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NEW INVARIANTS FOR COMPLEX MANIFOLDS AND RATIONAL SINGULARITIES

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Dedicated to Professor Stephen S.-T. Yau on the occasion of his sixtieth birthday.

Two new invariants $f^{(1,1)}$ and $g^{(1,1)}$ were introduced by Du and Yau for solving the complex Plateau problem. These invariants measure in some sense how far away the complex manifolds are from having global complex coordinates. In this paper, we study these two invariants further for rational surface singularities. We prove that these two invariants never vanish for rational surface singularities, which confirms Yau's conjecture for strict positivity of these two invariants. As an application, we solve regularity problem of the Harvey–Lawson solution to the complex Plateau problem for a strongly pseudoconvex compact rational CR manifold of dimension 3. We also construct resolution manifolds for rational triple points by means of local coordinates and show that $f^{(1,1)} = g^{(1,1)} = 1$ for rational triple points.

1. Introduction

Let M be a complex manifold of dimension n. It is a natural question to ask how far away this complex manifold is from having global complex coordinates. Together with Hing Sun Luk and Stephen Yau, the first author introduced in [Du et al. 2011] new biholomorphic invariants that give some measurements for this purpose.

It is well-known that if M is a complex submanifold in $\mathbb{C}^N$, then, given any global holomorphic p-form α on M, there exists a holomorphic p-form $\tilde{\alpha}$ on $\mathbb{C}^N$ such that the restriction of $\tilde{\alpha}$ to M is α. Obviously, $\tilde{\alpha}$ is a p-th wedge product of holomorphic 1-forms. However, for non-Stein complex manifolds, such as the resolution manifolds of singularities, the situation is totally different. In particular,

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for surface singularities, we considered in [Du et al. 2011; Du and Yau 2012] two invariants $f^{(1,1)}$ and $g^{(1,1)}$ and showed that these two invariants are strictly positive for some special singularities. So the resolution manifolds of these special singularities do not have the above properties.

A direct application of the positivity of $g^{(1,1)}$ is for solving one of the most fundamental questions in complex geometry: the complex Plateau problem [Du and Yau 2012]. Given a strongly pseudoconvex CR manifold X in $\mathbb{C}^N$, the problem asks when X is the boundary of a complex manifold V in $\mathbb{C}^N$. By the beautiful work of Harvey and Lawson [1975], as well as [Yau 1981b; Luk and Yau 1998a], X is a boundary of a complex variety V with only isolated singularities if X is contained in the boundary of a strictly pseudoconvex domain in $\mathbb{C}^N$. Thus from the complex Plateau problem point of view, it is very desirable to introduce a numerical invariant for isolated singularities which never vanishes. However many numerical invariants such as the geometric genus p_g, the arithmetic genus p_a and the irregularity q vanish on rational singularities. In [Du et al. 2011; Du and Yau 2012] this idea was used to introduce two invariants $f^{(1,1)}$ and $g^{(1,1)}$ for isolated surface singularities. The invariant $g^{(1,1)}$ was used in the latter article to solve the regularity problem of the Harvey–Lawson solution to the complex Plateau problem.

Those two articles provided a detailed study of $f^{(1,1)}$ and $g^{(1,1)}$. Yau has the following conjecture:

Conjecture. For all normal surface singularities, the invariants $f^{(1,1)}$ and $g^{(1,1)}$ are strictly positive.

Du and Yau showed that these two numerical invariants are strictly positive when the surface singularities have a $\mathbb{C}^*$-action. They also gave explicit calculations for $f^{(1,1)}$ and $g^{(1,1)}$ for rational double points and cyclic quotient singularities and proved that they are exactly 1. In this paper, we shall prove that for rational surface singularities, $f^{(1,1)}$ and $g^{(1,1)}$ also never vanish. So, our results in this paper confirm the conjecture.

Theorem 2.8. *For rational surface singularities,* $f^{(1,1)} = g^{(1,1)} \geq 1$.

As an application, we solve the regularity problem of the Harvey–Lawson solution to the complex Plateau problem for a strongly pseudoconvex compact rational CR manifold of dimension 3.

Theorem 3.9. *Let X be a strongly pseudoconvex compact rational CR manifold of dimension* 3. *Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is a boundary of the complex submanifold $V \subset D - X$ with boundary regularity if and only if $g^{(1,1)}(X) = 0$.*

We also construct resolution manifolds for rational triple points by local coordinates. By using these local coordinates, we give explicit calculations of $f^{(1,1)}$ and

$g^{(1,1)}$ for rational triple points and prove that they are also 1.

Theorem 4.4. *For 2-dimensional rational triple points,* $f^{(1,1)} = g^{(1,1)} = 1$.

2. Invariants of singularities

Let V be a n-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities. Yau [1982] considered four kinds of sheaves of germs of holomorphic p-forms:

(1) $\overline{\Omega}_V^p := \pi_* \Omega_M^p$, where $\pi : M \to V$ is a resolution of singularities of V.

(2) $\overline{\overline{\Omega}}_V^p := \theta_* \Omega^p_{V \setminus V_{\text{sing}}}$ where $\theta : V \setminus V_{\text{sing}} \to V$ is the inclusion map and V_{sing} is the singular set of V.

(3) $\Omega_V^p := \Omega^p_{\mathbb{C}^N} / \mathcal{K}^p$, where $\mathcal{K}^p = \{ f\alpha + dg \wedge \beta : \alpha \in \Omega^p_{\mathbb{C}^N}, \beta \in \Omega^{p-1}_{\mathbb{C}^N}, f, g \in \mathcal{I} \}$ and $\mathcal{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.

(4) $\tilde{\Omega}_V^p := \Omega^p_{\mathbb{C}^N} / \mathcal{H}^p$, where $\mathcal{H}^p = \{ \omega \in \Omega^p_{\mathbb{C}^N} : \omega|_{V \setminus V_{\text{sing}}} = 0 \}$.

Ω_V^p is the Grauert–Grothendieck sheaf of germs of holomorphic p-form on V. In case V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\overline{\overline{\Omega}}_V^n$. Clearly Ω_V^p, $\tilde{\Omega}_V^p$ are coherent. $\overline{\Omega}_V^p$ is a coherent sheaf because π is a proper map. $\overline{\overline{\Omega}}_V^p$ is also a coherent sheaf by Theorem A of [Siu 1970].

In [Du et al. 2011] and [Du and Yau 2012], another two sheaves $\overline{\Omega}_V^{1,1}$ and $\overline{\overline{\Omega}}_V^{1,1}$ were considered:

Definition 2.1. Let $(V, 0)$ be a 2-dimensional Stein analytic space with an isolated singularity at 0. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as its exceptional set. Define a sheaf of germs $\overline{\Omega}_V^{1,1}$ as the sheaf associated to the presheaf

$$U \mapsto \langle \Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1) \rangle,$$

where U is an open set of V and $\langle \Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1) \rangle$ represents the module generated by elements in $\Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1)$ over the ring $\Gamma(\pi^{-1}(U), \mathbb{O}_M)$.

Definition 2.2. Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Define a sheaf of germs $\overline{\overline{\Omega}}_V^{1,1}$ by the sheaf associated to the presheaf

$$U \mapsto \langle \Gamma(U, \overline{\overline{\Omega}}_V^1) \wedge \Gamma(U, \overline{\overline{\Omega}}_V^1) \rangle,$$

where U is an open set of V.

Du and Yau showed that these two new sheaves are coherent and found the relation between $\overline{\Omega}_V^{1,1}$ (respectively, $\overline{\overline{\Omega}}_V^{1,1}$) and $\overline{\Omega}_V^2$ (respectively, $\overline{\overline{\Omega}}_V^2$) by short exact sequence as follows:

Lemma 2.3 [Du and Yau 2012]. *Let* $(V,0)$ *be a* 2*-dimensional Stein space with an isolated singularity at* 0*. Let* $\pi:(M,A)\to(V,0)$ *be a resolution of the singularity with* A *as its exceptional set. Then* $\overline{\Omega}_V^{1,1}$ *is coherent and there is a short exact sequence*

$$0\longrightarrow\overline{\Omega}_V^{1,1}\longrightarrow\overline{\Omega}_V^{2}\longrightarrow\mathcal{F}^{(1,1)}\longrightarrow 0, \tag{2-1}$$

where $\mathcal{F}^{(1,1)}$ *is a sheaf supported on the singular point of* V*. Let*

$$F^{(1,1)}(M):=\Gamma(M,\Omega_M^2)/\langle\Gamma(M,\Omega_M^1)\wedge\Gamma(M,\Omega_M^1)\rangle; \tag{2-2}$$

then, $\dim\mathcal{F}_0^{(1,1)}=\dim F^{(1,1)}(M)$.

Lemma 2.4 [Du and Yau 2012]. *Let* V *be a* 2*-dimensional Stein space with* 0 *as its only singular point. Let* $\pi:(M,A)\to(V,0)$ *be a resolution of the singularity with* A *as its exceptional set. Then* $\overline{\overline{\Omega}}_V^{1,1}$ *is coherent and there is a short exact sequence*

$$0\longrightarrow\overline{\overline{\Omega}}_V^{1,1}\longrightarrow\overline{\overline{\Omega}}_V^{2}\longrightarrow\mathcal{G}^{(1,1)}\longrightarrow 0 \tag{2-3}$$

where $\mathcal{G}^{(1,1)}$ *is a sheaf supported on the singular point of* V*. Let*

$$G^{(1,1)}(M\backslash A):=\Gamma(M\backslash A,\Omega_M^2)/\langle\Gamma(M\backslash A,\Omega_M^1)\wedge\Gamma(M\backslash A,\Omega_M^1)\rangle; \tag{2-4}$$

then, $\dim\mathcal{G}_0^{(1,1)}=\dim G^{(1,1)}(M\backslash A)$.

They defined local invariants of singularities which are independent of resolution:

Definition 2.5. Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi:(M,A)\to(V,0)$ be a resolution of the singularity with A as its exceptional set. Let

$$f^{(1,1)}(0):=\dim\mathcal{F}_0^{(1,1)}=\dim F^{(1,1)}(M), \tag{2-5}$$

$$g^{(1,1)}(0):=\dim\mathcal{G}_0^{(1,1)}=\dim G^{(1,1)}(M\backslash A). \tag{2-6}$$

We will omit 0 in $f^{(1,1)}(0)$ and $g^{(1,1)}(0)$ if the context is clear.

The first author and Yau conjectured that for all normal surface singularities, the invariants $f^{(1,1)}$ and $g^{(1,1)}$ are strictly positive. This conjecture was confirmed when the singularities are with $\mathbb{C}^*$-action.

Theorem 2.6 [Du and Yau 2012]. *Let* V *be a* 2*-dimensional Stein space with* 0 *as its only normal singular point with* $\mathbb{C}^*$*-action. Then* $f^{(1,1)}\geq 1$.

Theorem 2.7 [Du et al. 2011]. *Let* V *be a* 2*-dimensional Stein space with* 0 *as its only normal singular point with* $\mathbb{C}^*$*-action. Then* $g^{(1,1)}\geq 1$.

Theorem 2.7 is the crucial part for the solution of the classical complex Plateau problem.

We will show that these two invariants are strictly positive for rational surface singularities. So these two invariants tell the difference between smoothness and singularity more precisely than the geometry genus does in some sense.

Theorem 2.8. *For rational surface singularities,* $f^{(1,1)} = g^{(1,1)} \geq 1$.

Proof. Let $(V, 0)$ be a 2-dimensional Stein space with 0 as its only rational singularity. So, the geometry genus p_g and the irregularity q are both 0. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as its exceptional set. Then

$$\frac{\Gamma(M\backslash A, \Omega_M^2)}{\langle \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1)\rangle} = \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle},$$

and

$$g^{(1,1)} = f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle}.$$

From [Yau 1981a], the canonical bundle K_M is generated by its global sections in a neighborhood of the exceptional set for rational surface singularities. So, there exists $\omega \in \Gamma(M, \Omega_M^2)$ such that ω does not vanish along some irreducible component A_k of A. As the singularity is rational, A_k is a smooth rational curve. Take a tubular neighborhood U of A_k such that $U \subset M$. By the proof of Proposition 3.9 in [Du et al. 2011], we know that the elements in $\langle \Gamma(U, \Omega_U^1) \wedge \Gamma(U, \Omega_U^1)\rangle$ vanish along A_k. Since $\Gamma(M, \Omega_M^1) \subset \Gamma(U, \Omega_U^1)$, the elements in $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle$ also vanish along A_k. Therefore $\omega \in \Gamma(M, \Omega_M^2) \backslash \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle$, i.e.,

$$g^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)\rangle} \geq 1. \qquad \square$$

3. The complex Plateau problem for 3-dimensional rational CR manifolds

Yau [1981b] solved the classical complex Plateau problem for the case $n \geq 3$.

Theorem 3.1 [ibid.]. *Let X be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$ for $n \geq 3$ in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^{n+1}$. Then X is the boundary of the complex submanifold $V \subset D - X$ if and only if the Kohn–Rossi cohomology groups $H_{KR}^{p,q}(X)$ are zero for $1 \leq q \leq n-2$.*

Luk and Yau [2012] introduced the so-called s-invariant in order to solve the complex Plateau problem as $n = 2$. But they could not give even a sufficient condition on the boundary such that it decides the smoothness in the interior.

Theorem 3.2 [Luk and Yau 2007]. *Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. If the holomorphic* *de Rham cohomology* $H_h^2(X)$ *is* 0, *then* X *is a boundary of a complex variety* V *in* D *with boundary regularity, and* V *has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing* s*-invariant.*

Corollary 3.3 [ibid.]. *Let* X *be a strongly pseudoconvex compact CR manifold of dimension* 3. *Suppose that* X *is contained in the boundary of a strongly pseudoconvex bounded domain* D *in* $\mathbb{C}^3$. *If the holomorphic De Rham cohomology* $H_h^2(X)$ *is* 0, *then* X *is a boundary of a complex variety* V *in* D *with boundary regularity, and* V *has only isolated quasihomogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star-shaped and all the curves are rational.*

Du and Yau [2012] used a new invariant $g^{(1,1)}$ for singularities to generate a new CR invariant $g^{(1,1)}(X)$.

Definition 3.4. Suppose X is a compact connected strongly pseudoconvex CR manifold of real dimension 3. Put

$$G^{(1,1)}(X) := \mathcal{S}^2(X)/\langle \mathcal{S}^1(X) \wedge \mathcal{S}^1(X)\rangle, \tag{3-1}$$

where $\mathcal{S}^p$ denotes the holomorphic sections of $\bigwedge^p(\widehat{T}(X)^*)$ and $\widehat{T}(X)^*$ is the holomorphic cotangent bundle of X. Then we set

$$g^{(1,1)}(X) := \dim G^{(1,1)}(X). \tag{3-2}$$

Lemma 3.5 [Du and Yau 2012]. *Let* X *be a compact connected strongly pseudoconvex CR manifold of real dimension* 3, *which bounds a bounded strongly pseudoconvex variety* V *with only isolated singularities* $\{0_1, \dots, 0_k\}$ *in* $\mathbb{C}^N$. *Then* $g^{(1,1)}(X) = \sum_i g^{(1,1)}(0_i)$.

Note that this invariant $g^{(1,1)}(X)$ can be calculated on X directly. In [ibid.], we use this CR invariant to give the sufficient and necessary condition for the variety bounded by X to be smooth if $H_h^2(X) = 0$:

Theorem 3.6 [ibid.]. *Let* X *be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension* 3. *Suppose that* X *is contained in the boundary of a strongly pseudoconvex bounded domain* D *in* $\mathbb{C}^N$ *with* $H_h^2(X) = 0$. *Then* X *is the boundary of the complex submanifold up to normalization* $V \subset D - X$ *with boundary regularity if and only if* $g^{(1,1)}(X) = 0$.

Theorem 3.7 [ibid.]. *Let* X *be a strongly pseudoconvex compact CR manifold of dimension* 3. *Suppose that* X *is contained in the boundary of a strongly pseudoconvex bounded domain* D *in* $\mathbb{C}^3$ *with* $H_h^2(X) = 0$. *Then* X *is the boundary of the complex submanifold* $V \subset D - X$ *if and only if* $g^{(1,1)}(X) = 0$.

In this paper, we solve the complex Plateau problem for a strongly pseudoconvex compact rational CR manifold of dimension 3, as a corollary of Theorem 2.8.

Recall the definition of a 3-dimensional rational CR manifold [Luk and Yau 1998b]:

Definition 3.8. Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3. Let V be the normal variety such that the boundary of V is X and V has isolated singularities at $\{0_1, \dots, 0_m\}$. Let $\pi : M \to V$ be a resolution of the singularities of V. Let U_i be a strongly pseudoconvex neighborhood of 0_i, for $1 \le i \le m$, such that the U_i are pairwise disjoint. Then

$$p_g(X) = \sum_{i=1}^{m} \dim \Gamma(U_i - \{0_i\}, \Omega^2)/L^2(U_i - \{0_i\}, \Omega^2).$$

If $p_g(X) = 0$, we call the CR manifold rational.

From the similar proof of Lemma 3.9 in [Du and Yau 2012], we know that the invariant $p_g(X)$ is also decided by the holomorphic sections of holomorphic cotangent bundle of X. So it is also a CR invariant.

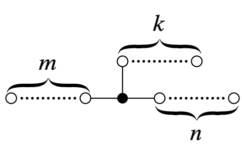
Theorem 3.9. *Let X be a strongly pseudoconvex compact rational CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is a boundary of the complex submanifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.*

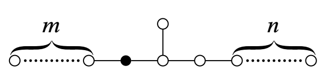
Proof. It is clear that rational CR manifolds can bound varieties with only rational singularities. Then from Theorem 2.8 and Lemma 3.5, we obtain our conclusion. □

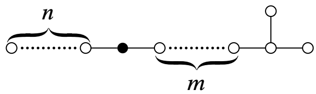
4. Explicit calculation of new invariants for special rational triple points

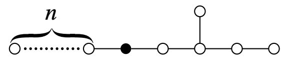
Du et al. [2011] calculated $f^{(1,1)}$ and $g^{(1,1)}$ for rational double points and quotient singularities. In this section we will calculate these two invariants for rational triple points. We suppose that V is a 2-dimensional Stein space with 0 as its only normal singularity and that V is contractible to 0.

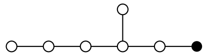
Artin [1966] classified the dual graphs of rational triple points of dimension 2 into nine classes, and proved that each rational triple point can be embedded into $\mathbb{C}^4$. Tyurina [1968a] gave explicitly three defining equations for each singularity. Tyurina [1968b] also proved that a rational triple point is determined uniquely by its dual graph ([Laufer 1973] totally gave all the dual graphs of singularities with such property). So, isomorphically, there are nine rational triple points, for which we use the notations defined in [Chen et al. 2007]:

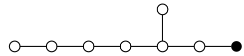
$A_{m,n,k}$: 

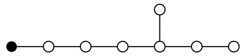
$B_{m,n}$: 

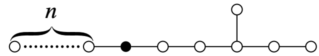
$C_{m,n}$:


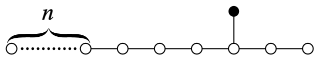
$D_{n,5}$:


$E_{6,0}$:


$E_{7,0}$:


$E_{0,7}$:


$F_{n,6}$:


$G_{n,0}$:


where $\circ$ is a (-2)-curve and $\bullet$ is a (-3)-curve.

In the following computation, we shall use explicit resolutions $\pi : M \to V$ of $A_{m,n,k}$, $B_{m,n}$, $C_{m,n}$, $D_{n,5}$, $E_{6,0}$, $E_{7,0}$, $E_{0,7}$, $F_{n,6}$ and $G_{n,0}$ to compute our new invariants. Note that for these rational singularities of dimension 2, the irregularity q is 0, so $f^{(1,1)} = g^{(1,1)}$. In order to calculate our new invariants for these rational singularities, we must know all the holomorphic 1-forms and holomorphic 2-forms on M. In general, the difficulty for calculating these two invariants is that the holomorphic 1-forms on the resolution manifolds are hard to express. But for rational singularities, we can use the following proposition to simplify the calculation. Campana and Flenner [2002] gave a proof of the following proposition by using mixed Hodge structure theory, which is of independent interest. We give a short proof here:

Proposition 4.1. *If* $(V, 0)$ *is a rational isolated singularity of dimension* 2 *and* M *is a resolution of the singularity*, *then* $H_h^1(M) = H_h^2(M) = 0$.

Proof. We recall the similar proof in [Du and Yau 2010]. Let $\pi : M \to V$ be a good resolution of the singularity. Let $\pi^{-1}(0) = A = \bigcup A_i$, $1 \le i \le n$, be the irreducible decomposition of the exceptional set A.

We have the spectral sequence

$$E_1^{p,q} = H^q(M, \Omega_M^p) \Rightarrow \boldsymbol{H}^{p+q}(M, \Omega_M^\bullet) \cong H^{p+q}(M, \mathbb{C}). \tag{4-1}$$

The spectral sequence induces an exact sequence of small-order terms

$$0 \to H_h^1(M) \to H^1(M, \mathbb{C}) \to E_2^{0,1} \to H_h^2(M) \to H^2(M, \mathbb{C}) \to E_2^{1,1} \to 0, \tag{4-2}$$

where

$$E_2^{0,1} = \ker(H^1(M, \mathbb{O}_M) \to H^1(M, \Omega_M^1)), \tag{4-3}$$

$$E_2^{1,1} = \operatorname{coker}(H^1(M, \mathbb{O}_M) \to H^1(M, \Omega_M^1)). \tag{4-4}$$

So

$$h_h^1(M) - h^1(M) + \dim E_2^{0,1} - h_h^2(M) + h^2(M) - \dim E_2^{1,1} = 0. \tag{4-5}$$

Since

$$\dim E_2^{0,1} - \dim E_2^{1,1} = h^1(M, \mathcal{O}_M) - h^1(M, \Omega_M^1), \tag{4-6}$$

we have

$$h_h^1(M) - h^1(M) - h_h^2(M) + h^2(M) + h^1(M, \mathcal{O}_M) - h^1(M, \Omega_M^1) = 0. \tag{4-7}$$

From [Wahl 1985], we know that

$$h^1(M, \Omega_M^1) = \gamma + q + n = p_g - g - b - \alpha - \beta + n \tag{4-8}$$

and

$$h^1(M) = \dim H^1(A, \mathbb{C}) = 2g + b, \quad h^2(M) = n, \quad h^1(M, \mathcal{O}_M) = p_g.$$

So

$$h_h^2(M) - h_h^1(M) = \alpha + \beta - g. \tag{4-9}$$

From [van Straten and Steenbrink 1985], we know that

$$p_g = q + g + b + \alpha + \beta + \gamma.$$

As $p_g = 0$, $q = g = b = \alpha = \beta = \gamma$. So $h^1(M) = 2g + b = 0$. From (4-2) and (4-9), we get that $h_h^2(M) = h_h^1(M) = 0$. □

Remark 4.2. In fact, from the above proof, we can get $h_h^2(M) = h_h^1(M) = 0$ if

$$E_2^{0,1} = \ker(H^1(M, \mathcal{O}_M) \to H^1(M, \Omega_M^1)) = 0.$$

So, rational singularity is a special case.

Now we can use holomorphic functions and holomorphic 2-forms to express holomorphic 1-forms on the resolution manifold from the following lemma:

Lemma 4.3. *If $(V, 0)$ is rational isolated singularity of dimension 2 and $\pi : M \to V$ is a resolution, then for any $\xi \in \Gamma(M, \Omega_M^1)$ and $\zeta \in d^{-1}(d\xi)$, there exists an $f \in \Gamma(M, \mathcal{O}_M)$ such that $\xi = \zeta + d(f)$, where d is the exterior differential operator.*

Proof. From Proposition 4.1 above, we have the exact sequence

$$0 \longrightarrow \Gamma(M, \mathcal{O}_M) \xrightarrow{d} \Gamma(M, \Omega_M^1) \xrightarrow{d} \Gamma(M, \Omega_M^2) \longrightarrow 0.$$

For $\xi \in \Gamma(M, \Omega_M^1)$ and any $\zeta \in d^{-1}(d\xi)$, $d(\xi - \zeta) = 0$. So there exists $f \in \Gamma(M, \mathcal{O}_M)$ such that $\xi = \zeta + d(f)$. □

From the lemma above, we see that in order to get holomorphic 1-forms on M, we only need to calculate holomorphic functions and holomorphic 2-forms on M.

Laufer [1971] constructed local coordinates for the resolution manifolds of cyclic quotient singularities such that one can calculate everything explicitly on the manifolds. Now we are going to construct local coordinates for the resolution manifolds of rational triple points to calculate our new invariants because of the tautness of rational triple points (“tautness” means that the singularity is determined by its dual graph).

An explicit resolution $\pi : M \to V$ can be given in terms of coordinates and transition functions on M for each type as follows:

Type $E_{6,0}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0, 1, \dots, 4,\\ W_t &= \{(u_t, v_t) : u_t^2 v_t \neq -1\}, && t = 5, 6,\\ W_7 &= \{(u_7, v_7)\} : u_7^5 v_7^2 \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t,\\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le 3, \qquad \begin{cases} u_5 = 1/(u_4 v_4),\\ v_5 = u_4 v_4^2 (1 - u_4), \end{cases}$$

$$\begin{cases} u_6 = 1/(v_4(1 - u_4)),\\ v_6 = u_4 v_4^2 (1 - u_4), \end{cases} \qquad \begin{cases} u_7 = 1/v_6,\\ v_7 = u_6 v_6^3. \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_7$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le 4,\\ C_5 &= \{v_3 = 1\} \cup \{u_4 = 1\} \cup \{v_5 = 0\},\\ C_6 &= \{u_4 = 0\} \cup \{v_6 = 0\},\\ C_7 &= \{u_6 = 0\} \cup \{v_7 = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} u_4^a v_4^b (1 - u_4)^c &= u_3^b v_3^{-a+2b-c} (v_3 - 1)^c\\ &= u_2^{-a+2b-c} v_2^{-2a+3b-2c} (u_2 v_2^2 - 1)^c\\ &= u_1^{-2a+3b-2c} v_1^{-3a+4b-3c} (u_1^2 v_1^3 - 1)^c\\ &= u_0^{-3a+4b-3c} v_0^{-4a+5b-4c} (u_0^3 v_0^4 - 1)^c\\ &= u_5^{-b+2c} v_5^c (1 + u_5^2 v_5)^{-a+b-c}\\ &= u_6^{2a-b} v_6^a (1 + u_6^2 v_6)^{-a+b-c}\\ &= u_7^{5a-3b} v_7^{2a-b} (1 + u_7^5 v_7^2)^{-a+b-c}, \end{aligned}$$

such that

$$\begin{cases} a,b,c \geq 0, \\ 5b \geq 4(a+c), \\ 2c \geq b, \\ 5a \geq 3b. \end{cases} \tag{4-10}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_4 \wedge dv_4 \\ &= -\frac{du_5 \wedge dv_5}{1+u_5^2 v_5} = \frac{du_6 \wedge dv_6}{1+u_6^2 v_6} = \frac{u_7 du_7 \wedge dv_7}{1+u_7^5 v_7^2}, \end{aligned}$$

such that

$$\begin{cases} a,b,c \geq 0, \\ 5b \geq 4(a+c), \\ 2c \geq b, \\ 5a+1 \geq 3b. \end{cases} \tag{4-11}$$

Type $E_{7,0}$:

Coordinate charts:

$$\begin{aligned} &W_t = \{(u_t, v_t)\}, && t = 0,1,\dots,5, \\ &W_t = \{(u_t, v_t) : u_t^2 v_t \neq -1\}, && t = 6,7, \\ &W_8 = \{(u_8, v_8)\} : u_8^5 v_8^2 \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \leq t \leq 4, \qquad \begin{cases} u_6 = 1/(u_5 v_5), \\ v_6 = u_5 v_5^2 (1-u_5), \end{cases}$$

$$\begin{cases} u_7 = 1/(v_5(1-u_5)), \\ v_7 = u_5 v_5^2 (1-u_5), \end{cases} \qquad \begin{cases} u_8 = 1/v_7, \\ v_8 = u_7 v_7^3. \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_8$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \leq t \leq 5, \\ C_6 &= \{v_4 = 1\} \cup \{u_5 = 1\} \cup \{v_6 = 0\}, \\ C_7 &= \{u_5 = 0\} \cup \{v_7 = 0\}, \\ C_8 &= \{u_7 = 0\} \cup \{v_8 = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} u_5^a v_5^b (1-u_5)^c &= u_4^b v_4^{-a+2b-c}(v_4-1)^c \\ &= u_3^{-a+2b-c} v_3^{-2a+3b-2c}(u_3 v_3^2-1)^c \\ &= u_2^{-2a+3b-2c} v_2^{-3a+4b-3c}(u_2^2 v_2^3-1)^c \\ &= u_1^{-3a+4b-3c} v_1^{-4a+5b-4c}(u_1^3 v_1^4-1)^c \\ &= u_0^{-4a+5b-4c} v_0^{-5a+6b-5c}(u_0^4 v_0^5-1)^c \\ &= u_6^{-b+2c} v_6^c (1+u_6^2 v_6)^{-a+b-c} \\ &= u_7^{2a-b} v_7^a (1+u_7^2 v_7)^{-a+b-c} \\ &= u_8^{5a-3b} v_8^{2a-b}(1+u_8^5 v_8^2)^{-a+b-c}, \end{aligned}$$

such that

$$\begin{cases} a,b,c \geq 0, \\ 6b \geq 5(a+c), \\ 2c \geq b, \\ 5a \geq 3b. \end{cases} \tag{4-12}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_5 \wedge dv_5 \\ &= -\frac{du_6 \wedge dv_6}{1+u_6^2 v_6} = \frac{du_7 \wedge dv_7}{1+u_7^2 v_7} = \frac{u_8 du_8 \wedge dv_8}{1+u_8^5 v_8^2}, \end{aligned}$$

such that

$$\begin{cases} a,b,c \geq 0, \\ 6b \geq 5(a+c), \\ 2c \geq b, \\ 5a+1 \geq 3b. \end{cases} \tag{4-13}$$

Type $E_{0,7}$:

Coordinate charts:

$$\begin{aligned} &W_t = \{(u_t, v_t)\}, && t = 0,1,\ldots,5, \\ &W_t = \{(u_t, v_t) : u_t^2 v_t \neq -1\}, && t = 6,7, \\ &W_8 = \{(u_8, v_8)\} : u_8^3 v_8^2 \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_1 = 1/v_0, \\ v_1 = u_0 v_0^3, \end{cases} \qquad \begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 1 \le t \le 4,$$

$$\begin{cases} u_6 = 1/(u_5 v_5), \\ v_6 = u_5 v_5^2 (1-u_5), \end{cases} \qquad \begin{cases} u_7 = 1/(v_5(1-u_5)), \\ v_7 = u_5 v_5^2 (1-u_5), \end{cases} \qquad \begin{cases} u_8 = 1/v_7, \\ v_8 = u_7 v_7^2. \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_8$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le 5, \\ C_6 &= \{v_4 = 1\} \cup \{u_5 = 1\} \cup \{v_6 = 0\}, \\ C_7 &= \{u_5 = 0\} \cup \{v_7 = 0\}, \\ C_8 &= \{u_7 = 0\} \cup \{v_8 = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} u_5^a v_5^b (1-u_5)^c &= u_4^b v_4^{-a+2b-c} (v_4 - 1)^c \\ &= u_3^{-a+2b-c} v_3^{-2a+3b-2c} (u_3 v_3^2 - 1)^c \\ &= u_2^{-2a+3b-2c} v_2^{-3a+4b-3c} (u_2^2 v_2^3 - 1)^c \\ &= u_1^{-3a+4b-3c} v_1^{-4a+5b-4c} (u_1^3 v_1^4 - 1)^c \\ &= u_0^{-4a+5b-4c} v_0^{-9a+11b-9c} (u_0^4 v_0^9 - 1)^c \\ &= u_6^{-b+2c} v_6^c (1 + u_6^2 v_6)^{-a+b-c} \\ &= u_7^{2a-b} v_7^a (1 + u_7^2 v_7)^{-a+b-c} \\ &= u_8^{3a-2b} v_8^{2a-b} (1 + u_8^3 v_8^2)^{-a+b-c}, \end{aligned}$$

such that

$$\begin{cases} a, b, c \ge 0, \\ 11b \ge 9(a+c), \\ 2c \ge b, \\ 3a \ge 2b. \end{cases} \tag{4-14}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= v_0 du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_5 \wedge dv_5 \\ &= -\frac{du_6 \wedge dv_6}{1 + u_6^2 v_6} = \frac{du_7 \wedge dv_7}{1 + u_7^2 v_7} = \frac{du_8 \wedge dv_8}{1 + u_8^5 v_8^2}, \end{aligned}$$

such that

$$
\begin{cases} a,b,c \geq 0, \\ 11b+1 \geq 9(a+c), \\ 2c \geq b, \\ 3a \geq 2b. \end{cases} \tag{4-15}
$$

Type $G_{n,0}$:

Coordinate charts:

$$
\begin{aligned}
W_t &= \{(u_t, v_t)\}, && t = 0,1,\dots,n+1, \\
W_t &= \{(u_t, v_t) : u_t^3 v_t^2 \neq -1\}, && t = n+2, n+4, \\
W_{n+3} &= \{(u_{n+3}, v_{n+3})\} : u_{n+3}^2 v_{n+3} \neq -1\}.
\end{aligned}
$$

Transition functions:

$$
\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \leq t \leq n, \qquad \begin{cases} u_{n+2} = 1/(u_{n+1} v_{n+1}), \\ v_{n+2} = u_{n+1}^2 v_{n+1}^3 (1-u_{n+1}), \end{cases}
$$

$$
\begin{cases} u_{n+3} = 1/(v_{n+1}(1-u_{n+1})), \\ v_{n+3} = u_{n+1} v_{n+1}^2 (1-u_{n+1}), \end{cases} \qquad \begin{cases} u_{n+4} = 1/v_{n+3}, \\ v_{n+4} = u_{n+3} v_{n+3}^2. \end{cases}
$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \dots \cup C_{n+4}$, where

$$
\begin{aligned}
C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \leq t \leq n+1, \\
C_{n+2} &= \{v_n = 1\} \cup \{u_{n+1} = 1\} \cup \{v_{n+2} = 0\}, \\
C_{n+3} &= \{u_{n+1} = 0\} \cup \{v_{n+3} = 0\}, \\
C_{n+4} &= \{u_{n+3} = 0\} \cup \{v_{n+4} = 0\}.
\end{aligned}
$$

A holomorphic function on M can be generated by the following forms:

$$
\begin{aligned}
u_{n+1}^a v_{n+1}^b (1-u_{n+1})^c &= u_t^{(n-t+1)b-(n-t)(a+c)} v_t^{(n-t+2)b-(n-t+1)(a+c)} \\
&\quad \cdot (u_t^{n-t} v_t^{n-t+1} - 1)^c \\
&= u_{n+2}^{-b+3c} v_{n+2}^c (1 + u_{n+2}^3 v_{n+2})^{-a+b-c} \\
&= u_{n+3}^{2a-b} v_{n+3}^a (1 + u_{n+3}^2 v_{n+3})^{-a+b-c} \\
&= u_{n+4}^{3a-2b} v_{n+4}^{2a-b} (1 + u_{n+4}^3 v_{n+4}^2)^{-a+b-c},
\end{aligned}
$$

where $0 \le t \le n$, such that

$$\begin{cases} a,b,c \ge 0, \\ (n+2)b \ge (n+1)(a+c), \\ 3c \ge b, \\ 3a \ge 2b. \end{cases} \tag{4-16}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_{n+1} \wedge dv_{n+1} \\ &= -\frac{u_{n+2}du_{n+2} \wedge dv_{n+2}}{1+u_{n+2}^3 v_{n+2}} = \frac{du_{n+3} \wedge dv_{n+3}}{1+u_{n+3}^2 v_{n+3}} = \frac{du_{n+4} \wedge dv_{n+4}}{1+u_{n+4}^3 v_{n+4}^2}, \end{aligned}$$

such that

$$\begin{cases} a,b,c \ge 0, \\ (n+2)b \ge (n+1)(a+c), \\ 3c+1 \ge b, \\ 3a \ge 2b. \end{cases} \tag{4-17}$$

Type $D_{n,5}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, && t = 0,1,\dots,n+3, \\ W_t &= \{(u_t, v_t) : u_t^2 v_t \ne -1\}, && t = n+4, n+5, n+7, \\ W_{n+6} &= \{(u_{n+6}, v_{n+6})\} : u_{n+6}^3 v_{n+6}^2 \ne -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le n-1 \text{ and } t = n+1, n+2, n+5, n+6,$$

$$\begin{cases} u_{n+1} = 1/v_n, \\ v_{n+1} = u_n v_n^3, \end{cases} \qquad \begin{cases} u_{n+4} = 1/(u_{n+3} v_{n+3}), \\ v_{n+4} = u_{n+3} v_{n+3}^2 (1-u_{n+3}), \end{cases}$$

$$\begin{cases} u_{n+5} = 1/(v_{n+3}(1-u_{n+3})), \\ v_{n+5} = u_{n+3} v_{n+3}^2 (1-u_{n+3}). \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{n+7}$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le n+3 \text{ and } t = n+6, n+7, \\ C_{n+4} &= \{v_{n+2} = 1\} \cup \{u_{n+3} = 1\} \cup \{v_{n+4} = 0\}, \\ C_{n+5} &= \{u_{n+3} = 0\} \cup \{v_{n+5} = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$
\begin{aligned}
&u_{n+3}^{a}v_{n+3}^{b}(1-u_{n+3})^{c}\\
&\qquad = u_{n+2}^{b}v_{n+2}^{-a+2b-c}(v_{n+2}-1)^{c}\\
&\qquad = u_{n+1}^{-a+2b-c}v_{n+1}^{-2a+3b-2c}(u_{n+1}v_{n+1}^{2}-1)^{c}\\
&\qquad = u_{t}^{(4n-4t+3)b-(3n-3t+2)(a+c)}v_{t}^{(4n-4t+7)b-(3n-3t+5)(a+c)}\\
&\qquad\quad \cdot(u_{t}^{3n-3t+2}v_{t}^{3n-3t+5}-1)^{c}\\
&\qquad = u_{n+4}^{-b+2c}v_{n+4}^{c}(1+u_{n+4}^{2}v_{n+4})^{-a+b-c}\\
&\qquad = u_{n+5}^{2a-b}v_{n+5}^{a}(1+u_{n+5}^{2}v_{n+5})^{-a+b-c}\\
&\qquad = u_{n+6}^{3a-2b}v_{n+6}^{2a-b}(1+u_{n+6}^{3}v_{n+6}^{2})^{-a+b-c}\\
&\qquad = u_{n+7}^{4a-3b}v_{n+7}^{3a-2b}(1+u_{n+7}^{4}v_{n+7}^{3})^{-a+b-c},
\end{aligned}
$$

where $0\le t\le n$, such that

$$
\begin{cases}
a,b,c\ge 0,\\
(4n+7)b\ge(3n+5)(a+c),\\
2c\ge b,\\
4a\ge 3b.
\end{cases}
\tag{4-18}
$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$
\begin{aligned}
\varphi_0 &= u_0^{n}v_0^{n+1}du_0\wedge dv_0 = u_1^{n-1}v_1^{n}du_1\wedge dv_1\\
&= \cdots = u_{n-1}v_{n-1}^{2}du_{n-1}\wedge dv_{n-1} = u_n du_n\wedge dv_n = du_{n+1}\wedge dv_{n+1}\\
&= du_{n+2}\wedge dv_{n+2} = du_{n+3}\wedge dv_{n+3} = -\frac{du_{n+4}\wedge dv_{n+4}}{1+u_{n+4}^{3}v_{n+4}}\\
&= \frac{du_{n+5}\wedge dv_{n+5}}{1+u_{n+5}^{2}v_{n+5}} = \frac{du_{n+6}\wedge dv_{n+6}}{1+u_{n+6}^{3}v_{n+6}^{2}} = \frac{du_{n+7}\wedge dv_{n+7}}{1+u_{n+7}^{4}v_{n+7}^{3}},
\end{aligned}
$$

such that

$$
\begin{cases}
a,b,c\ge 0,\\
(4n+7)b+n+1\ge(3n+5)(a+c),\\
2c\ge b,\\
4a\ge 3b.
\end{cases}
\tag{4-19}
$$

Type $F_{n,6}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, & t &= 0, 1, \dots, n+4, \\ W_t &= \{(u_t, v_t) : u_t^2 v_t \neq -1\}, & t &= n+5, n+6, \\ W_{n+7} &= \{(u_{n+7}, v_{n+7})\} : u_{n+7}^3 v_{n+7}^2 \neq -1\}, \\ W_{n+8} &= \{(u_{n+8}, v_{n+8})\} : u_{n+8}^4 v_{n+8}^3 \neq -1\}. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \leq t \leq n-1 \text{ and } t = n+1, n+2, n+3, n+6, n+7,$$

$$\begin{cases} u_{n+1} = 1/v_n, \\ v_{n+1} = u_n v_n^3, \end{cases} \qquad \begin{cases} u_{n+5} = 1/(u_{n+4} v_{n+4}), \\ v_{n+5} = u_{n+4} v_{n+4}^2 (1 - u_{n+4}), \end{cases}$$

$$\begin{cases} u_{n+6} = 1/(v_{n+4}(1 - u_{n+4})), \\ v_{n+6} = u_{n+4} v_{n+4}^2 (1 - u_{n+4}). \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{n+8}$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\} \quad 1 \leq t \leq n+4 \text{ and } t = n+7, n+8, \\ C_{n+5} &= \{v_{n+3} = 1\} \cup \{u_{n+4} = 1\} \cup \{v_{n+5} = 0\}, \\ C_{n+6} &= \{u_{n+4} = 0\} \cup \{v_{n+6} = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} &u_{n+4}^a v_{n+4}^b (1 - u_{n+4})^c \\ &\quad = u_{n+3}^b v_{n+3}^{-a+2b-c} (v_{n+3} - 1)^c \\ &\quad = u_{n+2}^{-a+2b-c} v_{n+2}^{-2a+3b-2c} (u_{n+2} v_{n+2}^2 - 1)^c \\ &\quad = u_{n+1}^{-2a+3b-2c} v_{n+1}^{-3a+4b-3c} (u_{n+1}^2 v_{n+1}^3 - 1)^c \\ &\quad = u_t^{(5n-5t+4)b-(4n-4t+3)(a+c)} v_t^{(5n-5t+9)b-(4n-4t+7)(a+c)} \\ &\qquad \cdot (u_t^{4n-4t+3} v_t^{4n-4t+7} - 1)^c \\ &\quad = u_{n+5}^{-b+2c} v_{n+5}^c (1 + u_{n+5}^2 v_{n+5})^{-a+b-c} \\ &\quad = u_{n+6}^{2a-b} v_{n+6}^a (1 + u_{n+6}^2 v_{n+6})^{-a+b-c} \\ &\quad = u_{n+7}^{3a-2b} v_{n+7}^{2a-b} (1 + u_{n+7}^3 v_{n+7}^2)^{-a+b-c} \\ &\quad = u_{n+8}^{4a-3b} v_{n+8}^{3a-2b} (1 + u_{n+8}^4 v_{n+8}^3)^{-a+b-c}, \end{aligned}$$

where $0 \le t \le n$, such that

$$\begin{cases} a, b, c \ge 0, \\ (5n+9)b \ge (4n+7)(a+c), \\ 2c \ge b, \\ 4a \ge 3b. \end{cases} \tag{4-20}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned} \varphi_0 &= u_0^n v_0^{n+1} du_0 \wedge dv_0 = u_1^{n-1} v_1^n du_1 \wedge dv_1 \\ &= \cdots = u_{n-1} v_{n-1}^2 du_{n-1} \wedge dv_{n-1} = u_n du_n \wedge dv_n \\ &= du_{n+1} \wedge dv_{n+1} = \cdots = du_{n+4} \wedge dv_{n+4} \\ &= -\frac{du_{n+5} \wedge dv_{n+5}}{1+u_{n+5}^3 v_{n+5}} = \frac{du_{n+6} \wedge dv_{n+6}}{1+u_{n+6}^2 v_{n+6}} \\ &= \frac{du_{n+7} \wedge dv_{n+7}}{1+u_{n+7}^3 v_{n+7}^2} = \frac{du_{n+8} \wedge dv_{n+8}}{1+u_{n+8}^4 v_{n+8}^3}, \end{aligned}$$

such that

$$\begin{cases} a, b, c \ge 0, \\ (5n+9)b + n + 1 \ge (4n+7)(a+c), \\ 2c \ge b, \\ 4a \ge 3b. \end{cases} \tag{4-21}$$

Type $C_{m,n}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, & t &= 0, 1, \ldots, m+n+2, \\ W_t &= \{(u_t, v_t) : u_t^2 v_t \ne -1\}, & t &= m+n+3, m+n+4. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le n-1 \text{ and } n+1 \le t \le m+n+1,$$

$$\begin{cases} u_{n+1} = 1/v_n, \\ v_{n+1} = u_n v_n^3, \end{cases} \qquad \begin{cases} u_{m+n+3} = 1/(u_{m+n+2} v_{m+n+2}), \\ v_{m+n+3} = u_{m+n+2} v_{m+n+2}^2 (1-u_{m+n+2}), \end{cases}$$

$$\begin{cases} u_{m+n+4} = 1/(v_{m+n+2}(1-u_{m+n+2})), \\ v_{m+n+4} = u_{m+n+2} v_{m+n+2}^2 (1-u_{m+n+2}). \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{m+n+4}$, where

$$C_t = \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le m+n+2,$$
$$C_{m+n+3} = \{v_{m+n+1} = 1\} \cup \{u_{m+n+2} = 1\} \cup \{v_{m+n+3} = 0\},$$
$$C_{m+n+4} = \{u_{m+n+2} = 0\} \cup \{v_{m+n+4} = 0\}.$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned}
u_{m+n+2}^a v_{m+n+2}^b &(1-u_{m+n+2})^c \\
&= u_t^{(m+n+2-t)b-(m+n+1-t)(a+c)} v_t^{(m+n+3-t)b-(m+n+2-t)(a+c)} \\
&\quad \cdot (u_t^{m+n+1-t} v_t^{m+n+2-t} - 1)^c \\
&= u_s^{[(m+3)(n-s)+m+2)]b-[(m+3)(n-s)+m+1)](a+c)} \\
&\quad \cdot v_s^{[(m+3)(n-s)+m+2)]b-[(m+3)(n-s)+m+1)](a+c)} \\
&\quad \cdot (u_s^{(m+3)(n-s)+m+2)} v_s^{(m+3)(n-s)+2m+3)} - 1)^c \\
&= u_{m+n+3}^{-b+2c} v_{m+n+3}^c (1 + u_{m+n+3}^2 v_{m+n+3})^{-a+b-c} \\
&= u_{m+n+4}^{2a-b} v_{m+n+4}^a (1 + u_{m+n+4}^2 v_{m+n+4})^{-a+b-c},
\end{aligned}$$

where $n+1 \le t \le m+n+1$, $0 \le s \le n$, such that

$$\begin{cases} a, b, c \ge 0, \\ (mn+3m+2n+5)b \ge (mn+2m+2n+3)(a+c), \\ 2c \ge b, \\ 2a \ge b. \end{cases} \tag{4-22}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned}
\varphi_0 &= u_0^n v_0^{n+1} du_0 \wedge dv_0 = u_1^{n-1} v_1^n du_1 \wedge dv_1 \\
&= \cdots = u_{n-1} v_{n-1}^2 du_{n-1} \wedge dv_{n-1} = u_n du_n \wedge dv_n = du_{n+1} \wedge dv_{n+1} \\
&= \cdots = du_{m+n+2} \wedge dv_{m+n+2} = -\frac{du_{m+n+3} \wedge dv_{m+n+3}}{1 + u_{m+n+3}^2 v_{m+n+3}} \\
&= \frac{du_{m+n+4} \wedge dv_{m+n+4}}{1 + u_{m+n+4}^2 v_{m+n+4}},
\end{aligned}$$

such that

$$\begin{cases} a, b, c \ge 0, \\ (mn+3m+2n+5)b + n + 1 \ge (mn+2m+2n+3)(a+c), \\ 2c \ge b, \\ 2a \ge b. \end{cases} \tag{4-23}$$

Type $B_{m,n}$:

Coordinate charts:

$$\begin{aligned} W_t &= \{(u_t, v_t)\}, & t &= 0, 1, \dots, m+2, \\ W_{m+3} &= \{(u_{m+3}, v_{m+3}) \,:\, u_{m+3}^2 v_{m+3} \neq -1\}, \\ W_t &= \{(u_t, v_t) \,:\, u_t^{t-m-2} v_t^{t-m-3} \neq -1\}, & m+4 &\le t \le m+n+4. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad 0 \le t \le m-1, t = m+1, \text{and } m+4 \le t \le m+n+3,$$

$$\begin{cases} u_{m+1} = 1/v_m, \\ v_{m+1} = u_m v_m^3, \end{cases} \qquad \begin{cases} u_{m+3} = 1/(u_{m+2} v_{m+2}), \\ v_{m+3} = u_{m+2} v_{m+2}^2 (1 - u_{m+2}), \end{cases}$$

$$\begin{cases} u_{m+4} = 1/(v_{m+2}(1 - u_{m+2})), \\ v_{m+4} = u_{m+2} v_{m+2}^2 (1 - u_{m+2}). \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \cdots \cup C_{m+n+4}$, where

$$\begin{aligned} C_t &= \{u_{t-1} = 0\} \cup \{v_t = 0\}, \quad 1 \le t \le m+2 \text{ and } m+5 \le t \le m+n+4, \\ C_{m+3} &= \{v_{m+1} = 1\} \cup \{u_{m+2} = 1\} \cup \{v_{m+3} = 0\}, \\ C_{m+4} &= \{u_{m+2} = 0\} \cup \{v_{m+4} = 0\}. \end{aligned}$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned} & u_{m+2}^a v_{m+2}^b (1 - u_{m+2})^c \\ &\qquad = u_{m+1}^b v_{m+1}^{-a+2b-c} (v_{m+1} 1)^c \\ &\qquad = u_t^{(3m-3t+2)b-(2m-2t+1)(a+c)} v_t^{(3m-3t+5)b-(2m-2t+3)(a+c)} \\ &\qquad\quad \cdot (u_t^{2m-2t+1} v_t^{2m-2t+3} - 1)^c \\ &\qquad = u_{m+3}^{-b+2c} v_{m+3}^c (1 + u_{m+3}^2 v_{m+3})^{-a+b-c} \\ &\qquad = u_s^{(t-m-2)a-(t-m-3)b} v_s^{(t-m-3)a-(t-m-4)b} (u_s^{t-m-2} v_s^{t-m-3} - 1)^c, \end{aligned}$$

where $0 \le t \le m$, $m+4 \le s \le m+n+4$, such that

$$\begin{cases} a, b, c \ge 0, \\ (3m+5)b \ge (2m+3)(a+c), \\ 2c \ge b, \\ (n+2)a \ge (n+1)b. \end{cases} \tag{4-24}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned}\varphi_0 &= u_0^m v_0^{m+1} du_0 \wedge dv_0 = u_1^{m-1} v_1^m du_1 \wedge dv_1 \\ &= \cdots = u_{m-1} v_{m-1}^2 du_{m-1} \wedge dv_{m-1} = u_m du_m \wedge dv_m \\ &= du_{m+1} \wedge dv_{m+1} = du_{m+2} \wedge dv_{m+2} \\ &= -\frac{du_{m+3} \wedge dv_{m+3}}{1 + u_{m+3}^2 v_{m+3}} \\ &= \frac{du_t \wedge dv_t}{1 + u_t^{t-m-2} v_t^{t-m-3}},\end{aligned}$$

where $m+4 \le t \le m+n+4$, such that

$$\begin{cases} a, b, c \ge 0, \\ (3m+5)b + m + 1 \ge (2m+3)(a+c), \\ 2c \ge b, \\ (n+2)a \ge (n+1)b. \end{cases} \tag{4-25}$$

Type $A_{m,n,k}$:

Coordinate charts:

$$\begin{aligned} &W_t = \{(u_t, v_t)\}, && t = 0, 1, \ldots, m+1, \\ &W_t = \{(u_t, v_t) : u_t^{t-m} v_t^{t-m-1} \ne -1\}, && m+2 \le t \le m+n+1, \\ &W_t = \{(u_t, v_t) : u_t^{t-m-n} v_t^{t-m-n-1} \ne -1\}, && m+n+2 \le t \le m+n+k+1. \end{aligned}$$

Transition functions:

$$\begin{cases} u_{t+1} = 1/v_t, \\ v_{t+1} = u_t v_t^2, \end{cases} \quad \begin{matrix} 0 \le t \le m-1, \ m+2 \le t \le m+n, \text{ or} \\ m+n+2 \le t \le m+n+k, \end{matrix}$$

$$\begin{cases} u_{m+1} = 1/v_m, \\ v_{m+1} = u_m v_m^3, \end{cases}$$

$$\begin{cases} u_{m+2} = 1/(v_{m+1}(1-u_{m+1})), \\ v_{m+2} = u_{m+1} v_{m+1}^2 (1-u_{m+1}), \end{cases}$$

$$\begin{cases} u_{m+n+2} = 1/(u_{m+1} v_{m+1}), \\ v_{m+n+2} = u_{m+1} v_{m+1}^2 (1-u_{m+1}). \end{cases}$$

Exceptional set: $A=\pi^{-1}(0)=C_1\cup\cdots\cup C_{m+n+k+1}$, where

$$C_t=\{u_{t-1}=0\}\cup\{v_t=0\},\quad t\neq m+n+2,$$
$$C_{m+n+2}=\{v_m=1\}\cup\{u_{m+1}=1\}\cup\{v_{m+n+2}=0\}.$$

A holomorphic function on M can be generated by the following forms:

$$\begin{aligned}
&u_{m+1}^a v_{m+1}^b(1-u_{m+1})^c\\
&\quad=u_t^{(2m-2t+1)b-(m-t)(a+c)}v_t^{(2m-2t+3)b-(m-t+1)(a+c)}(u_t^{m-t}v_t^{m-t+1}-1)^c\\
&\quad=u_s^{(s-m)a-(s-m-1)b}v_s^{(s-m-1)a-(s-m-2)b}(u_s^{s-m}v_s^{s-m-1}-1)^c,\\
&\quad=u_r^{(r-m-n)c-(r-m-n-1)b}v_r^{(r-m-n-1)c-(r-m-n-2)b}\\
&\qquad\cdot(u_r^{r-m-n}v_r^{r-m-n-1}-1)^c,
\end{aligned}$$

where $0\le t\le m$, $m+2\le s\le m+n+1$, $m+n+2\le r\le m+n+k+1$, such that

$$\begin{cases}a,b,c\ge 0,\\ (2m+3)b\ge(m+1)(a+c),\\ (n+1)a\ge nb,\\ (k+1)c\ge kb.\end{cases}\tag{4-26}$$

A holomorphic 2-form can be written as $f\varphi_0$, where f is a holomorphic function on M and

$$\begin{aligned}
\varphi_0&=u_0^m v_0^{m+1}du_0\wedge dv_0=u_1^{m-1}v_1^m du_1\wedge dv_1\\
&=\cdots=u_{m-1}v_{m-1}^2du_{m-1}\wedge dv_{m-1}=u_m du_m\wedge dv_m=du_{m+1}\wedge dv_{m+1}\\
&=\frac{du_t\wedge dv_t}{1+u_t^{t-m}v_t^{t-m-1}}\\
&=-\frac{du_s\wedge dv_s}{1+u_s^{s-m-n}v_s^{s-m-n-1}},
\end{aligned}$$

where $m+2\le t\le m+n+1$, $m+n+2\le s\le m+n+k+1$, such that

$$\begin{cases}a,b,c\ge 0,\\ (2m+3)b+m+1\ge(m+1)(a+c),\\ (n+1)a\ge nb,\\ (k+1)c\ge kb.\end{cases}\tag{4-27}$$

Theorem 4.4. *For* 2*-dimensional rational triple points*, $f^{(1,1)}=g^{(1,1)}=1$.

Proof. We consider only type $E_{6,0}$ as an example; the other singularities are similar. From the above calculation, we know that the holomorphic functions on M are generated by a base $\{u_4^a v_4^b(1-u_4)^c\}$ satisfying (4-10), and holomorphic 2-forms are generated by a base $\{u_4^a v_4^b (1-u_4)^c du_4 \wedge dv_4\}$ satisfying (4-11). For every holomorphic 2-form $\omega = u_4^a v_4^b (1-u_4)^c du_4 \wedge dv_4$ on M, we consider

$$\xi = -\frac{u_4^a v_4^{b+1}(1-u_4)^c}{b+1} du_4.$$

So, ξ defines a holomorphic 1-form on W_0 and $d\xi = \omega$. In fact, we need to check that under all changes of coordinate charts, ξ transforms to define a holomorphic 1-form in each coordinate chart:

$$\begin{aligned}
u_4^a v_4^{b+1}(1-u_4)^c du_4 &= -u_3^{b+1} v_3^{-a+2b-c}(v_3-1)^c dv_3 \\
&= -u_2^{-a+2b-c} v_2^{-2a+3b-2c+1}(u_2v_2^2-1)^c du_2 \\
&\quad\ -2u_2^{-a+2b-c+1} v_2^{-2a+3b-2c}(u_2v_2^2-1)^c dv_2 \\
&= -2u_1^{-2a+3b-2c} v_1^{-3a+4b-3c+1}(u_1^2v_1^3-1)^c du_1 \\
&\quad\ -3u_1^{-2a+3b-2c+1} v_1^{-3a+4b-3c}(u_1^2v_1^3-1)^c dv_1 \\
&= -3u_0^{-3a+4b-3c} v_0^{-4a+5b-4c+1}(u_0^3v_0^4-1)^c du_0 \\
&\quad\ -4u_0^{-3a+4b-3c+1} v_0^{-4a+5b-4c}(u_0^3v_0^4-1)^c dv_0 \\
&= -2u_5^{-b+2c} v_5^{c+1}(1+u_5^2v_5)^{-a+b-c-1} du_5 \\
&\quad\ -u_5^{-b+2c+1} v_5^{c}(1+u_5^2v_5)^{-a+b-c-1} dv_5 \\
&= 2u_6^{2a-b} v_6^{a+1}(1+u_6^2v_6)^{-a+b-c-1} du_6 \\
&\quad\ +u_6^{2a-b+1} v_6^{a}(1+u_6^2v_6)^{-a+b-c-1} dv_6 \\
&= 5u_7^{5a-3b+1} v_7^{2a-b+1}(1+u_7^5v_7^2)^{-a+b-c-1} du_7 \\
&\quad\ +2u_7^{5a-3b+2} v_7^{2a-b}(1+u_7^5v_7^2)^{-a+b-c-1} dv_7.
\end{aligned}$$

By Lemma 4.3, we can get general expressions for elements in $\Gamma(M, \Omega_M^1)$. Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = \langle u_4 \wedge v_4 \rangle,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = 1. \qquad \square$$

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