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$. Let $\mathfrak{b}_0 = \mathfrak{b} \cap \mathfrak{g}_0$, so that $\mathfrak{b}_0$ is a Borel subalgebra of $\mathfrak{g}_0$ that satisfies $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$.

3C. ***Tables and s-tables.*** We use the definitions and notation regarding frames, tables, s-frames and s-tables from [Brown and Goodwin 2013a, §4]. Below we explain how these are used to label highest weight modules for $U(\mathfrak{g}, e)$.

For this purpose, we let W_r be the Weyl group of type B_r, which acts on $\{0, \pm 1, \dots, \pm r\}$ in the natural way. We denote the standard generators of W_r by $\bar{s}_i = (i, i+1)(-i, -i-1)$, for $i = 1, \dots, r-1$. Let $\bar{S}_r$ be the subgroup of W_r generated by $\bar{s}_i$ for $i = 1, \dots, r-1$.

Given $\sigma \in W_r$, we define $\sigma \cdot P$ to be the diagram obtained from P by permuting rows according to σ, so that $\sigma \cdot P$ is an s-frame (recall that an s-frame is a collection of connected rows of boxes in the plane arranged symmetrically around the origin). We recall that by an *s-table with frame* $\sigma \cdot P$, we mean a skew-symmetric (with respect to the origin) filling of $\sigma \cdot P$ with complex numbers. Then we define $\mathrm{sTab}(\sigma \cdot P)$ to be the set of s-tables with frame $\sigma \cdot P$ such that all entries are integers if $\mathfrak{g} = \mathfrak{sp}_{2n}$, and either all entries are in $\mathbb{Z}$ or all entries are in $\frac{1}{2} + \mathbb{Z}$ if $\mathfrak{g} = \mathfrak{so}_{2n+1}$.

We let $\sigma \cdot K$ be the s-table obtained from K by permuting rows according to σ. Now given $A \in \mathrm{sTab}(\sigma \cdot P)$, we define $\lambda_A = \sum_{i=1}^{n} a_i \epsilon_i$, where a_i is the entry of A in the same box as i in $\sigma \cdot K$. In this way we get an identification of $\mathrm{sTab}(\sigma \cdot P)$ with the set of integral weights in $\mathfrak{t}^*$; we write $\mathfrak{t}^*_{\mathbb{Z}}$ for the set of integral weights of $\mathfrak{t}$.

The *row equivalence class* of an s-table is the set of s-tables that can be created by permuting entries within rows. We let $\mathrm{sRow}(\sigma \cdot P)$ denote the set of row equivalence classes of $\mathrm{sTab}(\sigma \cdot P)$. Then $\mathrm{sRow}(\sigma \cdot P)$ is identified naturally with $\mathfrak{t}^*_{\mathbb{Z}}/W_0$, where W_0 is the Weyl group of $\mathfrak{g}_0$. Let $\mathrm{sTab}^{\leq}(\sigma \cdot P)$ denote the elements of $\mathrm{sTab}(\sigma \cdot P)$ that have nondecreasing rows. Then every element of that $\mathrm{sRow}(\sigma \cdot P)$ contains a unique element of $\mathrm{sTab}^{\leq}(\sigma \cdot P)$.

We label the rows of $\sigma \cdot K$ with $1, \dots, r, 0, -r, \dots, -1$ from top to bottom. Now, we define $\mathfrak{q}_\sigma$ to be generated by the by f_{ij} for which the row of $\sigma \cdot K$ in which i appears is above or equal to the row containing j. Then $\mathfrak{q}_\sigma$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$, so we can use it to define the irreducible highest weight modules $L(\Lambda, \mathfrak{q}_\sigma)$, for $\Lambda \in \mathfrak{t}^*/W_0$ as defined in Section 2C.

Given $\Lambda \in \mathfrak{t}^*_{\mathbb{Z}}/W_0$, there is a unique $A \in \mathrm{sTab}^{\leq}(\sigma \cdot P)$ whose row equivalence class $\bar{A} \in \mathrm{sRow}(P)$ is identified with Λ as above. We let $L_\sigma(A)$ denote $L(\Lambda, \mathfrak{q}_\sigma)$.

Let $\mathfrak{u}_\sigma$ be the nilradical of $\mathfrak{q}_\sigma$, and define $\mathfrak{b}_\sigma = \mathfrak{b}_0 \oplus \mathfrak{u}_\sigma$, which is a Borel subalgebra of $\mathfrak{g}$. We write $L_\sigma(\lambda_A)$ for the irreducible highest weight $U(\mathfrak{g})$-module with respect to $\mathfrak{b}_\sigma$, with highest weight $\lambda_A - \rho_\sigma$, where ρ_σ is the half-sum of positive roots for $\mathfrak{b}_\sigma$.

Now Theorem 2.3 can be restated in our present notation as follows:

Theorem 3.3. *Let $\sigma \in W_r$ and $A \in \mathrm{sTab}^{\leq}(\sigma \cdot P)$. Then* $(\mathrm{Ann}_{U(\mathfrak{g},e)} L_\sigma(A))^\dagger = \mathrm{Ann}_{U(\mathfrak{g})} L_\sigma(\lambda_A)$.

We are mainly interested in the case where $\sigma = 1$. Here we have $\mathfrak{q}_\sigma = \mathfrak{q}$, and we write $L(A)$ instead of $L_1(A)$ and $L(\lambda_A)$ instead of $L_1(\lambda_A)$ for $A \in \mathrm{sTab}(P)$.

Thanks to Theorem 3.3, our goal of classifying the finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules and understanding the component group action on these modules can be broken down to answering the following questions:

(1) For which $A \in \mathrm{sTab}^{\leq}(P)$ is the associated variety of $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_A)$ equal to $\overline{G \cdot e}$?

(2) Given $A \in \mathrm{sTab}^{\leq}(P)$ such that $L(A)$ is finite-dimensional, which $B \in \mathrm{sTab}^{\leq}(P)$ satisfy $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_A) = \mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_B)$?

In the case that $\boldsymbol{p}$ has three parts, we answer these two questions in Sections 5 and 6. The key ingredients in answering the first question are the Robinson–Schensted and Barbasch–Vogan algorithms explained in Section 4A and Section 4C. For the second question, we use Vogan's τ-equivalence on integral weights of $\mathfrak{g}$, which is explained in Section 4D.

In moving from the three row case to the general case, a key role is played by the different choices of highest weight theories determined by the different parabolic subalgebras $\mathfrak{q}_\sigma$ for $\sigma \in W_r$. This dependence follows easily from the results for the case of even-multiplicity nilpotent elements established in [Brown and Goodwin 2013c], which hold in the present situation; the key result for us is Proposition 4.6. We also require the explicit description of the action of the component group on the set of finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules in terms of s-tables, which is given in Proposition 7.1. The proof of Theorem 1.2 for the general case is then dealt with in Section 7B.

3D. ***The component group.*** Recall that C denotes the component group of the centralizer of e in G. Here we take G to be the adjoint group of $\mathfrak{g}$, so G is either SO_{2n+1} or PSp_{2n}.

A specific realization of C is given as follows: Let $0 < p_{i_1} < \cdots < p_{i_s}$ be the maximal distinct parts of $\boldsymbol{p}$ such that $p_{i_j} \neq p_0$ and p_{i_j} is odd (respectively even) when $\mathfrak{g} = \mathfrak{so}_{2n+1}$ (respectively $\mathfrak{sp}_{2n}$); by maximal, we mean that if $p_k = p_{i_j}$, then $k \leq i_j$. Define the matrices $c_1, \ldots, c_s$ corresponding to $p_{i_1}, \ldots, p_{i_s}$ for $p_{i_k} \neq p_0$ by setting

$$c_k = \sum_{\substack{-n \leq i, j \leq n \\ \mathrm{col}(i)=\mathrm{col}(j) \\ \mathrm{row}(i)=i_k \\ \mathrm{row}(j)=-i_k}} \mathrm{sign}(\mathrm{col}(i))(e_{i,j} + e_{j,i}) + \sum_{\substack{-n \leq i \leq n \\ \mathrm{row}(i) \neq \pm i_k}} e_{i,i}.$$

Then one can calculate that c_k centralizes e. Furthermore, the argument used in [Brown 2011, Section 6] can be adapted to show that the images of $c_1, \ldots, c_s$ in C generate $C \cong \mathbb{Z}_2^s$.

As mentioned in Section 2B, there is an action of C on $\mathrm{Prim}\, U(\mathfrak{g}, e)$, and thus on isomorphism classes of irreducible modules, and, as explained in [Brown and Goodwin 2013a, §2.3], this can be seen as "twisting" modules by elements of C (up to isomorphism). Given an irreducible $U(\mathfrak{g}, e)$-module L and $b \in C$, we write $b \cdot L$ for the twisted module; we note that this is a minor abuse of notation as $b \cdot L$ is only defined up to isomorphism.

4. Some combinatorics for s-tables

4A. ***The Robinson–Schensted algorithm.*** We use the formulation of the Robinson–Schensted algorithm from [Brown and Goodwin 2013a, §4]. We denote the Robinson–Schensted algorithm by RS and recall that it takes as input a word of integers (or more generally complex numbers) or a table and outputs a tableau.

There are two lemmas about the Robinson–Schensted algorithm that we use repeatedly in the sequel. We state them below for convenience; they can be found in [Fulton 1997, §3]. For a word w, we define $\ell(w, k)$ to be the maximum possible sum of the lengths of k disjoint weakly increasing subsequences of w, and $\wp(w, k)$ to be the maximum possible sum of the lengths of k disjoint strictly decreasing subsequences of w. We write $\mathrm{part}(T)$ to denote the partition underlying a tableau T.

Lemma 4.1. *Let w be a word of integers and let $\boldsymbol{q} = (q_1 \geq \cdots \geq q_n) = \mathrm{part}(\mathrm{RS}(w))$. Then for all $k \geq 1$, $\ell(w, k) = q_1 + \cdots + q_k$.*

Lemma 4.2. *Let w be a word of integers and let $\boldsymbol{q}^T = (q_1^* \geq \cdots \geq q_n^*)$ be the dual partition to $\boldsymbol{q} = \mathrm{part}(\mathrm{RS}(w))$. Then for all $k \geq 1$, $\wp(w, k) = q_1^* + \cdots + q_k^*$.*

An elementary fact about the Robinson–Schensted algorithm, required later, is stated in Lemma 4.3 below; it is easily deduced from Lemma 4.1. Suppose u, w are words of integers and a, b are integers such that $a > b$; then we say the transposition of the word $uabw$ to $ubaw$ is a *larger-smaller transposition*. Also, we refer the reader to [Fulton 1997, §2] for the definition of Knuth equivalences.

Lemma 4.3. *If u and w are words of integers and w can be obtained from u by a sequence of Knuth equivalences and larger-smaller transpositions, then* $\mathrm{part}(\mathrm{RS}(u)) \leq \mathrm{part}(\mathrm{RS}(w))$.

The following theorem extends Theorem 4.6 of [Brown and Goodwin 2013a] and is important for us later. In the statement, P is the symmetric pyramid for the partition $\boldsymbol{p}$, as in the previous section. Also, recall we defined the notion of a justified row-equivalent to column-strict s-table in the introduction.

Theorem 4.4. *Let $A, B \in \mathrm{sTab}^{\leq}(P)$. Then*:

(i) *A is justified row-equivalent to column-strict if and only if* $\mathrm{part}(\mathrm{RS}(A)) = \boldsymbol{p}$.

(ii) *If* $\mathrm{part}(\mathrm{RS}(A)) = \boldsymbol{p}$, *then* $\mathrm{RS}(A) = \mathrm{RS}(B)$ *if and only if* $A = B$.

Proof. Part (i) can be proved in the same way as [Brown and Goodwin 2013a, Theorem 4.6]. We just need to check the proof still holds if A has an odd number of rows and the middle row of A is not A's longest row. The only thing to check is that there is a sequence of row swaps that transforms A into a tableau such that the convexity conditions required by Lemma 4.9 of the same reference are satisfied, which is clear.

To prove (ii), we simply note that each row swap from the sequence of row swaps from (i) which turns A into a tableau is invertible. □

Lastly in this section we give the following theorem, which is important later on:

Theorem 4.5. *Let $A, B \in \mathrm{sTab}^{c}(P)$. Suppose that* $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_A) = \mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_B)$. *Then $A = B$.*

Proof. First, we need to briefly explain some of the results of Garfinkle [1990; 1993]. Section 2 of [Garfinkle 1990] defines the map $L : W_n \to \mathrm{Dom}_n$, where Dom_n denotes the set of *domino tableaux for W_n* (see the appendix to this paper for more information on domino tableaux). Section 5 of the same work defines the map $S : \mathrm{Dom}_n \to \mathrm{sDom}_n$, where sDom_n denotes the set of domino tableaux for W_n of special shape (a domino tableau has special shape if its underlying partition is the Jordan type of a special nilpotent element of $\mathfrak{g}$). Furthermore, S restricted to sDom is the identity map.

For $\lambda \in \mathfrak{t}^*$, let $\mathrm{Prim}_\lambda U(\mathfrak{g})$ denote the primitive ideals of $U(\mathfrak{g})$ of central character λ. Suppose $\lambda \in \mathfrak{t}^*$ is antidominant and integral. Now, Theorem 3.5.11 of [Garfinkle 1993] states that the map $\mathrm{cl} : \mathrm{Prim}_\lambda U(\mathfrak{g}) \to \mathrm{sDom}(n)$ given by $\mathrm{cl}(\mathrm{Ann}\, L(w\lambda)) = S(L(w))$ is a bijection.

Next we need to know that Garfinkle's map L gives the same result as the Robinson–Schensted algorithm. This is provided in the appendix by Proposition A.4, which is simply a rephrasing of [van Leeuwen 1996, Proposition 4.2.3]. More specifically, the Robinson–Schensted algorithm outputs a tableau. There is a canonical way to associate a tableau that has been outputted by the Robinson–Schensted algorithm with a domino tableau (namely the algorithm DT from the appendix). Now, Proposition A.4 says that if we identify the output of the Robinson–Schensted algorithm with a domino tableau, then the result is the same as we would get from Garfinkle's L algorithm.

Now, to prove the theorem, note that since A and B are justified row-equivalent to column-strict, by Theorem 4.4, $\mathrm{part}(\mathrm{RS}(A)) = \mathrm{part}(\mathrm{RS}(B)) = \boldsymbol{p}$. Since

$$\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_A) = \mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_B),$$

the above discussion allows us to deduce that $\mathrm{RS}(A) = \mathrm{RS}(B)$. Now, the theorem follows from Theorem 4.4. □

4B. ***Row swapping.*** In the proof of Theorem 4.4 above, we have mentioned the row swapping operations $s_i\star$ on tables, as defined in [Brown and Goodwin 2013a, §4;2013c, §4]. An important ingredient for the definition of these row swapping operations is the notion of best fitting as defined in [Brown and Goodwin 2013a, §4], which we use repeatedly in the following.

We also require the operations $\bar{s}_i\star$ for s-tables, and we use the notation from [Brown and Goodwin 2013c, §5]. Recall that for $\sigma \in W_r$ and an s-table $A \in \mathrm{sTab}^{\leq}(\sigma \cdot P)$, either $\bar{s}_i \star A$ is undefined or it is an element of $\mathrm{sTab}^{\leq}(\bar{s}_i\sigma \cdot P)$. These operations can be extended to operations by elements of $\overline{S}_r$; the proof of Proposition 5.5(i) of the same reference goes through in our situation to show that this is well defined.

The following proposition is a version of [Brown and Goodwin 2013c, Proposition 5.3(ii)] in the present setting, and its proof adapts immediately:

Proposition 4.6. *Let $\sigma \in W_r$, $\tau \in \overline{S}_r$ and $A \in \mathrm{sTab}^{\leq}(\sigma \cdot P)$. Suppose that $\tau \star A$ is defined. Then $L_\sigma(A) \cong L_{\tau\sigma}(\tau \star A)$.*

Also, we state the following lemma, as it is key in the proof of Theorem 1.2. It is [Brown and Goodwin 2013a, Lemma 5.11], adapted to our situation, and the same proof holds. In the statement, A_r^1 denotes the table formed by rows 1 to r of A.

Lemma 4.7. *For $A \in \mathrm{sTab}^{\leq}(P)$, suppose that $L(A)$ is finite-dimensional, and let $\tau \in \overline{S}_r$. Then A_r^1 is justified row-equivalent to column-strict and $\tau \star A$ is defined.*

4C. ***The Barbasch–Vogan algorithm.*** The Barbasch–Vogan algorithm [1982] takes as input λ, an integral weight for a classical Lie algebra of type B or C, and outputs $\mathrm{BV}(\lambda)$, the Jordan type of the associated variety of $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda)$. Below we recall the description of it given in [Brown and Goodwin 2013a, §5.2]. We note that there is a version of it for type D, but we do not require that here.

We need to define the *content of a partition*. Let $\boldsymbol{q} = (q_1 \leq q_2 \leq \cdots \leq q_m)$ be a partition. By inserting 0 at the beginning if necessary, we may assume that m is odd. Let $(s_1, \dots, s_k)$, $(t_1, \dots, t_l)$ be such that

$$\{q_1, q_2 + 1, q_3 + 2, \dots, q_r + r - 1\} = \{2s_1, \dots, 2s_k, 2t_1 + 1, \dots, 2t_l + 1\}$$

(as unordered lists). Now, we define the content of $\boldsymbol{q}$ to be the unordered list

$$\operatorname{content}(\boldsymbol{q}) = \{s_1, \dots, s_k, t_1, \dots, t_l\}.$$

Algorithm:

Input: $\lambda = \sum_{i=1}^{n} a_i \epsilon_i$ an integral weight in $\mathfrak{t}^*$.

Step 1: Calculate $\boldsymbol{q} = \operatorname{part}(\operatorname{RS}(a_1, \dots, a_n, -a_n, \dots, -a_1))$.

Step 2: Calculate $\operatorname{content}(\boldsymbol{q})$.
Let $(u_1 \leq \cdots \leq u_{2k+1})$ be the sorted list with the same entries as $\operatorname{content}(\boldsymbol{q})$.
For $i = 1, \dots, k+1$, let $s_i = u_{2i-1}$.
For $i = 1, \dots, k$, let $t_i = u_{2i}$.

Step 3: Form the list $(2s_1 + 1, \dots, 2s_{k+1} + 1, 2t_1, \dots, 2t_k)$.
In either case, let $(v_1 < \cdots < v_k)$ be this list after sorting.

Output: $\operatorname{BV}(\lambda) = \boldsymbol{q}' = (v_1, v_2 - 1, \dots, v_{2k+1} - 2k)$.

We note that the output partition $\boldsymbol{q}'$ (potentially with an extraneous zero at the beginning) is the Jordan type of a *special* nilpotent orbit of $\mathfrak{g}$; this was proved in [Barbasch and Vogan 1982].

For our purposes in this paper, we also need a modified version of the algorithm to use in the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$. This modified version is denoted by BV'. It works in exactly the same way as BV, except that in Step 1 instead of calculating $\operatorname{RS}(a_1, \dots, a_n, -a_n, \dots, -a_1)$ we calculate $\operatorname{RS}(a_1, \dots, a_n, 0, -a_n, \dots, -a_1)$.

In Corollary A.7, in the appendix to this paper, it is proved that

$$\operatorname{BV}(\lambda) = \operatorname{BV}'(\lambda)$$

for $\lambda \in \mathfrak{t}^*$ in the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$. This proof of this is entirely combinatorial and may be of independent interest so it is has been placed in an appendix. In light of this, we redefine $\operatorname{BV}(\lambda)$, so that it is the old $\operatorname{BV}(\lambda)$ in the case $\mathfrak{g} = \mathfrak{sp}_{2n}$ and is $\operatorname{BV}'(\lambda)$ in the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$.

For convenience of reference later in this paper we state the following theorem from [Barbasch and Vogan 1982]:

Theorem 4.8. *Let $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$. Then the associated variety to* $\operatorname{Ann}_{U(\mathfrak{g})} L(\lambda)$ *is equal to the nilpotent G-orbit with Jordan type given by* $\operatorname{BV}(\lambda)$.

4D. *The τ-equivalence.* The Barbasch–Vogan algorithm is used to find the associated variety of $\operatorname{Ann}_{U(\mathfrak{g})}(L(\lambda))$; however, in order to determine the action of the component group we need to be able to determine when $\operatorname{Ann}_{U(\mathfrak{g})} L(\mu) = \operatorname{Ann}_{U(\mathfrak{g})} L(\lambda)$. This can be done using the τ-equivalence. This is an equivalence relation on the set of integral weights of $\mathfrak{t}$.

Recall our realization of $\mathfrak{g}$ and its Borel subalgebra $\mathfrak{b}$ defined in Section 3A, and recall that Φ^+ is the system of positive roots for $\mathfrak{g}$ defined from $\mathfrak{b}$ and $\mathfrak{t}$. Let Δ be the base of Φ corresponding to Φ^+. Also, for $\alpha \in \Phi$, let $s_\alpha \in W$ denote the corresponding reflection in the Weyl group W of $\mathfrak{g}$ with respect to $\mathfrak{t}$. For $w \in W$, let

$$S(w) = \{\alpha \in \Phi^+ \mid w\alpha \notin \Phi^+\}.$$

Now let

$$\tau(w) = S(w) \cap \Delta.$$

Suppose that $\lambda \in \mathfrak{t}^*$ is an integral antidominant weight. Let $\alpha \in \Delta$ and $w \in W$. Suppose that $\alpha \in \tau(w^{-1})$ satisfies $\tau(w^{-1}s_\alpha) \not\subseteq \tau(w^{-1})$. Then

$$\operatorname{Ann}_{U(\mathfrak{g})} L(s_\alpha w\lambda) = \operatorname{Ann}_{U(\mathfrak{g})} L(w\lambda)$$

by [Joseph 1977, Theorem 5.1]; see also [Barbasch and Vogan 1982, Proposition 15]. With this in mind, we define the τ-equivalence on integral weights to be the equivalence relation generated by declaring that

$$\lambda_1 \sim^\tau \lambda_2$$

if there exist an antidominant integral weight λ', and elements $w \in W$ and $\alpha \in \Delta$ such that $\lambda_1 = w\lambda'$, $\lambda_2 = s_\alpha w\lambda'$ and $\tau(w^{-1}s_\alpha) \not\subseteq \tau(w^{-1})$. In fact, the next theorem states that the τ-equivalence is a complete invariant on primitive ideals:

Theorem 4.9 [Garfinkle 1993, Theorem 3.5.9]. *Let $\lambda, \mu \in \mathfrak{t}^*$ be integral weights. Then $\lambda \sim^\tau \mu$ if and only if $\operatorname{Ann}_{U(\mathfrak{g})} L(\lambda) = \operatorname{Ann}_{U(\mathfrak{g})} L(\mu)$.*

We identify the weight $\sum_{i=1}^n a_i\epsilon_i \in \mathfrak{t}^*$ with the list $(a_1, \dots, a_n)$. Then one can check that the τ-equivalence is generated by the following three relations:

(R1) $(a_1, \dots, a_n) \sim^\tau (b_1, \dots, b_n)$ if $(a_1, \dots, a_n) \sim^K (b_1, \dots, b_n)$;

(R2) $(a_1, \dots, a_n) \sim^\tau (a_1, \dots, a_{n-1}, -a_n)$ if $|a_{n-1}| < |a_n|$;

(R3) $(a_1, \dots, a_n) \sim^\tau (a_1, \dots, a_{n-2}, a_n, a_{n-1})$ if $a_{n-1}a_n < 0$.

In (R1), $\sim^K$ denotes Knuth equivalence, as defined in [Fulton 1997, §2].

The references for the results in this section often only deal with the case of regular weights. However, [Jantzen 1983, Lemma 5.6] implies that they are valid for nonregular weights too.

5. The three row case for $\mathfrak{g} = \mathfrak{sp}_{2n}$

Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ and suppose that $\boldsymbol{p}$ has three parts. Then we write $\boldsymbol{p} = (l^2, m)$, where l must be even if $l < m$. In this section we classify the finite-dimensional $U(\mathfrak{g}, e)$-modules, and we use the τ-equivalence to describe the component group action on these modules.

Let C be the component group of e, so

$$C = \begin{cases} \langle c \rangle \cong \mathbb{Z}_2 & \text{if } l \text{ is even and } l \neq m, \\ 1 & \text{otherwise.} \end{cases}$$

The lemma below deals with the (easy) cases where l is even and $l \leq m$, or l is odd (in which case $l > m$):

Lemma 5.1. *Suppose that $A \in \mathrm{sTab}^{\leq}(P)$ and l is even and $l \leq m$, or l is odd. Then $L(A)$ is finite-dimensional if and only if A is justified row-equivalent to column-strict. Furthermore, in the case that l is even and $l < m$, if $L(A)$ is finite-dimensional, then $c \cdot L(A) \cong L(A)$.*

Proof. First, we consider the case where l is even and $l \leq m$. So $\mathrm{content}(l, l, m) = (l/2, l/2, m/2 + 1)$. It is easy to see that the only partition with this content is (l, l, m). Therefore by Theorem 2.4 and Theorem 4.4 we have $L(A)$ is finite-dimensional if and only $\mathrm{part}(\mathrm{RS}(A)) = (l, l, m)$ if and only if A is justified row-equivalent to column-strict. Now the statement about the action of C follows from 4.5.

The case where l is odd is similar. □

So we are left to consider the case where $l > m$ and l is even. Below we explain the action of c on the s-tables corresponding to finite-dimensional $U(\mathfrak{g}, e)$-modules. We need to use the definition of the $\sharp$-special element of a list of integers, which is given in [Brown 2011, §6].

Let $B \in \mathrm{sTab}^{\leq}(P')$ be an s-table for some s-frame P' with an even number of rows. If the $\sharp$-element of the upper-middle row of B is defined, then we let $c'B$ denote the s-table $B' \in \mathrm{sTab}^{\leq}(P')$ where all the rows of B' are the same as B, except that in the upper-middle row the $\sharp$-element is replaced by its negative, and the corresponding change to the lower-middle row is also made; otherwise we say the $c'B$ is undefined.

Let $a_1, \ldots, a_l$ be the entries in the top row, and let $b_1, \ldots, b_{m/2}$ be the entries in the first half of the middle row of A. Let A' be the s-table with 4 rows of lengths $l, m/2, m/2, l$, where the top row has entries $a_1, \ldots, a_l$ and the row below the top row has entries $b_1, \ldots, b_{m/2}$.

The rows of A' are labelled by $1, 2, -2, -1$ from top to bottom. We have the row swapping operators $\bar{s}_i$ from Section 4B acting on A'; for convenience in this section we do not include the ⋆ in the notation. Let $B = \bar{s}_1 c' \bar{s}_1 A'$, provided that it is defined; otherwise, $c \cdot A$ is undefined.

Let $d_1, \dots, d_l$ be the entries in the top row of B and let $e_1, \dots, e_m$ be the entries in the row below the top row of B. If $e_1, \dots, e_m$ are not all negative, then we say that $c \cdot A$ is undefined. Otherwise we declare that $c \cdot A$ is the s-table with row lengths (l, m, l) where the top row has entries $d_1, \dots, d_l$ and the middle row has entries $e_1, \dots, e_m, -e_m, \dots, -e_1$.

For example, if

$$A = \begin{array}{cccc} 2 & 3 & 4 & 5 \\ & -1 & 1 & \\ -5 & -4 & -3 & -2 \end{array} \qquad \text{then} \qquad A' = \begin{array}{cccc} 2 & 3 & 4 & 5 \\ -1 & & & \\ 1 & & & \\ -5 & -4 & -3 & -2 \end{array}$$

So,

$$\bar{s}_1 A' = \begin{array}{cccc} 2 & & & \\ -1 & 3 & 4 & 5 \\ -5 & -4 & -3 & 1 \\ -2 & & & \end{array} \qquad \text{and} \qquad c' \bar{s}_1 A' = \begin{array}{cccc} 2 & & & \\ -5 & -1 & 3 & 4 \\ -4 & -3 & 1 & 5 \\ -2 & & & \end{array}$$

Hence

$$\bar{s}_1 c' \bar{s}_1 A' = \begin{array}{cccc} -5 & 2 & 3 & 4 \\ -1 & & & \\ 1 & & & \\ -4 & -3 & -2 & 5 \end{array} \qquad \text{so} \qquad c \cdot A = \begin{array}{cccc} -5 & 2 & 3 & 4 \\ & -1 & 1 & \\ -4 & -3 & -2 & 5 \end{array}$$

The next lemma follows from [Brown and Goodwin 2013a, Remark 5.8]:

Lemma 5.2. *Let $A \in \mathrm{sTab}^{\leq}(P)$ and suppose $c \cdot A$ is defined. Then* $\mathrm{word}(A) \sim^{\tau} \mathrm{word}(c \cdot A)$.

Our next goal is to prove that $c \cdot A$ is defined when A corresponds to a finite-dimensional $U(\mathfrak{g}, e)$-module:

Lemma 5.3. *Let $A \in \mathrm{sTab}^{\leq}(P)$. If $L(A)$ is finite-dimensional, then $c \cdot A$ is defined.*

Proof. Let $a_1, \dots, a_l$ be the top row of A and let $b_1, \dots, b_{m/2}$ be the first half of the middle row of A. Since $L(A)$ is finite-dimensional, we must have that $\mathrm{content}\big(\mathrm{part}(\mathrm{RS}(A))\big) = \mathrm{content}(l, l, m) = (m/2, l/2, l + 2/2)$. This gives that part(RS(A)) must be (l, l, m), $(l+1, l-1, m)$, or $(l, l-1, m+1)$. The last of these we can rule out by Lemma 4.1. Thus, part(RS(A)) = (l, l, m) or $(l+1, l-1, m)$. In either case, we note that $\bar{s}_1 A'$ is defined; otherwise we would have for some $i \geq 0$ that $a_{l/2-i} < b_{m/2-i}$, in which case we have the increasing subword

$$a_1, \dots, a_{l/2-i}, b_{m/2-i}, \dots, b_{m/2}, -b_{m/2}, \dots, -b_{m/2-i}, -a_{l/2-i}, \dots, -a_1$$

of length $l+2$, which contradicts Lemma 4.1.

Now, suppose that part(RS(A)) $= (l, l, m)$. Then by Theorem 4.4, A is row-equivalent to column-strict, so we have $a_i + a_{m-i+1} > 0$ for all i. Let $a'_1 \leq \cdots \leq a'_m$ be the elements from the top row that best fit over $b_1, \dots, b_{m/2}, -b_{m/2}, \dots, -b_1$. Let $a''_1, \dots, a''_{l-m}$ be the remaining elements of the top row. Then, for $i = 1, \dots, m/2$ we have that $-a'_{m-i+1} < b_i < a'_i$. Since A is row-equivalent to column-strict, we also have that $a''_i + a''_{l-m+1-i} > 0$ for all i. This shows that the $\sharp$-element of $a''_1, \dots, a''_{l-m}, b_1, \dots, b_{m/2}, a'_{m/2+1}, \dots, a'_m$ is defined and is greater than or equal to 0. It also implies that the elements of $(a''_1, \dots, a''_{l-m}, b_1, \dots, b_{m/2}, a'_{m/2+1}, \dots, a'_m)^\sharp$ that best fit under $a'_1, \dots, a'_{m/2}$ are all negative. Thus $c \cdot A$ is defined.

Now, suppose that part(RS(A)) $= (l+1, l-1, m)$. By [Brown and Goodwin 2013a, Lemma 5.6] the $\sharp$-element of row 2 of $\bar{s}_1 A'$ is defined; otherwise, we could find an increasing subword of length $l+2$. Also, the $\sharp$-element must be negative; otherwise, we could not find an increasing subword of length $l+1$ in word($\bar{s}_1 A'$), since the middle two rows of $\bar{s}_1 A'$ would then be column-strict.

Next, we need to prove that the action of $\bar{s}_1$ is defined on $c'\bar{s}_1 A'$. If it was not, then we could find two disjoint increasing strings of length $l+1$ in word($c'\bar{s}_1 A'$), which is a contradiction since word($c'\bar{s}_1 A'$) is τ-equivalent to word(A); compare Theorem 4.9.

Finally, we need to argue why the elements of row 2 of $c'\bar{s}_1 A'$ that best fit under row 1 are all negative. If one the best-fitting elements, say b, was positive, then we could form the decreasing chain $a, b, -b, -a$, where a is any element of row 1 of A' that is larger than b. This contradicts the fact that part(RS($\bar{s}_1 c' \bar{s}_1 A'$)) $= (l, l, m)$ or $(l+1, l-1, m)$. □

We are now ready for the main theorem of this section:

Theorem 5.4. *Suppose that l is even and $l > m$, and let $A \in \mathrm{sTab}^{\leq}(P)$. Then $L(A)$ is finite-dimensional if and only if A is C-conjugate to an s-table that is justified row-equivalent to column-strict. Furthermore, if $L(A)$ is finite-dimensional, then $c \cdot L(A) \cong L(c \cdot A)$.*

Proof. From the proof of the previous lemma we know that if $L(A)$ is finite-dimensional, then part(RS(B)) is (l, l, m) or $(l+1, l-1, m)$. In the former case, A is row-equivalent to column-strict by Theorem 4.4. In the latter case we can see that $c \cdot A$ is row-equivalent to column-strict immediately from Lemma 5.3 and the following observation: Suppose $B \in \mathrm{sTab}^{\leq}(P)$ is such that $\mathrm{part}(\mathrm{RS}(B)) = (l, l, m)$ or $(l+1, l-1, m)$ and the middle two rows of B' are row-equivalent to column-strict. Then $\mathrm{part}(\mathrm{RS}(B)) = (l, l, m)$. Indeed, if we left-justify the top two rows of B' and right-justify the bottom two rows then the resulting diagram is column-strict, so it is impossible to find an increasing chain of length $l+1$.

Now we prove the statement about the action of c. Suppose that $L(A)$ is finite-dimensional and assume that $\mathrm{part}(\mathrm{RS}(A)) = (l, l, m)$. We have that $c \cdot L(A) \cong L(B)$ for some B. If $\mathrm{part}(\mathrm{RS}(B)) = (l, l, m)$, then $A = B$ by Theorem 4.4, and in this case it follows that we have $c \cdot A = B$. If $\mathrm{part}(\mathrm{RS}(B)) = (l+1, l-1, m)$, then $\mathrm{part}(\mathrm{RS}(c \cdot B)) = (l, l, m)$. So by Lemma 5.2, $L(c \cdot B)$ and $L(A)$ are associated to the same primitive ideal of $U(\mathfrak{g})$. Now from this and the fact that $c \cdot B$ and A are both row-equivalent to column-strict, we can deduce, using Section 2B and Theorem 4.5, that $A = c \cdot B$. □

Last in this section we give the following lemma, which we need in the proof of Theorem 1.2:

Lemma 5.5. *If $A \in \mathrm{sTab}^{\leq}(P)$ is row-equivalent to column-strict, then* $\mathrm{word}(c \cdot A)$ *can be obtained from* $\mathrm{word}(A)$ *through a series of Knuth equivalences and larger-smaller transpositions. In particular,* $\mathrm{part}(\mathrm{RS}(A)) \leq \mathrm{part}(\mathrm{RS}(c \cdot A))$.

Proof. This is proven in [Brown and Goodwin 2013a, Remark 5.8]. □

6. The three row case for $\mathfrak{g} = \mathfrak{so}_{2n+1}$

Let $\mathfrak{g} = \mathfrak{so}_{2n}$ and suppose that $\boldsymbol{p}$ has three parts. Then we write $\boldsymbol{p} = (l^2, m)$, where l must be odd if $l > m$. In this section, we classify the finite-dimensional $U(\mathfrak{g}, e)$-modules, and we use the τ-equivalence to describe the component group action on these modules.

Let C be the component group of e, so

$$C = \begin{cases} \langle c \rangle \cong \mathbb{Z}_2 & \text{if } l \text{ is odd and } l \neq m, \\ 1 & \text{otherwise.} \end{cases}$$

The lemma below deals with the (easy) cases where $l > m$ (in which case l must be odd) or $l \leq m$ and is even. The proof is very similar to that of Lemma 5.1, so it is omitted.

Lemma 6.1. *Suppose that l is even, or l is odd and $l \geq m$. Let $A \in \mathrm{sTab}^{\leq}(P)$. Then $L(A)$ is finite-dimensional if and only if A is justified row-equivalent to column-strict. Furthermore, in the case that l is odd and $l > m$, if $L(A)$ is finite-dimensional, then $c \cdot L(A) \cong L(A)$.*

So we are left to consider the case where l is odd and $m > l$; in this case, we let $l = 2p+1$ and $m = 2q+1$, where $q > p$. In the next few paragraphs we set up the combinatorics to describe the action of c on elements of $\mathrm{sTab}^{\leq}(P)$ corresponding to finite-dimensional representations.

Let $A \in \mathrm{sTab}^{\leq}(P)$. Let $a_1, \dots, a_{2p+1}$ be the top row of A and let $b_1, \dots, b_q, 0, -b_q, \dots, -b_1$ be the middle row. From A, we define two tables, A^{L^+} and A^{L^-}, in the following manner: A^{L^+} is the left-justified three row table with row 1 equal to $a_1, \dots, a_{p+1}$, row 2 equal to $b_1, \dots, b_q$, and row 3 equal to $-a_{2p+1}, \dots, -a_{p+2}$. A^{L^-} is the left-justified three row table with row 1 equal to $a_1, \dots, a_p$, row 2 equal to $b_1, \dots, b_q$, and row 3 equal to $-a_{2p+1}, \dots, -a_{p+1}$.

We define the C-action on A in the following manner depending on the cases below. Here we use the row swapping operations for tables mentioned in Section 4B, and we omit the $\star$ in the notation for convenience.

Case 1: If A^{L^-} is row-equivalent to column-strict, then we define $c \cdot A = B$, where B is the unique s-table in $\mathrm{sTab}^{\leq}(P)$ such that $B^{L^+} = s_2 s_1 s_2 A^{L^-}$.

Case 2: If A^{L^-} is not row-equivalent to column-strict but A^{L^+} is row-equivalent to column-strict, then we define $c \cdot A = B$, where B is the unique s-table in $\mathrm{sTab}^{\leq}(P)$ such that $B^{L^-} = s_2 s_1 s_2 A^{L^+}$, provided that such an s-table exists; note that B exists precisely when $s_1 s_2 A^{L^+}$ contains only negative numbers in row 2, and this will *not* happen if A^{L^-} is row-equivalent to column-strict. If such a B does not exist, then we say that $c \cdot A$ is not defined.

Case 3: If neither A^{L^-} nor A^{L^+} is row-equivalent to column-strict, then we say that $c \cdot A$ is undefined.

For example, suppose that

$$A = \begin{array}{ccccc} & \boxed{-2} & \boxed{5} & \boxed{6} & \\ \boxed{-3} & \boxed{-1} & \boxed{0} & \boxed{1} & \boxed{3} \\ & \boxed{-6} & \boxed{-5} & \boxed{2} & \end{array}$$

Then

$$A^{L^+} = \begin{array}{cc} \boxed{-2} & \boxed{5} \\ \boxed{-3} & \boxed{-1} \\ \boxed{-6} & \end{array} \qquad \text{and} \qquad A^{L^-} = \begin{array}{cc} \boxed{-2} & \\ \boxed{-3} & \boxed{-1} \\ \boxed{-6} & \boxed{-5} \end{array}$$

Since A^{L^-} is column-strict, we are in Case 1. Now

$$s_2 s_1 s_2 A^{L^-} = \begin{array}{cc} \boxed{-2} & \boxed{-1} \\ \boxed{-6} & \boxed{-3} \\ \boxed{-5} & \end{array}$$

so

$$c \cdot A = \begin{array}{|c|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & -2 & -1 & 5 & \multicolumn{1}{c}{} \\ \hline -6 & -3 & 0 & 3 & 6 \\ \hline \multicolumn{1}{c|}{} & -5 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \end{array}$$

We need to prove that word(A) is τ-equivalent to word$(c \cdot A)$. To do this, we need the following lemmas:

Lemma 6.2. *Let $a, b_1, \dots, b_m$ be such that $a > 0$, $b_1 < \dots < b_m < 0$ and $-a < b_m$. Then*

$$(a, b_1, \dots, b_m) \sim^\tau (b_1, , \dots, b_m, -a).$$

Proof. By applying the Robinson–Schensted algorithm we see that $(a, b_1, \dots, b_m)$ is Knuth-equivalent to $(b_1, \dots, b_{m-1}, a, b_m)$. By applying the relations (R3) then (R2) from the definition of the τ-equivalence, we get that this is τ-equivalent to $(b_1, \dots, b_m, -a)$. □

For positive integers k, m, we define an operation $\mathrm{LT}_{k,m}$ on certain lists. Suppose that $(a_1, \dots, a_l, b_1, \dots, b_m)$ is a list such that $l \geq 2k-1$, $m \geq k$, $b_m < 0$, $a_{l-k} > 0$, and the table

$$B = \begin{array}{|c|c|c|c|c|c|c|} \cline{1-4} a_{l-2k+2} & a_{l-2k+1} & \dots & a_{l-k} & \multicolumn{3}{c}{} \\ \hline b_1 & b_2 & \dots & b_{k-1} & b_k & \dots & b_m \\ \hline -a_l & -a_{l-1} & \dots & -a_{l-k+2} & -a_{l-k+1} & \multicolumn{2}{c}{} \\ \cline{1-5} \end{array} \tag{6.3}$$

is row-equivalent to column-strict with increasing rows. We define

$$\mathrm{LT}_{k,m}(a_1, \dots, a_l, b_1, \dots, b_m)$$

to be the list $(a_1, \dots, a_{l-2k+1})$ concatenated with word(B). For example, if $A \in \mathrm{sTab}^{\leq}(P)$ is justified row-equivalent to column-strict, P has row lengths $(2p+1, 2q+1, 2p+1)$, and

$$\mathrm{word}(A) = (a_1, \dots, a_{2p+1}, b_1, \dots, b_{2q+1}, 0, -b_{2q+1}, \dots, -b_1, -a_{2p+1}, \dots, -a_1),$$

then $\mathrm{LT}_{p+1,2q+1}(a_1, \dots, a_{2p+1}, b_1, \dots, b_{2q+1}) = \mathrm{word}(A^{L^-})$.

We would also like to explicitly describe $\mathrm{LT}_{k,m}^{-1}$. This will be defined on lists of the form $(a_1, \dots, a_l, b_1, \dots, b_m, c_1, \dots, c_k)$, where $m \geq k$, $l \geq k-1$, $c_k < 0$, $-c_k > a_l$, $b_m < 0$, and the following table is row-equivalent to column-strict with increasing rows:

$$B = \begin{array}{|c|c|c|c|c|c|c|} \cline{1-4} a_{l-k+2} & a_{l-k+1} & \dots & a_l & \multicolumn{3}{c}{} \\ \hline b_1 & b_2 & \dots & b_{k-1} & b_k & \dots & b_m \\ \hline c_1 & c_2 & \dots & c_{k-1} & c_k & \multicolumn{2}{c}{} \\ \cline{1-5} \end{array} \tag{6.4}$$

Now

$$\mathrm{LT}_{k,m}^{-1}(a_1,\dots,a_l,c_1,\dots,c_k,b_1,\dots,b_m)=(a_1,\dots,a_l,-c_k,\dots,-c_1,b_1,\dots,b_m).$$

We say that $\mathrm{LT}_{k,m}^{-1}(a_1,\dots,a_l,b_1,\dots,b_m,c_1,\dots,c_k)$ is undefined if any of the above conditions is not met.

Lemma 6.5. *Let* $(a_1,\dots,a_l,b_1,\dots,b_m)$ *be a list on which* $\mathrm{LT}_{k,m}$ *is defined. Then*

$$(a_1,\dots,a_l,b_1,\dots,b_m)\sim^\tau \mathrm{LT}_{k,m}(a_1,\dots,a_l,b_1,\dots,b_m).$$

Proof. We may assume that $l=2k-1$. We proceed by induction on k. The case $k=1$ is given by Lemma 6.2. Now, since

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
a_1 & a_2 & \dots & a_{k-1} & \multicolumn{3}{c}{} \\
\hline
b_1 & b_2 & \dots & b_{k-1} & b_k & \dots & b_m \\
\hline
-a_{2k-1} & -a_{2k-2} & \dots & -a_{k+1} & -a_k & \multicolumn{2}{c}{} \\
\cline{1-5}
\end{array} \tag{6.6}$$

is row-equivalent to column-strict, we also have that

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
a_3 & a_4 & \dots & a_k & \multicolumn{3}{c}{} \\
\hline
b_1 & b_2 & \dots & b_{k-2} & b_{k-1} & \dots & b_m \\
\hline
-a_{2k-1} & -a_{2k-2} & \dots & -a_{k+2} & -a_{k+1} & \multicolumn{2}{c}{} \\
\cline{1-5}
\end{array}$$

is row-equivalent to column-strict. So by induction $(a_1,\dots,a_l,b_1,\dots,b_m)$ is τ-equivalent to $\mathrm{LT}_{k-1,m}(a_1,\dots,a_l)=(a_1,\dots,a_k,b_1,\dots,b_m,-a_{2k-1},\dots,-a_{k+1})$. Now let $b_{i_1},\dots,b_{i_{k-1}}$ be the elements of $b_1,\dots,b_m$ that best fit over $-a_{2k-1},\dots,$ $-a_{k+1}$. Thus

$$\begin{aligned}(a_1,\dots,a_k,b_1,\dots,b_m,-a_{2k-1},\dots,-a_{k+1})&\\ \sim^K (a_1,\dots,a_k,b_{i_1},\dots,b_{i_{k-1}},a'_1,\dots,a'_m),&\end{aligned} \tag{6.7}$$

where $\sim^K$ denotes Knuth equivalence and $(a'_1,\dots,a'_m)$ is the sorted list consisting of $-a_{2k-1},\dots,-a_{k+1}$ and $\{b_l\mid l\neq i_j \text{ for } j=1,\dots,k-1\}$. Now, from (6.6) we can see that $b_{i_1},\dots,b_{i_{k-1}}$ best fits under $a_1,\dots,a_{k-1}$, so

$$\begin{aligned}&(a_1,\dots,a_k,b_{i_1},\dots,b_{i_{k-1}},a'_1,\dots,a'_m)\\ &\qquad\sim^K (a_1,\dots,a_{k-1},b_{i_1},\dots,b_{i_{k-1}},a_k,a'_1,\dots,a'_m).\end{aligned}$$

We also get from (6.6) that $a'_m=-b_m$, so by Lemma 6.2 we have that

$$\begin{aligned}&(a_1,\dots,a_{k-1},b_{i_1},\dots,b_{i_{k-1}},a_k,a'_1,\dots,a'_m)\\ &\qquad\sim^\tau (a_1,\dots,a_{k-1},b_{i_1},\dots,b_{i_{k-1}},a'_1,\dots,a'_m,-a_k).\end{aligned}$$

Finally we can use the Knuth equivalence in (6.7) to get that this is Knuth-equivalent to

$$(a_1,\dots,a_{k-1},b_1,\dots,b_m,-a_{2k-1},\dots,-a_k). \qquad \square$$

Lemma 6.8. *Suppose that we are given a skew-symmetric word*

$$w = (a, b_1, \dots, b_m, c, 0, -c, -b_m, \dots, -b_1, -a)$$

such that $\operatorname{part}(\mathrm{RS}(a, b_1, \dots, b_m, c)) = (m, 1, 1)$, $b_1 < b_2 < \cdots < b_m < 0$, $c < 0$, *and* $-c > a$. *Then*

$$w \sim^K (a, b_1, \dots, b_m, -c, 0, c, -b_m, \dots, -b_1, -a).$$

Proof. Calculate

$$\mathrm{RS}(a, b_1, \dots, b_m, c, 0, -c, -b_m) \quad \text{and} \quad \mathrm{RS}(a, b_1, \dots, b_m, -c, 0, c, -b_m),$$

then observe that they are equal. □

Lemma 6.9. *Suppose that we are given a skew-symmetric word*

$$w = (a_1, \dots, a_l, b_1, \dots, b_m, c_1, \dots, c_k, 0, -c_k, \dots, -c_1, -b_m, \dots, -b_1, -a_l, \dots, -a_1)$$

such that $k \le l \le m$, $a_1 < a_2 < \cdots < a_l$, $b_1 < b_2 < \cdots < b_m < 0$, $c_1 < c_2 < \cdots < c_k < 0$, $-c_k > a_l$, *and* $\operatorname{part}(\mathrm{RS}(a_1, \dots, a_l, b_1, \dots, b_m, c_1, \dots, c_k)) = (m, l, k)$. *Then*

$$w \sim^K (a_1, \dots, a_l, b_1, \dots, b_m, -c_k, \dots, -c_1, 0, c_1, \dots, c_k, -b_m, \dots, -b_1, -a_l, \dots, -a_1)$$

and

$$(a_1, \dots, a_l, b_1, \dots, b_m, c_1, \dots, c_k) \sim^\tau (a_1, \dots, a_l, b_1, \dots, b_m, -c_k, \dots, -c_1).$$

Proof. We prove this by induction on k. The case $k = 1$ is given by Lemma 6.8 and condition (R2) in the definition of the τ-equivalence.

To prove the general case, first we best-fit $c_1, \dots, c_{k-1}$ under $b_1, \dots, b_m$, which gives that

$$(b_1, \dots, b_m, c_1, \dots, c_{k-1}) \sim^K (b_{i_1}, \dots, b_{i_{k-1}}, c'_1, \dots, c'_m). \tag{6.10}$$

Now we can best fit $b_{i_1}, \dots, b_{i_{k-1}}$ under $a_1, \dots, a_l$ to get that

$$(a_1, \dots, a_l, b_{i_1}, \dots, b_{i_{k-1}}) \sim^K (a_{i'_1}, \dots, a_{i'_{k-1}}, b'_1, \dots, b'_l). \tag{6.11}$$

Putting this all together, we get that w is Knuth-equivalent to

$$\begin{aligned}(a_{i'_1}, \dots, a_{i'_{k-1}}, b'_1, \dots, b'_l, c'_1, \dots, c'_m, c_k, 0,&\\ -c_k, -c'_m, \dots, -c'_1, -b'_l, \dots, -b'_1, -a_{i'_{k-1}}, \dots, -a_{i'_1}&).\end{aligned}$$

Since $\operatorname{part}(\mathrm{RS}(a_1, \dots, a_l, b_1, \dots, b_m, c_1, \dots, c_k)) = (m, l, k)$, we can deduce that $b'_l = a_l$ and $c'_m = b_m$. We can also use this to deduce that the element of $(b_1, \dots, b_m)$ that best fits over c_k is an element of $(c'_1, \dots, c'_m)$. Now, we apply

Lemma 6.8 to the part of this word between b'_l and $-b'_l$ to get that this is Knuth-equivalent to

$$(a_{i'_1}, \dots, a_{i'_{k-1}}, b'_1, \dots, b'_l, c'_1, \dots, c'_m, -c_k, 0, \\ c_k, -c'_m, \dots, -c'_1, -b'_l, \dots, -b'_1, -a_{i'_{k-1}}, \dots, -a_{i'_1}).$$

We can apply the Knuth equivalences in (6.10) and (6.11) to get that this is Knuth-equivalent to

$$(a_1, \dots, a_l, b_1, \dots, b_m, c_1, \dots, c_{k-1}, -c_k, 0, \\ c_k, -c_{k-1}, \dots, -c_1, -b_m, \dots, -b_1, -a_l, \dots, -a_1).$$

Now, we can best fit $b_{m-k+2}, \dots, b_m$ over $c_1, \dots, c_{k-1}, -c_k$ to get

$$(b_{m-k+2}, \dots, b_m, c_1, \dots, c_{k-1}, -c_k) \sim^K (b_{m-k+2}, \dots, b_m, -c_k, c_1, \dots, c_{k-1}).$$

Next, we can best fit $a_1, \dots, a_l$ over $b_1, \dots, b_m, -c_k$ to get that

$$(a_1, \dots, a_l, b_1, \dots, b_m, -c_k) \sim^K (a'_1, \dots, a'_m, -c_k, b_{j_1}, \dots, b_{j_l}). \tag{6.12}$$

So we have that

$$(a_1, \dots, a_l, b_1, \dots, b_m, c_1, \dots, c_{k-1}, -c_k, 0, \\ c_k, -c_{k-1}, \dots, -c_1, -b_m, \dots, -b_1, -a_l, \dots, -a_1),$$

and therefore w, is Knuth-equivalent to

$$(a'_1, \dots, a'_m, -c_k, b_{j_1}, \dots, b_{j_l}, c_1, \dots, c_{k-1}, 0, \\ -c_{k-1}, \dots, -c_1, -b_{j_l}, \dots, -b_{j_1}, c_k, -a'_m, \dots, -a'_1).$$

By induction this is Knuth-equivalent to

$$(a'_1, \dots, a'_m, -c_k, b_{j_1}, \dots, b_{j_l}, -c_{k-1}, \dots, -c_1, 0, \\ c_1, \dots, c_{k-1}, -b_{j_l}, \dots, -b_{j_1}, c_k, -a'_m, \dots, -a'_1).$$

Finally, by applying the Knuth equivalence (6.12), we get that this is Knuth-equivalent to

$$(a_1, \dots, a_l, , b_1, \dots, b_m, -c_k, \dots, -c_1, 0, c_1, \dots, c_k, -b_m, \dots, -b_1, -a_l, \dots, -a_1).$$

□

Theorem 6.13. *Let $A \in \mathrm{sTab}^{\leq}(P)$ be justified row-equivalent to column-strict. Let*

$$(a_1, \dots, a_{q+1}, b_1, \dots, b_p, -a_{2q+1}, \dots, -a_{q+2}) = \mathrm{word}(s_2 s_1 s_2 A^{L^-}).$$

Then

$$(a_1, \dots, a_{q+1}, b_1, \dots, b_p, -a_{2q+1}, \dots, -a_{q+2}, 0, \\ a_{q+2}, \dots, a_{2q+1}, -b_p, \dots, -b_1, -a_{q+1}, \dots, -a_1)$$

is Knuth-equivalent to $\operatorname{word}(c \cdot A)$. *In particular, this implies that* $\operatorname{word}(A)$ *is* τ*-equivalent to* $\operatorname{word}(c \cdot A)$.

Proof. By Lemma 6.9 we have that

$$(a_1, \dots, a_{q+1}, b_1, \dots, b_p, -a_{2q+1}, \dots, -a_{q+2}, 0, \\ a_{q+2}, \dots, a_{2q+1}, -b_p, \dots, -b_1, -a_{q+1}, \dots, -a_1)$$

is Knuth-equivalent to

$$(a_1, \dots, a_{q+1}, b_1, \dots, b_p, a_{q+2}, \dots, a_{2q+1}, 0, \\ -a_{2q+1}, \dots, -a_{q+2}, -b_p, \dots, -b_1, -a_{q+1}, \dots, -a_1).$$

Now, if $b_{i_1}, \dots, b_{i_{q+1}}$ best fits under $a_1, \dots, a_{q+1}$, then we get that this is Knuth-equivalent to

$$(a'_1, \dots, a'_p, a_{q+2}, \dots, a_{2q+1}, b_{i_1}, \dots, b_{i_{q+1}}, 0, \\ -b_{i_{q+1}}, \dots, -b_{i_1}, -a_{2q+1}, \dots, -a_{q+2}, -a'_p, \dots, -a'_1).$$

Note that $a'_p = a_{q+1}$ or $a'_p = b_j < 0$ for some j, so in either case we can best fit

$$b_{i_1}, \dots, b_{i_{q+1}}, 0, -b_{i_{q+1}}, \dots, -b_{i_3} \quad \text{under} \quad a'_1, \dots, a'_p, a_{q+2}, \dots, a_{2q+1}$$

to get that

$$(a'_1, \dots, a'_p, a_{q+2}, \dots, a_{2q+1}, b_{i_1}, \dots, b_{i_{q+1}}, 0, \\ -b_{i_{q+1}}, \dots, -b_{i_1}, -a_{2q+1}, \dots, -a_{q+2}, -a'_p, \dots, -a'_1)$$

is Knuth-equivalent to

$$(a_1, \dots, a_{2q+1}, b_1, \dots, b_m, 0, -b_{i_{q+1}}, \dots, -b_{i_1}, -a_{2q+1}, \dots, -a_{q+2}, -a'_p, \dots, -a'_1).$$

Now we can best fit

$$b_{m-q+2}, \dots, b_m, 0, -b_{i_{q+1}}, \dots, -b_{i_1} \quad \text{over} \quad -a_{2q+1}, \dots, -a_{q+2}, -a'_p, \dots, -a'_1$$

to get that

$$(a_1, \dots, a_{2q+1}, b_1, \dots, b_m, 0, -b_{i_{q+1}}, \dots, -b_{i_1}, -a_{2q+1}, \dots, -a_{q+2}, -a'_p, \dots, -a'_1)$$

is Knuth-equivalent to

$$(a_1, \dots, a_{2q+1}, b_1, \dots, b_m, 0, -b_m, \dots, -b_1, -a_{2q+1}, \dots, -a_1). \qquad \square$$

Our goal is to prove that $L(A)$ is finite-dimensional if and only if A is C-conjugate to a row-equivalent to column-strict diagram. The following lemmas build up to this:

Lemma 6.14. *Let* $A \in \operatorname{sTab}^{\le}(P)$. *If* A^{L^-} *is row-equivalent to column-strict, then so is* A.

Proof. Recall that A has row lengths given by $(2p+1, 2q+1, 2p+1)$. By permuting entries within rows, we can find p strictly decreasing columns of A^{L^-} Furthermore, the entry in the bottom row of A^{L^-} that is not in one of these columns must be negative. By putting this entry below 0 in A and its negation above 0, we can find a row equivalence class of A where every column left of 0 contains one of the decreasing columns from A^{L^-}, and every column right of zero is the reverse of the negation of one of the columns left of 0. Thus, every column in this element of the row equivalence class of A is strictly decreasing. □

Lemma 6.15. *Let* $A \in \mathrm{sTab}^{\leq}(P)$, *and let* $\boldsymbol{q} = \mathrm{part}(\mathrm{RS}(A))$. *If* $\mathrm{content}(\boldsymbol{q}) = \mathrm{content}(\boldsymbol{p})$, *then* $\boldsymbol{q} = (2q+1, 2p+1, 2p+1)$ *or* $\boldsymbol{q} = (2q+1, 2p+2, 2p)$.

Proof. Note that $\mathrm{content}(2q+1, 2p+1, 2p+1) = (p, p+1, q+1)$, and the only other partition with this content is $(2q, 2p+2, 2p+1)$. Now, by Lemma 4.1, $\mathrm{part}(\mathrm{RS}(A)) \geq (2q+1, 2p+1, 2p+1)$, thus $\mathrm{part}(\mathrm{RS}(A)) \neq (2q, 2p+2, 2p+1)$. □

By Theorem 4.4 we have that if $\mathrm{part}(\mathrm{RS}(A)) = (2q+1, 2p+1, 2p+1)$, then A is row-equivalent to column-strict. So we need only consider the case that $\mathrm{part}(\mathrm{RS}(A)) = (2q+1, 2p+2, 2p)$.

Lemma 6.16. *Let* $A \in \mathrm{sTab}^{\leq}(P)$ *with* $\mathrm{part}(\mathrm{RS}(A)) = (2q+1, 2p+2, 2p)$. *Then*:

(1) A^{L^+} *is row-equivalent to column-strict.*

(2) *The middle row of* $s_2s_1s_2A^{L^+}$ *contains only negative numbers.*

(3) *The negation of the element in the bottom-right position of* $s_2s_1s_2A^{L^+}$ *is larger than the element in the upper-right position of* $s_2s_1s_2A^{L^+}$. *Thus* $c \cdot A$ *is defined.*

(4) $c \cdot A$ *is row-equivalent to column-strict.*

Proof. Let $a_{-p}, \ldots, a_{-1}, a_0, a_1, \ldots, a_p$ be the increasing entries in the first row of A, and let $-b_q, \ldots, -b_1, 0, b_1, \ldots, b_q$ be the middle row of A.

First we prove that $a_{-p}, \ldots, a_0$ must best fit over $-b_q, \ldots, -b_1$. If it does not, then there must exists $i \in \{0, \ldots, p\}$ such that $a_{-(p-i)} < -b_{q-i}$. Thus we can form the following increasing string in $\mathrm{word}(A)$:

$$a_{-p}, \ldots, a_{-(p-i)}, -b_{q-i}, \ldots, -b_1, 0, b_1, \ldots, b_{q-i}, -a_{-(p-i)}, \ldots, -a_{-p}.$$

This string has length $2q+3$, which contradicts $\mathrm{part}(\mathrm{RS}(A)) = (2q+1, 2p+2, 2p)$.

Next we prove that $a_1, \ldots, a_p$ best fits over $b_1, \ldots, b_q$. If it does not, then there exists $i \in \{1, \ldots, p\}$ such that $a_i < b_i$. Thus we can form the following increasing string in $\mathrm{word}(A)$:

$$a_{-p}, \ldots, a_0, \ldots, a_i, b_i, \ldots, b_q.$$

This string has length $p+q+2$, and we can use it to find the following increasing string of length $2p+2q+4$ in $\mathrm{word}(A)$:

$$a_{-p}, \ldots, a_0, \ldots, a_i, b_i, \ldots, b_q, -b_q, \ldots, -b_i, -a_1, \ldots, -a_0, \ldots, -a_{-p}.$$

This contradicts part(RS$(A)) = (2q+1, 2p+2, 2p)$.

Now, we assume for a contradiction that A^{L^+} is not row-equivalent to column-strict. Let $j_0, \dots, j_p$ be positive integers such that $-b_{j_p}, \dots, -b_{j_0}$ best fit under $a_{-p}, \dots, a_0$. Let i be the smallest nonnegative integer such that $-b_{j_i} < -a_{i+1}$. Such an i must exist, since otherwise A^{L^+} will be row-equivalent to column-strict. Define $b_0 = 0$. Now let k be the smallest integer such that

(1) $0 \le k \le i$;

(2) $j_{i-l} = j_i - l$ if $0 < l \le k$;

(3) $j_{i-k-1} \neq j_{i-k} - 1$.

This implies that $a_{-(i-k)} < -b_{j_{i-k}-1}$. So we can form the following two disjoint increasing substrings in word(A):

$$a_{-p}, \dots, a_{-(i-k)}, -b_{j_{i-k}-1}, \dots, b_{-1}, 0, b_1, \dots, b_q$$

and

$$-b_q, \dots, -b_{j_i}, -a_{i+1}, \dots, -a_1, -a_0, -a_{-1}, \dots, -a_{-p}.$$

The first string has length $p - i + k + 1 + j_{i-k} - 1 + 1 + q = p + q - i + k + j_{i-k} + 1$. The second string has length $q - j_i + 1 + i + 1 + p + 1 = q + p - j_i + i + 3$. Thus, using the fact that $j_{i-k} = j_i - k$, the combined length of these two strings is $2q + 2p + 4$, which contradicts part(RS$(A)) = (2q+1, 2p+2, 2p)$. Thus A^{L^+} is row-equivalent to column-strict.

Finally we need to prove that the middle row of $s_2 s_1 s_2 A^{L^+}$ contains only negative numbers. Let $j_1, \dots, j_p$ be such that $-b_{j_p}, \dots, -b_{j_1}$ best fit over $-a_p, \dots, -a_1$. Now it is clear that all the numbers in the last row of $s_2 A^{L^+}$ are negative. Now let a' be the entry in the first row of A^{L^+} that does not best fit over $-b_{j_p}, \dots, -b_{j_1}$. If $a' > 0$, then since all the $-b_i$ are negative we must have that $a' = a_0$. In this case, for $i = 1, \dots, p$, $(a_{-i}, -b_{j_i}, -a_i)$ is a decreasing string in word(A^{L^+}) and in word(A). Furthermore, reversing and negating these strings yields a further p disjoint deceasing strings of length 3 in word(A). These and the string $(a_0, 0, -a_0)$ show that part(RS$(A))^T$ is larger than a partition of the form $(3^{2p+1}, *)$. This contradicts part(RS$(A)) = (2q+1, 2p+2, 2p)$. So we have that $a' < 0$, and furthermore the middle row of $s_1 s_2 A^{L^+}$ contains only negative numbers. Now since the last row of $s_1 s_2 A^{L^+}$ also contains all negative numbers, we have that the middle row of $s_2 s_1 s_2 A^{L^+}$ contains only negative numbers.

Now, let x be the element in the upper-right position of $s_2 s_1 s_2 A^{L^+}$ and let y be the element in the lower-right position. We need to show that $x < -y$. If $a_0 < 0$ then this is clear, since in this case every element of A^{L^+} is negative. When $a_0 > 0$, we need to consider the bottom row of $s_2 s_1 s_2 A^{L^+}$. This row will contain $-a_n, \dots, -a_1$ and also $-b_i$, where $-b_i$ is not one of the elements of $-b_m, \dots, -b_1$ that best fits over $-a_n, \dots, -a_1$. Let $-b_{k_n}, \dots, -b_{k_1}$ be as above, i.e., the elements which best fit over $-a_n, \dots, -a_1$. Note that $-b_{k_n}, \dots, -b_{k_1}$ are the elements in the middle row of $s_2 A^{L^+}$. Now, let a' be the element in the first row of A^{L^+} that is one of the elements which best fit over $-b_{k_n}, \dots, -b_{k_1}$, so the middle row of $s_1 s_s A^{L^+}$ contains $-a', -b_{k_n}, \dots, -b_{k_1}$. We have already proved that since $a_0 > 0$, $a' < 0$. So $a' < -b_{k_1}$, since otherwise a_0 would be the element that did not best fit over $-b_{k_n}, \dots, -b_{k_1}$. So $-b_i < a' < -b_{k_1}$. This implies that $-b_i < -a_1$, since otherwise $-b_{k_1}$ would not be the element that best fits over $-a_1$. Thus $-a_1$ is the element in the bottom-right position of $s_2 s_1 s_2 A^{L^+}$, and a_0 is the element in the upper-right position of $s_2 s_1 s_2 A^{L^+}$, and we already have that $a_0 < a_1$.

To see that $c \cdot A$ is row-equivalent to column-strict, simply note that $(c \cdot A)^{L^-} = s_2 s_1 s_2 A^{L^+}$ is row-equivalent to column-strict and apply Lemma 6.14. □

Now we can state the main theorem of this section, which is analogous to Theorem 5.4. The proof is very similar, where Lemma 6.16 plays the role of Lemma 5.3, and so is omitted.

Theorem 6.17. *Suppose that l is odd and $l > m$, and let $A \in \mathrm{sTab}^{\leq}(P)$. Then $L(A)$ is finite-dimensional if and only if A is C-conjugate to an s-table that is justified row-equivalent to column-strict. Furthermore, if $L(A)$ is finite-dimensional, then $c \cdot L(A) \cong L(c \cdot A)$.*

Last in this section, we give the following technical lemma, which is needed in the proof of Theorem 1.2:

Lemma 6.18. *If $A \in \mathrm{sTab}^{\leq}(P)$ is row-equivalent to column-strict, then* $\mathrm{word}(c \cdot A)$ *can be obtained from* $\mathrm{word}(A)$ *through a series of Knuth equivalences and larger-smaller transpositions. In particular,* $\mathrm{part}(\mathrm{RS}(A)) \leq \mathrm{part}(\mathrm{RS}(c \cdot A))$.

Proof. Let

$$(a_1, \dots, a_{2q+1}, b_1, \dots, b_p, 0, -b_p, \dots, -b_1, -a_{2q+1}, \dots, -a_1) = \mathrm{word}(A).$$

Due to Theorem 6.13, since

$$\mathrm{word}(A^{L^-}) = (a_1, \dots, a_q, b_1, \dots, b_p, -a_{2q+1}, \dots, -a_{q+1}),$$

it suffices to show that

$$\begin{aligned}(a_1, \dots, a_q, b_1, \dots, b_p, -a_{2q+1}, \dots, -a_{q+1}, 0,\\ a_{q+1}, \dots, a_{2q+1}, -b_p, \dots, -b_1, -a_q, \dots, -a_1)\end{aligned}$$

can be obtained from $\mathrm{word}(A)$ by a sequence of larger-smaller transpositions and Knuth equivalences. First, we can swap a_{2q+1} with its right neighbour and $-a_{2q+1}$ with its left neighbour repeatedly until we get a word with $a_{2q+1}, 0, -a_{2q+1}$ in the middle; then, we can swap a_{2q+1} with 0, then swap a_{2q+1} with $-a_{2q+1}$, then swap

0 with $-a_{2q+1}$ so that we have $-a_{2q+1}, 0, a_{2q+1}$ in the middle of our word. Now we can repeat this process with a_{2q} and a_{2q}, then a_{2q-1} and $-a_{2q-1}$, and so on. Eventually, since $a_{q+1} > 0$, we will get

$$(a_1, \dots, a_q, b_1, \dots, b_p, -a_{2q+1}, \dots, -a_{q+1}, 0, \\ a_{q+1}, \dots, a_{2q+1}, -b_p, \dots, -b_1, -a_q, \dots, -a_1). \qquad \square$$

7. The general case

Now we return to the case of general $\boldsymbol{p}$ as in (3.1). As usual, P is the symmetric pyramid of $\boldsymbol{p}$, with rows labelled $1, \dots, r, 0, -r, \dots, -1$ from top to bottom.

7A. ***The component group action.*** In this section, we describe the action of the component group C on the subset of $\mathrm{sTab}^{\leq}(P)$ corresponding to finite-dimensional $U(\mathfrak{g}, e)$-modules. The discussion here is completely analogous to the situation for even multiplicity nilpotent elements as described in [Brown and Goodwin 2013c, §5.5], so we are quite brief. We use the notation for the component group C from Section 3D.

The operation of c has been defined on three row s-tables in Section 5 and Section 6, and this can be extended to any s-table by just acting on the middle three rows. To define the action of the c_k, we proceed in exact analogy with [Brown and Goodwin 2013c, §5.5]. That is, we use row swapping operations $\bar{s}_i \star$ to move row i_k to row r, then we apply c, and then we apply the reverse row swaps. So for $A \in \mathrm{sTab}^{\leq}(P)$ and $\tau = \bar{s}_{i_k} \bar{s}_{i_k+1} \dots \bar{s}_{r-1} \in \overline{S}_r$ we have

$$c_k \cdot A = \tau^{-1} \star (c \cdot (\tau \star A)).$$

Of course, this will not be defined for all $A \in \mathrm{sTab}^{\leq}(P)$, but the following proposition can be proved in the same way as Proposition 5.5 of the same reference, and we require Proposition 4.6 for the proof:

Proposition 7.1. *Let $A \in \mathrm{sTab}^{\leq}(P)$, and suppose that $L(A)$ is finite-dimensional. Then $c_k \cdot A$ is defined and $L(c_k \cdot A) \cong c_k \cdot L(A)$.*

7B. ***Proof of main theorem.*** Now we are in a position to prove Theorem 1.2:

Proof of Theorem 1.2. The statement in the theorem about the component group action is given by Proposition 7.1.

Suppose that A is justified row-equivalent to column-strict. Then $L(A)$ is finite-dimensional by Theorems 4.4, 4.8 and 2.4, and thus $b \cdot L(A)$ is finite-dimensional for any $b \in C$ by Proposition 7.1.

We are left to prove that if $L(A)$ is finite-dimensional, then $A \in \mathrm{sTab}^{c}(P)$. We prove this by induction on r. The case $r = 0$ is trivial, and the case $r = 1$ is given by Lemmas 5.1 and 6.1 and Theorems 5.4 and 6.17.

Now, assume that $L(A)$ is finite-dimensional and $r \geq 2$. Using an inductive argument based on "Levi subalgebras" of $U(\mathfrak{g}, e)$, just as in the proof of [Brown and Goodwin 2013a, Theorem 5.13], we may assume that A^2_{-2} is justified row-equivalent to column-strict, where A^2_{-2} denotes the s-table obtained from A by removing rows 1 and -1. Also by Lemma 4.7 we have that A^1_r is justified row-equivalent to column-strict, where A^1_r is the table formed by rows 1 to r of A.

Therefore, we can permute entries in the left justification of A^2_{-2} so that all the columns are strictly decreasing. Furthermore, we can place each of the entries in row 1 of A over a column so each entry is larger than the entry immediately below it. Then we can place each of the entries of row -1 of A under a column in the left justification of A^2_{-2} so that each entry is smaller than the entry above it, and we can do this skew-symmetrically in the sense that if a is an entry in row 1 of A and a is placed over a column whose top entry is b, then we can place $-a$ under a column whose bottom entry is $-b$. Let A_l denote the resulting diagram.

Let $\boldsymbol{q} = \mathrm{part}(\mathrm{RS}(A))$. As explained below, the conditions above along with Theorem 4.8 give restrictions on the possibilities for $\boldsymbol{q}$. The proof is completed with combinatorial arguments that show that either $\boldsymbol{q} = \boldsymbol{p}$ or $i_1 = 1$, and that $\mathrm{part}(\mathrm{RS}(c_1 \cdot A)) = \boldsymbol{p}$. So by Theorem 4.4, either A or $c_1 \cdot A$ is row-equivalent to column-strict.

In the diagram A_l, let x be the number of columns that go through all the rows, let y be the number of columns that go through all the rows except the top row (so y is also the number of columns that go through all the rows except the bottom row), and let z be the number of columns that go through all the rows except the top and bottom row. Further, let u be the number of columns that go through all the rows except the middle row, let v be the number of columns that go through all the rows except the top row and the middle row (so v is also the number of columns that go through all the rows except the middle row and the bottom row), and let w be the number of columns that go through all the rows except the top, middle and bottom rows. Note that $x+y+u+v = p_1$ and $x+2y+z+u+2v+w = p_2$. So we have x strictly decreasing columns of length $2r+1$, $2y+u$ strictly decreasing columns of length $2r$, $z+2v$ strictly decreasing columns of length $2r-1$, and w strictly decreasing columns of length $2r-2$.

By counting the lengths of the other columns in A_l similarly, and using Lemma 4.2, we can conclude that

$$\boldsymbol{q}^T \geq ((2r+1)^x, (2r)^{2y+u}, (2r-1)^{z+2v}, (2r-2)^w, (2r-4)^{p_3-p_2}, \dots, 2^{p_{r-1}-p_r})$$

if $p_0 \leq p_{r-1}$, and

$$\begin{aligned}\boldsymbol{q}^T \geq \big((2r+1)^x, (2r)^{2y}, (2r-1)^z, (2r-3)^{p_3-p_2}, \dots, (2r-2k+5)^{p_{k-1}-p_{k-2}},\\ (2r-2k+3)^{p_0-p_k-1}, (2r-2k+2)^{p_k-p_0}, (2r-2k)^{p_{k+1}-p_k}, \dots, 2^{p_{r-1}-p_r}\big)\end{aligned}$$

if $p_0 > p_{r-1}$, where k is the number such that $p_i \geq p_0$ if and only if $i \geq k$ (note that in this case we have $u = v = w = 0$).

Thus we get that

$$\boldsymbol{q} \leq (p_r^2, p_{r-1}^2, \dots, p_2^2, p_2 - w, x + 2y + u, x) \quad \text{if } p_0 \leq p_{r-1}$$

and

$$\boldsymbol{q} \leq (p_r^2, p_{r-1}^2, \dots, p_{k-1}^2, p_0, p_k^2, \dots, p_2^2, p_1 - z, x) \quad \text{if } p_0 > p_{r-1}.$$

Since we also have $\boldsymbol{p}$ as a lower bound of $\boldsymbol{q}$, this implies that

$$\boldsymbol{q} = (p_r^2, p_{r-1}^2, \dots, p_2^2, a, b, c) \quad \text{if } p_0 \leq p_{r-1}$$

and

$$\boldsymbol{q} = (p_r^2, p_{r-1}^2, \dots, p_k^2, p_0, p_{k-1}^2, \dots, p_2^2, a, b) \quad \text{if } p_0 > p_{r-1},$$

for positive integers a, b, c. Since $\operatorname{content}(\boldsymbol{q}) = \operatorname{content}(\boldsymbol{p})$, we get a very limited number of possibilities for a, b, c, as explained below.

From now we restrict to the case $\mathfrak{g} = \mathfrak{sp}_{2n}$ in this proof, as the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$ is entirely similar; in some places we would require references from Section 6 rather than Section 5.

We know that p_0 must be even. If $p_0 < p_1$ and p_1 is even, then we must have $(a, b, c) = (p_1, p_1, p_0)$ or $(a, b, c) = (p_1 + 1, p_1 - 1, p_0)$. If $p_0 < p_1$ and p_1 is odd, then $(a, b, c) = (p_1, p_1, p_0)$. If $p_1 < p_0 < p_2$, then p_1 is even and $(a, b, c) = (p_0, p_1, p_1)$. Finally, if $p_0 > p_2$, then p_1 is even and $(b, c) = (p_1, p_1)$.

By Theorem 4.4, if $\boldsymbol{q} = \boldsymbol{p}$, then A is justified row-equivalent to column-strict, and we are done. So for the rest of this proof we will assume that $\boldsymbol{q} \neq \boldsymbol{p}$; so, we are assuming that $p_0 < p_r$, p_r is even, and

$$\boldsymbol{q} = (p_r^2, \dots, p_2^2, p_1 + 1, p_1 - 1, p_0). \tag{7.2}$$

It is be useful to record that

$$\boldsymbol{q}^T = \big((2r+1)^{p_0}, (2r)^{p_1 - p_0 - 1}, (2r-1)^2, (2r-2)^{p_2 - p_1 - 1}, (2r-4)^{p_3 - p_2}, \dots, 2^{p_{r-1} - p_r}\big). \tag{7.3}$$

Let $\sigma = s_{r-1} \dots \bar{s}_2 \bar{s}_1$ and $A' = \sigma \star A$; then $\operatorname{RS}(A') = \operatorname{RS}(A)$ by Proposition 4.6. Then the lengths of the middle three rows of A' are given by p_1, p_0, p_1. Let B be the middle three rows of A'.

We claim that $\operatorname{part}(\operatorname{RS}(B)) = (p_1 + 1, p_1 - 1, p_0)$. To see this, first we suppose that $\operatorname{part}(\operatorname{RS}(B)) = (p_r, p_r, p_0)$. Then B is justified row-equivalent to column-strict. Now, since $(A')_r^1$ is justified row-equivalent to column-strict, this allows us to find p_0 disjoint decreasing words of length $2r + 1$ that are disjoint from a further $p_1 - p_0$ disjoint decreasing words of length $2r$. Thus, by Lemma 4.2,

$\boldsymbol{q}^T \geq ((2r+1)^{p_0}, (2r)^{p_1-p_0}, *)$, which contradicts (7.3). Now we also cannot have that any part of part(RS(B)) is larger than $p_r + 1$, since then we could use the fact that all the rows of A' are increasing to conclude that part($\boldsymbol{q}$) would be strictly larger than a partition of the form

$$(p_1^2, p_2^2, \dots, p_{r-1}^2, p_r + 1, *),$$

which contradicts (7.2).

Now, we have by Theorem 5.4 that $\mathrm{part}(\mathrm{RS}(c \cdot B)) = (p_0, p_1, p_1)$. We also have by Lemma 5.5 that $\mathrm{part}(\mathrm{RS}(c_1 \cdot A)) \leq \mathrm{part}(\mathrm{RS}(A))$.

We need to argue that we can find enough descending chains of maximal or near maximal length in $c \cdot A'$ to force $\mathrm{RS}(c_1 \cdot A)$ to have shape $\boldsymbol{p}$. We have by Lemma 4.7 that $(c \cdot A')_r^1$ is justified row-equivalent to column-strict. Further, by Theorem 4.4 we have that $c \cdot B$ is justified row-equivalent to column-strict.

We can find p_0 descending strings of length 3 and $p_1 - p_0$ strings of length 2, and all these strings start in row r and end in row $-r$. Since $(c \cdot A')_r^1$ is justified row-equivalent to column-strict, it has p_1 strings of length r ending in row r, and $(c \cdot A')_{-1}^{-r}$ has p_1 strings of length r starting in row -1. So we can glue these strings together along their entries in rows 1 and -1 to obtain p_0 disjoint decreasing strings of length $2r+1$ that are disjoint from $p_1 - p_0$ disjoint decreasing strings of length $2r$. So if $\boldsymbol{q}' = \mathrm{part}(\mathrm{RS}(c \cdot A))$, we can conclude that $\boldsymbol{q}'^T \geq ((2r+1)^{p_0}, (2r)^{p_1-p_0}, *)$, which implies that $\boldsymbol{q}' = \boldsymbol{p}$, so $c_1 \cdot A$ is justified row-equivalent to column-strict, as required. □

Finally, this theorem along with Theorems 2.3 and 4.5 immediately imply the following classification of the primitive ideals with associated variety equal to $\overline{G \cdot e}$:

Corollary 7.4. *The set of primitive ideals with associated variety $\overline{G \cdot e}$ is equal to*

$$\{\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_A) \mid A \in \mathrm{sTab}^{\boldsymbol{c}}(P)\}.$$

Appendix: An alternative version of the Barbasch–Vogan algorithm

In this appendix, we consider the alternative version of the Barbasch–Vogan algorithm for $\mathfrak{so}_{2n+1}$, mentioned in Section 4C above. Our main result is Corollary A.7, which shows that this adapted version gives the same output as the original version. Below, we recall the algorithm, then, in the subsequent subsections construct the machinery required to prove Corollary A.7.

Some terminology and notation used in this section are as follows. By a *Young diagram* we mean a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths weakly decreasing. We often identify a Young diagram with its underlying partition. A tableau is a filling of a Young diagram by integers with weakly increasing rows and strictly decreasing columns. We write part(T) for the partition underlying a tableau T. The Robinson–Schensted algorithm is denoted by RS.

The algorithms. Let $\boldsymbol{q} = (q_1 \le q_2 \le \cdots \le q_m)$ be a partition. By inserting 0 at the beginning if necessary, we may assume that m is odd. Let $(s_1, \ldots, s_k)$, $(t_1, \ldots, t_l)$ be such that

$$\{q_1, q_2 + 1, q_3 + 2, \ldots, q_r + r - 1\} = \{2s_1, \ldots, 2s_k, 2t_1 + 1, \ldots, 2t_l + 1\}$$

(as unordered lists). Now we define the content of $\boldsymbol{q}$ to be the unordered list

$$\operatorname{content}(\boldsymbol{q}) = \{s_1, \ldots, s_k, t_1, \ldots, t_l\}.$$

We now state the Barbasch–Vogan algorithm [1982] for the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$ in purely combinatorial terms:

Algorithm:

Input: $\boldsymbol{a} = (a_1, \ldots, a_n, -a_n, \ldots, -a_1)$ a skew-symmetric string of integers.

Step 1: Calculate $\boldsymbol{q} = \operatorname{part}(\mathrm{RS}(a_1, \ldots, a_n, -a_n, \ldots, -a_1))$.

Step 2: Calculate $\operatorname{content}(\boldsymbol{q})$.
Let $(u_1 \le \cdots \le u_{2k+1})$ be the sorted list with the same entries as $\operatorname{content}(\boldsymbol{q})$.
For $i = 1, \ldots, k + 1$ let $s_i = u_{2i-1}$.
For $i = 1, \ldots, k$ let $t_i = u_{2i}$.

Step 3: Form the list $(2s_1 + 1, \ldots, 2s_{k+1} + 1, 2t_1, \ldots, 2t_k)$.
In either case, let $(v_1 < \cdots < v_k)$ be this list after sorting.

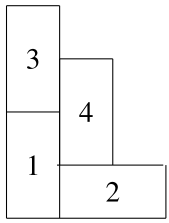
Output: $\mathrm{BV}(\boldsymbol{a}) = \boldsymbol{q}' = (v_1, v_2 - 1, \ldots, v_{2k+1} - 2k)$.

The modified version is denoted by BV' and works in exactly the same way as BV, except that in Step 1 it calculates $\mathrm{RS}(a_1, \ldots, a_n, 0, -a_n, \ldots, -a_1)$ instead of $\mathrm{RS}(a_1, \ldots, a_n, -a_n, \ldots, -a_1)$.

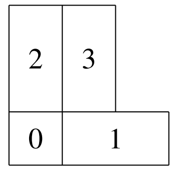
Domino tableaux. We require some facts about domino tableaux, which we collate below.

There are two types of domino tableaux: those with an even number of boxes and those with an odd number of boxes. A *domino tableau with an even number of boxes* is a Young diagram that has been tiled with 2×1 and 1×2 dominoes, where each domino is labelled with a positive integer, such that the rows are increasing and the columns are decreasing. A *domino tableau with an odd number of boxes* is the same as a domino tableau with an even number of boxes, except it also has a 1×1 box labelled with 0, which must necessarily occur in the lower-left position.

For example,



and



are domino tableaux.

Given a domino tableau R, we let $\mathrm{part}(R)$ denote the partition underlying R, i.e., the partition given by the row lengths of R. We say that a partition has *domino shape* if it is the underlying partition of a domino tableau.

The following lemma is straightforward to prove by induction:

Lemma A.1. *Let $\boldsymbol{p} = (p_1 \leq p_2 \leq \cdots \leq p_m)$ be a partition, where p_1 may be 0 and m is odd. Choose $r_1, \ldots, r_k$ and $s_1, \ldots, s_l$ so that*

$$\{p_1, p_2 + 1, \ldots, p_m + m - 1\} = \{2r_1, \ldots, 2r_k, 2s_1 + 1, \ldots, 2s_l + 1\}$$

(*as unordered lists*). *If $\boldsymbol{p}$ has domino shape and has an even number of boxes, then $k = l + 1$. If $\boldsymbol{p}$ has domino shape and has an odd number of boxes, then $k + 1 = l$.*

Let T be a tableau whose boxes are labelled by the integers $-n, \ldots, -1, 1, \ldots, n$ or the integers $-n, \ldots, -1, 0, 1, \ldots, n$. We recall an algorithm DT, which takes as input such a tableau and outputs a domino tableau; it was defined in [Barbasch and Vogan 1982]. To define $\mathrm{DT}(T)$, first note that $-n$ must occur in the lower-left corner of T. Swap $-n$ with the smaller of its neighbours that lie above or to the right of $-n$. Continue swapping $-n$ with its smaller neighbour that is either above or right of it. If the last number that $-n$ is swapped with is not n then we say that $\mathrm{DT}(T)$ is undefined. Otherwise, replace the squares with $-n$ and n with a domino containing n. Now repeat this procedure for $1 - n, 2 - n, \ldots, -1$, treating any squares that have been replaced with dominoes as if they were not present. If for any i the last number that $-i$ is replaced with is not i then $\mathrm{DT}(T)$ is undefined. Otherwise, we eventually get a domino tableau.

For example, suppose

$$T = \mathrm{RS}(-2, -3, 1, 0, -1, 3, 2) = \begin{array}{|c|c|c|} \hline 1 & \multicolumn{2}{c}{} \\ \hline -2 & 0 & 3 \\ \hline -3 & -1 & 2 \\ \hline \end{array}$$

Now, when we apply the above algorithm we first swap -3 with -2, then with 0, then with 3. Now, replace the boxes containing 3 and -3 with a domino containing 3.

This results in the following diagram:

<table>
<tr><td>1</td></tr>
<tr><td>0</td><td colspan="2">3</td></tr>
<tr><td>–2</td><td>–1</td><td>2</td></tr>
</table>

Now, -2 first swaps with -1, then 2, which results in the following diagram:

<table>
<tr><td>1</td></tr>
<tr><td>0</td><td colspan="2">3</td></tr>
<tr><td>–1</td><td colspan="2">2</td></tr>
</table>

Finally, -1 swaps with 0, then 1, and the resulting domino tableau is

(A.2) $\mathrm{DT}(T) =$

<table>
<tr><td rowspan="2">1</td></tr>
<tr><td colspan="2">3</td></tr>
<tr><td>0</td><td colspan="2">2</td></tr>
</table>

Let W_n denote the Weyl group of type B_n acting on $\{\pm 1, \dots, \pm n\}$ in the natural way. Then the image of $(-n, \dots, -1, 1, \dots, n)$ under the action of some $\sigma \in W_n$ is called a *signed permutation of* $(-n, \dots, -1, 1, \dots, n)$. A *signed permutation of* $(-n, \dots, -1, 0, 1, \dots, n)$ is defined similarly.

The next lemma follows from Proposition 2.3.3 and Theorem 4.1.1 in [van Leeuwen 1996]:

Lemma A.3. *Let* $\boldsymbol{a} = (a_1, \dots, a_n, -a_n, \dots, -a_1)$ *be a signed permutation of* $(-n, \dots, -1, 1, \dots, n)$ *and* $\boldsymbol{b} = (b_1, \dots, b_n, 0, -b_n, \dots, -b_1)$ *a signed permutation of* $(-n, \dots, -1, 0, 1, \dots, n)$. *Then* $\mathrm{DT}(\mathrm{RS}(\boldsymbol{a}))$ *and* $\mathrm{DT}(\mathrm{RS}(\boldsymbol{b}))$ *are defined.*

We may identify W_n with the signed permutations of $(-n, \dots, -1, 1, \dots, n)$ or the signed permutations of $(-n, \dots, -1, 0, 1, \dots, n)$. Under this identification, we consider the algorithms defined in [Garfinkle 1990, §2] to map a signed permutation of $(-n, \dots, -1, 1, \dots, n)$ or $(-n, \dots, -1, 0, 1, \dots, n)$ to a domino tableau. We denote these versions of Garfinkle's algorithm by G_0 and G_1 respectively.

Proposition A.4 [van Leeuwen 1996, Proposition 4.2.3].

(i) *If w is a signed permutation of* $(-n, \dots, -1, 1, \dots, n)$, *then* $\mathrm{DT}(\mathrm{RS}(w)) = \mathrm{G}_0(w)$.

(ii) *If w is a signed permutation of* $(-n, \dots, -1, 0, 1, \dots, n)$, *then* $\mathrm{DT}(\mathrm{RS}(w)) = \mathrm{G}_1(w)$.

Our aim is to show that part(RS$(a_1, \dots, a_n, -a_n, \dots, -a_1)$) has the same content as part(RS$(a_1, \dots, a_n, 0, -a_n, \dots, -a_1)$). We do this by exploiting the results in [Pietraho 2010], which explain how to relate $\mathrm{G}_0(a_1, \dots, a_n, -a_n, \dots, -a_1)$ and $\mathrm{G}_1(a_1, \dots, a_n, 0, -a_n, \dots, -a_1)$. To explain these results, we need to define the *cycles* of a standard domino tableau. This requires a few other definitions as well.

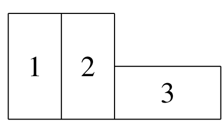
We define coordinates on a Young diagram by labelling its rows and columns. We declare that the bottom row is row 1, the row above the bottom is row 2, and so on. We declare that the left most column is column 1, the column to its right is column 2, and so on. Now we say the box in position (i, j) is *fixed* if $i + j$ is odd and the diagram has an even number of boxes or if $i + j$ is even and the diagram has an odd number of boxes.

Let R be a domino tableau, and let $D(k)$ be a domino with label k in R. If the fixed coordinate of $D(k)$ occurs in the lower box or right box of $D(k)$, let E denote the square below and to the right of the fixed coordinate of $D(k)$. If the fixed coordinate of $D(k)$ occurs in the upper box or left box of $D(k)$, let E denote the square above and to the left of the fixed coordinate of $D(k)$. We label E with the integer m determined via

$$m = \begin{cases} l & \text{if } E \text{ is a square in } R \text{ and } l \text{ is the label of } E\text{'s square in } R, \\ -1 & \text{if either coordinate of } E \text{ is } 0, \\ \infty & \text{if } E \text{ lies above or to the right of } R. \end{cases}$$

Now, we define $D'(k)$ to be a domino containing two squares, one in the fixed position of $D(k)$ and the other adjacent to E and such that the subdiagram containing $D'(k)$ and E has decreasing columns and increasing rows.

For example, if

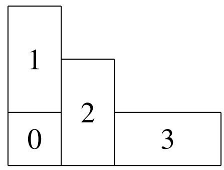
$R =$ 

then $D'(1)$ is a domino occupying positions $(2, 1)$ and $(3, 1)$, $D'(2)$ is a domino occupying positions $(1, 2)$ and $(1, 3)$, and $D'(3)$ is a domino occupying positions $(1, 4)$ and $(1, 5)$.

Suppose a domino tableau is labelled with $\{1, \dots, n\}$. We use this to generate an equivalence relation on $\{1, \dots, n\}$ via $i \sim j$ if $D(j)$ and $D'(i)$ share a box. The *cycles of a domino tableau* are the equivalence classes of this equivalence relation. For example, if R is as above, then the cycles of R are $\{1\}$ and $\{2, 3\}$.

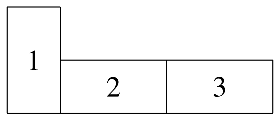
If R is a domino tableau with an even number of boxes and c is a cycle of R, then we can define a new domino tableau $R' = \mathrm{MT}(R, c)$ by replacing $D(k)$ with $D'(k)$ for every $k \in c$. This will remove one box and add one box to the underlying Young diagram of R. If the box removed is in position $(1, 1)$, then we put a box with 0 in position (1, 1) of R', so that we do in fact get another domino tableau. For example, if R is as above, then

$$\mathrm{MT}(R, \{1\}) =$$



and

$$\mathrm{MT}(R, \{2, 3\}) =$$



Observe that the operator MT removes a box and adds a box to the Young diagram underlying R, and that the removed box is either in position (1, 1), or is a *removable box* of R; that is, if it is removed, you still have a valid Young diagram.

A key feature of MT is that is does not change the content of the underlying partition:

Theorem A.5. *Let R be a domino tableau with an even number of boxes, let c be a cycle of R, and let $\boldsymbol{p} = \mathrm{part}(R)$ and $\boldsymbol{q} = \mathrm{part}(\mathrm{MT}(R, c))$. Then* $\mathrm{content}(\boldsymbol{p}) = \mathrm{content}(\boldsymbol{q})$.

Proof. First we rule out the case that $\boldsymbol{p}$ has an odd number (say, $2m + 1$) of parts and $\boldsymbol{q}$ has one more part than $\boldsymbol{p}$. Suppose for a contradiction that $\boldsymbol{q}$ has $2m+2$ parts, so the top row of $\mathrm{MT}(R, c)$ has one box. Let $D'(k)$ be the domino in $\mathrm{MT}(S, c)$ that covers this box. So the box in the fixed position of $D'(k)$ must be the box in position $(2m + 1, 1)$, which is a contradiction since $2m + 1 + 1$ is even.

Next we rule out the case that $\boldsymbol{p}$ has an even number (say, $2m$) of parts and $\boldsymbol{q}$ has one less part than $\boldsymbol{p}$. Suppose this is the case, so the top row $\boldsymbol{p}$ has length one, so there must be a domino $D(k)$ which occupies positions $(2m - 1, 1)$ and $(2m, 1)$ of $\boldsymbol{p}$. Now, $(2m, 1)$ is the fixed position of this domino, so $D'(k)$ will also have a box in position $(2m, 1)$; hence, $\boldsymbol{q}$ has at least $2m$ parts, which is a contradiction. Thus we have established that the number of integers in $\mathrm{content}(\boldsymbol{q})$ is the same as the number of integers in $\mathrm{content}(\boldsymbol{p})$.

Let $\boldsymbol{p} = (p_1 \leq \cdots \leq p_{2m+1})$, where p_1 may be 0. Now we consider the case that $\mathrm{MT}(R, c)$ has the same number of boxes as R. Let $\boldsymbol{q} = (q_1 \leq \cdots \leq q_{2m+1})$, where q_1 may be 0. So we must have that $q_i = p_i$ except for $i = j$ and $i = k$ for some integers j, k where $j \neq k$, and $q_j = p_j + 1$ and $q_k = p_k - 1$. By Lemma A.1, we have that one of $p_j + j - 1$, $p_k + k - 1$ must be even and one must be odd, because otherwise $(q_1, q_2 + 1, \ldots, q_{2m+1} + 2m)$ would not have one more even element than odd elements. The box at position (j, p_j) of $\mathrm{MT}(R, c)$ is the box that gets added to the Young diagram of R. Thus, this box is a box that is in $D'(k)$ but not in $D(k)$. This implies that this box is not the box in fixed position in $D'(k)$; thus, $p_j + j$ is even, so $p_j + j - 1$ must be odd, and $p_k + k - 1$ is even. This implies that $\boldsymbol{p}$ and $\boldsymbol{q}$ have the same content.

Now we consider the case that $\mathrm{MT}(R, c)$ has one more box than R. Let $\boldsymbol{q} = (q_1 \le \cdots \le q_{2m+1})$, where q_1 may be 0. So we must have that $q_i = p_i$ except for $i = j$ for some integer j, where $q_j = p_j + 1$. Note that $p_j + j - 1$ must be even since $(q_1, q_2 + 1, \ldots, q_{2m+1} + 2m)$ must have one more odd number than even number. This implies that $\boldsymbol{p}$ and $\boldsymbol{q}$ have the same content. □

For a list of cycles $c_1, \ldots, c_m$ of a domino tableau R with an even number of boxes, let $R_i = \mathrm{MT}(R_{i-1}, c_i)$, where $R_0 = R$. Now let $\mathrm{MT}(R, c_1, \ldots, c_m) = R_m$.

The following theorem is a less specific version of [Pietraho 2010, Theorem 3.1]:

Theorem A.6. *Let*

$$R = \mathrm{G}_0(a_1, \ldots, a_n, -a_n, \ldots, -a_1)$$

and

$$R' = \mathrm{G}_1(a_1, \ldots, a_n, 0, -a_n, \ldots, -a_1),$$

where $(a_1, \ldots, a_n, -a_n, \ldots, -a_1)$ *is a signed permutation of* $(-n, \ldots, -1, 1, \ldots, n)$. *Then there exist cycles* $c_1, \ldots, c_m$ *of* R *such that* $R' = \mathrm{MT}(R, c_1, \ldots, c_m)$.

Now we get the following corollary:

Corollary A.7. *Let* $\boldsymbol{a} = (a_1, \ldots, a_n, -a_n, \ldots, -a_1)$ *be a skew-symmetric string of integers. Then* $\mathrm{BV}(\boldsymbol{a}) = \mathrm{BV}'(\boldsymbol{a})$.

Proof. This follows from Proposition A.4 and Theorems A.5 and A.6 when $\boldsymbol{a}$ is a signed permutation of $(-n, \ldots, -1, 1, \ldots, n)$. The general case follows because $\boldsymbol{q} = \mathrm{RS}(a_1, \ldots, a_n, -a_n, \ldots, -a_1)$ and $\boldsymbol{q}' = \mathrm{RS}(a_1, \ldots, a_n, 0, -a_n, \ldots, -a_1)$ depend only on the relative order of the a_i, so we may replace $\boldsymbol{a}$ by a signed permutation without altering $\boldsymbol{q}$ or $\boldsymbol{q}'$. □

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NEW INVARIANTS FOR COMPLEX MANIFOLDS AND RATIONAL SINGULARITIES

RONG DU AND YUN GAO

Dedicated to Professor Stephen S.-T. Yau on the occasion of his sixtieth birthday.

Two new invariants $f^{(1,1)}$ and $g^{(1,1)}$ were introduced by Du and Yau for solving the complex Plateau problem. These invariants measure in some sense how far away the complex manifolds are from having global complex coordinates. In this paper, we study these two invariants further for rational surface singularities. We prove that these two invariants never vanish for rational surface singularities, which confirms Yau's conjecture for strict positivity of these two invariants. As an application, we solve regularity problem of the Harvey–Lawson solution to the complex Plateau problem for a strongly pseudoconvex compact rational CR manifold of dimension 3. We also construct resolution manifolds for rational triple points by means of local coordinates and show that $f^{(1,1)} = g^{(1,1)} = 1$ for rational triple points.

1. Introduction

Let M be a complex manifold of dimension n. It is a natural question to ask how far away this complex manifold is from having global complex coordinates. Together with Hing Sun Luk and Stephen Yau, the first author introduced in [Du et al. 2011] new biholomorphic invariants that give some measurements for this purpose.

It is well-known that if M is a complex submanifold in $\mathbb{C}^N$, then, given any global holomorphic p-form α on M, there exists a holomorphic p-form $\tilde{\alpha}$ on $\mathbb{C}^N$ such that the restriction of $\tilde{\alpha}$ to M is α. Obviously, $\tilde{\alpha}$ is a p-th wedge product of holomorphic 1-forms. However, for non-Stein complex manifolds, such as the resolution manifolds of singularities, the situation is totally different. In particular,

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for surface singularities, we considered in [Du et al. 2011; Du and Yau 2012] two invariants $f^{(1,1)}$ and $g^{(1,1)}$ and showed that these two invariants are strictly positive for some special singularities. So the resolution manifolds of these special singularities do not have the above properties.

A direct application of the positivity of $g^{(1,1)}$ is for solving one of the most fundamental questions in complex geometry: the complex Plateau problem [Du and Yau 2012]. Given a strongly pseudoconvex CR manifold X in $\mathbb{C}^N$, the problem asks when X is the boundary of a complex manifold V in $\mathbb{C}^N$. By the beautiful work of Harvey and Lawson [1975], as well as [Yau 1981b; Luk and Yau 1998a], X is a boundary of a complex variety V with only isolated singularities if X is contained in the boundary of a strictly pseudoconvex domain in $\mathbb{C}^N$. Thus from the complex Plateau problem point of view, it is very desirable to introduce a numerical invariant for isolated singularities which never vanishes. However many numerical invariants such as the geometric genus p_g, the arithmetic genus p_a and the irregularity q vanish on rational singularities. In [Du et al. 2011; Du and Yau 2012] this idea was used to introduce two invariants $f^{(1,1)}$ and $g^{(1,1)}$ for isolated surface singularities. The invariant $g^{(1,1)}$ was used in the latter article to solve the regularity problem of the Harvey–Lawson solution to the complex Plateau problem.

Those two articles provided a detailed study of $f^{(1,1)}$ and $g^{(1,1)}$. Yau has the following conjecture:

Conjecture. For all normal surface singularities, the invariants $f^{(1,1)}$ and $g^{(1,1)}$ are strictly positive.

Du and Yau showed that these two numerical invariants are strictly positive when the surface singularities have a $\mathbb{C}^*$-action. They also gave explicit calculations for $f^{(1,1)}$ and $g^{(1,1)}$ for rational double points and cyclic quotient singularities and proved that they are exactly 1. In this paper, we shall prove that for rational surface singularities, $f^{(1,1)}$ and $g^{(1,1)}$ also never vanish. So, our results in this paper confirm the conjecture.

Theorem 2.8. *For rational surface singularities*, $f^{(1,1)} = g^{(1,1)} \geq 1$.

As an application, we solve the regularity problem of the Harvey–Lawson solution to the complex Plateau problem for a strongly pseudoconvex compact rational CR manifold of dimension 3.

Theorem 3.9. *Let X be a strongly pseudoconvex compact rational CR manifold of dimension* 3. *Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is a boundary of the complex submanifold $V \subset D - X$ with boundary regularity if and only if $g^{(1,1)}(X) = 0$.*

We also construct resolution manifolds for rational triple points by local coordinates. By using these local coordinates, we give explicit calculations of $f^{(1,1)}$ and

$g^{(1,1)}$ for rational triple points and prove that they are also 1.

Theorem 4.4. *For* 2*-dimensional rational triple points*, $f^{(1,1)} = g^{(1,1)} = 1$.

2. Invariants of singularities

Let V be a n-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities. Yau [1982] considered four kinds of sheaves of germs of holomorphic p-forms:

(1) $\overline{\Omega}^p_V := \pi_* \Omega^p_M$, where $\pi : M \to V$ is a resolution of singularities of V.

(2) $\overline{\overline{\Omega}}^p_V := \theta_* \Omega^p_{V \setminus V_{\text{sing}}}$ where $\theta : V \setminus V_{\text{sing}} \to V$ is the inclusion map and V_{sing} is the singular set of V.

(3) $\Omega^p_V := \Omega^p_{\mathbb{C}^N} / \mathcal{K}^p$, where $\mathcal{K}^p = \{ f\alpha + dg \wedge \beta : \alpha \in \Omega^p_{\mathbb{C}^N}, \beta \in \Omega^{p-1}_{\mathbb{C}^N}, f, g \in \mathcal{I} \}$ and $\mathcal{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.

(4) $\widetilde{\Omega}^p_V := \Omega^p_{\mathbb{C}^N} / \mathcal{H}^p$, where $\mathcal{H}^p = \{ \omega \in \Omega^p_{\mathbb{C}^N} : \omega|_{V \setminus V_{\text{sing}}} = 0 \}$.

Ω^p_V is the Grauert–Grothendieck sheaf of germs of holomorphic p-form on V. In case V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\overline{\overline{\Omega}}^n_V$. Clearly Ω^p_V, $\widetilde{\Omega}^p_V$ are coherent. $\overline{\Omega}^p_V$ is a coherent sheaf because π is a proper map. $\overline{\overline{\Omega}}^p_V$ is also a coherent sheaf by Theorem A of [Siu 1970].

In [Du et al. 2011] and [Du and Yau 2012], another two sheaves $\overline{\Omega}^{1,1}_V$ and $\overline{\overline{\Omega}}^{1,1}_V$ were considered:

Definition 2.1. Let $(V, 0)$ be a 2-dimensional Stein analytic space with an isolated singularity at 0. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as its exceptional set. Define a sheaf of germs $\overline{\Omega}^{1,1}_V$ as the sheaf associated to the presheaf

$$U \mapsto \langle \Gamma(\pi^{-1}(U), \Omega^1_M) \wedge \Gamma(\pi^{-1}(U), \Omega^1_M) \rangle,$$

where U is an open set of V and $\langle \Gamma(\pi^{-1}(U), \Omega^1_M) \wedge \Gamma(\pi^{-1}(U), \Omega^1_M) \rangle$ represents the module generated by elements in $\Gamma(\pi^{-1}(U), \Omega^1_M) \wedge \Gamma(\pi^{-1}(U), \Omega^1_M)$ over the ring $\Gamma(\pi^{-1}(U), \mathbb{O}_M)$.

Definition 2.2. Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Define a sheaf of germs $\overline{\overline{\Omega}}^{1,1}_V$ by the sheaf associated to the presheaf

$$U \mapsto \langle \Gamma(U, \overline{\overline{\Omega}}^1_V) \wedge \Gamma(U, \overline{\overline{\Omega}}^1_V) \rangle,$$

where U is an open set of V.

Du and Yau showed that these two new sheaves are coherent and found the relation between $\overline{\Omega}^{1,1}_V$ (respectively, $\overline{\overline{\Omega}}^{1,1}_V$) and $\overline{\Omega}^2_V$ (respectively, $\overline{\overline{\Omega}}^2_V$) by short exact sequence as follows:

Lemma 2.3 [Du and Yau 2012]. *Let $(V,0)$ be a 2-dimensional Stein space with an isolated singularity at 0. Let $\pi:(M,A)\to(V,0)$ be a resolution of the singularity with A as its exceptional set. Then $\bar{\Omega}_V^{1,1}$ is coherent and there is a short exact sequence*

$$0\longrightarrow \bar{\Omega}_V^{1,1}\longrightarrow \bar{\Omega}_V^{2}\longrightarrow \mathcal{F}^{(1,1)}\longrightarrow 0, \tag{2-1}$$

where $\mathcal{F}^{(1,1)}$ is a sheaf supported on the singular point of V. Let

$$F^{(1,1)}(M):=\Gamma(M,\Omega_M^2)/\langle\Gamma(M,\Omega_M^1)\wedge\Gamma(M,\Omega_M^1)\rangle; \tag{2-2}$$

then, $\dim \mathcal{F}_0^{(1,1)}=\dim F^{(1,1)}(M)$.

Lemma 2.4 [Du and Yau 2012]. *Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi:(M,A)\to(V,0)$ be a resolution of the singularity with A as its exceptional set. Then $\bar{\bar{\Omega}}_V^{1,1}$ is coherent and there is a short exact sequence*

$$0\longrightarrow \bar{\bar{\Omega}}_V^{1,1}\longrightarrow \bar{\bar{\Omega}}_V^{2}\longrightarrow \mathcal{G}^{(1,1)}\longrightarrow 0 \tag{2-3}$$

where $\mathcal{G}^{(1,1)}$ is a sheaf supported on the singular point of V. Let

$$G^{(1,1)}(M\backslash A):=\Gamma(M\backslash A,\Omega_M^2)/\langle\Gamma(M\backslash A,\Omega_M^1)\wedge\Gamma(M\backslash A,\Omega_M^1)\rangle; \tag{2-4}$$

then, $\dim \mathcal{G}_0^{(1,1)}=\dim G^{(1,1)}(M\backslash A)$.

They defined local invariants of singularities which are independent of resolution:

Definition 2.5. Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi:(M,A)\to(V,0)$ be a resolution of the singularity with A as its exceptional set. Let

$$f^{(1,1)}(0):=\dim \mathcal{F}_0^{(1,1)}=\dim F^{(1,1)}(M), \tag{2-5}$$

$$g^{(1,1)}(0):=\dim \mathcal{G}_0^{(1,1)}=\dim G^{(1,1)}(M\backslash A). \tag{2-6}$$

We will omit 0 in $f^{(1,1)}(0)$ and $g^{(1,1)}(0)$ if the context is clear.

The first author and Yau conjectured that for all normal surface singularities, the invariants $f^{(1,1)}$ and $g^{(1,1)}$ are strictly positive. This conjecture was confirmed when the singularities are with $\mathbb{C}^*$-action.

Theorem 2.6 [Du and Yau 2012]. *Let V be a 2-dimensional Stein space with 0 as its only normal singular point with $\mathbb{C}^*$-action. Then $f^{(1,1)}\geq 1$.*

Theorem 2.7 [Du et al. 2011]. *Let V be a 2-dimensional Stein space with 0 as its only normal singular point with $\mathbb{C}^*$-action. Then $g^{(1,1)}\geq 1$.*

Theorem 2.7 is the crucial part for the solution of the classical complex Plateau problem.

We will show that these two invariants are strictly positive for rational surface singularities. So these two invariants tell the difference between smoothness and singularity more precisely than the geometry genus does in some sense.

Theorem 2.8. *For rational surface singularities,* $f^{(1,1)} = g^{(1,1)} \geq 1$.

Proof. Let $(V, 0)$ be a 2-dimensional Stein space with 0 as its only rational singularity. So, the geometry genus p_g and the irregularity q are both 0. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as its exceptional set. Then

$$\frac{\Gamma(M\backslash A, \Omega_M^2)}{\langle \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1)\rangle} = \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle},$$

and

$$g^{(1,1)} = f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle}.$$

From [Yau 1981a], the canonical bundle K_M is generated by its global sections in a neighborhood of the exceptional set for rational surface singularities. So, there exists $\omega \in \Gamma(M, \Omega_M^2)$ such that ω does not vanish along some irreducible component A_k of A. As the singularity is rational, A_k is a smooth rational curve. Take a tubular neighborhood U of A_k such that $U \subset M$. By the proof of Proposition 3.9 in [Du et al. 2011], we know that the elements in $\langle \Gamma(U, \Omega_U^1) \wedge \Gamma(U, \Omega_U^1)\rangle$ vanish along A_k. Since $\Gamma(M, \Omega_M^1) \subset \Gamma(U, \Omega_U^1)$, the elements in $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle$ also vanish along A_k. Therefore $\omega \in \Gamma(M, \Omega_M^2) \backslash \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle$, i.e.,

$$g^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle} \geq 1. \qquad \square$$

3. The complex Plateau problem for 3-dimensional rational CR manifolds

Yau [1981b] solved the classical complex Plateau problem for the case $n \geq 3$.

Theorem 3.1 [ibid.]. *Let X be a compact connected strongly pseudoconvex CR manifold of real dimension $2n - 1$ for $n \geq 3$ in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^{n+1}$. Then X is the boundary of the complex submanifold $V \subset D - X$ if and only if the Kohn–Rossi cohomology groups $H_{KR}^{p,q}(X)$ are zero for $1 \leq q \leq n - 2$.*

Luk and Yau [2012] introduced the so-called s-invariant in order to solve the complex Plateau problem as $n = 2$. But they could not give even a sufficient condition on the boundary such that it decides the smoothness in the interior.

Theorem 3.2 [Luk and Yau 2007]. *Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. If the holomorphic* *de Rham cohomology* $H_h^2(X)$ *is* 0, *then* X *is a boundary of a complex variety* V *in* D *with boundary regularity, and* V *has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing* s*-invariant.*

Corollary 3.3 [ibid.]. *Let* X *be a strongly pseudoconvex compact CR manifold of dimension* 3. *Suppose that* X *is contained in the boundary of a strongly pseudoconvex bounded domain* D *in* $\mathbb{C}^3$. *If the holomorphic De Rham cohomology* $H_h^2(X)$ *is* 0, *then* X *is a boundary of a complex variety* V *in* D *with boundary regularity, and* V *has only isolated quasihomogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star-shaped and all the curves are rational.*

Du and Yau [2012] used a new invariant $g^{(1,1)}$ for singularities to generate a new CR invariant $g^{(1,1)}(X)$.

Definition 3.4. Suppose X is a compact connected strongly pseudoconvex CR manifold of real dimension 3. Put

$$G^{(1,1)}(X) := \mathcal{S}^2(X)/\langle \mathcal{S}^1(X) \wedge \mathcal{S}^1(X)\rangle, \tag{3-1}$$

where $\mathcal{S}^p$ denotes the holomorphic sections of $\bigwedge^p(\widehat{T}(X)^*)$ and $\widehat{T}(X)^*$ is the holomorphic cotangent bundle of X. Then we set

$$g^{(1,1)}(X) := \dim G^{(1,1)}(X). \tag{3-2}$$

Lemma 3.5 [Du and Yau 2012]. *Let* X *be a compact connected strongly pseudoconvex CR manifold of real dimension* 3, *which bounds a bounded strongly pseudoconvex variety* V *with only isolated singularities* $\{0_1, \dots, 0_k\}$ *in* $\mathbb{C}^N$. *Then* $g^{(1,1)}(X) = \sum_i g^{(1,1)}(0_i)$.

Note that this invariant $g^{(1,1)}(X)$ can be calculated on X directly. In [ibid.], we use this CR invariant to give the sufficient and necessary condition for the variety bounded by X to be smooth if $H_h^2(X) = 0$:

Theorem 3.6 [ibid.]. *Let* X *be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension* 3. *Suppose that* X *is contained in the boundary of a strongly pseudoconvex bounded domain* D *in* $\mathbb{C}^N$ *with* $H_h^2(X) = 0$. *Then* X *is the boundary of the complex submanifold up to normalization* $V \subset D - X$ *with boundary regularity if and only if* $g^{(1,1)}(X) = 0$.

Theorem 3.7 [ibid.]. *Let* X *be a strongly pseudoconvex compact CR manifold of dimension* 3. *Suppose that* X *is contained in the boundary of a strongly pseudoconvex bounded domain* D *in* $\mathbb{C}^3$ *with* $H_h^2(X) = 0$. *Then* X *is the boundary of the complex submanifold* $V \subset D - X$ *if and only if* $g^{(1,1)}(X) = 0$.

In this paper, we solve the complex Plateau problem for a strongly pseudoconvex compact rational CR manifold of dimension 3, as a corollary of Theorem 2.8.

Recall the definition of a 3-dimensional rational CR manifold [Luk and Yau 1998b]:

Definition 3.8. Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3. Let V be the normal variety such that the boundary of V is X and V has isolated singularities at $\{0_1, \dots, 0_m\}$. Let $\pi : M \to V$ be a resolution of the singularities of V. Let U_i be a strongly pseudoconvex neighborhood of 0_i, for $1 \le i \le m$, such that the U_i are pairwise disjoint. Then

$$p_g(X) = \sum_{i=1}^{m} \dim \Gamma(U_i - \{0_i\}, \Omega^2)/L^2(U_i - \{0_i\}, \Omega^2).$$

If $p_g(X) = 0$, we call the CR manifold rational.

From the similar proof of Lemma 3.9 in [Du and Yau 2012], we know that the invariant $p_g(X)$ is also decided by the holomorphic sections of holomorphic cotangent bundle of X. So it is also a CR invariant.

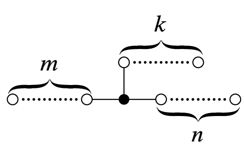
Theorem 3.9. *Let X be a strongly pseudoconvex compact rational CR manifold of dimension* 3. *Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is a boundary of the complex submanifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.*

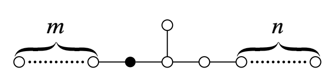
Proof. It is clear that rational CR manifolds can bound varieties with only rational singularities. Then from Theorem 2.8 and Lemma 3.5, we obtain our conclusion. □

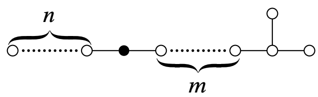
4. Explicit calculation of new invariants for special rational triple points

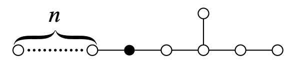
Du et al. [2011] calculated $f^{(1,1)}$ and $g^{(1,1)}$ for rational double points and quotient singularities. In this section we will calculate these two invariants for rational triple points. We suppose that V is a 2-dimensional Stein space with 0 as its only normal singularity and that V is contractible to 0.

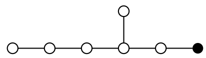
Artin [1966] classified the dual graphs of rational triple points of dimension 2 into nine classes, and proved that each rational triple point can be embedded into $\mathbb{C}^4$. Tyurina [1968a] gave explicitly three defining equations for each singularity. Tyurina [1968b] also proved that a rational triple point is determined uniquely by its dual graph ([Laufer 1973] totally gave all the dual graphs of singularities with such property). So, isomorphically, there are nine rational triple points, for which we use the notations defined in [Chen et al. 2007]:

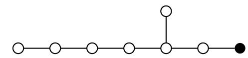
$A_{m,n,k}$: 

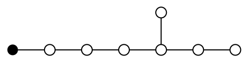
$B_{m,n}$: 

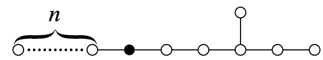
$C_{m,n}$: 

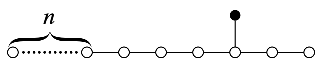
$D_{n,5}$: 

$E_{6,0}$: 

$E_{7,0}$: 

$E_{0,7}$: 

$F_{n,6}$: 

$G_{n,0}$: 

where $\circ$ is a (-2)-curve and $\bullet$ is a (-3)-curve.

In the following computation, we shall use explicit resolutions $\pi : M \to V$ of $A_{m,n,k}$, $B_{m,n}$, $C_{m,n}$, $D_{n,5}$, $E_{6,0}$, $E_{7,0}$, $E_{0,7}$, $F_{n,6}$ and $G_{n,0}$ to compute our new invariants. Note that for these rational singularities of dimension 2, the irregularity q is 0, so $f^{(1,1)} = g^{(1,1)}$. In order to calculate our new invariants for these rational singularities, we must know all the holomorphic 1-forms and holomorphic 2-forms on M. In general, the difficulty for calculating these two invariants is that the holomorphic 1-forms on the resolution manifolds are hard to express. But for rational singularities, we can use the following proposition to simplify the calculation. Campana and Flenner [2002] gave a proof of the following proposition by using mixed Hodge structure theory, which is of independent interest. We give a short proof here:

Proposition 4.1. *If* $(V, 0)$ *is a rational isolated singularity of dimension* 2 *and* M *is a resolution of the singularity*, *then* $H_h^1(M) = H_h^2(M) = 0$.

Proof. We recall the similar proof in [Du and Yau 2010]. Let $\pi : M \to V$ be a good resolution of the singularity. Let $\pi^{-1}(0) = A = \bigcup A_i$, $1 \le i \le n$, be the irreducible decomposition of the exceptional set A.

We have the spectral sequence

$$E_1^{p,q} = H^q(M, \Omega_M^p) \Rightarrow \boldsymbol{H}^{p+q}(M, \Omega_M^\bullet) \cong H^{p+q}(M, \mathbb{C}). \tag{4-1}$$

The spectral sequence induces an exact sequence of small-order terms

$$0 \to H_h^1(M) \to H^1(M, \mathbb{C}) \to E_2^{0,1} \to H_h^2(M) \to H^2(M, \mathbb{C}) \to E_2^{1,1} \to 0, \tag{4-2}$$

where

$$E_2^{0,1} = \ker(H^1(M, \mathbb{O}_M) \to H^1(M, \Omega_M^1)), \tag{4-3}$$

$$E_2^{1,1} = \operatorname{coker}(H^1(M, \mathbb{O}_M) \to H^1(M, \Omega_M^1)). \tag{4-4}$$

So

$$h_h^1(M) - h^1(M) + \dim E_2^{0,1} - h_h^2(M) + h^2(M) - \dim E_2^{1,1} = 0. \tag{4-5}$$

Since

$$\dim E_2^{0,1} - \dim E_2^{1,1} = h^1(M, \mathcal{O}_M) - h^1(M, \Omega_M^1), \tag{4-6}$$

we have

$$h_h^1(M) - h^1(M) - h_h^2(M) + h^2(M) + h^1(M, \mathcal{O}_M) - h^1(M, \Omega_M^1) = 0. \tag{4-7}$$

From [Wahl 1985], we know that

$$h^1(M, \Omega_M^1) = \gamma + q + n = p_g - g - b - \alpha - \beta + n \tag{4-8}$$

and

$$h^1(M) = \dim H^1(A, \mathbb{C}) = 2g + b, \quad h^2(M) = n, \quad h^1(M, \mathcal{O}_M) = p_g.$$

So

$$h_h^2(M) - h_h^1(M) = \alpha + \beta - g. \tag{4-9}$$

From [van Straten and Steenbrink 1985], we know that

$$p_g = q + g + b + \alpha + \beta + \gamma.$$

As $p_g = 0$, $q = g = b = \alpha = \beta = \gamma$. So $h^1(M) = 2g + b = 0$. From (4-2) and (4-9), we get that $h_h^2(M) = h_h^1(M) = 0$. □

Remark 4.2. In fact, from the above proof, we can get $h_h^2(M) = h_h^1(M) = 0$ if

$$E_2^{0,1} = \ker(H^1(M, \mathcal{O}_M) \to H^1(M, \Omega_M^1)) = 0.$$

So, rational singularity is a special case.

Now we can use holomorphic functions and holomorphic 2-forms to express holomorphic 1-forms on the resolution manifold from the following lemma:

Lemma 4.3. *If $(V, 0)$ is rational isolated singularity of dimension 2 and $\pi : M \to V$ is a resolution, then for any $\xi \in \Gamma(M, \Omega_M^1)$ and $\zeta \in d^{-1}(d\xi)$, there exists an $f \in \Gamma(M, \mathcal{O}_M)$ such that $\xi = \zeta + d(f)$, where d is the exterior differential operator.*

Proof. From Proposition 4.1 above, we have the exact sequence

$$0 \longrightarrow \Gamma(M, \mathcal{O}_M) \xrightarrow{d} \Gamma(M, \Omega_M^1) \xrightarrow{d} \Gamma(M, \Omega_M^2) \longrightarrow 0.$$

For $\xi \in \Gamma(M, \Omega_M^1)$ and any $\zeta \in d^{-1}(d\xi)$, $d(\xi - \zeta) = 0$. So there exists $f \in \Gamma(M, \mathcal{O}_M)$ such that $\xi = \zeta + d(f)$. □

From the lemma above, we see that in order to get holomorphic 1-forms on M, we only need to calculate holomorphic functions and holomorphic 2-forms on M.

Laufer [1971] constructed local coordinates for the resolution manifolds of cyclic quotient singularities such that one can calculate everything explicitly on the manifolds. Now we are going to construct local coordinates for the resolution manifolds of rational triple points to calculate our new invariants because of the tautness of rational triple points ("tautness" means that the singularity is determined by its dual graph).

An explicit resolution $\pi : M \to V$ can be given in terms of coordinates and transition functions on M for each type as follows:

Type $E_{6,0}$**:**

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0, 1, \dots, 4, \\ W_t &= \{(u_t, v_t) : u_t^2 v_t \neq -1\}, && t = 5, 6, \\ W_7 &= \{(u_7, v_7)\} : u_7^5 v_7^2 \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le 3, \qquad \begin{cases} u_5 = 1/(u_4 v_4), \\ v_5 = u_4 v_4^2 (1 - u_4), \end{cases}$$

$$\begin{cases} u_6 = 1/(v_4(1 - u_4)), \\ v_6 = u_4 v_4^2 (1 - u_4), \end{cases} \qquad \begin{cases} u_7 = 1/v_6, \\ v_7 = u_6 v_6^3. \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \dots \cup C_7$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le 4, \\ C_5 &= \{v_3 = 1\} \cup \{u_4 = 1\} \cup \{v_5 = 0\}, \\ C_6 &= \{u_4 = 0\} \cup \{v_6 = 0\}, \\ C_7 &= \{u_6 = 0\} \cup \{v_7 = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} u_4^a v_4^b (1 - u_4)^c &= u_3^b v_3^{-a+2b-c} (v_3 - 1)^c \\ &= u_2^{-a+2b-c} v_2^{-2a+3b-2c} (u_2 v_2^2 - 1)^c \\ &= u_1^{-2a+3b-2c} v_1^{-3a+4b-3c} (u_1^2 v_1^3 - 1)^c \\ &= u_0^{-3a+4b-3c} v_0^{-4a+5b-4c} (u_0^3 v_0^4 - 1)^c \\ &= u_5^{-b+2c} v_5^c (1 + u_5^2 v_5)^{-a+b-c} \\ &= u_6^{2a-b} v_6^a (1 + u_6^2 v_6)^{-a+b-c} \\ &= u_7^{5a-3b} v_7^{2a-b} (1 + u_7^5 v_7^2)^{-a+b-c}, \end{aligned}$$

such that

$$\begin{cases} a, b, c \geq 0, \\ 5b \geq 4(a+c), \\ 2c \geq b, \\ 5a \geq 3b. \end{cases} \tag{4-10}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_4 \wedge dv_4 \\ &= -\frac{du_5 \wedge dv_5}{1+u_5^2 v_5} = \frac{du_6 \wedge dv_6}{1+u_6^2 v_6} = \frac{u_7 du_7 \wedge dv_7}{1+u_7^5 v_7^2}, \end{aligned}$$

such that

$$\begin{cases} a, b, c \geq 0, \\ 5b \geq 4(a+c), \\ 2c \geq b, \\ 5a+1 \geq 3b. \end{cases} \tag{4-11}$$

Type $E_{7,0}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0, 1, \ldots, 5, \\ W_t &= \{(u_t, v_t) : u_t^2 v_t \neq -1\}, && t = 6, 7, \\ W_8 &= \{(u_8, v_8)\} : u_8^5 v_8^2 \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \leq t \leq 4, \qquad \begin{cases} u_6 = 1/(u_5 v_5), \\ v_6 = u_5 v_5^2 (1-u_5), \end{cases}$$

$$\begin{cases} u_7 = 1/(v_5(1-u_5)), \\ v_7 = u_5 v_5^2 (1-u_5), \end{cases} \qquad \begin{cases} u_8 = 1/v_7, \\ v_8 = u_7 v_7^3. \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_8$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \leq t \leq 5, \\ C_6 &= \{v_4 = 1\} \cup \{u_5 = 1\} \cup \{v_6 = 0\}, \\ C_7 &= \{u_5 = 0\} \cup \{v_7 = 0\}, \\ C_8 &= \{u_7 = 0\} \cup \{v_8 = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} u_5^a v_5^b (1-u_5)^c &= u_4^b v_4^{-a+2b-c}(v_4-1)^c \\ &= u_3^{-a+2b-c} v_3^{-2a+3b-2c}(u_3 v_3^2-1)^c \\ &= u_2^{-2a+3b-2c} v_2^{-3a+4b-3c}(u_2^2 v_2^3-1)^c \\ &= u_1^{-3a+4b-3c} v_1^{-4a+5b-4c}(u_1^3 v_1^4-1)^c \\ &= u_0^{-4a+5b-4c} v_0^{-5a+6b-5c}(u_0^4 v_0^5-1)^c \\ &= u_6^{-b+2c} v_6^c (1+u_6^2 v_6)^{-a+b-c} \\ &= u_7^{2a-b} v_7^a (1+u_7^2 v_7)^{-a+b-c} \\ &= u_8^{5a-3b} v_8^{2a-b} (1+u_8^5 v_8^2)^{-a+b-c}, \end{aligned}$$

such that

$$\begin{cases} a,b,c \geq 0, \\ 6b \geq 5(a+c), \\ 2c \geq b, \\ 5a \geq 3b. \end{cases} \tag{4-12}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_5 \wedge dv_5 \\ &= -\frac{du_6 \wedge dv_6}{1+u_6^2 v_6} = \frac{du_7 \wedge dv_7}{1+u_7^2 v_7} = \frac{u_8 du_8 \wedge dv_8}{1+u_8^5 v_8^2}, \end{aligned}$$

such that

$$\begin{cases} a,b,c \geq 0, \\ 6b \geq 5(a+c), \\ 2c \geq b, \\ 5a+1 \geq 3b. \end{cases} \tag{4-13}$$

Type $E_{0,7}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0, 1, \dots, 5, \\ W_t &= \{(u_t, v_t) : u_t^2 v_t \neq -1\}, && t = 6, 7, \\ W_8 &= \{(u_8, v_8)\} : u_8^3 v_8^2 \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_1 = 1/v_0, \\ v_1 = u_0 v_0^3, \end{cases} \qquad \begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 1 \le t \le 4,$$

$$\begin{cases} u_6 = 1/(u_5 v_5), \\ v_6 = u_5 v_5^2 (1-u_5), \end{cases} \qquad \begin{cases} u_7 = 1/(v_5(1-u_5)), \\ v_7 = u_5 v_5^2 (1-u_5), \end{cases} \qquad \begin{cases} u_8 = 1/v_7, \\ v_8 = u_7 v_7^2. \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_8$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le 5, \\ C_6 &= \{v_4 = 1\} \cup \{u_5 = 1\} \cup \{v_6 = 0\}, \\ C_7 &= \{u_5 = 0\} \cup \{v_7 = 0\}, \\ C_8 &= \{u_7 = 0\} \cup \{v_8 = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} u_5^a v_5^b (1-u_5)^c &= u_4^b v_4^{-a+2b-c} (v_4 - 1)^c \\ &= u_3^{-a+2b-c} v_3^{-2a+3b-2c} (u_3 v_3^2 - 1)^c \\ &= u_2^{-2a+3b-2c} v_2^{-3a+4b-3c} (u_2^2 v_2^3 - 1)^c \\ &= u_1^{-3a+4b-3c} v_1^{-4a+5b-4c} (u_1^3 v_1^4 - 1)^c \\ &= u_0^{-4a+5b-4c} v_0^{-9a+11b-9c} (u_0^4 v_0^9 - 1)^c \\ &= u_6^{-b+2c} v_6^c (1 + u_6^2 v_6)^{-a+b-c} \\ &= u_7^{2a-b} v_7^a (1 + u_7^2 v_7)^{-a+b-c} \\ &= u_8^{3a-2b} v_8^{2a-b} (1 + u_8^3 v_8^2)^{-a+b-c}, \end{aligned}$$

such that

$$\begin{cases} a, b, c \ge 0, \\ 11b \ge 9(a+c), \\ 2c \ge b, \\ 3a \ge 2b. \end{cases} \tag{4-14}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= v_0 du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_5 \wedge dv_5 \\ &= -\frac{du_6 \wedge dv_6}{1 + u_6^2 v_6} = \frac{du_7 \wedge dv_7}{1 + u_7^2 v_7} = \frac{du_8 \wedge dv_8}{1 + u_8^5 v_8^2}, \end{aligned}$$

such that

$$\begin{cases} a,b,c \geq 0, \\ 11b+1 \geq 9(a+c), \\ 2c \geq b, \\ 3a \geq 2b. \end{cases} \tag{4-15}$$

Type $G_{n,0}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0, 1, \ldots, n+1, \\ W_t &= \{(u_t, v_t) \,:\, u_t^3 v_t^2 \neq -1\}, && t = n+2, n+4, \\ W_{n+3} &= \{(u_{n+3}, v_{n+3})\} \,:\, u_{n+3}^2 v_{n+3} \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \leq t \leq n, \qquad \begin{cases} u_{n+2} = 1/(u_{n+1} v_{n+1}), \\ v_{n+2} = u_{n+1}^2 v_{n+1}^3 (1 - u_{n+1}), \end{cases}$$

$$\begin{cases} u_{n+3} = 1/(v_{n+1}(1 - u_{n+1})), \\ v_{n+3} = u_{n+1} v_{n+1}^2 (1 - u_{n+1}), \end{cases} \qquad \begin{cases} u_{n+4} = 1/v_{n+3}, \\ v_{n+4} = u_{n+3} v_{n+3}^2. \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{n+4}$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \leq t \leq n+1, \\ C_{n+2} &= \{v_n = 1\} \cup \{u_{n+1} = 1\} \cup \{v_{n+2} = 0\}, \\ C_{n+3} &= \{u_{n+1} = 0\} \cup \{v_{n+3} = 0\}, \\ C_{n+4} &= \{u_{n+3} = 0\} \cup \{v_{n+4} = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} u_{n+1}^a v_{n+1}^b (1 - u_{n+1})^c &= u_t^{(n-t+1)b-(n-t)(a+c)} v_t^{(n-t+2)b-(n-t+1)(a+c)} \\ &\quad \cdot (u_t^{n-t} v_t^{n-t+1} - 1)^c \\ &= u_{n+2}^{-b+3c} v_{n+2}^c (1 + u_{n+2}^3 v_{n+2})^{-a+b-c} \\ &= u_{n+3}^{2a-b} v_{n+3}^a (1 + u_{n+3}^2 v_{n+3})^{-a+b-c} \\ &= u_{n+4}^{3a-2b} v_{n+4}^{2a-b} (1 + u_{n+4}^3 v_{n+4}^2)^{-a+b-c}, \end{aligned}$$

where $0 \le t \le n$, such that

$$\begin{cases} a, b, c \ge 0, \\ (n+2)b \ge (n+1)(a+c), \\ 3c \ge b, \\ 3a \ge 2b. \end{cases} \tag{4-16}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_{n+1} \wedge dv_{n+1} \\ &= -\frac{u_{n+2}du_{n+2} \wedge dv_{n+2}}{1+u_{n+2}^3 v_{n+2}} = \frac{du_{n+3} \wedge dv_{n+3}}{1+u_{n+3}^2 v_{n+3}} = \frac{du_{n+4} \wedge dv_{n+4}}{1+u_{n+4}^3 v_{n+4}^2}, \end{aligned}$$

such that

$$\begin{cases} a, b, c \ge 0, \\ (n+2)b \ge (n+1)(a+c), \\ 3c+1 \ge b, \\ 3a \ge 2b. \end{cases} \tag{4-17}$$

Type $D_{n,5}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0, 1, \ldots, n+3, \\ W_t &= \{(u_t, v_t) : u_t^2 v_t \neq -1\}, && t = n+4, n+5, n+7, \\ W_{n+6} &= \{(u_{n+6}, v_{n+6})\} : u_{n+6}^3 v_{n+6}^2 \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le n-1 \text{ and } t = n+1, n+2, n+5, n+6,$$

$$\begin{cases} u_{n+1} = 1/v_n, \\ v_{n+1} = u_n v_n^3, \end{cases} \qquad \begin{cases} u_{n+4} = 1/(u_{n+3} v_{n+3}), \\ v_{n+4} = u_{n+3} v_{n+3}^2 (1-u_{n+3}), \end{cases}$$

$$\begin{cases} u_{n+5} = 1/(v_{n+3}(1-u_{n+3})), \\ v_{n+5} = u_{n+3} v_{n+3}^2 (1-u_{n+3}). \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{n+7}$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le n+3 \text{ and } t = n+6, n+7, \\ C_{n+4} &= \{v_{n+2} = 1\} \cup \{u_{n+3} = 1\} \cup \{v_{n+4} = 0\}, \\ C_{n+5} &= \{u_{n+3} = 0\} \cup \{v_{n+5} = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$
\begin{aligned}
u_{n+3}^{a} v_{n+3}^{b}(1-u_{n+3})^{c} \\
&= u_{n+2}^{b} v_{n+2}^{-a+2b-c}(v_{n+2}-1)^{c} \\
&= u_{n+1}^{-a+2b-c} v_{n+1}^{-2a+3b-2c}(u_{n+1}v_{n+1}^{2}-1)^{c} \\
&= u_{t}^{(4n-4t+3)b-(3n-3t+2)(a+c)} v_{t}^{(4n-4t+7)b-(3n-3t+5)(a+c)} \\
&\quad \cdot (u_{t}^{3n-3t+2} v_{t}^{3n-3t+5}-1)^{c} \\
&= u_{n+4}^{-b+2c} v_{n+4}^{c}(1+u_{n+4}^{2}v_{n+4})^{-a+b-c} \\
&= u_{n+5}^{2a-b} v_{n+5}^{a}(1+u_{n+5}^{2}v_{n+5})^{-a+b-c} \\
&= u_{n+6}^{3a-2b} v_{n+6}^{2a-b}(1+u_{n+6}^{3}v_{n+6}^{2})^{-a+b-c} \\
&= u_{n+7}^{4a-3b} v_{n+7}^{3a-2b}(1+u_{n+7}^{4}v_{n+7}^{3})^{-a+b-c},
\end{aligned}
$$

where $0 \le t \le n$, such that

$$
\begin{cases}
a, b, c \ge 0, \\
(4n+7)b \ge (3n+5)(a+c), \\
2c \ge b, \\
4a \ge 3b.
\end{cases} \tag{4-18}
$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$
\begin{aligned}
\varphi_0 &= u_0^{n} v_0^{n+1}\,du_0 \wedge dv_0 = u_1^{n-1} v_1^{n}\,du_1 \wedge dv_1 \\
&= \cdots = u_{n-1} v_{n-1}^{2}\,du_{n-1} \wedge dv_{n-1} = u_n\,du_n \wedge dv_n = du_{n+1} \wedge dv_{n+1} \\
&= du_{n+2} \wedge dv_{n+2} = du_{n+3} \wedge dv_{n+3} = -\frac{du_{n+4} \wedge dv_{n+4}}{1+u_{n+4}^{3}v_{n+4}} \\
&= \frac{du_{n+5} \wedge dv_{n+5}}{1+u_{n+5}^{2}v_{n+5}} = \frac{du_{n+6} \wedge dv_{n+6}}{1+u_{n+6}^{3}v_{n+6}^{2}} = \frac{du_{n+7} \wedge dv_{n+7}}{1+u_{n+7}^{4}v_{n+7}^{3}},
\end{aligned}
$$

such that

$$
\begin{cases}
a, b, c \ge 0, \\
(4n+7)b + n + 1 \ge (3n+5)(a+c), \\
2c \ge b, \\
4a \ge 3b.
\end{cases} \tag{4-19}
$$

Type $F_{n,6}$:

Coordinate charts:

$$
\begin{aligned}
W_t &= \{(u_t, v_t)\}, && t = 0, 1, \ldots, n+4,\\
W_t &= \{(u_t, v_t) : u_t^2 v_t \neq -1\}, && t = n+5, n+6,\\
W_{n+7} &= \{(u_{n+7}, v_{n+7})\} : u_{n+7}^3 v_{n+7}^2 \neq -1\},\\
W_{n+8} &= \{(u_{n+8}, v_{n+8})\} : u_{n+8}^4 v_{n+8}^3 \neq -1\}.
\end{aligned}
$$

Transition functions:

$$
\begin{cases} u_{t+1} = 1/v_t,\\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le n-1 \text{ and } t = n+1, n+2, n+3, n+6, n+7,
$$

$$
\begin{cases} u_{n+1} = 1/v_n,\\ v_{n+1} = u_n v_n^3, \end{cases} \qquad \begin{cases} u_{n+5} = 1/(u_{n+4} v_{n+4}),\\ v_{n+5} = u_{n+4} v_{n+4}^2 (1 - u_{n+4}), \end{cases}
$$

$$
\begin{cases} u_{n+6} = 1/(v_{n+4}(1 - u_{n+4})),\\ v_{n+6} = u_{n+4} v_{n+4}^2 (1 - u_{n+4}). \end{cases}
$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{n+8}$, where

$$
\begin{aligned}
C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\} \quad 1 \le t \le n+4 \text{ and } t = n+7, n+8,\\
C_{n+5} &= \{v_{n+3} = 1\} \cup \{u_{n+4} = 1\} \cup \{v_{n+5} = 0\},\\
C_{n+6} &= \{u_{n+4} = 0\} \cup \{v_{n+6} = 0\}.
\end{aligned}
$$

A holomorphic function on M can be generated by the following forms:

$$
\begin{aligned}
&u_{n+4}^a v_{n+4}^b (1 - u_{n+4})^c\\
&\qquad = u_{n+3}^b v_{n+3}^{-a+2b-c} (v_{n+3} - 1)^c\\
&\qquad = u_{n+2}^{-a+2b-c} v_{n+2}^{-2a+3b-2c} (u_{n+2} v_{n+2}^2 - 1)^c\\
&\qquad = u_{n+1}^{-2a+3b-2c} v_{n+1}^{-3a+4b-3c} (u_{n+1}^2 v_{n+1}^3 - 1)^c\\
&\qquad = u_t^{(5n-5t+4)b-(4n-4t+3)(a+c)} v_t^{(5n-5t+9)b-(4n-4t+7)(a+c)}\\
&\qquad\quad \cdot (u_t^{4n-4t+3} v_t^{4n-4t+7} - 1)^c\\
&\qquad = u_{n+5}^{-b+2c} v_{n+5}^c (1 + u_{n+5}^2 v_{n+5})^{-a+b-c}\\
&\qquad = u_{n+6}^{2a-b} v_{n+6}^a (1 + u_{n+6}^2 v_{n+6})^{-a+b-c}\\
&\qquad = u_{n+7}^{3a-2b} v_{n+7}^{2a-b} (1 + u_{n+7}^3 v_{n+7}^2)^{-a+b-c}\\
&\qquad = u_{n+8}^{4a-3b} v_{n+8}^{3a-2b} (1 + u_{n+8}^4 v_{n+8}^3)^{-a+b-c},
\end{aligned}
$$

where $0 \le t \le n$, such that

$$\text{(4-20)} \qquad \begin{cases} a, b, c \ge 0, \\ (5n+9)b \ge (4n+7)(a+c), \\ 2c \ge b, \\ 4a \ge 3b. \end{cases}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= u_0^n v_0^{n+1} du_0 \wedge dv_0 = u_1^{n-1} v_1^n du_1 \wedge dv_1 \\ &= \cdots = u_{n-1} v_{n-1}^2 du_{n-1} \wedge dv_{n-1} = u_n du_n \wedge dv_n \\ &= du_{n+1} \wedge dv_{n+1} = \cdots = du_{n+4} \wedge dv_{n+4} \\ &= -\frac{du_{n+5} \wedge dv_{n+5}}{1+u_{n+5}^3 v_{n+5}} = \frac{du_{n+6} \wedge dv_{n+6}}{1+u_{n+6}^2 v_{n+6}} \\ &= \frac{du_{n+7} \wedge dv_{n+7}}{1+u_{n+7}^3 v_{n+7}^2} = \frac{du_{n+8} \wedge dv_{n+8}}{1+u_{n+8}^4 v_{n+8}^3}, \end{aligned}$$

such that

$$\text{(4-21)} \qquad \begin{cases} a, b, c \ge 0, \\ (5n+9)b + n + 1 \ge (4n+7)(a+c), \\ 2c \ge b, \\ 4a \ge 3b. \end{cases}$$

Type $C_{m,n}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, & t &= 0, 1, \dots, m+n+2, \\ W_t &= \{(u_t, v_t) : u_t^2 v_t \ne -1\}, & t &= m+n+3, m+n+4. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le n-1 \text{ and } n+1 \le t \le m+n+1,$$

$$\begin{cases} u_{n+1} = 1/v_n, \\ v_{n+1} = u_n v_n^3, \end{cases} \qquad \begin{cases} u_{m+n+3} = 1/(u_{m+n+2} v_{m+n+2}), \\ v_{m+n+3} = u_{m+n+2} v_{m+n+2}^2 (1 - u_{m+n+2}), \end{cases}$$

$$\begin{cases} u_{m+n+4} = 1/(v_{m+n+2}(1 - u_{m+n+2})), \\ v_{m+n+4} = u_{m+n+2} v_{m+n+2}^2 (1 - u_{m+n+2}). \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{m+n+4}$, where

$$
\begin{aligned}
C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le m+n+2,\\
C_{m+n+3} &= \{v_{m+n+1} = 1\} \cup \{u_{m+n+2} = 1\} \cup \{v_{m+n+3} = 0\},\\
C_{m+n+4} &= \{u_{m+n+2} = 0\} \cup \{v_{m+n+4} = 0\}.
\end{aligned}
$$

A holomorphic function on M can be generated by the following forms:

$$
\begin{aligned}
&u_{m+n+2}^a v_{m+n+2}^b (1-u_{m+n+2})^c\\
&\qquad = u_t^{(m+n+2-t)b-(m+n+1-t)(a+c)} v_t^{(m+n+3-t)b-(m+n+2-t)(a+c)}\\
&\qquad\quad \cdot (u_t^{m+n+1-t} v_t^{m+n+2-t} - 1)^c\\
&\qquad = u_s^{[(m+3)(n-s)+m+2)]b-[(m+3)(n-s)+m+1)](a+c)}\\
&\qquad\quad \cdot v_s^{[(m+3)(n-s)+m+2)]b-[(m+3)(n-s)+m+1)](a+c)}\\
&\qquad\quad \cdot (u_s^{(m+3)(n-s)+m+2)} v_s^{(m+3)(n-s)+2m+3)} - 1)^c\\
&\qquad = u_{m+n+3}^{-b+2c} v_{m+n+3}^c (1+u_{m+n+3}^2 v_{m+n+3})^{-a+b-c}\\
&\qquad = u_{m+n+4}^{2a-b} v_{m+n+4}^a (1+u_{m+n+4}^2 v_{m+n+4})^{-a+b-c},
\end{aligned}
$$

where $n+1 \le t \le m+n+1$, $0 \le s \le n$, such that

$$
\begin{cases}
a, b, c \ge 0,\\
(mn+3m+2n+5)b \ge (mn+2m+2n+3)(a+c),\\
2c \ge b,\\
2a \ge b.
\end{cases}
\tag{4-22}
$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$
\begin{aligned}
\varphi_0 &= u_0^n v_0^{n+1} du_0 \wedge dv_0 = u_1^{n-1} v_1^n du_1 \wedge dv_1\\
&= \cdots = u_{n-1} v_{n-1}^2 du_{n-1} \wedge dv_{n-1} = u_n du_n \wedge dv_n = du_{n+1} \wedge dv_{n+1}\\
&= \cdots = du_{m+n+2} \wedge dv_{m+n+2} = -\frac{du_{m+n+3} \wedge dv_{m+n+3}}{1+u_{m+n+3}^2 v_{m+n+3}}\\
&= \frac{du_{m+n+4} \wedge dv_{m+n+4}}{1+u_{m+n+4}^2 v_{m+n+4}},
\end{aligned}
$$

such that

$$
\begin{cases}
a, b, c \ge 0,\\
(mn+3m+2n+5)b + n + 1 \ge (mn+2m+2n+3)(a+c),\\
2c \ge b,\\
2a \ge b.
\end{cases}
\tag{4-23}
$$

Type $B_{m,n}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0, 1, \dots, m+2, \\ W_{m+3} &= \{(u_{m+3}, v_{m+3}) : u_{m+3}^2 v_{m+3} \neq -1\}, \\ W_t &= \{(u_t, v_t) : u_t^{t-m-2} v_t^{t-m-3} \neq -1\}, && m+4 \le t \le m+n+4. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le m-1, t = m+1, \text{and } m+4 \le t \le m+n+3,$$

$$\begin{cases} u_{m+1} = 1/v_m, \\ v_{m+1} = u_m v_m^3, \end{cases} \qquad \begin{cases} u_{m+3} = 1/(u_{m+2} v_{m+2}), \\ v_{m+3} = u_{m+2} v_{m+2}^2 (1 - u_{m+2}), \end{cases}$$

$$\begin{cases} u_{m+4} = 1/(v_{m+2}(1 - u_{m+2})), \\ v_{m+4} = u_{m+2} v_{m+2}^2 (1 - u_{m+2}). \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \dots \cup C_{m+n+4}$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le m+2 \text{ and } m+5 \le t \le m+n+4, \\ C_{m+3} &= \{v_{m+1} = 1\} \cup \{u_{m+2} = 1\} \cup \{v_{m+3} = 0\}, \\ C_{m+4} &= \{u_{m+2} = 0\} \cup \{v_{m+4} = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} &u_{m+2}^a v_{m+2}^b (1 - u_{m+2})^c \\ &\qquad = u_{m+1}^b v_{m+1}^{-a+2b-c} (v_{m+1} 1)^c \\ &\qquad = u_t^{(3m-3t+2)b-(2m-2t+1)(a+c)} v_t^{(3m-3t+5)b-(2m-2t+3)(a+c)} \\ &\qquad\quad \cdot (u_t^{2m-2t+1} v_t^{2m-2t+3} - 1)^c \\ &\qquad = u_{m+3}^{-b+2c} v_{m+3}^c (1 + u_{m+3}^2 v_{m+3})^{-a+b-c} \\ &\qquad = u_s^{(t-m-2)a-(t-m-3)b} v_s^{(t-m-3)a-(t-m-4)b} (u_s^{t-m-2} v_s^{t-m-3} - 1)^c, \end{aligned}$$

where $0 \le t \le m$, $m+4 \le s \le m+n+4$, such that

$$\begin{cases} a, b, c \ge 0, \\ (3m+5)b \ge (2m+3)(a+c), \\ 2c \ge b, \\ (n+2)a \ge (n+1)b. \end{cases} \tag{4-24}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned}\varphi_0 &= u_0^m v_0^{m+1} du_0 \wedge dv_0 = u_1^{m-1} v_1^m du_1 \wedge dv_1 \\ &= \cdots = u_{m-1} v_{m-1}^2 du_{m-1} \wedge dv_{m-1} = u_m du_m \wedge dv_m \\ &= du_{m+1} \wedge dv_{m+1} = du_{m+2} \wedge dv_{m+2} \\ &= -\frac{du_{m+3} \wedge dv_{m+3}}{1 + u_{m+3}^2 v_{m+3}} \\ &= \frac{du_t \wedge dv_t}{1 + u_t^{t-m-2} v_t^{t-m-3}},\end{aligned}$$

where $m + 4 \le t \le m + n + 4$, such that

$$\begin{cases} a, b, c \ge 0, \\ (3m+5)b + m + 1 \ge (2m+3)(a+c), \\ 2c \ge b, \\ (n+2)a \ge (n+1)b. \end{cases} \tag{4-25}$$

Type $A_{m,n,k}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0, 1, \ldots, m+1, \\ W_t &= \{(u_t, v_t) : u_t^{t-m} v_t^{t-m-1} \neq -1\}, && m+2 \le t \le m+n+1, \\ W_t &= \{(u_t, v_t) : u_t^{t-m-n} v_t^{t-m-n-1} \neq -1\}, && m+n+2 \le t \le m+n+k+1. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, & 0 \le t \le m-1,\ m+2 \le t \le m+n, \text{ or} \\ v_{t+1} = u_t v_t^2, & m+n+2 \le t \le m+n+k, \end{cases}$$

$$\begin{cases} u_{m+1} = 1/v_m, \\ v_{m+1} = u_m v_m^3, \end{cases}$$

$$\begin{cases} u_{m+2} = 1/(v_{m+1}(1-u_{m+1})), \\ v_{m+2} = u_{m+1} v_{m+1}^2 (1-u_{m+1}), \end{cases}$$

$$\begin{cases} u_{m+n+2} = 1/(u_{m+1} v_{m+1}), \\ v_{m+n+2} = u_{m+1} v_{m+1}^2 (1-u_{m+1}). \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{m+n+k+1}$, where

$$C_t = \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad t \neq m+n+2,$$
$$C_{m+n+2} = \{v_m = 1\} \cup \{u_{m+1} = 1\} \cup \{v_{m+n+2} = 0\}.$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} u_{m+1}^a v_{m+1}^b (1-u_{m+1})^c \\ &= u_t^{(2m-2t+1)b-(m-t)(a+c)} v_t^{(2m-2t+3)b-(m-t+1)(a+c)} (u_t^{m-t} v_t^{m-t+1} - 1)^c \\ &= u_s^{(s-m)a-(s-m-1)b} v_s^{(s-m-1)a-(s-m-2)b} (u_s^{s-m} v_s^{s-m-1} - 1)^c, \\ &= u_r^{(r-m-n)c-(r-m-n-1)b} v_r^{(r-m-n-1)c-(r-m-n-2)b} \\ &\quad \cdot (u_r^{r-m-n} v_r^{r-m-n-1} - 1)^c, \end{aligned}$$

where $0 \leq t \leq m$, $m+2 \leq s \leq m+n+1$, $m+n+2 \leq r \leq m+n+k+1$, such that

$$\begin{cases} a, b, c \geq 0, \\ (2m+3)b \geq (m+1)(a+c), \\ (n+1)a \geq nb, \\ (k+1)c \geq kb. \end{cases} \tag{4-26}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= u_0^m v_0^{m+1} du_0 \wedge dv_0 = u_1^{m-1} v_1^m du_1 \wedge dv_1 \\ &= \cdots = u_{m-1} v_{m-1}^2 du_{m-1} \wedge dv_{m-1} = u_m du_m \wedge dv_m = du_{m+1} \wedge dv_{m+1} \\ &= \frac{du_t \wedge dv_t}{1 + u_t^{t-m} v_t^{t-m-1}} \\ &= -\frac{du_s \wedge dv_s}{1 + u_s^{s-m-n} v_s^{s-m-n-1}}, \end{aligned}$$

where $m+2 \leq t \leq m+n+1$, $m+n+2 \leq s \leq m+n+k+1$, such that

$$\begin{cases} a, b, c \geq 0, \\ (2m+3)b + m + 1 \geq (m+1)(a+c), \\ (n+1)a \geq nb, \\ (k+1)c \geq kb. \end{cases} \tag{4-27}$$

Theorem 4.4. *For 2-dimensional rational triple points*, $f^{(1,1)} = g^{(1,1)} = 1$.

Proof. We consider only type $E_{6,0}$ as an example; the other singularities are similar. From the above calculation, we know that the holomorphic functions on M are generated by a base $\{u_4^a v_4^b (1-u_4)^c\}$ satisfying (4-10), and holomorphic 2-forms are generated by a base $\{u_4^a v_4^b (1-u_4)^c du_4 \wedge dv_4\}$ satisfying (4-11). For every holomorphic 2-form $\omega = u_4^a v_4^b (1-u_4)^c du_4 \wedge dv_4$ on M, we consider

$$\xi = -\frac{u_4^a v_4^{b+1}(1-u_4)^c}{b+1} du_4.$$

So, ξ defines a holomorphic 1-form on W_0 and $d\xi = \omega$. In fact, we need to check that under all changes of coordinate charts, ξ transforms to define a holomorphic 1-form in each coordinate chart:

$$\begin{aligned}
u_4^a v_4^{b+1}(1-u_4)^c du_4 &= -u_3^{b+1} v_3^{-a+2b-c}(v_3-1)^c dv_3 \\
&= -u_2^{-a+2b-c} v_2^{-2a+3b-2c+1}(u_2 v_2^2 - 1)^c du_2 \\
&\quad - 2u_2^{-a+2b-c+1} v_2^{-2a+3b-2c}(u_2 v_2^2 - 1)^c dv_2 \\
&= -2u_1^{-2a+3b-2c} v_1^{-3a+4b-3c+1}(u_1^2 v_1^3 - 1)^c du_1 \\
&\quad - 3u_1^{-2a+3b-2c+1} v_1^{-3a+4b-3c}(u_1^2 v_1^3 - 1)^c dv_1 \\
&= -3u_0^{-3a+4b-3c} v_0^{-4a+5b-4c+1}(u_0^3 v_0^4 - 1)^c du_0 \\
&\quad - 4u_0^{-3a+4b-3c+1} v_0^{-4a+5b-4c}(u_0^3 v_0^4 - 1)^c dv_0 \\
&= -2u_5^{-b+2c} v_5^{c+1}(1+u_5^2 v_5)^{-a+b-c-1} du_5 \\
&\quad - u_5^{-b+2c+1} v_5^{c}(1+u_5^2 v_5)^{-a+b-c-1} dv_5 \\
&= 2u_6^{2a-b} v_6^{a+1}(1+u_6^2 v_6)^{-a+b-c-1} du_6 \\
&\quad + u_6^{2a-b+1} v_6^{a}(1+u_6^2 v_6)^{-a+b-c-1} dv_6 \\
&= 5u_7^{5a-3b+1} v_7^{2a-b+1}(1+u_7^5 v_7^2)^{-a+b-c-1} du_7 \\
&\quad + 2u_7^{5a-3b+2} v_7^{2a-b}(1+u_7^5 v_7^2)^{-a+b-c-1} dv_7.
\end{aligned}$$

By Lemma 4.3, we can get general expressions for elements in $\Gamma(M, \Omega_M^1)$. Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = \langle u_4 \wedge v_4 \rangle,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = 1. \qquad \square$$

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HOMOGENEITY GROUPS OF ENDS OF OPEN 3-MANIFOLDS

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For every finitely generated abelian group G, we construct an irreducible open 3-manifold M_G whose end set is homeomorphic to a Cantor set and whose homogeneity group is isomorphic to G. The end homogeneity group is the group of self-homeomorphisms of the end set that extend to homeomorphisms of the 3-manifold. The techniques involve computing the embedding homogeneity groups of carefully constructed Antoine-type Cantor sets made up of rigid pieces. In addition, a generalization of an Antoine Cantor set using infinite chains is needed to construct an example with integer homogeneity group. Results about the local genus of points in Cantor sets and about the geometric index are also used.

1. Introduction

Each Cantor set C in S^3 has for complement an open 3-manifold M^3 with end set C. Properties of the embedding of the Cantor set give rise to properties of the corresponding complementary 3-manifold M^3. See [Souto and Stover 2013], [Garity and Repovš 2013], and [Garity et al. 2014] for examples of this.

We investigate possible group actions on the end set C of the open 3-manifold M^3 in the following sense: the *homogeneity group of the end set* is the group of homeomorphisms of the end set C that extend to homeomorphisms of the open 3-manifold M^3. Referring specifically to the embedding of the Cantor set, this group can also be called the *embedding homogeneity group of the Cantor set*. See [Dijkstra 2010] and [van Mill 2011] for some other types of homogeneity.

The standardly embedded Cantor set is at one extreme here. The embedding homogeneity group is the full group of self-homeomorphisms of the Cantor set, an extremely rich group (there is such a homeomorphism taking any countable dense set to any other). Cantor sets with this full embedding homogeneity group are called *strongly homogeneously embedded*. See [Daverman 1979] for an example of a nonstandard Cantor set with this property.

At the other extreme are *rigidly embedded* Cantor sets, those Cantor sets for

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which only the identity homeomorphism extends. Shilepsky [1974] constructed Antoine-type [1920] rigid Cantor sets. Their rigidity is a consequence of Sher's result [1968] that if two Antoine Cantor sets are equivalently embedded, the stages of their defining sequences must match up exactly. In the last decade, new examples [Garity et al. 2006; 2011] of nonstandard Cantor sets were constructed that were both rigidly embedded and had simply connected complement. See [Wright 1986] for additional examples of rigidity.

These examples naturally lead to the question of which types of groups can arise as end homogeneity groups between the two extremes mentioned above. In this paper we show that for each finitely generated abelian group G, there is an irreducible open 3-manifold with end set homeomorphic to a Cantor set and end homogeneity group isomorphic to G. (See Corollary 6.3.)

The Cantor sets produced are *unsplittable*, in the sense that for each such C, no 2-sphere in the complement of C separates points of C. We produce these examples by constructing, for each natural number m greater than one, 3-manifolds with end homogeneity groups $\mathbb{Z}_m$, and by separately constructing 3-manifolds with end homogeneity group $\mathbb{Z}$. We then link the Cantor sets needed for a given abelian group in an unsplittable manner.

In Section 2, we give definitions and the basic results needed for the rest of the paper. In Section 3, we review the needed results about Antoine Cantor sets. In Section 4 we produce Cantor sets with embedding homogeneity group $\mathbb{Z}_m$. In Section 5 we produce Cantor sets with embedding homogeneity group $\mathbb{Z}$. Section 6 ties together the previous results and lists and proves the main theorems. Section 7 lists some remaining questions.

2. Preliminaries

Background. Refer to [Garity et al. 2005; 2006; 2014] for a discussion of Cantor sets in general and of rigid Cantor sets, and to [Željko 2005] for results about the local genus of points in Cantor sets and defining sequences for Cantor sets. The bibliographies in these papers contain additional references to results about Cantor sets. Two Cantor sets X and Y in S^3 are said to be *equivalent* if there is a self-homeomorphism of S^3 taking X to Y; otherwise they are *inequivalent*, or *inequivalently embedded*. A Cantor set C is *rigidly embedded* in S^3 if the only self-homeomorphism of C that extends to a homeomorphism of S^3 is the identity.

Geometric index. We list the results we need on geometric index. See [Schubert 1953] and [Garity et al. 2011] for more details.

If K is a link in the interior of a solid torus T, the *geometric index* of K in T, denoted by $\mathrm{N}(K, T)$, is defined as the minimum of $|K \cap D|$ over all meridional disks D of T intersecting K transversely. If T is a solid torus and M is a finite union of disjoint solid tori such that $M \subset \mathrm{Int}(T)$, then the geometric index $\mathrm{N}(M, T)$ of M in T is $\mathrm{N}(K, T)$, where K is a core of M.

Theorem 2.1 [Schubert 1953; Garity et al. 2011, Theorem 3.1]. *Let T_0 and T_1 be unknotted solid tori in S^3 with $T_0 \subset \mathrm{Int}(T_1)$ and $\mathrm{N}(T_0, T_1) = 1$. Then ∂T_0 and ∂T_1 are parallel*; *i.e., the manifold $T_1 - \mathrm{Int}(T_0)$ is homeomorphic to $\partial T_0 \times I$, where I is the closed unit interval* [0, 1].

Theorem 2.2 [Schubert 1953; Garity et al. 2011, Theorem 3.2]. *Let T_0 be a finite union of disjoint solid tori. Let T_1 and T_2 be solid tori such that $T_0 \subset \mathrm{Int}(T_1)$ and $T_1 \subset \mathrm{Int}(T_2)$. Then $\mathrm{N}(T_0, T_2) = \mathrm{N}(T_0, T_1) \cdot \mathrm{N}(T_1, T_2)$.*

There is one additional result we will need:

Theorem 2.3 [Schubert 1953; Garity et al. 2011, Theorem 3.3]. *Let T be a solid torus in S^3 and let $T_1, \dots, T_n$ be unknotted pairwise disjoint solid tori in T, each of geometric index 0 in T. Then the geometric index of $\bigcup_{i=1}^{n} T_i$ in T is even.*

Defining sequences and local genus. We review the definition and some basic facts from [Željko 2005] about the local genus of points in a Cantor set. See that work for a discussion of defining sequences.

Let $\mathfrak{D}(X)$ be the set of all defining sequences for a Cantor set $X \subset S^3$. Let $(M_i) \in \mathfrak{D}(X)$ be a specific defining sequence for an X. For $A \subset X$, denote by M_i^A the union of those components of M_i which intersect A. The *genus* $g(M_i^A)$ of M_i^A is the maximum genus of a component of M_i^A. Define

$$g_A(X; (M_i)) = \sup\{g(M_i^A) : i \geq 0\}, \quad g_A(X) = \inf\{g_A(X; (M_i)) : (M_i) \in \mathfrak{D}(X)\}.$$

The number $g_A(X)$ is called *the genus of the Cantor set X with respect to the subset A*. For $A = \{x\}$ we call the number $g_{\{x\}}(X)$ *the local genus of the Cantor set X at the point x* and denote it by $g_x(X)$.

Let x be an arbitrary point of a Cantor set X and $h\colon S^3 \to S^3$ a homeomorphism. Then the local genus $g_x(X)$ is the same as the local genus $g_{h(x)}(h(X))$. Also note that if $x \in C \subset C'$, then the local genus of x in C is less than or equal to the local genus of x in C'. See [Željko 2005, Theorem 2.4].

The following result is needed to show that certain points in our examples have local genus 2.

Theorem 2.4 [Željko 2005]. *Let $X, Y \subset S^3$ be Cantor sets and $p \in X \cap Y$. Suppose there exists a 3-ball B and a 2-disk $D \subset B$ such that*

(1) $p \in \mathrm{Int}(B)$, $\partial D = D \cap \partial B$, $D \cap (X \cup Y) = \{p\}$; *and*

(2) $X \cap B \subset B_X \cup \{p\}$ *and* $Y \cap B \subset B_Y \cup \{p\}$, *where B_X and B_Y are the components of $B - D$.*

Then $g_p(X \cup Y) = g_p(X) + g_p(Y)$.

Discussion and examples of ends and homogeneity groups. For background on Freudenthal compactifications and theory of ends, see [Dickman 1968; Freudenthal 1942; Siebenmann 1965]. For an alternate proof using defining sequences of the result that every homeomorphism of the open 3-manifold extends to a homeomorphism of its Freudenthal compactification, see [Garity and Repovš 2013].

At the end of the next section, we will discuss elements of the homogeneity group of a standard self-similar Antoine Cantor set. Note that removing n points from S^3 yields a reducible open 3-manifold with end homogeneity group the symmetric group on n elements. It is not immediately obvious how to produce examples that are irreducible, have a rich end structure (for example a Cantor set), and at the same time have specified abelian end homogeneity groups.

3. Properties of the Antoine Cantor set

An Antoine Cantor set is described by a defining sequence (M_i) as follows: Let M_0 be an unknotted solid torus in S^3. Let M_1 be a chain of at least four linked, pairwise disjoint, unknotted solid tori in M_0, as in Figure 1. Inductively, M_i consists of pairwise disjoint solid tori in S^3 and M_{i+1} is obtained from M_i by placing a chain of at least four linked, pairwise disjoint, unknotted solid tori in each component of M_i. If the diameter of the components goes to 0, the Antoine Cantor set is $C = \bigcap_{i=0}^{\infty} M_i$.

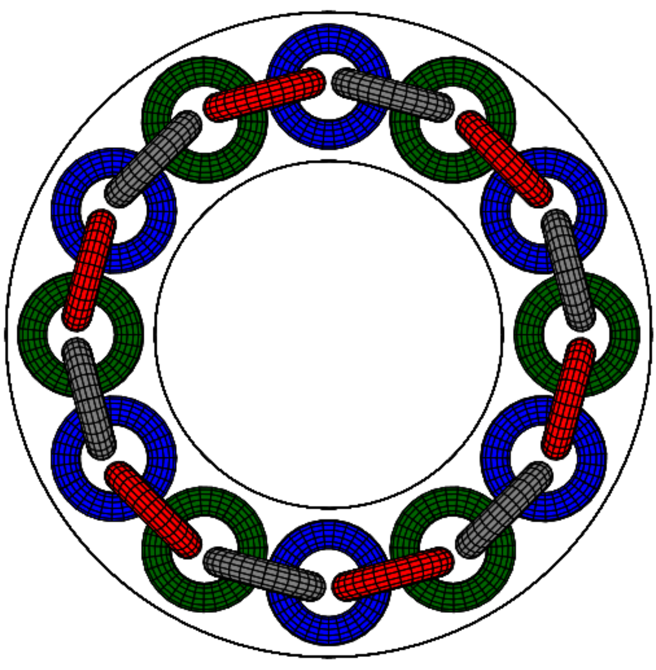

Figure 1. Antoine chain with $\mathbb{Z}_6$ group action.

We refer to [Sher 1968] for basic results and description of Antoine Cantor sets. The key result we shall need is the following:

Theorem 3.1 [Sher 1968, Theorems 1 and 2]. *Two Antoine Cantor sets in S^3, with defining sequences (M_i) and (N_i), respectively, are equivalently embedded if and only if there is a self-homeomorphism h of S^3 with $h(M_i) = N_i$ for each i.*

In particular, the number and adjacency of links in the chains must match up at each stage. Because we need a modification of this result for infinite chains in Section 5, we outline an alternative proof of the forward implication.

Proof of forward implication of Theorem 3.1. It suffices to show that if C has two Antoine defining sequences (M_i) and (N_i), then there is a homeomorphism h as in the theorem.

Step 1: There is a general position homeomorphism h_1, fixed on C, such that $h_1(\partial(M_1) \cup \partial(M_2))$ is in general position with $\partial(N_1) \cup \partial(N_2)$. The curves of intersection of $h_1(\partial(M_1) \cup \partial(M_2)) \cap (\partial(N_1) \cup \partial(N_2))$ can be eliminated by a homeomorphism h_2 also fixed on C, by a standard argument and the facts that any nontrivial curve on $\partial(M_i)$ does not bound a disk in the complement of C and that no 2-sphere separates the points of C. For details on the type of argument in this step, see [Sher 1968] or [Garity et al. 2011].

Step 2: Let T be a component of $h_2 \circ h_1(M_1)$ and assume T intersects a component S of N_1. Either $T \subset \mathrm{Int}(S)$ or $S \subset \mathrm{Int}(T)$. First assume $T \subset \mathrm{Int}(S)$. If the geometric index of T in S is 0, then since the other components of $h_2 \circ h_1(M_1)$ are linked to T by a finite chain, all components of $h_2 \circ h_1(M_1)$ are in the interior of S. This is a contradiction since there are points of C not in S. So the geometric index of T in S is greater than or equal to 1.

Note that T cannot be contained in any component of N_2 that is in S since these have geometric index 0 in S. So T contains all the components of N_2 that are in S. Each of these components has geometric index 0 in T, so the union of these components has an even geometric index in T by Theorem 2.3. This geometric index must then be 2 and the geometric index of T in S must be 1. Now there is a homeomorphism h_3, fixed on C and the complement of S, that takes T to S.

If instead $S \subset \mathrm{Int}(T)$, a similar argument shows there is a homeomorphism h_3, fixed on C and the complement of T, taking S to T. The net result is that it is possible to construct a homeomorphism h_3' taking the components of $h_2 \circ h_1(M_1)$ to the components of S. One now proceeds inductively, matching up further stages in the constructions, obtaining the desired homeomorphism h as a limit. □

Remark 3.2. A standard argument shows that an Antoine Cantor set cannot be separated by a 2-sphere. This is also true if the construction starts with a finite open chain of linked tori as in Figure 3.

Remark 3.3. Also note that the homeomorphism of Theorem 3.1 can be realized as the final stage of an isotopy since each of the homeomorphisms in the argument can be realized by an isotopy.

Homogeneity groups of Antoine Cantor sets. Let C be obtained by a standard Antoine construction where the same number of tori are used in tori of previous stages in each stage of the construction. For example, the Antoine pattern in Figure 1 with 24 smaller tori, each geometrically similar to the outer torus, can be repeated in each component at each stage of the construction.

We now consider some elements of the embedding homogeneity group of C. There is an obvious $\mathbb{Z}_{24}$ group action on the resulting Cantor set obtained by rotating and twisting the large torus. There is also a $\mathbb{Z}_{24} \oplus \mathbb{Z}_{24}$ action on C obtained by considering the first two stages, where we require each torus in the second stage to rotate the same amount. If we allow the tori in the second stage to rotate different amounts, we get an even larger group action by a wreath product of $\mathbb{Z}_{24}$ with itself. Considering more stages results in even more complicated group actions.

In addition to these group actions arising from rotating and twisting, there are also orientation-reversing $\mathbb{Z}_2$ actions that arise from reflecting through a horizontal plane (containing the core of the large torus) or through a vertical plane (containing meridians of the large torus).

From this we see that even for a simple self-similar Antoine Cantor set, the embedding homogeneity group is more complex than just the group of obvious rotations from the linking structure. In the next section we shall carefully combine certain Antoine constructions to produce a more rigid example with nontrivial end homogeneity group, in such a way that these kinds of orientation-reversing homeomorphisms are not possible, and that also restricts the possible rotations.

4. A Cantor set with embedding homogeneity group $\mathbb{Z}_m$

Fix an integer $m > 1$. We describe how to construct a Cantor set in S^3 with embedding homogeneity group $\mathbb{Z}_m$.

Construction 4.1. As in the previous section, let S_0 be an unknotted solid torus in S^3. Let $\{S_{(1,i)} : 1 \leq i \leq 4m\}$ be an Antoine chain of $4m$ pairwise disjoint linked solid tori in the interior of S_0 and let

$$S_1 = \bigcup_{i=1}^{4m} S_{(1,i)}.$$

See Figure 1 for the case when $m = 6$. Let C_j, $1 \leq j \leq 4$, be a rigid Antoine Cantor set with first stage $S_{(1,j)}$. Choose these four rigid Antoine Cantor sets so that they are inequivalently embedded in S^3. Let h be a homeomorphism of S^3, fixed on the complement of the interior of S_0, that takes $S_{(1,j)}$ to $S_{(1,j+4 \bmod 4m)}$ for $1 \le j \le 4m$. Require that h^m is the identity on each $S_{(1,i)}$.

For $4k < i \le 4k+4$, let C_i be the rigid Cantor set in $S_{(1,i)}$ given by $h^k(C_{i-4k})$. Note that this produces m copies of each of the rigid Cantor sets C_1, C_2, C_3, and C_4. Again, see Figure 1, where the coloring indicates the four classes of rigid Cantor sets. The Cantor set we are looking for is

$$C = \bigcup_{i=1}^{4m} C_i.$$

Theorem 4.2. *The Cantor set C from the previous construction has embedding homogeneity group $\mathbb{Z}_m$ and is unsplittable.*

Proof. Let $\ell : S^3 \to S^3$ be a homeomorphism taking C to C. We show that $\ell|_C = h^k|_C$ for some k, $1 \le k \le m$. By [Sher 1968], we may assume that ℓ takes each $S_{(1,i)}$ to some $S_{(1,j)}$, and so $\ell(C_i) = C_j$. Because of the distinct rigid Cantor sets involved, this is only possible if $j - i \equiv 0 \bmod 4$.

So assume that $\ell(S_{(1,1)}) = S_{(1,4k+1)}$. Then $\ell(S_{(1,2)})$ must be one of the two tori linked with $S_{(1,4k+1)}$, namely $S_{(1,4k)}$ or $S_{(1,4k+2)}$. Since $(4k-2) \not\equiv 0 \bmod 4$, $\ell(S_{(1,2)})$ must be $S_{(1,4k+2)}$. Continuing inductively, one sees that $\ell(S_{(1,i)}) = S_{(1,4k+i \bmod 4m)}$. Thus $\ell(C_i) = C_{4k+i \bmod 4m}$. But $h^k(C_i)$ is also $C_{4k+i \bmod 4m}$. Since these are rigid Cantor sets, $\ell|_{C_i} = h^k|_{C_i}$ for each i.

So the embedding homogeneity group of C is $\{h^k : 1 \le k \le p\} \simeq \mathbb{Z}_m$. By Remark 3.2, C is unsplittable. The assertion follows. $\square$

5. A Cantor set with embedding homogeneity group $\mathbb{Z}$

We now construct a Cantor set in S^3 with embedding homogeneity group $\mathbb{Z}$. This requires careful analysis of an infinite chain analogue of the Antoine construction.

Construction 5.1. Let S_0 be a pinched solid torus in S^3, i.e., the quotient of a solid torus with a meridional disk collapsed to a single point w. Let T_i, $i \in \mathbb{Z}$, be a countable collection of unknotted pairwise disjoint solid tori in S_0 such that each T_i is of simple linking type with both T_{i-1} and T_{i+1}, and is not linked with T_j, $j \ne i-1$ or $i+1$. Place the tori T_i so that the T_i, $i > 0$, and the T_i, $i < 0$, have w as a limit point as in Figure 2.

For $1 \le j \le 3$, let C_j be a rigid Antoine Cantor set with first stage T_j. Choose these three rigid Antoine Cantor sets so that they are inequivalently embedded in S^3. Let α be a homeomorphism of S^3, fixed on the complement of the interior of S_0, that takes T_j to T_{j+3} for $j \in \mathbb{Z}$.

For $3k < i \le 3k+3$, let C_i be the rigid Cantor set in T_i given by $\alpha^k(C_{i-3k})$. Note that this produces a countable number of copies of each of the rigid Cantor

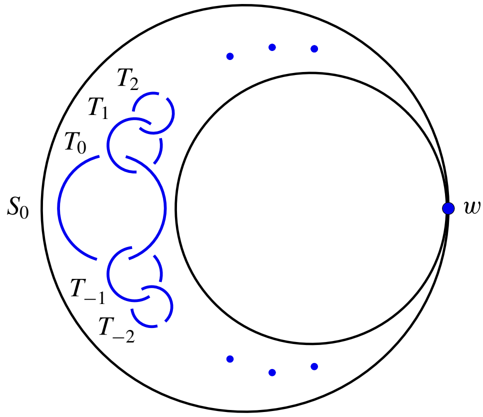

Figure 2. Infinite Antoine chain.

sets C_1, C_2, and C_3. Again, see Figure 2. The Cantor set we are looking for is

$$C = \overline{\bigcup_{i\in\mathbb{Z}} C_i} = \bigcup_{i\in\mathbb{Z}} C_i \cup \{w\}.$$

Note that C is a Cantor set since it is perfect, compact, and totally disconnected.

Theorem 5.2. *The Cantor set C from the previous construction has embedding homogeneity group $\mathbb{Z}$ and is unsplittable.*

Proof. It is clear from the construction that each point of $C - \{w\}$ has local genus 1. Theorem 2.4, applied to w and the Cantor sets $C_+ = \overline{\bigcup_{i>0} C_i}$ and $C_- = \overline{\bigcup_{i<0} C_i}$, shows that w has local genus 2 in C. Thus, any homeomorphism of S^3 that takes C to C must fix w.

Let h be such a homeomorphism of S^3 taking C to C. Let T_i' be the union of the linked tori in the Antoine chain at the second stage of the construction of C_i. Let

$$\Lambda_N = \bigcup_{i=-N}^{N} T_i, \quad \Gamma_N = \bigcup_{i=-N}^{N} C_i, \quad \text{and} \quad \Lambda_N' = \bigcup_{i=-N}^{N} T_i'.$$

Fix an integer $n \in \mathbb{Z}$. Since $h(T_n)$ does not contain w, there is a positive integer $N_1 > |n|$ such that $h(C_n) \subset \Gamma_{N_1}$. Similarly, there is a positive integer $N_2 > N_1$ such that $h^{-1}(\Gamma_{N_1}) \subset \Gamma_{N_2}$.

As in Step 1 in the proof of Theorem 3.1, there is a homeomorphism k of S^3 to itself, fixed on C, such that

$$k\big(h(\partial(\Lambda_{N_2+1}) \cup \partial(\Lambda'_{N_2+1}))\big) \cap \big(\partial(\Lambda_{N_2+1}) \cup \partial(\Lambda'_{N_2+1})\big) = \varnothing.$$

Fix a point p of C_n and let $k(h(p)) = h(p) = q \in C_m$. We will show that $k(h(C_n)) = h(C_n) = C_m$. Let $\ell = k \circ h$. Since $\ell(T_n) \cap T_m \neq \varnothing$, and since the boundaries do not intersect, either $\ell(T_n) \subset \mathrm{Int}(T_m)$ or $\mathrm{Int}(\ell(T_n)) \supset T_m$. We consider these cases separately.

Case I: $\ell(T_n) \subset \mathrm{Int}(T_m)$. If $\ell(T_n)$ has geometric index 0 in T_m, then $\ell(T_n)$ is contained in a cell in T_m and so it contracts in T_m. Since a contraction of $\ell(T_n)$ meets the boundary of the linked $\ell(T_{n+1})$, and since the boundary of $\ell(T_{n+1})$ is disjoint from the boundary of T_m, $\ell(T_{n+1}) \subset \mathrm{Int}(T_m)$. Continuing inductively, one finds that one of the following two situations occur when $\ell(T_n)$ has geometric index 0 in T_m:

Case Ia: Each $\ell(T_j)$, for $n \leq j \leq N_2$, is contained in T_m and has geometric index 0 there. It follows that $\ell(T_{N_2+1}) \subset \mathrm{Int}(T_m)$. But then $C_m \subset \Gamma_{N_1}$ and $h^{-1}(C_m) \cap C_{N_2+1} \neq \varnothing$, contradicting the choice of N_2.

Case Ib: There exists j with $n < j \leq N_2$ and such that $\ell(T_j)$ is contained in $\mathrm{Int}(T_m)$ and geometric index k in T_m, where $k > 1$. Then, by Theorem 2.2, $\ell(T_j)$ cannot be contained in any component of the next stage of the construction contained in T_m, since these have geometric index 0 in T_m. So some component of the next stage in T_m is contained in $\ell(T_j)$ and has geometric index 0 there by Theorem 2.2. Since the components of the next stage are linked, all components of the next stage in T_m are contained in $\ell(T_j)$. The geometric index of the union of the next stages of in T_m in $\ell(T_n)$ is even by Theorem 2.3 and cannot be equal to 0. Otherwise, by Theorem 2.2 the union of the next stages of T_m would have index 0 in T_m, which is a contradiction. So the geometric index of the union of the next stages of in T_m in $\ell(T_n)$ is at least 2. Then by Theorem 2.2, the geometric index of the union of the next stages of in T_m in T_m is at least 4, contradicting the fact that this geometric index is 2.

It follows that $\ell(T_j)$ has geometric index 1 in T_m and contains the union of the next stages contained in T_m. Since ℓ is a homeomorphism that takes C to C, it follows from the construction of C that $\ell(C_j) = C_m$. Since $\ell(p) \in C_m$, $\ell(T_n) \cap \ell(C_j) \neq \varnothing$, contradicting the fact that ℓ is a homeomorphism.

Thus, neither Case Ia nor Case Ib can occur. So the geometric index of $\ell(T_n)$ in T_m must be at least 1. Repeating the argument from Case Ib above with T_j replaced by T_n, we see that $\ell(T_n)$ has geometric index 1 in T_m and contains the union of the next stages contained in T_m. Since ℓ is a homeomorphism that takes C to C, it follows from the construction of C that $\ell(C_n) = C_m$ as desired.

Case II: $\mathrm{Int}(\ell(T_n)) \supset T_m$. Then $\ell^{-1}(T_m) \subset \mathrm{Int}(T_n)$. The argument from Case I can now be repeated, replacing ℓ by ℓ^{-1} and interchanging T_n and T_m. It follows that $\ell^{-1}(C_m) = C_n$ and so $\ell(C_n) = C_m$ as desired.

Since $\ell(C_n) = h(C_n) = C_m$, it must be the case that $(m - n) \equiv 0 \bmod 3$. Continuing as in the proof of the $\mathbb{Z}_m$ result (Theorem 4.2), we have that for each i,

$h(C_i) = C_{i+(m-n)}$. Recall that for the homeomorphism α from the construction of C, it is also the case that $\alpha^{(m-n)/3}(C_i) = C_{i+(m-n)}$. By the rigidity of these Cantor sets, it follows that $\alpha^{(m-n)/3}|_{C_i} = h|_{C_i}$. Thus the embedding homogeneity group of C is $\{\alpha^k : k \in \mathbb{Z}\} \simeq \mathbb{Z}$.

We now show that C is unsplittable. Assume that Σ is a 2-sphere in S^3 that separates C. Choose $\epsilon > 0$ so that the distance from Σ to C is greater than ϵ. Choose N so that each T_i, $|i| \geq N$, has diameter less than $\epsilon/6$ and is within $\epsilon/6$ of w. Since Σ separates C, $w \cup \bigcup_{|i|\geq N} T_i$ must be in one component of $S^3 - \Sigma$ and there must be points of C in the other component of $S^3 - \Sigma$. So $\bigcup_{|i|\leq N} T_i$ contains points in both components of $S^3 - \Sigma$.

Form an Antoine Cantor set C' related to C as follows. Use $\bigcup_{|i|\leq N} T_i$ as a part of the first stage of the construction. Complete the first stage of the construction by adding an unknotted solid torus T, linked to T_N and T_{-N}, that is within the $\epsilon/3$-neighborhood of w. For successive stages of the Antoine Cantor set C' in T_i, $|i| \leq N$, use the successive stages in forming the Cantor set $C_i \subset C$. For successive stages of the Antoine Cantor set C' in T, use any Antoine construction.

By construction and the properties of Σ, the 2-sphere Σ separates the Antoine Cantor set C', contradicting Remark 3.2. □

6. Main results

Given a finitely generated abelian group G, we use the results from the previous two sections to construct an unsplittable Cantor set C_G in S^3 with embedding homogeneity group G.

Construction 6.1. Let $G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k}$ be any finitely generated abelian group. Form a simple chain of $n+k$ pairwise disjoint unknotted solid tori. Figure 3 illustrates the case $n+k = 6$. Label the tori as $T_1, T_2, \ldots, T_{n+k}$ so that T_1 is only linked with T_2, T_{n+k} is only linked with T_{n+k-1}, and each T_i, for $2 \leq i \leq n+k-1$, is linked with T_{i-1} and T_{i+1}.

For $1 \leq i \leq n$, perform Construction 5.1 in T_i, treating a pinched version of T_i in the interior of T_i as the torus S_0 in Construction 5.1. Let w_i be the limit point corresponding to w in Construction 5.1. This yields a Cantor set C_i in T_i with embedding homogeneity group $\mathbb{Z}$. See Figure 4.

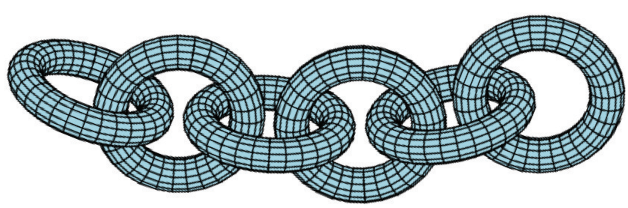

Figure 3. An Antoine chain containing C_G.

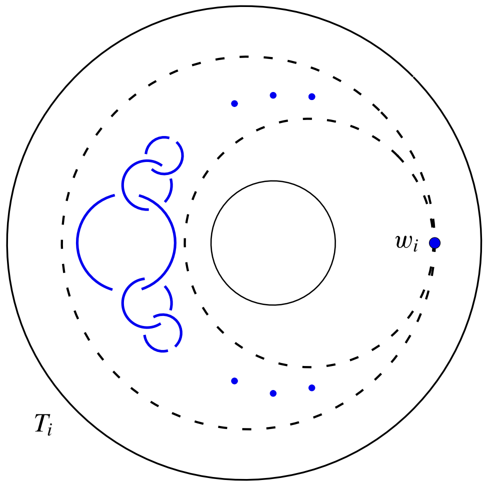

Figure 4. Pinched torus in T_i.

For $n+1 \le i \le n+k$, perform Construction 4.1 for the group $\mathbb{Z}_{m_{i-n}}$ in T_i. This yields a Cantor set C_i in T_i with embedding homogeneity group $\mathbb{Z}_{m_{i-n}}$. Choose all the rigid Cantor sets from Constructions 5.1 and 4.1 to be inequivalent.

Let

$$C_G = \bigcup_{i=1}^{n+k} C_i.$$

Theorem 6.2. *The Cantor set C_G constructed above has embedding homogeneity group G and is unsplittable.*

Proof. For $1 \le i \le n+k$, let h_i be a self-homeomorphism of S^3, fixed on the complement of T_i, such that $h_i|_{C_i}$ generates the embedding homeomorphism group of C_i ($\mathbb{Z}$ for $1 \le i \le n$ and $\mathbb{Z}_{m_{i-n}}$ for $n+1 \le i \le n+k$). Then

$$\left\{(h_1^{j_1} \circ h_2^{j_2} \circ \cdots \circ h_{n+k}^{j_{n+k}})\big|_{C_G}\right\} \simeq G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k}.$$

Let h be a homeomorphism of S^3 to itself taking C_G to C_G. We will show that $h|_{C_G} = (h_1^{j_1} \circ h_2^{j_2} \circ \cdots \circ h_{n+k}^{j_{n+k}})\big|_{C_G}$ for some choice of j_i.

<u>Step 1</u>: The homeomorphism h must take each C_i to itself. As in the proof of Theorem 5.2, there are exactly n points of genus 2 in C_G, one in each C_i, $1 \le i \le n$. These are the points $\{w_1, w_2, \ldots, w_n\}$. The homeomorphism must take this set of genus 2 points to itself.

Let T be one of the solid torus components of the first stage of the Antoine construction for some C_i, $1 \le i \le n+k$. As in the proofs of Theorems 4.2 and 5.2, after a general position adjustment, either $h(T)$ must lie in the interior of some solid torus component T' of the first stage of the Antoine construction for some C_j, or T' must lie in the interior of $h(T)$. A similar argument to that in Theorem 5.2 shows that $N(h(T), T') = 1$ or $N(T', h(T)) = 1$, and that $h(C_i \cap T) = C_j \cap T'$.

This same argument can be applied to all first stage tori in C_i, resulting in the fact that $h(C_i) = C_j$. Because of the inequivalence of the rigid Cantor sets used in the construction, $i = j$ and $h(C_i) = C_i$.

Step 2: For each i, $h|_{C_i} = h_i^{k(i)}|_{C_i}$ for some $k(i)$. By Step 1, we have that $h(C_i) = C_i$. It follows from the construction that $h|_{C_i} = h_i^{k(i)}$ for some $k(i)$. From this, it follows that $h|_{C_G} = (h_1^{j_1} \circ h_2^{j_2} \circ \cdots \circ h_{n+k}^{j_{n+k}})|_{C_G}$ for some choice of j_i.

Thus, the embedding homeomorphism group of C_G is isomorphic to G.

Step 3: C_G is unsplittable. Let Σ be a 2-sphere in S^3 separating C_G. As in the proof of Theorem 5.2, an Antoine Cantor set with first stage $\bigcup_{i=1}^{n+k} T_i$ can be formed so that Σ separates this Antoine Cantor set. This is a contradiction. (See Remark 3.2.) □

Corollary 6.3. *Let $G \simeq \mathbb{Z}^n \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k}$ be any finitely generated abelian group. There is a irreducible open 3-manifold M_G with the following properties*:

(a) *The Freudenthal compactification of M_G is S^3.*

(b) *The end set of M_G is homeomorphic to a Cantor set.*

(c) *The end homogeneity group of M_G is isomorphic to G.*

(d) *The genus of M_G at infinity is 2 at the n points corresponding to $\mathbb{Z}^n$ and is 1 otherwise.*

Proof. Let M_G be $S^3 - C_G$, where C_G is as in Construction 6.1. The end set of M_G is C_G and the end homogeneity group of M_G is isomorphic to the embedding homogeneity group of C_G. M_G is irreducible because C_G is unsplittable. Claims (b) and (c) now follow from Theorem 6.2, while (d) follows from the proof of that theorem. □

Remark 6.4. For each finitely generated abelian group G as above, there are uncountably many nonhomeomorphic 3-manifolds as in the corollary. This follows from varying the rigid Cantor sets used in the construction.

7. Questions

Question 7.1. If a finitely generated abelian group is infinite, is there an open 3-manifold with end homogeneity group G that is genus 1 at infinity?

Question 7.2. Given a finitely generated abelian group G, are there simply connected open 3-manifolds with end homogeneity group G?

Question 7.3. Is the mapping class group of the open 3-manifold M_G isomorphic to G?

Question 7.4. If G is a finitely generated nonabelian group, is there an open 3-manifold with end homogeneity group G?

Question 7.5. If G is a nonfinitely generated group, is there an open 3-manifold with end homogeneity group G?

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ON THE CONCIRCULAR CURVATURE OF A (κ, μ, ν)-MANIFOLD

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We study (κ, μ, ν)-contact metric 3-manifolds (a notion introduced by Koufogiorgos, Markellos and Papantoniou) that are Ricci flat, or are Einstein but not Sasakian, or satisfy $\nabla Z = 0$, where Z is the concircular curvature tensor, or satisfy $Z(\xi, X) \cdot Z = 0$, where ξ is the Reeb field, or satisfy $Z(\xi, X) \cdot S = 0$, where S is the Ricci tensor, or finally satisfy $R(\xi, X) \cdot Z = 0$, where R is the Riemannian curvature tensor.

1. Introduction

A contact metric manifold (M, ξ) is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y = R_0(X, Y)\xi, \tag{1-1}$$

where

$$R_0(X, Y)U = g(Y, U)X - g(X, U)Y, \quad X,\ Y,\ U \in \mathcal{X}(M). \tag{1-2}$$

There exist contact metric manifolds that satisfy the condition $R(X, Y)\xi = 0$; for example, the tangent sphere bundle of a flat Riemannian manifold admits a contact metric satisfying this condition. D. E. Blair, Th. Koufogiorgos and B. Papantoniou [Blair et al. 1995] generalized both this condition and the Sasakian case introducing the (κ, μ)-nullity distribution on a contact metric manifold

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{U \in T_pM \mid R(X, Y)U = (\kappa I + \mu h)R_0(X, Y)U\}$$

for all $X, Y \in \mathcal{X}(M)$, and $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-contact metric manifold. In particular we have

$$R(X, Y)\xi = (\kappa I + \mu h)R_0(X, Y)\xi, \quad X,\ Y \in \mathcal{X}(M), \tag{1-3}$$

with $\kappa \leq 1$ and if $\kappa = 1$ the structure is Sasakian. The full classification of these manifolds was given by E. Boeckx [2000]. If $\mu = 0$ we have the κ-nullity distribution and if $\xi \in N(\kappa)$ we have a $N(\kappa)$-contact metric manifold. Koufogiorgos

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and Ch. Tsichlias [2000] introduced the generalized (κ, μ)-contact metric manifolds, where κ and μ are real functions, and they gave several examples. Finally, the (κ, μ, ν)-contact metric manifolds have been introduced by Koufogiorgos, M. Markellos and V. Papantoniou [Koufogiorgos et al. 2008] where κ, μ, ν are smooth functions and the curvature tensor satisfies, for every $X, Y \in \mathcal{X}(M)$, the condition

$$R(X,Y)\xi = \kappa\big(\eta(Y)X - \eta(X)Y\big) + \mu\big(\eta(Y)hX - \eta(X)hY\big) + \nu\big(\eta(Y)\phi hX - \eta(X)\phi hY\big). \tag{1-4}$$

D. Perrone defined a H-contact metric manifold as a $(2n+1)$-dimensional contact metric manifold M whose characteristic vector field (or the Reeb vector field) ξ is a harmonic vector field. In [Perrone 2004], it was proved that $M(\eta, \xi, \phi, g)$ is an H-contact metric manifold if and only if ξ is an eigenvector of the Ricci operator Q. The class of H-contact metric manifolds includes several classes of contact metric manifolds such as Sasakian, η-Einstein, or even generalized (κ, μ)-contact metric manifolds. Perrone [2003] also showed that a contact metric 3-manifold M is a generalized (κ, μ)-contact metric manifold on an everywhere dense open subset of M if and only if its Reeb vector field ξ determines a harmonic map. In turn, Koufogiorgos, Markellos and Papantoniou proved that the (κ, μ, ν)-condition on a 3-dimensional contact metric manifold is equivalent to the Reeb vector field ξ being a harmonic vector field, at least on an open dense subset of the manifold [Koufogiorgos et al. 2008]. They proved also that these manifolds exist only in the dimension 3, whereas such a manifold does not exist in dimension greater than 3; hence, we restrict ourselves to dimension 3.

On the other hand, many geometers have studied the contact manifolds of constant curvature and their generalizations like the locally symmetric spaces ($\nabla R = 0$), Einstein spaces, the semisymmetric spaces ($R(\xi, X) \cdot R = 0$), Ricci semisymmetric spaces ($R(X, Y) \cdot S = 0$), Weyl semisymmetric spaces ($R(X, Y) \cdot C = 0$), where $R(X, Y)$ acts as a derivation respectively on R, S, C etc. For example, a contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature $+1$ or is 3-dimensional and flat [Blair 2002, pages 98–99; Olszak 1979]. S. Tanno [1969] showed that a semisymmetric K-contact manifold M^{2n+1} is locally isometric to the unit sphere $S^{2n+1}(1)$, and that for a K-contact manifold M^{2n+1} the following conditions are equivalent: (i) M is an Einstein manifold; (ii) M is Ricci-symmetric, that is, its Ricci tensor is parallel; (iii) M is Ricci semisymmetric, i.e., it satisfies the condition $R(X, Y) \cdot S = 0$; (iv) M is ξ-Ricci semisymmetric, that is, $R(\xi, Y) \cdot S = 0$.

Perrone [1992] showed that if ξ belongs to the κ-nullity distribution and if $R(\xi, Y) \cdot S = 0$, then the contact metric manifold is locally isometric to $E^{n+1} \times S^n(4)$ or is Sasaki–Einstein. M. M. Tripathi [2006] proved that a contact metric manifold

M^{2n+1} such that ξ belongs to the (κ, μ)-nullity distribution and $R(\xi, Y) \cdot S$ vanishes is either flat and 3-dimensional, or is locally isometric to $E^{n+1} \times S^n(4)$, or is a Sasaki–Einstein manifold. Finally, we studied in [Gouli-Andreou et al. 2012], together with Ph. J. Xenos, the (κ, μ, ν)-contact 3-manifolds in which certain curvature conditions are satisfied; for instance the Ricci tensor S is cyclic parallel, or η-parallel or $R(\xi, Y) \cdot S = 0$.

After the curvature tensor R and the Weyl conformal curvature tensor C, the *concircular curvature tensor* Z is the next most important (1,3)-type curvature tensor. It is defined on a Riemannian manifold (M^n, g) by Yano [1940a] (see also [Yano and Bochner 1953]) as

$$Z = R - \frac{r}{n(n-1)} R_0, \tag{1-5}$$

where R is the curvature tensor, R_0 is given by (1-2) and r the scalar curvature. We remark that Riemannian manifolds with vanishing Z are of constant curvature; thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. Z is an invariant of concircular transformations, which have important geometric and algebraic applications; see [Yano 1940a; 1940b; 1940c; 1940d; 1942; Vanhecke 1977]. Hence, Blair, J. S. Kim and Tripathi [Blair et al. 2005] started a study of the concircular curvature tensor on M^{2n+1} contact metric manifolds. They classified $N(\kappa)$-contact metric manifolds satisfying $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$ or $R(\xi, X) \cdot Z = 0$. Similarly, Tripathi and Kim [2004] classified M^{2n+1} (κ, μ)-contact manifolds with $Z(\xi, X) \cdot S = 0$.

This article is motivated by these studies, and is organized in the following way. In Section 2 we give some preliminaries on contact manifolds and the concircular curvature tensor. In Section 3 we present a brief account of (κ, μ, ν)-contact 3-manifolds while Section 4 contains some basic results. Finally, in Section 5 we study (κ, μ, ν)-contact metric 3-manifolds M satisfying any of these conditions:

(i) M is Ricci flat.

(ii) M is Einstein but not Sasakian.

(iii) $\nabla Z = 0$, where Z is the concircular curvature tensor.

(iv) $Z(\xi, X) \cdot Z = 0$, where $Z(\xi, X)$ acts as a derivation on Z.

(v) $Z(\xi, X) \cdot S = 0$, where $Z(\xi, X)$ acts as a derivation on S.

(vi) $R(\xi, X) \cdot Z = 0$, where $R(\xi, X)$ acts as a derivation on Z.

2. Preliminaries

By a *contact manifold* we mean a smooth manifold M^{2n+1}, endowed with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Then there is an underlying *contact* *metric structure* (η, ξ, ϕ, g) where g is a Riemannian metric (the *associated metric*), ϕ a global tensor of type (1,1) and ξ a unique global vector field (the *characteristic* or *Reeb vector field*). These structure tensors satisfy the equations

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2-1}$$

$$d\eta(X,Y) = g(X, \phi Y) = -g(\phi X, Y), \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2-2}$$

for all $X, Y \in \mathcal{X}(M)$. The associated metrics can be constructed by the polarization of $d\eta$ on the contact subbundle defined by $\eta = 0$. Denoting Lie differentiation by $\mathcal{L}$, we define for all $X \in \mathcal{X}(M)$ the (1,1)-tensor field

$$hX = \tfrac{1}{2}(\mathcal{L}_\xi \phi)X.$$

We give some basic equations for these tensor fields:

$$\begin{gathered} \phi\xi = h\xi = 0, \quad \eta \circ \phi = \eta \circ h = 0, \quad \nabla_\xi \phi = 0, \\ \operatorname{Tr} h = \operatorname{Tr}(h\phi) = 0, \quad h\phi = -\phi h. \end{gathered} \tag{2-3}$$

If X is an eigenvector of h corresponding to the eigenvalue λ, then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$ since h anticommutes with ϕ:

$$hX = \lambda X \quad \Rightarrow \quad h\phi X = -\lambda \phi X, \tag{2-4}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2-5}$$

$$(\nabla_X \eta)(Y) = -g(\phi X + \phi h X, Y), \tag{2-6}$$

where ∇ is the Levi-Civita connection of g. We also denote by R the corresponding Riemann curvature tensor field given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, by S the Ricci tensor field of type (0, 2), by Q the Ricci operator, which is the corresponding endomorphism field, by r the scalar curvature and by H the ϕ-sectional curvature.

A contact metric manifold for which ξ is a Killing field is called a *K-contact* manifold. A contact metric manifold is K-contact if and only if $h = 0$. A contact structure on M^{2n+1} implies an almost complex structure on the product manifold $M^{2n+1} \times \mathbb{R}$. If this structure is integrable, then the contact metric manifold is said to be *Sasakian*. A K-contact structure is Sasakian only in dimension 3, and this fails in higher dimensions. More details on contact manifolds can be found in [Blair 2002].

We restrict ourselves to the 3-dimensional case. Let (M, ϕ, ξ, η, g) be a 3-dimensional contact metric manifold and U the open subset of points $p \in M$ where $h \neq 0$ in a neighborhood of p and U_0 the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. For any point $p \in U \cup U_0$ there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p. On U we put $he = \lambda e$, where λ is a nonvanishing smooth function which is supposed positive. From (2-4) we have $h\phi e = -\lambda \phi e$.

Lemma 2.1 [Gouli-Andreou and Xenos 1998]. *On U we have*

$$\begin{aligned}
&\nabla_\xi e = a\phi e, && \nabla_e e = b\phi e, && \nabla_{\phi e} e = -c\phi e + (\lambda-1)\xi,\\
&\nabla_\xi \phi e = -ae, && \nabla_e \phi e = -be + (1+\lambda)\xi, && \nabla_{\phi e}\phi e = ce,\\
&\nabla_\xi \xi = 0, && \nabla_e \xi = -(1+\lambda)\phi e, && \nabla_{\phi e}\xi = (1-\lambda)e,
\end{aligned}$$

where a is a smooth function and

$$\begin{aligned}
b &= \frac{1}{2\lambda}[(\phi e\cdot\lambda)+A] \quad \text{with} \quad A = S(\xi, e),\\
c &= \frac{1}{2\lambda}[(e\cdot\lambda)+B] \quad \text{with} \quad B = S(\xi, \phi e).
\end{aligned} \tag{2-7}$$

Lemma 2.1 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$ yield

$$\begin{aligned}
[e, \phi e] &= \nabla_e\phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi,\\
[e, \xi] &= \nabla_e \xi - \nabla_\xi e = -(a+\lambda+1)\phi e,\\
[\phi e, \xi] &= \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a-\lambda+1)e.
\end{aligned} \tag{2-8}$$

Definition 2.2. Let M^3 be a 3-dimensional contact metric manifold and let $h = \lambda h^+ - \lambda h^-$ be the spectral decomposition of h on U. If

$$\nabla_{h^- X} h^- X = [\xi, h^+ X]$$

for all vector fields X on M^3 and all points of an open subset W of U, and if $h = 0$ on the points of M^3 which do not belong to W, then the manifold is said to be a *semi-K contact* manifold.

From Lemma 2.1 and the relations (2-8), the condition above leads to $[\xi, e] = 0$ when $X = e$ and to $\nabla_{\phi e}\phi e = 0$ when $X = \phi e$. Hence on a semi-K contact manifold we have $a+\lambda+1 = c = 0$. If we apply the deformation $e \to \phi e$, $\phi e \to e$, $\xi \to -\xi$, $\lambda \to -\lambda$, $b \to c$ and $c \to b$ then the contact metric structure remains the same. Hence the condition for a 3-dimensional contact metric manifold to be semi-K contact is equivalent to $a - \lambda + 1 = b = 0$.

Definition 2.3 [Blair 2002, page 105; Okumura 1962]. A contact metric manifold M is said to be *η-Einstein* if the Ricci tensor S satisfies the condition $S = \alpha g + \beta\eta\otimes\eta$, where α and β are smooth functions on M. In particular, if $\beta = 0$, then M becomes an *Einstein manifold*.

Definition 2.4. A Riemannian manifold (M^n, g) is called *Ricci flat* if its Ricci tensor vanishes identically.

Since the Ricci operator Q in dimension 3 determines completely the curvature of the contact manifold, the vanishing of Q implies the vanishing of the Riemannian curvature tensor. Hence, the class of Ricci flat manifolds is a hyperclass of the flat manifolds, or equivalently a flat manifold is certainly *Ricci flat*, while a *Ricci flat* manifold is an Einstein manifold.

Definition 2.5. A Riemannian manifold (M^m, g), $m \geq 3$, is called *pseudosymmetric* in the sense of R. Deszcz [1992] if at every point of M the curvature tensor R satisfies the equation $R(X, Y) \cdot R = L\{(X \wedge Y) \cdot R\}$ where $(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y$ for all vectors fields X, Y, Z on M, the dot means that $R(X, Y)$ and $X \wedge Y$ act as derivations on R, and L is a smooth function.

If L is constant, then M is a pseudosymmetric manifold of constant type while if $L = 0$ then M is a *semisymmetric* manifold.

Definition 2.6. A Riemannian manifold (M^n, g) is called *concircularly symmetric* if the concircular tensor Z satisfies the condition $\nabla Z = 0$.

All manifolds are assumed connected and all manifolds and maps are assumed smooth (class C^∞) unless otherwise stated. Finally, differentiation will be denoted by "( )".

3. (κ, μ, ν)-contact metric manifolds

A (κ, μ, ν)*-contact metric manifold* is defined in [Koufogiorgos et al. 2008] by (1-4) where κ, μ, ν are smooth functions on M. If $\nu = 0$ we have a generalized (κ, μ)-contact metric manifold [Koufogiorgos and Tsichlias 2000] and if additionally κ, μ are constants then the manifold is a contact metric (κ, μ)-space [Blair et al. 1995; Boeckx 2000]. Moreover in [Koufogiorgos et al. 2008] and [Koufogiorgos and Tsichlias 2000] it is proved respectively that for a (κ, μ, ν) or a generalized (κ, μ)-contact metric manifold M^{2n+1} of dimension greater than 3, the functions κ, μ are constants and ν is the zero function. We recall some lemmas and equations:

Lemma 3.1 [Koufogiorgos et al. 2008]. *For every point p of a (κ, μ, ν)-contact metric manifold M^{2n+1} with $\kappa(p) < 1$, there exists an open neighborhood U of p and orthonormal local vector fields $X_i, \phi X_i, \xi, i = 1, \dots, n$, defined on U such that*

$$hX_i = \lambda X_i, \qquad h\phi X_i = -\lambda \phi X_i, \qquad h\xi = 0$$

for $i = 1, \dots, n$, where $\lambda = \sqrt{1 - \kappa}$.

From now on, we will call the vector fields of Lemma 3.1 a local *h-basis*.

On any (κ, μ, ν)-contact metric manifold we have

$$h^2 = (\kappa - 1)\phi^2, \quad \kappa \leq 1, \tag{3-1}$$

$$(\xi \cdot \kappa) = 2\nu(\kappa - 1). \tag{3-2}$$

For the 3-dimensional case we have for the Ricci operator Q

$$Q=\left(\tfrac{1}{2}r-\kappa\right)I+\left(-\tfrac{1}{2}r+3\kappa\right)\eta\otimes\xi+\mu h+\nu\phi h, \tag{3-3}$$

$$Q\phi-\phi Q=2\nu h-2\mu\phi h, \tag{3-4}$$

$$r=4\kappa+2H, \tag{3-5}$$

where r is the scalar curvature and H is the ϕ-sectional curvature. From now on, we suppose $\kappa<1$ everywhere on M^3 and we use X, Y, U to denote arbitrary elements of $\mathcal{X}(M)$. We have

$$r=\frac{1}{\lambda}\Delta\lambda-(\xi\cdot\nu)-\frac{\|\operatorname{grad}\lambda\|^2}{\lambda^2}+2(\kappa-\mu), \tag{3-6}$$

where Δ is the Laplace operator and for the gradient of a function f we have

$$g(\operatorname{grad} f, X)=X(f)=df(X), \tag{3-7}$$

$$(\xi\cdot r)=2(\xi\cdot\kappa),\quad (\xi\cdot H)=-(\xi\cdot\kappa). \tag{3-8}$$

For a 3-dimensional (κ,μ)-contact metric manifold, that is, for constant κ,μ we have (see [Blair et al. 1995] and [Markellos 2009])

$$r=2(\kappa-\mu), \tag{3-9}$$

(3-10)
$$\begin{aligned}R(X,Y)U=\mu[&g(Y,U)hX-g(X,U)hY+g(hY,U)X-g(hX,U)Y]\\&+\nu[g(Y,U)\phi hX-g(X,U)\phi hY+g(\phi hY,U)X-g(\phi hX,U)Y]\\&+(\kappa-H)[g(Y,U)\eta(X)-g(X,U)\eta(Y)]\xi\\&+(\kappa-H)[\eta(Y)\eta(U)X-\eta(X)\eta(U)Y]\\&+H[g(Y,U)X-g(X,U)Y],\end{aligned}$$

$$\begin{aligned}(\nabla_X h)Y=&-\tfrac{1}{2(1-\kappa)}g(hX,Y)\operatorname{grad}\kappa-\tfrac{1}{2(1-\kappa)}g(hX,\phi Y)\phi(\operatorname{grad}\kappa)\\&+[(1-\kappa)g(X,\phi Y)+g(hX,\phi Y)-\nu g(hX,Y)]\xi\\&+\eta(Y)[(\kappa-1)\phi X+h\phi X]+\eta(X)[\mu h\phi Y+\nu hY],\end{aligned} \tag{3-11}$$

$$(\nabla_X\phi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{3-12}$$

while $(\nabla_X\phi h)Y=(\nabla_X\phi)hY+\phi(\nabla_X h)Y$ is calculated from (3-11) and (3-12):

$$\begin{aligned}(\nabla_X\phi h)Y=&[g(X+hX,hY)+\nu g(hX,\phi Y)]\xi\\&-\tfrac{1}{2(1-\kappa)}g(hX,Y)\phi(\operatorname{grad}\kappa)+\tfrac{1}{2(1-\kappa)}g(hX,\phi Y)\operatorname{grad}\kappa\\&+\eta(Y)[(\kappa-1)\phi^2X+hX]+\eta(X)[\mu hY+\nu\phi hY].\end{aligned} \tag{3-13}$$

From (3-3) and (3-5) we calculate the Ricci tensor $S(X,Y) = g(QX,Y)$:

$$S(X,Y) = (\kappa + H)g(X,Y) + (\kappa - H)\eta(X)\eta(Y) + \mu g(hX,Y) + \nu g(\phi hX,Y); \tag{3-14}$$

hence,

$$S(hX,Y) = (\kappa + H)g(hX,Y) - \mu(\kappa - 1)[g(X,Y) - \eta(X)\eta(Y)] + \nu(\kappa - 1)g(X,\phi Y), \tag{3-15}$$

$$S(\phi hX,Y) = (\kappa + H)g(\phi hX,Y) - \nu(\kappa - 1)[g(X,Y) - \eta(X)\eta(Y)] + \mu(\kappa - 1)g(\phi X,Y). \tag{3-16}$$

4. Some auxiliary results

Equation (1-5) gives for the 3-dimensional case and for all $X, Y, U \in \mathcal{X}(M)$

$$Z(X,Y)U = R(X,Y)U - \tfrac{1}{6}rR_0(X,Y)U, \tag{4-1}$$

where R_0 is given by (1-2) and hence

$$R_0(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{4-2}$$

while (1-4) for a (κ,μ,ν)*-contact metric manifold* is written in the form

$$R(X,Y)\xi = (\kappa I + \mu h + \nu\phi h)R_0(X,Y)\xi, \tag{4-3}$$

which is equivalent to

$$R(\xi,X) = R_0(\xi,(\kappa I + \mu h + \nu\phi h)X). \tag{4-4}$$

From (4-3) we get

$$R(\xi,X)\xi = \kappa(\eta(X)\xi - X) - \mu hX - \nu\phi hX. \tag{4-5}$$

Proposition 4.1. *In a* (κ,μ,ν)*-contact metric manifold* M^3, *the concircular curvature tensor* Z *satisfies*

$$Z(X,Y)\xi = \Big(\big(\kappa - \tfrac{1}{6}r\big)I + \mu h + \nu\phi h\Big)R_0(X,Y)\xi, \tag{4-6}$$

$$Z(\xi,X) = \big(\kappa - \tfrac{1}{6}r\big)R_0(\xi,X) + \mu R_0(\xi,hX) + \nu R_0(\xi,\phi hX). \tag{4-7}$$

Consequently, we have

$$Z(\xi,X)\xi = \big(\kappa - \tfrac{1}{6}r\big)(\eta(X)\xi - X) - \mu hX - \nu\phi hX, \tag{4-8}$$

$$\eta(Z(X,Y)\xi) = 0, \tag{4-9}$$

$$\eta(Z(\xi,X)Y) = \big(\kappa - \tfrac{1}{6}r\big)\big(g(X,Y) - \eta(X)\eta(Y)\big) + \mu g(hX,Y) + \nu g(\phi hX,Y). \tag{4-10}$$

Proof. Equations (4-1), (4-3), (4-4) lead us to conclude equations (4-6) and (4-7). Equation (4-7) implies (4-8) while (4-6) and (4-7) imply (4-9) and (4-10) respectively by virtue of (2-3). □

Proposition 4.2. *In a (κ, μ, ν)-contact metric manifold M^3 we have*

$$S(Z(\xi, X)Y, \xi) = 2\kappa\big(\kappa - \tfrac{1}{6}r\big)(g(X, Y) - \eta(X)\eta(Y)) + 2\kappa\mu g(hX, Y) + 2\kappa\nu g(\phi hX, Y), \tag{4-11}$$

$$S(Z(\xi, X)\xi, Y) = 2\kappa\big(\kappa - \tfrac{1}{6}r\big)\eta(X)\eta(Y) - \big(\kappa - \tfrac{1}{6}r\big)S(X, Y) - \mu S(hX, Y) - \nu S(\phi hX, Y). \tag{4-12}$$

Proof. For a (κ, μ, ν)-contact metric manifold M^3 we obtain from (3-14)

$$S(X, \xi) = 2\kappa\eta(X). \tag{4-13}$$

From (4-7), (4-10), (4-13) we get (4-11), while (4-8) and (4-13) yield (4-12). □

Proposition 4.3. *Let M^3 be a non-Sasakian (κ, μ, ν)-contact metric manifold.*

(i) *If M^3 satisfies*

$$\nu(\kappa - H) = 0, \tag{4-14}$$

$$\mu(\kappa - H) = 0, \tag{4-15}$$

$$\tfrac{1}{3}(\kappa - H)^2 + (\kappa - 1)(\mu^2 + \nu^2) = 0, \tag{4-16}$$

then the manifold is either flat or locally isometric to SU(2) *or* SL(2, R), *where these two Lie groups are equipped with a left invariant metric.*

(ii) *If M^3 satisfies*

$$\nu H = 0, \tag{4-17}$$

$$\mu H = 0, \tag{4-18}$$

$$\kappa(\kappa - H) + (\kappa - 1)(\mu^2 + \nu^2) = 0, \tag{4-19}$$

then the manifold is a generalized (κ, μ)-contact metric manifold with $(\xi \cdot \mu) = 0$.

Proof. (i) Let M be a 3-dimensional (κ, μ, ν)-contact metric manifold with $\kappa < 1$ everywhere. We suppose that there is a point $p \in M$ where $\nu \neq 0$. The continuity of this function implies that there is a neighborhood $F_p \subseteq M$ of p, where $\nu \neq 0$ everywhere in F_p or by virtue of (4-14), $\kappa - H = 0$. Differentiating this equation with respect to ξ and using (3-8) and (3-2) we conclude that $\kappa = 1$ everywhere in F_p, which is a contradiction since $F_p \subseteq M$. Hence, $\nu = 0$ everywhere in M and M is a generalized (κ, μ)-contact metric manifold.

Similarly we suppose that there is a point $p \in M$ where $\kappa - H \neq 0$. There is a neighborhood $F_p \subseteq M$ of p, where $\kappa - H \neq 0$ everywhere in F_p or by virtue of

(4-15), $\mu = 0$. Setting $\mu = \nu = 0$ in (4-16) we are led to $\frac{1}{3}(\kappa - H)^2 = 0$ which is a contradiction in F_p. Hence $\kappa - H = 0$ everywhere in M and from (4-16), $\mu = 0$. Since in a generalized (κ, μ)-contact metric manifold the constancy of one of the κ or μ implies the constancy of the other [Koufogiorgos and Tsichlias 2000], we can conclude that κ is constant in this $N(\kappa)$-contact metric manifold. From (3-4) and because $\mu = \nu = 0$ we get $Q\phi = \phi Q$; by [Blair et al. 1990, Theorem 3.3] and the main theorem of [Blair and Chen 1992] such a manifold is either Sasakian, flat, locally isometric to a left invariant metric on the Lie group SU(2) with $\kappa > 0$, or SL(2, R) with $\kappa < 0$. Finally, we can remark that the equations $\kappa - H = 0$ and (3-5) give $r = 6\kappa$, $\kappa < 1$, and hence r is constant.

(ii) We suppose that there is a point $p \in M$ where $\nu \neq 0$. Then there is a neighborhood $F_p \subseteq M$ of p, where $\nu \neq 0$ everywhere in F_p or by virtue of (4-17), $H = 0$. Differentiating this equation with respect to ξ and using (3-8) and (3-2) we conclude that $\kappa = 1$ everywhere in F_p, which is a contradiction since $F_p \subseteq M$. Hence, $\nu = 0$ everywhere in M and M is a generalized (κ, μ)-contact metric manifold.

For (4-18), we suppose that there is a point $p \in M$ where $H \neq 0$. There is a neighborhood $F_p \subseteq M$ of p, where $H \neq 0$ everywhere in F_p or by virtue of (4-18), $\mu = 0$. Since μ is constant, κ is also constant and hence from (3-5) and (3-9), $H = -\kappa - \mu$ or more explicitly $H = -\kappa$. From (4-19) and because $\mu = \nu = 0$ we get $\kappa = 0$ and obviously $H = 0$, which is a contradiction in F_p. Hence $H = 0$ everywhere in M and from (4-19), $\kappa^2 + (\kappa - 1)\mu^2 = 0$. Differentiating this equation with respect to ξ and by virtue of (3-2) and $\nu = 0$ we conclude $(\xi \cdot \mu) = 0$, while (3-5) implies $r = 4\kappa$ with $\kappa < 1$. □

Remark 4.4. The generalized (κ, μ)-contact metric manifolds in dimension 3 with $\kappa < 1$ (equivalently $\lambda \neq 0$) and $(\xi \cdot \mu) = 0$ have been studied by T. Koufogiorgos and C. Tsichlias [2008]. They proved in [2008, Theorem 4.1] that at any point of $P \in M$, precisely one of the following relations is valid: $\mu = 2(1 + \sqrt{1 - \kappa})$, or $\mu = 2(1 - \sqrt{1 - \kappa})$, while there exists a chart (U,(x,y,z)) with $P \in U \subseteq M$ such that the functions κ, μ depend only on z and the tensors fields η, ξ, ϕ, g take a suitable form. We can also add that such a manifold according to Definition 2.2 is a semi-K contact manifold.

Theorem 4.5 [Blair 2002, page 101]. *Let M^{2n+1} be a contact metric manifold satisfying the condition $R(X, Y)\xi = 0$. Then M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

5. Main results

Theorem 5.1. *A non-Sasakian Ricci flat 3-dimensional (κ, μ, ν)-contact metric manifold is flat.*

Proof. Since the manifold M is Ricci flat, from (4-13) we have

$$0 = S(\xi, \xi) = 2\kappa,$$

or $\kappa = 0$. Then, (3-2) yields $\nu = 0$, so M is a generalized (κ, μ)-contact metric manifold with $\kappa = 0$. In a generalized (κ, μ)-contact metric manifold the constancy of one of κ or μ implies the constancy of the other [Koufogiorgos and Tsichlias 2000], so μ is also constant. We set $\kappa = \nu = 0$ in (3-14) and by virtue of (3-5) and (3-9) we have

$$S(X, Y) = \mu[g(hX, Y) - g(X, Y) + \eta(X)\eta(Y)] \tag{5-1}$$

for all $X, Y \in \mathcal{X}(M)$. For any point $p \in M$ we consider a local orthonormal *h-basis* as in Lemma 3.1. In the last equation we set (i) $X = Y = e$ and (ii) $X = Y = \phi e$ and since we have a Ricci flat manifold we get respectively

$$0 = S(e, e) = \mu(\lambda - 1),$$
$$0 = S(\phi e, \phi e) = \mu(-\lambda - 1).$$

By adding these equations we see that $\mu = 0$, and Theorem 4.5 completes the proof. □

Remark 5.2. For a Sasakian 3-manifold, from Equation (3-14) with $\kappa = 1$ and $h = 0$, setting $X = Y = \xi$ yields $S(\xi, \xi) = 2$ and hence a Sasakian manifold cannot be Ricci flat.

Theorem 5.3. *A non-Sasakian Einstein 3-dimensional (κ, μ, ν)-contact metric manifold is flat.*

Proof. Since the manifold is Einstein, Equation (3-3) gives

$$\left(\tfrac{1}{2}r - \kappa\right)X + \left(-\tfrac{1}{2}r + 3\kappa\right)\eta(X)\xi + \mu hX + \nu\phi hX = aX. \tag{5-2}$$

For any point $p \in U$ as in Lemma 3.1 we consider a local orthonormal *h-basis* and we set in (5-2) $X = \xi$, $X = e$ and $X = \phi e$. We obtain respectively

$$2\kappa = a, \quad \nu = 0,$$
$$\tfrac{1}{2}r - \kappa + \lambda\mu = a, \quad \tfrac{1}{2}r - \kappa - \lambda\mu = a.$$

We have a generalized (κ, μ)-contact metric manifold with $\kappa < 1$ or equivalently $\lambda \neq 0$. From the two last equations we get $\mu = 0$ and hence κ is constant [Koufogiorgos and Tsichlias 2000]. In a 3-dimensional (κ, μ)-contact metric manifold $r = 2(\kappa - \mu)$. By substituting r in the last equation we obtain $a = 0$ or equivalently $\kappa = 0$, and Theorem 4.5 completes the proof. □

Remark 5.4. According to [Yano and Kon 1984, Proposition 3.3, page 38], a 3-dimensional Einstein manifold M is a space of constant curvature. Hence, a Sasaki–Einstein 3-manifold, since it has constant curvature, must have curvature 1.

Theorem 5.5. *If M is a 3-dimensional concircularly symmetric (κ, μ, ν)-contact metric manifold, then M is either flat or locally isometric to the sphere $S^3(1)$.*

Proof. We consider the open subsets of M:

$$U_1 = \{p \in M : \kappa = 1 \text{ in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \text{ in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is an open and dense subset of M.

In the case where $M = U_1$ the manifold is a Sasakian concircularly symmetric manifold.

Next, we assume that U_2 is not empty. Differentiating (4-1) and using (1-2), (2-1), (2-2), (2-5), (2-6), (3-7), (3-10), (3-11), (3-13), with $\kappa < 1$ everywhere, it follows that

$$\begin{aligned}
(\nabla_W Z)(X, Y)U &= [(W\cdot H) - \tfrac{1}{6}(W\cdot r)][g(Y, U)X - g(X, U)Y] \\
&\quad + [(W\cdot\kappa) - (W\cdot H)][g(Y, U)\eta(X) - g(X, U)\eta(Y)]\xi \\
&\quad + [(W\cdot\kappa) - (W\cdot H)][\eta(Y)\eta(U)X - \eta(X)\eta(U)Y] \\
&\quad + (W\cdot\mu)[g(Y, U)hX - g(X, U)hY + g(hY, U)X - g(hX, U)Y] \\
&\quad + (W\cdot\nu)[g(Y, U)\phi hX - g(X, U)\phi hY + g(\phi hY, U)X - g(\phi hX, U)Y] \\
&\quad + (\kappa - H)\big\{[g(Y, U)g(W + hW, \phi X) - g(X, U)g(W + hW, \phi Y)]\xi \\
&\qquad + [\eta(Y)X - \eta(X)Y]g(W + hW, \phi U) \\
&\qquad + [g(W + hW, \phi Y)X - g(W + hW, \phi X)Y]\eta(U) \\
&\qquad - [g(Y, U)\eta(X) - g(X, U)\eta(Y)](\phi W + \phi hW)\big\} \\
&\quad + \mu\Big[\big\{\tfrac{1}{2(\kappa-1)}g(hW, X)\operatorname{grad}\kappa + \tfrac{1}{2(\kappa-1)}g(hW, \phi X)\phi(\operatorname{grad}\kappa) \\
&\qquad + [(1-\kappa)g(W, \phi X) + g(hW, \phi X) - \nu g(hW, X)]\xi \\
&\qquad + \eta(X)[(\kappa - 1)\phi W + h\phi W] + \eta(W)(\mu h\phi X + \nu hX)\big\}g(Y, U) \\
&\quad - \big\{\tfrac{1}{2(\kappa-1)}g(hW, Y)\operatorname{grad}\kappa + \tfrac{1}{2(\kappa-1)}g(hW, \phi Y)\phi(\operatorname{grad}\kappa) \\
&\qquad + [(1-\kappa)g(W, \phi Y) + g(hW, \phi Y) - \nu g(hW, Y)]\xi \\
&\qquad + \eta(Y)[(\kappa - 1)\phi W + h\phi W] + \eta(W)(\mu h\phi Y + \nu hY)\big\}g(X, U) \\
&\quad + \big\{\tfrac{1}{2(\kappa-1)}g(hW, Y)(U\cdot\kappa) - \tfrac{1}{2(\kappa-1)}g(hW, \phi Y)(\phi U\cdot\kappa) \\
&\qquad + [(1-\kappa)g(W, \phi Y) + g(hW, \phi Y) - \nu g(hW, Y)]\eta(U) \\
&\qquad + \eta(Y)g((\kappa-1)\phi W + h\phi W, U) + \eta(W)g(\mu h\phi Y + \nu hY, U)\big\}X
\end{aligned}$$

$$
\begin{aligned}
&-\Big\{\tfrac{1}{2(\kappa-1)}g(hW,X)(U\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi X)(\phi U\cdot\kappa)\\
&\qquad+[(1-\kappa)g(W,\phi X)+g(hW,\phi X)-\nu g(hW,X)]\eta(U)\\
&\qquad+\eta(X)g((\kappa-1)\phi W+h\phi W,U)+\eta(W)g(\mu h\phi X+\nu hX,U)\Big\}Y\Big]\\
&+\nu\Big[\Big\{\tfrac{1}{2(\kappa-1)}g(hW,X)\phi(\operatorname{grad}\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi X)\operatorname{grad}\kappa\\
&\qquad+[g(W+hW,hX)+\nu g(hW,\phi X)]\xi\\
&\qquad+\eta(X)[(\kappa-1)\phi^2W+hW]+\eta(W)[\mu hX+\nu\phi hX]\Big\}g(Y,U)\\
&-\Big\{\tfrac{1}{2(\kappa-1)}g(hW,Y)\phi(\operatorname{grad}\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi Y)\operatorname{grad}\kappa\\
&\qquad+[g(W+hW,hY)+\nu g(hW,\phi Y)]\xi\\
&\qquad+\eta(Y)[(\kappa-1)\phi^2W+hW]+\eta(W)[\mu hY+\nu\phi hY]\Big\}g(X,U)\\
&+\Big\{\tfrac{-1}{2(\kappa-1)}g(hW,Y)(\phi U\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi Y)(U\cdot\kappa)\\
&\qquad+[g(W+hW,hY)+\nu g(hW,\phi Y)]\eta(U)\\
&\qquad+\eta(Y)g((\kappa-1)\phi^2W+hW,U)+\eta(W)g(\mu hY+\nu\phi hY,U)\Big\}X\\
&-\Big\{\tfrac{-1}{2(\kappa-1)}g(hW,X)(\phi U\cdot\kappa)-\tfrac{1}{2(\kappa-1)}g(hW,\phi X)(U\cdot\kappa)\\
&\qquad+[g(W+hW,hX)+\nu g(hW,\phi X)]\eta(U)\\
&\qquad+\eta(X)g((\kappa-1)\phi^2W+hW,U)+\eta(W)g(\mu hX+\nu\phi hX,U)\Big\}Y\Big].
\end{aligned}
$$

In this equation, we set $W=\xi$ and by virtue of (2-1), (2-3), (3-8) we obtain

$$
\begin{aligned}
(5\text{-}3)\quad (\nabla_\xi Z)(X,Y)U&=2(\xi\cdot\kappa)[g(Y,U)\eta(X)-g(X,U)\eta(Y)]\xi\\
&-\tfrac{4}{3}(\xi\cdot\kappa)[g(Y,U)X-g(X,U)Y]\\
&+(\xi\cdot\mu)[g(Y,U)hX-g(X,U)hY+g(hY,U)X-g(hX,U)Y]\\
&+(\xi\cdot\nu)[g(Y,U)\phi hX-g(X,U)\phi hY+g(\phi hY,U)X-g(\phi hX,U)Y]\\
&+\mu\big\{g(Y,U)(\mu h\phi X+\nu hX)-g(X,U)(\mu h\phi Y+\nu hY)\\
&\quad+g(\mu h\phi Y+\nu hY,U)X-g(\mu h\phi X+\nu hX,U)Y\big\}\\
&+\nu\big\{g(Y,U)(\mu hX+\nu\phi hX)-g(X,U)(\mu hY+\nu\phi hY)\\
&\quad+g(\mu hY+\nu\phi hY,U)X-g(\mu hX+\nu\phi hX,U)Y\big\}.
\end{aligned}
$$

For any point $p\in U_2$ we consider a local orthonormal *h-basis* as in Lemma 3.1. We set in (5-3): $X=U=e$, $Y=\phi e$ which yields

$$(\nabla_\xi Z)(e,\phi e)e=\tfrac{4}{3}(\xi\cdot\kappa)\phi e.$$

Since the manifold is concircularly symmetric we conclude that

$$(\xi\cdot\kappa)=0,$$

or equivalently, by virtue of (3-2), $\nu = 0$. We set in (5-3): $X = e$, $Y = U = \xi$ and $\nu = 0$, and get

$$(\nabla_\xi Z)(e, \xi)\xi = \lambda[(\xi \cdot \mu)e - \mu^2 \phi e].$$

The manifold is concircularly symmetric and hence $\mu = 0$. The constancy of μ implies the constancy of κ [Koufogiorgos and Tsichlias 2000] and finally [Blair et al. 2005, Theorem 5.2] completes the proof. □

Theorem 5.6. *Let M a 3-dimensional (κ, μ, ν)-contact metric manifold. If the concircular curvature tensor Z satisfies the condition $Z(\xi, X) \cdot Z = 0$, then M is either Sasakian ($\kappa = 1$), flat or locally isometric to either* SU(2) *or* SL(2, *R*), *where these two Lie groups are equipped with a left invariant metric and they have constant scalar curvature $r = 6\kappa$ ($\kappa < 1$).*

Proof. We consider the open subsets of M:

$$U_1 = \{p \in M : \kappa = 1 \text{ in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \text{ in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is open and dense subset of M.

In the case where $M = U_1$ the manifold is Sasakian and then according to [Blair et al. 2005, Theorem 4.1], it has constant curvature 1.

Next, we assume that U_2 is not empty. Note that the condition $Z(\xi, X) \cdot Z = 0$ implies $(Z(\xi, U) \cdot Z)(X, Y)\xi = 0$ or more explicitly

$$Z(\xi, U)Z(X, Y)\xi - Z(Z(\xi, U)X, Y)\xi - Z(X, Z(\xi, U)Y)\xi - Z(X, Y)Z(\xi, U)\xi = 0$$

which by virtue of (1-1), (1-4), (2-3), (4-1), (4-6), (4-7), (4-8), (4-9), (4-10) yields

$$\begin{aligned}
\text{(5-4)}\quad 0 =\; & \mu\big(\kappa - \tfrac{1}{6}r\big)[\eta(Y)g(hU, X) - \eta(X)g(hU, Y)]\xi \\
& + \mu^2[\eta(Y)g(hU, hX) - \eta(X)g(hU, hY)]\xi \\
& + \nu\big(\kappa - \tfrac{1}{6}r\big)[\eta(Y)g(\phi hU, X) - \eta(X)g(\phi hU, Y)]\xi \\
& + \nu^2[\eta(Y)g(\phi hU, \phi hX) - \eta(X)g(\phi hU, \phi hY)]\xi \\
& + \big(\kappa - \tfrac{1}{6}r\big)^2 g(U, X)Y + \mu\big(\kappa - \tfrac{1}{6}r\big)g(hU, X)Y + \nu\big(\kappa - \tfrac{1}{6}r\big)g(\phi hU, X)Y \\
& + \mu\big(\kappa - \tfrac{1}{6}r\big)g(U, X)hY + \mu^2 g(hU, X)hY + \mu\nu g(\phi hU, X)hY \\
& + \nu\big(\kappa - \tfrac{1}{6}r\big)g(U, X)\phi hY + \mu\nu g(hU, X)\phi hY + \nu^2 g(\phi hU, X)\phi hY \\
& - \big(\kappa - \tfrac{1}{6}r\big)^2 g(U, Y)X - \mu\big(\kappa - \tfrac{1}{6}r\big)g(hU, Y)X - \nu\big(\kappa - \tfrac{1}{6}r\big)g(\phi hU, Y)X \\
& - \mu\big(\kappa - \tfrac{1}{6}r\big)g(U, Y)hX - \mu^2 g(hU, Y)hX - \mu\nu g(\phi hU, Y)hX \\
& - \nu\big(\kappa - \tfrac{1}{6}r\big)g(U, Y)\phi hX - \mu\nu g(hU, Y)\phi hX - \nu^2 g(\phi hU, Y)\phi hX \\
& + \big(\kappa - \tfrac{1}{6}r\big)Z(X, Y)U + \mu Z(X, Y)hU + \nu Z(X, Y)\phi hU.
\end{aligned}$$

For any point $p \in U_2$ we consider a local orthonormal *h-basis* as in Lemma 3.1. In (5-4) we set $X = U = e$, $Y = \phi e$, and by virtue of (2-3), (2-4) we obtain

$$\left[\left(\kappa-\tfrac{1}{6}r\right)^2-\lambda^2(\mu^2+\nu^2)\right]\phi e+\left(\kappa-\tfrac{1}{6}r\right)Z(e,\phi e)e+\mu Z(e,\phi e)he \\ +\nu Z(e,\phi e)\phi he=0. \tag{5-5}$$

Equation (4-1) by virtue of (1-2), (2-4) and (3-10) yields

$$\begin{aligned} Z(e,\phi e)e&=\left(-H+\tfrac{1}{6}r\right)\phi e,\\ Z(e,\phi e)he&=\lambda\left(-H+\tfrac{1}{6}r\right)\phi e,\\ Z(e,\phi e)\phi he&=\lambda\left(H-\tfrac{1}{6}r\right)e. \end{aligned} \tag{5-6}$$

Substituting (5-6) in (5-5) we obtain

$$\nu\lambda\left(H-\tfrac{1}{6}r\right)e+\left[\left(\kappa-\tfrac{1}{6}r\right)(\kappa-H)-\lambda^2(\mu^2+\nu^2)-\lambda\mu\left(H-\tfrac{1}{6}r\right)\right]\phi e=0,$$

and hence

$$\nu\lambda\left(H-\tfrac{1}{6}r\right)=0, \tag{5-7}$$

$$\left(\kappa-\tfrac{1}{6}r\right)(\kappa-H)-\lambda^2(\mu^2+\nu^2)-\lambda\mu\left(H-\tfrac{1}{6}r\right)=0. \tag{5-8}$$

In (5-4) we set $X = e$, $Y = U = \phi e$, and by virtue of (2-3), (2-4) we obtain

$$\left[-\left(\kappa-\tfrac{1}{6}r\right)^2+\lambda^2(\mu^2+\nu^2)\right]e+\left(\kappa-\tfrac{1}{6}r\right)Z(e,\phi e)\phi e \\ +\mu Z(e,\phi e)h\phi e+\nu Z(e,\phi e)\phi h\phi e=0. \tag{5-9}$$

Equation (4-1) by virtue of (1-2), (2-4) and (3-10) yields

$$\begin{aligned} Z(e,\phi e)\phi e&=\left(H-\tfrac{1}{6}r\right)e,\\ Z(e,\phi e)h\phi e&=\lambda\left(-H+\tfrac{1}{6}r\right)e,\\ Z(e,\phi e)\phi h\phi e&=\lambda\left(-H+\tfrac{1}{6}r\right)e. \end{aligned} \tag{5-10}$$

Substituting the equations (5-10) in (5-9) we obtain

$$\left[-\left(\kappa-\tfrac{1}{6}r\right)(\kappa-H)+\lambda^2(\mu^2+\nu^2)-\lambda\mu\left(H-\tfrac{1}{6}r\right)\right]e-\nu\lambda\left(H-\tfrac{1}{6}r\right)\phi e=0,$$

and hence, in addition to (5-7), we get

$$-\left(\kappa-\tfrac{1}{6}r\right)(\kappa-H)+\lambda^2(\mu^2+\nu^2)-\lambda\mu\left(H-\tfrac{1}{6}r\right)=0. \tag{5-11}$$

Since we work in U_2 where $\kappa \neq 1$ (more precisely $\kappa < 1$) or equivalently $\lambda \neq 0$, the equations (5-7), (5-8) and (5-11) by virtue of (3-5) yield the equations (4-14), (4-15) and (4-16). Finally Proposition 4.3 completes the proof. □

Corollary 5.7. *Let M be a 3-dimensional (κ,μ,ν)-contact metric manifold. If the concircular curvature tensor Z satisfies the condition $Z(\xi, X)\cdot Z = 0$, then M is a pseudosymmetric manifold, in the sense of Deszcz, of constant type.*

Proof. From [Blair et al. 1990, Proposition 3.2] this manifold is an η-Einstein and then [Cho and Inoguchi 2005, Proposition 1.2] completes the proof. □

Theorem 5.8. *Let M be a 3-dimensional (κ, μ, ν)-contact metric manifold. If the concircular curvature tensor Z satisfies the condition $Z(\xi, X) \cdot S = 0$, then M is either Sasakian ($\kappa = 1$), flat or locally isometric to either* SU(2) *or* SL(2, R), *where these two Lie groups are equipped with a left invariant metric and they have constant scalar curvature $r = 6\kappa$ ($\kappa < 1$).*

Proof. We consider the open subsets of M:

$$U_1 = \{p \in M : \kappa = 1 \text{ in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \text{ in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is an open and dense subset of M.

In the case where $M = U_1$, the manifold is Sasakian and according to [Tripathi and Kim 2004, Theorem 1.4], it has constant curvature 1.

Next, we assume that U_2 is not empty; we work in U_2 where $\kappa < 1$ everywhere. The condition $Z(\xi, X) \cdot S = 0$ or equivalently

$$0 = (Z(\xi, X) \cdot S)(Y, W) = Z(\xi, X) \cdot S(Y, W) - S(Z(\xi, X)Y, W) - S(Y, Z(\xi, X)W)$$

implies

$$S(Z(\xi, X)Y, W) + S(Y, Z(\xi, X)W) = 0 \tag{5-12}$$

which in view of (4-11) and (4-12) yields

$$\begin{aligned}\left(\kappa - \tfrac{1}{6}r\right)[S(X, Y) - 2\kappa g(X, Y)] + \mu[S(hX, Y) - 2\kappa g(hX, Y)]\\ +\nu[S(\phi hX, Y) - 2\kappa g(\phi hX, Y)] = 0.\end{aligned} \tag{5-13}$$

For any point $p \in U_2$ we consider an h-basis. In (5-13) setting (i) $X = Y = e$, (ii) $X = Y = \phi e$ and (iii) $X = e$ and $Y = \phi e$, and by virtue of (3-14), (3-15) and (3-16), we obtain respectively

$$\left(\kappa - \tfrac{1}{6}r\right)(H - \kappa + \lambda\mu) + \mu(\lambda H - \lambda\kappa - \mu\kappa + \mu) - \nu^2(\kappa - 1) = 0, \tag{5-14}$$

$$\left(\kappa - \tfrac{1}{6}r\right)(H - \kappa - \lambda\mu) + \mu(-\lambda H + \lambda\kappa - \mu\kappa + \mu) - \nu^2(\kappa - 1) = 0, \tag{5-15}$$

and (4-14). By virtue of (3-5) and by subtracting (5-15) from (5-14) we obtain (4-15), while by adding equations (5-14) and (5-15) we get (4-16). Proposition 4.3 completes the proof. □

Corollary 5.9. *Let M be a 3-dimensional (κ, μ, ν)-contact metric manifold. If the concircular curvature tensor Z satisfies the condition $Z(\xi, X) \cdot S = 0$, then M is a pseudosymmetric manifold, in the sense of Deszcz, of constant type.*

Proof. From [Blair et al. 1990, Proposition 3.2] this manifold is an η-Einstein and then [Cho and Inoguchi 2005, Proposition 1.2] completes the proof. □

Theorem 5.10. *Let $M^3(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold satisfying the condition $R(\xi, X) \cdot Z = 0$. Then, there are at most two open subsets of M^3 for which their union is an open and dense subset of M^3, and each of them as an open submanifold of M^3 is either (a) a Sasakian manifold or (b) a semi-K generalized (κ, μ)-contact metric manifold with $(\xi \cdot \mu) = 0$ and $r = 4\kappa$.*

Proof. We consider the open subsets of M:

$$U_1 = \{p \in M : \kappa = 1 \text{ in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \text{ in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is open and dense in M.

In the case where $M = U_1$, the manifold is Sasakian and according to [Blair et al. 2005, Theorem 4.3], it has constant curvature 1.

Next, we assume that U_2 is not empty. Firstly, we remark that the condition $R(\xi, X) \cdot Z = 0$ implies $(R(\xi, U) \cdot Z)(X, Y)\xi = 0$ or more explicitly

$$R(\xi, U)Z(X, Y)\xi - Z(R(\xi, U)X, Y)\xi - Z(X, R(\xi, U)Y)\xi - Z(X, Y)R(\xi, U)\xi = 0$$

which by virtue of (1-1), (1-4), (2-3), (3-10), (4-1), (4-9) yields

$$\begin{aligned}
0 = {} & \mu\kappa[\eta(Y)g(U, hX) - \eta(X)g(U, hY)]\xi \\
& + \nu\kappa[\eta(Y)g(U, \phi hX) - \eta(X)g(U, \phi hY)]\xi \\
& + \mu^2[\eta(Y)g(hU, hX) - \eta(X)g(hU, hY)]\xi \\
& + \nu^2[\eta(Y)g(\phi hU, \phi hX) - \eta(X)g(\phi hU, \phi hY)]\xi \\
& + \kappa\big(\kappa - \tfrac{1}{6}r\big)g(U, X)Y + \kappa\mu g(U, X)hY + \kappa\nu g(U, X)\phi hY \\
& - \kappa\big(\kappa - \tfrac{1}{6}r\big)g(U, Y)X - \kappa\mu g(U, Y)hX - \kappa\nu g(U, Y)\phi hX \\
& + \mu\big(\kappa - \tfrac{1}{6}r\big)g(hU, X)Y + \mu^2 g(hU, X)hY + \mu\nu g(hU, X)\phi hY \\
& - \mu\big(\kappa - \tfrac{1}{6}r\big)g(hU, Y)X - \mu^2 g(hU, Y)hX - \mu\nu g(hU, Y)\phi hX \\
& + \nu\big(\kappa - \tfrac{1}{6}r\big)g(\phi hU, X)Y + \mu\nu g(\phi hU, X)hY + \nu^2 g(\phi hU, X)\phi hY \\
& - \nu\big(\kappa - \tfrac{1}{6}r\big)g(\phi hU, Y)X - \mu\nu g(\phi hU, Y)hX - \nu^2 g(\phi hU, Y)\phi hX \\
& + \kappa Z(X, Y)U + \mu Z(X, Y)hU + \nu Z(X, Y)\phi hU.
\end{aligned} \tag{5-16}$$

For any point $p \in U_2$ we consider a local orthonormal *h-basis* as in Lemma 3.1. In (5-16) we set $X = U = e$, $Y = \phi e$ and by virtue of (2-3), (2-4) we obtain

$$\begin{aligned}
\tfrac{1}{6}r\nu\lambda e + \big[\kappa^2 - \tfrac{1}{6}r\kappa - \lambda^2(\mu^2 + \nu^2) - \tfrac{1}{6}r\lambda\mu\big]\phi e + \kappa Z(e, \phi e)e & \\
+ \mu Z(e, \phi e)he + \nu Z(e, \phi e)\phi he = 0&,
\end{aligned}$$

which by (5-6) gives

$$\nu\lambda He+[\kappa(\kappa-H)-\lambda^2(\mu^2+\nu^2)-\lambda\mu H]\phi e=0,$$

and hence

$$\nu\lambda H=0, \tag{5-17}$$

$$\kappa(\kappa-H)-\lambda^2(\mu^2+\nu^2)-\lambda\mu H=0. \tag{5-18}$$

In (5-16) we set $X=e$, $Y=U=\phi e$, and by virtue of (2-3), (2-4) we obtain

$$\begin{aligned}\left[-\kappa^2+\tfrac{1}{6}r\kappa+\lambda^2(\mu^2+\nu^2)-\tfrac{1}{6}r\lambda\mu\right]e-\tfrac{1}{6}r\lambda\nu\phi e+\kappa Z(e,\phi e)\phi e&\\ +\mu Z(e,\phi e)h\phi e+\nu Z(e,\phi e)\phi h\phi e&=0\end{aligned}$$

which by virtue of (5-10) yields

$$[-\kappa(\kappa-H)+\lambda^2(\mu^2+\nu^2)-\lambda\mu H]e-\nu\lambda H\phi e=0,$$

and hence, in addition from (5-17), we get

$$-\kappa(\kappa-H)+\lambda^2(\mu^2+\nu^2)-\lambda\mu H=0. \tag{5-19}$$

Since we work in U_2 where $\kappa<1$ or equivalently $\lambda\neq 0$, the equations (5-17), (5-18) and (5-19) yield the equations (4-17), (4-18) and (4-19) and hence Proposition 4.3 completes the proof. Our open submanifold U_2 is a generalized (κ,μ)-contact metric 3-manifold with $(\xi\cdot\mu)=0$ and according to Remark 4.4 this submanifold is a semi-K contact manifold.

We have proved:

(a) If $M=U_1$ then M is Sasakian with $\kappa=1$.

(b) If $M=U_2$ then M is a semi-K generalized (κ,μ)-contact metric manifold with $\kappa<1$, $(\xi\cdot\mu)=0$ and $r=4\kappa$.

(c) If $U_1\neq\varnothing$ and $U_2\neq\varnothing$, the union $U_1\cup U_2$ is open and dense in M; also, $\kappa=1$ in U_1 and $\kappa<1$ in U_2. The function κ is continuous in U_1 and in U_2. □

Remark 5.11. According to Proposition 4.3 and [Blair 2002, Theorem 7.5, p. 101]. U_2 becomes flat when $\mu=0$ since Equation (4-19) yields $\kappa=0$.

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GENUSES OF CLUSTER QUIVERS OF FINITE MUTATION TYPE

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In this paper, we study the distribution of the genuses of cluster quivers of finite mutation type. First, we prove that in the 11 exceptional cases, the distribution of genuses is 0 or 1. Next, we consider the relationship between the genus of an oriented surface and that of cluster quivers from this surface. It is verified that the genus of an oriented surface is an upper bound for the genuses of cluster quivers from this surface. Furthermore, for any nonnegative integer *n* and a closed oriented surface of genus *n*, we show that there always exist a set of punctures and a triangulation of this surface such that the corresponding cluster quiver from this triangulation is exactly of genus *n*.

1. Introduction

Cluster quivers are a valuable notion in the theory of cluster algebras, first introduced in the famous paper [Fomin and Zelevinsky 2002]. Since then this subject has been studied extensively by many mathematicians. The original motivation was to give a combinatorial characterization of dual canonical bases in the theory of quantum groups, and for the study of total positivity for algebraic groups. Now cluster algebras are connected to various fields of mathematics such as representation theory, Poisson geometry, algebraic geometry, Lie theory, combinatorics and so on. One knows that cluster algebras are commutative algebras equipped with a distinguished set of generators, i.e., cluster variables.

Two types of cluster algebras are of special interest: those of finite type, and those of finite mutation type. The former is a special case of the latter. Cluster algebras of finite type were completely classified in [Fomin and Zelevinsky 2003], and skew-symmetric cluster algebras of finite mutation type were completely classified in [Felikson et al. 2012]. The classification of cluster algebras of finite type is identical to the Cartan–Killing classification of semisimple Lie algebras and finite root systems. For a cluster algebra of finite type, there is a one-to-one correspondence

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between the set of cluster variables and the set of almost positive roots (consisting of positive roots and negative simple roots). Additionally, the classification of skew-symmetric cluster algebras (equivalently, the classification of cluster quivers) of finite mutation type tells us that almost all skew-symmetric cluster algebras (equivalently, cluster quivers) of this type come from triangulations of surfaces, except for 11 exceptional cases.

Given an oriented 2-dimensional Riemann surface S with boundary ∂S, let $M \subset S$ be a finite set of *marked points* such that each connected boundary component contains at least one such point. Marked points in the interior of S are called *punctures*. The pair (S, M) is simply called a *surface*. An *arc* [Fomin et al. 2008] is the homotopy class of a curve γ in S whose endpoints come from M, such that:

- γ does not intersect itself, except that its endpoints may coincide;
- except for the endpoints, γ is disjoint from M and ∂S;
- γ does not cut out an unpunctured monogon or an unpunctured digon.

An *ideal triangulation* T is a maximal set of noncrossing (i.e., there are no intersections in the interior of S) arcs. For the details of the construction of cluster quivers from triangulations of surfaces, see Section 2B.

In this paper, all surfaces we consider are oriented surfaces; all subgraphs and subquivers are full.

In topological graph theory, the *genus* of a graph is the minimal genus of the surfaces where the graph can be drawn without crossings. The *genus* of a quiver is defined to be that of its underlying graph. When discussing the genus of a quiver, one only needs to consider its simple underlying graph (without multiple edges and orientation). A graph (respectively, quiver) is *planar* if it is of genus 0. It is well known that genus is a topological invariant for surfaces, as well as for topological graphs. A natural question is to find out the relation between the genus of a surface and that of a cluster quiver from this surface. As an answer, we have the main conclusion in this paper:

Theorem 1.1. (i) *For a triangulation T of a surface S with genus g, let g' be the genus of the cluster quiver Q associated with T. Then $g' \leq g$.*

(ii) *Furthermore, for any nonnegative integer n and a closed oriented surface S_n of genus n, there exists a set of marked points M on S_n and an ideal triangulation P_n of S_n such that the corresponding cluster quiver T_n of P_n has genus n.*

From this result, we know that the genus of a surface is in fact an upper bound for the genuses of cluster quivers from the triangulations of this surface; moreover, any nonnegative integer n can be reached as the genus of some cluster quiver from surface.

The paper is organized as follows. The requisite background on cluster quivers, their mutation, and triangulations of surfaces are presented in Section 2. In Section 2A, we give the basic definitions of matrix mutation and quiver mutation. We mention the fact that skew-symmetric matrices are in bijection with cluster quivers, and also that matrix mutation and quiver mutation are compatible. In Section 2B, we recall some basic definitions and properties of triangulations of surfaces from [Fomin et al. 2008]. We recapitulate how to obtain a cluster quiver from a surface triangulation and the compatibility between quiver mutations and flips of triangulations. A cluster quiver comes from a surface if and only if it is block-decomposable. At the end of this subsection, we restate the classification of skew-symmetric cluster algebras of finite mutation type.

Section 3 mainly deals with the genuses of cluster quivers of finite mutation type. In Section 3A, we give the table of genus distribution of the 11 exceptional quivers by utilizing Keller's quiver mutation in Java [Keller 2006]. In Section 3B, we first prove Theorem 1.1(i) which states that the genus of a surface is an upper bound for the genuses of cluster quivers obtained by triangulations of this surface. From this result, one can easily see that genus is a mutation invariant for cluster quivers from the surface of genus 0. As another application of this result, we give a sufficient condition for two quivers not to be mutation equivalent. Part (ii) of Theorem 1.1 is proved by constructing a graph R_n, using topological graph theory for genus n and the classification theorem of compact surfaces in algebraic topology.

2. Preliminaries

2A. ***Cluster quiver and its mutation.*** The notion of skew-symmetric matrix or equivalently of cluster quiver is crucial in the theory of cluster algebras. In the definition of cluster algebras, the most important ingredient is the so-called *seed mutation*. For our purpose in this paper, we only introduce *matrix mutation* (an important part of seed mutation) so as to understand the motivation of cluster quivers. For the details of the definitions of seed mutation and cluster algebras, we refer to [Fomin and Zelevinsky 2003].

Suppose $B = (b_{ij})$ is an $n \times n$ integer matrix. For $1 \le k \le n$, a *matrix mutation* μ_k at direction k transforms B into a new matrix $B' = (b'_{ij})$ where b'_{ij} is defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Here, all matrices we consider are *skew-symmetric*. It is easy to see that matrix mutation transforms a skew-symmetric matrix into another one.

Given an $n \times n$ skew-symmetric matrix $B = (b_{ij})$, we can construct a quiver Q without loops and 2-cycles as follows: the vertex set is $\{1, 2, \dots, n\}$ (the set of row and column indices of the matrix B), and the number of arrows from i to j is defined to be b_{ij} if $b_{ij} > 0$.

Definition 2.1. A quiver without loops and 2-cycles is said to be a *cluster quiver*.

There is a one-to-one correspondence between the set of skew-symmetric matrices and the set of cluster quivers. In fact, given a cluster quiver Q with n vertices, one can construct a skew-symmetric matrix $B = (b_{ij})$ defined by $b_{ij} = \#\{i \to j\} - \#\{j \to i\}$, where $\#\{i \to j\}$ denotes the number of arrows from i to j. According to this one-to-one correspondence, *quiver mutation* can be deduced from matrix mutation.

Definition 2.2. Suppose Q is a cluster quiver with vertex set $Q_0 = \{1, 2, \dots, n\}$. For $k \in Q_0$, a *quiver mutation* μ_k at vertex k transforms Q into Q', where Q' is obtained by the following three steps:

(1) For every path $i \to k \to j$, add a new arrow $i \to j$.

(2) Reverse all arrows incident with k.

(3) Delete all 2-cycles.

One can easily see that the resulting quiver Q' is also a cluster quiver. Matrix mutation and quiver mutation are compatible in the following sense: given any $k \in \{1, 2, \dots, n\}$, $\mu_k(Q_B) = Q_{\mu_k(B)}$ and $\mu_k(B_Q) = B_{\mu_k(Q)}$.

It is easy to verify that both matrix mutation and quiver mutation are involutions, i.e., $\mu_k^2 = 1$. If $Q' = \mu_{k_1}\mu_{k_2} \dots \mu_{k_l}(Q)$ for some $k_1, k_2, \dots, k_l \in \{1, 2, \dots, n\}$, we will say that Q and Q' are *mutation equivalent*. Obviously, this is an equivalence relation on the set of isomorphism classes of cluster quivers with n vertices. A cluster quiver (respectively, skew-symmetric cluster algebra constructed from this quiver) is said to be of *finite mutation type* if the number of quivers in its mutation-equivalence class is finite. Cluster quivers of this type were completely classified in [Felikson et al. 2012]. We will restate this classification theorem in Section 2B.

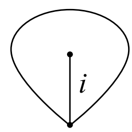
2B. ***Cluster quivers from surfaces.*** Given a surface (S, M), the number of arcs in any triangulation of (S, M) is a constant. The following lemma gives the formula to calculate the number of arcs in a triangulation.

Lemma 2.3 [Fomin et al. 2008]. *For a triangulation of a surface, the following formula holds*:

$$n = 6g + 3b + 3p + c - 6, \tag{1}$$

where n is the number of arcs, g is the genus of the surface, b is the number of connected boundary components, p is the number of punctures, and c is the number of marked points on the boundary.

The arcs of an ideal triangulation cut the surface S into *ideal triangles*. The three sides of an ideal triangle do not have to be distinct, i.e., we allow *self-folded* triangles, like this:



Given an ideal triangulation T, there is an associated signed adjacency matrix $B(T)$ (see [Fomin et al. 2008, §4]). Suppose the arcs in T are labeled by the numbers $1, 2, \ldots, n$, and let the rows and columns of $B(T)$ be numbered from 1 to n. For an arc i, let $\pi_T(i)$ denote the arc defined as follows: if there is a self-folded ideal triangle in T folded along i (see figure above), then $\pi_T(i)$ is its remaining side; otherwise, we set $\pi_T(i) = i$.

For each non-self-folded triangle $\triangle$, define the $n \times n$ integer matrix $B^{\triangle} = (b_{ij}^{\triangle})$ by setting

$$b_{ij}^{\triangle} = \begin{cases} 1 & \text{if side } \pi_T(j) \text{ immediately follows } \pi_T(i) \text{ in } \triangle \text{ going clockwise;} \\ -1 & \text{if side } \pi_T(i) \text{ immediately follows } \pi_T(j) \text{ in } \triangle \text{ going clockwise;} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $B = B(T) = (b_{ij})$ is defined by

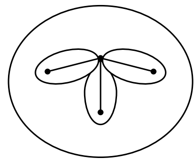
$$B = \sum_{\triangle} B^{\triangle}, \tag{2}$$

where the sum is taken over all non-self-folded triangles $\triangle$. It is easy to verify that $B(T)$ is skew-symmetric, and that all its entries are equal to 0, 1, -1, 2 or -2. Therefore, given a triangulation T, we can first associate a skew-symmetric matrix $B(T)$ to T and then obtain a cluster quiver Q corresponding to $B(T)$, just as in Section 2A. The corresponding cluster quiver Q_B of $B = B(T)$ is said to *come from a surface*. Correspondingly, the cluster algebra defined by Q_B is also said to *come from a surface*.

A *flip* is a transformation of an ideal triangulation T into a new triangulation T' obtained by replacing an arc γ with a unique different arc γ' and leaving other arcs unchanged. Flips of triangulation and matrix mutation are compatible in the sense of the following proposition.

Proposition 2.4 [Fomin et al. 2008, Proposition 4.8]. *Suppose that the triangulation $\bar{T}$ is obtained from T by a flip replacing an arc k. Then $B(\bar{T}) = \mu_k(B(T))$.*

According to [Fomin et al. 2008, Remark 4.2], all triangulations that we are interested in can be obtained by gluing together a number of puzzle pieces, except for one case: the triangulation of the 4-punctured sphere obtained by gluing three self-folded triangles to respective sides of an ordinary triangle:



There are three types of puzzle pieces:

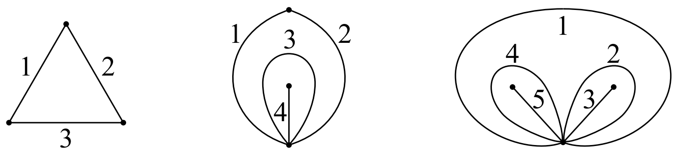
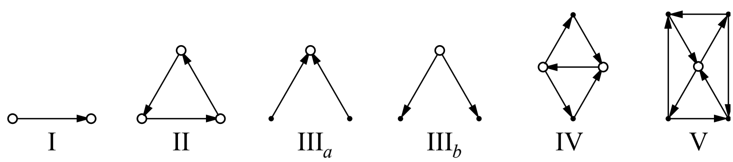


Figure 1. The three types of puzzle pieces.

These three types of puzzle pieces correspond to blocks of type I–V below, depending on whether the outer sides are lying on the boundary (for the details, see the proof of Theorem 13.3 in [Fomin et al. 2008]).



The vertices marked by open circles in this figure are called *outlets*.

Definition 2.5 [Fomin et al. 2008]. A quiver is said to be *block-decomposable* if it can be obtained from a collection of disjoint blocks by the following procedure:

(1) Take a partial matching of the combined set of outlets (matching an outlet to itself or to another outlet from the same block is not allowed).

(2) Glue the outlets in each pair of the matching.

(3) Remove all 2-cycles.

According to [Fomin et al. 2008, Theorem 13.3], a cluster quiver comes from a surface if and only if it is block-decomposable.

The following theorem gives a complete classification of skew-symmetric cluster algebras of finite mutation type.

Lemma 2.6 [Felikson et al. 2012]. *A skew-symmetric cluster algebra $\mathscr{A}$ of rank n is of finite mutation type if and only if $\mathscr{A}$ comes from a surface* ($n \geq 3$), *or $n \leq 2$, or $\mathscr{A}$ is one of the* 11 *exceptional types shown in* Figure 2 (that is, $\mathscr{A}$ has a cluster quiver at one of these types).

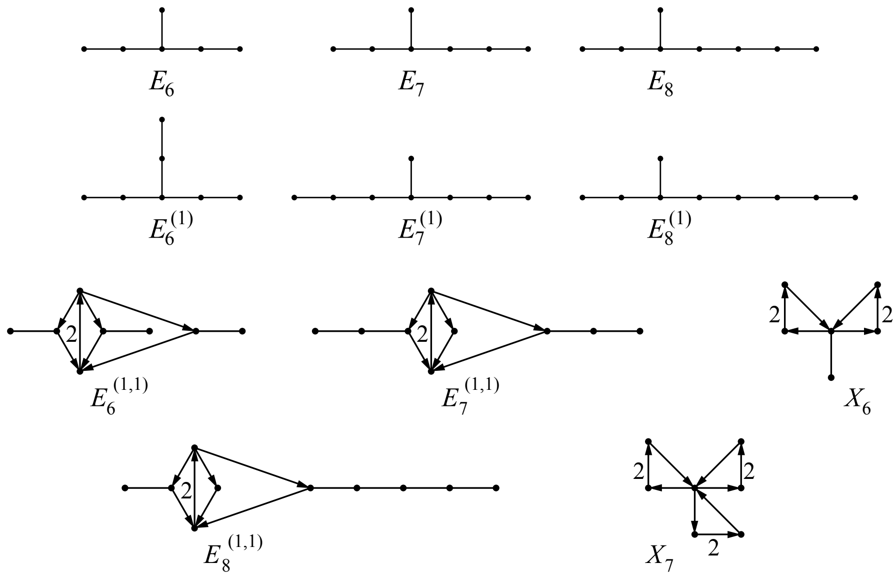


Figure 2. The eleven exceptional types.

3. Genus distribution of cluster quivers of finite mutation type

3A. ***Genuses of exceptional cluster quivers.*** Table 1 in this section gives the genus distribution of the 11 exceptional cluster quivers in the classification of cluster quivers of finite mutation type. Our main tool is Keller's quiver mutation in Java [Keller 2006]. To obtain the table, we note the following facts:

(1) E_6, E_7, E_8, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ are trees. According to Lemma 1.1 of [Vatne 2010], any orientations on the same tree are mutation equivalent.

(2) E_6, E_7, E_8 and $E_8^{(1)}$ are full subgraphs of the underlying graph of $E_8^{(1,1)}$; $E_6^{(1)}$ is a full subgraph of the underlying graph of $E_6^{(1,1)}$; $E_7^{(1)}$ is a full subgraph of the underlying graph of $E_7^{(1,1)}$. Since any quiver mutation-equivalent to a full subquiver of Q must be a full subquiver of some Q' that is mutation-equivalent to Q, we first test the mutation classes of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ in order to see their genus distribution.

(3) To see the genus of a quiver, we only need to see its underlying graph. Hence when doing the quiver mutation in Java due to Keller [2006], we can choose the mutation class under graph isomorphism. This can greatly cut down the number of quivers in the mutation class that we have to consider.

(4) We check the quivers in the mutation classes of $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ and find they are all planar. So are the other exceptional cluster quivers of type E.

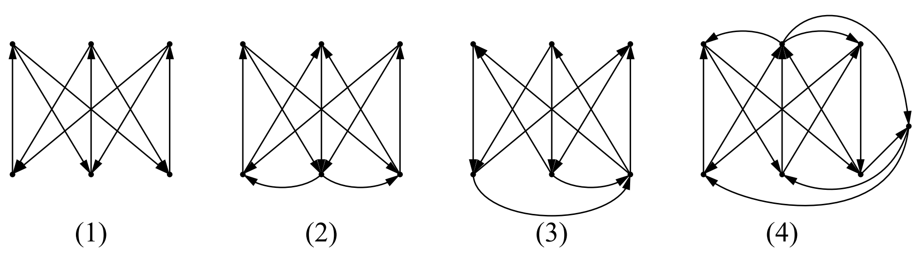
Type	Total number	Number of genus 0	Number of genus 1
E_6	21	21	0
E_7	112	112	0
E_8	391	391	0
$E_6^{(1)}$	52	52	0
$E_7^{(1)}$	338	338	0
$E_8^{(1)}$	1935	1935	0
$E_6^{(1,1)}$	27	27	0
$E_7^{(1,1)}$	217	217	0
$E_8^{(1,1)}$	1886	1886	0
X_6	4	1	3
X_7	2	1	1

Table 1. Statistics on exceptional cluster quivers of different types.

In Table 1, the *total number* means the number of quivers in the mutation class up to quiver isomorphism, and the *number of genus* 0 (respectively, 1) means the number of quivers (up to quiver isomorphism) in the mutation class whose genus is 0 (respectively, 1).

From the table, one can easily see that the genus of the quiver of type E is invariant under quiver mutation, but the genus of the quiver of type X will vary under quiver mutation.

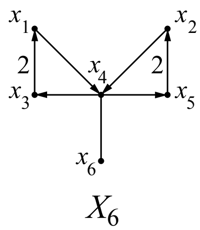
Proposition 3.1. *There are exactly four nonplanar cluster quivers of exceptional finite mutation types that have genus* 1:

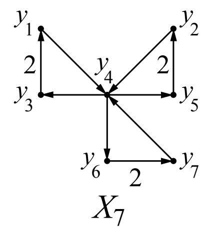


Quivers (1), (2), *and* (3) *are in the mutation-equivalence class of* X_6, *and quiver* (4) *is in the mutation-equivalence class of* X_7.

Proof.

- Quiver (1) is obtained from X_6 by mutation on the vertices x_4 and x_6, the vertex labeling being as shown on the top of the next page.
- Quiver (2) is obtained from X_6 by mutation on the vertex x_4.
- Quiver (3) is obtained from X_6 by mutation on the vertices x_4 and x_3.



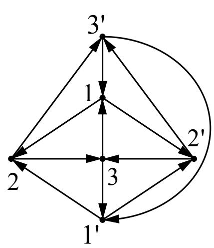


- Quiver (4) is obtained from X_7 by mutation on the vertex y_4. $\square$

3B. ***Proof of the main conclusion.*** We will begin by proving the first part of the theorem, i.e., that the genuses of cluster quivers obtained from the triangulations of a surface are not greater than that of the surface.

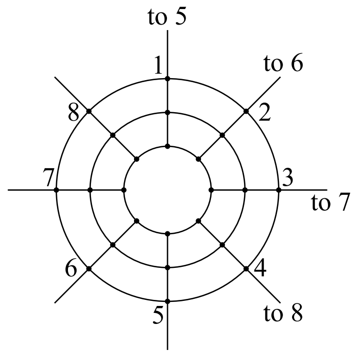
Proof of Theorem 1.1(i). By the correspondence of puzzle pieces and blocks, each puzzle piece corresponds to a block of type I–V. For each puzzle piece, we put its corresponding block into the face bounded by it. If two puzzle pieces have a common edge, then we glue the two vertices corresponding to the common edge between these two blocks. Hence we obtain the quiver Q of T in this way, and moreover the underlying graph of Q can be drawn without self-crossings on the surface S. We then have $g' \leq g$ by definition of the genus of a quiver.

To complete the proof of the theorem, we should consider the only exceptional case the triangulation of which cannot be obtained by gluing the puzzle pieces. Let T be the triangulation of the 4-punctured sphere obtained by gluing three self-folded triangles to respective sides of an ordinary triangle. The corresponding cluster quiver of T can be obtained by gluing four blocks of type II, as follows:



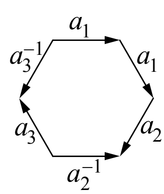
In this figure, for $i = 1, 2$ and 3, i and i' denote the corresponding vertices of two arcs in the same self-folded triangles. Obviously it is a planar quiver, and hence in this case $g' = g = 0$. This completes the proof. $\square$

To prove Theorem 1.1(ii), we need some preliminaries. First, we borrow from [Gross and Tucker 1987, Example 3.4.2] a class of graphs with arbitrary large genus. For each positive integer n, the graph R_n is constructed by taking $n+1$ concentric cycles consisting of $4n$ edges each, together with $4n^2$ inner edges connecting the $n+1$ cycles to each other and $2n$ outer edges adjoining antipodal vertices on the outermost cycle. Here is the graph R_2:



It was shown in [Gross and Tucker 1987] that R_n is of genus n.

Secondly, recall that the classification theorem for compact (or closed) surfaces (see, for example, [Massey 1977, Chapter 1, Theorem 5.1]) asserts that any compact surface is homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes. Any compact surface can be considered as the quotient space of a polygon with directed edges identified in pairs. There is a convenient way to indicate which paired edges are to be identified in such a polygon. We give a letter (for example, $a, b, c, \dots$) to each pair of edges, different pairs receiving different letters. Starting at a definite vertex, we traverse the boundary of the polygon either clockwise or counterclockwise. If the arrow on an edge points in the same traversing direction, we put no exponent (or the exponent $+1$) on the letter for that edge; otherwise, we write the letter for that edge with the exponent -1. For example, the string $a_1a_1a_2a_2^{-1}a_3a_3^{-1}$ indicates the same identifications as this figure:



The various surfaces can then be described by the following strings (see [Massey 1977, §5]):

(1) The sphere: aa^{-1}.

(2) The connected sum of n tori: $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_nb_na_n^{-1}b_n^{-1}$.

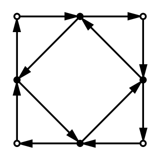
(3) The connected sum of n projective planes: $a_1a_1a_2a_2\dots a_na_n$.

Given a polygon, if the letter designating a certain pair of edges occurs with both exponents $+1$ and -1 in the symbol, then this pair of edges is said to be of the *first kind*; otherwise the pair is said to be of the *second kind*. From the proof of Theorem 5.1 in [Massey 1977], we know that if all the pairs of edges are of the first kind, then the resulting surface is oriented; if there exists a pair of edges of the second kind, then the resulting surface is nonoriented. Moreover, since the pair of adjacent edges of the first kind can be eliminated, the resulting surface of a $4n$-gon with pairs all of the first kind is an oriented surface with genus at most n.

To prepare for the proof of Theorem 1.1(ii), we first prove a lemma.

Lemma 3.2. *For an arbitrary nonnegative integer n, there always exists a block-decomposable cluster quiver T_n such that the genus $g(T_n)$ of T_n satisfies $g(T_n) \geq n$.*

Proof. Given a graph R_n as above, label the $n+1$ cycles from innermost to outermost by 1 to $n+1$. For each $i \in \{1, 2, \ldots, n\}$, there are $4n$ rectangles between the i-th cycle and the $(i+1)$-st cycle. For the outermost cycle, there exist $2n$ rectangles between the $(n+1)$-st cycle and itself. Two rectangles are said to be *neighbors* if they share a common edge; otherwise, they are said to be *distant*. It is easy to observe that there are $4n^2 + 2n$ rectangles in R_n. Given any rectangle A in R_n, we first choose four rectangles distant from A but having a common vertex with A. We repeat this process for each of these four rectangles; continuing this process, we will obtain a maximal set of mutually distant rectangles. This is denoted by $\mathcal{S}$. This set contains $2n^2 + n$ rectangles. The other $2n^2 + n$ rectangles form another maximal set of mutually distant rectangles. This is denoted by $\mathcal{T}$. Trivially, the two sets $\mathcal{S}$ and $\mathcal{T}$ are independent of the choice of the original rectangle A. Consider the set $\mathcal{S}$: each rectangle in $\mathcal{S}$ can be obtained by gluing four blocks of type II as shown on the right.



For the innermost cycle, there are $2n$ edges which do not lie in any rectangles of $\mathcal{S}$. We can then substitute one block of type IV for each such edge. For all these $2n$ edges, we need $2n$ blocks of type IV.

In summary, we obtain a quiver T_n by gluing $8n^2 + 4n$ blocks of type II and $2n$ blocks of type IV. According to the construction of T_n, obviously, R_n is a subgraph of the underlying graph of T_n. Therefore, the genus $g(T_n)$ is at least $g(R_n) = n$. Figure 3 on the next page illustrates the case $n = 2$. □

Proof of Theorem 1.1(ii). We will use the fact that the quiver T_n given in the proof of Lemma 3.2 can be obtained from a closed surface of genus n. By Lemma 3.2, $g(T_n) \geq n$. It is easy to check that T_n is a uniquely block-decomposable quiver and hence T_n can be uniquely encoded by its corresponding triangulation, that is, blocks of type II are encoded by puzzle pieces of the first type (see the left graph in Figure 1) and blocks of type IV are encoded by puzzle pieces of the second type (see the middle graph in Figure 1). In order to draw T_n, we first draw a planar quiver T'_n which has $4n$ unglued outlets. After gluing these $4n$ outlets in pairs, one obtains T_n, where each pair consists of one outlet and its opposite one. See Figure 3 for an

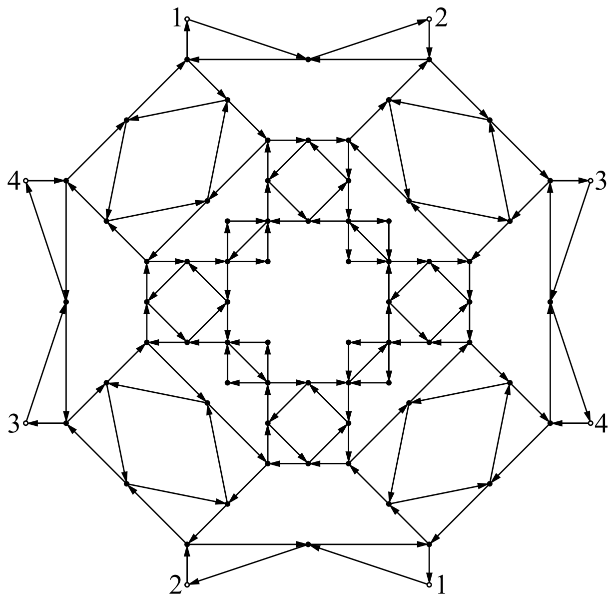
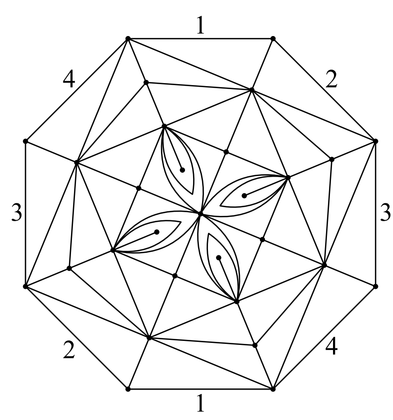


Figure 3. Quiver corresponding to the graph T_2. Vertices labeled by the same numbers should be glued together.

illustration of the case $n = 2$. Now we will construct a closed surface S_n of genus n and a triangulation P_n of S_n such that the corresponding cluster quiver is T_n.

We will chase T_n from innermost to outermost. Blocks of type II and type IV are encoded by puzzle pieces of the first and second types, respectively. For the outermost $4n$ oriented triangles in T'_n, we let each of them correspond to a puzzle piece of the first type. Thus we obtain a $4n$-gon with a triangulation. Denote this $4n$-gon with triangulation by S'_n. Then we can obtain a closed oriented surface S_n by identifying the edges of S'_n in pairs and gluing all outermost vertices into one, and then obtain a triangulation P_n of S_n such that its corresponding quiver is exactly T_n. For the case T_2, its corresponding S'_2 is given on the right.



To obtain S_2 and P_2, one only needs to glue the edges labeled by the same number in pairs and to glue all 8 outermost vertices into one.

By the proof of the classification theorem of compact surfaces in [Massey 1977], the genus of S_n is at most n.

Since T_n is obtained from a triangulation of S_n, by Theorem 1.1(i), $g(T_n) \leq n$.

On the other hand, by Lemma 3.2, $g(T_n) \geq n$. Hence, $g(T_n) = n$.

For the genus $g(S_n)$ of S_n, since $n = g(T_n) \leq g(S_n) \leq n$, we also have $g(S_n) = n$.

Then Theorem 1.1(ii) easily follows from the fact that all closed oriented surfaces with the same genus are homeomorphic. □

3C. ***Applications and further problems.*** As an application of Theorem 1.1(i), we give two corollaries.

Corollary 3.3. *Let S be a surface of genus* 0 *and M a set of marked points of S. Given any triangulation T of (S, M), suppose Q is the associated cluster quiver. Then all quivers in the mutation-equivalence class of Q are of genus* 0.

Besides the cluster quivers of type E in Section 3A, this corollary gives another class of cluster quivers of finite mutation type whose genuses are invariant under mutation.

Corollary 3.4. *Let S be a surface of genus g, with M its set of marked points. For any triangulation T of (S, M), let Q be its corresponding quiver and let Q' be another cluster quiver of genus g' such that $g' > g$. Then Q and Q' are not mutation equivalent.*

Proof. According to Proposition 12.3 in [Fomin et al. 2008], all quivers in the mutation-equivalence class of Q are the corresponding quivers of some triangulations of (S, M). Hence, by Theorem 1.1(i), the genuses of these quivers are not greater than g. Hence Q' is not in the mutation-equivalence class of Q, that is, Q and Q' are not mutation equivalent. □

This corollary gives us a necessary condition for two quivers with the same number of vertices, one coming from a triangulation of a surface and the other nonplanar, to be mutation equivalent.

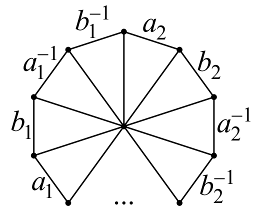
Remark 3.5. An easy calculation shows that the number of marked points on the closed surface S_n in the proof of Theorem 1.1(ii) is $4n^2 + 2n + 2$. For example, in the case $n = 2$, one can easily see that there are 22 marked points on S_2; here the outermost 8 marked points in S'_2 (see figure at the bottom of page 144) are glued into one.

Theorem 1.1(ii) tells us that, given a closed surface S of genus n, the upper bound of genuses of quivers from triangulations of S given in part (i) of Theorem 1.1 can be reached.

On the other hand, the lower bound 0 of genuses can also be reached; that is, given any closed oriented surface S with genus n, there always exists a triangulation T of S such that the corresponding cluster quiver Q of T is planar.

In fact, if the closed surface is a sphere, this obviously holds by Corollary 3.3; whereas if the closed surface S is of genus $n \geq 1$, it is homeomorphic to the connected sum of n tori. In this case, the symbol of the corresponding polygon is

$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_nb_na_n^{-1}b_n^{-1}$. A triangulation T of S with two punctures is shown on the right. For this triangulation, the outer $4n$ vertices in fact come from the same puncture and the only inner vertex is the other puncture. One can easily check that the corresponding cluster quiver Q of T is planar.



Restricting the discussion to the torus, we reach the following conclusion:

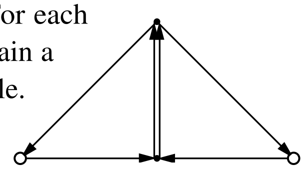
Proposition 3.6. *For a given cluster quiver Q from the torus S with p punctures, there exists at least one planar quiver in the mutation-equivalence class of Q.*

Proof. According to Proposition 12.3 in [Fomin et al. 2008], the corresponding quivers from all triangulations of S are mutually mutation equivalent. Hence, we only need to find a triangulation T of S such that its corresponding quiver is planar.

For the convenience of describing the desired triangulation, we first restate how a torus is constructed. Given two circles C and C', assume the radius of C is greater than that of C'. Let the center of C' run along C for one round; then a torus is built. The circle C is called a *basic circle* for this torus.

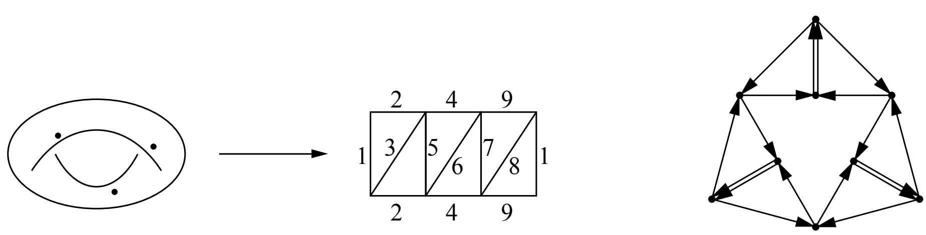
For the torus S with p punctures, we construct a triangulation T as follows:

For each puncture, construct a closed arc on S perpendicular to the basic circle such that its two endpoints coincide at the puncture; we have p such arcs. These p arcs cut down the torus into p pieces of cylinders. For each cylinder, drawing an arc between two punctures, we obtain a rectangle. Moreover, we draw a diagonal in this rectangle. The corresponding quiver from such a rectangle with its diagonal is shown on the right.



All p such rectangles with diagonal are arranged continuously together to form a graph. The quiver Q of T is obtained by gluing p pieces of such quivers along the outlets. Obviously, it is a planar quiver. □

For example, in the case $p = 3$, the triangulation and the corresponding cluster quiver are as follows, where the numbers $1, \dots, 9$ label the arcs:



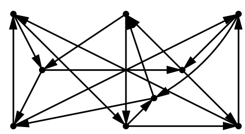
Since both the upper and lower bounds for genuses of cluster quivers from closed surfaces can be attained, based on Theorem 1.1 and Proposition 3.6 we propose these further interesting problems:

Problem 3.7. For any closed surface S with genus n and $0 \leq i \leq n$, does there exist a certain number of punctures and an ideal triangulation $T^{(i)}$ of S such that the corresponding cluster quiver Q_i from $T^{(i)}$ is of genus i?

Problem 3.8. Given a closed surface S of genus n, find the minimal number of punctures on S with the property that there exists an ideal triangulation T of S such that the corresponding cluster quiver Q_n of T is of genus exactly n.

For the case of the torus, we know at least one planar quiver in each mutation-equivalence class according to Proposition 3.6. Hence, for a given number of punctures we can check the corresponding mutation-equivalence class of this planar quiver by Keller's quiver mutation in Java [Keller 2006]. Since the genus of a quiver has nothing to do with the orientations of the arrows, we can choose the mutation-equivalence class under graph isomorphism when doing quiver mutation in Java.

For the cases $p = 1$ and $p = 2$, all quivers in their two mutation-equivalence classes are planar. When $p = 3$, there exists exactly one quiver of genus 1 in the mutation class:



Therefore, the answer to Problem 3.8 for the case of the torus is $p = 3$, which is much smaller than the number $4 \times 1^2 + 2 \times 1 + 2 = 8$ of punctures given in Remark 3.5 when constructing T_1 from the torus.

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TAUT FOLIATIONS IN KNOT COMPLEMENTS

TAO LI AND RACHEL ROBERTS

We show that for any nontrivial knot in S^3, there is an open interval containing zero such that a Dehn surgery on any slope in this interval yields a 3-manifold with taut foliations. This generalizes a theorem of Gabai on zero frame surgery.

1. Introduction

A transversely orientable codimension-one foliation $\mathscr{F}$ of a 3-manifold M is called *taut* [Gabai 1991] if every leaf of $\mathscr{F}$ intersects some closed transverse curve. The existence of a taut foliation in a 3-manifold M provides much interesting topological information about both M and objects embedded in M. If a closed 3-manifold M contains a taut foliation, either M is finitely covered by $S^2 \times S^1$ or M is irreducible [Novikov 1965; Reeb 1952; Rosenberg 1968]. If a closed 3-manifold M contains a taut foliation, then its fundamental group is infinite [Haefliger 1962; Novikov 1965; Gabai and Oertel 1989] and acts nontrivially on interesting 1-dimensional objects (see, for example, [Thurston 1998; Calegari and Dunfield 2003; Palmeira 1978; Roberts et al. 2003]), and its universal cover is $\mathbb{R}^3$ [Palmeira 1978]. Taut foliations can be perturbed to interesting contact structures [Eliashberg and Thurston 1998; Kazez and Roberts 2014] and hence can be used to obtain Heegaard–Floer information [Ozsváth and Szabó 2004b]. In this paper we seek to add to the understanding of the existence of taut foliations by describing a new construction of taut foliations.

Let k be a nontrivial knot in S^3. In his proof of the Property R conjecture, Gabai [1987b] showed that the knot exterior $M = S^3 \setminus \operatorname{int} N(k)$ has a taut foliation whose restriction to the torus ∂M is a collection of circles of slope 0. Thus a zero frame Dehn surgery on k yields a closed 3-manifold that admits a taut foliation obtained by adding disks along the boundary circles of the taut foliation of M. In this paper, we extend Gabai's theorem from zero frame surgery to any slope in an interval that contains 0. Although we restrict attention to knots in S^3, the approach described in this paper applies more generally to manifolds $(M, \partial M)$ with boundary a nonempty

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union of tori and for which there exists a well-groomed sutured manifold hierarchy which meets each component of ∂M only in essential simple closed curves.

Theorem 1.1. *Let k be a nontrivial knot in S^3. Then there is an interval $(-a, b)$, where $a > 0$ and $b > 0$, such that for any slope $s \in (-a, b)$, the knot exterior $M = S^3 \setminus \operatorname{int}(N(k))$ has a taut foliation whose restriction to the torus ∂M is a collection of circles of slope s. Moreover, by attaching disks along the boundary circles, the foliation can be extended to a taut foliation in $M(s)$, where $M(s)$ is the manifold obtained by performing Dehn surgery to k with surgery slope s.*

A group G is called *left-orderable* if there is a total order on G which is invariant under left multiplication. We thank Liam Watson for calling our attention to the following results.

Corollary 1.2. *Let k be a hyperbolic knot in S^3 and let $M(1/n)$ denote the manifold obtained by $1/n$ Dehn filling along k. Then there is some number $N = N(k)$ such that $\pi_1(M(1/n))$ is left-orderable whenever $|n| > N$.*

Proof. The surgered manifold $M(1/n)$ is a homology S^3 and, by Thurston's hyperbolic Dehn surgery theorem [Thurston 1982], atoroidal when $|n|$ is sufficiently large (or, equivalently, when $1/n$ is sufficiently small). Moreover, by Theorem 1.1, $M(1/n)$ contains a transversely oriented taut foliation whenever $1/n$ is sufficiently close to 0. It therefore follows from [Calegari and Dunfield 2003, Corollary 7.6] that $\pi_1(M(1/n))$ is left-orderable. □

Ozsváth and Szabó [2004c; 2004d] defined the Heegaard–Floer homology group $\widehat{HF}(Y)$ of a 3-manifold Y. In [Ozsváth and Szabó 2005], they define L-spaces as follows.

Definition 1.3 [Ozsváth and Szabó 2005, Definition 1.1]. A closed three-manifold is called an *L-space* if $H_1(Y; \mathbb{Q}) = 0$ and $\widehat{HF}(Y)$ is a free abelian group of rank $|H_1(Y; \mathbb{Z})|$.

L-spaces are therefore the closed 3-manifolds with the simplest possible Heegaard–Floer homology groups and the following is an important open question:

Question 1.4 [Ozsváth and Szabó 2004a, Question 11]. Is there a topological characterization of L-spaces (i.e., one that makes no reference to Floer homology)?

Ozsváth and Szabó proposed the following partial answer to this question:

Conjecture 1.5 [Hedden and Levine 2012, Conjecture 1]. If Y is an irreducible homology sphere that is an L-space, then Y is homeomorphic to either S^3 or the Poincaré homology sphere.

Approaches to understanding L-spaces have included investigations into the following two questions. Are L-spaces exactly those irreducible rational homology

3-spheres which contain no transversely oriented taut foliation? Are L-spaces exactly those irreducible rational homology 3-spheres which have non-left-orderable fundamental groups? (See [Boyer et al. 2012] for a nice survey.)

Conjecture 1.6 [Boyer et al. 2012, Conjecture 1]. An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.

With Conjecture 1.6 in mind, we compare Corollary 1.2 with the following result, which appears in various contexts [Ozsváth and Szabó 2004b, Corollary 1.3; Ghiggini 2008, Corollary 1.5], but is stated most conveniently as [Hedden and Watson 2010, Proposition 5].

Proposition 1.7 [Ozsváth and Szabó 2004b; Hedden and Watson 2010]. *Suppose k is a nontrivial knot in S^3 and let $M(1/n)$ denote the manifold obtained by $1/n$ Dehn filling along k. If $M(1/n)$ is an L-space, then either $n = 1$ and k is the right-handed trefoil or $n = -1$ and k is the left-handed trefoil.*

It follows that Conjecture 1.5 holds for 3-manifolds obtained by surgery on knots in S^3. And it follows from Corollary 1.2 and Proposition 1.7 that Conjecture 1.6 holds for 3-manifolds obtained by $1/n$ surgery on the complement of hyperbolic knots when $|n|$ is sufficiently large.

In Theorem 1.1, the interval $(-a, b)$ depends both on the knot k and on the sutured manifold decomposition in [Gabai 1987b]. In [Roberts 2001a; 2001b], it is shown that if k is a fibered hyperbolic knot (not necessarily in S^3), then this interval can always be chosen to contain $(-1, \infty)$, $(-\infty, 1)$, or $(-\infty, \infty)$. Related results appear in [Dasbach and Li 2004; Delman and Roberts 1999; Roberts 1995]. Moreover, the values of a and b in a maximal such interval $(-a, b)$ reveal information about the pseudo-Anosov monodromy and hence the geometry of M.

Question 1.8. Let k be a nontrivial knot in S^3, and let $a > 0$ and $b > 0$. What is the maximal interval $(-a, b)$ such that for any slope $s \in (-a, b)$, the knot exterior $M = S^3 \setminus \mathrm{int}(N(k))$ has a taut foliation whose restriction to the torus ∂M is a collection of circles of slope s, and the foliation can be extended to a taut foliation in $M(s)$ by attaching disks along the boundary circles, where $M(s)$ is the manifold obtained by performing Dehn surgery to k with surgery slope s?

Conjecture 1.9. Such a maximal interval will always contain $(-1, 1)$.

The proof of the main theorem uses theorems in [Li 2002; 2003] on branched surfaces to generalize the approach of [Roberts 2001a] to nonfibered knots. We first use Gabai's [1983; 1987a; 1987b] sutured manifold decomposition to construct a branched surface B. Then, after first splitting B as necessary, we add in some product disks to get a new branched surface that carries more laminations which extend to taut foliations. The key point in the construction is to add branch sectors so that the new branched surface does not contain any sink disk. By [Li 2002; 2003], this means that the branched surface carries a lamination.

2. Laminar branched surfaces

Definition 2.1. A *branched surface* B in M is a union of finitely many compact smooth surfaces, glued together to form a compact subspace (of M) locally modeled on Figure 1, left (ignore the arrows in the picture for now).

Given a branched surface B embedded in a 3-manifold M, we denote by $N(B)$ a regular neighborhood of B, as shown in Figure 1, right. One can regard $N(B)$ as an interval bundle over B. We denote by $\pi : N(B) \to B$ the projection that collapses every interval fiber to a point. As shown in Figure 1, right, the boundary of $N(B)$ consists of two parts: the horizontal boundary $\partial_h N(B)$ which is transverse to the I-fibers of $N(B)$, and the vertical boundary $\partial_v N(B)$ which is the union of subarcs of the I-fibers. The *branch locus* of B is $L = \{b \in B : b$ does not have a neighborhood in B homeomorphic to $\mathbb{R}^2\}$. We call the closure (under the path metric) of each component of $B \backslash L$ a *branch sector* of B. L is a collection of smooth immersed curves in B. Let Z be the union of double points of L. We associate with every component of $L \backslash Z$ a normal vector (in B) pointing in the direction of the cusp, as shown in Figure 1, left. We call it the *branch direction* of this arc. Let D be a disk branch sector of B. We call D a *sink disk* if the branch direction of every smooth arc in its boundary points into the disk and $D \cap \partial M = \varnothing$. We call D a *half sink disk* if $\partial D \cap \partial M \neq \varnothing$ and the branch direction of each arc in $\partial D \backslash \partial M$ points into D. Note that $\partial D \cap \partial M$ might not be connected.

Laminar branched surfaces were introduced in [Li 2002] as a branched surface with the usual properties in [Gabai and Oertel 1989] plus a condition that there is no sink disk. The notion of laminar branched surface was slightly extended to

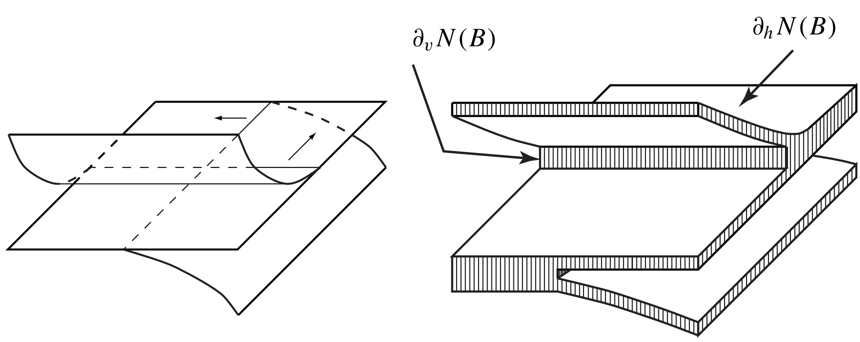

Figure 1. Left: a branched surface B. Right: a regular neighborhood $N(B)$.

branched surfaces with boundary, by adding a requirement that there is no half sink disk [Li 2003]. Note that if a branched surface has no half sink disk, then one can arbitrarily split the branched surface near its boundary train track without creating any sink disk. This plus Theorem 1 of [Li 2002] implies the following theorem from [Li 2003]. Note that the condition that there is no sink disk basically guarantees that the branched surface carries a lamination and the other conditions in [Gabai and Oertel 1989] imply that the lamination is an essential lamination.

Theorem 2.2 [Li 2003, Theorem 2.2]. *Let M be an irreducible and orientable 3-manifold whose boundary is an incompressible torus. Suppose B is a laminar branched surface and $\partial M \backslash \partial B$ is a union of bigons. Then, for any rational slope $s \in \mathbb{Q} \cup \infty$ that can be realized by the train track ∂B, if B does not carry a torus that bounds a solid torus in $M(s)$, then B fully carries a lamination $\mathscr{L}$ whose boundary consists of loops of slope s and $\mathscr{L}$ can be extended to an essential lamination in $M(s)$.*

3. Sutured manifold decompositions

Gabai [1983] introduced the notions of sutured manifold and sutured manifold decomposition. We will state basic definitions and theorems as needed for this paper but we refer the reader to [Gabai 1983; 1987a; 1987b] for a more detailed description. The papers [Altman 2012; Cantwell and Conlon 2012; Juhász 2008] and book [Candel and Conlon 2003] also provide nice descriptions of some of Gabai's sutured manifold theory. In this paper, we will use branched surfaces to describe sutured manifolds and sutured manifold decompositions.

Definition 3.1 [Gabai 1983, Definition 2.6]. A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a *suture*, that is, a homologically nontrivial oriented simple closed curve. We denote the set of sutures by $s(\gamma)$.

Finally, every component of $R(\gamma) = \partial M \backslash \operatorname{int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M \backslash \operatorname{int}(\gamma)$ whose normal vectors point out of (into) M. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$; that is, if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture.

Roughly speaking, a sutured manifold is a 3-manifold together with extra information about ∂M. Given a sufficiently nice surface S properly embedded in a sutured manifold (M, γ), it is important to be able to cut M open along S while keeping track of corresponding boundary information. This is captured in the following definition.

Definition 3.2 [Gabai 1983, Definition 3.1]. Let (M, γ) be a sutured manifold and S a properly embedded surface in M such that every component λ of $S \cap \gamma$ satisfies one of these three conditions:

(1) λ is a properly embedded nonseparating arc in γ.

(2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A \cap s(\gamma)$.

(3) λ is a homotopically nontrivial curve in a toral component T of γ, and if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

The surface S defines a *sutured manifold decomposition*

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma'),$$

where $M' = M \backslash \operatorname{int}(N(S))$ and

$$\begin{aligned} \gamma' &= (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma)), \\ R'_+(\gamma') &= \big((R_+(\gamma) \cap M') \cup S'_+\big) \backslash \operatorname{int}(\gamma'), \\ R'_-(\gamma') &= \big((R_-(\gamma) \cap M') \cup S'_-\big) \backslash \operatorname{int}(\gamma'), \end{aligned}$$

where S'_+ and S'_- are those components of $\partial N(S) \cap M'$ whose normal vectors point out of and into M', respectively.

Definition 3.3 [Gabai 1987a, Definition 0.2]. A sutured manifold decomposition

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$$

is called *well-groomed* if for each component V of $R(\gamma)$, $S \cap V$ is a union of parallel, coherently oriented, nonseparating closed curves and arcs.

Definition 3.4 [Gabai 1987b, Definition 3.2]. Let

$$(M, \partial M) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

be a sequence of sutured manifold decompositions where ∂M is a nonempty union of tori. Define $E_0 = \partial M$. Define E_i to be the union of those components of $E_{i-1} \backslash \operatorname{int}(N(S_i))$ which are annuli and tori (i.e., if M_i is viewed as a submanifold of M, then E_i consists of those components of γ_i which are contained in ∂M). The components of E_i are called the *boundary sutures* of γ_i.

Definition 3.5. Let (M, γ) and (N, τ) be sutured manifolds. We will call (M, γ) a *sutured submanifold* of (N, τ), and write $(M, \gamma) \subset (N, \tau)$, if M is a union of components of N and $\gamma = \tau \cap M$.

If $(M, \gamma) \subset (N, \tau)$, then we write $(N, \tau) \backslash (M, \gamma)$ to denote the sutured manifold $(N \backslash M, \tau \backslash \gamma)$.

Theorem 3.6 [Gabai 1987b, Lemmas 3.6 and 5.1]. *Let k be a knot in S^3. There is a well-groomed sutured manifold sequence*

$$(M,\gamma) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n) = (S\times I, \partial S\times I)$$

of

$$(M,\gamma) = (S^3 \backslash \operatorname{int}(N(k)), \partial N(k))$$

such that $\partial S_i \cap \partial N(k)$ is a (possibly empty) union of circles for each i, $1 \le i \le n$, S_1 is a minimal genus Seifert surface, and S is a compact (not necessarily connected) oriented surface.

Sutured manifold decompositions determine branched surfaces. As described by Gabai in [1987b, Construction 4.6] (and detailed further in [Cantwell and Conlon 2012]), a sutured manifold decomposition sequence corresponds to building a (finite depth) branched surface, starting with S_1 and successively adding the S_i's. To see this, inductively construct a sequence of transversely oriented branched surfaces. Let $B_1 = S_1$. So we may view M_1 as $M\backslash \operatorname{int}(N(B_1))$, where $N(B_1)$ is a fibered neighborhood of B_1. As a sutured manifold (M_1,γ_1), its suture γ_1 is the annulus $\overline{\partial M \backslash N(B_1)}$ and the two components of $\partial_h N(B_1)$ are the plus and minus boundaries $R_+(\gamma_1)$ and $R_-(\gamma_1)$ of the sutured manifold. We may view $R_+(\gamma_1)$ and $R_-(\gamma_1)$ as lying on the plus and minus sides of S_1 respectively and we assign a normal direction for $B_1 = S_1$ pointing from the plus side to the minus side.

Suppose we have constructed a branched surface B_k using the surfaces $S_1,\dots,S_k$ in the sutured manifold decomposition, such that $M\backslash \operatorname{int}(N(B_k)) = M_k$ and the suture γ_k of (M_k,γ_k) consists of $\partial_v N(B_k)$ and a collection of annuli in the boundary torus ∂M. Now we consider the sutured manifold decomposition

$$(M_k,\gamma_k) \overset{S_{k+1}}{\rightsquigarrow} (M_{k+1},\gamma_{k+1}).$$

The surface S_{k+1} has a normal vector. Then we can deform $B_k \cup S_{k+1}$ into a branched surface B_{k+1} as follows:

(1) For each component of ∂S_{k+1} that is not totally inside $\partial_v N(B_k)$, we can deform $B_k \cup S_{k+1}$ near ∂S_{k+1} as in Figure 2, left, so that the normal directions of B_k and S_{k+1} are compatible in the newly constructed branched surface.

(2) For each component c of ∂S_{k+1} lying inside a suture $\partial_v N(B_k)$, we first slightly isotope S_{k+1} by pushing c into $R_\pm(\gamma_k) \subset \partial_h N(B_k)$, then as shown in Figure 2, right, we can deform $B_k \cup S_{k+1}$ near c into a branched surface. By the requirement of the normal directions in the sutured manifold decomposition, the normal directions of B_k and S_{k+1} are compatible in the newly constructed branched surface.

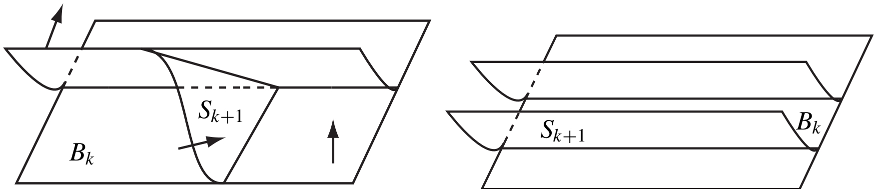

Figure 2. $B_k \cup S_{k+1}$ is deformed near ∂S_{k+1}. Left: the normal directions of B_k and S_{k+1} are compatible. Right: the neighborhoods of each suture are branched surfaces.

It follows from the definition of sutured manifold decomposition [Gabai 1983] that $M \setminus \text{int}(N(B_{k+1})) = M_{k+1}$ and the suture γ_{k+1} of (M_{k+1}, γ_{k+1}) consists of $\partial_v N(B_{k+1})$ and a collection of annuli in the boundary torus ∂M. We will sometimes use the notation

$$B_{k+1} = B_{(M_{k+1},\gamma_{k+1})} = B_{\langle S_1;S_2;\ldots;S_{k+1}\rangle}.$$

In summary, there is a map from the set of sutured manifold decomposition sequences to the set of properly embedded branched surfaces given by

$$(S_1, S_2, \ldots, S_l) \mapsto B_{\langle S_1;S_2;\ldots;S_l\rangle},$$

and a (forgetful) map from the set of properly embedded branched surfaces to the set of sutured 3-manifolds given by

$$B \mapsto (M_B, \gamma_B) = (M \setminus \text{int}(N(B)), \partial_v N(B) \cup E'),$$

where $E' \subset \partial M$ satisfies $E' = E$, the set of boundary sutures, if B intersects ∂M only in longitudes. For future reference, it is useful to highlight that under this correspondence, $\partial_h N(B)$ corresponds naturally to $R_+(\gamma_B) \cup R_-(\gamma_B)$.

4. The construction

Modifying the sutured manifold hierarchy. Given a well-groomed sutured manifold hierarchy satisfying the conclusions of Theorem 3.6, we can inductively construct the sequence of branched surfaces $B_1, \ldots, B_n$ corresponding to the sutured manifold decomposition. The branched surface B_n in the end has the properties that (1) $M \setminus \text{int}(N(B_n))$ is a product and (2) ∂B_n is a collection of circles in ∂M of slope 0. In particular, any taut foliation carried by B_n will also necessarily meet ∂M only in simple closed curves of slope 0.

To obtain a branched surface carrying taut foliations realizing an open interval of boundary slopes about 0, it is necessary to modify the sutured manifold hierarchy, or, equivalently, the sequence of branched surfaces B_k. In this section, we describe one way of doing this. We break the process into two steps.

As a first step, we slightly modify the sutured manifold hierarchy by adding some parallel copies of the surfaces S_k. Equivalently, we modify the sequence of branched surfaces B_k by adding some parallel copies of the surfaces S_k. This operation is equivalent to a splitting of the branched surface. As a second (and final) step, we further modify the sutured manifold hierarchy by adding carefully chosen product disks.

Before giving a precise description of these steps, we introduce some terminology. Let B be a transversely oriented branched surface and let F be a component of $\partial_h N(B)$. The boundary of F has two parts: $\partial F \cap \partial M$ and $\partial F \cap \partial_v N(B)$. We call $\partial F \cap \partial_v N(B)$ the *internal boundary* of F. Let L be the branch locus of B. Let L_F be the closure of $\pi^{-1}(L) \cap \mathrm{int}(F)$, where $\pi : N(B) \to B$ is the map collapsing each interval fiber to a point. So L_F is a trivalent graph properly embedded in F. We call L_F the *projection* of the branch locus to F. Each arc in L_F has a normal direction induced from the branch direction of L.

Definition 4.1. Let F be a component of $\partial_h N(B)$ with $\partial F \cap \partial M \neq \varnothing$ and let η be an arc properly embedded in F. If F has nonempty internal boundary, we require that η connects $\partial F \cap \partial M$ to the internal boundary of F. Choose η so that it intersects L_F transversely and only at points in the interior of edges of L_F (namely, it misses all triple points). Since η is transverse to L_F, the induced branch direction of L_F gives a direction along η for each point in $\eta \cap L_F$. We say η is *good* if these induced directions are coherent along η and all point away from an endpoint of η that lies in ∂M.

We say F is *good* if F satisfies the following properties:

(1) The closure of each component D of $F \backslash L_F$ has a boundary arc with induced branch direction (from L_F) pointing out of D.

(2) If F has internal boundary, then there is a set of disjoint good arcs, denoted by Γ_F, connecting each component of $\partial F \cap \partial M$ to the internal boundary of F.

(3) If F has no internal boundary (in which case, F must be a Seifert surface of the knot exterior), then there is a properly embedded nonseparating good arc in F, which we also denote by Γ_F.

Lemma 4.2. *Let B be a branched surface. If each component of $\partial_h N(B)$ is good, then B does not contain any sink disk or half sink disk.*

Proof. Let F be a component of $\partial_h N(B)$ and let L_F be as above. Let P be the closure (under path metric) of a component of $F \backslash L_F$. So P can be viewed as a copy of a branch sector of B. It follows from part (1) of Definition 4.1 that B has no sink disk or half sink disk. □

Definition 4.3. We say the branched surface B is *good* if

(1) every component of $\partial_h N(B)$ is good, and

(2) the arc systems Γ_F as described in (2) and (3) in Definition 4.1 can be chosen so that the projections $\pi(\Gamma_F)$, as F ranges over all components of $\partial_h N(B)$, are disjoint in B.

Note that these good arcs Γ_F will be the arcs along which we will attach product disks.

Step 1: Splitting B_n. Next we will describe the first modification of a sutured manifold decomposition sequence satisfying the conclusions of Theorem 3.6.

Lemma 4.4. *Let k be a nontrivial knot in S^3 and $M = S^3\backslash \mathrm{int}(N(k))$ the knot exterior. Let*

$$(M, \partial M) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n) = (S \times I, \partial S \times I)$$

be a well-groomed sutured manifold hierarchy that satisfies the conclusions of Theorem 3.6. Then there exists a well-groomed sutured manifold hierarchy

$$(M, \gamma) \overset{S_1}{\rightsquigarrow} (M_1', \gamma_1') \overset{R_1'}{\rightsquigarrow} (M_1'', \gamma_1'') \overset{S_2}{\rightsquigarrow} (M_2', \gamma_2') \overset{R_2'}{\rightsquigarrow} (M_2'', \gamma_2'') \overset{S_3}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n', \gamma_n')$$

which also satisfies the conclusions of Theorem 3.6. Moreover, the branched surfaces $B_l' = B_{(M_l', \gamma_l')}$, $1 \le l \le n$, satisfy the conditions:

(1) *$\partial B_l' \cap \partial M$ is a collection of simple closed curves of slope 0 in ∂M for each l.*

(2) *(M_l, γ_l) is a sutured submanifold of (M_l', γ_l') and $(M_l', \gamma_l')\backslash(M_l, \gamma_l)$ is a product sutured manifold for each l.*

(3) *Every branched surface B_l' is good.*

(4) *No B_l' carries a torus.*

(5) *(M_n', γ_n') is a product sutured manifold $(S' \times I, \partial S' \times I)$.*

Proof. First note that, in the sutured manifold hierarchy above, each R_i' is a parallel copy of some components of $R_+(\gamma_i') \cup R_-(\gamma_i')$.

We proceed by induction on l. Since k is nontrivial and hence S_1 has genus at least one, the branched surface $B_1' = S_1$ is easily seen to satisfy conditions (1)–(4). So suppose we have constructed

$$(M, \gamma) \overset{S_1}{\rightsquigarrow} (M_1', \gamma_1') \overset{R_1'}{\rightsquigarrow} (M_1'', \gamma_1'') \overset{S_2}{\rightsquigarrow} (M_2', \gamma_2') \overset{R_2'}{\rightsquigarrow} (M_2'', \gamma_2'') \overset{S_3}{\rightsquigarrow} \cdots \overset{S_l}{\rightsquigarrow} (M_l', \gamma_l')$$

satisfying the conclusions of Theorem 3.6 and such that the corresponding branched surfaces $B_i' = B_{(M_i', \gamma_i')}$ satisfy the conditions (1)–(4) for all i, $1 \le i \le l$.

By condition (2), (M_l, γ_l) is a sutured submanifold of (M_l', γ_l'). Let $R_+'(\gamma_l)$ and $R_-'(\gamma_l)$ be parallel copies of $R_+(\gamma_l)$ and $R_-(\gamma_l)$, chosen to be properly embedded in

$(M_l, \gamma_l) \subset (M'_l, \gamma'_l)$ and with boundary lying in $E_l \cup A(\gamma_l)$ (see Definition 3.1 and Definition 3.4). Set $R'_l = R'_+(\gamma_l) \cup R'_-(\gamma_l)$. We first consider the sutured manifold decomposition $(M'_l, \gamma'_l) \overset{R'_l}{\rightsquigarrow} (M''_l, \gamma''_l)$. By the definition of R'_l, this decomposition only adds some product complementary regions. Set $B''_l = B_{(M''_l, \gamma''_l)}$. The change from B'_l to B''_l is basically the addition of branch sectors corresponding to R'_l, and this operation creates some product complementary regions. See Figure 3, left, for a schematic picture. We may view (M_l, γ_l) as a subset of (M''_l, γ''_l), and consider the sutured manifold decompositions

$$(M'_l, \gamma'_l) \overset{R'_l}{\rightsquigarrow} (M''_l, \gamma''_l) \overset{S_{l+1}}{\rightsquigarrow} (M'_{l+1}, \gamma'_{l+1}),$$

where we now view S_{l+1} as lying in $(M_l, \gamma_l) \subset (M''_l, \gamma''_l)$. Certainly B'_{l+1} satisfies conditions (1) and (2).

Consider condition (3). We begin by considering a component F of $\partial_h N(B''_l)$. The surface F can be classified as one of the following 3 types (see Figure 3, right):

(1) F can be viewed as a component G of $\partial_h N(B'_l)$, as illustrated in Figure 3, right. Since the new branch sectors are attached to B'_l along cusp circles, L_F is obtained from L_G by adding curves parallel to curves in L_G with coherent induced branch direction, where L_G is the projection of the branch locus of

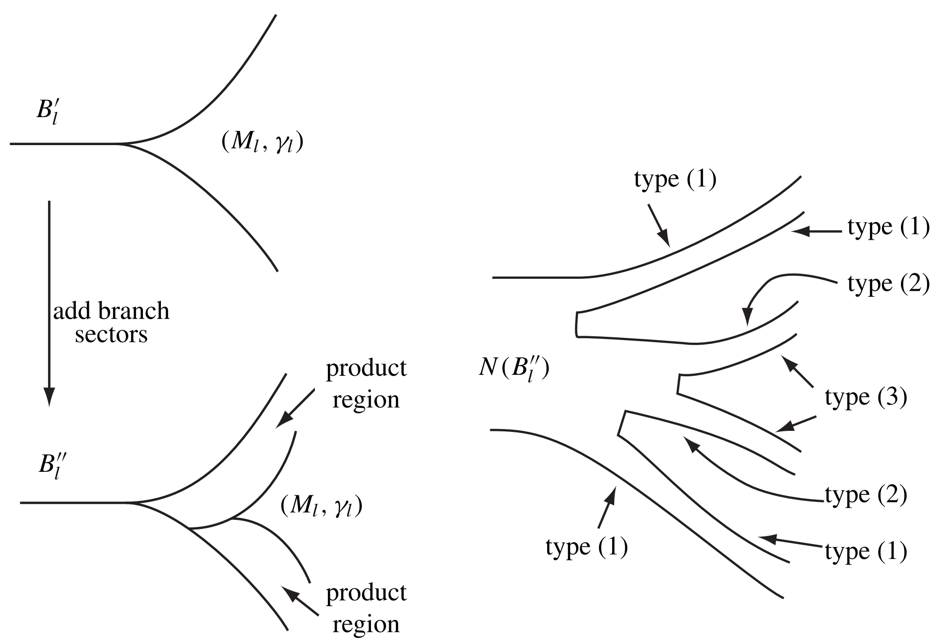

Figure 3. Left: adding branch sectors. Right: three different classifications of a component of $\partial_h N(B''_l)$.

B'_l to G. Since the branch directions are coherent, adding such parallel curves to L_G does not affect the good arcs in G. Thus in this case F is good with respect to B''_l with the same set of good arcs as G.

(2) F is a horizontal boundary component for a newly created product complementary region and $\pi(F)$ contains part of the branch sectors added to B'_l, as illustrated in Figure 3, right. In this case, each component of L_F consists of a circle C parallel to the internal boundary and with induced branch direction pointing to the internal boundary and possibly a collection of essential arcs in the annulus between C and the internal boundary.

(3) F is in the boundary of the sutured submanifold $(M_l, \gamma_l) \subset (M''_l, \gamma''_l)$. In this case, $L_F = \varnothing$.

Next consider how $\partial_h N(B'_{l+1})$ is related to $\partial_h N(B''_l)$. Let H be a component of $\partial_h N(B'_{l+1})$. Then either H can be viewed as a component of $\partial_h N(B''_l)$ or H contains a subset of one side of S_{l+1}. Our goal is to find a set of good arcs for each component H of $\partial_h N(B'_{l+1})$, so that the projections of the good arcs in B'_{l+1} are disjoint.

Case (a). H is not a component of $\partial_h N(B''_l)$

In this case, H is contained in the union of one side of S_{l+1} and $F \backslash \partial S_{l+1}$, where F is a component of $\partial_h N(B''_l)$ of type (3). By our construction, $L_F = \varnothing$. Moreover, on the other side of F, there is a corresponding component F' of $\partial_h N(B''_l)$ of type (2) such that $\pi(F) \cap \pi(F') \neq \varnothing$ in the branched surface B''_l. Adding S_{l+1} to B''_l does not affect F', so we may also view F' as a component of $\partial_h N(B'_{l+1})$. Next we choose good arcs for both H and F'.

First note that since the original sutured manifold decomposition is well-groomed, ∂S_{l+1} is homologically nontrivial in $H_1(F, \partial F)$. There is a simple closed curve η in F transverse to S_{l+1}, as shown in Figure 4 (note that the arrows in Figure 4 on ∂S_{l+1} denote the branch direction at ∂S_{l+1}), such that the algebraic intersection

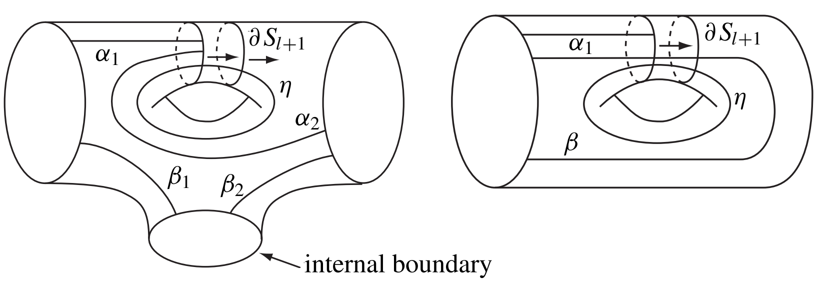

Figure 4. Left: arcs connecting each component of $\partial F \cap M$ to a component of ∂S_{l+1}. Right: arcs connecting each component of $\partial F' \cap \partial M$ to the internal boundary of F'.

number of η and ∂S_{l+1} is equal to $|\eta \cap \partial S_{l+1}|$ (this is equivalent to saying that the normal direction of ∂S_{l+1} at $\eta \cap \partial S_{l+1}$, induced from the branch direction of B'_{l+1}, are coherent along η).

Recall that H can be viewed as the union of one side of S_{l+1} and $F \backslash \partial S_{l+1}$. We first consider the components $\theta_1, \ldots, \theta_p$ of $\partial H \cap \partial M$ that are not in F (i.e., each θ_i can be viewed as a component of $\partial S_{l+1} \cap \partial M$). We can find an arc γ_i connecting θ_i to the internal boundary of H such that γ_i either is totally in (one side of) S_{l+1} or consists an arc in S_{l+1} and an arc in F parallel to a subarc of η. Moreover, we can choose these arcs γ_i to be disjoint in H.

Now we consider the components of $\partial F \cap \partial M$ (which are viewed as components of $\partial H \cap \partial M$). It is easy to see from our construction that there is a collection of disjoint good arcs $\alpha_1, \ldots, \alpha_q$ in F (see the arcs α_1 and α_2 in Figure 4, left), such that (1) these arcs α_j connect each component of $\partial F \cap \partial M$ to a component of ∂S_{l+1}, and (2) these arcs α_j are disjoint from the curve η describe above.

It follows from our construction that these arcs γ_i and α_j form a set of good arcs Γ_H for H.

Next we consider the component F' of $\partial_h N(B''_l)$ on the other side of F. F' is a type (2) component of $\partial_h N(B''_l)$, and we may view F' as a component of $\partial_h N(B'_{l+1})$. Moreover, we view F' as a parallel copy of F and view the curves ∂S_{l+1}, η and α_j described above as curves in F'. We have two slightly different situations. The first is that F' (and hence F) has nonempty internal boundary, and the second is that F' has no internal boundary.

If F' has nonempty internal boundary, then there are arcs $\beta_1, \ldots, \beta_r$ in F' (see the arcs β_1 and β_2 in Figure 4, left), such that (1) the arcs β_k connect each component of $\partial F' \cap \partial M$ to the internal boundary of F', and (2) the arcs β_k are disjoint from η, ∂S_{l+1} and the arcs α_j. The arcs β_k form a set of good arcs $\Gamma_{F'}$ for F'. Moreover, since each β_k is disjoint from η and the arcs α_j, the projections $\pi(\Gamma_H)$ and $\pi(\Gamma_{F'})$ of the good arcs Γ_H and $\Gamma_{F'}$ for H and F' respectively are disjoint in B'_{l+1}.

If F' does not have internal boundary (in which case F' must be a Seifert surface of the knot exterior), then as shown in Figure 4, right, there is an arc β properly embedded in F' such that (1) β is disjoint from η and the arcs α_j and (2) the intersection of β with ∂S_{l+1} is minimal up to isotopy. Since the original sutured manifold is well-groomed, the requirement (2) implies that the algebraic intersection number of β and ∂S_{l+1} is equal to $|\beta \cap \partial S_{l+1}|$. Thus β is a good arc for F'. Since β is chosen to be disjoint from η and each α_j, the projections of $\pi(\beta)$ and $\pi(\Gamma_H)$ on B'_{l+1} are disjoint.

<u>Case (b)</u>. H is a component of $\partial_h N(B''_l)$

In this case, either L_H is unchanged by the decomposition by S_{l+1} or H is the surface F' of type (2) considered in Case (a). In Case (a), we have already constructed a set of good arcs for the type (2) surface F', so we may assume that L_H is unchanged by the decomposition by S_{l+1}. Since H (viewed as a component of $\partial_h N(B_l''))$ is good in B_l'', H is good in B_{l+1}'. Furthermore, the projections of the good arcs in Case (a) and the good arcs (from the induction) of H in this case are disjoint in B_{l+1}'.

So B_{l+1}' is good. It remains to show that B_{l+1}' does not carry any torus. Since B_l' does not carry any torus and B_l'' can be obtained by splitting B_l', B_l'' does not carry any torus. Suppose B_{l+1}' carries a torus T. Then T can be expressed as the union of some copies of S_{l+1} and a surface in $N(B_l'')$ transverse to the I-fibers. Moreover, the transverse orientation of the branched surface induces a compatible normal orientation for T. Since the original sutured manifold decomposition sequence is well-groomed, $\partial S_{l+1} \cap R_\pm(\gamma_l)$ is a collection of homologically nontrivial curves in $H_1(R_\pm(\gamma_l), \partial R_\pm(\gamma_l))$. Thus there is a component F of $\partial_h N(B_l'')$, such that $T \cap F$ (with the induced orientation) is homologically nontrivial in F. However, since T is a torus in S^3, T is homologically trivial and this is impossible.

Therefore, B_{l+1}' satisfies properties (1)–(4) of the lemma and we can inductively construct the sutured manifold hierarchy and corresponding sequence of branched surfaces as claimed. $\square$

Step 2: Adding product disks. Let B_n' be the good branched surface constructed in the proof of Lemma 4.4. It follows from the conditions on the sutured manifold hierarchy and our construction above that $\partial B_n'$ consists of circles of slope 0 in the torus ∂M. In this section, we will add some product disks and modify B_n' to get a laminar branched surface carrying more laminations.

As $M \setminus \operatorname{int}(N(B_n'))$ is a product, we may suppose $M \setminus \operatorname{int}(N(B_n')) = S \times I$, where S is a compact and possibly disconnected surface. Let $S_+ = S \times \{0\}$ and $S_- = S \times \{1\}$. So $\partial_h N(B_n') = S_+ \cup S_-$. It is possible to decompose $S \times I$ as the disjoint union

$$S \times I = (F \times I) \cup (G \times I),$$

where F is the union of the components of S without internal boundary. Thus $\partial F \subset \partial M$ and each component of G has nonempty internal boundary. Moreover, each component of F must be a Seifert surface in the knot exterior. Note that, since we take parallel copies of surfaces in the horizontal boundary in each step of the sutured manifold decompositions (see Lemma 4.4), $F \neq \varnothing$. Furthermore, $G = \varnothing$ only if k is fibered.

Let $m = |\partial S_\pm \cap \partial M|$ be the number of components of the noninternal boundary $S_\pm \cap \partial M$. Since B_n' is good, there is a collection of pairwise disjoint good arcs in S_+, denoted by $\eta_1, \dots, \eta_m$, and a collection of pairwise disjoint good arcs in S_-, denoted by $\delta_1, \dots, \delta_m$, such that $\pi\left(\bigcup_i \eta_i\right) \cap \pi\left(\bigcup_i \delta_i\right) = \varnothing$ (in B_n') and each component of $\partial S_\pm \cap \partial M$ has exactly one incident good arc η_i and one incident good arc δ_i attached to it. After relabeling as necessary, we may assume that for $1 \le i \le r$, η_i and δ_i lie in $F \times \{0, 1\}$, while for $r+1 \le i \le m$, η_i and δ_i lie in $G \times \{0, 1\}$. It follows that each η_i and each δ_i, $1 \le i \le r$, has both endpoints lying on ∂M while each η_i and δ_i, $r+1 \le i \le m$, has exactly one endpoint lying on ∂M.

Consider first $F \times [0, 1]$. Recall that each component of F is a Seifert surface of the knot exterior. Let F_1 be any component of F and relabel as necessary so that $\eta_1 \subset F_1 \times \{0\}$ and $\delta_1 \subset F_1 \times \{1\}$. By [Roberts 2001a, Lemma 4.4], there is a sequence of simple arcs

$$\alpha_0 = \eta_1, \alpha_1, \dots, \alpha_l = \delta_1$$

such that $\alpha_i \cap \alpha_{i+1} = \varnothing$ and a regular neighborhood of $\alpha_i \cup \alpha_{i+1} \cup \partial F_1$ in F_1 is a twice-punctured torus for each i, $1 \le i \le l$. For $1 \le i \le l$, let F_1 induce a consistent orientation on each $F_1 \times \left\{\frac{i}{l+1}\right\}$ and orient the disks $\alpha_i \times \left[\frac{i}{l+1}, \frac{i+1}{l+1}\right]$ arbitrarily. Add branch sectors to B_n' as prescribed by the following sequence of sutured manifold decompositions:

$$(M_n', \gamma_n') \overset{A}{\rightsquigarrow} (M_{n+1}', \gamma_{n+1}') \overset{B}{\rightsquigarrow} (M_{F_1}, \gamma_{F_1}),$$

where

$$A = F_1 \times \left\{\tfrac{1}{l+1}, \dots, \tfrac{l}{l+1}\right\} \quad \text{and} \quad B = \bigcup_i \left(\alpha_i \times \left[\tfrac{i}{l+1}, \tfrac{i+1}{l+1}\right]\right).$$

Repeat for each remaining component of F and let (M_F, γ_F) denote the resulting sutured manifold. Set $B_F = B_{(M_F, \gamma_F)}$. Notice that the conditions satisfied by the arcs α_i guarantee that B_F is laminar.

Now consider $G \times I$. Let G_1 be a component of G and let $p = |\partial G_1 \cap \partial M|$. Let $\{C_1, \dots, C_p\}$ be a listing of the components of $G_1 \cap \partial M$. After relabeling as necessary, we may assume $\eta_{r+1}, \dots, \eta_{r+p}$ lie in $G_1 \times \{0\}$ and $\delta_{r+1}, \dots, \delta_{r+p}$ lie in $G_1 \times \{1\}$, with $\{\eta_{r+i}(0), \delta_{r+i}(0)\} \subset C_i$ for each $1 \le i \le p$.

Lemma 4.5. *Let $\{\alpha_1, \dots, \alpha_p\}$ and $\{\beta_1, \dots, \beta_p\}$ each be a set of pairwise disjoint arcs properly embedded in G_1 with $\{\alpha_i(0), \beta_i(0)\} \subset C_i$ and $\{\alpha_i(1), \beta_i(1)\} \subset \partial G \backslash \{C_1, \dots, C_p\}$, the internal boundary of G_1. Let $s = \left|\bigcup_i \alpha_i \cap \bigcup_i \beta_i\right|$. Then either $s = 0$ or there is a set $\{\gamma_1, \dots, \gamma_p\}$ of pairwise disjoint arcs properly embedded in G_1 with $\gamma_i(0) \in C_i$, $\gamma_i(1) \in \partial G_1 \backslash \{C_1, \dots, C_p\}$, such that*

$$\max\left\{\left|\bigcup_i \alpha_i \cap \bigcup_i \gamma_i\right|, \left|\bigcup_i \beta_i \cap \bigcup_i \gamma_i\right|\right\} < s.$$

Proof. Suppose $s \neq 0$. Relabeling as necessary, we may assume that α_1 and $\bigcup_i \beta_i$ intersect. Choose z to be the point in $\alpha_1 \cap \bigcup_i \beta_i$ that is furthest along α_1. So there are j, t_0, t_1 such that $z = \alpha_1(t_0) = \beta_j(t_1)$ and $\alpha_1(t_0, 1] \cap \bigcup_i \beta_i = \varnothing$. Let γ_j be the concatenation of the two arcs $\beta_j[0, t_1]$ and $\alpha_1[t_0, 1]$, perturbed slightly so that it intersects α_1 transversely and minimally. For $i \neq j$, set $\gamma_i = \beta_i$. Then $\left|\bigcup_i \alpha_i \cap \bigcup_i \gamma_i\right| < \left|\bigcup_i \alpha_i \cap \bigcup_i \beta_i\right|$ and $\left|\bigcup_i \gamma_i \cap \bigcup_i \beta_i\right| = 0$. □

The next corollary follows immediately.

Corollary 4.6. *There are sets of arcs* $\mathcal{A}_i = \{\alpha_1^i, \dots, \alpha_p^i\}$, $1 \le i \le q$, *such that*

(1) *for each* i, *the arcs in* $\mathcal{A}_i$ *are pairwise disjoint and properly embedded in* G_1, $\alpha_j^i(0) \in C_j$, *and* $\alpha_j^i(1) \in \partial G_1 \backslash \{C_1, \dots, C_p\}$, $j = 1, \dots, p$,

(2) $\mathcal{A}_0 = \{\eta_{r+1}, \dots, \eta_{r+p}\}$ *and* $\mathcal{A}_{q+1} = \{\delta_{r+1}, \dots, \delta_{r+p}\}$, *and*

(3) $\bigcup_j \alpha_j^i \cap \bigcup_j \alpha_j^{i+1} = \varnothing$ *for each* i.

For $1 \le i \le q$, let G_1 induce a consistent orientation on each $G_1 \times \left\{\frac{i}{q+1}\right\}$. Orient the disks $\alpha_j^i \times \left[\frac{i}{q+1}, \frac{i+1}{q+1}\right]$ so that the orientation induced on their boundaries agrees with the orientation of α_j^i (which is the orientation from its starting point in ∂M to its ending point in the internal boundary). Add branch sectors to B_F as given by the following sequence of sutured manifold decompositions:

$$(M_F, \gamma_F) \overset{A}{\rightsquigarrow} (M'_F, \gamma'_F) \overset{B}{\rightsquigarrow} (M_{G_1}, \gamma_{G_1}),$$

where

$$A = G_1 \times \left\{\frac{1}{q+1}, \dots, \frac{q}{q+1}\right\} \quad \text{and} \quad B = \bigcup_{i,j} \left(\alpha_j^i \times \left[\frac{i}{q+1}, \frac{i+1}{q+1}\right]\right).$$

Repeat for each remaining component of G and let (M_G, γ_G) denote the resulting sutured manifold. Set $B_G = B_{(M_G, \gamma_G)}$. Notice that the conditions satisfied by the arcs α_j^i guarantee that B_G is laminar.

By Lemma 4.4, B'_n does not carry any torus. Therefore, any branched surface obtained by splitting B'_n also cannot carry a torus. And finally, any (closed) torus carried by B_G but not by this splitting of B'_n would necessarily pass through one of the added disk branches and hence would necessarily have nonempty boundary. Thus B_G does not carry a torus.

Noting that for each product disk in the above construction, its two normal directions give two ways of deforming it into a branched surface, let B'_G denote the branched surface obtained from B_G by reversing the orientations of the disks $\alpha_j^i \times \left[\frac{i}{q+1}, \frac{i+1}{q+1}\right]$. Notice that B'_G is also laminar, has only product complementary regions, and does not carry a torus.

Hence we have laminar branched surfaces B_G and B'_G with only product complementary regions and which do not carry a torus. We may therefore apply Theorem 2.2 to conclude the existence of taut foliations realizing any boundary slope carried by $B_G \cap \partial M$ or $B'_G \cap \partial M$. It remains to compute these boundary slopes.

The boundary train tracks. Let τ denote the train track $B_G \cap \partial M$ and let τ' denote the train track $B'_G \cap \partial M$.

Lemma 4.7. *Together*, τ *and* τ' *realize all slopes in* $(-a, b)$ *for some* $a, b > 0$.

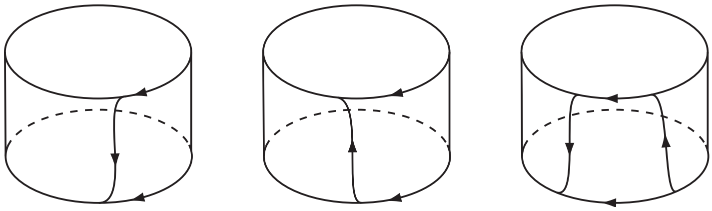

Figure 5. Train tracks that realize all slopes in $(-a, b)$ for $a, b > 0$.

Proof. Consider an annular component A_G of $\partial G_1 \times \left[\frac{i}{q+1}, \frac{i+1}{q+1}\right]$. The train tracks τ and τ' restricted to A_G have the form indicated in parts (a) and (b), respectively, of Figure 5. Similarly, consider an annular component A_F of $\partial F_1 \times \left[\frac{i}{l+1}, \frac{i+1}{l+1}\right]$. Recall that each $F_1 \times \left\{\frac{i}{l+1}\right\}$ is a Seifert surface and the good arc for F_1 has both endpoints on the circle ∂F_1. Thus both τ and τ' restricted to A_F have the form indicated in Figure 5(c). Call all such nonlongitudinal branches of τ or τ' *vertical*.

Since all vertical branches of τ (or τ', respectively) are of one of the three types shown in Figure 5, it follows that τ (or τ') is a train track obtained by concatenating pieces of the types of Figure 5(a) or (c) (or (b) or (c), respectively). Examples are shown in Figure 6. Notice that τ and τ' are orientable and measurable; namely, they admit a transverse measure [Hatcher 1988, page 66; Penner and Harer 1992, page 86]. Assign weights x, y, and $x + y$ to the vertical branches of τ and τ' as indicated in Figure 6; namely, vertical branches in $G \times I$ regions are weighted x, the compatibly oriented branches in $F \times I$ regions are weighted $x + y$, and the remaining branches in $F \times I$ regions are weighted y. Then assign weights from $\{1, 1 + x, 1 + y, 1 + x + y\}$ to the remaining branches of τ and τ' to obtain a measure μ on τ and a measure μ' on τ'.

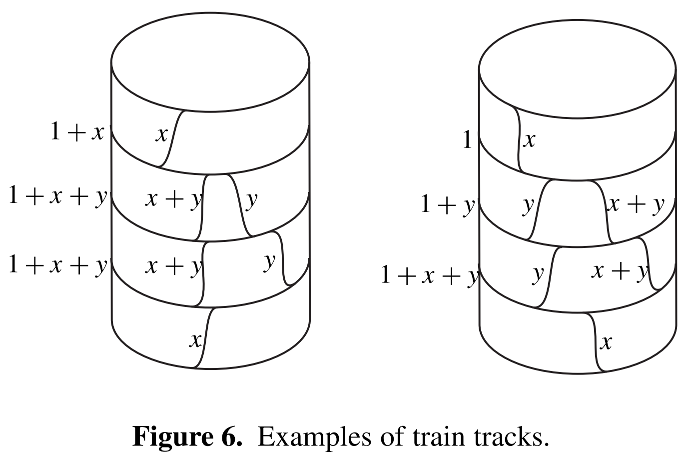

Figure 6. Examples of train tracks.

Recall that if γ is a simple closed curve in a torus, then the slope of γ is given in standard coordinates by

$$\text{slope}(\gamma) = \frac{\langle \lambda, \gamma \rangle}{\langle \gamma, m \rangle}, \tag{1}$$

where $\langle\, , \rangle$ denotes algebraic intersection number and λ is the longitude and m is the meridian of the knot k in S^3.

Applying (1) to the measured train tracks (τ, μ) and (τ', μ') while letting x, y range over all values $0 < y \ll x$, we see that (τ, μ) and (τ', μ') together carry all boundary slopes in some open interval $(-a, b)$ about 0. □

By Theorem 2.2, if τ (or τ') fully carries a curve of slope s, then B_G (or B'_G, respectively) fully carries an essential lamination whose boundary consists of loops of slope s in ∂M. Moreover, this lamination extends to an essential lamination in $M(s)$. Since $M \setminus \text{int}(N(B_G))$ and $M \setminus \text{int}(N(B'_G))$ consist of product regions, such essential laminations can be extended to taut foliations. This proves Theorem 1.1.

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ON THE SET OF MAXIMAL NILPOTENT SUPPORTS OF SUPERCUSPIDAL REPRESENTATIONS

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Let G be a quasisplit reductive group over a p-adic field k, T a maximal unramified anisotropic torus of $G(k)$, and χ a character of $T(k)$ satisfying certain conditions. Assume the residue characteristic p of k is large enough. It was shown by DeBacker and Reeder that the irreducible supercuspidal representation π_χ of $G(k)$ associated to $(T(k), \chi)$ is generic if and only if $\mathcal{B}(T, k)$ is a special vertex of $\mathcal{B}(G, k)$. We compute the set of maximal nilpotent support $\mathcal{N}_{\mathrm{wh,max}}(\pi_\chi)$ when $\mathcal{B}(T, k)$ is not a special point in $\mathcal{B}(G, k)$.

1. Introduction

Let k be a p-adic field and ψ a nontrivial character of k. Let $\boldsymbol{G}$ be a split orthogonal or symplectic group over k, $\boldsymbol{\mathfrak{g}}$ the Lie algebra of $\boldsymbol{G}$, $G = \boldsymbol{G}(k)$, and $\mathfrak{g} = \boldsymbol{\mathfrak{g}}(k)$. Let $\mathfrak{g}_{\mathrm{nil}}$ be the set of nilpotent elements in $\mathfrak{g}$ upon which G acts by the adjoint action. Let O be an orbit in $\mathfrak{g}_{\mathrm{nil}}/G$, $z \in O$, and let $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ be a Lie algebra homomorphism with

$$\phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = z.$$

Identify a scalar $t \in k$ with the diagonal matrix $\mathrm{diag}(t, t^{-1}) \in \mathfrak{sl}_2(k)$. For $j \in \mathbb{Z}$, let

$$\mathfrak{g}_j = \{Y \in \mathfrak{g} \mid \mathrm{Ad} \circ \phi(t)(Y) = itY \text{ for all } t \in k\}.$$

Then $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, $z \in \mathfrak{g}_{-2}$.

Let $N_{\geq 2}$ (resp. $N_{\geq 1}$) be the unipotent subgroup of G with Lie algebra $\mathfrak{n}_{\geq 2} = \bigoplus_{j \geq 2} \mathfrak{g}_j$ (resp. $\mathfrak{n}_{\geq 1} = \bigoplus_{j \geq 1} \mathfrak{g}_j$) and $\psi_z(n) = \psi(\mathrm{tr}(z \log n))$ be a character of $N_{\geq 2}$. Let S_z be the irreducible representation of $N_{\geq 1}$ whose restriction to $N_{\geq 2}$ is a multiple of ψ_z. Let π be an irreducible representation of G; following [Mœglin and Waldspurger 1987], let $\mathcal{N}_{\mathrm{wh}}(\pi)$ be the subset of nilpotent orbits such that $O \in \mathcal{N}_{\mathrm{wh}}(\pi)$ if and only if $\mathrm{Hom}_{N_{\geq 1}}(\pi, S_z) \neq 0$ for any $z \in O$. Let $\mathcal{N}_{\mathrm{wh,max}}(\pi)$ be the subset of maximal elements in $\mathcal{N}_{\mathrm{wh}}(\pi)$ with respect to the inclusion relation of closure of orbits.

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Keywords: supercuspidal representations, Bruhat–Tits building, nilpotent orbits.

On the other hand, let $\boldsymbol{T}$ be a maximal K-split anisotropic torus of $\boldsymbol{G}$; here, K is the maximal unramified extension of k. Then $T = \boldsymbol{T}(k)$ is a maximal unramified anisotropic torus of G. Let χ be a character of T satisfying certain conditions described in [Adler 1998] or [Reeder 2008]. There is a supercuspidal irreducible representation π_χ of G associated to (T, χ). Identify $\mathcal{B}(\boldsymbol{T}, k)$ as a point in $\mathcal{B}(\boldsymbol{G}, k)$. In [DeBacker and Reeder 2010], it was shown that π_χ is generic (that is, $\mathcal{N}_{\mathrm{wh}}(\pi_\chi)$ contains a regular nilpotent orbit) if and only if $\mathcal{B}(T, k)$ is a special point in $\mathcal{B}(G, k)$. In [Barbasch and Moy 1997], it was shown that if χ is of depth zero, the character of π_χ can be expanded as linear combination of orbital integrals over elements in $N_{\mathrm{wh}}(\pi_\chi)$.

For those (T, χ) with $\mathcal{B}(\boldsymbol{T}, k)$ nonspecial (that is, when $\mathrm{rank}(G)$ is large enough for $\mathcal{B}(G)$ to contain nonspecial vertices), we show in Theorem 3.2 that if χ is of *positive* depth, there is one element in $\mathcal{N}_{\mathrm{wh,max}}(\pi_\chi)$ which is related to $\mathcal{B}(\boldsymbol{T}, k)$. Note that in this case the supercuspidal representation π_χ is of *positive integral depth*. We also apply this theorem to irreducible representations in Π'_φ, the L-packet of φ, where φ is the Langlands parameter of π_χ.

This article is organized as follows: in Section 2, preliminary notation are recalled, including vertices in Bruhat–Tits building, L-packet of positive-depth supercuspidal representations [Reeder 2008], classification of maximal unramified anisotropic tori [DeBacker 2006], and classification of rational nilpotent orbits [Waldspurger 2001]. We also show by example in the Appendix how to choose a particular element from a rational nilpotent orbit. The main theorems are stated and proved in Section 3.

2. Preliminary

2A. ***Notation.*** Let k be a nonarchimedean local field of characteristic 0 with residue field $\mathfrak{f}$, and let p be the characteristic of $\mathfrak{f}$. Let $\mathfrak{O}$ be the ring of integers of k and $\mathfrak{P}$ the maximal ideal of $\mathfrak{O}$. Let K be the maximal unramified field extension of k and $\mathfrak{F}$ the residue field of K. Let ν be the normalized valuation of k and ν_K the extension of ν to K. Let ψ be an additive character of k with conductor $\mathfrak{P}$, and denote the character of $\mathfrak{f} = \mathfrak{O}/\mathfrak{P}$ derived from ψ by ψ also.

Throughout this paper, assume p is large enough that p is a good prime in the sense in [Carter 1972].

Let W be a finite-dimensional vector space over k, $\langle\cdot,\cdot\rangle$ a nondegenerate bilinear form on W, and $d = \dim_k(W)$. Assume that

$$\langle v, w\rangle = \epsilon_W \langle w, v\rangle \quad \text{for all } v, w \in W,$$

with $\epsilon_W = \pm 1$. Let $\boldsymbol{G}$ be the reductive group defined over k with

$$\boldsymbol{G} = \begin{cases} \mathbf{SO}(W) & \text{if } \epsilon_W = 1, \\ \mathbf{Sp}(W) & \text{if } \epsilon_W = -1. \end{cases}$$

Throughout this paper, assume that W has a k-basis $\{e_1, \dots, e_d\}$ satisfying

$$\langle e_j, e_k\rangle = \begin{cases} 0 & \text{if } j+k \neq d+1, \\ 1 & \text{if } j+k = d+1, j \leq k. \end{cases}$$

Then $\boldsymbol{G}$ is a connected split reductive group over k with finite center. Where no confusion will result, denote $\boldsymbol{G}$ by $\mathbf{SO}(d), \mathbf{Sp}(d)$ for $\epsilon_W = 1, -1$, respectively.

Let $J_W = (a_{i,j})$ be the matrix of degree d such that ${}^tJ_W = \epsilon_W J_W$ and

$$a_{j,k} = \delta_{j,d+1-k} \quad \text{for } j \leq k.$$

Let $\bar{k}$ be the algebraic closure of k and $R \subset \bar{k}$ a commutative k-algebra. Then $\boldsymbol{G}(R)$, the set of R-rational points of $\boldsymbol{G}$, is identified with the set of R-valued matrices g of degree d satisfying

$${}^tgJ_Wg = J_W, \quad \det(g) = 1.$$

Let $\boldsymbol{\mathfrak{g}}$ be the Lie algebra of $\boldsymbol{G}$; then $\boldsymbol{\mathfrak{g}}(R)$ is identified with the set of R-valued matrices g of degree d satisfying

$${}^tgJ_W + J_Wg = 0.$$

2B. ***Vertices of Bruhat–Tits building of G.*** Let $G = \boldsymbol{G}(k)$ and $\mathfrak{g} = \boldsymbol{\mathfrak{g}}(k)$. Let $\mathcal{B}(G) = \mathcal{B}(\boldsymbol{G}, k)$ be the Bruhat–Tits building of G. For $x \in \mathcal{B}(G)$, let G_x be the parahoric subgroup attached to x and $G_{x,+}$ the prounipotent radical of G_x. Let G_x be the connected reductive group defined over $\mathfrak{f}$ such that $G_x/G_{x,+}$ is the group of $\mathfrak{f}$-rational points of G_x. If F is a G-facet of $\mathcal{B}(G)$ and $x \in F$, let $G_F = G_x$, $G_{F,0+} = F_{x,0+}$, and $\mathsf{G}_F = \mathsf{G}_x$.

Let $\boldsymbol{S}$ be the maximal k-split torus of $\boldsymbol{G}$ containing all diagonal matrices in $\boldsymbol{G}$, $\boldsymbol{B}$ the Borel subgroup of $\boldsymbol{G}$ containing all upper triangular matrices in $\boldsymbol{G}$, $S = \boldsymbol{S}(k)$, and $B = \boldsymbol{B}(k)$. Let Φ be the set of roots of G with respect to S, Φ^+ the set of positive roots of G with respect to B, and $\Delta \subset \Phi^+$ the subset of simple roots of Φ^+. Let $\boldsymbol{\mathfrak{s}}$ be the Lie algebra of $\boldsymbol{S}$; then $\mathfrak{s} = \boldsymbol{\mathfrak{s}}(k)$ consists of all diagonal matrices in $\mathfrak{g}$. By taking differentials, roots in Φ are identified with linear functions on $\mathfrak{s}$.

Identify $\mathfrak{s}$ with k^n by the following isomorphism:

$$s = \operatorname{diag}(c_1, \dots, c_d) \in \mathfrak{s} \mapsto (c_1, \dots, c_n) \in k^n;$$

here, $n = [d/2]$. For $i = 1, \dots, n$, the i-th coordinate function e_i on k^n is identified with a linear function on $\mathfrak{s}$, still denoted by e_i. Let γ, α_i $(i = 1, \dots, n)$ be positive roots as follows:

$$\begin{array}{llll} & \alpha_i = e_i - e_{i+1}, & i = 1, \dots, n; & \\ & \alpha_n = e_n, & \gamma = e_1 + e_2, & \text{if } \boldsymbol{G} = \mathbf{SO}(2n+1); \\ \text{or} & \alpha_n = e_{n-1} + e_n, & \gamma = e_1 + e_2, & \text{if } \boldsymbol{G} = \mathbf{SO}(2n); \\ \text{or} & \alpha_n = 2e_n, & \gamma = 2e_1, & \text{if } \boldsymbol{G} = \mathbf{Sp}(2n). \end{array}$$

Then $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and γ is the highest root in Φ with respect Δ.

Let Φ_{af} be the set of affine roots of G with respect to S. As a subset of affine functions on $\mathfrak{s}$,

$$\Phi_{\mathrm{af}} = \{\alpha + m \mid \alpha \in \Phi, m \in \mathbb{Z}\}.$$

Let $\alpha_0 = 1 - \gamma \in \Phi_{\mathrm{af}}$ and $\Sigma = \Delta \cup \{\alpha_0\}$. Then every affine root is an integral combination of elements in Σ.

Let $X^*(S)$ be the character group of S, $X_*(S)$ the dual group of $X^*(S)$, and

$$\mathfrak{a} := X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $A = A(S)$ be the underlying affine space of $\mathfrak{a}$. Then A is an apartment in $\mathcal{B}(G)$. By fixing a hyperspecial point $o \in A$, one can identify A with $\mathfrak{a}$ and elements in Φ_{af} with affine functions on $\mathfrak{a}$.

Let C be the fundamental chamber of A defined by

$$C = \{z \in A \mid 0 < \alpha(z) < 1 \text{ for all } \alpha \in \Sigma\}.$$

For $\alpha \in \Phi_{\mathrm{af}}$, let $H_\alpha = \{z \in A \mid \alpha(z) = 0\}$. Then the H_α $(\alpha \in \Sigma)$ are walls of $\overline{C}$. For $0 \le i \le n$, let $y_i \in \overline{C}$, such that $\{y_i\} = \bigcap_{\substack{\alpha \in \Sigma \\ \alpha \neq \alpha_i}} H_{\alpha_j}$. Then the y_i $(i = 0, \dots, n)$ are vertices of $\overline{C}$. Let

$$I_{\mathrm{nsp}} = \begin{cases} \{2, \dots, n\} & \text{if } \boldsymbol{G} = \mathbf{SO}(2n+1), \\ \{2, \dots, n-2\} & \text{if } \boldsymbol{G} = \mathbf{SO}(2n), \\ \{1, \dots, n-1\} & \text{if } \boldsymbol{G} = \mathbf{Sp}(2n). \end{cases} \tag{1}$$

Then y_i is not a special vertex (see [Tits 1979]) for all $i \in I_{\mathrm{nsp}}$, and

$$\mathsf{G}_{y_i}(\mathfrak{f}) \simeq \begin{cases} \mathrm{SO}(2i, \mathfrak{f}) \times \mathrm{SO}(2n-2i+1, \mathfrak{f}) & \text{if } \boldsymbol{G} = \mathbf{SO}(2n+1), \\ \mathrm{SO}(2i, \mathfrak{f}) \times \mathrm{SO}(2n-2i, \mathfrak{f}) & \text{if } \boldsymbol{G} = \mathbf{SO}(2n), \\ \mathrm{Sp}(2i, \mathfrak{f}) \times \mathrm{Sp}(2n-2i, \mathfrak{f}) & \text{if } \boldsymbol{G} = \mathbf{Sp}(2n). \end{cases}$$

2C. ***On the stable conjugacy classes of maximal tori.*** If $\boldsymbol{T}$ is a maximal K-split k-torus of $\boldsymbol{G}$ defined over k, then $T = \boldsymbol{T}(k)$ is a maximal unramified torus of G [DeBacker 2006]. In this case, let $\mathcal{B}(T) = \mathcal{B}(\boldsymbol{T}, k)$. By [Adler 1998], choose a $\mathrm{Gal}(K/k)$-equivariant embedding of $\mathcal{B}(\boldsymbol{T}, K)$ into $\mathcal{B}(\boldsymbol{G}, K)$; then $\mathcal{B}(T)$ is identified with a subset of $\mathcal{B}(G)$:

$$\mathcal{B}(T) = \mathcal{B}(\boldsymbol{T}, K)^{\Gamma} \subset \mathcal{B}(\boldsymbol{G}, K)^{\Gamma} = \mathcal{B}(G).$$

DeBacker [2006] defines a set I^m and an equivalence relation "$\sim$" on I^m, so that there is a one-to-one and onto correspondence between $I^m/\sim$ and the set of G-conjugacy classes of unramified maximal tori in G. Elements in I^m are of the form (F, T), where F is an arbitrary G-facet in $\mathcal{B}(G)$ and T is a maximal minisotropic $\mathfrak{f}$-torus in G_F. Let $C(F,\mathsf{T})$ be the G-conjugacy class of maximal unramified tori in G corresponding to the equivalence class in I^m containing (F,T).

Let $o \in \mathscr{B}(G)$ be one of the special points chosen in Section 2B, to which we associate a conjugacy class of a maximal anisotropic $\mathfrak{f}$-torus in G_o and a conjugacy class in $\boldsymbol{W}(\mathsf{G}_o)$ (see [DeBacker 2006; Carter 1985]). Here $\boldsymbol{W}(\mathsf{G}_o)$ is the Weyl group of G_o. Let T_o (resp. w_o) be a representative of the conjugacy class of a maximal anisotropic $\mathfrak{f}$ torus (resp. the $\boldsymbol{W}(\mathsf{G}_o)$-conjugacy class). Then $(\{o\},\mathsf{T}_o) \in I^m$. Take $T = \boldsymbol{T}(k) \in C(\{o\},\mathsf{T}_o)$; then T is a maximal unramified anisotropic k-torus in G (see [DeBacker 2006]).

Let $\mathscr{S}(\mathsf{T}_o)$ be the subset of I^m consisting of elements (F,T) such that if $\boldsymbol{W}(\mathsf{G}_F)$ is identified with a subgroup of $\boldsymbol{W}(\mathsf{G}_o)$, then ${}^{\boldsymbol{W}(\mathsf{G}_F)}w_F \cap {}^{\boldsymbol{W}(\mathsf{G}_o)}w_o \neq \varnothing$, where w_F is a representative of the $\boldsymbol{W}(\mathsf{G}_F)$-conjugacy class corresponding to T. Then $\mathscr{S}(\mathsf{T}_o)$ depends only on the conjugacy class of w_o in $\boldsymbol{W}(\mathsf{G}_o)$. In fact, $\mathscr{S}(\mathsf{T}_o)$ is the set of G-conjugacy classes of maximal unramified anisotropic tori in the stable conjugacy class of T in G, which is the stable conjugacy class of maximal unramified tori in G corresponding to w_o [ibid., Corollary 4.3.2]. Let "$\sim$" be the equivalence relation on $\mathscr{S}(\mathsf{T}_o)$ inherited from I^m.

We briefly recall the classification of conjugacy classes in $\boldsymbol{W}(\mathsf{G}_o)$. Since G_o is split special orthogonal group or symplectic group over $\mathfrak{f}$,

$$\boldsymbol{W}(\mathsf{G}_o) \simeq \begin{cases} S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n & \text{if } \mathsf{G}_o = \mathrm{SO}(2n+1) \text{ or } \mathrm{Sp}(2n), \\ S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^{n-1} & \text{if } \mathsf{G}_o = \mathrm{SO}(2n), n \geq 2. \end{cases}$$

Here S_n is the n-th symmetric group. Conjugacy classes in $\boldsymbol{W}(\mathsf{G}_o)$ are parametrized by the set of pairs of partitions $(\boldsymbol{\lambda},\boldsymbol{\mu})$ with $S(\boldsymbol{\lambda}) + S(\boldsymbol{\mu}) = n$; moreover, if $\mathsf{G}_o = \mathrm{SO}(2n)$, $c(\mu)$ is even [Carter 1972, Propositions 24, 25]. Here, terminology in [Waldspurger 2001] is used: for a partition $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n,\dots)$,

$$S(\boldsymbol{\lambda}) = \sum_{i=1}^{\infty} \lambda_i, \quad c(\boldsymbol{\lambda}) = |\{i \geq 1 \mid \lambda_i \neq 0\}|.$$

In particular, conjugacy classes of anisotropic maximal tori in $\mathsf{G}_o(\mathfrak{f})$ are parametrized by the subset consisting of $(\varnothing,\boldsymbol{\mu})$, with $S(\boldsymbol{\mu}) = n$; if $\mathsf{G}_o = \mathrm{SO}(2n)$, $c(\boldsymbol{\mu})$ is even.

Assume $(\varnothing,\boldsymbol{\mu})$ corresponds to the conjugacy class of w_o in $W(\mathsf{G}_o)$, and write

$$\boldsymbol{\mu} = (\mu_1,\dots,\mu_s), \quad \mu_1 \geq \cdots \geq \mu_s \geq 1,$$

so that $S(\boldsymbol{\mu}) = n$, and s is even if $G = \mathrm{SO}(2n)$. Let

$$\mathscr{S}(\boldsymbol{\mu}) = \{\boldsymbol{\mu}' = (\mu_{j_1},\dots,\mu_{j_{s-2m}}) \mid \text{for some } 1 \leq j_1 < j_2 < \cdots < j_{s-2m}, 0 \leq 2m \leq s\}, \quad \text{if } G = \mathrm{SO}(2n+1) \text{ or } \mathrm{SO}(2n);$$

$$\mathscr{S}(\boldsymbol{\mu}) = \{\boldsymbol{\mu}' = (\mu_{j_1},\dots,\mu_{j_{s-m}}) \mid \text{for some } 1 \leq j_1 < j_2 < \cdots < j_{s-m}, 0 \leq m \leq s\}, \quad \text{if } G = \mathrm{Sp}(2n);$$

For $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, define

$$i := i_{\boldsymbol{\mu}'} := i(\boldsymbol{\mu}') := S(\boldsymbol{\mu}) - S(\boldsymbol{\mu}').$$

Then ${}^{\boldsymbol{W}(\mathsf{G}_o)}w_o \cap \boldsymbol{W}(\mathsf{G}_{y_i}) \neq \varnothing$. Here ${}^{\boldsymbol{W}(\mathsf{G}_o)}w_o$ is the conjugacy class of w_o and $\boldsymbol{W}(\mathsf{G}_{y_i})$ is the Weyl group of G_{y_i} identified as a subgroup of $\boldsymbol{W}(\mathsf{G}_o)$. By [DeBacker 2006, Corollary 4.3.2], there is a maximal anisotropic torus $\mathsf{T}_{\boldsymbol{\mu}'}$ in $\mathsf{G}_{y_i}(\mathfrak{f})$ that is $\mathsf{G}_o(\mathfrak{f})$-conjugate to T_o. Hence $(\{y_{i(\boldsymbol{\mu}')}\}, \mathsf{T}_{\boldsymbol{\mu}'}) \in \mathcal{S}(\mathsf{T}_o)$.

Take $T_{\boldsymbol{\mu}'} \in C(\{y_{i(\boldsymbol{\mu}')}\}, \mathsf{T}_{\boldsymbol{\mu}'})$; then $T_{\boldsymbol{\mu}'}$ is a maximal unramified anisotropic torus in G stably conjugate to T and $\mathcal{B}(T_{\boldsymbol{\mu}'}) = \{y_{i_{\boldsymbol{\mu}'}}\}$. In particular, $\boldsymbol{\mu} \in \mathcal{S}(\boldsymbol{\mu})$. Take $T_{\boldsymbol{\mu}} = T$. Conversely, all G-conjugacy classes in the stable conjugacy class of T have a representative of this form.

Lemma 2.1. *The set* $\{(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'}) \mid \boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})\}$ *is a complete set of representatives of* $\mathcal{S}(\mathsf{T}_o)/\sim$.

Proof. It remains to show that the pairs $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ are not equivalent to one another, for $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$. If $i_{\boldsymbol{\mu}'} = i_{\boldsymbol{\mu}''}$ for distinct $\boldsymbol{\mu}', \boldsymbol{\mu}'' \in \mathcal{S}(\boldsymbol{\mu})$, then by the choice of $\mathsf{T}_{\boldsymbol{\mu}'}$ and $\mathsf{T}_{\boldsymbol{\mu}''}$, $\mathsf{T}_{\boldsymbol{\mu}'}$ is not conjugate to $\mathsf{T}_{\boldsymbol{\mu}''}$ in $\mathsf{G}_{y_{i_{\boldsymbol{\mu}'}}}$; therefore $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ is not equivalent to $(\{y_{i_{\boldsymbol{\mu}''}}\}, \mathsf{T}_{\boldsymbol{\mu}''})$.

If $i_{\boldsymbol{\mu}'} \neq i_{\boldsymbol{\mu}''}$ for $\boldsymbol{\mu}', \boldsymbol{\mu}'' \in \mathcal{S}(\boldsymbol{\mu})$, we will show $y_{i_{\boldsymbol{\mu}'}}$ is not associated to $y_{i_{\boldsymbol{\mu}''}}$. As a consequence, $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ is not equivalent to $(\{y_{i_{\boldsymbol{\mu}''}}\}, \mathsf{T}_{\boldsymbol{\mu}''})$.

The case for $\boldsymbol{G} = \mathbf{Sp}(2n)$ is trivial, since the vertices $y_0, y_1, \ldots, y_n$ of $\bar{C}$ are not associated to each other.

If $\boldsymbol{G} = \mathbf{SO}(2n+1)$, among all vertices $y_0, y_1, \ldots, y_n$ of $\bar{C}$, y_0 is associated to y_1, and $y_0, y_2, \ldots, y_n$ are not associated to each other. For $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, if $i_{\boldsymbol{\mu}'} \neq 0$, then $i_{\boldsymbol{\mu}'} \geq 2$. As a result, $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ is not equivalent to $(\{y_{i_{\boldsymbol{\mu}''}}\}, \mathsf{T}_{\boldsymbol{\mu}''})$.

If $\boldsymbol{G} = \mathbf{SO}(2n)$, among all vertices $y_0, y_1, \ldots, y_n$, y_0 is associated to y_1, y_{n-1} is associated to y_n, and $y_0, y_2, \ldots, y_{n-2}, y_n$ are not associated to each other. For $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, if $i_{\boldsymbol{\mu}'} \neq 0$, then $i_{\boldsymbol{\mu}'} \neq 1$, $i_{\boldsymbol{\mu}'} \neq n-1$. Then $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ is not equivalent to $(\{y_{i_{\boldsymbol{\mu}''}}\}, \mathsf{T}_{\boldsymbol{\mu}''})$. □

2D. *L-packet.* Keep the notation of the previous subsection. Let $\mathfrak{t}_{\boldsymbol{\mu}}$ (resp. $\mathfrak{t}_{\boldsymbol{\mu}}(K)$) be the Lie algebra of $T_{\boldsymbol{\mu}}$ (resp. $\boldsymbol{T}_{\boldsymbol{\mu}}(K)$). For $s \in \mathbb{Z}$, let $\mathfrak{t}_{\boldsymbol{\mu},s}$ (resp. $T_{\boldsymbol{\mu},s}$) be the s-th filtration of $\mathfrak{t}_{\boldsymbol{\mu}}$ (resp. $T_{\boldsymbol{\mu}}$) [Adler 1998]. Let r be a positive integer, $X_{\boldsymbol{\mu}}$ a good element in $\mathfrak{t}_{\boldsymbol{\mu},-r}$ (i.e., $X_{\boldsymbol{\mu}} \in \mathfrak{t}_{-r}$), and for every root α of $\boldsymbol{T}_{\boldsymbol{\mu}}(K)$ in $\boldsymbol{G}(K)$, assume $\mathrm{d}\alpha(X_{\boldsymbol{\mu}}) \neq 0$. Let $\chi_{\boldsymbol{\mu}}$ be a character of $T_{\boldsymbol{\mu}}$ satisfying $\chi_{\boldsymbol{\mu}}|_{T_{\boldsymbol{\mu},r+1}} = 1$,

$$\chi_{\boldsymbol{\mu}}(\exp_o(Y)) = \psi(\mathrm{tr}(X_{\boldsymbol{\mu}} Y)) \quad \text{for all } Y \in \mathfrak{t}_{\boldsymbol{\mu},r}.$$

Here $\exp_o$ is the mock exponential map defined in [Adler 1998].

Let $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}}$ be the supercuspidal representation constructed by using $\chi_{\boldsymbol{\mu}}$ and $X_{\boldsymbol{\mu}}$, $\varphi : \mathcal{W}_k \to {}^L G$ be the L-parameter of $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}}$ (see [Adler 1998; Reeder 2008]), where

$\mathcal{W}_k$ is the Weil group of k. For $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, let $g \in \boldsymbol{G}(K)_o$ be an element such that $T_{\boldsymbol{\mu}'}(k) = {}^g T_{\boldsymbol{\mu}}(k)$; then $X_{\boldsymbol{\mu}'} = {}^g X_{\boldsymbol{\mu}}$ is a good element in $\mathfrak{t}_{\boldsymbol{\mu}',-r}$. Define a depth r character $\chi_{\boldsymbol{\mu}'}$ of $T_{\boldsymbol{\mu}'}$ by $\chi_{\boldsymbol{\mu}'} := {}^g \chi_{\boldsymbol{\mu}'}$; then,

$$\chi_{\boldsymbol{\mu}'}(\exp_{y_{i(\boldsymbol{\mu}')}}(Y)) = \psi(\operatorname{tr} X_{\boldsymbol{\mu}'} Y) \quad \text{for all } Y \in \mathfrak{t}_{\boldsymbol{\mu}',r}.$$

Let $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}$ be the supercuspidal representation of G constructed by using $\chi_{\boldsymbol{\mu}'}$ and $X_{\boldsymbol{\mu}'}$. Then:

Theorem 2.2 [Reeder 2008]. *The set $\Pi'(\varphi) = \{\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'} \mid \boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})\}$ is the L-packet associated to φ.*

The main result of this paper concerns nilpotent orbits supporting representations in $\Pi'(\varphi)$. Prior to the statement of the main theorems, we recall the classification of k-rational nilpotent orbits in $\mathfrak{g}$ [Waldspurger 2001, §I.6] and define a partition $\boldsymbol{\lambda}^i$ for every $i \in I_{\text{nsp}}$.

2E. *Nilpotent orbits.* Let $\boldsymbol{\lambda} = (\lambda_i)_{i \in \mathbb{N}}$ be a sequence of nonnegative integers such that $\lambda_j = 0$ for j sufficiently large. Define

$$S(\boldsymbol{\lambda}) = \sum_{j \geq 1} \lambda_j, \quad c(\boldsymbol{\lambda}) = |\{j \geq 1 \mid \lambda_j \neq 0\}|, \quad c_i(\boldsymbol{\lambda}) = |\{j \mid \lambda_j = i\}| \text{ for all } i \in \mathbb{N}.$$

If $\lambda_1 \geq \lambda_2 \geq \cdots$, $\boldsymbol{\lambda}$ is called a partition. Let $\mathcal{P}$ be the set of all partitions and $\mathcal{P}(n)$ the subset of all $\boldsymbol{\lambda} \in \mathcal{P}$ such that $S(\boldsymbol{\lambda}) = n$. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}$, let $\boldsymbol{\lambda} \cup \boldsymbol{\mu}$ be the unique partition such that $c_i(\boldsymbol{\lambda} \cup \boldsymbol{\mu}) = c_i(\boldsymbol{\lambda}) + c_i(\boldsymbol{\mu})$ for all $i \in \mathbb{N}$.

Let W be the vector space defined in Section 2A and $d = \dim_k W$. If $\epsilon_W = 1$, let $\mathcal{P}(W)$ be the set of partitions $\boldsymbol{\lambda} \in \mathcal{P}(d)$ such that c_i is even for all even i. If $\epsilon_W = -1$, let $\mathcal{P}(W)$ be the set of partitions $\boldsymbol{\lambda} \in \mathcal{P}(d)$ so that c_i is even for all odd i. Let $\mathrm{Nil}_I(W)$ be the set of $(\boldsymbol{\lambda}, (q_i))$ with $\boldsymbol{\lambda} \in \mathcal{P}(W)$, and let q_i, $i \in \mathbb{N}$, be quadratic forms satisfying these conditions:

- If $\epsilon_W = 1$, q_i is a nondegenerate quadratic form on k^{c_i} for i odd, $q_i = 0$ for i even, moreover the quadratic form $\bigoplus_{i \in \mathbb{N}} q_i$ has the same anisotropic kernel as q_W; here, q_W is the quadratic form on W defined by $q_W(v) = \langle v, v\rangle$.
- If $\epsilon_W = -1$, q_i is a nondegenerate quadratic form on k^{c_i} for i even, $q_i = 0$ for i odd.

Definition 2.3. $(\boldsymbol{\lambda}, (q_i)) \in \mathrm{Nil}_I(W)$ is called exceptional if $\epsilon_W = 1$, $4 \mid d$, and λ_i is even for all $i \in \mathbb{N}$. In this case, $q_i = 0$ for all $i \in \mathbb{N}$.

Definition 2.4. • If $\epsilon_W = -1$, let $\mathrm{Nil}(W) = \mathrm{Nil}_I(W)$;

- If $\epsilon_W = 1$, $4 \nmid d$, let $\mathrm{Nil}(W) = \mathrm{Nil}_I(W)$;
- If $\epsilon_W = 1$, $4 \mid d$, let $\mathrm{Nil}(W)$ be the set consisting all nonexceptional $(\boldsymbol{\lambda}, (q_i)) \in \mathrm{Nil}_I(W)$ and $(\boldsymbol{\lambda}, (q_i), \varepsilon)$ with $(\boldsymbol{\lambda}, (q_i))$ exceptional, $\varepsilon = \pm 1$.

By [Waldspurger 2001], there is a bijective correspondence between $\mathrm{Nil}(W)$ and $\mathfrak{g}_{\mathrm{nil}}/G$, the set of k-rational nilpotent orbits. Define a partial order on $\mathcal{P}(n)$: for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}(n)$, $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$ if and only if for all $j \geq 1$, $\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$.

Definition 2.5. Define a partial order on the set of nilpotent orbits in $\mathfrak{g}$: $O_1 \geq O_2$ if and only if $\overline{O}_1 \supset \overline{O}_2$. Here the closure is taken with respect to the usual topology in $\mathfrak{g}$.

Lemma 2.6. *Let O_1, O_2 be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}, (q_i))$ or $(\boldsymbol{\lambda}, \varnothing, \varepsilon)$ and $(\boldsymbol{\mu}, (q_i'))$ or $(\boldsymbol{\mu}, \varnothing, \varepsilon')$ respectively. If $O_1 > O_2$, then $\boldsymbol{\lambda} > \boldsymbol{\mu}$.*

Proof. The proof is similar to that of Theorem 6.2.5 of [Collingwood and McGovern 1993]. Take arbitrary $X \in O_1$, $Y \in O_2$, with O_1, O_2 corresponding to $(\boldsymbol{\lambda}, (q_i))$ or $(\boldsymbol{\lambda}, \varnothing, \varepsilon)$ and $(\boldsymbol{\mu}, (q_i'))$ or $(\boldsymbol{\mu}, \varnothing, \varepsilon')$ respectively. If $O_1 > O_2$, then $\overline{O}_1 \supsetneq \overline{O}_2$,

$$\mathrm{rank}(X^k) > \mathrm{rank}(Y^k) \quad \text{for all } k \geq 1,$$

since the condition that rank of a matrix be strictly less than a fixed number is a closed condition for the usual topology. Now $\boldsymbol{\lambda} > \boldsymbol{\mu}$ by of [ibid., Lemma 6.2.2], $\square$

Example 2.7. Regular nilpotent orbits in $\mathfrak{g}_{\mathrm{nil}}$ are those corresponding to:

- $([2n+1], q_{2n+1})$, if $\epsilon_W = 1$, $d = 2n+1$. Here q_{2n+1} is the nondegenerate quadratic form on k defined by $q_{2n+1}(x) = x^2$.
- $([2n-1, 1], (q_{2n-1}, q_1))$, if $\epsilon_W = 1$, $d = 2n$. Here q_{2n-1}, q_1 are nondegenerate quadratic forms on k such that $q_{2n-1} \oplus q_1 \simeq q'$, where q' is the quadratic form on k^2 defined by $q'(x, y) = 2xy$ for all $x, y \in k$.
- $([2n], q_{2n})$, if $\epsilon_W = -1$, $d = 2n$. Here q_{2n} is a nondegenerate quadratic form on k.

Let I_{nsp} be the set defined in (1). For $i \in I_{\mathrm{nsp}}$, let $\boldsymbol{\lambda}^i = \boldsymbol{\mu}' \cup \boldsymbol{\mu}''$ with

$$\begin{aligned}
&\boldsymbol{\mu}' = [2i-1, 1], && \boldsymbol{\mu}'' = [2n-2i+1], && \text{if } \epsilon_W = 1, d = 2n+1;\\
&\boldsymbol{\mu}' = [2i-1, 1], && \boldsymbol{\mu}'' = [2n-2i-1, 1], && \text{if } \epsilon_W = 1, d = 2n;\\
&\boldsymbol{\mu}' = [2i], && \boldsymbol{\mu}'' = [2n-2i], && \text{if } \epsilon_W = -1, d = 2n.
\end{aligned}$$

For $i \notin I_{\mathrm{nsp}}$, let

$$\boldsymbol{\lambda}^i = \begin{cases} [d] & \text{if } \epsilon_W = 1,\ d = 2n+1,\\ [d-1, 1] & \text{if } \epsilon_W = 1,\ d = 2n,\\ [d] & \text{if } \epsilon_W = -1,\ d = 2n. \end{cases}$$

Lemma 2.8. *Let $i \in I_{\mathrm{nsp}}$. Let O', O^i be nilpotent orbits in $\mathfrak{g}_{\mathrm{nil}}$ corresponding to $(\boldsymbol{\lambda}', (q_j'))$ or $(\boldsymbol{\lambda}', \varnothing, \varepsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$. Assume $O' > O^i$. Then*:

- *If* $\boldsymbol{G} = \mathbf{SO}(2n+1)$, *then* $\boldsymbol{\lambda}' = [2n+1]$ *or* $[m, 2n-m, 1]$ *for some odd* $m > \max(2i-1, 2n-2i+1)$.
- *If* $\boldsymbol{G} = \mathbf{SO}(2n)$ *and* $i \neq n/2$, *then* $\boldsymbol{\lambda}' = [m, 2n-m]$ *for some odd* $m \geq \max(2i-1, 2n-2i-1)$, *or* $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ *for some odd* $m > \max(2i-1, 2n-2i-1)$.
- *If* $\boldsymbol{G} = \mathbf{SO}(2n)$ *and* $i = n/2$, *then* $\boldsymbol{\lambda}' = [n^2]$, *or*
$$\boldsymbol{\lambda}' = [m, 2n-m] \text{ or } [m, 2n-m-2, 1^2]$$
for some odd $m > \max(2i-1, 2n-2i-1)$.
- *If* $\boldsymbol{G} = \mathbf{Sp}(2n)$, *then* $\boldsymbol{\lambda}' = [m, 2n-m]$ *for some even* $m > \max(2i, 2n-2i)$.

Proof. Assume $\boldsymbol{\lambda}' = [\lambda'_1, \lambda'_2, \dots] \in \mathcal{P}(W)$, with $\lambda'_1 \geq \lambda'_2 \geq \cdots$. By Lemma 2.6, if $O' > O^i$, then $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$.

Assume $\boldsymbol{G} = \mathbf{SO}(2n+1)$, $\boldsymbol{\lambda}^i = [2i-1, 1] \cup [2n-2i+1]$. First, assume $2i-1 \geq 2n-2i+1$, $\boldsymbol{\lambda}^i = [2i-1, 2n-2i+1, 1]$.

By definition, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$ if and only if $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$ and
$$\lambda'_1 \geq 2i-1, \quad \lambda'_1 + \lambda'_2 \geq 2n, \quad \lambda'_1 + \lambda'_2 + \lambda'_3 = 2n+1.$$
Then $\lambda'_3 = 0$ or $\lambda'_3 = 1$. If $\lambda'_3 = 0$, $\lambda'_2 = 0$, then $\boldsymbol{\lambda}' = [2n+1] > \boldsymbol{\lambda}^i$. If $\lambda'_3 = 0$, $\lambda'_2 \neq 0$, then $\boldsymbol{\lambda}' = [\lambda'_1, 2n+1-\lambda'_1] \notin \mathcal{P}(W)$, which contradicts the assumption $\boldsymbol{\lambda}' \in \mathcal{P}(W)$.

If $\lambda'_3 = 1$, $\boldsymbol{\lambda}' = [m, 2n-m, 1]$ for some $m \geq 2i-1$. If $m = 2i-1$, then $\boldsymbol{\lambda}' = \lambda^i$, which contradicts the assumption $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$. Hence $m > 2i-1$. If m is even, then $c_m(\boldsymbol{\lambda}')$ is even and $2n-m = m$; hence $m = n$, and $\boldsymbol{\lambda}' = [n^2, 1]$. On the other hand, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$, $2i-1 = 2n-2i+1 = n = m$, which contradicts $m > 2i-1$. In conclusion, $\boldsymbol{\lambda}' = [m, 2n-m, 1]$ for some odd $m > 2i-1$.

Similarly, if $2n-2i-1 \geq 2i-1$, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i = [2n-2i-1, 2i-1, 1]$, then $\boldsymbol{\lambda}' = [m, 2n-m, 1]$ for some odd $m > 2n-2i+1$. This concludes the proof for $\boldsymbol{G} = \mathbf{SO}(2n+1)$.

Assume $\boldsymbol{G} = \mathbf{SO}(2n)$, $\boldsymbol{\lambda}^i = [2i-1, 1] \cup [2n-2i-1, 1]$. First, assume $2i-1 > 2n-2i-1$, $\boldsymbol{\lambda}^i = [2i-1, 2n-2i-1, 1^2]$.

By definition, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$ if and only if $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$ and
$$\lambda'_1 \geq 2i-1, \quad \lambda'_1+\lambda'_2 \geq 2n-2, \quad \lambda'_1+\lambda'_2+\lambda'_3 \geq 2n-1, \quad \lambda'_1+\lambda'_2+\lambda'_3+\lambda'_4 = 2n.$$
Then $\lambda'_4 = 0$ or $\lambda'_4 = 1$. Assume $\lambda'_4 = 0$; then, $\lambda'_3 = 0$ or $\lambda'_3 = 1$. If $\lambda'_3 = 1$, $\lambda'_4 = 0$, then λ'_1 and λ'_2 have different parity, so $\boldsymbol{\lambda}' \notin \mathcal{P}(W)$. If $\lambda'_3 = \lambda'_4 = 0$, then $\boldsymbol{\lambda}' = [m, 2n-m]$ with $m \geq 2i-1$. If m is even, then $c_m(\boldsymbol{\lambda}')$ is even, $m = 2n-m = n$. Hence $m = n > 2i-1 > 2n-2i-1$, which has no solution since the second inequality requires $2i-1 > n-1$. In conclusion, if $\lambda'_4 = 0$, then $\boldsymbol{\lambda}' = [m, 2n-m]$ for some odd $m \geq 2i-1$.

If $\lambda'_4 = 1$, then $\lambda'_3 = 1$, $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some $m \geq 2i-1$. If $m = 2i-1$, then $\boldsymbol{\lambda}' = \boldsymbol{\lambda}^i$ which contradicts the assumption $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$. Hence $m > 2i-1$. If m is even, then $c_m(\boldsymbol{\lambda}')$ is even, $m = 2n-m-2 = n-1$. Hence $m = n-1 > 2i-1 > 2n-2i-1$, which has no solution since the second inequality requires $2i-1 > n-1$. In conclusion, if $\lambda'_4 = 1$, then $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some odd $m > 2i-1$.

Similarly, if $2n-2i-1 > 2i-1$, then $\boldsymbol{\lambda}' = [m, 2n-m]$ for some odd $m \geq \max(2i-1, 2n-2i-1)$, or $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some odd $m > \max(2i-1, 2n-2i-1)$.

Assume now $2i-1 = 2n-2i-1$. Then n is even, $i = n/2$, and $\boldsymbol{\lambda}^i = [(n-1)^2, 1^2]$. Assume $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$, $\lambda \in \mathcal{P}(W)$. Then

$$\lambda'_1 \geq n-1, \quad \lambda'_1 + \lambda'_2 \geq 2n-2, \quad \lambda'_1 + \lambda'_2 + \lambda'_3 \geq 2n-1, \quad \lambda'_1 + \lambda'_2 + \lambda'_3 + \lambda'_4 = 2n.$$

If $\lambda'_1 = n-1$, then $\lambda'_2 = n-1$, $\boldsymbol{\lambda}' = [(n-1)^2, 1^2] = \boldsymbol{\lambda}^i$, contradicting the assumption $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$. Hence $\lambda'_1 \geq n$. If λ'_1 is even, then $c_{\lambda'_1}$ is even, $\lambda'_1 = \lambda'_2 = n$, and $\boldsymbol{\lambda} = [n^2]$. If $m = \lambda'_1 > n$ is odd, then $m > \max(2i-1, 2n-2i-1) = n-1$ and $\boldsymbol{\lambda}' = [m, 2n-m]$ or $[m, 2n-m-2, 1^2]$. This concludes the proof for $\boldsymbol{G} = \mathbf{SO}(2n)$.

Assume $\boldsymbol{G} = \mathbf{Sp}(2n)$. Without loss of generality, assume $2i \geq 2n-2i$; i.e., $i \geq n/2$. Then $\boldsymbol{\lambda}^i = [2i, 2n-2i]$. By definition, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$ if and only if $\lambda' \neq \boldsymbol{\lambda}^i$ and

$$\lambda'_1 \geq 2i, \qquad \lambda'_1 + \lambda'_2 = 2n.$$

Hence $\boldsymbol{\lambda} = [\lambda'_1, 2n-\lambda'_1]$. If $\lambda'_1 = 2i$, then $\lambda'_2 = 2n-2i$, $\boldsymbol{\lambda}' = \boldsymbol{\lambda}^i$, which contradicts the assumption $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$. Hence $\lambda'_1 > 2i \geq n$. If λ'_1 is odd, then $c_{\lambda'_1}\boldsymbol{\lambda}'$ is even, $\lambda'_1 = \lambda'_2 = n$, which contradicts $\lambda'_1 > n$. As a result, $\boldsymbol{\lambda}' = [m, 2n-m]$ with $m = \lambda'_1 > 2i$ even. This concludes the proof for $\boldsymbol{G} = \mathbf{Sp}(2n)$. □

2F. ***Nilpotent support.*** Let O' be a rational nilpotent orbit in $\mathfrak{g}/G$ and fix an element $z \in O'$. Let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$; i.e., let there be a Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ such that

$$z = \phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right), \quad h = \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \quad z' = \phi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

For $i \in \mathbb{Z}$, let $\mathfrak{g}_i = \{Z \in \mathfrak{g} \mid \mathrm{Ad}(h)(Z) = iZ\}$. Then $z \in \mathfrak{g}_{-2}$ and $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$.

Define nilpotent subalgebras $\mathfrak{n}'_{\geq 1}, \mathfrak{n}'_{\geq 2}$ of $\mathfrak{g}$ and unipotent subgroups $N'_{\geq 1}, N'_{\geq 2}$ of G as follows:

$$\begin{aligned} \mathfrak{n}'_{\geq 1} &= \bigoplus_{i \geq 1} \mathfrak{g}_i, \quad N'_{\geq 1} = \exp(\mathfrak{n}'_{\geq 1}), \\ \mathfrak{n}'_{\geq 2} &= \bigoplus_{i \geq 2} \mathfrak{g}_i, \quad N'_{\geq 2} = \exp(\mathfrak{n}'_{\geq 2}). \end{aligned} \tag{2}$$

Let ψ_z be the character of $N'_{\geq 2}$ defined by

$$\psi_z(Z) = \psi \circ \mathrm{tr}(z \cdot \log Z) \quad (Z \in N'_{\geq 2}). \tag{3}$$

Then $\mathrm{Ker}(\psi_z)$ is a subgroup of $N'_{\geq 2}$. If $\mathfrak{n}'_{\geq 1} = \mathfrak{n}'_{\geq 2}$, so $N'_{\geq 1} = N'_{\geq 2}$, let S_z be the character ψ_z of $N'_{\geq 1}$. If $\mathfrak{n}'_{\geq 1} \neq \mathfrak{n}'_{\geq 2}$, then $\mathfrak{g}_1 \neq 0$ and $N'_{\geq 1}/\,\mathrm{Ker}(\psi_z)$ is isomorphic to a Heisenberg group over $\mathfrak{f}$ with center $N'_{\geq 2}/\,\mathrm{Ker}(\psi_z)$. In this case, let S_z be the irreducible representation of $N'_{\geq 1}$ whose restriction to $N'_{\geq 2}$ is a multiple of ψ_z.

Definition 2.9. Keep the notation above. Following [Mœglin and Waldspurger 1987], denote by $\mathcal{N}_{\mathrm{wh}}(\pi)$ the set of all nilpotent orbits O' in $\mathfrak{g}/G$ such that, for some smooth irreducible representation π of G, we have $\mathrm{Hom}_{N'_{\geq 1}}(\pi, S_z) \neq 0$. Let $\mathcal{N}_{\mathrm{wh,max}}(\pi)$ be the subset of maximal elements in $\mathcal{N}_{\mathrm{wh}}(\pi)$ with respect to the inclusion relation of closure of orbits.

3. Main theorems

The main results of this paper are the following theorems, whose proofs are given starting on page 185 and page 192, respectively.

Theorem 3.1. *Let $\pi \in \Pi'(\varphi)$. Assume $\pi = \pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}$ for some $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, $i = i_{\boldsymbol{\mu}'}$. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. Take arbitrary $z \in O'$. Then*

$$\mathrm{Hom}_{N'_{\geq 1}}(\pi, S_z) = 0.$$

Theorem 3.2. *Let $\pi \in \Pi'(\varphi)$. Assume $\pi = \pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}$ for some $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu}), i = i_{\boldsymbol{\mu}'}$. Then there is a nilpotent orbit O^i corresponding to $(\boldsymbol{\lambda}^i, (q_j))$ such that $O^i \in \mathcal{N}_{\mathrm{wh,max}}(\pi)$.*

If $i \notin I_{\mathrm{nsp}}$, then y_i is special. In this case, Theorem 3.1 is void and Theorem 3.2 is proved in [DeBacker and Reeder 2010].

The subset $\boldsymbol{\Gamma_z}$ of $\boldsymbol{\Phi^+}$. Assume now $i \in I_{\mathrm{nsp}}$; that is, $\mathrm{rank}(G)$ is large enough for I_{nsp} to be nonempty. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. In this subsection, we will choose a particular element $z \in O'$ such that

$$N'_{\geq 2} \subset B, \quad N'_{\geq 4} \subset B. \tag{4}$$

Here B is the Borel subgroup consisting of upper triangular matrices in G and $N'_{\geq j}$ is the object defined in Section 2F for any $\mathfrak{sl}_2$ triple $\{z, h, z'\}$ attached to z in $\mathfrak{g}$. Let $\Gamma'_z \subset \Phi^+$ be the subset of positive roots such that $\alpha \in \Gamma'_z$ if and only if the root space $\mathfrak{u}_\alpha \subset \mathfrak{n}'_{\geq 4}$, and let

$$\Gamma_z := \Phi^+ \backslash \Gamma'_z. \tag{5}$$

The following notation is used frequently: let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s)$ be a sequence of positive integers such that $d = \sum_{j=1}^{s} \nu_j$. Then every matrix $a \in \mathfrak{gl}(d, k)$ can be written in blocks $a = (a_{j,\ell})_{j,\ell \le s}$, with $a_{jj} \in \mathfrak{gl}(\nu_j, k)$. Let A_j be an arbitrary $\nu_{j+1} \times \nu_j$ matrix for $1 \le j \le s-1$, and let $z(\boldsymbol{\nu}; A_1, \ldots, A_{s-1}) = (z_{j,\ell})_{j,\ell \le s}$ be the nilpotent element in $\mathfrak{gl}(d, k)$ such that

$$z_{j,\ell} = \begin{cases} A_\ell & j = \ell + 1, \\ 0_{\nu_j \times \nu_\ell} & j \ne \ell + 1. \end{cases}$$

Assume $\boldsymbol{G} = \mathbf{SO}(2n+1)$. By Lemma 2.8, $\boldsymbol{\lambda}' = [2n+1]$ or $[m, 2n-m, 1]$ with m odd and $m > \max(2i-1, 2n-2i+1)$.

First, assume $\boldsymbol{\lambda}' = [2n+1]$, $q'_{2n+1} = q_{2n+1}$ as in Example 2.7. Let

$$z = z(\boldsymbol{\nu}; 1, \ldots, 1, -1, \ldots, -1), \tag{6}$$

with $\boldsymbol{\nu} = (1^{2n+1})$ a regular nilpotent element in $\mathfrak{g}$. Let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and $\mathfrak{g}_j, \mathfrak{n}'_{\ge j}, N'_{\ge j}$ the objects defined in Section 2F. Then, we naturally have

$$N'_{\ge 2} = \{n = (n_{j,\ell})_{j,\ell \le 2n+1} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \ge \ell\} \subset B,$$
$$N'_{\ge 4} = \{n = (n_{j,\ell})_{j,\ell \le 2n+1} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \ge \ell - 1\} \subset B.$$

Let Γ_z be the subset of Φ^+ defined in (5); then,

$$\Gamma_z = \{\alpha_j \mid j = 1, \ldots, n\}. \tag{7}$$

Second, assume $m = 2n-1$. Then $\boldsymbol{\lambda}' = [2n-1, 1^2]$, q'_{2n-1} is a nondegenerate quadratic form on k, identified with a nonzero element in $k^\times$, and q'_1 is a nondegenerate quadratic form on k^2, such that $q'_{2n-1} \oplus q'_1$ is isometric to the quadratic form on k^3

$$(u, v, w) \mapsto 2uw + v^2 \quad (u, v, w \in k).$$

Let

$$z = z(\boldsymbol{\nu}; 1, 1, \ldots, 1, A^*, A, -1, \ldots, -1), \tag{8}$$

with $\boldsymbol{\nu} = (1^{n-1}, 3, 1^{n-1})$,

$$A^* = (a_m, b_m, c_m)^t, \quad A = -(c_m, b_m, a_m),$$

such that $AA^* = -q'_{2n-1}$. Then $z \in O'$, as shown in the Appendix.

Let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and $\mathfrak{g}_j, \mathfrak{n}'_{\ge j}, N'_{\ge j}$ the objects defined in Section 2F. Let $s = s(\boldsymbol{\nu}) = 2n-1 = m$. It is shown in the Appendix that

$$N'_{\ge 2} = \{n = (n_{j,\ell})_{j,\ell \le s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \ge \ell\},$$
$$N'_{\ge 4} = \{n = (n_{j,\ell})_{j,\ell \le s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \ge \ell - 1\};$$

that is, (4) is satisfied. Let Γ_z be the subset of Φ^+ defined in (5); then,

$$\Gamma_z = \{\alpha_j \mid j = 1, \dots, n-2\} \cup \{e_{n-1} \pm e_n\} \cup \{e_{n-1}, e_n\}. \tag{9}$$

Here the α_j $(j = 0, 1, \dots n)$ are simple roots defined in Section 2B.

Third, assume $m < 2n-1$. Then $\boldsymbol{\lambda}' = [m, 2n-m, 1]$, and q'_m, q'_{2n-m}, q'_1 are nondegenerate quadratic forms on k such that $q'_m \oplus q'_{2n-m} \oplus q'_1$ is isometric to quadratic form $(u, v, w) \mapsto 2uw + v^2$ $(u, v, w \in k)$. Let

$$z = z(\boldsymbol{\nu}; 1 \dots, 1, a^*, 1_2, \dots, 1_2, A^*, A, -1_2, \dots, -1_2, a, -1, \dots, -1), \tag{10}$$

with $\boldsymbol{\nu} = (1^{m-n}, 2^{n-(m+1)/2}, 3, 2^{n-(m+1)/2}, 1^{m-n})$, $a^* = (1, 0)^t$, $a = -(0, 1)$,

$$A^* = \begin{pmatrix} a_m & a_{2n-m} \\ b_m & b_{2n-m} \\ c_m & c_{2n-m} \end{pmatrix}, \quad A = -\begin{pmatrix} c_{2n-m} & b_{2n-m} & a_{2n-m} \\ c_m & b_m & a_m \end{pmatrix},$$

such that

$$AA^* = -\begin{pmatrix} 0 & q'_{2n-m} \\ q'_m & 0 \end{pmatrix}.$$

Working as in the Appendix, given $z \in O'$, let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and let $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ be the objects defined in Section 2F. Let $s = s(\nu) = m$; then, (4) is satisfied:

$$N'_{\geq 2} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B,$$
$$N'_{\geq 4} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5); then,

$$\begin{aligned} \Gamma_z = &\{\alpha_j \mid j = 1, \dots, m-n\} \cup \{e_{m-n} - e_{m-n+2}\} \\ &\cup \{\alpha_{m-n+2j-1} \mid j = 1, \dots, n-(m+1)/2\} \\ &\cup \bigcup_{j=1}^{n-\frac{m+3}{2}} \{e_{m-n+2j-1} - e_{m-n+2j+1}, e_{m-n+2j-1} - e_{m-n+2j+2}\} \\ &\cup \bigcup_{j=1}^{n-\frac{m+3}{2}} \{e_{m-n+2j} - e_{m-n+2j+1}, e_{m-n+2j} - e_{m-n+2j+2}\} \\ &\cup \{e_{n-2} \pm e_n\} \cup \{e_{n-1} \pm e_n\} \cup \{e_{n-2}, e_{n-1}, e_n\}. \end{aligned} \tag{11}$$

Assume $\boldsymbol{G} = \mathbf{SO}(2n)$. By Lemma 2.8, $\boldsymbol{\lambda}'$ is one of $[n^2]$, $[m, 2n-m]$, or $[m, 2n-m-2, 1^2]$ for some odd $m \geq \max(2i-1, 2n-2i-1)$.

First, assume $m = 2n-3$ and $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2] = [2n-3, 1^3]$. Then q'_{2n-3} and q'_1 are nondegenerate quadratic forms on k and k^3, respectively, such that $q'_{2n-3} \oplus q'_1$ is isometric to the quadratic form on k^4 defined by $(u, v, w, x) = 2ux + 2vw$ $(u, v, w, x \in k)$. Let $\boldsymbol{\nu} = (1^{n-2}, 4, 1^{n-2})$, $s = s(\boldsymbol{\nu}) = 2n-3 = m$, and $z = z(\boldsymbol{\nu}; 1, \dots, 1, A^*, A, -1, \dots, -1)$, with

$$A^* = (a_{2n-3}, b_{2n-3}, c_{2n-3}, d_{2n-3})^t, \quad A = -(d_{2n-3}, c_{2n-3}, b_{2n-3}, a_{2n-3})$$

satisfying $AA^* = -q'_{2n-3}$. Similar to that in the Appendix, $z \in O'$. Let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ the objects defined in Section 2F. Then

$$N'_{\geq 2} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B,$$
$$N'_{\geq 4} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5); then,

(12) $\Gamma_z = \{\alpha_j \mid j = 1, \dots, n-3\} \cup \{e_{n-2} \pm e_{n-1}\} \cup \{e_{n-2} \pm e_n\} \cup \{e_{n-1} \pm e_n\}$.

Second, assume $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some odd $m < 2n-3$, $m > \max(2i-1, 2n-2i-1)$. Since $m > 2n-m-2 > 1$, q'_m, q'_{2n-m-2} are quadratic forms on k and q'_1 is a quadratic form on k^2 such that $q'_m \oplus q'_{2n-m-2} \oplus q'_1$ is isometric to the quadratic form on k^4 defined by

$$(u, v, w, x) = 2ux + 2vw \quad (u, v, w, x \in k).$$

Let $\boldsymbol{\nu} = (1^{m-n+1}, 2^{n-\frac{m+3}{2}}, 4, 2^{n-\frac{m+3}{2}}, 1^{m-n+1})$, $s = s(\nu) = m$, and

$$z = z(\boldsymbol{\nu}; 1, \dots, 1, a^*, 1_2, \dots, 1_2, A^*, A, -1_2, \dots, -1_2, a, -1, \dots, -1),$$

with $a^* = (1, 0)^t, a = -(0, 1)$,

$$A^* = \begin{pmatrix} a_m & a_{2n-m-2} \\ b_m & b_{2n-m-2} \\ c_m & c_{2n-m-2} \\ d_m & c_{2n-m-2} \end{pmatrix}, A = -\begin{pmatrix} d_{2n-m-2} & c_{2n-m-2} & b_{2n-m-2} & a_{2n-m-2} \\ d_m & c_m & b_m & a_m \end{pmatrix},$$

such that

$$AA^* = -\begin{pmatrix} 0 & q'_{2n-m-2} \\ q'_m & 0 \end{pmatrix}.$$

Working as in the Appendix, given $z \in O'$, let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and let $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ be the objects defined in Section 2F. Then

$$N'_{\geq 2} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B,$$
$$N'_{\geq 4} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5); then,

$$
\begin{aligned}
(13)\quad \Gamma_z =& \{\alpha_j \mid j=1,\dots,m-n+1\} \cup \{e_{m-n+1}-e_{m-n+3}\} \\
&\cup \{\alpha_{m-n+1+2j-1} \mid j=1,\dots,n-(m+3)/2\} \\
&\cup \bigcup_{j=1}^{n-\frac{m+5}{2}} \{e_{m-n+1+2j-1}-e_{m-n+1+2j+1}, e_{m-n+1+2j-1}-e_{m-n+1+2j+2}\} \\
&\cup \bigcup_{j=1}^{n-\frac{m+5}{2}} \{e_{m-n+1+2j}-e_{m-n+1+2j+1}, e_{m-n+1+2j}-e_{m-n+1+2j+2}\} \\
&\cup \{e_{n-3}\pm e_{n-1}, e_{n-3}\pm e_n\} \cup \{e_{n-2}\pm e_{n-1}, e_{n-2}\pm e_n\} \\
&\cup \{e_{n-1}\pm e_n\}.
\end{aligned}
$$

Third, assume $\boldsymbol{\lambda}' = [m, 2n-m]$ for some odd $m \geq n$. If $m > n$, then q'_m, q'_{2n-m} are quadratic forms on k such that $q'_m \oplus q'_{2n-m}$ is isometric to the quadratic form on k^2 defined by $(u, w) \mapsto 2uw$. If $m = n$ is odd, then $\boldsymbol{\lambda}' = [n^2]$, and q'_n is the quadratic form on k^2 isometric to the quadratic form on k^2 defined by $(u, w) \mapsto 2uw$.

Let $\boldsymbol{\nu} = (1^{m-n}, 2^{2n-m}, 1^{m-n})$, $s = s(\boldsymbol{\nu}) = m$, and

$$
z = \begin{cases} z(\boldsymbol{\nu}; 1_2, \dots, 1_2, A^*, A, -1_2, \dots, -1_2), & m = n, \\ z(\boldsymbol{\nu}; 1, \dots, 1, a^*, 1_2, \dots, 1_2, A^*, A, -1_2, \dots, -1_2, a, -1, \dots, -1), & m > n, \end{cases}
$$

with $a^* = (1, 0)^t, a = -(0, 1)$,

$$
A^* = \begin{pmatrix} a_m & a_{2n-m} \\ b_m & b_{2n-m} \end{pmatrix}, \quad A = -\begin{pmatrix} b_{2n-m} & a_{2n-m} \\ b_m & a_m \end{pmatrix},
$$

satisfying

$$
AA^* = -\begin{cases} \begin{pmatrix} 0 & q'_{2n-m} \\ q'_m & 0 \end{pmatrix} & \text{if } m > n, \\ -\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \text{if } m = n. \end{cases}
$$

Working as in the Appendix, given $z \in O'$, let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and let $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ be the objects defined in Section 2F. Then

$$
\begin{aligned}
N'_{\geq 2} &= \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B, \\
N'_{\geq 4} &= \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.
\end{aligned}
$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5); then,

$$\begin{aligned}(14)\qquad \Gamma_z =&\{\alpha_j \mid j=1,\dots,m-n\}\cup\{e_{m-n}-e_{m-n+2}\}\\ &\cup\{\alpha_{m-n+2j-1}\mid j=1,\dots,n-(m+1)/2\}\\ &\cup\bigcup_{j=1}^{n-\frac{m+3}{2}}\{e_{m-n+2j-1}-e_{m-n+2j+1},e_{m-n+2j-1}-e_{m-n+2j+2}\}\\ &\cup\bigcup_{j=1}^{n-\frac{m+3}{2}}\{e_{m-n+2j}-e_{m-n+2j+1},e_{m-n+1+2j}-e_{m-n+2j+2}\}\\ &\cup\{e_{n-1}\pm e_n\}\cup\{e_{n-2}\pm e_n\}.\end{aligned}$$

Fourth, assume n is even and $\boldsymbol{\lambda}'=[n^2]$. Let $\boldsymbol{\nu}=(2^n)$,

$$z=z(\nu;1_2,\dots,1_2,A,-1_2,\dots,-1_2), \tag{15}$$

with $A=\operatorname{diag}(1,-1)$. Working as in the Appendix, take $z_\varepsilon\in O'_\varepsilon$, where O'_ε is the nilpotent orbit corresponding to $(\boldsymbol{\lambda}',\varnothing,\varepsilon)$ for some $\varepsilon=1$ or -1. Let $\{z_\varepsilon,h_\varepsilon,z_\varepsilon\}$ be an $\mathfrak{sl}_2$ triple attached to z_ε in $\mathfrak{g}$, and $\mathfrak{g}_j$, $\mathfrak{n}'_{\geq j}$, $N'_{\geq j}$ the objects defined in Section 2F. Then

$$N'_{\geq 2}=\{u=(u_{j,\ell})_{j,\ell\leq n}\in\mathfrak{g}\mid u_{j,\ell}=0_{\nu_j\times\nu_\ell}\text{ if }j\geq\ell\}\subset B,$$
$$N'_{\geq 4}=\{u=(u_{j,\ell})_{j,\ell\leq n}\in\mathfrak{g}\mid u_{j,\ell}=0_{\nu_j\times\nu_\ell}\text{ if }j\geq\ell-1\}\subset B.$$

Let $\Gamma_{z_\varepsilon}\subset\Phi^+$ be the subset of positive roots defined in (5) for z_ε; then,

$$\begin{aligned}(16)\qquad \Gamma_{z_\epsilon}=&\{\alpha_{2j-1}\mid j=1,\dots,n/2-1\}\cup\{e_{n-1}\pm e_n\}\\ &\cup\bigcup_{j=1}^{\frac{n}{2}-1}\{e_{2j-1}-e_{2j+1},e_{2j-1}-e_{2j+2},e_{2j}-e_{2j+1},e_{2j}-e_{2j+2}\}.\end{aligned}$$

Let $w_0=(a_{\ell,\ell'})_{2n\times 2n}$ be the element in $\mathbf{O}(2n)$ satisfying

$$\begin{cases}a_{n,n+1}=a_{n+1,n}=a_{j,j}=1 & \text{if } 1\leq j\leq 2n,\ j\neq n,\ j\neq n+1,\\ a_{\ell,\ell'}=0 & \text{otherwise.}\end{cases}$$

Let $z_{-\varepsilon}=w_0z_\varepsilon w_0^{-1}$; then $z_{-\varepsilon}\in O'_{-\epsilon}$, where $O'_{-\epsilon}$ is the nilpotent orbit corresponding to $(\boldsymbol{\lambda}',\phi,-\varepsilon)$. Let $\{z_{-\varepsilon},h_{-\varepsilon},z_{-\varepsilon}\}$ be an $\mathfrak{sl}_2$ triple attached to $z_{-\varepsilon}$ in $\mathfrak{g}$ and $\mathfrak{g}''_j$, $\mathfrak{n}''_{\geq j}$, $N''_{\geq j}$ the objects defined in Section 2F. Then

$$N''_{\geq 2}=w_0N'_{\geq 2}w_0^{-1}\subset B,\quad N''_{\geq 4}=w_0N'_{\geq 4}w_0^{-1}\subset B.$$

Let $\Gamma_{z_{-\varepsilon}}\subset\Phi^+$ be the subset of positive roots defined in (5) for $z_{-\varepsilon}$, then

$$\Gamma_{z_{-\varepsilon}}=\{e_{n-3}+e_n,e_{n-2}+e_n\}\cup\Gamma_{z_\varepsilon}\setminus\{e_{n-3}-e_n,e_{n-2}-e_n\}. \tag{17}$$

Assume $\boldsymbol{G} = \mathbf{Sp}(2n)$. By Lemma 2.8, $\boldsymbol{\lambda}' = [m, 2n-m]$ for some even $m > \max(2i, 2n-2i)$. Then $m > 2n-m$, and q'_m, q'_{2n-m} are nondegenerate quadratic forms on k. Let $\boldsymbol{\nu} = (1^{m-n}, 2^{2n-m}, 1^{m-n})$, $s = s(\boldsymbol{\nu}) = m$, and

$$z = z(\boldsymbol{\nu}; 1, \dots, 1, a^*, 1_2, \dots, 1_2, A, -1_2, \dots, -1_2, a, -1, \dots, -1),$$

with $a^* = (1,0)^t, a = -(0,1)$, $A = \left(\begin{smallmatrix} b & a \\ a & c \end{smallmatrix}\right)$, such that $q'_m \oplus q'_{2n-m}$ is isometric to the quadratic form given by the symmetric matrix A.

Working as in the Appendix, given $z \in \mathcal{O}'$, let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and let $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ be the objects defined in Section 2F. Then

$$N'_{\geq 2} = \{u = (u_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid u_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B,$$
$$N'_{\geq 4} = \{u = (u_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid u_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5) for z; then,

$$\begin{aligned}
(18) \qquad \Gamma_z =& \{\alpha_j \mid j = 1, \dots, m-n\} \cup \{e_{m-n} - e_{m-n+2}\} \\
&\cup \{\alpha_{m-n+2j-1} \mid j = 1, \dots, n-(m)/2\} \\
&\cup \bigcup_{j=1}^{n-\frac{m}{2}-1} \{e_{m-n+2j-1} - e_{m-n+2j+1}, e_{m-n+2j-1} - e_{m-n+2j+2}\} \\
&\cup \bigcup_{j=1}^{n-\frac{m}{2}-1} \{e_{m-n+2j} - e_{m-n+2j+1}, e_{m-n+1+2j} - e_{m-n+2j+2}\} \\
&\cup \{e_{n-1} + e_n, 2e_{n-1}, 2e_n\}.
\end{aligned}$$

Proof of Theorem 3.1. We keep the notation used so far in this section and in Section 2B. For $i \in I_{\text{nsp}}$, let

$$\Sigma_i = \{\alpha_j \mid j = 1, \dots, n, j \neq i\} \cup \{-\gamma\},$$

which is a set of simple roots of a root subsystem of Φ. Let $\mathcal{O}', \mathcal{O}^i$ be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $\mathcal{O}' > \mathcal{O}^i$. Let $z \in \mathcal{O}', \Gamma_z \subset \Phi^+$ be as defined (6), (8), (10), (15), and set $\Gamma'_z = \Phi^+ \backslash \Gamma_z$.

Lemma 3.3. *Let w be a Weyl element of G such that $w^{-1}(\Sigma_i) \subset \Phi^+$. Then $w^{-1}(\Sigma_i) \cap \Gamma'_z \neq \varnothing$.*

Proof. First assume $\boldsymbol{G} = \mathbf{SO}(2n+1)$. Then $-\gamma = -e_1 - e_2, \alpha_j = e_j - e_{j+1}$ for $j = 1, \dots, n-1$, and $\alpha_n = e_n$. Let w be a Weyl element of G such that $w^{-1}(\Sigma_i) \subset \Phi^+$; then, there is a permutation σ of $\{1, 2, \dots, n\}$ satisfying $\sigma(1) > \sigma(2) > \cdots > \sigma(i)$,

$\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$, such that

$$w^{-1}(e_j) = \begin{cases} \pm e_{\sigma(1)} & \text{if } j = 1, \\ -e_{\sigma(j)} & \text{if } 2 \le j \le i, \\ e_{\sigma(j)} & \text{if } i+1 \le j \le n. \end{cases} \tag{19}$$

Assume on the contrary that $w^{-1}(\Sigma_i) \cap \Gamma_z' = \varnothing$; then

$$w^{-1}(\Sigma_i) \subset \Gamma_z. \tag{20}$$

If $i = n$, then $\boldsymbol{\lambda}^i = [2n-1, 1^2]$, $\Sigma_n = \{\alpha_j \mid 1 \le j < n\} \cup \{-\gamma\}$. Then by Lemma 2.8, $\boldsymbol{\lambda}' = [2n+1]$ and $q_{2n+1}' = q_{2n+1}$, and by (7), $\Gamma_z = \{\alpha_j \mid j = 1, \dots, n\}$. If w satisfies (19) and (20), then $\sigma(j) = n+1-j$,

$$w^{-1}(e_1) = \pm e_n, \qquad w^{-1}(e_j) = -e_{n+1-j} \quad (1 < j \le n).$$

As a result, $w^{-1}(\Sigma_n) = \{\alpha_j \mid 1 \le j < n\} \cup \{e_{n-1} + e_n\} \not\subset \Gamma_z$, which contradicts (20). Hence $w^{-1}(\Sigma_n) \cap \Gamma_z' \neq \varnothing$.

If $i < n$, by Lemma 2.8, $\boldsymbol{\lambda}' = [2n+1]$ or $[m, 2n-m, 1]$ for some odd $m > \max(2i-1, 2n-2i+1)$. Let w be a Weyl element satisfying (19) and (20). Since $\pm e_1 - e_2, e_n \in \Sigma_i$, we have

$$w^{-1}(\pm e_1 - e_2) = e_{\sigma(2)} \pm e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(e_n) = e_{\sigma(n)} \in \Gamma_z. \tag{21}$$

If $\boldsymbol{\lambda}' = [2n+1]$, then $\Gamma_z = \{\alpha_j \mid 1 \le j \le n\}$ and $e_{\sigma(2)} + e_{\sigma(1)} \notin \Gamma_z$, which contradicts (21).

If $\boldsymbol{\lambda}' = [m, 2n-m, 1]$, $m = 2n-1$, then Γ_z is the set in (9). By (21), $\sigma(2) = n-1$, $\sigma(1) = n$, while $\sigma(n) = n$ or $n-1$, which is impossible since σ is a permutation.

If $\boldsymbol{\lambda}' = [m, 2n-m, 1]$, $m < 2n-1$, then Γ_z is the set in (11). By (21), $\sigma(1) = n$, $\{\sigma(2), \sigma(n)\} = \{n-2, n-1\}$. If $e_2 - e_3, e_{n-1} - e_n \in \Sigma_i$, then by (20),

$$w^{-1}(e_2 - e_3) = e_{\sigma(3)} - e_{\sigma(2)} \in \Gamma_z, \quad w^{-1}(e_{n-1} - e_n) = e_{\sigma(n-1)} - e_{\sigma(n)} \in \Gamma_z.$$

Then $\{\sigma(3), \sigma(n-1)\} = \{n-4, n-3\}$. Since $m > \max(2i-1, 2n-2i+1)$, we have

$$n - \frac{m+1}{2} < \min(n-i, i-1),$$

so the procedure can be repeated $n - \frac{m+1}{2}$ times. Then, for $\ell = 2, \dots, n - \frac{m-1}{2}$,

$$\{\sigma(\ell), \sigma(n+2-\ell)\} = \{n - 2(\ell-1), n - 2(\ell-1) + 1\}.$$

In particular, for $\ell_0 = n - \frac{m-1}{2}$ and $n + 2 - \ell_0 = \frac{m+3}{2}$,

$$\{\sigma(\ell_0), \sigma(n+2-\ell_0)\} = \Big\{\sigma(\ell_0), \sigma\Big(\frac{m+3}{2}\Big)\Big\} = \{m-n+1, m-n+2\}.$$

Since $m > 2i - 1$, we have $m > 2n - 2i + 1$,

$$\ell_0 = n - \frac{m-1}{2} < i, \quad i+1 < \frac{m+3}{2} = n+2-\ell_0, \quad e_{\ell_0} - e_{\ell_0+1}, e_{\frac{m+1}{2}} - e_{\frac{m+3}{2}} \in \Sigma_i.$$

By (20),

$$w^{-1}(e_{\ell_0} - e_{\ell_0+1}) = e_{\sigma(\ell_0+1)} - e_{\sigma(\ell_0)} \in \Gamma_z,$$
$$w^{-1}(e_{\frac{m+1}{2}} - e_{\frac{m+3}{2}}) = e_{\sigma(\frac{m+1}{2})} - e_{\sigma(\frac{m+3}{2})} \in \Gamma_z.$$

Then $\sigma(\ell_0 + 1) = \sigma(\frac{1}{2}(m+1)) = m - n$, which contradicts the assumption that σ is a permutation, for $\ell_0 + 1 \le i$, $(m+1)/2 \ge i+1$, $\ell_0 + 1 \ne (m+1)/2$. Hence $w^{-1}(\Sigma_i) \cap \Gamma'_z \ne \varnothing$, concluding the proof for $\boldsymbol{G} = \mathbf{SO}(2n+1)$.

Assume now $\boldsymbol{G} = \mathbf{SO}(2n)$; then we have $-\gamma = -e_1 - e_2$, $\alpha_j = e_j - e_{j+1}$ for $j = 1, \dots, n-1$, and $\alpha_n = e_{n-1} + e_n$. Let w be a Weyl element of G such that $w^{-1}(\Sigma_i) \subset \Phi^+$; then, there is a permutation σ of $\{1, 2, \dots, n\}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ satisfying $\sigma(1) > \sigma(2) > \dots > \sigma(i)$, $\sigma(i+1) < \sigma(i+2) < \dots < \sigma(n)$, $(-1)^{i-1}\varepsilon_1\varepsilon_2 = 1$, such that

$$w^{-1}(e_j) = \begin{cases} \varepsilon_1 e_{\sigma(1)} & \text{if } j = 1, \\ -e_{\sigma(j)} & \text{if } 2 \le j \le i, \\ e_{\sigma(j)} & \text{if } i+1 \le j \le n-1, \\ \varepsilon_2 e_{\sigma(n)} & \text{if } j = n. \end{cases} \tag{22}$$

Assume on the contrary that $w^{-1}(\Sigma_i) \cap \Gamma'_z = \varnothing$; then

$$w^{-1}(\Sigma_i) \subset \Gamma_z. \tag{23}$$

By Lemma 2.8, $\boldsymbol{\lambda}'$ is of the form $[m, 2n-m-2, 1^2]$ or $[m, 2n-m]$.

Assume first $m = 2n-3 > \max(2i-1, 2n-2i-1)$, $\boldsymbol{\lambda}' = [2n-3, 1^3]$; then Γ_z is the set in (12). Since $i \in I_{\mathrm{nsp}}$, I_{nsp} is nonempty and $n \ge 4$. Hence $1, 2, n-1, n$ are four distinct numbers. On the other hand, $\pm e_1 - e_2, e_{n-1} \pm e_n \in \Sigma_i$, so by (23),

$$w^{-1}(\pm e_1 - e_2) = e_{\sigma(2)} \pm \epsilon_1 e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(e_{n-1} \pm e_n) = e_{\sigma(n-1)} \pm \epsilon_2 e_{\sigma(n)} \in \Gamma_z.$$

Hence the cardinality of $\{\sigma(1), \sigma(2), \sigma(n-1), \sigma(n)\}$ is 3, which contradicts the assumption that σ is a permutation.

Second, assume $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some odd m with $m < 2n-3$, $m > \max(2i-1, 2n-2i-1)$. Then Γ_z is the set in (13). Since $\pm e_1 - e_2, e_{n-1} \pm e_n \in \Sigma_i$, we have, by (23),

$$w^{-1}(\pm e_1 - e_2) = e_{\sigma(2)} \pm \epsilon_1 e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(e_{n-1} \pm e_n) = e_{\sigma(n-1)} \pm \epsilon_2 e_{\sigma(n)} \in \Gamma_z.$$

Then $\{\sigma(1), \sigma(n)\} = \{n-1, n\}$ and $\{\sigma(2), \sigma(n-1) = \{n-2, n-3\}$. If $e_2 - e_3$, $e_{n-2} - e_{n-1} \in \Sigma_i$, then by (23),

$$w^{-1}(e_2 - e_3) = e_{\sigma(3)} - e_{\sigma(2)} \in \Gamma_z, \quad w^{-1}(e_{n-2} - e_{n-1}) = e_{\sigma(n-2)} - e_{\sigma(n-1)} \in \Gamma_z.$$

Then $\{\sigma(3),\sigma(n-2)\}=\{n-5,n-4\}$. Since $m>2i-1$, $m>2n-2i-1$,

$$n-\frac{m+3}{2}<\min(i-1,n-i-1),$$

the procedure can be repeated $n-\frac{m+3}{2}$ times. Then for $\ell=1,2,\dots,n-\frac{m+1}{2}$,

$$\{\sigma(\ell),\sigma(n+1-\ell)\}=\{n-2(\ell-1),n-2(\ell-1)-1\}.$$

In particular, for $\ell_0=n-\frac{m+1}{2}$, we have $n+1-\ell_0=\frac{m+3}{2}$,

$$\{\sigma(\ell_0),\sigma(n+1-\ell_0)\}=\left\{\sigma(\ell_0),\frac{m+3}{2}\right\}=\{m-n+3,m-n+2\}.$$

Since $m>2i-1$, we have $m>2n-2i-1$,

$$\ell_0=n-\frac{m+1}{2}<i,\quad i+1<\frac{m+3}{2}=n+1-\ell_0,\quad e_{\ell_0}-e_{\ell_0+1},e_{\frac{m+1}{2}}-e_{\frac{m+3}{2}}\in\Sigma_i.$$

By (23),

$$w^{-1}(e_{\ell_0}-e_{\ell_0+1})=e_{\sigma(\ell_0+1)}-e_{\sigma(\ell_0)}\in\Gamma_z,$$
$$w^{-1}(e_{\frac{m+1}{2}}-e_{\frac{m+3}{2}})=e_{\sigma(\frac{m+1}{2})}-e_{\sigma(\frac{m+3}{2})}\in\Gamma_z.$$

Then $\sigma(\ell_0+1)=\sigma(\frac{1}{2}(m+1))=m-n+1$, which contradicts the assumption that σ is a permutation, for $\ell_0+1\le i$, $\frac{1}{2}(m+1)\ge i+1$, $\ell_0+1\ne\frac{1}{2}(m+1)$.

Third, assume $\boldsymbol{\lambda}'=[m,2n-m]$ for some odd $m\ge\max(2i-1,2n-2i+1)$. Then Γ_z is the set in (14). Since $\pm e_1-e_2, e_{n-1}\pm e_n\in\Sigma_i$, we have, by (23),

$$w^{-1}(\pm e_1-e_2)=e_{\sigma(2)}\pm\epsilon_1 e_{\sigma(1)}\in\Gamma_z,\quad w^{-1}(e_{n-1}\pm e_n)=e_{\sigma(n-1)}\pm\epsilon_2 e_{\sigma(n)}\in\Gamma_z.$$

Then $\sigma(1)=\sigma(n)=n$, which contradicts the assumption that σ is a permutation.

Fourth, assume n is even and $\boldsymbol{\lambda}'=[n^2]$. Then Γ_z is either the set in (16) or the set in (17). Since $\pm e_1-e_2, e_{n-1}\pm e_n$ belong to Σ_i, by (23),

$$w^{-1}(\pm e_1-e_2)=e_{\sigma(2)}\pm\epsilon_1 e_{\sigma(1)}\in\Gamma_z,\quad w^{-1}(e_{n-1}\pm e_n)=e_{\sigma(n-1)}\pm\epsilon_2 e_{\sigma(n)}\in\Gamma_z.$$

Then $\sigma(1)=\sigma(n)=n$, which contradicts the assumption that σ is a permutation. Hence $w^{-1}(\Sigma_i)\cap\Gamma'_z\ne\varnothing$. This concludes the proof for $\boldsymbol{G}=\mathbf{SO}(2n)$.

Assume now $\boldsymbol{G}=\mathbf{Sp}(2n)$; then we have $-\gamma=-2e_1$, $\alpha_j=e_j-e_{j+1}$ for $j=1,\dots,n-1$, and $\alpha_n=2e_n$. Since $w^{-1}(\Sigma_i)\subset\Phi^+$, there is a permutation σ of $\{1,2,\dots,n\}$, satisfying $\sigma(1)>\sigma(2)>\cdots>\sigma(i)$, $\sigma(i+1)<\sigma(i+2)<\cdots<\sigma(n)$, such that

$$w^{-1}(e_j)=\begin{cases}-e_{\sigma(j)} & \text{if } 1\le j\le i,\\ e_{\sigma(j)} & \text{if } i+1\le j\le n.\end{cases}\tag{24}$$

By Lemma 2.8, $\boldsymbol{\lambda}'=[m,2n-m]$ for some even $m>\max(2i,2n-2i)$. Then Γ_z

is the set in (18). Assume on the contrary that $w^{-1}(\Sigma_i) \cap \Gamma'_z = \varnothing$; then

$$w^{-1}(\Sigma_i) \subset \Gamma_z.$$

Since $-2e_1, 2e_n \in \Sigma_i$, we have

$$w^{-1}(-2e_1) = 2e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(2e_n) = 2e_{\sigma(n)} \in \Gamma_z.$$

Then $\{\sigma(1), \sigma(n)\} = \{n-1, n\}$. If $e_1 - e_2, e_{n-1} - e_n \in \Sigma_i$,

$$w^{-1}(e_1 - e_2) = e_{\sigma(2)} - e_{\sigma(1)} \in \Gamma_{O'}, \quad w^{-1}(e_{n-1} - e_n) = e_{\sigma(n-1)} - e_{\sigma(n)} \in \Gamma_{O'}.$$

Then $\{\sigma(2), \sigma(n-1)\} = \{n-3, n-2\}$. Since $m > 2i$ and $m > 2n - 2i$, we have

$$n - \frac{m}{2} < \max(i, n-i),$$

the above procedure can be repeated $n - \frac{m}{2}$ times. Then for $\ell = 1, 2, \dots, n - \frac{m}{2}$,

$$\{\sigma(\ell), \sigma(n+1-\ell)\} = \{n - 2(\ell-1), n - 2(\ell-1) - 1\}.$$

In particular, for $\ell_0 = n - \frac{m}{2}$ and $n + 1 - \ell_0 = \frac{m}{2} + 1$, we have

$$\{\sigma(\ell_0), \sigma(n+1-\ell_0)\} = \left\{\sigma(\ell_0), \sigma\left(\frac{m}{2} + 1\right)\right\} = \{m - n + 1, m - n + 2\}.$$

Since $m > 2i$, $m > 2n - 2i$,

$$\ell_0 = n - \frac{m}{2} < i, \quad i + 1 < \frac{m}{2} + 1 = n + 1 - \ell_0, \quad e_{\ell_0} - e_{\ell_0+1}, e_{\frac{m}{2}} - e_{\frac{m}{2}+1} \in \Sigma_i.$$

By assumption,

$$\begin{aligned} w^{-1}(e_{\ell_0} - e_{\ell_0+1}) &= e_{\sigma(\ell_0+1)} - e_{\sigma(\ell_0)} \in \Gamma_{O'}, \\ w^{-1}(e_{\frac{m}{2}} - e_{\frac{m}{2}+1}) &= e_{\sigma(\frac{m}{2})} - e_{\sigma(\frac{m}{2}+1)} \in \Gamma_{O'}. \end{aligned}$$

Then $\sigma(\ell_0 + 1) = \sigma(m/2) = m - n$. But $i \geq \ell_0 + 1 \neq m/2 > i$, which contradicts the assumption that σ is a permutation. Hence $w^{-1}(\Sigma_i) \subset \Phi^+$. This conclude the proof for $\boldsymbol{G} = \mathbf{Sp}(2n)$. □

Let $A = A(S)$ be the apartment of $\mathscr{B}(G)$ defined by the maximal split torus S of G; see Section 2B. Let r be a positive integer. $F \subset A$ is called an r-facet if F is connected and there is a finite subset Φ_F of Φ_{af} such that

$$\psi(x) = r \quad \text{for all } x \in F,\ \psi \in \Phi_F.$$

Here Φ_{af} is the set of affine roots associated to S. For more details on r-facets, see [DeBacker 2002]. Since r is integer, the r-facet is in fact the usual facet.

Lemma 3.4. *For $i \in I_{\mathrm{nsp}}$, let w be a Weyl element satisfying $w^{-1}(\Sigma_i) \subset \Phi^+$. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. Let $z \in O'$ be the nilpotent element in* (6), (8), (10), (15), *and let $r > 0$ a positive integer. Then there is an r-facet F such that $y_i \in \partial F$ and*

$$(wN'_{\geq 4}w^{-1} \cap G_{y_i,r})G_{y_i,r+} \supset G_{F,r+}.$$

Here y_i is the vertex of the fundamental chamber C defined in Section 2B and $N'_{\geq j}$ is the object defined in Section 2F for any $\mathfrak{sl}_2$ triple $\{z, h, z'\}$ attached to z in $\mathfrak{g}$.

Proof. Let $\Gamma_z \subset \Phi^+$ be the set defined in (7), (9), (11), (13), and set $\Gamma'_z = \Phi^+ \backslash \Gamma_z$. By Lemma 3.3, $w^{-1}(\Sigma_i) \cap \Gamma'_z \neq \varnothing$. Take $\beta \in \Sigma_i$ such that $w^{-1}(\beta) \in \Gamma'_z$ and, let x_β be an arbitrary point in the apartment $\mathscr{A}$ such that $0 < \beta(x_\beta) < \frac{1}{2}$ and $\alpha(x_\beta) = 0$ for all $\alpha \in \Sigma_i$ distinct from β. Let F be the smallest r-facet containing x_β. Then $y_i \in \partial F$ and F satisfies the requirement of the lemma.

In fact, let Φ_i be the root subsystem generated by Σ_i and Φ_i^+ the subset of positive roots of Φ_i generated by Σ_i. Then by definition

$$\mathfrak{g}_{F,r+} := \mathfrak{g}_{x_\beta,r+} = \Bigg(\prod_{\substack{\delta \in \Phi_i \\ \delta(x_\beta) > \delta(y_i)}} u_{\delta,r} \Bigg) + \mathfrak{g}_{y_i,r+} \subset \mathfrak{g}_{y_i,r}.$$

Note that the following sets are the same:

$$\begin{aligned}\{\delta \in \Phi_i \mid \delta(x_\beta) > \delta(y_i)\} &= \{\delta \in \Phi_i^+ \mid \delta - \beta \in \Phi_i^+\} \\ &= \{\delta \in \Phi_i^+ \mid \delta \in \beta + \Phi_i^+\} \\ &= \{w(\alpha) \in \Phi_i^+ \mid \alpha \in w^{-1}(\beta) + w^{-1}(\Phi_i^+)\}.\end{aligned}$$

By Lemma 3.3, $w^{-1}(\beta) \in \Gamma'_z$; that is, the root space $\mathfrak{u}_{w^{-1}(\beta)} \subset \mathfrak{n}'_{\geq 4}$. On the other hand, since $w^{-1}(\Sigma_i) \subset \Phi^+$, $w^{-1}(\Phi_i^+) \subset \Phi^+$. For all $\delta \in \Phi^+$, $\mathfrak{u}_\delta \in \mathfrak{n}'_{\geq 0}$ (see Appendix), so $u_\alpha \subset \mathfrak{n}'_{\geq 4}$ for all $\alpha \in \Phi^+ \cap (w^{-1}(\beta) + w^{-1}(\Sigma_i))$.

Hence $\mathfrak{g}_{F,r+} \subset w\mathfrak{n}'_{\geq 4}w^{-1} \cap \mathfrak{g}_{y_i,r} + \mathfrak{g}_{y_i,r+}$, and thus

$$(wN'_{\geq 4}w^{-1} \cap G_{y_i,r})G_{y_i,r+} \supset G_{F,r+}. \qquad \square$$

Proposition 3.5. *Let $\pi = \pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'} \in \Pi'(\varphi)$ be an irreducible representation defined in Section 2D such that $i = i(\boldsymbol{\mu}') \in I_{\mathrm{nsp}}$. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. Let $z \in O'$ be the nilpotent element in* (6), (8), (10), (15), *and let $N'_{\geq j}$ be the object defined in Section 2F for any $\mathfrak{sl}_2$ triple $\{z, h, z'\}$ attached to z in $\mathfrak{g}$.*

Let $N' = N'_{\geq 2}$ and ψ_z the character of N' defined in (3). *Let v be a representative of a double coset in $G_{y_i} \backslash G/N'$ and ψ_z^v the character of $vN'v^{-1} \cap G_{y_i}$ defined as* *follows*: *for all* $x \in vN'v^{-1} \cap G_{y_i}$,

$$\psi_z^v(x) := \psi_z(v^{-1}xv). \tag{25}$$

Let $r > 0$ *be a positive integer. Then there is an* r*-facet* F *such that* $y_i \in \partial F$ *and*

$$(vN'v^{-1} \cap G_{y_i,r})G_{y_i,r+}/G_{y_i,r+} \supset G_{F,r+}/G_{y_i,r+}, \quad \psi_z^v|_{G_{F,r+}} = 1.$$

Proof. Let S, B be the split torus and the Borel subgroup of G defined in Section 2B and U the unipotent subgroup of B. Let v be a representative of $G_{y_i}\backslash G/N'$; then,

$$v = w \cdot a \cdot u$$

for some Weyl element w of G such that $w^{-1}(\Sigma_i) \subset \Phi^+$, $a \in S$, and $u \in U/N'$, where Σ_i is the set defined in Lemma 3.3 (see [Reeder 1997]).

Note that a, u normalize N', and let $\psi' = \psi_z^{au}$, the character of N' defined in (25) with v replaced by au. By Lemma 3.4, there is an r-facet F with $y_i \in \partial F$ such that

$$(vN'v^{-1} \cap G_{y_i,r})G_{y_i,r+} \supset (wN'_{\geq 4}w^{-1} \cap G_{y_i,r})G_{y_i,r+} \supset G_{F,r+}.$$

For all $x \in G_{F,r+}$,

$$v^{-1}xv \in (au)^{-1}w^{-1}[wN'_{\geq 4}w^{-1}]wau \subset (au)^{-1}N'_{\geq 4}au = N_{\geq 4}.$$

By the definition of ψ_z, $\psi_z^v(x) = \psi_z(v^{-1}xv) = 1$. □

We can now conclude the proof of Theorem 3.1. By the discreteness criterion in [DeBacker and Reeder 2010, Lemma 2.4],

$$\chi(\pi) := \{x \in \mathcal{B}(G) \mid V_\pi^{G_{x,r+}} \neq 0\} = G.y_i,$$

and the $G_{y_i,r}/G_{y_i,r+}$-module $V_\pi^{G_{y_i,r+}}$ is cuspidal; i.e., for any r-facet F with $y_i \in \partial F$,

$$(V_\pi^{G_{y_i,r+}})^{\mathsf{L}_F} = 0. \tag{26}$$

Here $\mathsf{L}^F = G_{F,r+}/G_{y_i,r+}$ and V_π is the representation space of π.

Assume on the contrary $\mathrm{Hom}_{N'}(\pi, \psi_z) \neq 0$. By the construction of π in [Adler 1998], $\pi = c - \mathrm{Ind}_{G_{y_i}}^G(\Xi)$ for some irreducible representation Ξ of G_{y_i}. Let V_Ξ be the space of Ξ. Then

$$\mathrm{Hom}_{N'}(\pi, \psi_z) = \prod_{v \in G_{y_i}\backslash G/N'} \mathrm{Hom}_{vN'v^{-1} \cap G_{y_i}}(\Xi, \psi_z^v),$$

and there is some $v \in G_{y_i}\backslash G/N'$ such that $\mathrm{Hom}_{vN'v^{-1}\cap G_{y_i}}(\Xi, \psi_z^v) \neq 0$. Then

$$\mathrm{Hom}_{vN'v^{-1}\cap G_{y_i,r}}(\Xi, \psi_z^v) \neq 0.$$

Applying Proposition 3.5, there is an r-facet F such that $y_i \in \partial F$ and $V_{\Xi}^{G_{F,r+}} \neq 0$. Then $V_{\pi}^{G_{F,r+}} \neq 0$, which contradicts the discreteness criterion (26). $\square$

Proof of Theorem 3.2. Let $\bar{\mathfrak{f}}$ be the algebraic closure of $\mathfrak{f}$. Assume the characteristic p of $\mathfrak{f}$ is large enough that p is a good prime in the sense of [Carter 1972].

Keep the notation of Proposition 3.5. Then $i = i(\boldsymbol{\mu}') \in I_{\mathrm{nsp}}$ and $G_{y_i,r}/G_{y_i,r+} = \mathfrak{g}_1(\mathfrak{f}) \times \mathfrak{g}_2(\mathfrak{f})$, with $\mathfrak{g}_1 = \mathfrak{so}(2i, f)$ or $\mathfrak{sp}(2i, f)$ (see Section 2B). Let $\bar{\xi}_j \in \mathfrak{g}_j(\mathfrak{f})$ ($j = 1, 2$) be regular nilpotent elements and $\{\bar{\xi}_j, \bar{h}_j, \bar{\xi}'_j\}$ an $\mathfrak{sl}_2$ triple in $\mathfrak{g}_j(\mathfrak{f})$ attached to $\bar{\xi}_j$. Let

$$\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2), \quad \bar{h} = (\bar{h}_1, \bar{h}_2), \quad \bar{\xi}' = (\bar{\xi}_1, \bar{\xi}_2).$$

Then $(\bar{\xi}, \bar{h}, \bar{\xi}')$ is an $\mathfrak{sl}_2$ triple in $\mathfrak{g}_1(\mathfrak{f}) \times \mathfrak{g}_2(\mathfrak{f})$.

Recall that if $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, $i = i_{\boldsymbol{\mu}'} \in I_{\mathrm{nsp}}$, then $\mathsf{T} := \mathsf{T}_{\boldsymbol{\mu}'} = \mathsf{T}_1 \times \mathsf{T}_2$ is a maximal anisotropic torus in G_{y_i}. Let $T := T_{\boldsymbol{\mu}'}$ be the maximal anisotropic unramified torus in G associated to $(y_i, \mathsf{T}_{\boldsymbol{\mu}'})$ in Section 2C. Let $X = X_{\boldsymbol{\mu}'} \in \mathfrak{t} = \mathrm{Lie}(T)$ be the good element of depth $-r$ defining $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}$, whose image under the natural projection

$$\mathfrak{g}_{y_i,-r} \to \mathfrak{g}_{y_i,-r}/\mathfrak{g}_{y_i,-r+} \simeq \mathfrak{g}_1 \times \mathfrak{g}_2.$$

is denoted by $\bar{X} = (\bar{X}_1, \bar{X}_2)$. Since X is a good element in $\mathfrak{t}$ with $C_G(X) = T$, $\bar{X}_j$ is a regular semisimple element in $\mathrm{Lie}(\mathsf{T}_j)(\mathfrak{f})$ for $j = 1, 2$.

Let $O_{\bar{X}_j}$ be the orbit of $\bar{X}_j$ in $\mathfrak{g}_j(\bar{\mathfrak{f}})/\mathsf{G}_j(\bar{\mathfrak{f}})$. By [Slodowy 1980, §7.4, Corollary 2], the Slodowy slice

$$\bar{V}_j := \bar{\xi}_j + C_{\mathfrak{g}_j(\bar{\mathfrak{f}})}(\bar{\xi}'_j) \tag{27}$$

intersects $O_{\bar{X}_j}$ at a unique $\mathfrak{f}$-rational point $\bar{X}'_j \in \mathfrak{g}_j(\mathfrak{f})$.

Since X is good, $C_{\mathsf{G}_j(\bar{\mathfrak{f}})}(\bar{X}_j)$ is connected [Carter 1985, Theorem 3.5.3]. Then there is a $\bar{g}_j \in \mathsf{G}_j(\mathfrak{f})$ such that $\mathrm{Ad}(\bar{g}_j)(X_j) = \bar{X}'_j$ [Digne and Michel 1991, §3.25]. Moreover $\mathsf{T}'_j = C_{\mathsf{G}_j}(\bar{X}'_j) = \mathrm{Ad}(\bar{g}_j)(\mathsf{T}_j)$ is a maximal anisotropic torus of $\mathsf{G}_j(\mathfrak{f})$, with $\mathsf{G}_j(\mathfrak{f})$-conjugate to T_j. Let $\bar{g} = (\bar{g}_1, \bar{g}_2) \in \mathsf{G}(\mathfrak{f})$; then, $\mathrm{Ad}(\bar{g})(\mathsf{T}_1 \times \mathsf{T}_2) = \mathsf{T}' := \mathsf{T}'_1 \times \mathsf{T}'_2$.

Let $g \in G_{y_i,0} - g_{y_i,0+}$ such that g projects to $\bar{g}$, $T' := \mathrm{Ad}(g)(T)$, and $X' := \mathrm{Ad}(g)(X) \in \mathfrak{t}'$. Then T' is the maximal unramified torus in G, associated to (y_i, T'), X' is a good element in $\mathfrak{g}_{y_i,-r} \backslash \mathfrak{g}_{y_i,-r+}$, whose image under the natural projection in G_{y_i} is $\bar{X}' = (\bar{X}'_1, \bar{X}'_2)$. Note that $\bar{X}' \in \bar{V}_1(\mathfrak{f}) \times \bar{V}_2(\mathfrak{f})$, where

$$\bar{V}_1(\mathfrak{f}) = \bar{\xi}_1 + C_{\mathfrak{g}_1(\mathfrak{f})}(\bar{\xi}'_1), \quad \bar{V}_2(\mathfrak{f}) = \bar{\xi}_2 + C_{\mathfrak{g}_2(\mathfrak{f})}(\bar{\xi}'_2)$$

are sets of $\mathfrak{f}$-rational points of $\bar{V}_1, \bar{V}_2$ respectively. Without loss of generality, assume $X = X'$. Then the natural image $\bar{X}$ of X in $\mathfrak{g}_{y_i,-r}/\mathfrak{g}_{y_i,-r+}$ belongs to $\bar{V}_1(\mathfrak{f}) \times \bar{V}_2(\mathfrak{f})$.

By [DeBacker 2002, Corollary 4.3.2], let $(\xi, h, \xi') \in \mathfrak{g}_{y_i,-r} \times \mathfrak{g}_{y_i,0} \times \mathfrak{g}_{y_i,r}$ be an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$ such that $\{\xi, h, \xi'\}$ lifts $\{\bar{\xi}, \bar{h}, \bar{\xi}'\}$ respectively and $O' = \mathrm{Ad}(G)(\xi)$ the nilpotent orbit of ξ in $\mathfrak{g}$. By the choice of $\{\xi, h, \xi'\}$, $O' = O^i$ is a nilpotent orbit corresponding to $(\boldsymbol{\lambda}^i, (q_j))$. Let $N'_{\geq j}$ be the object defined in Section 2F for the triple $\{\xi, h, \xi'\}$ attached to ξ in $\mathfrak{g}$.

We can now conclude the proof of Theorem 3.2. Let $N' = N'_{\geq 2}$ and let S_ξ be the character ψ_ξ of N':

$$S_\xi(\exp Y) = \psi \circ \mathrm{tr}(\xi Y), \quad Y \in \mathrm{Lie}(N').$$

On the other hand, by the construction in [Adler 1998], $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'} = c - \mathrm{Ind}_{G_{y_i}}^{G(k)}(\Xi)$, while $\Xi = \mathrm{Ind}_{TJ}^{G_{y_i}}(\sigma_\chi)$. Here

$$J = \exp_{y_i}(\mathfrak{J}), \qquad \mathfrak{J} = \mathfrak{t}_{y_i,r} + \mathfrak{t}^{\perp}_{y_i,\frac{r}{2}},$$
$$J^+ = \exp_{y_i}(\mathfrak{J}^+), \quad \mathfrak{J}^+ = \mathfrak{t}_{y_i,r} + \mathfrak{t}^{\perp}_{y_i,\frac{r}{2}+},$$

with $\mathfrak{t}^\perp$ the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to the killing form. Here TJ and TJ^+ are subgroups of G, since T normalizes J and J^+, and σ_χ is the irreducible representation of TJ such that $\sigma_\chi|_{TJ^+}$ is a multiple of χ, where χ is the character of TJ^+ extending $\chi_{\boldsymbol{\mu}'}$ on T, such that

$$\chi(\exp_{y_i} Y) = \psi(\mathrm{tr}(X \cdot Y)) \quad \text{for all } Y \in \mathfrak{J}^+.$$

Note that T is anisotropic and $N' \cap TJ = N' \cap J \supset N' \cap J^+$, while $N' \cap J / N' \cap J^+$ is an isotropic subspace over $\mathfrak{f}$ with respect to the nondegenerate symplectic form defined on J/J^+ by $(n, n') \mapsto \psi_\xi([\log n, \log n'])$. On the other hand, since $\overline{X} \in \overline{V}_1(\mathfrak{f}) \times \overline{V}_2(\mathfrak{f})$, $\chi|_{J^+ \cap N'} = \psi_\xi|_{J^+ \cap N'}$. By the definition of σ_χ,

$$\mathrm{Hom}_{N' \cap TJ}(\sigma_\chi, \psi_\xi) = \mathrm{Hom}_{N' \cap J}(\sigma_\chi, \psi_\xi) \neq 0.$$

Apply Lemma 3.6 below with G_1 replaced by G_{y_i}, G_2 by $N' \cap G_{y_i}$, and H_1 by TJ; then,

$$\mathrm{Hom}_{N' \cap G_{y_i}}(\Xi, \psi_\xi) \neq 0. \tag{28}$$

Since $\mathrm{Hom}_{N'}(\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}, S_\xi) = \prod_{v \in G_{y_i} \backslash G / N'} \mathrm{Hom}_{vN'v^{-1} \cap G_{y_i}}(\Xi, \psi_\xi^v)$, by (28),

$$\mathrm{Hom}_{N'}(\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}, \psi_\xi) \neq 0.$$

Hence $O' \in \mathcal{N}_{\mathrm{wh}}(\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'})$. Combining with Theorem 3.1, $O' \in \mathcal{N}_{\mathrm{wh,max}}(\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'})$. □

Lemma 3.6. *Let G_1 be a compact subgroup, and H_1, G_2 open compact subgroups of G_1. Let (σ, V_σ) (resp. (ξ, V_ξ)) be a smooth representation of H_1 (resp. G_2). If $\mathrm{Hom}_{H_1 \cap G_2}(\sigma, \xi) \neq 0$, then $\mathrm{Hom}_{G_2}(\mathrm{Ind}_{H_1}^{G_1} \sigma, \xi) \neq 0$.*

Proof. The proof is similar to that of Proposition 2.1 in [Arthur 2008]. Consider a nonzero $A \in \mathrm{Hom}_{H_1 \cap G_2}(\sigma, \xi)$, and define $J_A \in \mathrm{Hom}_{G_2}(\mathrm{Ind}_{H_1}^{G_1}\sigma, \xi)$ as follows: for arbitrary $\phi \in \mathrm{Ind}_{H_1}^{G_1}\sigma$,

$$J_A\phi = \sum_{H_1 \cap G_2 \backslash G_2} \xi(g')^{-1} A(\phi(g')) \in V_\xi.$$

For all $g \in G_2$,

$$\begin{aligned} J_A(\mathrm{Ind}\sigma)(g)\phi &= \sum_{H_1 \cap G_2 \backslash G_2} \xi(g')^{-1} A(\mathrm{Ind}\sigma(g)\phi)(g') \\ &= \sum_{H_1 \cap G_2 \backslash G_2} \xi(g')^{-1} A\phi(g'g) \\ &= \xi(g) J_A\phi. \end{aligned}$$

Take some $v \in V_\sigma$ such that $Av \neq 0$. Define $\phi_v(g) = \sigma(h)v$ if $g = h \in H_1$ and $\phi_v(g) = 0$ if $g \notin H_1$. Then $\phi_v \in \mathrm{Ind}_{H_1}^{G_1}\sigma$, and $J_A\phi_v = Av \neq 0$, so J_A is a nonzero element in $\mathrm{Hom}_{G_2}(\mathrm{Ind}_{H_1}^{G_1}\sigma, \xi)$. □

Appendix: Rational nilpotent orbits

In this section, we show by example how to choose a particular element from a rational nilpotent orbit parametrized by $(\boldsymbol{\lambda}, (q_j))$.

Let W be a $(2n+1)$-dimensional symmetric k-space as defined in Section 2A, with bilinear form q_W. Let z be a nonzero nilpotent element in $\mathfrak{g} = \mathfrak{so}(W) \subset \mathfrak{gl}(W)$, and set $G = \mathbf{SO}(k, W)$. Let $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ be a Lie algebra homomorphism with

$$\phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = z.$$

Identify a scalar $t \in k$ with the diagonal matrix $\mathrm{diag}(t, t^{-1}) \in \mathfrak{sl}_2(k)$. As in [Mœglin 1996], for $i \in \mathbb{Z}$, let

$$\begin{aligned} \mathfrak{g}(i) &= \{Y \in \mathfrak{g} \mid \mathrm{Ad} \circ \phi(t)(Y) = itY \text{ for all } t \in k\}, \\ W(i) &= \{v \in W \mid \phi(t)(v) = itv \text{ for all } t \in k\}. \end{aligned}$$

Then $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$, $W = \bigoplus_{i \in \mathbb{Z}} W(i)$.

Assume the orbit $O = \mathrm{Ad}(G)(z)$ of z is parametrized by $(\boldsymbol{\lambda}, (q_i))$ with $\boldsymbol{\lambda} = [m, 2n-m, 1]$, where $m > n$ is an odd number. For $i = 1, \ldots, 2n+1$, let

$$W_i = \mathrm{Ker}(z^i)/(\mathrm{Ker}(z^{i-1}) + z\,\mathrm{Ker}(z^{i+1})). \tag{29}$$

Then by [Waldspurger 2001, §I.6], $\dim W_i = c_i(\boldsymbol{\lambda})$ and q_i is the nondegenerate quadratic form on W_i defined by

$$q_i(\bar{v}, \bar{v}') = (-1)^{[\frac{i-1}{2}]} q_W(z^{i-1}v, v') \quad (\bar{v}, \bar{v}' \in W_i), \tag{30}$$

where v (resp. v') is an inverse image of $\bar{v}$ (resp. $\bar{v}'$) in $\mathrm{Ker}(z^i)$.

Assume $m = 2n-1$; in this case $\boldsymbol{\lambda} = [2n-1, 1^2]$, $c_1(\boldsymbol{\lambda}) = 2$, $c_{2n-1}(\boldsymbol{\lambda}) = 1$. Then $\dim W_1 = 2$ and $\dim W_m = 1$. By (29), let $v_1, v_1' \in \mathrm{Ker}\, z$, $v_m \in \mathrm{Ker}\, z^m$ such that

$$\begin{aligned} \mathrm{Ker}\, z &= z\, \mathrm{Ker}\, z^2 \oplus k v_1 \oplus k v_1', \\ \mathrm{Ker}\, z^m &= (\mathrm{Ker}\, z^{m-1} + z\, \mathrm{Ker}\, z^{m+1}) \oplus k v_m. \end{aligned}$$

Let $\bar{v}_1, \bar{v}_1'$ be the natural images of v_1, v_1' in W_1 and $\bar{v}_m$ that of v_m in W_m. Without loss of generality, assume $\bar{v}_1, \bar{v}_1'$ are orthogonal to each other under q_1; then $q_1 = \langle q_1(\bar{v}_1, \bar{v}_1), q_1(\bar{v}_1', \bar{v}_1')\rangle$,

$$q_m = \langle q_m(\bar{v}_m, \bar{v}_m)\rangle = (-1)^{\frac{m-1}{2}} q_W(z^{m-1}v_m, v_m). \tag{31}$$

In the following, identify q_m with $q_m(\bar{v}_m, \bar{v}_m)$.

Through $\phi : \mathfrak{sl}_2 \to \mathfrak{so}(W) \subset \mathfrak{gl}(W)$, W is a representation space of $\mathfrak{sl}_2$. In fact, since O_X corresponds to $(\boldsymbol{\lambda}, (q_i))$, $W \simeq V_m \oplus V_1 \oplus V_1$, where V_j is the irreducible representation of $\mathfrak{sl}_2$ of dimension j. By the representation theory of $\mathfrak{sl}_2$, $v_1, v_1' \in W(0)$ and $v_m \in W(m-1)$. Modifying by elements in $z\,\mathrm{Ker}\, z^2$, we can assume further that the subspace generated by v_1, v_1' is $V_1 \oplus V_1$.

Then ${}_{0\neq} z^\ell(v_m) \in W(m-1-2\ell)$ for all $\ell = 1, \dots, m-1$, and

$$\begin{aligned} W(m-1) &= k v_m, \\ W(m-3) &= k z v_m, \\ \vdots \quad & \quad \vdots \\ W(2) &= k z^{n-2} v_m, \\ W(0) &= k z^{n-1} v_m \oplus k v_1 \oplus k v_1', \\ W(-2) &= k z^n v_m, \\ \vdots \quad & \quad \vdots \\ W(-(m-1)) &= k z^{m-1} v_m. \end{aligned}$$

For $j = 1, \dots, m$, let $F_j = \bigoplus_{\ell \le -(m-1)+2(j-1)} W(\ell)$ be a subspace of W. Then

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m = W,$$

and $zF_j = F_{j-1}$ for $j = 1, \dots, m$. Take a basis of W such that

$$\begin{aligned} e_1 &= v_m, & e_{-1} &= (-1)^{\frac{m-1}{2}} q_m^{-1} z^{m-1} v_m, \\ e_2 &= z v_m, & e_{-2} &= (-1)^{\frac{m-3}{2}} q_m^{-1} z^{m-2} v_m, \\ &\vdots & &\vdots \\ e_{n-1} &= z^{n-2} v_m, & e_{-(n-1)} &= (-1) q_m^{-1} z^{n+1} v_m. \end{aligned}$$

By (30), $q_W(e_i, e_j) = 0$ unless $i + j = 0$, and $q_W(e_i, e_{-i}) = 1$. Note that $W(0)$ has orthogonal decomposition

$$W(0) = kz^{n-1} v_m \oplus kv_1 \oplus kv_1'$$

under $q_W|_{W(0)}$. By (30), $q_W(z^{n-1} v_m, z^{n-1} v_m) = q_m(\bar{v}_m, \bar{v}_m)$, $q_W(v_1, v_1) = q_1(\bar{v}_1, \bar{v}_1)$, and $q_W(v_1', v_1') = q_1(\bar{v}_1', \bar{v}_1')$. By (31),

$$\begin{aligned} q_W|_{W(0)} &= \langle q_W(z^{\frac{m-1}{2}} v_m, z^{\frac{m-1}{2}} v_m), q_W(v_1, v_1), q_W(v_1', v_1') \rangle \\ &= \langle q_m(\bar{v}_m, \bar{v}_m), q_1(\bar{v}_1, \bar{v}_1), q_1(\bar{v}_1', \bar{v}_1') \rangle \\ &= q_m \oplus q_1. \end{aligned}$$

Because $q_1 \oplus q_m$ has the same anisotropic kernel as W, let e_n, e_0, e_{-n} be a basis of $W(0)$ such that

$$q_W(e_n, e_{-n}) = 1, \quad q_W(e_0, e_0) = 1, \quad q_W(e_n, e_0) = q_W(e_{-n}, e_0) = 0.$$

Then $e_1, e_2, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}$ is a basis of W, under which the matrix of q_W is J_W (defined in Section 2A), and the matrix of z is the lower triangular block matrix

$$\begin{pmatrix} 0 & & & & & & & & \\ 1 & 0 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 1 & 0 & & & & & \\ & & & A^* & 0_3 & & & & \\ & & & & A & 0 & & & \\ & & & & & -1 & 0 & & \\ & & & & & & \ddots & \ddots & \end{pmatrix},$$

with

$$A^* = \begin{pmatrix} a_m \\ b_m \\ c_m \end{pmatrix}, \quad A = -\begin{pmatrix} c_m & b_m & a_m \end{pmatrix},$$

where (a_m, b_m, c_m) are the coordinates of $z^{n-1} v_m$ in the basis $\{e_n, e_0, e_{-n}\}$. Note that in this case, $AA^* = -q_m$.

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THE NATURAL FILTRATIONS OF FINITE-DIMENSIONAL MODULAR LIE SUPERALGEBRAS OF WITT AND HAMILTONIAN TYPE

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We study the natural filtrations of the finite-dimensional modular Lie superalgebras $W(n, m)$ and $H(n, m)$. In particular, the natural filtrations which are invariant relative to the automorphisms of the Lie superalgebras are employed in order to characterize the Lie superalgebras themselves.

1. Introduction

In mathematics, a Lie superalgebra is a generalization of a Lie algebra including a $\mathbb{Z}_2$-grading. Lie superalgebras are also important in theoretical physics where they are used to describe the mathematics of supersymmetry [Varadarajan 2004]. Although many structural features of Lie superalgebras over fields of characteristic zero (see [Kac 1977; Scheunert 1979]) are well understood, there seem to be very few general results on modular Lie superalgebras. In particular, the classification problem is still open for the finite-dimensional simple Lie superalgebras over fields of positive characteristic (see [Bouarroudj and Leites 2006; Zhang 1997] for example). The treatment of modular Lie superalgebras necessitates different techniques which are set forth in [Kochetkov and Leites 1992; Petrogradski 1992]. Elduque [2007] obtained two new simple modular Lie superalgebras. These Lie superalgebras share the property that their even parts are orthogonal Lie algebras and the odd parts are their spin modules. In [Zhao 2010] modular representations of basic classical Lie superalgebras were studied. The Lie superalgebras of Cartan type play an extremely important role in the study of modular Lie superalgebras. Recent works on them can be found in [Chen and Liu 2011; Yuan et al. 2011; Zhang and Fu 2002].

It is well known that filtration techniques are of great importance in the structure and the classification theories of Lie (super)algebras (see [Block and Wilson 1988; Strade 1993; Kac 1977; Scheunert 1979]). We know that the simple Lie

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(super)algebras of Cartan type possess various natural filtration structures. For the filtration structures, the invariance may be used to make an insight for the intrinsic properties and the automorphism groups of those Lie (super)algebras. The natural filtrations of finite-dimensional modular Lie algebras of Cartan type were proved to be invariant in [Kac 1974; Kostrikin and Shafarevich 1969]. The finite-dimensional simple modular Lie superalgebras W, S, and H of Cartan type were defined in [Zhang 1997] and their natural filtrations were investigated in [Zhang and Fu 2002; Zhang and Nan 1998]. Recently, the natural filtrations of odd Hamiltonian superalgebras and special odd Hamiltonian superalgebras of formal vector fields were investigated in [Ren et al. 2012].

The finite-dimensional modular Lie superalgebras $W(n, m)$ and $H(n, m)$ were first introduced in [Awuti and Zhang 2008] and [Ren et al. 2011], respectively. In these papers, their derivation superalgebras were also determined. The starting point of our studies is the investigation of the ad-nilpotent elements of $W(n, m)$. Then the natural filtration of $W(n, m)$ is proved to be invariant by the determined ad-nilpotent elements. In the case of $H(n, m)$, the invariance of the natural filtration is studied by the methods of minimal dimension of image spaces and the derivation superalgebras. In view of the above invariance of the natural filtrations we describe the intrinsic properties of these modular Lie superalgebras.

This paper is arranged as follows. A brief summary of the relevant concepts and notations in finite-dimensional modular Lie superalgebras $W(n, m)$ and $H(n, m)$ is presented in Section 2. In Section 3, by using the ad-nilpotent elements of the Lie superalgebras $W(n, m)$, we show that the natural filtration of $W(n, m)$ is invariant under their automorphisms. In Section 4, the intrinsic properties with respect to the natural filtration of finite-dimensional modular Lie superalgebras $H(n, m)$ are investigated. Besides, the isomorphic relation between $H(n, m)$ and $H(n', m')$ is also proved by the method of the natural filtration.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ denotes an algebraic closed field of characteristic $p > 2$, n is an integer greater than 1. Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of integers, positive integers and nonnegative integers. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the residue class ring of integers modulo 2.

Let $\Lambda(n)$ be the Grassmann algebra over $\mathbb{F}$ in n variables $x_1, x_2, \dots, x_n$. Set $\mathbb{B}_k = \{\langle i_1, i_2, \dots, i_k\rangle \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}$ and $\mathbb{B}(n) = \bigcup_{k=0}^{n} B_k$, where $B_0 = \varnothing$. For $u = \langle i_1, i_2, \dots, i_k\rangle \in \mathbb{B}(k)$, set $|u| = k$, $\{u\} = \{i_1, i_2, \dots, i_k\}$, and $x^u = x_{i_1} x_{i_2} \cdots x_{i_k}$ ($|\varnothing| = 0, x^{\varnothing} = 1$). Then $\{x^u | u \in \mathbb{B}(n)\}$ is an $\mathbb{F}$-basis of $\Lambda(n)$.

Let Π denote the prime field of $\mathbb{F}$, that is, $\Pi = \{0, 1, \dots, p-1\}$. Suppose that $\{z_1, z_2, \dots, z_m\}$ is a Π-linearly independent finite subset of $\mathbb{F}$. Let $G =$

$\{\sum_{i=1}^{m} \lambda_i z_i \mid \lambda_i \in \Pi\}$. Then G is an additive subgroup of $\mathbb{F}$. Let $\mathbb{F}[y_1, y_2, \ldots, y_m]$ be the truncated polynomial algebra satisfying $y_i^p = 1$ for all $i = 1, 2, \ldots, m$. For every element $\lambda = \sum_{i=1}^{m} \lambda_i z_i \in G$, define $y^\lambda = y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_m^{\lambda_m}$. Then $y^\lambda y^\eta = y^{\lambda+\eta}$ for all $\lambda, \eta \in G$. Let $\mathbb{T}(m)$ denote $\mathbb{F}[y_1, y_2, \ldots, y_m]$. Then $\mathbb{T}(m) = \{\sum_{\lambda \in G} a_\lambda y^\lambda \mid a_\lambda \in \mathbb{F}\}$. We denote the tensor product by $\mathcal{U} = \Lambda(n) \otimes \mathbb{T}(m)$. Then $\mathcal{U}$ is an associative superalgebra with $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathbb{T}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$, that is, $\mathcal{U} = \mathcal{U}_{\bar{0}} \oplus \mathcal{U}_{\bar{1}}$, where $\mathcal{U}_{\bar{0}} = \Lambda(n)_{\bar{0}} \otimes \mathbb{T}(m)$ and $\mathcal{U}_{\bar{1}} = \Lambda(n)_{\bar{1}} \otimes \mathbb{T}(m)$.

For $f \in \Lambda(n)$ and $\alpha \in \mathbb{T}(m)$, we abbreviate $f \otimes \alpha$ as $f\alpha$. Then the elements $x^u y^\lambda$ with $u \in \mathbb{B}(n)$ and $\lambda \in G$ form an $\mathbb{F}$-basis of $\mathcal{U}$. It is easy to see that $\mathcal{U} = \bigoplus_{i=0}^{n} \mathcal{U}_i$ is a $\mathbb{Z}$-graded superalgebra, where $\mathcal{U}_i = \operatorname{span}_{\mathbb{F}}\{x^u y^\lambda \mid u \in \mathbb{B}(n), |u| = i, \lambda \in G\}$. In particular, $\mathcal{U}_0 = \mathbb{T}(m)$ and $\mathcal{U}_n = \operatorname{span}_{\mathbb{F}}\{x^\pi y^\lambda \mid \lambda \in G\}$, where $\pi := \langle 1, 2, \ldots, n\rangle \in \mathbb{B}(n)$.

In this paper, if $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a superalgebra (or $\mathbb{Z}_2$-graded linear space), let $\operatorname{hg}(A) = A_{\bar{0}} \cup A_{\bar{1}}$; that is, $\operatorname{hg}(A)$ is the set of all $\mathbb{Z}_2$-homogeneous elements of A. If $\deg x$ occurs in some expression, we regard x as a $\mathbb{Z}_2$-homogeneous element and $\deg x$ as the $\mathbb{Z}_2$-degree of x. Let $A = \bigoplus_{i=-r}^{n} A_i$ be a $\mathbb{Z}$-graded superalgebra. If $x \in A_i$, we call x a $\mathbb{Z}$-homogeneous element, i the $\mathbb{Z}$-degree of x and set $\operatorname{zd}(x) = i$. If $y \in A$, let $\mu(y)$ denote the nonzero $\mathbb{Z}$-homogeneous part of y with the least $\mathbb{Z}$-degree.

Let $\operatorname{pl}(A) = \operatorname{pl}_{\bar{0}}(A) \oplus \operatorname{pl}_{\bar{1}}(A)$ denote the general linear Lie superalgebra of the $\mathbb{Z}_2$-graded space A. For $\varphi \in \operatorname{pl}_\theta(A)$ with $\theta \in \mathbb{Z}_2$, if

$$\varphi(xy) = \varphi(x)y + (-1)^{\theta \deg x} x\varphi(y)$$

for all $x \in \operatorname{hg}(A)$ and $y \in A$, then φ is called a derivation of A with $\mathbb{Z}_2$-degree θ. Let $\operatorname{Der}_\theta A$ denote the set of all derivations of A with $\mathbb{Z}_2$-degree θ. Then $\operatorname{Der} A = \operatorname{Der}_{\bar{0}} A \oplus \operatorname{Der}_{\bar{1}} A$ is a subalgebra of $\operatorname{pl}(A)$ (see [Scheunert 1979]), which is called the derivation superalgebra of A.

Set $\mathrm{Y} = \{1, 2, \ldots, n\}$. Given $i \in \mathrm{Y}$, let $\partial/\partial x_i$ be the partial derivative on $\Lambda(n)$ with respect to x_i. For $i \in Y$, let D_i be the linear transformation on $\mathcal{U}$ such that $D_i(x^u y^\lambda) = (\partial x^u/\partial x_i)y^\lambda$ for all $u \in \mathbb{B}(n)$ and $\lambda \in G$. Then $D_i \in \operatorname{Der}_{\bar{1}} \mathcal{U}$ for all $i \in \mathrm{Y}$ since $\partial/\partial x_i \in \operatorname{Der}_{\bar{1}}(\Lambda(n))$.

Suppose that $u \in \mathbb{B}_k \subseteq \mathbb{B}(n)$ and $i \in \mathrm{Y}$. When $i \in \{u\}$, we denote the uniquely determined element of $\mathbb{B}_{k-1}$ satisfying $\{u - \langle i\rangle\} = \{u\} \setminus \{i\}$ by $u - \langle i\rangle$, and denote the number of integers less than i in $\{u\}$ by $\tau(u, i)$. When $i \notin \{u\}$, we set $\tau(u, i) = 0$ and $x^{u-\langle i\rangle} = 0$. Therefore, $D_i(x^u) = (-1)^{\tau(u,i)} x^{u-\langle i\rangle}$ for any $i \in \mathrm{Y}$ and $u \in \mathbb{B}(n)$.

We define $(fD)(g) = fD(g)$ for $f, g \in \operatorname{hg}(\mathcal{U})$ and $D \in \operatorname{hg}(\operatorname{Der} \mathcal{U})$. Since the multiplication of $\mathcal{U}$ is supercommutative, it follows that fD is a derivation of $\mathcal{U}$. Let

$$W(n, m) = \operatorname{span}_{\mathbb{F}}\{x^u y^\lambda D_i \mid u \in \mathbb{B}(n), \lambda \in G, i \in \mathrm{Y}\}.$$

Then $W(n,m)$ is a finite-dimensional Lie superalgebra contained in $\operatorname{Der}\mathcal{U}$. A direct computation shows that

$$[fD_i, gD_j] = fD_i(g)D_j - (-1)^{\deg fD_i \deg gD_j} gD_j(f)D_i, \tag{2-1}$$

where $f, g \in \operatorname{hg}(\mathcal{U})$ and $i, j \in \mathrm{Y}$.

Let $D_H : \mathcal{U} \to W(n,m)$ be the linear mapping such that for every $f \in \operatorname{hg}(\mathcal{U})$, $D_H(f) = \sum_{i=1}^n f_i D_i$, where $f_i = (-1)^{\deg f} D_i(f)$. It is easy to see that D_H is an even linear mapping and $D_i(f_j) = -D_j(f_i)$ for all $i, j \in \mathrm{Y}$. Let $\overline{H}(n,m) = \{D_H(f) \mid f \in \mathcal{U}\}$ and $H(n,m) = \bigl\{D_H(f) \mid f \in \bigoplus_{i=0}^{n-1} \mathcal{U}_i i\bigr\}$. Then $H(n,m)$ is a finite-dimensional Hamiltonian Lie superalgebra, with a $\mathbb{Z}$-gradation $H(n,m) = \bigoplus_{i=-1}^{n-3} H_i(n,m)$, where $H_i(n,m) = \{D_H(x^u y^\lambda) \mid u \in \mathbb{B}(n), |u| = i+2, \lambda \in G\}$. It was shown in [Ren et al. 2011] that $H(n,m)$ is a subalgebra of $W(n,m)$ and that

$$[D_H(f), D_H(g)] = D_H\biggl(\sum_{i=1}^n (-1)^{\deg f} D_i(f) D_i(g)\biggr), \tag{2-2}$$

$$[D_j, D_H(f)] = D_H(D_j(f)), \tag{2-3}$$

where $f, g \in \operatorname{hg}(\mathcal{U})$ and $j \in Y$.

Let $\Theta := T(m)^m = T(m) \times \cdots \times T(m)$. For every $\theta = (h_1(y), \ldots, h_m(y)) \in \Theta$, we define $\tilde{\theta}\colon G \to T(m)$ by $\tilde{\theta}(\lambda) = \sum_{j=1}^m \lambda_j h_j(y)$ for $\lambda = \sum_{j=1}^m \lambda_j z_j \in G$. It is easy to check that $\tilde{\theta}(\lambda+\eta) = \tilde{\theta}(\lambda) + \tilde{\theta}(\eta)$ for $\lambda, \eta \in G$. For every $\theta \in \Theta$, let $D_\theta\colon H(n,m) \to H(n,m)$ be the linear mapping given by $D_\theta D_H(x^u y^\lambda) = \tilde{\theta}(\lambda) D_H(x^u y^\lambda)$ for $x^u y^\lambda \in \mathcal{U}$. A direct verification shows that $D_\theta \in \operatorname{Der}_{\bar{0}} H$ for all $\theta \in \Theta$. Put $\Omega := \{D_\theta \mid \theta \in \Theta\}$.

3. The natural filtration of $W(n, m)$

In this section, W always denotes Lie superalgebras $W(n,m)$. Then $W = \bigoplus_{k=-1}^{n-1} W_k$ is $\mathbb{Z}$-graded, where $W_k = \operatorname{span}_{\mathbb{F}}\{x^u y^\lambda D_j \mid |u| = k+1, j \in \mathrm{Y}\}$.

Adopting the notion of [Jin 1992], the element x of Lie superalgebra L is called ad-nilpotent if $\operatorname{ad} x$ is a nilpotent linear transformation. The set of all ad-nilpotent elements of L is denoted by $\operatorname{nil}(L)$. Let $L_{(j)} = \bigoplus_{k \ge j} L_k$; then $\{L_{(j)} \mid j \ge -1\}$ is the natural filtration of L. If L is $\mathbb{Z}$-graded and finite-dimensional, then $L_{-1} \subseteq \operatorname{nil}(L)$ and $L_{(1)} \subseteq \operatorname{nil}(L)$.

Let $M_n(\mathbb{F})$ denote the set of all $n \times n$ matrices over $\mathbb{F}$. Notice that $\dim T(m) = p^m$. Without loss of generality, we may suppose that $\{y_1, \ldots, y_{p^m}\}$ is a standard $\mathbb{F}$-basis of $T(m)$. If

$$z = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in W_0,$$

where $a_{ijq} \in \mathbb{F}$, let

$$\rho(z) = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_{p^m} \end{pmatrix}_{np^m \times np^m}$$

where $A_q = (a_{ijq})_{n\times n} \in M_n(\mathbb{F})$.

Lemma 3.1. *Suppose that* $z = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in W_0$. *If* z *is ad-nilpotent, then* $\rho(z)$ *is a nilpotent matrix.*

Proof. Let Γ be the representation of W_0 with values in W_{-1}. Then $\Gamma(z) = \operatorname{ad} z$ and the matrix of $\Gamma(z)$ over the basis $\{y_1 D_1, \dots, y_1 D_n, \dots, y_{p^m} D_1, \dots, y_{p^m} D_n\}$ of W_{-1} is

$$A = \begin{pmatrix} -(A_1)^t & & \\ & \ddots & \\ & & -(A_{p^m})^t \end{pmatrix}_{np^m \times np^m}$$

where $A_q = (a_{ijq})_{n\times n} \in M_n(\mathbb{F})$. Since z is ad-nilpotent, the representation $\Gamma(z)$ is a nilpotent linear transformation. This implies that A is nilpotent. Therefore, $\rho(z) = -A^t$ is a nilpotent matrix. □

Lemma 3.2. *Let* $z = \sum_{i=k}^{n-1} z_i$, *where* $z_i \in W_i$ *and* $k \le n-1$. *If* $z \in \operatorname{nil}(W)$ *and* $k \ge 0$, *then* $z_k \in \operatorname{nil}(W)$.

Proof. Suppose that $z = z_k + z'$, where $z_k \in W_k$ and $z' \in \bigoplus_{i=k+1}^{n-1} W_i \subseteq W_{(k+1)}$. Since $z \in \operatorname{nil}(W)$, we may assume that $(\operatorname{ad} z)^t = 0$. Let x is a $\mathbb{Z}$-homogeneous element of W with $\mathbb{Z}$-degree i. Then $(\operatorname{ad} z)^t(x) = 0$. On the other hand,

$$(\operatorname{ad} z)^t(x) = (\operatorname{ad}(z_k + z'))^t(x) = (\operatorname{ad} z_k)^t(x) + h,$$

which implies $(\operatorname{ad} z_k)^t(x) + h = 0$. It is easy to see that $(\operatorname{ad} z_k)^t(x) \in W_{(kt+i)}$ and $h \in W_{(kt+i+1)} = \bigoplus_{j \ge kt+i+1} W_j$. Thus $(\operatorname{ad} z_k)^t(x) = 0$. Since x is an arbitrary $\mathbb{Z}$-homogeneous element of W, we have $(\operatorname{ad} z_k)^t(W) = 0$. Then $(\operatorname{ad} z_k)^t = 0$, that is, $z_k \in \operatorname{nil}(W)$. □

Suppose that E_{ij} denotes the $n \times n$ matrix whose (i, j) element is 1 and otherwise are zero. Obviously,

$$E_{ij} E_{kl} = \delta_{jk} E_{il}, \tag{3-1}$$

where δ_{jk} is the Kronecker delta.

If $z = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in W_0$, where $a_{ijq} \in \mathbb{F}$, then

$$\rho(z) = \sum_{i,j=1}^{n} a_{ij1} E_{ij} + \sum_{i,j=n+1}^{2n} a_{ij2} E_{ij} + \cdots + \sum_{i,j=n(p^m-1)+1}^{np^m} a_{ijp^m} E_{ij}. \tag{3-2}$$

Let $\Delta = \{z \in \operatorname{nil}(W) \mid \operatorname{ad} z(W) \subseteq \operatorname{nil}(W)\}$.

Lemma 3.3. *Suppose that* $z=\sum_{i=-1}^{n-1} z_i$, *where* $z_i \in W_i$. *If* $z\in\Delta$, *then* $z_{-1}=z_0=0$.

Proof. Suppose that $0\neq z_{-1}=\sum_{i=1}^{n}\sum_{q=1}^{p^m} a_{iq}y_qD_i$, where $a_{iq}\in\mathbb{F}$. Let $a_{jq}\neq 0$ and $j,l\in \mathrm{Y}$ such that j,l are distinct. We may assume that $d=[z_{-1},x_lx_jD_l]$. A direct calculation shows that

$$d=\left[\sum_{i=1}^{n}\sum_{q=1}^{p^m} a_{iq}y_qD_i,\, x_lx_jD_l\right]=\sum_{q=1}^{p^m}(a_{lq}x_jy_qD_l-a_{jq}x_ly_qD_l).$$

By (3-1) and (3-2), we have

$$\begin{aligned}(\rho(d))^t &= (-1)^t(a_{j1})^tE_{ll}+(-1)^{t-1}a_{l1}(a_{j1})^{t-1}E_{jl}\\ &\quad+(-1)^t(a_{(j+n)2})^tE_{(l+n)(l+n)}+(-1)^{t-1}a_{(l+n)2}(a_{(j+n)1})^{t-1}E_{(j+n)(l+n)}\\ &\quad+\cdots\\ &\quad+(-1)^t(a_{(j+p^m-n)p^m})^tE_{(l+p^m-n)(l+p^m-n)}\\ &\quad+(-1)^{t-1}a_{(l+p^m-n)p^m}(a_{(j+p^m-n)p^m})^{t-1}E_{(j+p^m-n)(l+p^m-n)}.\end{aligned}$$

Since $(a_{j1})^t\neq 0$, we have $(\rho(d))^t\neq 0$. So $\rho(d)$ is not a nilpotent matrix. By Lemma 3.1, it shows that $d\notin \mathrm{nil}(W)$. By Lemma 3.2, we have $[z,x_lx_jD_l]\notin\mathrm{nil}(W)$. Then $z\notin\Delta$. This contradicts $z\in\Delta$, and proves our assertion that $z_{-1}=0$.

Assume that $z_0\neq 0$. Let $z_0=\sum_{i,j=1}^{n}\sum_{q=1}^{p^m} a_{ijq}x_iy_qD_j$, $a_{ijq}\in\mathbb{F}$ and

(3-3) $$l=\min\{i\mid a_{ij\lambda}\neq 0,\ i,j\in \mathrm{Y}\},$$

(3-4) $$t=\min\{j\mid a_{ij\lambda}\neq 0,\ i,j\in \mathrm{Y}\}.$$

(i) Suppose that $l\leq t$. Let

(3-5) $$k=\max\{j\mid a_{lj\lambda}\neq 0,\ j\in \mathrm{Y}\}.$$

Then $a_{lkq}\neq 0$. It is easy to see that $t\leq k$. Since $l\leq t$, we have $l\leq k$. Therefore,

$$z_0=\sum_{j=t}^{k}\sum_{q=1}^{p^m} a_{ljq}x_ly_qD_j+\sum_{i=l+1}^{n}\sum_{j=t}^{n}\sum_{q=1}^{p^m} a_{ijq}x_iy_qD_j.$$

Assume that $l=k$. It follows from $t\leq k$ that $t\leq l$. Then $t=l$, which implies that

$$z_0=\sum_{q=1}^{p^m} a_{llq}x_ly_qD_l+\sum_{i=l+1}^{n}\sum_{j=t}^{n}\sum_{q=1}^{p^m} a_{ijq}x_iy_qD_j.$$

Therefore,

$$\begin{aligned}\rho(z_0) &= a_{ll1}E_{ll} + \sum_{i=l+1}^{n}\sum_{j=t}^{n} a_{ij1}E_{ij}\\ &\quad + a_{(l+n)(l+n)2}E_{(l+n)(l+n)} + \sum_{i=l+1+n}^{2n}\sum_{j=t+n}^{2n} a_{ij2}E_{ij}\\ &\quad + \cdots\\ &\quad + a_{(l+n(p^m-1))(l+n(p^m-1))p^m}E_{(l+n)(l+n)}\\ &\quad + \sum_{i=l+1+n(p^m-1)}^{np^m}\sum_{j=t+n(p^m-1)}^{np^m} a_{ijp^m}E_{ij}\\ &= \begin{pmatrix} \begin{smallmatrix} A_1 & \\ B_1 & C_1 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} A_{p^m} & \\ B_{p^m} & C_{p^m} \end{smallmatrix} \end{pmatrix}_{np^m\times np^m},\end{aligned}$$

where $A_k = a_{(l+(k-1)n)(l+(k-1)n)q}E_{(l+(k-1)n)(l+(k-1)n)}$ is an $(l+(k-1)n)$-by-$(l+(k-1)n)$ matrix and $q \in \{1, \dots, p^m\}$. Since $a_{ll1} \neq 0$, we have A_1 is not a nilpotent matrix. Then $\rho(z_0)$ is not a nilpotent matrix and $z_0 \notin \mathrm{nil}(W)$. Lemma 3.2 shows that $z \notin \mathrm{nil}(W)$. This is in contradiction with $z \in \Delta$; thus $l < k$.

Suppose that $h \in \mathrm{Y}$ and $h \neq l, k$. Let $d = [z_0, x_k D_l]$. From (2-1), we obtain

$$d = \sum_{q=1}^{p^m}\Big(a_{lkq}x_l y_q D_l + \sum_{i=l+1}^{n} a_{ikq}x_i y_q D_l - \sum_{j=t}^{k} a_{ljq}x_k y_q D_j\Big).$$

Since $l < k$, $\rho(d)$ also has the form

$$\begin{pmatrix} \begin{smallmatrix} A_1 & \\ B_1 & C_1 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} A_{p^m} & \\ B_{p^m} & C_{p^m} \end{smallmatrix} \end{pmatrix}_{np^m\times np^m}.$$

It follows from $a_{lkq} \neq 0$ that A_1 is not a nilpotent matrix. Then $\rho(d)$ is not nilpotent. So $z \notin \mathrm{nil}(W)$ and $[z, x_k D_l] \notin \mathrm{nil}(W)$. This is in contradiction with $z \in \Delta$.

(ii) Suppose that $t < l$. Let $k = \max\{i \mid a_{it\lambda} \neq 0\}$ and $d' = [z, x_t D_k]$. Imitating (i), we may prove that $\rho(d')$ is also not nilpotent. The desired result follows. □

Lemma 3.4. (i) *If $z \in W_0 \cap \mathrm{nil}(W)$ and $h \in W_{(1)}$, then $z + h \in \mathrm{nil}(W)$.*

(ii) *If i, j are distinct elements of Y, then $x_i y^\lambda D_j \in \mathrm{nil}(W)$ for all $\lambda \in G$.*

(iii) *If i, j, k are distinct elements of* Υ, *then* $ax_j y^\lambda D_k + bx_i y^\eta D_k \in \mathrm{nil}(W)$ *and* $x_i x_j y^\lambda D_k \in \Delta$, *where* $a, b \in \mathbb{F}$ *and* λ, η *are arbitrary elements of* G.

Proof. (i) A direct verification shows that $\{\mathrm{ad}\, z\} \cup \{\mathrm{ad}\, W_{(1)}\}$ is a weakly closed subset of nilpotent elements of $\mathrm{pl}(W)$. It was shown in [Jacobson 1962, Theorem 1 of Chapter II] that each element of $\mathrm{span}_{\mathbb{F}}(\{\mathrm{ad}\, z\} \cup \{\mathrm{ad}\, W_{(1)}\})$ is a nilpotent linear transformation of W. Then $\mathrm{ad}\, z + \mathrm{ad}\, h$ is nilpotent. So $z + h$ is ad-nilpotent.

(ii) To prove $(\mathrm{ad}\, x_i y^\lambda D_j)^p = 0$, we may assume without loss of generality that $i < j$. Set η is an arbitrary element of G. If $k \neq i$, then

$$\begin{aligned}(\mathrm{ad}\, x_i y^\lambda D_j)^2 (x^u y^\eta D_k) &= [x_i y^\lambda D_j, [x_i y^\lambda D_j, x^u y^\eta D_k]] \\ &= (-1)^{\tau(u,j)} [x_i y^\lambda D_j, x_i x^{u-\langle j \rangle} y^{\lambda+\eta} D_k] \\ &= 0.\end{aligned}$$

In the case of $k = i$, we have

$$\begin{aligned}(\mathrm{ad} x_i y^\lambda D_j)^3 (x^u y^\eta D_k) &= [x_i y^\lambda D_j, [x_i y^\lambda D_j, [x_i y^\lambda D_j, x^u y^\eta D_i]]] \\ &= [x_i y^\lambda D_j, [x_i y^\lambda D_j, (-1)^{\tau(u,j)} x_i x^{u-\langle j \rangle} y^\lambda D_i - x^u y^{\lambda+\eta} D_j]] \\ &= (-1)^{\tau(u,j)} [x_i y^\lambda D_j, -x_i x^{u-\langle j \rangle} y^\lambda D_j - x_i x^{u-\langle j \rangle} y^{2\lambda+\eta} D_j] \\ &= 0.\end{aligned}$$

For $p > 2$ we get $(\mathrm{ad}\, x_i y^\lambda D_j)^p (x^u y^\eta D_k) = 0$. Therefore $(\mathrm{ad}\, x_i y^\lambda D_j)^p (W) = 0$. This yields $(\mathrm{ad}\, x_i y^\lambda D_j)^p = 0$. Thus $x_i y^\lambda D_j \in \mathrm{nil}(W)$.

(iii) According to (ii) and $[x_j y^\lambda D_k, x_i y^\eta D_k] = 0$, $\{\mathrm{ad}\, x_j y^\lambda D_k, \mathrm{ad}\, x_i y^\eta D_k\}$ is a weakly closed subset of nilpotent elements of pl(W). So $ax_j y^\lambda D_k + bx_i y^\eta D_k \in \mathrm{nil}(W)$, where $a, b \in \mathbb{F}$.

Suppose that $l \in \Upsilon \setminus \{i, j, k\}$. Then $x_i x_j y^\lambda D_k \in W_{(1)} \subseteq \mathrm{nil}(W)$. Let $z = \sum_{i=-1}^{n-1} z_i$, where $z_i \in W_i$. Without loss, we may assume that $[x_i x_j y^\lambda D_k, z] = f_0 + f_1$, where $f_0 = [x_i x_j y^\lambda D_k, z_{-1}] \in W_0$ and $f_1 \in W_{(1)}$. Let $z_{-1} = \sum_{l=1}^n \sum_{\eta \in G} a_{l\eta} y^\eta D_l$. Then

$$f_0 = [x_i x_j y^\lambda D_k, \sum_{l=1}^n \sum_{\eta \in G} a_{l\eta} y^\eta D_l] = -\sum_{\eta \in G} (a_{i\eta} x_j y^{\lambda+\eta} D_k + a_{j\eta} x_i y^{\lambda+\eta} D_k).$$

It follows that $f_0 \in W_0 \cap \mathrm{nil}(W)$. Statement (i) shows that $f_0 + f_1 \in \mathrm{nil}(W)$. We finally obtain $x_i x_j y^\lambda D_k \in \Delta$ for all $\lambda \in G$. □

Let $Q = \{z \in \mathrm{nil}(W) \mid \mathrm{ad}\, z(\Delta) \subseteq \Delta\}$.

Lemma 3.5. $$Q = W_{(1)}.$$

Proof. By the definition of Δ, we have $W_{(2)} \subseteq \Delta$. Lemma 3.3 show that $\Delta \subseteq W_{(1)}$. Then $[W_{(1)}, \Delta] \subseteq [W_{(1)}, W_{(1)}] \subseteq W_{(2)} \subseteq \Delta$. Thus $W_{(1)} \subseteq Q$.

Next we will prove $Q \subseteq W_{(1)}$. Let $z \in Q$ and $z = \sum_{i=-1}^{n-1} z_i$, where $z_i \in W_i$. Assume that $z_{-1} = \sum_{l=1}^{n} \sum_{\lambda \in G} a_{l\lambda} y^{\lambda} D_l \neq 0$, $a_{l\lambda} \in \mathbb{F}$. Without loss of generality, we may suppose that $a_i \neq 0$. Let $d = x_i x_j y^{\eta} D_k$, where i, j, k are distinct elements of Y and η is an arbitrary element of G. By Lemma 3.4 (iii), we have $d \in \Delta$. Let $[z, d] = h_0 + h_1$, where $h_0 = [z_{-1}, d] \in W_0$ and $h_1 \in W_{(1)}$. Since $a_i \neq 0$, we have $h_0 = \sum_{\lambda \in G} (a_{i\lambda} x_j y^{\lambda+\eta} D_k - a_{j\lambda} x_i y^{\lambda+\eta} D_k) \neq 0$. Lemma 3.3 implies that $h_0 + h_1 \notin \Delta$. It is a contradiction to $z \in Q$. Hence $z_{-1} = 0$.

Assume that $0 \neq z_0 = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ij\lambda} x_i y_q D_j$, $a_{ijq} \in \mathbb{F}$ and suppose that l and t are as the definitions in (3-3) and (3-4). We may suppose that $l \leq t$ (the proof is similar to the case $t < l$) and let k be as definition in (3-5). Similar to the first part of the proof in Lemma 3.3, we have $l < k$. Suppose that $h \in \mathrm{Y} \setminus \{l, k, t\}$ and $d_1 = x_k x_h D_l$. Lemma 3.4 (iii) shows that $d_1 \in \Delta$. Let $[z, d_1] = g_1 + g_2$, where $g_1 = [z_0, d_1] \in W_1$ and $g_2 \in W_{(2)}$. Using (2-1), we have

$$g_1 = \sum_{q=1}^{p^m} \left(a_{lkq} x_l x_h y_q D_l - \sum_{i=l+1}^{n} a_{ihq} x_i x_k y_q D_l - \sum_{j=t}^{k} a_{ljq} x_k x_h y_q D_j \right).$$

If $h < t$, then $a_{ihq} = 0$ in the above equality, where $i \in \mathrm{Y} \setminus \{1, \ldots, l-1\}$. Thus

$$[D_h, g_1] = -\sum_{q=1}^{p^m} \left(a_{lkq} x_l y_q D_l + \sum_{i=l+1}^{n} a_{ihq} x_i y_q D_l + a_{hhq} x_k y_q D_l - a_{ljq} x_k y_q D_j \right).$$

By (3-2), the matrix $\rho([D_h, g_1])$ has the form

$$\begin{pmatrix} \begin{matrix} A_1 & \\ B_1 & C_1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} A_{p^m} & \\ B_{p^m} & C_{p^m} \end{matrix} \end{pmatrix}_{np^m \times np^m}$$

as in Lemma 3.3. Since $a_{lkq} \neq 0$, A_1 is not a nilpotent matrix. This implies that $\rho([D_h, g_1])$ is not nilpotent. Hence $[D_h, g_1] \notin \mathrm{nil}(W)$. Lemma 3.2 shows that $[D_h, g_1 + g_2] \notin \mathrm{nil}(W)$, that is, $[D_h, g_1 + g_2] \notin \Delta$. It is contradict with $z \in Q$. Thus $z_0 = 0$. Therefore, $z \in W_{(1)}$ and $Q \subseteq W_{(1)}$. □

It is easy to verify that Δ and Q are invariant subspaces under the automorphisms of W. According to Lemma 3.5, $W_{(1)}$ is also invariant under the automorphisms of W. Since

$$W_{(0)} = \{x \in W \mid [x, W_{(1)}] \subseteq W_{(1)}\}, \tag{3-6}$$

$$W_{(i)} = \{x \in W_{i-1} \mid [x, W] \subseteq W_{(i-1)}\}, \quad i \geq 1, \tag{3-7}$$

we easily obtain the following theorem.

Theorem 3.6. *The natural filtration of W is invariant under automorphisms of W.*

Let $\mathfrak{W}_i = W_{(i)}/W_{(i+1)}$ for $-1 \le i \le n-1$. Then $\mathfrak{W}_i$ is a $\mathbb{Z}$-graded space. Suppose that $\mathfrak{W} := \bigoplus_{i=-1}^{n-1} \mathfrak{W}_i$; then $\mathfrak{W}$ is also a $\mathbb{Z}$-graded space. Let $x + W_{(i+1)} \in \mathfrak{W}_i$ and $y + W_{(j+1)} \in \mathfrak{W}_j$. Define

$$[x + W_{(i+1)}, y + W_{(j+1)}] := [x, y] + W_{(i+j+1)}.$$

It follows from $[\mathfrak{W}_i, \mathfrak{W}_j] \subseteq \mathfrak{W}_{i+j}$ that the operator [,] on $\mathfrak{W}$ is well-defined. There exists a linear expansion such that $\mathfrak{W}$ has a operator [,]. A direct verification shows that $\mathfrak{W}$ is a Lie superalgebra with respect to the operator [,]. The Lie superalgebras $\mathfrak{W}$ is called a Lie superalgebra induced by the natural filtration of W.

Lemma 3.7. $$\mathfrak{W} \cong W.$$

Proof. Let $\phi\colon W \to \mathfrak{W}$ is a linear map such that $\phi(x) = x + W_{(i+1)}$, where $x \in W_{(i)} \setminus W_{(i+1)}$. A direct verification shows that ϕ is a homomorphism of Lie superalgebras. Suppose that $y \in \ker\phi$. If $y \neq 0$, then there exists $i \geq -1$ such that $y \in W_{(i)} \setminus W_{(i+1)}$. Since $\phi(y) = 0$, we have $y + W_{(i+1)} = 0$. Hence $y \in W_{(i+1)}$. That shows that $y = 0$. Thus, $\ker\phi = 0$. Therefore, ϕ is a monomorphism. It follows from W is finite-dimensional that ϕ is an isomorphism. □

The definition of ϕ shows that, for $i \geq -1$

$$\phi(W_i) = \{x + W_{(i+1)} \mid x \in W_i\} = \{x + W_{(i+1)} \mid x \in W_{(i)}\} = W_{(i)}/W_{(i+1)} = \mathfrak{W}_i. \tag{3-8}$$

Suppose that m, n, m', n' are elements of $\mathbb{N}$ greater than 1. Similar to W, the Lie superalgebra $W(n', m')$ will be simply denoted by W'. According to the definitions of Δ, Q, and $\mathfrak{W}$ in W, we define Δ', Q', and $\mathfrak{W}'$ in W' using the same method.

Proposition 3.8. *Suppose that $W \cong W'$ and σ is an isomorphism from W to W'. Then $\sigma(W_{(i)}) = W'_{(i)}$ for all $i \geq -1$.*

Proof. It is clear that $\sigma(W_{(-1)}) = W'_{(-1)}$ and $\sigma(\mathrm{nil}(W)) = \mathrm{nil}(W')$. A direct verification shows that $\sigma(\Delta) = \Delta'$. Hence $\sigma(Q) = Q'$. By virtue of Lemma 3.5, we have $Q = W_{(1)}$ and $Q' = W'_{(1)}$. Thus $\sigma(W_{(1)}) = W'_{(1)}$. By (3-6) and (3-7), the desired result $\sigma(W_{(i)}) = W'_{(i)}$ for all $i \geq -1$ is obtained. □

Lemma 3.9. *Suppose that $W \cong W'$ and σ is an isomorphism from W to W'. Then σ induces an isomorphism $\tilde{\sigma}$ from $\mathfrak{W}$ to $\mathfrak{W}'$ such that $\tilde{\sigma}(\mathfrak{W}_i) = \mathfrak{W}'_i$ for all $i \geq -1$.*

Proof. Define a linear map $\tilde{\sigma} : \mathfrak{W} \to \mathfrak{W}'$ such that

$$\tilde{\sigma}(x + W_{(i+1)}) = \sigma(x) + W'_{(i+1)},$$

where $x + W_{(i+1)} \in \mathfrak{W}_i$. Because of Proposition 3.8, the definition of $\tilde{\sigma}$ is reasonable and

$$\begin{aligned}\tilde{\sigma}([x+W_{(i+1)}, y+W_{(j+1)}]) &= \sigma([x,y]) + W'_{(i+j+1)} \\ &= [\sigma(x)+W'_{(i+1)}, \sigma(y)+W'_{(j+1)}] \\ &= [\tilde{\sigma}(x+W'_{(i+1)}), \tilde{\sigma}(y+W'_{(j+1)})].\end{aligned}$$

Thus $\tilde{\sigma}$ is a homomorphism from $\mathfrak{W}$ to $\mathfrak{W}'$. Clearly, $\tilde{\sigma}(\mathfrak{W}_i) = \mathfrak{W}'_i$ for all $i \geq -1$. It shows that $\tilde{\sigma}$ is an epimorphism.

Suppose that $y \in \ker\tilde{\sigma}$; then $y \in \mathfrak{W}$. So we may suppose that $y = \sum_{i=-1}^{n-1} y_i$ and $y_i \in \mathfrak{W}_i$. Since $\mathfrak{W}_i = W_{(i)}/W_{(i+1)}$, let $y_i = z_i + W_{(i+1)}$, where $z_i \in W_{(i)}$. Hence $\tilde{\sigma}(y_i) = \sigma(z_i) + W'_{(i+1)}$. It follows from $\tilde{\sigma}(y) = 0$ that $\sum_{i=-1}^{n-1}\tilde{\sigma}(y_i) = 0$. Thus $\tilde{\sigma}(y_i) = 0$, that is, $\sigma(z_i) + W'_{(i+1)} = 0$. This shows $\sigma(z_i) \in W'_{(i+1)}$. By Proposition 3.8, we have $z_i \in \sigma^{-1}(W'_{(i+1)}) = W_{(i+1)}$. Then $y_i = z_i + W_{(i+1)} = 0$ for $-1 \leq i \leq n-1$. Therefore, $y = 0$ and $\ker\tilde{\sigma} = 0$. Consequently, $\tilde{\sigma}$ is an isomorphism induced by σ such that $\tilde{\sigma}(\mathfrak{W}_i) = \mathfrak{W}'_i$ for all $i \geq -1$. □

Theorem 3.10. *$W \cong W'$ if and only if $m = m'$ and $n = n'$.*

Proof. Since the sufficiency is obvious, it suffices to prove the necessity. Suppose that $\phi\colon W \to \mathfrak{W}$ is the isomorphism given in the proof of Lemma 3.7. Similarly, there also exists the $\phi'\colon W' \to \mathfrak{W}'$. According to (3-8) and Lemma 3.9, we have

$$\phi(W_i) = \mathfrak{W}_i, \quad \phi'(W'_i) = \mathfrak{W}'_i, \quad \tilde{\sigma}(\mathfrak{W}_i) = \mathfrak{W}'_i$$

for $-1 \leq i \leq n-1$. Let $\psi = (\phi')^{-1}\tilde{\sigma}\phi$. Then

$$\psi(W_i) = (\phi')^{-1}\tilde{\sigma}\phi(W_i) = (\phi')^{-1}\tilde{\sigma}(\mathfrak{W}_i) = (\phi')^{-1}(\mathfrak{W}'_i) = W_i.$$

In particular, $\psi(W_{-1}) = W'_{-1}$ and $\psi(W_0) = W'_0$. Since $\dim W_{-1} = \dim W'_{-1}$, we get $np^m = n'p^{m'}$. By virtue of the definition of W_i, we have

$$W_0 = \mathrm{span}_{\mathbb{F}}\{x_i y^{\lambda} D_j \in W \mid i, j \in \mathrm{Y}, \lambda \in G\}.$$

Thus $\dim W_0 = n^2 p^m$. By the same method used in W_0, we may obtain $\dim W'_0 = n'^2 p^{m'}$. According to $\dim W_0 = \dim W'_0$ and $np^m = n'p^{m'}$, we have $n = n'$ and $m = m'$. In conclusion, the proof is completed. □

4. The natural filtration of $H(n, m)$

In this section we will investigate the question of the natural filtration of the Lie superalgebras $H(n,m)$. For convenience, $H(n,m)$, $\overline{H}(n,m)$ and $H_i(n,m)$ will be simply denoted by H, $\overline{H}$ and H_i.

Let $H_{(j)} = \bigoplus_{i \geq j} H_i$. Then

$$H = H_{(-1)} \supseteq H_{(0)} \supseteq H_{(1)} \supseteq \cdots \supseteq H_{(n-3)} \supseteq H_{(n-2)} = 0$$

is a descending filtration of H, which is called the natural filtration of H. We also call $\{H_{(k)} \mid k \in \mathbb{Z}\}$ a filtration of H for short, where $H_{(k)} = H$ if $k \leq -1$ and $H_{(k)} = 0$ if $k \geq n-2$.

Lemma 4.1. *Let $f_i = g_i + h_i$, where $f_i, g_i, h_i \in \mathcal{U}$ and $i = 1, \dots, k$. If the set $\{g_i \mid i = 1, \dots, k\}$ is linearly independent and*

$$\operatorname{span}_{\mathbb{F}}\{g_i \mid i = 1, \dots, k\} \cap \operatorname{span}_{\mathbb{F}}\{h_i \mid i = 1, \dots, k\} = 0,$$

then $\{f_i \mid i = 1, \dots, k\}$ is linearly independent.

Proof. If $\sum\limits_{i=1}^{k} a_i f_i = 0$, $a_i \in \mathbb{F}$, then $\sum\limits_{i=1}^{k} a_i g_i = -\sum\limits_{i=1}^{k} a_i h_i$. This shows that

$$\sum_{i=1}^{k} a_i g_i \in \operatorname{span}_{\mathbb{F}}\{g_i \mid i = 1, \dots, k\} \cap \operatorname{span}_{\mathbb{F}}\{h_i \mid i = 1, \dots, k\} = 0.$$

Since $\{g_i \mid i = 1, \dots, k\}$ is linearly independent, we obtain $a_i = 0$, $i = 1, \dots, k$. □

Lemma 4.2. *If $h_1, h_2, \dots, h_k \in H \setminus \{0\}$. If $\{h_i \mid i = 1, \dots, k\}$ is linearly dependent, then so is $\{\mu(h_i) \mid i = 1, \dots, k\}$.*

Proof. Since $\{h_i \mid i = 1, \dots, k\}$ is linearly dependent, there exist $a_1, \dots, a_k \in \mathbb{F}$ such that $\sum_{i=1}^{k} a_i h_i = 0$ and some a_i is not zero. We may suppose that $a_1, \dots, a_s \neq 0$ and $a_{s+1} = \cdots = a_k = 0$, where $1 \leq s \leq k$. Let

$$\varepsilon = \min\{\operatorname{zd}(\mu(h_i)) \mid i = 1, \dots, s\}.$$

Without loss of generality, we may suppose that $\operatorname{zd}(\mu(h_i)) = \varepsilon$ for $i = 1, \dots, t$ and $\operatorname{zd}(\mu(h_i)) > \varepsilon$ for $i = t+1, \dots, s$. It follows from $\sum_{i=1}^{k} a_i h_i = 0$ that $\sum_{i=1}^{k} a_i \mu(h_i) = 0$. Since $a_1, \dots, a_t \neq 0$, we obtain that $\{\mu(h_i) \mid i = 1, \dots, t\}$ is linearly dependent. Hence so is $\{\mu(h_i) \mid i = 1, \dots, k\}$. □

Lemma 4.3. *Let $g_1, g_2, \dots, g_k \in \mathcal{U}$. If $\operatorname{zd}(\mu(g_i)) \geq 1$, $i = 1, \dots, k$, then $\{g_i \mid i = 1, \dots, k\}$ is linearly dependent if and only if $\{D_H(g_i) \mid i = 1, \dots, k\}$ is.*

Proof. If $\{g_i \mid i = 1, \dots, k\}$ is linearly dependent, there exist $a_1, \dots, a_k \in \mathbb{F}$, not all zero, such that $\sum_{i=1}^{k} a_i g_i = 0$. Clearly, $D_H\big(\sum_{i=1}^{k} a_i g_i\big) = \sum_{i=1}^{k} a_i D_H(g_i) = 0$. Hence $\{D_H(g_i) \mid i = 1, \dots, k\}$ is linearly dependent.

Conversely, we consider the sufficiency. Without loss of generality, we may suppose that $g = x^u y^\lambda$ for $u \in \mathbb{B}(n)$ and $\lambda \in G$ such that $D_H(g) = 0$. Then

$$D_H(x^u y^\lambda) = \sum_{i=1}^{n} (-1)^{|u|} D_i(x^u) y^\lambda D_i.$$

Hence $D_i(x^u) = 0$, which shows that $|u| = 0$. Thus $\ker(D_H) = \mathbb{T}(m)$. Since the set $\{D_H(g_i) \mid i = 1, \dots, k\}$ is linearly dependent, there exist $a_1, \dots, a_k \in \mathbb{F}$, not all zero, such that $\sum_{i=1}^{k} a_i D_H(g_i) = 0$. Then $D_H\big(\sum_{i=1}^{k} a_i g_i\big) = 0$. Hence $\sum_{i=1}^{k} a_i g_i \in \mathbb{T}(m)$. Notice that $\mathrm{zd}(\mu(g_i)) \geq 1$, $i = 1, \ldots, k$; thus $\sum_{i=1}^{k} a_i g_i = 0$, showing that $\{g_i \mid i = 1, \ldots, k\}$ is linearly dependent. □

For a superderivation D of a Lie superalgebra L. Set $I(D) = \dim(\mathrm{Im}(D))$. If T is a subset of superderivations of L, we define $I(T) = \min\{I(D) \mid 0 \neq D \in T\}$.

Theorem 4.4. *Suppose that* $T = \mathrm{ad}(\mathrm{hg}(\overline{H}))|_H$, *then* $I(T) \geq np^m$. *Besides,* $I(\mathrm{ad}\, D_H(g)) = np^m$ *if and only if* $0 \neq D_H(g) \in \mathrm{span}_{\mathbb{F}}\{D_H(x^\pi y^\lambda) \mid \lambda \in G\}$, *where* $D_H(g) \in \mathrm{hg}(\overline{H})$.

Proof. For any $h \in \mathrm{hg}(\overline{H})$ we write $\mathrm{ad}\, h|_H$ simply as $\mathrm{ad}\, h$. A direct calculation shows that

$$[D_H(x^\pi y^\lambda), D_H(x^v y^\eta)] = D_H\left(\sum_{i=1}^{n} (-1)^n D_i(x^\pi y^\lambda) D_i(x^v y^\eta)\right)$$

for $v \in \mathbb{B}(n)$ and $\lambda, \eta \in G$.

In the case of $|v| \geq 2$ we have

$$D_i(x^v y^\eta) = (-1)^{\tau(v,i)} x^{v-\langle i\rangle} y^\eta, \quad D_i(x^\pi y^\lambda) = (-1)^{\tau(\pi,i)} x^{\pi-\langle i\rangle} y^\lambda.$$

Clearly, $\{v - \langle i\rangle\} \in \{\pi - \langle i\rangle\}$. Then $[D_H(x^\pi y^\lambda), D_H(x^v y^\eta)] = 0$ in this case.

In the case of $|v| = 1$ we may suppose that $x^v y^\eta = x_i y^\eta$ for some $i \in \mathrm{Y}$. Then

$$\begin{aligned}[D_H(x^\pi y^\lambda), D_H(x_i y^\eta)] &= D_H\left(\sum_{j=1}^{n} (-1)^n D_j(x^\pi y^\lambda) D_j(x_i y^\eta)\right)\\ &= D_H\big((-1)^{n+\tau(\pi,i)} x^{\pi-\langle i\rangle} y^{\lambda+\eta}\big).\end{aligned}$$

Since $\{x^{\pi-\langle i\rangle} y^{\lambda+\eta} \mid i \in \mathrm{Y}, \lambda, \eta \in G\}$ is a linearly independent set, Lemma 4.3 shows that $\{[D_H(x^\pi y^\lambda), D_H(x_i y^\eta)] \mid i \in \mathrm{Y}, \lambda, \eta \in G\}$ is linearly independent. Thus $I(\mathrm{ad}\, D_H(g)) = np^m$.

Next we will consider the converse inclusion. Assume that $D_H(g_0) \in \mathrm{hg}(\overline{H})$ and $D_H(g_0) \notin \mathrm{span}_{\mathbb{F}}\{D_H(x^\pi y^\lambda) \mid \lambda \in G\}$. We want to prove that $I(\mathrm{ad}\, D_H(g_0)) > np^m$. Suppose that $\mu(D_H(g_0)) = D_H(g)$. By Lemma 4.2, it suffices to prove that $I(\mathrm{ad}\, D_H(g)) > np^m$.

Let $g = x^u y^\lambda$, where $u \in \mathbb{B}(n)$ and $\lambda \in G$. Then $1 \leq |u| < n$. There exist $v \in \mathbb{B}(n)$ and $\eta \in G$ such that $D_H(x^v y^\eta) \in H$. Then

$$[D_H(x^u y^\lambda), D_H(x^v y^\eta)] = D_H\left(\sum_{i=1}^{n} (-1)^{|u|} D_i(x^u y^\lambda) D_i(x^v y^\eta)\right).$$

(1) Suppose that $|u| = 1$ and $x^u y^\lambda = x_i y^\lambda$ for some $i \in \mathrm{Y}$, then

$$[D_H(x_i y^\lambda), D_H(x^v y^\eta)] = -(-1)^{\tau(v,i)} D_H\left(x^{v-\langle i\rangle} y^{\lambda+\eta}\right).$$

Considering $x^{v-\langle i\rangle}$, the following statements hold:

$$\begin{cases} D_H(x^{v-\langle i\rangle}y^{\lambda+\eta})=0 & \text{if } |v|=0 \text{ or } 1,\\ \dim\{x^{v-\langle i\rangle}\}=\mathrm{C}^1_{n-1} & \text{if } |v|=2,\\ \dim\{x^{v-\langle i\rangle}\}=\mathrm{C}^2_{n-1} & \text{if } |v|=3,\\ \qquad\vdots & \\ \dim\{x^{v-\langle i\rangle}\}=\mathrm{C}^{n-1}_{n-1} & \text{if } |v|=n.\end{cases}$$

Therefore,

$$\dim\{x^{v-\langle i\rangle}y^{\lambda+\eta}\}=(\mathrm{C}^1_{n-1}+\mathrm{C}^2_{n-1}+\cdots+\mathrm{C}^{n-1}_{n-1})p^m=(2^{n-1}-1)p^m>np^m.$$

(2) If $1<|u|<n$, then we suppose that $|u|=l$.

For $|v|=2$ we may suppose that $x^vy^\eta=x_jx_ky^\eta$, where j,k are distinct elements of Υ and η is an arbitrary element of G. A direct calculation shows that $[D_H(x^uy^\lambda),D_H(x_jx_ky^\eta)]$ equals

$$D_H\left((-1)^{|u|}((-1)^{\tau(u,j)}x^{u-\langle j\rangle}x_k-(-1)^{\tau(u,k)}x^{u-\langle k\rangle}x_j)y^{\lambda+\eta}\right).$$

Consider $\Upsilon=(-1)^{\tau(u,j)}x^{u-\langle j\rangle}x_k-(-1)^{\tau(u,k)}x^{u-\langle k\rangle}x_j$. Then the following statements hold:

$$\begin{cases} \Upsilon=0 & \text{if } j,k\in u \text{ or } j,k\notin u,\\ \Upsilon=(-1)^{\tau(u,j)}x^{(u\setminus\{j\})\cup\{k\}} & \text{if } j\in u \text{ and } k\notin u,\\ \Upsilon=(-1)^{\tau(u,k)}x^{(u\setminus\{k\})\cup\{j\}} & \text{if } k\in u \text{ and } j\notin u.\end{cases}$$

Thus $\dim\Upsilon=l(n-l)$.

For $|v|=1$ we may suppose that $x^vy^\eta=x_iy^\eta$ for some $i\in\Upsilon$. Then

$$[D_H(x_iy^\lambda),D_H(x_iy^\eta)]=(-1)^{|u|+\tau(u,i)}D_H(x^{u-\langle i\rangle}y^{\lambda+\eta}),\quad i\in\Upsilon.$$

Hence $\dim(x^{u-\langle i\rangle})=|u|=l$. It is clear to see that $l(n-l)+l>n$. Therefore, $I(\mathrm{ad}\,D_H(g))\geq(l(n-l)+l)p^m>np^m$. □

Theorem 4.5. $I(\mathrm{hg}(\mathrm{Der}\,H))=np^m$. *Moreover, for* $D\in\mathrm{hg}(\mathrm{Der}(H))$, *we have* $I(D)=np^m$ *if and only if D is nonzero and lies in* $\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}D_H(x^\pi y^\lambda)\mid\lambda\in G\}$.

Proof. By virtue of Theorem 4.4, we have $I(\mathrm{ad}\,D_H(x^\pi y^\lambda))=np^m$. By [Ren et al. 2011, Proposition 3.7], we obtain

$$\mathrm{Der}\,H=\mathrm{ad}(\overline{H}+\mathbb{F}y^\lambda h)\oplus\Omega,$$

where $h=\sum_{i=1}^n x_iD_i$ and $\lambda\in G$. Hence $I(\mathrm{hg}(\mathrm{Der}\,H))\leq np^m$. Let $D\in\mathrm{hg}(\mathrm{Der}\,H)$ and $I(D)\leq np^m$. Without loss of generality, we may suppose that

$$D=\mathrm{ad}\,D_H(g)+a\,\mathrm{ad}\,y^\lambda h+\sum_{\theta\in\Theta}b_\theta D_\theta,$$

where $a, b_\theta \in \mathbb{F}$ and $D_H(g) \in \mathrm{hg}(\overline{H})$. Then

$$\begin{aligned} &D(D_H(x^u y^\eta)) \\ &\quad = [D_H(g), D_H(x^u y^\eta)] + a\Big[\sum_{i=1}^{n} x_i y^\lambda D_i, D_H(x^u y^\eta)\Big] + \sum_{\theta\in\Theta} b_\theta D_\theta(D_H(x^u y^\eta)) \end{aligned}$$

for all $u \in \mathbb{B}(n)$ and $\eta \in G$.

Next we will prove that a and b_θ are all zero for all $\theta \in \Theta$.

First of all we consider the coefficient a. A direct calculation shows that

$$\begin{aligned} a\Big[\sum_{i=1}^{n} x_i y^\lambda D_i, D_H(x^u y^\eta)\Big] &= \sum_{i,j=1}^{n} (-1)^{|u|} a[x_i y^\lambda D_i, D_j(x^u) y^\eta D_j] \\ &= \sum_{i,j=1}^{n} (-1)^{|u|} a[x_i y^\lambda D_i, (-1)^{\tau(u,j)} x^{u-\langle j\rangle} y^\eta D_j] \\ &= \begin{cases} -\displaystyle\sum_{j=1}^{n} (-1)^{|u|+\tau(u,j)} a x^{u-\langle j\rangle} y^{\lambda+\eta} D_j & \text{if } i = j, \\ (n-1)\displaystyle\sum_{j=1}^{n} (-1)^{|u|+\tau(u,j)} a x^{u-\langle j\rangle} y^{\lambda+\eta} D_j & \text{if } i \neq j. \end{cases} \end{aligned}$$

Using the similar discussion in Theorem 4.4, we obtain

$$\dim\big(\mathrm{span}_{\mathbb{F}}\{x^{u-\langle j\rangle} y^{\lambda+\eta} D_j\}\big) > np^m$$

for given $j \in u$. Since $n > 1$, we have $a = 0$.

Secondly, the other coefficient b_θ will be considered. For any $u \in \mathbb{B}(n)$ and $\eta \in G$, we have

$$\begin{aligned} b_\theta D_\theta(D_H(x^u y^\eta)) &= \tilde{\theta}(\eta) D_H(b_\theta x^u y^\eta) \\ &= \sum_{i=1}^{n} (-1)^{|u|} b_\theta D_i(x^u) D_i \tilde{\theta}(\eta) y^\eta \\ &= b_\theta \sum_{i=1}^{n} (-1)^{|u|+\tau(u,i)} x^{u-\langle i\rangle} D_i \tilde{\theta}(\eta) y^\eta. \end{aligned}$$

By the equality above and the similar discussion in Theorem 4.4, we have

$$\dim(\mathrm{span}_{\mathbb{F}}\{x^{u-\langle i\rangle} \tilde{\eta}(\mu) y^\eta D_j\}) > np^m.$$

Hence $b_\theta = 0$ for all $\theta \in \Theta$. Therefore, $D = \mathrm{ad}\, D_H(g)$. It follows from Theorem 4.4 that $I(\mathrm{hg}(\mathrm{Der}\, H)) = np^m$. In particular, $I(D) = np^m$ if and only if

$$0 \neq D \in \mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_H(x^\pi y^\lambda) \mid \lambda \in G\}. \qquad \square$$

We adopt the notations n', m' in Section 3 and let $H' = H(n', m')$ and $G' = \left\{\sum_{i=1}^{m'} \lambda_i z_i \mid \lambda_i \in \Pi, i = 1, \ldots, m'\right\}$.

Proposition 4.6. *Let*

$$R = \mathrm{span}_{\mathbb{F}}\{D_H(x^u y^\lambda) \in H \mid u \in \mathbb{B}(n), |u| \geq 2, \lambda \in G\},$$
$$R' = \mathrm{span}_{\mathbb{F}}\{D_{H'}(x^u y^\lambda) \in H' \mid u \in \mathbb{B}(n'), |u| \geq 2, \lambda \in G'\}.$$

If σ is an isomorphism from H to H', then $\sigma(R) = R'$.

Proof. It is easy to see that the map $\xi\colon D \to \sigma D \sigma^{-1}$ is a bijection. Then ξ is an isomorphism from $\mathrm{Der}\, H$ to $\mathrm{Der}\, H'$. Thus $I(\mathrm{hg}(\mathrm{Der}\, H)) = I(\mathrm{hg}(\mathrm{Der}\, H'))$. According to Theorem 4.5, we have

$$\sigma(\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_H(x^\pi y^\lambda)\})\sigma^{-1} = \mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_H(x^{\pi'} y^{\lambda'})\},$$

where $\pi' = \{1, \ldots, n'\} \in \mathbb{B}(n')$, $\lambda \in G$, and $\lambda' \in G'$. Note that

$$[D_H(x^\pi y^\lambda), D_H(x^u y^\eta)] = D_H\left(\sum_{i=1}^{n} (-1)^n D_i(x^\pi y^\lambda) D_i(x^u y^\eta)\right).$$

for $u \in \mathbb{B}(n)$ and $\lambda, \eta \in G$. If $|u| \geq 2$, then $D_i(x^u y^\eta) = (-1)^{\tau(u,i)} x^{u-\langle i\rangle} y^\eta$ and $D_i(x^\pi y^\lambda) = (-1)^{\tau(\pi,i)} x^{\pi-\langle i\rangle} y^\lambda$. Since $\{u - \langle i\rangle\} \in \{\pi - \langle i\rangle\}$, we have

$$[D_H(x^\pi y^\lambda), D_H(x^u y^\eta)] = 0.$$

Hence

$$R = \{h \in H \mid (\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_H(x^\pi y^\lambda)\})(h) = 0\}.$$

Similarly, $R' = \{h \in H' \mid (\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_{H'}(x^{\pi'} y^{\lambda'})\})(h) = 0\}$. Then

$$\begin{aligned}
(\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_{H'}(x^{\pi'} y^{\lambda'})\})(\sigma(R)) &= \sigma(\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_H(x^\pi y^\lambda)\})\sigma^{-1}(\sigma(R)) \\
&= \sigma(\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_H(x^\pi y^\lambda)\})(R) \\
&= \sigma(\mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_H(x^\pi y^\lambda)\})(R) \\
&= \sigma(0) \\
&= 0.
\end{aligned}$$

Thus $\sigma(R) \subseteq R'$. By the same method above, we have $\sigma^{-1}(R') \subseteq R$. Hence $R' \subseteq \sigma(R)$. In conclusion, $\sigma(R) = R'$. $\square$

Lemma 4.7. *Let $H = H_{(-1)} \supseteq H_{(0)} \supseteq \cdots \supseteq H_{(n-3)} \supseteq H_{(n-2)} = 0$ be the natural filtration of H. Then*

$$H_{(0)} = R, \quad H_{(i)} = \{h \in H_{(i-1)} \mid [h, H] \subseteq H_{(i-1)}\} \text{ for } i \geq 1.$$

Similarly, for the natural filtration of H',

$$H'_{(0)} = R', \quad H'_{(i)} = \{h \in H'_{(i-1)} \mid [h, H'] \subseteq H'_{(i-1)}\} \text{ for } i \geq 1.$$

Proof. Suppose that $M = \{h \in H_{(i-1)} \mid [h, H] \subseteq H_{(i-1)}\}$. Note that $H_{(i)} \subseteq H_{(i-1)}$ and $[H_{(i)}, H] = [H_{(i)}, H_{(-1)}] \subseteq H_{(i-1)}$. Then the inclusion relations show that $H_{(i)} \subseteq M$.

Conversely, if $h \in M$, then $h \in H_{(i-1)}$. So we may suppose that $h = \sum_{j=i-1}^{n-3} h_j$, where $h_j \in H_j$. Let $h_{i-1} = \sum_k a_k D(x^{u_k} y^{\lambda_k})$, where $a_k \in \mathbb{F}$, $u_k \in \mathbb{B}(n)$, $|u_k| = i - 1 + 2 = i + 1 \geq 2$, and $\lambda_k \in G$.

If $h_{i-1} = 0$, then $h \in H_{(i)}$. Therefore, the desired result follows in this case.

If $h_{i-1} \neq 0$, then it follows from $h \in M$ that $[h, H_{-1}] \subseteq H_{(i-1)}$. Hence $[h_{i-1}, H_{-1}] = 0$, that is,

$$\left[\sum_k a_k D(x^{u_k} y^{\lambda_k}), D_H(x_i y^{\eta})\right] = 0$$

for all $i \in \mathrm{Y}$ and $\eta \in G$. As $|u_k| \geq 2$, there exists $i \in \mathrm{Y}$ such that

$$D_H((-1)^{|u_k|} D_i(x^{u_k} y^{\lambda_k})) \neq 0.$$

Hence $a_k = 0$ which is in contradiction with $h_{i-1} \neq 0$.

The considerations above show that $M \subseteq H_{(i)}$. Therefore,

$$H_{(i)} = \{h \in H_{(i-1)} \mid [h, H] \subseteq H_{(i-1)}\} \quad \text{for } i \geq 1.$$

Similarly, $H'_{(i)} = \{h \in H'_{(i-1)} \mid [h, H'] \subseteq H'_{(i-1)}\}$ for $i \geq 1$. □

Proposition 4.8. *Suppose that $H \cong H'$ and σ is an isomorphism from H to H', then $\sigma(H_{(i)}) = H'_{(i)}$ for all $i \geq -1$.*

Proof. If $i = 0$, then $H_{(0)} = R$ and $H'_{(0)} = R'$. Proposition 4.6 shows that $\sigma(H_{(0)}) = H'_{(0)}$.

If $i = -1$, then $H_{(-1)} = H$ and $H'_{(-1)} = H'$. Hence $\sigma(H_{(-1)}) = H'_{(-1)}$.

Next we use induction on i. Assume that $\sigma(H_{(i)}) = H'_{(i)}$ for $i \geq 1$. By Lemma 4.7, for $h \in H_{(i+1)}$, we have $h \in H_{(i)}$ as well as $[h, H] \subseteq H_{(i)}$. Since $h \in H_{(i)}$, the induction hypothesis yields $\sigma(h) \in H'_{(i)}$. Then

$$\sigma([h, H]) = [\sigma(h), \sigma(H)] \subseteq [H'_{(i)}, H'] \subseteq H'_{(i)}.$$

By Lemma 4.7, we have $\sigma(h) \in H'_{(i+1)}$. This implies that $\sigma(H_{(i+1)}) \subseteq H'_{(i+1)}$.

For any $h' \in H'_{(i+1)}$, we want to prove that $h' \in \sigma(H_{(i+1)})$. The fact $h' \in H'_{(i)} = \sigma(H_{(i)})$ ensures that there exists $h \in H_{(i)}$ such that $\sigma(h) = h'$. It is easy to see that $[h', H'] \subseteq H'_{(i)} = \sigma(H_{(i)})$. Since $[h', H'] = [\sigma(h), \sigma(H)] = \sigma[h, H]$, we have $[h, H] \in H_{(i)}$. Then $h \in H_{(i+1)}$, that is, $h' \in \sigma(H_{(i+1)})$. Consequently, $\sigma(H_{(i)}) = H'_{(i)}$ for all $i \geq -1$. □

Theorem 4.9. *The natural filtration of H is invariant under the automorphisms of H.*

Proof. It is a direct conclusion of Proposition 4.8. □

Imitating the definition of $\mathfrak{W}_i$ in W, we let $\mathfrak{H}_i = H_{(i)}/H_{(i+1)}$ for $-1 \leq i \leq n-3$. Suppose that $\mathfrak{H} := \bigoplus_{i=-1}^{n-3} \mathfrak{H}_i$, then $\mathfrak{H}$ is a $\mathbb{Z}$-graded space. Let $x + H_{(i+1)} \in \mathfrak{H}_i$ and $y + H_{(j+1)} \in \mathfrak{H}_j$. We define

$$[x + H_{(i+1)}, y + H_{(j+1)}] := [x, y] + H_{(i+j+1)}.$$

It is easy to see that the operator [,] on $\mathfrak{H}$ is well-defined. There exists a linear expansion such that $\mathfrak{H}$ has a operator [,]. A direct verification shows that $\mathfrak{H}$ is a Lie superalgebra with respect to the operator [,]. The Lie superalgebras $\mathfrak{H}$ is called a Lie superalgebra induced by the natural filtration of H.

By the similar methods used to prove Propositions 3.7 and 3.9, the following lemmas are easy to obtain.

Lemma 4.10. $\mathfrak{H} \cong H$.

Lemma 4.11. *Suppose that $H \cong H'$ and σ is an isomorphism from H to H', then σ induces an isomorphism $\tilde{\sigma}$ from $\mathfrak{H}$ to $\mathfrak{H}'$ such that $\tilde{\sigma}(\mathfrak{H}_i) = \mathfrak{H}'_i$ for all $i \geq -1$.*

Theorem 4.12. *$H \cong H'$ if and only if $m = m'$ and $n = n'$.*

Proof. Since the sufficiency is obvious, it suffices to prove the necessity. Using the similar methods in the proof of Theorem 3.10, we have $\dim H_{-1} = \dim H'_{-1}$ and $\dim H_0 = \dim H'_0$. It follows from $W_{-1} = H_{-1}$ that $np^m = n'p^{m'}$. By virtue of the definition of H_i, we have

$$H_0 = \operatorname{span}_{\mathbb{F}}\{D_H(x_i x_j y^\lambda) \in H \mid i, j \in \mathrm{Y}, \lambda \in G\}.$$

Thus $\dim H_0 = \mathrm{C}_n^2 p^m = \frac{1}{2}n(n-1)p^m$. Similarly, $\dim H'_0 = \frac{1}{2}n'(n'-1)p^{m'}$. According to $\dim H_0 = \dim H'_0$ and $np^m = n'p^{m'}$, we have $n = n'$ and $m = m'$. Consequently, the desired result follows. □

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FREE BROWNIAN MOTION AND FREE CONVOLUTION SEMIGROUPS: MULTIPLICATIVE CASE

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We consider a pair of probability measures μ, ν on the unit circle such that $\Sigma_\lambda(\eta_\nu(z)) = z/\eta_\mu(z)$. We prove that the same type of equation holds for any $t \geq 0$ when we replace ν by $\nu \boxtimes \lambda_t$ and μ by $\mathbb{M}_t(\mu)$, where λ_t is the free multiplicative analogue of the normal distribution on the unit circle of $\mathbb{C}$ and $\mathbb{M}_t$ is the map defined by Arizmendi and Hasebe. These equations are a multiplicative analogue of equations studied by Belinschi and Nica. In order to achieve this result, we study infinite divisibility of the measures associated with subordination functions in multiplicative free Brownian motion and multiplicative free convolution semigroups. We use the modified $\mathcal{S}$-transform introduced by Raj Rao and Speicher to deal with the case that ν has mean zero. The same type of the result holds for convolutions on the positive real line. In the end, we give a new proof for some Biane's results on the densities of the free multiplicative analogue of the normal distributions.

1. Introduction

Let $\mathcal{M}_\mathbb{R}$ be the set of probability measures on $\mathbb{R}$. For every $t \geq 0$, Belinschi and Nica [2008b] defined a family of maps $\mathbb{B}_t : \mathcal{M}_\mathbb{R} \to \mathcal{M}_\mathbb{R}$ by setting

$$\mathbb{B}_t(\mu) = (\mu^{\boxplus(t+1)})^{\uplus 1/(t+1)}, \quad \mu \in \mathcal{M}_\mathbb{R}.$$

These maps have several remarkable properties. For any $t \geq 0$, $\mathbb{B}_t$ is an endomorphism of $(\mathcal{M}_{\mathbb{R}^+}, \boxtimes)$, where $\mathcal{M}_{\mathbb{R}^+}$ is the set of probability measures on $[0, +\infty)$ and $\boxtimes$ is free multiplicative convolution. $\{\mathbb{B}_t\}_{t\geq 0}$ is a semigroup and $\mathbb{B}_1$ is the Boolean to free Bercovici–Pata bijective map.

The maps $\mathbb{B}_t$ have strong connections with $\boxplus$-infinite divisibility. They are also connected to free Brownian motion and additive free convolution semigroups. For $\mu \in \mathcal{M}_\mathbb{R}$, we denote by G_μ the Cauchy transform of μ and by F_μ the reciprocal Cauchy transform of μ. Given a pair of probability measures $\mu, \nu \in \mathcal{M}_\mathbb{R}$ such that

$$G_\nu(z) = z - F_\mu(z), \quad z \in \mathbb{C}^+,$$

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we have

$$G_{\nu\boxplus\gamma_t}(z) = z - F_{\mathbb{B}_t(\mu)}(z), \quad t > 0,\ z \in \mathbb{C}^+, \tag{1-1}$$

where γ_t is the semicircular distribution with variance t. This result was generalized to the multi-variable case in [Belinschi and Nica 2008a; 2009; Nica 2009]. An equivalent form of (1-1) was used to prove the superconvergence theorem in [Wang 2010]. Anshelevich [2010; 2011a; 2011b; 2012] generalized the above correspondence of $\mu \leftrightarrow \nu$ and $\mathbb{B}_t(\mu) \leftrightarrow \nu \boxplus \gamma_t$ to the context of two-state probability spaces. Motivated by these generalizations and applications, we study in this article the analogue of these equations for multiplicative free convolution.

Throughout this article, we denote by T the unit circle of $\mathbb{C}$, by $\mathcal{M}_{\mathrm{T}}$ the set of probability measures on T, and by $\mathcal{M}_*$ the set of probability measures on $\mathbb{C}$ with nonzero mean. We also set

$$\mathcal{M}_{\mathrm{T}}^* = \{\mu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_* : \eta_\mu(z) \neq 0 \text{ for all } z \in \mathbb{D}\backslash\{0\}\}.$$

It was shown in [Belinschi and Bercovici 2005] that one can define multiplicative free convolution power $\mu^{\boxtimes t}$ for $\mu \in \mathcal{M}_{\mathrm{T}}^*$ and $t > 1$.

In [Arizmendi and Hasebe 2013], a family of maps $\mathbb{M}_t$, which is the analogue of the semigroup $\mathbb{B}_t$, was defined for the probability measures in $\mathcal{M}_{\mathrm{T}}^*$. The definition of $\mathbb{M}_t$ there was more general; we only need a simpler form defined as follows. Given $\mu \in \mathcal{M}_{\mathrm{T}}^*$ having positive mean, then for $t \geq 0$, the map $\mathbb{M}_t$ is defined by

$$\mathbb{M}_t(\mu) = (\mu^{\boxtimes(t+1)})^{\uplus 1/(t+1)},$$

where the convolution power $\mu^{\boxtimes(t+1)}$ and the measure $\mathbb{M}_t(\mu)$ are chosen in a way such that they have positive means.

We then state one of our main theorems.

Theorem 1.1. *Given a pair of probability measures $\mu \in \mathcal{M}_{\mathrm{T}}^*$ and $\nu \in \mathcal{M}_{\mathrm{T}}$ such that*

$$\Sigma_\lambda(\eta_\nu(z)) = \frac{z}{\eta_\mu(z)}, \quad z \in \mathbb{D}, \tag{1-2}$$

we have

$$\Sigma_\lambda(\eta_{\nu\boxtimes\lambda_t}(z)) = \frac{z}{\eta_{\mathbb{M}_t(\mu)}(z)}, \quad z \in \mathbb{D}, \tag{1-3}$$

where λ_t is the analogue of the normal distribution on T with $\Sigma_{\lambda_t}(z) = \exp((t/2)(1+z)/(1-z))$ and $\lambda = \lambda_1$.

In order to prove Theorem 1.1, we consider two semigroups $\nu \boxtimes \lambda_t$ and $\mu^{\boxtimes(t+1)}$ for all $t \geq 0$. It is well-known that $\eta_{\nu\boxtimes\lambda_t}$ and $\eta_{\mu^{\boxtimes(t+1)}}$ are subordinated to η_ν and η_μ respectively. We prove that the subordination functions are η-transforms of some $\boxtimes$-infinitely divisible probability measures on T. It turns out that the equation

$\Sigma_\lambda(\eta_\nu(z)) = z/\eta_\mu(z)$ means that the subordination function of $\eta_{\nu\boxtimes\lambda_t}$ with respect to η_ν and the subordination function of $\eta_{\mu^{\boxtimes(t+1)}}$ with respect to η_μ are the same. The proof of Theorem 1.1 will be given in Subsection 3.5.

Given $\mu \in \mathcal{M}_{\mathbb{T}}$, we prove that if $\mu^{\boxtimes t}$ can be defined and $\eta_{\mu^{\boxtimes t}}$ is subordinated to η_μ for all $t > 1$, then $\mu \in \mathcal{M}_{\mathbb{T}}^*$; in addition, we prove that for nontrivial measures $\mu \in \mathcal{M}_{\mathbb{T}}$ and $\nu \in \mathcal{I}\mathcal{D}(\boxtimes, \mathbb{T})$, the density functions of the measures $\mu \boxtimes \nu_t$ and $\mu^{\boxtimes t}$ converge to $1/2\pi$ uniformly as $t \to \infty$.

To deal with the case that $\nu \in \mathcal{M}_{\mathbb{T}} \backslash \mathcal{M}_*$, we use the modified $\mathcal{S}$-transform [Arizmendi 2012; Rao and Speicher 2007] to study subordination functions. In this case, the subordination function of $\eta_{\nu\boxtimes\lambda_t}$ with respect to η_ν is generally not unique. However, we can prove that there exists a unique subordination function satisfying certain properties (see Theorem 3.11). Let ρ_t be the measure associated with this subordination function of $\eta_{\nu\boxtimes\lambda_t}$ with respect to η_ν, we have that $\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\nu(z))$.

Similar results to Theorem 1.1 for multiplicative convolution on $\mathcal{M}_{\mathbb{R}^+}$ are also valid. The proof for this case is much simpler because of the uniqueness of multiplicative convolution powers and the uniqueness of subordination functions.

Finally, we give a new proof for some results concerning the density functions of the free multiplicative analogue of the normal distributions studied by Biane [1997c], and we obtain some new results. For example, for λ_t ($t > 0$) the free multiplicative analogue of the normal distributions on $\mathbb{T}$, we prove that λ_t is unimodal.

This article is organized as follows. After this introductory section, we describe some backgrounds in the additive case in Section 2. In Section 3, we consider multiplicative free and multiplicative Boolean convolution on $\mathcal{M}_{\mathbb{T}}$, and prove our main theorems. Section 4 is devoted to studying multiplicative free and multiplicative Boolean convolution on $\mathcal{M}_{\mathbb{R}^+}$. The regularity properties of the free multiplicative analogue of the normal distributions are discussed in Section 5.

2. Background: additive case

Additive free convolution and additive Boolean convolution. For a measure $\mu \in \mathcal{M}_{\mathbb{R}}$, we define the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ by

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{1}{z-t}\, d\mu(t), \quad z \in \mathbb{C}^+.$$

We set

$$F_\mu(z) = \frac{1}{g\mu(z)}, \quad z \in \mathbb{C}^+,$$

so that $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ is analytic.

The following result characterizes those functions which are reciprocal Cauchy transforms of probability measures.

Proposition 2.1 [Bercovici and Voiculescu 1993]. *Let $F : \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic function. The following assertions are equivalent*:

(1) *There exists a probability measure μ on $\mathbb{R}$ such that $F(z) = F_\mu(z)$ in $\mathbb{C}^+$.*

(2) *There exists $a \in \mathbb{R}$, and a finite positive measure ρ on $\mathbb{R}$ such that*

$$F(z) = a + z + \int_{-\infty}^{+\infty} \frac{1+tz}{t-z}\, d\rho(t) \quad \text{for all } z \in \mathbb{C}^+.$$

(3) $\lim_{y\to+\infty} F(iy)/iy = 1$.

It was proved in [Bercovici and Voiculescu 1993] that F_μ is invertible in some domain. More precisely, for two positive numbers M and N, we set

$$\Gamma_{M,N} = \{z \in \mathbb{C}^+ : |x| < My,\ y > N\}.$$

Then for any $M > 0$, there exists $N > 0$ such that the left inverse F_μ^{-1} of F_μ is defined in $\Gamma_{M,N}$, and then we can define the Voiculescu transform of μ by

$$\varphi_\mu(z) = F_\mu^{-1}(z) - z,$$

for $z \in \Gamma_{M,N}$. For any two measures $\mu, \nu \in \mathcal{M}_\mathbb{R}$, we have

$$\varphi_{\mu\boxplus\nu}(z) = \varphi_\mu(z) + \varphi_\nu(z) \tag{2-1}$$

in any truncated cone $\Gamma_{M,N}$ where φ_μ, φ_ν and $\varphi_{\mu\boxplus\nu}$ are defined. This remarkable result was proved in [Voiculescu 1986] for compactly supported measures and then extended to general cases in [Bercovici and Voiculescu 1993; Maassen 1992].

Given $\nu \in \mathcal{M}_\mathbb{R}$, we say that ν is $\boxplus$-infinitely divisible if for every positive integer n, there exists a probability measure $\nu_{1/n} \in \mathcal{M}_\mathbb{R}$ such that

$$\nu = \underbrace{\nu_{1/n} \boxplus \nu_{1/n} \boxplus \cdots \boxplus \nu_{1/n}}_{n \text{ times}}.$$

It is known [Bercovici and Voiculescu 1993; Maassen 1992; Voiculescu 1986] that a probability measure ν on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if its Voiculescu transform φ_ν has an analytic extension defined on $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$. We denote by $\mathcal{ID}(\boxplus, \mathbb{R})$ the set of all $\boxplus$-infinitely divisible probability measures on the real line. If $\nu \in \mathcal{ID}(\boxplus, \mathbb{R})$, then for every $t > 0$, there exists a probability measure ν_t such that $\varphi_{\nu_t}(z) = t\varphi_\nu(z)$ for z in the common domain of φ_ν and φ_{ν_t}.

Proposition 2.2. *Let ν is $\boxplus$-infinitely divisible, and let $H(z) = z + \varphi_\nu(z)$. Then*

$$H(F_\nu(z)) = z \tag{2-2}$$

for $z \in \mathbb{C}^+$. The set $U := \{z \in \mathbb{C}^+ : \Im H(z) > 0\}$ is a simply connected domain with boundary which is a simple curve and H maps $\mathbb{C}^+$ conformally onto U. Moreover, *the boundary ∂U is the graph of a function and the function H is continuous up to the real axis.*

Proof. The first part of the assertion appears in [Bercovici and Voiculescu 1993; Voiculescu 1986], and the second part of the assertion follows from the fact that H satisfies the conditions of [Belinschi and Bercovici 2005, Proposition 4.7]. The last part of the assertion is due to [Chistyakov and Götze 2013, Lemma 3.3; Belinschi and Bercovici 2005, Proposition 4.7]. □

Additive Boolean convolution was introduced in [Speicher and Woroudi 1997]. For $\mu \in \mathcal{M}_{\mathbb{R}}$, we set $E_\mu(z) = z - F_\mu(z)$. For $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, the additive Boolean convolution $\mu \uplus \nu$ is characterized by the identity

$$E_\rho(z) = E_\mu(z) + E_\nu(z) \quad \text{for } z \in \mathbb{C}^+.$$

We can also consider the infinite divisibility with respect to additive Boolean convolution. It turns out that every $\mu \in \mathcal{M}_{\mathbb{R}}$ is $\uplus$-infinitely divisible; see [Speicher and Woroudi 1997]. We denote by $\mathcal{ID}(\uplus, \mathbb{R})$ the set of all $\uplus$-infinitely divisible probability measures on the real line.

Infinite divisibility and subordination functions. Given $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, it is known that $F_{\mu \boxplus \nu}$ is subordinated to F_μ and F_ν, and by Proposition 2.1, we can also regard these subordination functions as the reciprocal Cauchy transforms of probability measures on $\mathbb{R}$.

Definition 2.3. For $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, the subordination distribution [Anshelevich 2012; Lenczewski 2007; Nica 2009] $\mu \boxminus \nu$ is defined to be the unique probability measure in $\mathcal{M}_{\mathbb{R}}$ such that $F_{\mu \boxplus \nu}(z) = F_\nu(F_{\mu \boxminus \nu}(z))$.

Many subordination distributions in semigroups related to free convolution are infinitely divisible; see [Anshelevich 2012; Nica 2009].

Proposition 2.4. *Let $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$.*

(1) $\varphi_{\mu \boxminus \nu}(z) = (\varphi_\mu \circ F_\nu)(z)$.

(2) *If $\mu \in \mathcal{ID}(\boxplus, \mathbb{R})$, then $\mu \boxminus \nu \in \mathcal{ID}(\boxplus, \mathbb{R})$. In particular, $\gamma_t \boxminus \nu \in \mathcal{ID}(\boxplus, \mathbb{R})$ and $\varphi_{\gamma_t \boxminus \nu}(z) = tG_\nu(z)$, where γ_t is the semicircular distribution with variance t.*

(3) *If $\nu = \mu \boxplus \nu'$ for $\nu' \in \mathcal{M}_{\mathbb{R}}$, then $\mu \boxminus \nu \in \mathcal{ID}(\boxplus, \mathbb{R})$. In particular, $\mu \boxminus \mu \in \mathcal{ID}(\boxplus, \mathbb{R})$, and $\varphi_{\mu \boxminus \mu}(z) = z - F_\mu(z)$.*

Proof. Part (1) is Lemma 1 of [Anshelevich 2012]. Note that $\varphi_{\gamma_t}(z) = t/z$ and $(\varphi_\mu \circ F_\mu)(z) = z - F_\mu(z)$; parts (2) and (3) follow from part 1 and Lemma 2 of [Anshelevich 2012]. See also [Chistyakov and Götze 2011, Corollary 2.3]. □

The following result was inspired by a question of Michael Anshelevich [2012], to whom I am grateful for sending me an updated version of his paper.

Lemma 2.5. *Given $\tau, \rho \in \mathcal{M}_{\mathbb{R}}$, if $\tau \boxplus \rho \in \mathcal{ID}(\boxplus, \mathbb{R})$, then $\rho \boxplus \tau^{\boxplus t}$ is defined for all $t \geq 0$ in the sense that $\varphi_\rho + t\varphi_\tau$ is the Voiculescu transform of a positive measure. Moreover, $F_{\rho \boxplus (\tau^{\boxplus t})} = F_\rho(F_{(\tau \boxplus \rho)^{\boxplus t}}(z))$.*

Proof. Let $\sigma = \tau \boxplus \rho$ and $\sigma_t = \sigma^{\boxplus t}$. By Proposition 2.1, there exists a unique probability measure $\mu_t \in \mathcal{M}_{\mathbb{R}}$ such that

$$F_{\mu_t} = F_\rho(F_{\sigma_t}(z)).$$

We claim that $\varphi_{\mu_t}(z) = \varphi_\rho(z) + t\varphi_\tau(z)$. Indeed, by Proposition 2.4, we have

$$F_{\sigma_t}^{-1}(z) - z = t \cdot \varphi_\sigma(z) = t \cdot \varphi_\tau(F_\rho(z)),$$

and we thus obtain

$$\begin{aligned}\varphi_{\mu_t}(F_\rho(z)) &= F_{\mu_t}^{-1}(F_\rho(z)) - F_\rho(z) = F_{\sigma_t}^{-1}(z) - F_\rho(z) = F_{\sigma_t}^{-1}(z) - z + z - F_\rho(z) \\ &= t \cdot \varphi_\tau(F_\rho(z)) + F_\rho^{-1}(F_\rho(z)) - F_\rho(z).\end{aligned}$$

By analytic continuation, we conclude that

$$\varphi_{\mu_t}(z) = t \cdot \varphi_\tau(z) + F_\rho^{-1}(z) - z = \varphi_\rho(z) + t \cdot \varphi_\tau(z),$$

which completes the proof. □

Remark. There are examples $\rho, \tau \in \mathcal{M}_{\mathbb{R}}$ such that $\tau \boxplus \rho \in \mathcal{ID}(\boxplus, \mathbb{R})$ but τ does not lie in $\mathcal{ID}(\boxplus, \mathbb{R})$ and is not a summand of ρ; see [Anshelevich 2012].

Combining Proposition 2.4 and Lemma 2.5, we can reconstruct Nica–Speicher free convolution semigroups [Belinschi and Bercovici 2004; Nica and Speicher 1996] as follows.

Theorem 2.6. *Given $\mu \in \mathcal{M}_{\mathbb{R}}$, the measure $\mu^{\boxplus t} \in \mathcal{M}_{\mathbb{R}}$ is defined by $\varphi_{\mu^{\boxplus t}}(z) = t\varphi_\mu(z)$ for all $t > 1$. Moreover, there exists an analytic map $\omega_t : \mathbb{C}^+ \to \mathbb{C}^+$ such that for $z \in \mathbb{C}^+$ the following conditions are satisfied:*

- $F_{\mu^{\boxplus t}}(z) = F_\mu(\omega_t(z))$.
- $\omega_t = F_{(\mu \boxplus \mu)^{\boxplus (t-1)}}(z)$.
- $\varphi_{(\mu \boxplus \mu)^{\boxplus (t-1)}} = (t-1)(z - F_\mu(z))$ *for all* $t > 1$.

Let $H_t(z) = z + (t-1)(z - F_\mu(z))$. By Proposition 2.2 and Theorem 2.6, we know that H_t is the left inverse of ω_t such that $H_t(\omega_t(z)) = z$ for $z \in \mathbb{C}^+$. Therefore, for $t > 1$, we can write

$$\omega_t(z) = z + \left(1 - \frac{1}{t}\right)(F_{\mu^{\boxplus t}}(z) - z), \quad z \in \mathbb{C}^+. \tag{2-3}$$

We deduce from (2-3) and the definition of ω_t in Theorem 2.6 that, for $t > 0$,

$$z - F_{(\mu \boxplus \mu)^{\boxplus t}}(z) = \left(1 - \frac{1}{t+1}\right)(z - F_{\mu^{\boxplus (t+1)}}(z)),$$

which implies that

$$(\mu \boxplus \mu)^{\boxplus t}(z) = (\mu^{\boxplus(t+1)})^{\uplus(t/(t+1))}. \tag{2-4}$$

Two formulas related to free Brownian motion. Given $\mu \in \mathcal{M}_{\mathbb{R}}$, we construct subordination functions ω_t as in Theorem 2.6. Let $\sigma_t = (\mu \boxplus \mu)^{\boxplus t} \in \mathcal{M}_{\mathbb{R}}$; then $\omega_{t+1} = F_{\sigma_t}(z)$ for $t > 0$. Given $\nu \in \mathcal{M}_{\mathbb{R}}$, let $\rho_t = \gamma_t \boxplus \nu$ and let $F_t = F_{\rho_t}(z)$ for all $t > 0$. From Proposition 2.4 and Theorem 2.6, we know that ρ_t and σ_t are $\boxplus$-infinitely divisible and their Voiculescu transforms are given by $\varphi_{\rho_t}(z) = tG_\nu$ and $\varphi_{\sigma_t}(z) = t(z - F_\mu(z))$. By comparing Voiculescu transform of ρ_t with Voiculescu transform of σ_t, we deduce that $F_t = \omega_{t+1}$ for some $t > 0$ if and only if $G_\nu(z) = z - F_\mu(z)$.

For $t > 0$, Belinschi and Nica [2008b] constructed a transformation $\mathbb{B}_t : \mathcal{M}_{\mathbb{R}} \to \mathcal{M}_{\mathbb{R}}$ such that

$$\mathbb{B}_t(\mu) = (\mu^{\boxplus(1+t)})^{\uplus 1/(1+t)} \quad \text{for } \mu \in \mathcal{M}_{\mathbb{R}}.$$

They also showed that $\mathbb{B}_t$ is a semigroup and $\mathbb{B}_2 = \mathbb{B}$, where

$$\mathbb{B} : \mathcal{ID}(\uplus, \mathbb{R}) \to \mathcal{ID}(\boxplus, \mathbb{R})$$

is the bijective map from the $\uplus$-infinitely divisible distributions to the $\boxplus$-infinitely divisible distributions, discovered in the seminal [Bercovici and Pata 1999].

Theorem 2.7 [Belinschi and Nica 2008b]. *Let μ and ν be a pair of probability measures on the real line such that*

$$G_\nu(z) = z - F_\mu(z), \quad z \in \mathbb{C}^+. \tag{2-5}$$

Then we have

$$G_{\nu \boxplus \gamma_t}(z) = z - F_{\mathbb{B}_t(\mu)}(z), \quad t > 0,\ z \in \mathbb{C}^+.$$

Remark. Maassen [1992] has shown that, given $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ satisfying (2-5), μ has mean zero and variance one. Conversely, if $\mu \in \mathcal{M}_{\mathbb{R}}$ has mean zero and variance one, then there exists a unique $\nu \in \mathcal{M}_{\mathbb{R}}$ satisfying (2-5).

Given $\tau \in \mathcal{ID}(\boxplus, \mathbb{R})$ and $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, we compare free Lévy process $\nu \boxplus \tau^{\boxplus t}$ and free convolution semigroup $\mu^{\boxplus(t+1)}$. If $\varphi_\tau(F_\nu(z)) = z - F_\mu(z)$, then $\tau \boxplus \nu = \mu \boxplus \mu$, which implies that subordination function of $F_{\nu \boxplus (\tau^{\boxplus t})}$ to F_ν is the same as the subordination function of $F_{\mu^{\boxplus(t+1)}}$ to F_μ. The following theorem generalizes Theorem 2.7. The argument is similar to the proof of Theorem 1.6 in [Belinschi and Nica 2008b] (see also the proof of Lemma 3 in [Anshelevich 2012]); therefore we omit the proof.

Theorem 2.8. *Given $\tau \in \mathcal{ID}(\boxplus, \mathbb{R})$, and let μ and ν be a pair of probability measures on the real line such that*

$$\varphi_\tau(F_\nu(z)) = z - F_\mu(z), \quad z \in \mathbb{C}^+.$$

Then we have

$$\varphi_\tau(F_{\nu\boxplus(\tau^{\boxplus t})}(z)) = z - F_{\mathbb{B}_t(\mu)}(z), \quad t > 0,\ z \in \mathbb{C}^+.$$

Remark. Let $\tau = \gamma_{a,b}$ be the semicircular distribution with mean a and variance b, and let μ, ν be a pair of probability measures on the real line such that

$$\varphi_\tau(F_\nu(z)) = z - F_\mu(z).$$

We first compute

$$F_\mu(z) = z - \varphi_\tau(F_\nu(z)) = z - \left(a + \frac{b}{z}\right) \circ F_\nu(z) = z - a - bG_\nu(z). \tag{2-6}$$

Then, by Theorem 2.8,

$$\begin{aligned} F_{\mathbb{B}_t(\mu)}(z) &= z - \varphi_\tau(F_{\nu\boxplus\tau^{\boxplus t}}(z)) \\ &= z - \left(\left(a + \frac{b}{z}\right) \circ F_{\nu\boxplus\gamma_{a,b}^{\boxplus t}}\right)(z) = z - a - bG_{\nu\boxplus\gamma_{a,b}^{\boxplus t}}(z). \end{aligned} \tag{2-7}$$

By (2-7) and the definition of Boolean convolution, we obtain

$$F_{(\mathbb{B}_t(\mu))^{\uplus t}}(z) = z - ta - tbG_{\nu\boxplus\gamma_{a,b}^{\boxplus t}}(z). \tag{2-8}$$

Equations (2-6), (2-7) and (2-8) were studied in [Anshelevich 2012]. As shown there (Proposition 1 and Example 1), we have $(\mathbb{B}_t(\mu))^{\uplus t} \in \mathcal{ID}(\boxplus, \mathbb{R})$, and $(\mathbb{B}_t(\mu))^{\uplus t} = (\tau \boxplus \nu)^{\boxplus t} = (\mu \boxplus \mu)^{\boxplus t}$. In fact, for all $\mu \in \mathcal{M}_\mathbb{R}$, we can deduce from (2-4) and the identity $(\mathbb{B}_t(\mu))^{\uplus t} = (\mu^{\boxplus(1+t)})^{\uplus t/(1+t)}$ that $(\mathbb{B}_t(\mu))^{\uplus t}$ is the measure associated with the subordination function of $\mu^{\boxplus(1+t)}$ with respect to μ: $(\mathbb{B}_t(\mu))^{\uplus t} = (\mu \boxplus \mu)^{\boxplus t}$.

3. Multiplicative free convolution and multiplicative Boolean convolution on $\mathcal{M}_\mathrm{T}$

Given any two probability measures μ, ν on T, the unit circle of $\mathbb{C}$, we can define their multiplicative free convolution. We first recall the calculation of the multiplicative free convolution of two measures on T with nonzero means. Given $\mu \in \mathcal{M}_\mathrm{T}$, we define

$$\psi_\mu(z) = \int_\mathrm{T} \frac{tz}{1 - tz}\, d\mu(t)$$

and set $\eta_\mu(z) = \psi_\mu(z)/(1 + \psi_\mu(z))$. The following proposition characterizes the η-transforms of probability measures on T.

Proposition 3.1 [Belinschi and Bercovici 2005]. *If $\eta : \mathbb{D} \to \mathbb{C}$ is an analytic function, the following assertions are equivalent.*

(1) *There exists a probability measure $\mu \in \mathcal{M}_\mathrm{T}$ such that $\eta = \eta_\mu$.*

(2) $\eta(0) = 0$, *and* $|\eta(z)| < 1$ *holds for all* $z \in \mathbb{D}$.

If $\mu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, then $\eta'_\mu(0) = \int_{\mathrm{T}} t \, d\mu(t) \neq 0$. Therefore, the inverse η_μ^{-1} is defined in a neighborhood of zero. We set $\Sigma_\mu(z) = \eta_\mu^{-1}(z)/z$. Given $\mu, \nu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, their multiplicative free convolution, which is denoted by $\mu \boxtimes \nu$, is the unique probability measure in $\mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$ such that

$$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z) \tag{3-1}$$

for z in a neighborhood of zero.

It is known (see [Biane 1998; Belinschi and Bercovici 2007]) that there exist two analytic functions $\omega_1, \omega_2 : \mathbb{D} \to \mathbb{D}$ such that

(1) $\omega_1(0) = \omega_2(0) = 0$,

(2) $\eta_{\mu\boxtimes\nu}(z) = \eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z))$.

A probability measure $\mu \in \mathcal{M}_{\mathrm{T}}$ is said to be $\boxtimes$-infinitely divisible if for any positive integer n, there exists $\mu_n \in \mathcal{M}_{\mathrm{T}}$ such that $\mu = (\mu_n)^{\boxtimes n} = \mu_n \boxtimes \cdots \boxtimes \mu_n$. It is shown in [Bercovici and Voiculescu 1992] that if $\mu \in \mathcal{M}_{\mathrm{T}} \backslash \mathcal{M}_*$ is $\boxtimes$-infinitely divisible, then μ is the Haar measure on T; and $\mu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$ is $\boxtimes$-infinitely divisible if and only if there exists a function

$$u(z) = \alpha i + \int_{\mathrm{T}} \frac{1+tz}{1-tz} \, d\sigma(t), \tag{3-2}$$

such that $\Sigma_\mu(z) = \exp(u(z))$, where $\alpha \in \mathbb{R}$ and σ is a finite positive measure on T. Equation (3-2) is the analogue of the Lévy–Khintchine formula for multiplicative free convolution on T. The analogue of the normal distribution in this context is given by $\Sigma_{\lambda_t}(z) = \exp((t/2)(1+z)/(1-z))$. Denote by $\mathcal{ID}(\boxtimes, \mathrm{T})$ the set of all $\boxtimes$-infinitely divisible measures on T.

Lemma 3.2. *Let $\mu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$ be $\boxtimes$-infinitely divisible.*

(1) *The function $H(z) = z\Sigma_\mu(z)$ is the left inverse of $\eta_\mu(z)$, that is $H(\eta_\mu(z)) = z$ for all $z \in \mathbb{D}$.*

(2) *The function η_μ extends to be a continuous function on $\overline{\mathbb{D}}$, and η_μ is one-to-one on $\overline{\mathbb{D}}$.*

(3) *The set $\{z \in \mathbb{D} : |z\Sigma_\mu(z)| < 1\}$ is a simply connected domain which coincides with $\{\eta_\mu(z) : z \in \mathbb{D}\}$, and its boundary is $\eta_\mu(T)$ which is a simple closed curve.*

Proof. Observing that $H(\eta_\mu(z)) = z$ is valid in a neighborhood of zero, we obtain assertion (1) by analytic continuation.

Note that $H : \mathbb{D} \to \mathbb{C}$ satisfies the conditions in [Belinschi and Bercovici 2005, Proposition 4.5] and thus assertions (2) and (3) hold. □

Multiplicative free Brownian motion. For $\mu \in \mathcal{M}_{\mathbb{T}}$ and $t > 0$, we study the multiplicative free convolution $\mu \boxtimes \lambda_t$. We first concentrate on the case when μ has nonzero mean. The case when μ has mean zero will be studied in Subsection 3.2.

We start with the following result which is the multiplicative version of [Biane 1997b, Lemma 1].

Lemma 3.3. *Given $\mu, \nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, we have*

$$\eta_\mu(z) = \eta_{\mu\boxtimes\nu}(z \cdot \Sigma_\nu(\eta_\mu(z)))$$

for z in a neighborhood of zero.

Proof. From (3-1), we find that

$$\frac{\eta^{-1}_{\mu\boxtimes\nu}(z)}{z} = \frac{\eta^{-1}_{\mu}(z)}{z} \cdot \frac{\eta^{-1}_{\nu}(z)}{z} \tag{3-3}$$

for z in a neighborhood of zero, which we denote by D_0. We choose a subdomain $D_1 \subset D_0$ such that $\eta_\mu(D_1) \subset D_0$. Replacing z by $\eta_\mu(z)$ in (3-3), we obtain

$$\frac{\eta^{-1}_{\mu\boxtimes\nu}(\eta_\mu(z))}{\eta_\mu(z)} = \frac{\eta^{-1}_{\mu}(\eta_\mu(z))}{\eta_\mu(z)} \cdot \frac{\eta^{-1}_{\nu}(\eta_\mu(z))}{\eta_\mu(z)} = \frac{z}{\eta_\mu(z)} \cdot \frac{\eta^{-1}_{\nu}(\eta_\mu(z))}{\eta_\mu(z)} \tag{3-4}$$

for $z \in D_1$. Note that $\eta_\nu^{-1}(z) = z\Sigma_\nu(z)$ for $z \in D_0$, and we then rewrite (3-4) as

$$\eta^{-1}_{\mu\boxtimes\nu}(\eta_\mu(z)) = z\Sigma_\nu(\eta_\mu(z)). \tag{3-5}$$

Applying $\eta_{\mu\boxtimes\nu}$ on both sides of (3-5) yields

$$\eta_\mu(z) = \eta_{\mu\boxtimes\nu}(z\Sigma_\nu(\eta_\mu(z)))$$

for z in a neighborhood of zero D_1. □

For any $t > 0$, we denote by $\eta_t : \mathbb{D} \to \mathbb{D}$ the subordination function of $\mu \boxtimes \lambda_t$ with respect to μ. Since $\eta_t : \mathbb{D} \to \mathbb{D}$ is analytic and $\eta_t(0) = 0$, Proposition 3.1 implies the existence of a probability measure ρ_t such that $\eta_{\rho_t}(z) = \eta_t(z)$.

Lemma 3.4. *The measure ρ_t is $\boxtimes$-infinitely divisible and its Σ-transform is*

$$\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\mu(z)).$$

Proof. Define analytic function $\Phi_t : \mathbb{D} \to \mathbb{C}$ by $\Phi_t(z) := z\Sigma_{\lambda_t}(\eta_\mu(z))$ for all $t > 0$. By Lemma 3.3, we have

$$\eta_\mu(z) = \eta_{\mu\boxtimes\lambda_t}(z\Sigma_{\lambda_t}(\eta_\mu(z))) = \eta_{\mu\boxtimes\lambda_t}(\Phi_t(z))$$

which implies that

$$\eta_{\mu\boxtimes\lambda_t}(z) = \eta_\mu(\eta_t(z)) = \eta_{\mu\boxtimes\lambda_t}(\Phi_t(\eta_t(z))).$$

Since $\eta_{\mu\boxtimes\lambda_t}$ is invertible in a neighborhood of zero, we have $\Phi_t(\eta_t(z)) = z$ in a neighborhood of zero.

We thus obtain $\eta_{\rho_t}^{-1}(z) = \eta_t^{-1}(z) = \Phi_t(z)$ holds for z in a neighborhood of zero, which yields that

$$\Sigma_{\rho_t}(z) = \frac{\eta_{\rho_t}^{-1}(z)}{z} = \Sigma_{\lambda_t}(\eta_\mu(z)). \tag{3-6}$$

By the definition of the ψ- and η-transforms, we have

$$\Sigma_{\lambda_t}(\eta_\mu(z)) = \exp\left(\frac{t}{2}\int_{\mathbb{T}} \frac{1+\xi z}{1-\xi z}\, d\mu(\xi)\right). \tag{3-7}$$

The real part of the integrand in (3-7) is positive for all $z \in \mathbb{D}$; thus the assertion follows from (3-6) and [Bercovici and Voiculescu 1992, Theorem 6.7]. □

By (3-6), the right hand side of (3-7) is the Lévy–Khintchine representation of ρ_t. We can also write η_t in terms of λ_t and $\mu \boxtimes \lambda_t$. Replacing z by $\eta_{\mu\boxtimes\lambda_t}(z)$ in the equation

$$\frac{\eta_{\mu\boxtimes\lambda_t}^{-1}(z)}{z} = \frac{\eta_\mu^{-1}(z)}{z} \cdot \frac{\eta_{\lambda_t}^{-1}(z)}{z},$$

we obtain

$$\frac{z}{\eta_{\mu\boxtimes\lambda_t}(z)} = \frac{\eta_t(z)}{\eta_{\mu\boxtimes\lambda_t}(z)} \cdot \Sigma_{\lambda_t}(\eta_{\mu\boxtimes\lambda_t}(z)),$$

which shows that

$$\eta_t(z) = \frac{z}{\Sigma_{\lambda_t}(\eta_{\mu\boxtimes\lambda_t}(z))}. \tag{3-8}$$

Modified $\mathcal{S}$-transform and subordination functions. Given $\mu \in \mathcal{M}_{\mathbb{T}} \backslash \mathcal{M}_*$ and $\nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, it is known from [Biane 1998] that $\eta_{\mu\boxtimes\nu}$ is subordinated to η_μ and η_ν. The subordination function for this case is generally not unique (see Example 3.5 below). However, we show that there is a nice subordination function, which we call the principal subordination function, uniquely determined by certain conditions. Using the principal subordination function, results related to subordination function in the case $\mu, \nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$ can be extended to the case where $\mu \in \mathcal{M}_{\mathbb{T}} \backslash \mathcal{M}_*$ and $\nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$.

Let us first give an example which illustrates the non-uniqueness of subordination functions.

Example 3.5. For $k \in \mathbb{N}$, and let $\lambda^{(k)} = 1/k \sum_{n=0}^{k-1} \delta_{z_n}$, where $z_n = e^{2\pi i n/k}$. We have $\psi_{\lambda^{(k)}}(z) = z^k/(1-z^k)$ and $\eta_{\lambda^{(k)}}(z) = z^k$. Given $\nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, if $\omega : \mathbb{D} \to \mathbb{D}$ is a subordination function of $\eta_{\lambda^{(k)}\boxtimes\nu}$ with respect to $\eta_{\lambda^{(k)}}$, then $\omega^{(n)}(z) := e^{2\pi i n/k}\omega(z)$ is also a subordination function of $\eta_{\lambda^{(k)}\boxtimes\nu}$ with respect to $\eta_{\lambda^{(k)}}$ for all integer $0 < n < k$.

We now introduce the modified $\mathcal{S}$-transform. Given two free random variables x and y in a W*-probability space $(\mathcal{A}, \phi)$, such that $\phi(x) = 0$ and $\phi(y) \neq 0$, we can not directly apply Voiculescu's $\mathcal{S}$-transform (Σ-transform) to calculate the distribution of xy. N. Raj Rao and R. Speicher [2007] introduce a new transform, which we call the modified $\mathcal{S}$-transform, to deal with this case. They apply the modified $\mathcal{S}$-transform to study the distribution of xy where x, y are free self-adjoint random variables such that $\phi(x) = 0$, $\phi(y) \neq 0$. For nonzero self-adjoint operator x, we have $\phi(x^2) \neq 0$. Assume that $\phi(x) = \cdots = \phi(x^{k-1}) = 0$ and $\phi(x^k) \neq 0$. Arizmendi [2012] observe that we can calculate the distribution of xy using the idea in [Rao and Speicher 2007]. We present the details of their work for reader's convenience.

We first recall some definitions. For $\mu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, we have $\psi_\mu(0) = 0$ and $\psi'_\mu(0) \neq 0$. It follows that there exists a function $\chi_\mu(z)$, which is analytic in a neighborhood of zero, such that

$$\psi_\mu(\chi_\mu(z)) = \chi_\mu(\psi_\mu(z)) = z$$

for sufficiently small z. The usual $\mathcal{S}$-transform is defined by

$$\mathcal{S}_\mu(z) = \frac{z+1}{z}\chi_\mu(z).$$

We then have

$$\Sigma_\mu(z) = \mathcal{S}_\mu\left(\frac{z}{1-z}\right), \qquad \eta_\mu^{-1}(z) = \chi\left(\frac{z}{1-z}\right).$$

We set

$$\mathcal{M}_{\mathbb{T}}^k = \left\{\mu \in \mathcal{M}_{\mathbb{T}} : \int_{\mathbb{T}} t^n \, d\mu(t) = 0 \text{ for } 1 \leq n < k, \text{ and } \int_{\mathbb{T}} t^k \, d\mu(t) \neq 0\right\}.$$

Then, for $\mu \in \mathcal{M}_{\mathbb{T}}^k$, we have

$$\text{(3-9)} \qquad \begin{cases} \psi'_\mu(0) = \cdots = \psi_\mu^{(k-1)}(0) = 0 = \eta'_\mu(0) = \cdots = \eta_\mu^{(k-1)}(0), \\ \psi_\mu^{(k)}(0) \neq 0 \text{ and } \eta_\mu^{(k)} \neq 0. \end{cases}$$

For $\mu \in \mathcal{M}_{\mathbb{T}}^k$, $\nu \in \mathcal{M}_{\mathbb{T}} \cap \mathcal{M}_*$, from the definition of free independence, we deduce that $\mu \boxtimes \nu \in \mathcal{M}_{\mathbb{T}}^k$.

We recall the following classical result in complex analysis; see, for example, [Hille 1959].

Theorem 3.6. *If $f(z)$ is holomorphic in $|z| < R$, and suppose that*

$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0, \qquad f^{(k)} \neq 0.$$

Then for small values of $w \neq 0$ the equation

$$f(z) = w$$

has k roots $z_1(w), \dots, z_k(w)$, *which tend to zero when* w *tends to zero. Moreover, there exists a function* $g(w)$, *holomorphic for* w *sufficiently small with* $g(0)=0$ *and* $g'(0)\neq 0$, *such that for any fixed small values* $w\neq 0$,

$$z_j(w)=g(\omega^j w^{1/k}),\quad \omega=e^{2\pi i/k},\quad 0\le \arg\omega^{1/k}<\frac{2\pi}{k},$$

if we put those roots in a certain order.

Remark. The converse of Theorem 3.6 is also true. More precisely, given a function $g(w)$ that is holomorphic for small enough w and satisfies $g(0)=0$ and $g'(0)\neq 0$, if we set

$$z_j(w)=g(\omega^j w^{1/k}),\quad \omega=e^{2\pi i/k},\quad 0\le \arg\omega^{1/k}<\frac{2\pi}{k}$$

for $j=1,\dots,k$, then $z_1(w),\dots,z_k(w)$ are the roots of the equation

$$F^k(z)=w,$$

where F is a holomorphic function defined in a neighborhood of the zero such that $F(g(w))=w$.

For $j=1,\dots,k$, denote $D_{j,r}=\{\omega^j z : 0\le \arg(z)<2\pi/k, |z|<r\}$. We record the following result for convenience.

Proposition 3.7. *Under the assumption of Theorem 3.6, we have* $z_j(f(z))=z$ *for* $z\in g(D_{j,r})$ *for* r *sufficiently small.*

Given $\mu\in\mathcal{M}_{\mathbb{T}}^k$, and by Theorem 3.6, we know that there exist k functions represented by the power series in $z^{1/k}$ such that

$$\psi_\mu(\chi_\mu^{(j)}(z))=z, \tag{3-10}$$

for z sufficiently small. Moreover, there exists a function $g_\mu(w)$ holomorphic in a neighborhood of the zero, such that for $j=1,\dots,k$,

$$\chi_\mu^{(j)}(z)=g_\mu(\omega^j z^{1/k}),$$

where $\omega=e^{2\pi i/k}$, $0\le\arg z^{1/k}<2\pi/k$.

Definition 3.8. Given $\mu\in\mathcal{M}_{\mathbb{T}}^k$. Let $\chi_\mu^{(j)}$ be the inverse function of ψ_μ in (3-10), the modified $\mathcal{S}$-transform of μ is k functions $\mathcal{S}_\mu^{(1)}(z),\dots,\mathcal{S}_\mu^{(k)}(z)$, such that for $j=1,\dots,k$,

$$\mathcal{S}_\mu^{(j)}(z)=\chi_\mu^{(j)}(z)\cdot\frac{1+z}{z}.$$

Given $\mu\in\mathcal{M}_{\mathbb{T}}^k$ and $\nu\in\mathcal{M}_{\mathbb{T}}\cap\mathcal{M}_*$, we set

$$\mathcal{S}^{(j)}(z)=\mathcal{S}_\mu^{(j)}(z)\cdot\mathcal{S}_\nu(z)$$

and compute

$$\chi^{(j)}(z) = \mathcal{S}^{(j)}(z)\cdot\frac{z}{1+z} = \mathcal{S}_\mu^{(j)}(z)\cdot\mathcal{S}_\nu(z)\cdot\frac{z}{1+z} = \chi_\mu^{(j)}(z)\cdot\mathcal{S}_\nu(z) = g_\mu(\omega^j z^{1/k})\cdot\mathcal{S}_\nu(z) = g(\omega^j z^{1/k}), \tag{3-11}$$

where $g(z) = g_\mu(z)\cdot\mathcal{S}_\nu(z^k)$ is a function such that $g(0)=0,\ g'(0)\neq 0$. From the remark after Theorem 3.6, we deduce that for different j, there exists the same left inverse ψ such that $\psi(\chi^{(j)}(z)) = z$. Therefore, we have the following proposition.

Proposition 3.9. *Given $\mu\in\mathcal{M}_{\mathrm{T}}^k$ and $\nu\in\mathcal{M}_{\mathrm{T}}\cap\mathcal{M}_*$, for $1\le j\le k$, let*

$$\mathcal{S}^{(j)}(z) = \mathcal{S}_\mu^{(j)}(z)\cdot\mathcal{S}_\nu(z),\qquad \chi^{(j)}(z) = \mathcal{S}^{(j)}(z)\cdot\frac{z}{1+z}.$$

Then there exists a unique holomorphic function ψ defined in a neighborhood of the zero such that

$$\psi((\chi^{(j)})(z)) = z.$$

Theorem 3.10 [Rao and Speicher 2007; Arizmendi 2012]. *Given $\mu\in\mathcal{M}_{\mathrm{T}}^k$ and $\nu\in\mathcal{M}_{\mathrm{T}}\cap\mathcal{M}_*$, we have*

$$\mathcal{S}_{\mu\boxtimes\nu}^{(j)}(z) = \mathcal{S}_\mu^{(j)}(z)\cdot\mathcal{S}_\nu(z),\qquad j=1,\dots,k, \tag{3-12}$$

where the modified $\mathcal{S}$-transforms are listed in a certain order.

Because of Proposition 3.9 and Theorem 3.10, for fixed $\mu\in\mathcal{M}_{\mathrm{T}}^k$ and $\nu\in\mathcal{M}_{\mathrm{T}}\cap\mathcal{M}_*$, we denote

$$\psi(z) = \psi_{\mu\boxtimes\nu}(z)\quad\text{and}\quad \chi^{(j)}(z) = \chi_{\mu\boxtimes\nu}^{(j)}(z), \tag{3-13}$$

and we also denote $g(z) = g_\mu(z)\cdot\mathcal{S}_\nu(z^k)$ as in (3-11).

Given $\mu\in\mathcal{M}_{\mathrm{T}}^k$, $\nu\in\mathcal{M}_{\mathrm{T}}\cap\mathcal{M}_*$, we set $\iota_\mu^{(j)}(z) = \chi_\mu^{(j)}(z/(1-z))$ and $\iota_\nu(z) = \chi_\nu(z/(1-z))$. Theorem 3.10 implies that

$$\chi_{\mu\boxtimes\nu}^{(j)}(z) = \chi_\mu^{(j)}(z)\cdot\chi_\nu(z)\cdot\frac{1+z}{z}. \tag{3-14}$$

We also have $\chi_\mu^{(j)}(z) = g_\mu(\omega^j z^{1/k})$ and $\chi_{\mu\boxtimes\nu}^{(j)}(z) = g_\mu(\omega^j z^{1/k})\cdot\mathcal{S}_\nu(z) = g(\omega^j z^{1/k})$. Substituting z by $\psi_{\mu\boxtimes\nu}(z)$ in (3-14), and applying Proposition 3.7, we find that

$$z = \chi_{\mu\boxtimes\nu}^{(j)}(\psi_{\mu\boxtimes\nu}(z)) = \chi_\mu^{(j)}(\psi_{\mu\boxtimes\nu}(z))\cdot\chi_\nu(\psi_{\mu\boxtimes\nu}(z))\cdot\frac{1+\psi_{\mu\boxtimes\nu}(z)}{\psi_{\mu\boxtimes\nu}(z)},$$

where $z\in g(D_{j,r})$ for r sufficiently small. Thus

$$z\eta_{\mu\boxtimes\nu} = \iota_\mu^{(j)}(\eta_{\mu\boxtimes\nu})\cdot\iota_\nu(\eta_{\mu\boxtimes\nu}) \tag{3-15}$$

holds in the same domain.

We can now utilize the argument in [Belinschi and Bercovici 2007] to prove the existence of subordination function of $\eta_{\mu\boxtimes\nu}$ with respect to η_μ for $\mu \in \mathcal{M}_{\mathrm{T}}^k$, $\nu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$. Note that part of the following result is known in [Biane 1998].

Theorem 3.11. *Given $\mu \in \mathcal{M}_{\mathrm{T}}^k$, $\nu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, there exists two unique analytic functions $\omega_1, \omega_2 : \mathbb{D} \to \mathbb{D}$ such that*

(1) $\omega_1(0) = \omega_2(0) = 0$;

(2) $\eta_{\mu\boxtimes\nu}(z) = \eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z))$,

(3) $\omega_1(z)\omega_2(z) = z\eta_{\mu\boxtimes\nu}(z)$ *for all* $z \in \mathbb{D}$.

Proof. Since $\eta_\mu(0) = 0$, $\eta_\nu(0) = 0$, we can write $\eta_\mu(z) = zf_1(z)$, $\eta_\nu(z) = zf_2(z)$ for two analytic functions $f_1, f_2 : \mathbb{D} \to \mathbb{D}$. Fix $1 \le j \le k$, set $\omega_1(z) = \iota_\mu^{(j)}(\eta_{\mu\boxtimes\nu}(z))$, $\omega_2(z) = \iota_\nu(\eta_{\mu\boxtimes\nu}(z))$ defined in $g(D_{j,r})$ for r sufficiently small.

By (3-15), we have

$$z\eta_{\mu\boxtimes\nu}(z) = \omega_1(z)\omega_2(z)$$

for $z \in g(D_{j,r})$. We thus obtain

$$\omega_1(z) = \frac{z\eta_{\mu\boxtimes\nu}}{\omega_2(z)} = \frac{z\eta_\nu(\omega_2(z))}{\omega_2(z)} = zf_2(\omega_2(z)).$$

Similarly, we have $\omega_2(z) = zf_1(\omega_1(z))$ for $z \in g(D_{j,r})$. Regarding $\omega_1(z), \omega_2(z)$ as Denjoy–Wolff points, the same argument as in [Belinschi and Bercovici 2007] implies ω_1, ω_2 can be extended analytically to $\mathbb{D}$. By the uniqueness of Denjoy–Wolff points, ω_1, ω_2 does not depend on the choice of j.

By the definitions of $\iota_\mu^{(j)}, \iota_\nu$, we have $\eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z)) = \eta_{\mu\boxtimes\nu}$ for z in $g(D_{j,r})$. Thus (2) and (3) hold by analytic continuation. Since $\eta_\nu'(0) \neq 0$, η_ν is locally invertible near the origin and therefore ω_2 is unique. Finally (3) implies the uniqueness of ω_1. □

Since $\mu \in \mathcal{M}_{\mathrm{T}}^k$ and $\mu \boxtimes \nu \in \mathcal{M}_{\mathrm{T}}^k$, we have $\omega_1'(0) \neq 0$, where ω_1 is given in Theorem 3.11.

Definition 3.12. For $\mu \in \mathcal{M}_{\mathrm{T}}^k$, $\nu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, the subordination ω_1 satisfying the relations (1), (2) and (3) in Theorem 3.11 is called the *principal* subordination function of $\eta_{\mu\boxtimes\nu}$ with respect to η_μ. The measure $\rho \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$ satisfying $\eta_\rho(z) = \omega_1(z)$ is called the *principal* subordination distribution of $\eta_{\mu\boxtimes\nu}$ with respect to η_μ.

Note that for $\mu, \nu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, the principal subordination function of $\eta_{\mu\boxtimes\nu}$ with respect to η_μ is the usual subordination function.

The following result might be obtained by approximation. We provide a direct proof.

Corollary 3.13. *Given $\mu \in \mathcal{M}_{\mathrm{T}}^k$, $\nu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, let ρ be the principal subordination distribution of $\eta_{\mu \boxtimes \nu}$ with respect to η_μ, we have*

$$\Sigma_\rho(z) = \Sigma_\nu(\eta_\mu(z)).$$

In particular, if $\nu \in \mathcal{ID}(\boxtimes, \mathrm{T})$, we have $\rho \in \mathcal{ID}(\boxtimes, \mathrm{T})$.

Proof. By choosing a sequence $\mu_n \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$ such that μ_n converges to μ weakly, Lemma 3.3 implies

$$\eta_\mu(z) = \eta_{\mu \boxtimes \nu}(z \Sigma_\nu(\eta_\mu(z)))$$

for z in a neighborhood of zero.

Set $\Phi(z) = z \Sigma_\nu(\eta_\mu(z)) = z \cdot \mathcal{S}_\nu(\psi_\mu(z))$, and we thus have

$$\eta_{\mu \boxtimes \nu}(z) = \eta_\mu(\omega_1(z)) = \eta_{\mu \boxtimes \nu}(\Phi(\omega_1(z))) = \eta_{\mu \boxtimes \nu}(\Phi(\eta_\rho(z))). \tag{3-16}$$

Fix $1 \le j \le k$, we claim that if $z \in g(D_{j,r})$, then $\Phi(\omega_1(z)) = z$. Indeed, for $0 \le \arg(w^{1/k}) < 2\pi/k$ and $z = g(\omega^j w^{1/k}) = \chi^{(j)}(w)$, using the construction of ω_1 in Theorem 3.11, we have

$$\omega_1(z) = \iota_\mu^{(j)}(\eta_{\mu \boxtimes \nu}(z)) = \chi_\mu^{(j)}(\psi_{\mu \boxtimes \nu}(z)). \tag{3-17}$$

From (3-11) and (3-13), we have

$$\chi^{(j)}(w) = g(\omega^j w^{1/k})) \quad \text{and} \quad \psi_{\mu \boxtimes \nu}(\chi^{(j)}(w)) = w. \tag{3-18}$$

Equations (3-17) and (3-18) imply that $\omega_1(g(\omega^j w^{1/k}))) = \chi_\mu^{(j)}(w)$.

Note that $\psi_\mu(\omega_1(g(\omega^j w^{1/k}))) = \psi_{\mu \boxtimes \nu}(g(\omega^j w^{1/k})) = w$. Thus we obtain

$$\Phi(\omega_1(z)) = \Phi(\omega_1(g(\omega^j w^{1/k}))) = \chi_\mu^{(j)}(w) \mathcal{S}_\nu(w) = \chi^{(j)}(w) = z.$$

The above claim, (3-16) and Proposition 3.7 imply

$$z = \Phi(\omega_1(z)) = \Phi(\eta_\rho(z))$$

for $z \in g(D_{j,r})$. We conclude that $\Sigma_\rho(z) = \Phi(z)/z = \Sigma_\nu(\eta_\mu(z))$ for z in a small neighborhood of zero by applying the above argument for all $1 \le j \le k$.

If $\nu \in \mathcal{ID}(\boxtimes, \mathrm{T})$, then by [Bercovici and Voiculescu 1993, Theorem 6.7], there exists an analytic function $u(z)$ defined in $\mathbb{D}$ such that $\Sigma_\nu(z) = \exp(u(z))$ and $\Re u(z) \ge 0$ for all $z \in \mathbb{D}$. Thus $\Sigma_\nu(\eta_\mu(z)) = \exp(u(\eta_\mu(z)))$ and $\Re(u(\eta_\mu(z))) \ge 0$ for all $z \in \mathbb{D}$, and then the second assertion follows from [Bercovici and Voiculescu 1993, Theorem 6.7]. $\square$

Remark. If $k = 1$, noticing that $\mathcal{M}_{\mathrm{T}}^k = \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, the modified $\mathcal{S}$-transform is the usual $\mathcal{S}$-transform. We see that Corollary 3.14 holds when $\mu, \nu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$.

The following result is the multiplicative analogue of Lemma 2.6.

Proposition 3.14. *Given $\rho, \tau \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, let σ be a measure in $\mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$ such that $\eta_{\rho\boxtimes\tau}(z) = \eta_\rho(\eta_\sigma(z))$. If $\sigma \in \mathcal{ID}(\boxtimes, \mathrm{T})$, then $\rho \boxtimes \tau^{\boxtimes t}$ can be defined for all $t \geq 0$ in the sense that $\Sigma_{\rho\boxtimes(\tau^{\boxtimes t})}(z) = \Sigma_\rho(z)(\Sigma_\tau(z))^t$.*

Proof. For $t > 0$, there exists $\mu_t \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$ such that

$$\eta_{\mu_t}(z) = \eta_\rho(\eta_{\sigma^{\boxtimes t}}).$$

Using a similar argument as in the proof of Lemma 2.6 and applying Corollary 3.13, we can find that $\Sigma_{\mu_t}(z) = \Sigma_\rho(z)(\Sigma_\tau(z))^t$. □

Semigroups related to multiplicative free convolution. Recall that

$$\mathcal{M}_{\mathrm{T}}^* = \{\mu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_* : \eta_\mu(z) \neq 0 \text{ for all } z \in \mathbb{D}\backslash\{0\}\}.$$

Let $\mu \in \mathcal{M}_{\mathrm{T}}^*$ and $t > 1$ be given, and let u be an analytic function such that $z/(\eta_\mu(z)) = e^{u(z)}$ for z in a neighborhood of zero. Set $H_t(z) = ze^{(t-1)u(z)} = z[z/(\eta_\mu(z))]^{t-1}$. It is shown in [Belinschi and Bercovici 2005] that H_t has a right inverse $\omega_t : \mathbb{D} \to \mathbb{D}$ such that $H_t(\omega(z)) = z$, and there exists a probability measure $\mu^{\boxtimes t} \in \mathcal{M}_{\mathrm{T}}^*$ such that

(1) $\eta_{\mu^{\boxtimes t}}(z) = \eta_\mu(\omega_t(z))$ and $\Sigma_{\mu^{\boxtimes t}}(z) = (\Sigma_\mu(z))^t$,

(2) $\omega_t(z) = \eta_{\mu^{\boxtimes t}}(z)[z/\eta_{\mu^{\boxtimes t}}(z)]^{1/t}$ for $z \in \mathbb{D}$, where the power is chosen so that the equation holds.

Observe that for each $t > 0$, by Proposition 3.1, there exists a probability measure $\sigma_t \in \mathcal{M}_{\mathrm{T}}$ such that $\eta_{\sigma_t}(z) = \omega_{t+1}(z)$. It turns out that σ_t is $\boxtimes$-infinitely divisible and its Σ-transform is $\Sigma_{\sigma_t}(z) = [z/\eta_\mu(z)]^t$, which can be obtained by applying the same argument as in the proof of Lemma 3.4.

The following result is a partial converse of [Belinschi and Bercovici 2005, Theorem 3.5].

Theorem 3.15. *Given $\mu \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$, assume that for any $t > 1$, there exists a probability measure $\mu_t \in \mathcal{M}_{\mathrm{T}}$ such that*

$$\Sigma_{\mu_t}(z) = (\Sigma_\mu(z))^t. \tag{3-19}$$

Assume in addition that μ_t is subordinated with respect to μ for all $t > 1$. Then $\eta_\mu(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$, that is $\mu \in \mathcal{M}_{\mathrm{T}}^*$.

Proof. For each $t > 1$, we denote by ω_t the subordination function of μ_t to μ. Observing that $\mu_t \in \mathcal{M}_*$ and $\omega_t'(0) \neq 0$, for each $t > 1$, there exists a probability measure $\sigma_{t-1} \in \mathcal{M}_{\mathrm{T}} \cap \mathcal{M}_*$ such that $\eta_{\sigma_{t-1}}(z) = \omega_t(z)$. We rewrite (3-19) as

$$\frac{\eta_{\mu_t}^{-1}(z)}{z} = \left[\frac{\eta_\mu^{-1}(z)}{z}\right]^t \tag{3-20}$$

for z in a neighborhood of zero.

Note that $\omega_t^{-1}(z)=\eta_{\mu_t}^{-1}(\eta_\mu(z))$ for z in a neighborhood of zero. Replacing z by $\eta_\mu(z)$ in (3-20), we obtain

$$\frac{\omega_t^{-1}(z)}{\eta_\mu(z)}=\frac{\eta_{\mu_t}^{-1}(\eta_\mu(z))}{\eta_\mu(z)}=\left[\frac{\eta_\mu^{-1}(\eta_\mu(z))}{\eta_\mu(z)}\right]^t=\left[\frac{z}{\eta_\mu(z)}\right]^t,$$

which implies

$$\frac{\omega_t^{-1}(z)}{z}=\left[\frac{z}{\eta_\mu(z)}\right]^{t-1}.$$

Given $t>0$, we thus have $\Sigma_{\sigma_t}(z)=[z/\eta_\mu(z)]^t$ for z in a neighborhood of zero. Therefore σ_t is $\boxtimes$-infinitely divisible.

By [Bercovici and Voiculescu 1992, Theorem 6.7], there exists an analytic function $u(z)$ in $\mathbb{D}$ such that $\Re u(z)\geq 0$ if $z\in\mathbb{D}$ and $\Sigma_{\sigma_1}(z)=\exp(u(z))$. We thus obtain $z/\eta_\mu(z)=\exp(u(z))$, which implies that $\eta_\mu(z)\neq 0$ for all $z\in\mathbb{D}$. □

It was pointed out in [Belinschi and Bercovici 2005] that $\mu^{\boxtimes t}$ is only determined up to a rotation by a multiple of $2\pi t$. Note that ω_t and σ_t are determined by the choice of $\mu^{\boxtimes t}$.

Multiplicative Boolean convolution and the Bercovici–Pata bijection. Multiplicative Boolean convolution on $\mathbb{T}$ was studied by Franz [2008]. Let $\mu\in\mathcal{M}_{\mathbb{T}}$, and we set $k_\mu(z)=z/\eta_\mu(z)$. Given two probability measures $\mu,\nu\in\mathcal{M}_{\mathbb{T}}$, their multiplicative Boolean convolution $\mu\mathbin{\underline{\times}}\nu$ is a probability measure on $\mathbb{T}$ such that

$$k_{\mu\mathbin{\underline{\times}}\nu}(z)=k_\mu(z)k_\nu(z)$$

for all $z\in\mathbb{D}$.

A probability $\mu\in\mathcal{M}_{\mathbb{T}}$ is said to be $\mathbin{\underline{\times}}$-infinitely divisible, if for any positive integer n, there exists $\mu_n\in\mathcal{M}_{\mathbb{T}}$ such that $\mu=(\mu_n)^{\mathbin{\underline{\times}}n}$. Let P_0 be the Haar measure. It is shown in [Franz 2008] that $\mu\in\mathcal{M}_{\mathbb{T}}\backslash\{P_0\}$ is $\mathbin{\underline{\times}}$-infinitely divisible if and only if $\eta_\mu'(0)\neq 0$ and $\eta_\mu\neq 0$ for all $z\in\mathbb{D}\backslash\{0\}$, that is $\mu\in\mathcal{M}_{\mathbb{T}}^*$, which is equivalent to

$$k_\mu(z)=\exp\left(bi+\int_{\mathbb{T}}\frac{1+\xi z}{1-\xi z}\,d\,\tau_\mu(\xi)\right), \tag{3-21}$$

where $b\in\mathbb{R}$ and τ_μ is a finite measure on $\mathbb{T}$. Equation (3-21) is the analogue of the Lévy–Khintchine formula in this context.

The multiplicative Bercovici–Pata bijection from $\mathbin{\underline{\times}}$ to $\boxtimes$ was studied in [Wang 2008]. Denote the set of all $\mathbin{\underline{\times}}$-infinitely divisible measures on $\mathbb{T}$ by $\mathcal{ID}(\mathbin{\underline{\times}},\mathbb{T})$, and the multiplicative Bercovici–Pata bijection from $\mathbin{\underline{\times}}$ to $\boxtimes$ by $\mathbb{M}$. Then we have $k_\mu(z)=\Sigma_{\mathbb{M}(\mu)}(z)$.

Given $\mu\in\mathcal{ID}(\mathbin{\underline{\times}},\mathbb{T})\backslash P_0=\mathcal{M}_{\mathbb{T}}^*$, let ω_2 be the subordination function of $\mu^{\boxtimes 2}$ with respect to μ, and let σ be the probability measure on $\mathbb{T}$ such that $\eta_\sigma(z)=\omega_2(z)$.

Then σ is $\boxtimes$-infinitely divisible and its Σ-transform is $\Sigma_\sigma(z) = z/\eta_\mu(z) = k_\mu(z)$. Therefore, σ is the same as $\mathbb{M}(\mu)$. Since $P_0 \boxtimes P_0 = P_0$ and $\eta_{P_0} = z$, the subordination function of $P_0 \boxtimes P_0$ with respect to P_0 is the identity map z, and the measure associated with the identity map z is P_0. To summarize:

Corollary 3.16. *Given $\mu \in \mathcal{ID}(\Cup\!\!\!\!\times, \mathrm{T})$, let ω_2 be the subordination function of $\mu^{\boxtimes 2}$ with respect to μ, and let σ be the probability measure on* T *such that $\eta_\sigma(z) = \omega_2(z)$. Then $\sigma = \mathbb{M}(\mu)$, where $\mathbb{M}$ is the multiplicative Bercovici–Pata bijection from $\Cup\!\!\!\!\times$ to $\boxtimes$.*

Proposition 3.17. *If $\mu \in \mathcal{M}_\mathrm{T}$, the following are equivalent.*

(1) $\mu \in \mathcal{ID}(\mathrm{T}, \boxtimes)$.

(2) $\delta_\beta \Cup\!\!\!\!\times \mu \in \mathcal{ID}(\mathrm{T}, \boxtimes)$ *for any* $\beta \in \mathrm{T}$.

Proof. It is enough to prove that (1) implies (2) for $\mu \in \mathcal{M}_\mathrm{T} \cap \mathcal{M}_*$. Observing that $\eta_{\delta_\beta \Cup\!\!\!\times \mu}(z) = \beta \cdot \eta_\mu(z)$, we thus have

$$\Sigma_{\delta_\beta \Cup\!\!\!\times \mu}(z) = \frac{\eta^{-1}_{\delta_\beta \Cup\!\!\!\times \mu}(z)}{z} = \frac{\eta^{-1}_\mu(\bar\beta z)}{z} = \bar\beta \cdot \Sigma_\mu(\bar\beta z).$$

The result follows from the Lévy–Khintchine formula for the multiplicative free convolution on T. $\square$

An analogue of equations studied by Belinschi and Nica. In this subsection, we prove our Theorem 1.1. Recall that λ_t is the free multiplicative analogue of the normal distribution on T, the unit circle of $\mathbb{C}$, with $\Sigma_{\lambda_t}(z) = \exp((t/2)(1+z)/(1-z))$ and we set $\lambda = \lambda_1$. For $\mu \in \mathcal{M}_\mathrm{T}$, we denote $m_1(\mu) = \int_\mathrm{T} \xi \, d\mu(\xi)$.

Proposition 3.18. *Given $\mu \in \mathcal{M}^*_\mathrm{T}$ and a finite measure τ on* T, *define an analytic map u by*

$$u(z) = bi + \int_\mathrm{T} \frac{1+\xi z}{1-\xi z}\, d\tau(\xi), \qquad z \in \mathbb{D}, \tag{3-22}$$

where $b \in [0, 2\pi)$. If $k_\mu(z) = z/\eta_\mu(z) = \exp u(z)$, then $b = \arg 1/m_1(\mu) \in [0, 2\pi)$ and $\tau(\mathrm{T}) = \ln |1/m_1(\mu)|$. In particular, there exists a probability measure $\nu \in \mathcal{M}_\mathrm{T}$ such that $k_\mu(z) = \Sigma_\lambda(\eta_\nu(z))$ if and only if $m_1(\mu) = e^{-1/2}$.

Proof. By definition, we have

$$k_\mu(0) = \lim_{z\to 0} \frac{z}{\eta_\mu(z)} = \frac{1}{\eta'_\mu(0)} = \frac{1}{m_1(\mu)}. \tag{3-23}$$

Since $u(0) = bi + \tau(\mathrm{T})$, we obtain

$$b = \arg\left(\frac{1}{m_1(\mu)}\right) \quad \text{and} \quad \tau(\mathrm{T}) = \ln\left|\frac{1}{m_1(\mu)}\right|. \tag{3-24}$$

The first assertion follows.

By (3-7), we have

$$\Sigma_\lambda(\eta_\nu(z)) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{1+\xi z}{1-\xi z}\, d\nu(\xi)\right).$$

Noticing that k_μ has the Herglotz representation as (3-21), we conclude that $k_\mu(z)$ can be written in the form of $\Sigma_\lambda(\eta_\nu(z))$ for a probability measure ν on $\mathbb{T}$ if and only if $\ln(1/m_1(\mu)) = 1/2$. This implies the second half of the assertion. $\square$

For $\mu \in \mathcal{ID}(\uplus, \mathbb{T})\backslash P_0 = \mathcal{M}_{\mathbb{T}}^*$ with $m_1(\mu) > 0$, let $u(z)$ be the analytic function satisfying $k_\mu(z) = \exp(u(z))$ and $u(0) > 0$. Given $t > 1$, let $H_t(z) = z\exp((t-1)u(z))$, and denote its right inverse by $\omega_t : \mathbb{D} \to \mathbb{D}$ with $\omega_t(0) = 0$. We define (see [Belinschi and Bercovici 2005]) $\mu^{\boxtimes t}$ by the relation

$$\eta_{\mu^{\boxtimes t}}(z) = \eta_\mu(\omega_t(z)). \tag{3-25}$$

Then we see that $H_t'(0) > 0$, $\omega_t'(0) > 0$ and that

$$m_1(\mu^{\boxtimes t}) = \eta'_{\mu^{\boxtimes t}}(0) > 0.$$

For $t > 0$, we also define $\mu^{\uplus t}$ by the relation

$$k_{\mu^{\uplus t}}(z) = \exp(tu(z)). \tag{3-26}$$

For this choice of the Boolean convolution power, we have

$$m_1(\mu^{\uplus t}) > 0.$$

Definition 3.19. Given $\mu \in \mathcal{M}_{\mathbb{T}}^*$ such that $m_1(\mu) > 0$, we define a family of maps $\{\mathbb{M}_t\}_{t\neq 0}$ by

$$\mathbb{M}_t(\mu) = (\mu^{\boxtimes(t+1)})^{\uplus 1/(t+1)},$$

where we choose $\mu^{\boxtimes(t+1)}$ and $\mathbb{M}_t(\mu)$ in a way such that they have positive means.

The next result is a special case of [Arizmendi and Hasebe 2013, Theorem 4.4].

Lemma 3.20. *Given $\mu \in \mathcal{M}_{\mathbb{T}}^*$ with $m_1(\mu) > 0$, the following assertions are true.*

(1) $\mathbb{M}_{t+s}(\mu) = \mathbb{M}_t(\mathbb{M}_s(\mu))$ *for all* $t, s \geq 0$.

(2) $\mathbb{M}_1(\mu) = \mathbb{M}(\mu)$.

Proof of Theorem 1.1. We set

$$u(z) = \frac{1}{2}\int_{\mathbb{T}} \frac{1+\xi z}{1-\xi z}\, d\nu(\xi). \tag{3-27}$$

By (3-7) and the assumption (1-2), we have

$$\frac{z}{\eta_\mu(z)} = \Sigma_\lambda(\eta_\nu(z)) = \exp(u(z)). \tag{3-28}$$

By Proposition 3.18, we see that $m_1(\mu) > 0$. We therefore can choose the multiplicative convolution power $\mu^{\boxtimes(t+1)}$ such that $m_1(\mu^{\boxtimes(t+1)}) > 0$.

Let η_t be the principal subordination function of $\nu \boxtimes \lambda_t$ with respect to ν and ω_{t+1} be the subordination function of $\mu^{\boxtimes(t+1)}$ with respect to μ. Let $\rho_t, \sigma_t \in \mathcal{M}_{\mathbb{T}}$ such that $\eta_{\rho_t} = \eta_t$ and $\eta_{\sigma_t} = \omega_{t+1}$.

By Corollary 3.13, (1-2) implies that $\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\mu(z)) = \exp(tu(z))$. From the choice of $\mu^{\boxtimes(t+1)}$, the function $H_{t+1}(z) := z\exp(tu(z))$ is the left inverse of ω_{t+1} such that $H_{t+1}(\omega_{t+1}(z)) = z$ for all $z \in \mathbb{D}$, which implies that

$$\Sigma_{\sigma_t}(z) = \exp(tu(z)). \tag{3-29}$$

We thus obtain that $\rho_t = \sigma_t$ and $\eta_t = \omega_{t+1}$.

Replacing z by η_t in (1-2), we obtain

$$\begin{aligned}\Sigma_\lambda(\eta_{\nu\boxtimes\lambda_t}(z)) &= \Sigma_\lambda(\eta_\nu((\eta_t(z)))) = \frac{\eta_t(z)}{\eta_\mu(\eta_t(z))} \\ &= \frac{\omega_{t+1}(z)}{\eta_\mu(\omega_{t+1}(z))} = \frac{\omega_{t+1}(z)}{\eta_{\mu^{\boxtimes(t+1)}}(z))} = \left(\frac{z}{\eta_{\mu^{\boxtimes(t+1)}}(z)}\right)^{1/t+1}.\end{aligned} \tag{3-30}$$

On the other hand, by the definition of $\mathbb{M}_t$, we have

$$\frac{z}{\eta_{\mathbb{M}_t(\mu)}(z)} = \frac{z}{\eta_{(\mu^{\boxtimes(t+1)})\uplus(1/(t+1))}(z)} = \left(\frac{z}{\eta_{\mu^{\boxtimes(t+1)}}(z)}\right)^{1/t+1},$$

completing the proof of Theorem 1.1. □

Some examples and applications. We start with the multiplicative analogues of examples studied in [Anshelevich 2010; 2012; Arizmendi and Hasebe 2013; Belinschi and Nica 2008b]. We define the set

$$(\mathrm{A}) = \{\mu \in \mathcal{M}_{\mathbb{T}}^* : m_1(\mu) = e^{-1/2}\}.$$

By (3-22), the set $\mathcal{M}_{\mathbb{T}}$ is in one-to-one correspondence with the set (A) via the bijection $\nu \leftrightarrow \mu$, such that $\Sigma_\lambda(\eta_\nu(z)) = z/\eta_\mu(z)$.

Definition 3.21. The bijective map $\Lambda : \mathcal{M}_{\mathbb{T}} \to (\mathrm{A})$ is defined by

$$\Sigma_\lambda(\eta_\nu(z)) = \frac{z}{\eta_{\Lambda[\nu]}(z)} \quad \text{for all } \nu \in \mathcal{M}_{\mathbb{T}}.$$

Using the Λ notation, Theorem 1.1 implies that

$$\Lambda[\nu \boxtimes \lambda_t] = \mathbb{M}_t[\Lambda(\nu)] \quad \text{for all } \nu \in \mathcal{M}_{\mathbb{T}}.$$

Example 3.22. Let δ_1 be the Dirac measure at 1, and let $\mu = \Lambda[\delta_1]$, we have $z/\eta_\mu(z) = \Sigma_\lambda(\eta_{\delta_1}(z)) = \exp(\frac{1}{2}(1+z)/(1-z))$. For $t \geq 0$, Theorem 1.1 implies that

$$\frac{z}{\eta_{\mathbb{M}_t(\mu)}(z)} = \Sigma_\lambda(\eta_{\delta_1 \boxtimes \lambda_t}(z)) = \Sigma_\lambda(\eta_{\lambda_t}(z)).$$

In particular, when $t = 1$,

$$\eta_{\mathbb{M}_1(\mu)}(z) = \frac{z}{\Sigma_\lambda(\eta_{\lambda_1}(z))} = \eta_{\lambda_1}(z),$$

where we used the equality $((z\Sigma_\lambda) \circ \eta_\lambda)(z) = z$ and $\lambda = \lambda_1$. Therefore, $\mathbb{M}_1(\mu)$ is the free multiplicative analogue of the normal distribution on T.

Example 3.23. More generally, we consider $\lambda_{b,t} = \delta_b \boxtimes \lambda_t$ and $\mu_{b,t} = \Lambda[\lambda_{b,t}]$. Then

$$\Sigma_\lambda(\eta_{\lambda_{b,t}}(z)) = \frac{z}{\eta_{\mu_{b,t}}(z)} \quad \text{for } t \neq 0.$$

On the other hand, Theorem 1.1 implies that

$$\Sigma_\lambda(\eta_{\lambda_{b,t_1+t_2}}(z)) = \Sigma_\lambda(\eta_{\lambda_{b,t_1} \boxtimes \lambda_2}(z)) = \frac{z}{\eta_{\mathbb{M}_{t_2}(\mu_{b,t_1})}(z)} \quad \text{for } t_1, t_2 \geq 0,$$

which yields that $\mu_{b,t_1+t_2} = \mathbb{M}_{t_2}(\mu_{b,t_1})$ for $t_1, t_2 \geq 0$.

We would like to provide another example which covers part of [Arizmendi and Hasebe 2013, Example 4.10].

Example 3.24. Let P_0 be the Haar measure on T. Then by the free independence, $P_0 \boxtimes \lambda_t = P_0$. We set $\mu = \Lambda[P_0]$, and we have

$$\Sigma_\lambda(\eta_{P_0 \boxtimes \lambda_t}(z)) = \Sigma_\lambda(\eta_{P_0}(z)) = \frac{z}{\eta_\mu(z)},$$

which implies that $\mathbb{M}_t(\mu) = \mu$ for all $t \geq 0$. To calculate the distribution of μ, we note that $\eta_{P_0} \equiv 0$, which shows that $\eta_\mu = e^{-1}z$, and thus $\psi_\mu(z) = z/e - z$. Using the identity

$$\frac{1}{\pi}\left(\psi_\mu(z) + \frac{1}{2}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(e^{-it}),$$

and Stieltjes's inversion formula, we obtain

$$\mu(dt) = \frac{1}{2\pi}\frac{1-e^{-2}}{1+e^{-2}-2e^{-1}\cos(t)}\, dt, \quad 0 \leq t \leq 2\pi.$$

We then give some applications of results concerning infinity divisibility of the measures associated with subordination functions. For $\mu \in \mathcal{M}_{\mathrm{T}}$, we say μ is nontrivial if it is not a Dirac measure at a point on T.

Lemma 3.25. *Given $\sigma \in \mathcal{ID}(\boxtimes, \mathrm{T})$ which is non-trivial, and $0 < \epsilon < 1$, there exists a positive number $n(\epsilon)$ such that*

$$\eta_{\sigma_t}(\overline{\mathbb{D}}) \subset \mathbb{D}_\epsilon = \{z = re^{i\theta} : 0 \le r \le \epsilon, 0 \le \theta < 2\pi\}$$

for any $t > n(\epsilon)$, where $\sigma_t = \sigma^{\boxtimes t}$.

Proof. If $\sigma = P_0$, the Haar measure on T, the result is trivial. If $\sigma \neq P_0$ is nontrivial, then by [Bercovici and Voiculescu 1992, Theorem 6.7], there exists a finite positive measure ν on T with $\nu(\mathrm{T}) > 0$, $\alpha \in \mathbb{R}$, and an analytic function u defined by

$$u(z) = i\alpha + \int_{\mathrm{T}} \frac{1+\xi z}{1-\xi z}\, d\nu(\xi) \quad \text{for } z \in \mathbb{D},$$

such that $\Sigma_\sigma(z) = \exp(u(z))$. We choose $\sigma_t \in \mathcal{M}_{\mathrm{T}}$ satisfying $\Sigma_{\sigma_t}(z) = \exp(tu(z))$. Noticing that other choices of the multiplicative free convolution power of σ can be obtained from σ_t by a rotation of a multiple of $2\pi t$, it is enough to prove the assertion for σ_t.

We set $\Phi_{\sigma_t} = z\Sigma_{\sigma_t}(z)$; then, by Lemma 3.2, we have

$$\Phi_{\sigma_t}^{-1}(\mathbb{D}) = \eta_{\sigma_t}(\mathbb{D}).$$

For $z = re^{i\theta} \in \mathbb{D}$, we calculate

$$\begin{aligned} |\Phi_{\sigma_t}(z)| &= r\exp\left(t\int_{\mathrm{T}} \frac{1-r^2}{|1-\xi z|^2}\, d\nu(\xi)\right) \\ &\ge r\exp\left(t\int_{\mathrm{T}} \frac{1-r^2}{|1+r|^2}\, d\nu(\xi)\right) = r\exp\left(t\cdot\nu(\mathrm{T})\frac{1-r}{1+r}\right). \end{aligned} \tag{3-31}$$

Since

$$\lim_{t\to\infty} r\exp\left(t\cdot\nu(\mathrm{T})\frac{1-r}{(1+r)}\right) = \infty,$$

we deduce that for any $0 < \epsilon < 1$, there exists a positive number $n(\epsilon)$ such that, for all $t > n(\epsilon)$, we have

$$|\Phi_{\sigma_t}(z)| > 1 \quad \text{for } |z| = \epsilon.$$

By Lemma 3.2, $\Phi_{\sigma_t}(\mathbb{D})$ is a simply connected domain which contains zero, which implies that

$$\eta_{\sigma_t}(\mathbb{D}) = \Phi_{\sigma_t}^{-1}(\mathbb{D}) \subset \mathbb{D}_\epsilon, \text{ for } t > n(\epsilon).$$

The assertion follows because η_{σ_t} extends to a continuous function on $\overline{\mathbb{D}}$. □

For $\mu \in \mathcal{M}_{\mathrm{T}}$, we have

$$\frac{1}{2\pi}\left(\frac{1+\eta_\mu(z)}{1-\eta_\mu(z)}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(e^{-i\theta}), \quad z \in \mathbb{D}.$$

The real part of this function is the Poisson integral of the measure $d\mu(e^{-i\theta})$, we can recover μ by Stieltjes's inversion formula. The functions

$$\frac{1}{2\pi}\Re\left(\frac{1+\eta_\mu(re^{i\theta})}{1-\eta_\mu(re^{i\theta})}\right)=\frac{1}{2\pi}\frac{1-|\eta_\mu(re^{i\theta})|^2}{|1-\eta_\mu(re^{i\theta})|^2} \tag{3-32}$$

converge to the density of $\mu(e^{-i\theta})$ a.e. relative to Lebesgue measure, and they converge to infinity a.e. relative to the singular part of this measure.

Proposition 3.26. *Given $\mu \in \mathcal{M}_{\mathbb{T}}$ and $\sigma \in \mathcal{ID}(\boxtimes, \mathbb{T})$ which is nontrivial, let μ_t be the unique probability measure on* T *such that*

$$\eta_{\mu_t}(z)=\eta_\mu(\eta_{\sigma_t}(z)).$$

Then we have

$$\lim_{t\to\infty}\sup_{\theta\in[0,2\pi]}\left|\frac{d\mu_t(e^{i\theta})}{d\theta}-\frac{1}{2\pi}\right|=0,$$

where $d\mu_t(e^{i\theta})/d\theta$ is the density function of μ_t at $e^{i\theta}$ with respect to Lebesgue measure.

Proof. *Given $0<\epsilon<1$, by Lemma 3.25, there exists $n(\epsilon)>0$ such that $\eta_{\sigma_t}(e^{i\theta})<\epsilon$ for $t\geq n(\epsilon)$, which yields that $\eta_{\mu_t}(z)$ extends continuously to $\overline{\mathbb{D}}$. Thus*

$$|\eta_{\mu_t}(e^{i\theta})|=|\eta_\mu(\eta_{\sigma_t}(e^{i\theta}))|\leq|\eta_{\sigma_t}(e^{i\theta})|<\epsilon,$$

which implies that

$$\frac{1-\epsilon}{1+\epsilon}=\frac{1-\epsilon^2}{|1+\epsilon|^2}\leq\frac{1-|\eta_{\mu_t}(e^{i\theta})|^2}{|1-\eta_{\mu_t}(e^{i\theta})|^2}\leq\frac{1}{|1-\epsilon|^2}. \tag{3-33}$$

Since ϵ is arbitrary, combining (3-32) *with* (3-33), *we prove our assertion.* □

Corollary 3.27. *Given $\mu \in \mathcal{M}_{\mathbb{T}}$ and a nontrivial measure $\nu \in \mathcal{ID}(\boxtimes, \mathbb{T})$, the density functions of the measures $\mu\boxtimes\nu_t$ converge to $1/2\pi$ uniformly as $t\to\infty$; if $\mu\in\mathcal{M}_{\mathbb{T}}$ is nontrivial, the density functions of the measures $\mu^{\boxtimes t}$ converge to $1/2\pi$ uniformly as $t\to\infty$.*

Proof. Noticing Corollary 3.13, Propositions 3.14, 3.26 and Subsection 3.3, we only need to prove the case of $\mu^{\boxtimes t}$ for $\mu\notin\mathcal{M}_{\mathbb{T}}^*$. We point out that the measures are nontrivial imply that the subordination distributions involved are nontrivial.

For $\mu\in\mathcal{M}_{\mathbb{T}}\backslash\mathcal{M}_*$, we have $\mu^{\boxtimes n}=P_0$, where P_0 is the Haar measure on T. Thus the assertion is true for this case. For $\mu\in\mathcal{M}_{\mathbb{T}}\cap\mathcal{M}_*$, but $\mu\notin\mathcal{M}_{\mathbb{T}}^*$, it is shown in [Belinschi and Bercovici 2005] that $\mu\boxtimes\mu\in\mathcal{M}_{\mathbb{T}}^*$; thus this case reduces to the case when $\mu\in\mathcal{M}_{\mathbb{T}}^*$. This finishes the proof. □

4. Multiplicative convolution on $\mathcal{M}_{\mathbb{R}^+}$

Multiplicative free convolution on $\mathcal{M}_{\mathbb{R}^+}$**.** We are interested in the probability measures on the positive real line $\mathbb{R}^+$, which are different from the Dirac measure at zero, we thus set

$$\mathcal{M}^*_{\mathbb{R}^+} = \mathcal{M}_{\mathbb{R}^+} \backslash \{\delta_0\}.$$

Given $\mu \in \mathcal{M}^*_{\mathbb{R}^+}$, we define

$$\psi_\mu(z) = \int_0^{+\infty} \frac{tz}{1-tz}\, d\mu(t),$$

and $\eta_\mu(z) = \psi_\mu(z)/(1+\psi_\mu(z))$. The transform η_μ is characterized by the following proposition; see [Belinschi and Bercovici 2005].

Proposition 4.1. *Let $\eta : \mathbb{C}\backslash\mathbb{R}^+ \to \mathbb{C}$ be an analytic function such that $\eta(\bar{z}) = \overline{\eta(z)}$ for all $z \in \mathbb{C}\backslash\mathbb{R}^+$. Then the following two conditions are equivalent.*

(1) $\eta = \eta_\mu$ *for some* $\mu \in \mathcal{M}^*_{\mathbb{R}^+}$.

(2) $\eta(0-) = 0$ *and* $\arg(\eta(z)) \in [\arg z, \pi)$ *for all* $z \in \mathbb{C}^+$.

It can be shown that η_μ is invertible in some neighborhood of $(-\infty, 0)$, and we set $\Sigma_\mu(z) = \eta_\mu^{-1}(z)/z$ where η_μ^{-1} is defined in some neighborhood of $(\alpha, 0)$. Given two measures $\mu, \nu \in \mathcal{M}^*_{\mathbb{R}^+}$, the multiplicative free convolution of μ and ν is the probability measure $\mu \boxtimes \nu$ in $\mathcal{M}^*_{\mathbb{R}^+}$ such that

$$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z)$$

in some neighborhood of $(\alpha, 0)$, where these functions are defined.

It is known from [Belinschi and Bercovici 2007; Biane 1998] that there exist two analytic functions $\omega_1, \omega_2 : \mathbb{C}\backslash\mathbb{R}^+ \to \mathbb{C}\backslash\mathbb{R}^+$ such that

(1) $\omega_j(0-) = 0$ for $j = 1, 2$,

(2) for any $\lambda \in \mathbb{C}^+$, we have $\omega_j(\bar{\lambda}) = \overline{\omega_j(\lambda)}$ for $j = 1, 2$,

(3) $\eta_{\mu\boxtimes\nu}(z) = \eta_\mu(\omega_1(z)) = \eta_\nu(\omega_2(z))$ for $z \in \mathbb{C}\backslash\mathbb{R}^+$.

For simplicity, we say that ω_1 (resp. ω_2) is the subordination function of $\mu \boxtimes \nu$ with respect to μ (resp. ν), and $\mu \boxtimes \nu$ is subordinated to μ and ν.

The analogy of the Lévy–Khintchine in this setting was proved in [Bercovici and Voiculescu 1992; 1993]. A measure $\mu \in \mathcal{M}_{\mathbb{R}^+}$ is $\boxtimes$-infinitely divisible if and only if $\Sigma_\mu(z) = \exp(u(z))$, with

$$u(z) = a - bz + \int_0^{+\infty} \frac{1+tz}{z-t}\, d\sigma(t),$$

where $b \in \mathbb{R}$ and σ is a finite positive measure on $\mathbb{R}^+$. The analogue of the normal distribution in this context is given by $\Sigma_{\lambda_t}(z) = \exp((t/2)(z+1)/(z-1))$.

Lemma 4.2. *If $\mu, \nu \in \mathcal{M}_{\mathbb{R}^+}^*$, we have*

$$\eta_\mu(z) = \eta_{\mu\boxtimes\nu}(z\Sigma_\nu(\eta_\mu(z)))$$

in some neighborhood of interval $(\alpha, 0)$.

The proof of Lemma 4.2 is identical to the proof of Lemma 3.3, therefore we omit the details.

For any $t > 0$, assume that $\eta_t : \mathbb{D} \to \mathbb{D}$ is the subordination function of $\mu \boxtimes \lambda_t$ with respect to μ, by Lemma 4.2 and the characterization of η-transform, there exists a probability measure ρ_t in $\mathcal{M}_{\mathbb{R}^+}^*$ such that $\eta_{\rho_t}(z) = \eta_t(z)$. The argument in the proof of Lemma 3.4 implies the following result.

Proposition 4.3. *The measure ρ_t is $\boxtimes$-infinitely divisible and its Σ-transform is* $\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\mu(z))$, *and*

$$\Sigma_{\lambda_t}(\eta_\mu(z)) = \exp\biggl(\frac{t}{2}\int_0^{+\infty} \frac{1+\xi z}{\xi z - 1}\, d\mu(\xi)\biggr).$$

We now discuss free convolution semigroups. Given $t > 1$, it is proved in [Belinschi and Bercovici 2005] that one can define $\mu^{\boxtimes t} \in \mathcal{M}_{\mathbb{R}^+}^*$ such that $\Sigma_{\mu^{\boxtimes t}}(z) = (\Sigma_\mu(z))^t$ for $z < 0$ sufficiently close to zero. Similar to the case of $\mathcal{M}_{\mathbb{T}}$, $\mu^{\boxtimes t}$ is subordinated with respect to μ and we denote the subordination function by ω_t. By [Belinschi and Bercovici 2005, Theorem 2.6] and the characterization of η-transform, there exists a probability $\sigma_t \in \mathcal{M}_{\mathbb{R}^+}^*$ such that $\eta_{\sigma_t}(z) = \omega_{t+1}$ for all $t > 0$. Moreover, σ_t is $\boxtimes$-infinitely divisible and its Σ-transform is $\Sigma_{\sigma_t}(z) = [z/\eta_\mu(z)]^t$.

Multiplicative Boolean convolution on $\mathcal{M}_{\mathbb{R}^+}$ and the semigroup $\mathbb{M}_t$. Bercovici [2006] proved that the multiplicative Boolean convolution does not preserve $\mathcal{M}_{\mathbb{R}^+}$. But we can still define $\mu^{\utimes t}$ for $\mu \in \mathcal{M}_{\mathbb{R}^+}$ and $0 \le t \le 1$ as follows. Let $k_\mu(z) = z/\eta_\mu(z)$, the Boolean convolution power $\mu^{\utimes t}$ is defined by

$$k_{\mu^{\utimes t}}(z) = (k_\mu(z))^t.$$

Definition 4.4 [Arizmendi and Hasebe 2013]. A family of maps from $\mathcal{M}_{\mathbb{R}^+}$ to itself is defined by

$$\mathbb{M}_t(\mu) = (\mu^{\boxtimes(t+1)})^{\utimes(1/(t+1))}.$$

It is also shown in [Arizmendi and Hasebe 2013] that $\mathbb{M}_{t+s} = \mathbb{M}_t \circ \mathbb{M}_s$ for $t, s \ge 0$.

Analogous equations. Given a pair of probability measures $\nu, \mu \in \mathcal{M}_{\mathbb{R}^+}$, we also consider, as in the case $\mathcal{M}_{\mathbb{T}}$, the semigroups $\nu \boxtimes \lambda_t$ and $\mu^{\boxtimes(t+1)}$, the subordination functions η_t and ω_{t+1}, and their associated probability measures ρ_t, σ_t for all $t > 0$. Since $\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\nu(z))$ and $\Sigma_{\sigma_t}(z) = [z/\eta_\mu(z)]^t$, we deduce that $\eta_t = \omega_{t+1}$ if and only if

$$\Sigma_\lambda(\eta_\nu(z)) = \frac{z}{\eta_\mu(z)}.$$

Applying the same argument as in the proof of the Theorem 1.1, we obtain the following result.

Theorem 4.5. *Given a pair of probability measures $\mu, \nu \in \mathcal{M}_{\mathbb{R}^+}$ such that*

$$\Sigma_\lambda(\eta_\nu(z)) = \frac{z}{\eta_\mu(z)}, \quad z \in \mathbb{C}^+, \tag{4-1}$$

we have $\Sigma_\lambda(\eta_{\nu\boxtimes\lambda_t}(z)) = \dfrac{z}{\eta_{\mathbb{M}_t(\mu)}(z)}$, $z \in \mathbb{C}^+$.

5. A description of the analogue of the normal distribution

Biane [1997a; 1997c] studied free Brownian motion and proved many important results. In this section, we give a new proof for the density functions of the free multiplicative analogue of the normal distributions, which was first obtained in [Biane 1997c] (See also [Demni and Hmidi 2012] for a different approach). Some results are new. For example, we show that λ_t is unimodal for the circle case; and we show that $\Phi_\lambda^{-1}(\mathbb{C}^+)$ contains infinitely many connected components where λ is the free multiplicative analogue of the normal distribution on the positive half line with $\Sigma_\lambda(z) = \exp((z+1)/(z-1))$. We also give a description of the boundaries Ω_t, Ω (defined below), we observe that $\partial\Omega_t$ can be parametrized by θ and $\partial\Omega$ can be parametrized by r.

The circle case. Let $\lambda_t \in \mathcal{M}_{\mathrm{T}}$ be the analogue of the normal distribution such that $\Sigma_{\lambda_t}(z) = \exp((t/2)(1+z)/(1-z))$. We set $\Phi_t(z) = z\Sigma_{\lambda_t}(z)$, and let $\Omega_t = \{z \in \mathbb{D} : |\Phi_t(z)| < 1\}$. By Lemma 3.2, η_{λ_t} extends continuously to the unit circle T, Ω_t is simply connected and bounded by a simple closed curve, and we have $\partial\Omega_t = \eta_{\lambda_t}(\mathrm{T})$.

Observe that for $t \neq 4$, Φ_t has zeros of order one at $z_1(t) = (2-t+\sqrt{t^2-4t})/2$ and $z_2(t) = (2-t-\sqrt{t^2-4t})/2$. Φ_4 has a zero of order two at -1; and for all t, Φ_t has an essential singularity at 1, and no other zeros and singularities. For $0 < t < 4$, $z_1(t), z_2(t) \in \mathrm{T}$ and $z_2(t) = \overline{z_1(t)}$, we let $\theta_1(t) \in (0, \pi)$ and $\theta_2(t) \in (\pi, 2\pi)$ such that $z_1(t) = e^{i\theta_1(t)}$ and $z_2(t) = e^{i\theta_2(t)}$. We have $z_1(4) = z_2(4) = -1$ and for $t > 4$, $z_1(t) \in (-1, 0)$ and $z_2(t) \in (-\infty, -1)$.

We define

$$g_t(r,\theta) = r\exp\left(\frac{t}{2}\frac{1-r^2}{1-2r\cos\theta+r^2}\right) = |\Phi_t(z)|$$

for $z = re^{i\theta}$. The unit circle is parametrized by $\mathrm{T} = \{e^{i\theta} : 0 \le \theta < 2\pi\}$.

Lemma 5.1. *For $0 < t < 4$, $\partial\Omega_t = \{z = e^{i\theta} : \theta_1(t) \le \theta \le \theta_2(t)\} \cup \mathcal{L}_{1,t} \cup \mathcal{L}_{2,t}$, where $\mathcal{L}_{1,t}$ is an analytic curve, and $\mathcal{L}_{1,t}$ is in $\mathbb{D} \cap \mathbb{C}^+$ except one of its endpoints, and $\mathcal{L}_{2,t}$ is the reflection of $\mathcal{L}_{1,t}$ about x-axis. $\mathcal{L}_{1,t}$ can be parametrized by $\gamma_t(u)$ $(0 \le u \le 1)$ such that $\gamma_t(0) \in \mathbb{R}$, $\gamma_t(1) = z_1(t)$ and $\gamma_t(u) \subset \mathbb{D} \cap \mathbb{C}^+$ for $0 < u < 1$. Moreover, $|\gamma_t(u)|$ is an increasing function of u on the interval* [0, 1].

Proof. Observing that $\Phi_t(\bar{z}) = \overline{\Phi_t(z)}$, we see that $\partial\Omega_t$ is symmetric with respect to x-axis. Since Ω_t is simply connected and $\partial\Omega_t$ is a simple closed curve, $\partial\Omega_t$ intersects x-axis at two points.

Restricting Φ_t to real numbers, we find that $\Phi_t(\mathbb{R}) \subset \mathbb{R}$, and that Φ_t is an increasing function on $(-1, 1)$ since $\Phi_t'(z)$ is positive for $z \in (-1, 1)$. From $\Phi_t(-1) = -1$ and $\lim_{z\to 1^-} \Phi_t(z) = +\infty$, we deduce that

$$\Phi_t^{-1}((-1, 1)) = (-1, x(t)), \tag{5-1}$$

where $x(t)$ is the unique solution of the equation $\Phi_t(z) = 1$ for $z \in (-1, 1)$. The fact that $\Phi_t'(z) \neq 0$ for $z \neq z_t(t), z_2(t)$ implies that Φ_t is locally invertible for $z \neq z_1(t), z_2(t)$. Combining the fact that $\Phi_t(\mathrm{T}\backslash\{1\}) \subset \mathrm{T}$, we obtain that

$$\{e^{i\theta} : \theta_1(t) \leq \theta \leq \theta_2(t)\} \subset \partial\Omega_t$$

and $\partial\Omega_t$ has corners of opening $\pi/2$ at $z_1(t)$ and $z_2(t)$.

Since Φ_t is a conformal mapping from Ω_t to $\mathbb{D}$, by the symmetry $\Phi_t(\bar{z}) = \overline{\Phi_t(z)}$ and (5-1), noticing that $\Phi_t'(0) = 1$, we thus deduce that $\Phi_t(\Omega_t \cap \mathbb{C}^+) \subset \overline{\mathbb{D}} \cap \mathbb{C}^+$. Since $\partial\Omega_t$ is a simple closed curve, $z_1(t)$ and $x(t)$ are connected by $\partial\Omega_t$. It is clear that $\partial\Omega_t\backslash\{e^{i\theta} : \theta_1(t) \leq \theta \leq \theta_2(t)\}$ does not intersect with T, we thus assume the curve $\gamma_t = \{\gamma_t(u) : 0 \leq u \leq 1\}$ is the part of Ω_t which connects $z_1(t)$ and $x(t)$ such that $\gamma_t(0) = x(t)$, $\gamma_t(1) = z_1(t)$ and $\gamma_t(u) \in \mathbb{D}$ for $0 < u < 1$.

We claim that $|\gamma_t(u)|$ is an increasing function of u on the interval $[0, 1]$. For given $0 < r < 1$, we define the function of θ by

$$g_{t,r}(\theta) = g_t(r, \theta) = |\Phi_t(re^{i\theta})|.$$

Then $g_{t,r}$ is a strictly decreasing function of θ on the interval $[0, \pi]$. From the fact that Ω_t is simply connected, we deduce that, for $z_0 \in \overline{\Omega_t} \cap \mathbb{D} \cap \mathbb{C}^+$, the arc

$$\{re^{i\theta} : |r| = |z_0|, \arg z_0 < \theta \leq \pi\} \subset \Omega_t. \tag{5-2}$$

Given $0 < u_1 < u_2 < 1$, we need to prove that $|\gamma_t(u_1)| < |\gamma_t(u_2)|$. Since $[0, x(t)] \subset \overline{\Omega_t}$, we obtain from (5-2) that

$$\{re^{i\theta} : 0 \leq r \leq x(t), 0 < \theta \leq \pi\} \subset \Omega_t, \tag{5-3}$$

which shows that $|\gamma_t(u_1)| > x(t)$. Suppose that $|\gamma_t(u_1)| \geq |\gamma_t(u_2)|$. There exists $0 < u_1' \leq u_1$ such that $|\gamma_t(u_1')| = |\gamma_t(u_2)|$. If $\arg(\gamma_t(u_1')) > \arg(\gamma_t(u_2))$, then by (5-2), $\gamma_t(u_2) \in \Omega_t$ and thus $\gamma_t(u_2) \notin \partial\Omega_t$; if $\arg(\gamma_t(u_1')) < \arg(\gamma_t(u_2))$, then $\gamma_t(u_1') \in \Omega_t$ and thus $\gamma_t(u_1') \notin \partial\Omega_t$. For both cases, we obtain a contradiction. Thus $|\gamma_t(u_1)| < |\gamma_t(u_2)|$ and our claim is proved. $\square$

For $t > 0$, we let $x_1(t) \in (0, 1)$ be the unique solution of the equation $\Phi_t(z) = 1$ for $z \in (0, 1)$. For $0 < t \leq 4$ we let $x_2(t) = -1$; for $t > 4$, we let $x_2(t) \in (-1, 0)$ be the unique solution of the equation $\Phi_t(z) = -1$ for $z \in (-1, 0)$.

Lemma 5.2. *For $t \geq 4$, $\partial\Omega_t = \mathcal{L}_{1,t} \cup \mathcal{L}_{2,t}$, where $\mathcal{L}_{1,t}$ is an analytic curve, and $\mathcal{L}_{1,t}$ is in $\mathbb{D} \cap \mathbb{C}^+$ except its endpoints, and $\mathcal{L}_{2,t}$ is the reflection of $\mathcal{L}_{1,t}$ about x-axis. $\mathcal{L}_{1,t}$ can be parametrized by $\gamma_t(u)$ $(0 \leq u \leq 1)$ such that $\gamma_t(0) = x_1(t)$, $\gamma_t(1) = x_2(t)$ and $\gamma_t(u) \subset \mathbb{D} \cap \mathbb{C}^+$ for $0 < u < 1$. Moreover, $|\gamma_t(u)|$ is an increasing function of u on the interval* $[0, 1]$.

Proof. Recall that Φ_4 has a zero of order two at -1. For all $t > 4$, $z_2(t) < -1$ and $z_1 \in (-1, 0)$. The assertion follows from the similar arguments in the proof Lemma 5.1. □

From the proof of Lemmas 5.1 and 5.2, for $t > 0$, we have $\Phi_t^{-1}((-1, 1)) = (x_2(t), x_1(t))$. Moreover, $x_1(t) = \min\{|z| : z \in \partial\Omega_t\}$ and $-x_2(t) = \max\{|z| : z \in \partial\Omega_t\}$.

Remark. In fact, for any $t > 0$, from the equation

$$g_t(r, \theta) = 0, \quad 0 < r < 1, \ 0 \leq \theta \leq \pi,$$

we can prove that $dr/d\theta > 0$ for $0 < \theta < \pi$, which implies that if $z \in \partial\Omega_t$, the entire radius $\{rz : 0 \leq r < 1\}$ is contained in Ω_t. Therefore, $\partial\Omega_t$ can be parametrized by θ.

Lemma 5.3. *Using the same notations in Lemmas 5.1 and 5.2, for $t > 0$, the function $|1 - \gamma_t(u)|$ is an increasing function of u on* $[0, 1]$.

Proof. We only prove the case when $0 < t < 4$, the proof for other cases are similar. Noticing that $|1 - re^{i\theta}|^2 = 1 - 2\cos\theta + r^2$, since $|\gamma_t(u)|$ is an increasing function of u, to prove the assertion, we only need to prove that for the implicit function $r\exp((t/2)(1 - r^2)/h) = 1$ of r and h, the value of h increases when r increases on $(0, 1)$. From this equation, we have $h = h(r) = -(t/2)(1 - r^2)/(\ln r)$. One can check that $h'(r) > 0$ for $0 < r < 1$, therefore h is an increasing function of r. □

Theorem 5.4. *Denote by A_t the support of λ_t.*

(1) *For $t > 0$, the measure λ_t has no singular part, and its density function is an analytic function. $A_{t_1} \subset A_{t_2}$ if $t_1 < t_2 < 4$. $A_t \subsetneq \mathrm{T}$ for $0 < t < 4$ and $A_t = \mathrm{T}$ for $t \geq 4$.*

(2) *The measure λ_t is unimodal for all $t > 0$ and its density is maximal at $z = 1$ and is minimal at $z = -1$.*

(3) *The density fucntion $d\lambda_t/d\theta$ converges uniformly to $1/(2\pi)$ as $t \to \infty$.*

Proof. Since $z = 1$ is not in the closure of $\Omega_t = \eta_{\lambda_t}(\mathbb{D})$, the singular part of λ_t vanishes. From the analyticity of Φ_t or a general theorem in [Belinschi and Bercovici 2005], the density function is analytic.

For $0 < t < 4$, set $a_1(t) = \Phi_t(z_1(t))$, $a_2(t) = \Phi_t(z_2(t))$. Note that $\eta_{\lambda_t}(\Phi_t(z)) = z$ for $z \in \overline{\Omega_t}$. From (3-32) we see that A_t is the closed arc on T with endpoints $a_1(t), a_2(t)$ which contains 1. Thus, to prove that $A_{t_1} \subset A_{t_2}$, it is enough to prove that $\arg(a_1(t))$ is an increasing function of t. A direct computation shows that $|z_1(t) - 1|^2 = t$ and $\arg(\Sigma_{\lambda_t}(z_1(t))) = \Im z_1(t) = \sqrt{t(4-t)}/2$. We thus have

$$\arg(a_1(t)) = \Im z_1(t) + \arg(z_1(t)) = \sin(\theta_1(t)) + \theta_1(t).$$

From $z_1(t) = (2 - t + \sqrt{t^2 - 4t})/2$ we see that $\theta_1(t)$ is an increasing function of t. The function $\theta \to \sin(\theta) + \theta$ is an increasing function on $(0, \pi)$. Thus $\arg(a_1(t))$ is an increasing function of t and (1) is proved.

To prove (2), recall that a probability measure is unimodal if its density with respect to Lebesgue measure has a unique local maximum. η_{λ_t} extends continuously to T, we thus have

$$\frac{d\lambda_t(e^{-i\theta})}{d\theta} = \frac{1}{2\pi} \frac{1 - |\eta_{\lambda_t}(e^{i\theta})|^2}{|1 - \eta_{\lambda_t}(e^{i\theta})|^2}. \tag{5-4}$$

We first prove the case when $0 < t < 4$. From $\eta_{\lambda_t}(\Phi_t(z)) = z$ for $z \in \overline{\Omega_t}$ and $\eta_{\lambda_t}(1) = x(t)$, to prove λ_t is unimodal, by the boundary correspondence, it is enough to show that the function f_t of u defined by

$$f_t(u) := \frac{1 - |\gamma_t(u)|^2}{|1 - \gamma_t(u)|^2},$$

is a decreasing function on $[0, 1]$ and is maximal at 0. Since $\gamma_t(u) \in \partial\Omega_t$, we have $|\Phi_t(\gamma_t(u))| = 1$. In other words, we have

$$|\gamma_t(u)| \exp\left(\frac{t}{2} f_t(u)\right) = |\gamma_t(u)| \exp\left(\frac{t}{2} \frac{1 - |\gamma_t(u)|^2}{|1 - \gamma_t(u)|^2}\right) = 1. \tag{5-5}$$

As we shown in Lemma 5.1 that the function $|\gamma_t(u)|$ is an increasing function of u, from (5-5), we deduce that f_t is a decreasing function of u and $\max\{f_t\} = f_t(0)$. By the symmetric property of the function Φ_t in Lemma 5.1, the density function is symmetric with respect to x-axis as well. Thus the density of λ_t has only one local maximum at $\Phi_t(\gamma_t(0)) = \Phi_t(x_1(t)) = 1$.

The proof for the case $t \ge 4$ is similar. In this case $A_t = \mathrm{T}$ and $\max\{f_t\} = f_t(0)$ and $\min\{f_t\} = f_t(1)$. Part (3) is a consequence of Corollary 3.27. □

Remark. From the proof of Theorem 5.4, we see that, for $t < 4$,

$$\arg(a_1(t)) = \theta_1(t) + \sin(\theta_1(t)) = \tfrac{1}{2}\sqrt{t(4-t)} + \arccos\left(1 - \frac{t}{2}\right),$$

which implies a known result in [Biane 1997c], namely

$$A_t = \left\{e^{i\theta} : -\tfrac{1}{2}\sqrt{t(4-t)} - \arccos\left(1 - \frac{t}{2}\right) \le \theta \le \tfrac{1}{2}\sqrt{t(4-t)} + \arccos\left(1 - \frac{t}{2}\right)\right\}.$$

The positive half line case. Let $\lambda \in \mathcal{M}_{\mathbb{R}^+}$ be the analogue of the normal distribution such that $\Sigma_\lambda(z) = \exp((z+1)/(z-1))$.

We restate [Bercovici and Voiculescu 1993, Proposition 6.14] in terms of η and Σ transforms as follows.

Lemma 5.5. *Let μ be a $\boxtimes$-infinitely divisible measure on $\mathbb{R}^+$, and set $\Phi_\mu(z) := z\Sigma_\mu(z)$.*

(1) *We have $\Phi_\mu\left(\eta_\mu(z)\right) = z$ for every $z \in \mathbb{C}^+$.*

(2) *The set $\{\eta_\mu(z) : z \in \mathbb{C}^+\} = \Omega$, where Ω is the component of the set $\{z \in \mathbb{C}^+ : \Im(\Phi_\mu(z)) > 0\}$ whose boundary contains the left half line $(-\infty, 0)$. Moreover, $\eta_\mu(\Phi_\mu(z)) = z$ for $z \in \Omega$.*

We set $\Phi_\lambda(z) = z\exp((z+1)/(z-1))$. The following lemma is elementary.

Lemma 5.6. *Φ_λ has zero of order one at $2-\sqrt{3}$ and $2+\sqrt{3}$, and Φ_λ has an essential singularity at 1. These are the only zeros and singularities of Φ_λ.*

Theorem 5.7. *The measure λ has no singular part. The support of this measure is the closure of its interior, and this interior has only one connected component.*

Proof. By [Bercovici and Voiculescu 1992, Theorem 7.5], the measure λ has compact support on $\mathbb{R}^+$.

Let Ω be the component of $\{z \in \mathbb{C}^+ : \Im(\Phi_\lambda(z)) > 0\}$ whose boundary contains $(-\infty, 0)$. By Lemma 5.5, $\eta_\lambda : \mathbb{C}^+ \to \Omega$ is a conformal map and Φ_λ is its inverse map; thus Ω is simply connected. By Lemma 5.6, $\partial\Omega$ is locally analytic. A general theorem in complex analysis tells us that η_λ extends continuously to $\mathbb{C}^+ \cup \mathbb{R}$ and it establishes a homeomorphism between the real axis and $\partial\Omega$. We continue to denote by η_λ and Φ_λ their extensions.

We claim that

$$\partial\Omega = \left(-\infty, 2-\sqrt{3}\right] \cup \left[2+\sqrt{3}, +\infty\right) \cup \mathcal{L},$$

where $\mathcal{L}$ is an analytic and open curve in $\mathbb{C}^+$ with endpoints $2-\sqrt{3}$ and $2+\sqrt{3}$. We denote $\gamma(t) = \eta_\lambda(t)$, $t \in \mathbb{R}$ be a parametrization of $\partial\Omega$. Set $t_1 = \Phi_\lambda(2-\sqrt{3}) > 0$ and $t_2 = \Phi_\lambda(2+\sqrt{3}) > 0$. Then $\eta_\lambda(t_1) = 2-\sqrt{3}$ and $\eta_\lambda(t_2) = 2+\sqrt{3}$, and $\mathcal{L} = \{\gamma(t)\}_{t_1<t<t_2}$. Note that

(1) $(-\infty, 0) \subset \partial\Omega$,

(2) $\Phi'_\lambda(x) > 0$ for all $x \in (-\infty, 2-\sqrt{3})$.

From this we deduce that $(-\infty, 2-\sqrt{3}) \subset \partial\Omega$. Lemma 5.6 tells us Φ_λ has a zero of order one at $2-\sqrt{3}$, therefore $\partial\Omega$ has a corner of opening $\pi/2$ at $2-\sqrt{3}$. Note that $\Phi'_\lambda(x) > 0$ for all $x \in (2+\sqrt{3}, +\infty)$, thus $(2+\sqrt{3}, +\infty) \subset \partial\Omega$, and $\partial\Omega$ has a corner of opening $\pi/2$ at $2+\sqrt{3}$.

It remains to prove that $\mathscr{L} \cap \mathbb{R} = \varnothing$. First we show $1 \notin \mathscr{L}$. Suppose that is the case, and suppose $\gamma(t_0) = 1$ where $t_1 < t_0 < t_2$, by continuity, we have

$$\gamma(t) \exp\left(\frac{\gamma(t)+1}{\gamma(t)-1}\right) = \Phi_\lambda(\gamma(t)) = \Phi_\lambda(\eta_\lambda(t)) = t$$

for all $t \in \mathbb{R}$. Therefore in a small neighborhood of t_0, we have

$$\frac{\gamma(t)+1}{\gamma(t)-1} = \ln \frac{t}{\gamma(t)}.$$

The left side of the above equation blows up, while the right hand side is bounded. This contradiction tells us that $1 \notin \mathscr{L}$. Now suppose $\mathscr{L}$ touches the real axis at $x_0 \in (2-\sqrt{3}, 1) \cup (1, 2+\sqrt{3})$. Since Ω is connected, it is not hard to see that x_0 must be a critical point of Φ_λ. This is not possible by Lemma 5.6. We therefore proved that $\mathscr{L} \subset \mathbb{C}^+$ and the claim.

From the definitions of the Cauchy transform and η-transform, one can easily check that

$$G_\lambda\left(\frac{1}{z}\right) = \frac{z}{1-\eta_\lambda(z)}.$$

From the above equation we know that G_λ extends to be a continuous function on $\mathbb{C} \cup \mathbb{R}$, and $\{x \in \mathbb{R} : \Im(G_\mu(x)) > 0\} = (1/t_2, 1/t_1)$. By the Stieltjes inverse formula, we deduce that the support of λ is $(1/t_2, 1/t_1)$. From the analyticity of the curve $\mathscr{L} \subset \mathbb{C}^+$, we conclude that λ has positive and analytic density in the interior of its support. $\square$

We are interested in the level curves of the function

$$f(r,\theta) = \theta - \frac{2r\sin\theta}{1-2r\cos\theta + r^2} = \arg(\Phi_\lambda(z)), \tag{5-6}$$

where $z = r^{i\theta} \in \mathbb{C}^+$. For $t \leq 0$, set $\gamma_t = \{z = re^{i\theta} \in \mathbb{C}^+ : f(r,\theta) = t\}$.

Proposition 5.8. (A) *γ_0 is a simple open curve with endpoints $2-\sqrt{3}, 2+\sqrt{3}$ and $\gamma_0 = \mathscr{L}$.*

(B) *γ_t is a simple open curve which starts at $z = 1$ and ends at $z = 1$ as well for all $t < 0$.*

Denote by Ω_0 the open domain bounded $\gamma_0 \cup [2-\sqrt{3}, 2+\sqrt{3}]$. For all $t < 0$, denote by Ω_t the open domain bounded $\gamma_t \cup \{1\}$.

(C) *For $t_1 < t_2 \leq 0$, we have $\Omega_{t_1} \subset \Omega_{t_2}$; and for all $t_0 \leq 0$, $\Omega_{t_0} = \cup_{t<t_0}\Omega_t$.*

Proof. Given $\theta \in (0,\pi)$, we define a function of r by $f_\theta(r) = f(r,\theta)$ for $r \in (0,+\infty)$. We first note that $f(r,\theta) < \theta < \pi$ and observe that

$$\lim_{r\to+\infty} f_\theta(r) = \theta.$$

Thus $\{z = re^{i\theta} : f(r,\theta) > 0,\ 0 < \theta < \pi\} \subset \Phi^{-1}(\mathbb{C}^+)$.

Given $\theta \in (0,\pi)$ and $t \le 0$, the equation $f(r,\theta) = t$ is equivalent to the quadratic equation

$$h_\theta(r) := (\theta - t)r^2 - (2(\theta - t)\cos\theta + 2\sin\theta)r + \theta - t = 0 \tag{5-7}$$

with discriminant $d(\theta,t) = [2(\theta-t)\cos\theta + 2\sin\theta]^2 - 4(\theta-t)^2$. We then rewrite $d(\theta,t)$ as follows.

$$d(\theta,t) = 4(1-\cos^2(\theta))\left[\frac{\sin\theta}{1+\cos\theta} + \theta - t\right]\cdot\left[\frac{\sin\theta}{1-\cos\theta} - \theta + t\right], \tag{5-8}$$

The first two factors in (5-8) are never zero for $\theta \in (0,\pi)$; thus only the last factor in (5-8) matters to determine the sign of $d(\theta,t)$. We consider the function k by $k(\theta) = \sin\theta/(1-\cos\theta) - \theta$ for $\theta \in (0,\pi)$, and calculate

$$k'(\theta) = \frac{1}{\cos\theta - 1} - 1 < 0, \tag{5-9}$$

which implies that k is a decreasing function of θ. For $t \le 0$, we now set $d_t(\theta) := d(\theta,t)$. We then deduce that $d_t(\theta) = 0$ has exactly one solution, which we denote by θ_t, and $d_t(\theta) > 0$ if and only if $0 < \theta < \theta_t$. Therefore, the half line $r = \theta$ intersects with γ_t at two points if and only if $0 < \theta < \theta_t$ and the half line $r = \theta_t$ is tangent to γ_t. Moreover, $\theta_{t_1} < \theta_{t_2}$ if $t_1 < t_2 \le 0$.

For the solutions of the equation $f(r,\theta) = 0$, one can check as $\theta \to 0$, r satisfying the equation $r^2 - 4r + 1$. Given $t < 0$, for the solutions of the equation $f(r,\theta) = t$, we can easily see that r tend to 1 as $\theta \to 0$. Now (A) and (B) follow from this observation.

Given $\theta \in (0,\pi)$, from (5-6), we see that the function $f_\theta(r)$ defined by $f_\theta(r) = f(r,\theta)$ has exactly one local minimum at $r = 1$. $f_\theta(r)$ is a decreasing function of r on $(0, 1)$ and an increasing function of r on $(1,\infty)$. Therefore, if the half line $r = \theta$ intersects γ_t at two points, one of them is inside the unit circle of $\mathbb{C}$ and the other one is outside the unit circle. We conclude that (C) is valid. □

It is interesting to compare the next result with Proposition 2.2 and Lemma 3.2.

Corollary 5.9. *We have $\Phi_\lambda^{-1}(\mathbb{C}^+) = \Omega \cup_{k=1}^{\infty} (\Omega_{(2k-1)\pi} \backslash \Omega_{(2k-2)\pi})$. Moreover, Ω and $\Omega_{(2k-1)\pi} \backslash \Omega_{(2k-2)\pi}$ $(k = 1, 2, \ldots)$ are all connected components of Φ_λ. In particular, $\Phi_\lambda^{-1}(\mathbb{C}^+)$ has infinitely many connected components.*

We would like to point out that for $z = re^{i\theta} \in \mathscr{L} = \gamma_0$, the curve $\mathscr{L}$ can be parametrized by r. Noticing (5-7) and (5-8), we first observe the following equivalences:

$$d(\theta,0) = 0 \iff \theta\cos\theta + \sin\theta = \theta \iff r = 1. \tag{5-10}$$

By (5-9), we see that (5-10) has exactly one solution θ_0 for $\theta \in (0, \pi)$. By differentiating the equation $f(r, \theta) = 0$, we obtain

$$\frac{d\theta}{dr} = \frac{2\theta\cos\theta + 2\sin\theta - 2\theta r}{r^2 + 2\theta\sin\theta - 4\cos\theta + 1}. \tag{5-11}$$

Thus, $d\theta/dr = 0$ if and only if $r = (\theta\cos\theta + \sin\theta)/\theta$. Fix θ, the equation $f_\theta(r) = 0$ is equivalent to the quadratic equation $\theta r^2 - (2\theta\cos\theta + 2\sin\theta)r + \theta = 0$, from which we deduce that $r = (\theta\cos\theta + \sin\theta)/\theta$ if and only if $d(r, 0) = 0$. From (5-11) and continuity of $d\theta/dr$, we see that $d\theta/dr > 0$ for $0 < \theta < \theta_0$, $r < 1$ and $d\theta/dr < 0$ for $0 < \theta < \theta_0$, $r > 1$. Therefore, for the solutions of the equation $f(r, \theta) = 0$, θ is a function of r and the curve $\mathscr{L}$ can be parametrized by r.

Denote by g the density function of λ. From the equation $G_\lambda(1/x) = x/(1 - \eta_\lambda(x))$, we obtain the following formula for the density function of λ.

Proposition 5.10. *Given $z = re^{i\theta} \in \gamma_0 = \mathscr{L}$, we have*

$$g(1/x) = \theta\Phi_\lambda(z) = r\theta\exp\left(\frac{r^2 - 1}{1 - 2r\cos\theta + r^2}\right),$$

where $x = \Phi_\lambda(z)$.

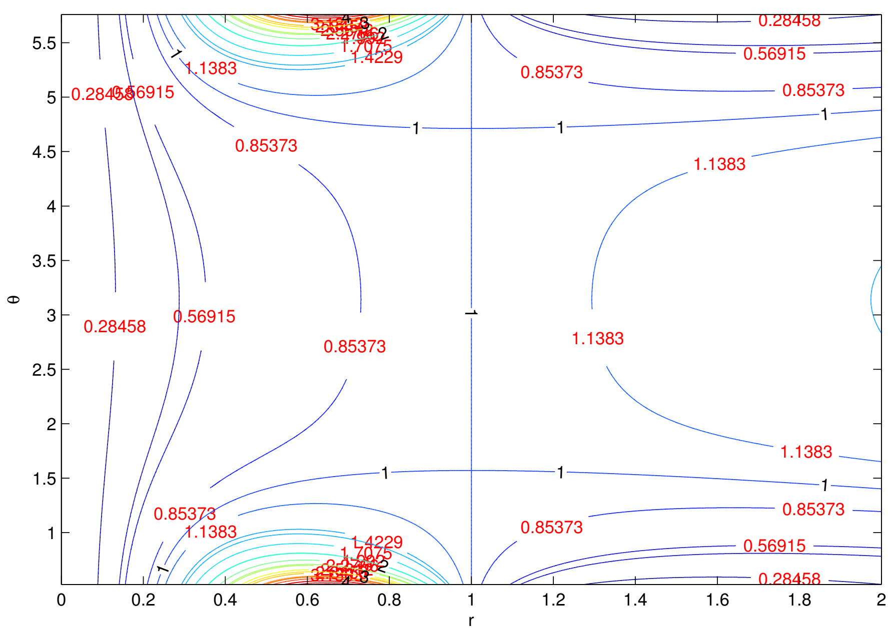

Figure 1. Level curves of $g_2(r, \theta) = |\Phi_2(re^{i\theta})|$. The vertical axis indicates θ, and the horizontal axis indicates r.

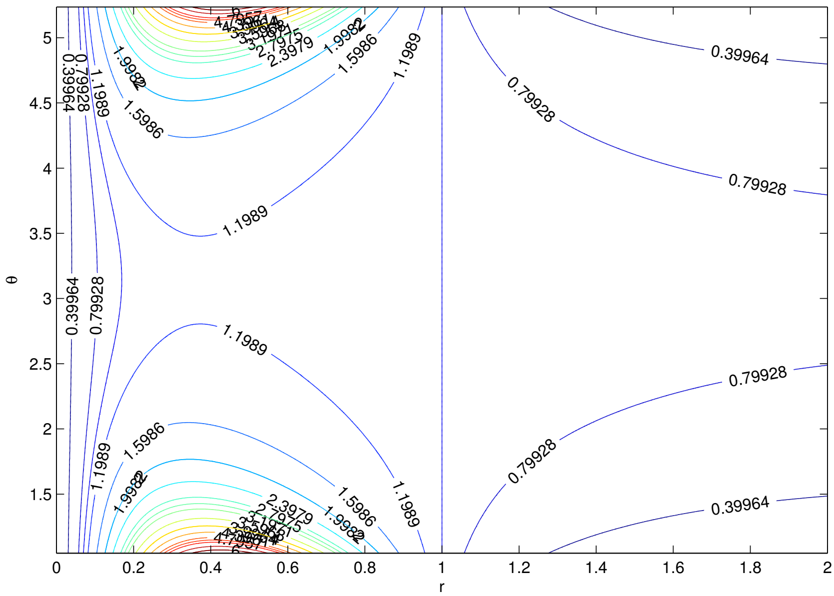

Figure 2. Level curves of $g_5(r,\theta) = |\Phi_5(re^{i\theta})|$. The vertical axis indicates θ, and the horizontal axis indicates r.

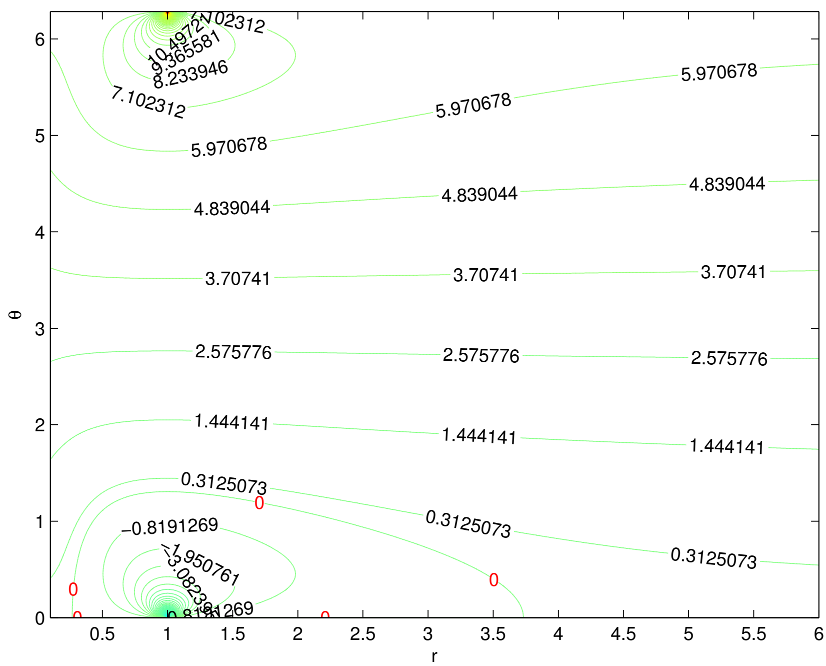

Figure 3. Level curves of $f(r,\theta) = \arg(\Phi_\lambda(re^{i\theta}))$. The vertical axis indicates θ, and the horizontal axis indicates r.

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