Pacific Journal of Mathematics

ON THE CONCIRCULAR CURVATURE OF A (κ, μ, ν)-MANIFOLD

FLORENCE GOULI-ANDREOU AND EVAGGELIA MOUTAFI

Volume 269 No. 1 May 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 269, No. 1, 2014
dx.doi.org/10.2140/pjm.2014.269.113

ON THE CONCIRCULAR CURVATURE OF A (κ, μ, ν)-MANIFOLD

FLORENCE GOULI-ANDREOU AND EVAGGELIA MOUTAFI

We study (κ, μ, ν)-contact metric 3-manifolds (a notion introduced by Koufogiorgos, Markellos and Papantoniou) that are Ricci flat, or are Einstein but not Sasakian, or satisfy $\nabla Z = 0$, where Z is the concircular curvature tensor, or satisfy $Z(\xi, X) \cdot Z = 0$, where ξ is the Reeb field, or satisfy $Z(\xi, X) \cdot S = 0$, where S is the Ricci tensor, or finally satisfy $R(\xi, X) \cdot Z = 0$, where R is the Riemannian curvature tensor.

1. Introduction

A contact metric manifold (M, ξ) is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y = R_0(X, Y)\xi, \tag{1-1}$$

where

$$R_0(X, Y)U = g(Y, U)X - g(X, U)Y, \quad X,\ Y,\ U \in \mathcal{X}(M). \tag{1-2}$$

There exist contact metric manifolds that satisfy the condition $R(X, Y)\xi = 0$; for example, the tangent sphere bundle of a flat Riemannian manifold admits a contact metric satisfying this condition. D. E. Blair, Th. Koufogiorgos and B. Papantoniou [Blair et al. 1995] generalized both this condition and the Sasakian case introducing the (κ, μ)-nullity distribution on a contact metric manifold

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{U \in T_pM \mid R(X, Y)U = (\kappa I + \mu h)R_0(X, Y)U\}$$

for all $X, Y \in \mathcal{X}(M)$, and $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-contact metric manifold. In particular we have

$$R(X, Y)\xi = (\kappa I + \mu h)R_0(X, Y)\xi, \quad X,\ Y \in \mathcal{X}(M), \tag{1-3}$$

with $\kappa \leq 1$ and if $\kappa = 1$ the structure is Sasakian. The full classification of these manifolds was given by E. Boeckx [2000]. If $\mu = 0$ we have the κ-nullity distribution and if $\xi \in N(\kappa)$ we have a $N(\kappa)$-contact metric manifold. Koufogiorgos

MSC2010: primary 53C15, 53C25, 53D10; secondary 53C35.
Keywords: contact metric manifold, (κ, μ, ν)-contact metric manifolds, η-Einstein, Ricci flat, Sasakian manifold, concircular curvature tensor, pseudosymmetric manifold.

and Ch. Tsichlias [2000] introduced the generalized (κ, μ)-contact metric manifolds, where κ and μ are real functions, and they gave several examples. Finally, the (κ, μ, ν)-contact metric manifolds have been introduced by Koufogiorgos, M. Markellos and V. Papantoniou [Koufogiorgos et al. 2008] where κ, μ, ν are smooth functions and the curvature tensor satisfies, for every $X, Y \in \mathcal{X}(M)$, the condition

$$R(X, Y)\xi = \kappa\big(\eta(Y)X - \eta(X)Y\big) + \mu\big(\eta(Y)hX - \eta(X)hY\big) + \nu\big(\eta(Y)\phi hX - \eta(X)\phi hY\big). \tag{1-4}$$

D. Perrone defined a H-contact metric manifold as a $(2n+1)$-dimensional contact metric manifold M whose characteristic vector field (or the Reeb vector field) ξ is a harmonic vector field. In [Perrone 2004], it was proved that $M(\eta, \xi, \phi, g)$ is an H-contact metric manifold if and only if ξ is an eigenvector of the Ricci operator Q. The class of H-contact metric manifolds includes several classes of contact metric manifolds such as Sasakian, η-Einstein, or even generalized (κ, μ)-contact metric manifolds. Perrone [2003] also showed that a contact metric 3-manifold M is a generalized (κ, μ)-contact metric manifold on an everywhere dense open subset of M if and only if its Reeb vector field ξ determines a harmonic map. In turn, Koufogiorgos, Markellos and Papantoniou proved that the (κ, μ, ν)-condition on a 3-dimensional contact metric manifold is equivalent to the Reeb vector field ξ being a harmonic vector field, at least on an open dense subset of the manifold [Koufogiorgos et al. 2008]. They proved also that these manifolds exist only in the dimension 3, whereas such a manifold does not exist in dimension greater than 3; hence, we restrict ourselves to dimension 3.

On the other hand, many geometers have studied the contact manifolds of constant curvature and their generalizations like the locally symmetric spaces ($\nabla R = 0$), Einstein spaces, the semisymmetric spaces ($R(\xi, X) \cdot R = 0$), Ricci semisymmetric spaces ($R(X, Y) \cdot S = 0$), Weyl semisymmetric spaces ($R(X, Y) \cdot C = 0$), where $R(X, Y)$ acts as a derivation respectively on R, S, C etc. For example, a contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature $+1$ or is 3-dimensional and flat [Blair 2002, pages 98–99; Olszak 1979]. S. Tanno [1969] showed that a semisymmetric K-contact manifold M^{2n+1} is locally isometric to the unit sphere $S^{2n+1}(1)$, and that for a K-contact manifold M^{2n+1} the following conditions are equivalent: (i) M is an Einstein manifold; (ii) M is Ricci-symmetric, that is, its Ricci tensor is parallel; (iii) M is Ricci semisymmetric, i.e., it satisfies the condition $R(X, Y) \cdot S = 0$; (iv) M is ξ-Ricci semisymmetric, that is, $R(\xi, Y) \cdot S = 0$.

Perrone [1992] showed that if ξ belongs to the κ-nullity distribution and if $R(\xi, Y) \cdot S = 0$, then the contact metric manifold is locally isometric to $E^{n+1} \times S^n(4)$ or is Sasaki–Einstein. M. M. Tripathi [2006] proved that a contact metric manifold

M^{2n+1} such that ξ belongs to the (κ, μ)-nullity distribution and $R(\xi, Y) \cdot S$ vanishes is either flat and 3-dimensional, or is locally isometric to $E^{n+1} \times S^n(4)$, or is a Sasaki–Einstein manifold. Finally, we studied in [Gouli-Andreou et al. 2012], together with Ph. J. Xenos, the (κ, μ, ν)-contact 3-manifolds in which certain curvature conditions are satisfied; for instance the Ricci tensor S is cyclic parallel, or η-parallel or $R(\xi, Y) \cdot S = 0$.

After the curvature tensor R and the Weyl conformal curvature tensor C, the *concircular curvature tensor* Z is the next most important (1,3)-type curvature tensor. It is defined on a Riemannian manifold (M^n, g) by Yano [1940a] (see also [Yano and Bochner 1953]) as

$$Z = R - \frac{r}{n(n-1)} R_0, \tag{1-5}$$

where R is the curvature tensor, R_0 is given by (1-2) and r the scalar curvature. We remark that Riemannian manifolds with vanishing Z are of constant curvature; thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. Z is an invariant of concircular transformations, which have important geometric and algebraic applications; see [Yano 1940a; 1940b; 1940c; 1940d; 1942; Vanhecke 1977]. Hence, Blair, J. S. Kim and Tripathi [Blair et al. 2005] started a study of the concircular curvature tensor on M^{2n+1} contact metric manifolds. They classified $N(\kappa)$-contact metric manifolds satisfying $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$ or $R(\xi, X) \cdot Z = 0$. Similarly, Tripathi and Kim [2004] classified M^{2n+1} (κ, μ)-contact manifolds with $Z(\xi, X) \cdot S = 0$.

This article is motivated by these studies, and is organized in the following way. In Section 2 we give some preliminaries on contact manifolds and the concircular curvature tensor. In Section 3 we present a brief account of (κ, μ, ν)-contact 3-manifolds while Section 4 contains some basic results. Finally, in Section 5 we study (κ, μ, ν)-contact metric 3-manifolds M satisfying any of these conditions:

(i) M is Ricci flat.

(ii) M is Einstein but not Sasakian.

(iii) $\nabla Z = 0$, where Z is the concircular curvature tensor.

(iv) $Z(\xi, X) \cdot Z = 0$, where $Z(\xi, X)$ acts as a derivation on Z.

(v) $Z(\xi, X) \cdot S = 0$, where $Z(\xi, X)$ acts as a derivation on S.

(vi) $R(\xi, X) \cdot Z = 0$, where $R(\xi, X)$ acts as a derivation on Z.

2. Preliminaries

By a *contact manifold* we mean a smooth manifold M^{2n+1}, endowed with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Then there is an underlying *contact* *metric structure* (η, ξ, ϕ, g) where g is a Riemannian metric (the *associated metric*), ϕ a global tensor of type (1,1) and ξ a unique global vector field (the *characteristic* or *Reeb vector field*). These structure tensors satisfy the equations

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2-1}$$

$$d\eta(X,Y) = g(X,\phi Y) = -g(\phi X,Y), \quad g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2-2}$$

for all $X, Y \in \mathcal{X}(M)$. The associated metrics can be constructed by the polarization of $d\eta$ on the contact subbundle defined by $\eta = 0$. Denoting Lie differentiation by $\mathcal{L}$, we define for all $X \in \mathcal{X}(M)$ the (1,1)-tensor field

$$hX = \tfrac{1}{2}(\mathcal{L}_\xi \phi)X.$$

We give some basic equations for these tensor fields:

$$\begin{gathered} \phi\xi = h\xi = 0, \quad \eta \circ \phi = \eta \circ h = 0, \quad \nabla_\xi \phi = 0, \\ \operatorname{Tr} h = \operatorname{Tr}(h\phi) = 0, \quad h\phi = -\phi h. \end{gathered} \tag{2-3}$$

If X is an eigenvector of h corresponding to the eigenvalue λ, then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$ since h anticommutes with ϕ:

$$hX = \lambda X \quad \Rightarrow \quad h\phi X = -\lambda\phi X, \tag{2-4}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2-5}$$

$$(\nabla_X \eta)(Y) = -g(\phi X + \phi h X, Y), \tag{2-6}$$

where ∇ is the Levi-Civita connection of g. We also denote by R the corresponding Riemann curvature tensor field given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, by S the Ricci tensor field of type (0, 2), by Q the Ricci operator, which is the corresponding endomorphism field, by r the scalar curvature and by H the ϕ-sectional curvature.

A contact metric manifold for which ξ is a Killing field is called a *K-contact* manifold. A contact metric manifold is K-contact if and only if $h = 0$. A contact structure on M^{2n+1} implies an almost complex structure on the product manifold $M^{2n+1} \times \mathbb{R}$. If this structure is integrable, then the contact metric manifold is said to be *Sasakian*. A K-contact structure is Sasakian only in dimension 3, and this fails in higher dimensions. More details on contact manifolds can be found in [Blair 2002].

We restrict ourselves to the 3-dimensional case. Let (M, ϕ, ξ, η, g) be a 3-dimensional contact metric manifold and U the open subset of points $p \in M$ where $h \neq 0$ in a neighborhood of p and U_0 the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. For any point $p \in U \cup U_0$ there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p. On U we put $he = \lambda e$, where λ is a nonvanishing smooth function which is supposed positive. From (2-4) we have $h\phi e = -\lambda\phi e$.

Lemma 2.1 [Gouli-Andreou and Xenos 1998]. *On U we have*

$$\nabla_\xi e = a\phi e, \quad \nabla_e e = b\phi e, \quad \nabla_{\phi e} e = -c\phi e + (\lambda-1)\xi,$$
$$\nabla_\xi \phi e = -ae, \quad \nabla_e \phi e = -be + (1+\lambda)\xi, \quad \nabla_{\phi e}\phi e = ce,$$
$$\nabla_\xi \xi = 0, \quad \nabla_e \xi = -(1+\lambda)\phi e, \quad \nabla_{\phi e}\xi = (1-\lambda)e,$$

where a is a smooth function and

$$b = \frac{1}{2\lambda}[(\phi e\cdot\lambda) + A] \quad \textit{with} \quad A = S(\xi, e),$$
$$c = \frac{1}{2\lambda}[(e\cdot\lambda) + B] \quad \textit{with} \quad B = S(\xi, \phi e). \tag{2-7}$$

Lemma 2.1 and the formula $[X,Y] = \nabla_X Y - \nabla_Y X$ yield

$$[e,\phi e] = \nabla_e\phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi,$$
$$[e,\xi] = \nabla_e \xi - \nabla_\xi e = -(a+\lambda+1)\phi e,$$
$$[\phi e,\xi] = \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a-\lambda+1)e. \tag{2-8}$$

Definition 2.2. Let M^3 be a 3-dimensional contact metric manifold and let $h = \lambda h^+ - \lambda h^-$ be the spectral decomposition of h on U. If

$$\nabla_{h^-X} h^- X = [\xi, h^+ X]$$

for all vector fields X on M^3 and all points of an open subset W of U, and if $h = 0$ on the points of M^3 which do not belong to W, then the manifold is said to be a *semi-K contact* manifold.

From Lemma 2.1 and the relations (2-8), the condition above leads to $[\xi, e] = 0$ when $X = e$ and to $\nabla_{\phi e}\phi e = 0$ when $X = \phi e$. Hence on a semi-K contact manifold we have $a+\lambda+1 = c = 0$. If we apply the deformation $e \to \phi e$, $\phi e \to e$, $\xi \to -\xi$, $\lambda \to -\lambda$, $b \to c$ and $c \to b$ then the contact metric structure remains the same. Hence the condition for a 3-dimensional contact metric manifold to be semi-K contact is equivalent to $a-\lambda+1 = b = 0$.

Definition 2.3 [Blair 2002, page 105; Okumura 1962]. A contact metric manifold M is said to be *η-Einstein* if the Ricci tensor S satisfies the condition $S = \alpha g + \beta\eta\otimes\eta$, where α and β are smooth functions on M. In particular, if $\beta = 0$, then M becomes an *Einstein manifold*.

Definition 2.4. A Riemannian manifold (M^n, g) is called *Ricci flat* if its Ricci tensor vanishes identically.

Since the Ricci operator Q in dimension 3 determines completely the curvature of the contact manifold, the vanishing of Q implies the vanishing of the Riemannian curvature tensor. Hence, the class of Ricci flat manifolds is a hyperclass of the flat manifolds, or equivalently a flat manifold is certainly *Ricci flat*, while a *Ricci flat* manifold is an Einstein manifold.

Definition 2.5. A Riemannian manifold (M^m, g), $m \geq 3$, is called *pseudosymmetric* in the sense of R. Deszcz [1992] if at every point of M the curvature tensor R satisfies the equation $R(X, Y)\cdot R = L\{(X \wedge Y)\cdot R\}$ where $(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y$ for all vectors fields X, Y, Z on M, the dot means that $R(X, Y)$ and $X \wedge Y$ act as derivations on R, and L is a smooth function.

If L is constant, then M is a pseudosymmetric manifold of constant type while if $L = 0$ then M is a *semisymmetric* manifold.

Definition 2.6. A Riemannian manifold (M^n, g) is called *concircularly symmetric* if the concircular tensor Z satisfies the condition $\nabla Z = 0$.

All manifolds are assumed connected and all manifolds and maps are assumed smooth (class C^∞) unless otherwise stated. Finally, differentiation will be denoted by "( )".

3. (κ, μ, ν)-contact metric manifolds

A (κ, μ, ν)*-contact metric manifold* is defined in [Koufogiorgos et al. 2008] by (1-4) where κ, μ, ν are smooth functions on M. If $\nu = 0$ we have a generalized (κ, μ)-contact metric manifold [Koufogiorgos and Tsichlias 2000] and if additionally κ, μ are constants then the manifold is a contact metric (κ, μ)-space [Blair et al. 1995; Boeckx 2000]. Moreover in [Koufogiorgos et al. 2008] and [Koufogiorgos and Tsichlias 2000] it is proved respectively that for a (κ, μ, ν) or a generalized (κ, μ)-contact metric manifold M^{2n+1} of dimension greater than 3, the functions κ, μ are constants and ν is the zero function. We recall some lemmas and equations:

Lemma 3.1 [Koufogiorgos et al. 2008]. *For every point p of a (κ, μ, ν)-contact metric manifold M^{2n+1} with $\kappa(p) < 1$, there exists an open neighborhood U of p and orthonormal local vector fields $X_i, \phi X_i, \xi$, $i = 1, \ldots, n$, defined on U such that*

$$hX_i = \lambda X_i, \qquad h\phi X_i = -\lambda\phi X_i, \qquad h\xi = 0$$

for $i = 1, \ldots, n$, where $\lambda = \sqrt{1-\kappa}$.

From now on, we will call the vector fields of Lemma 3.1 a local *h-basis*.

On any (κ, μ, ν)-contact metric manifold we have

(3-1) $$h^2 = (\kappa - 1)\phi^2, \quad \kappa \leq 1,$$

(3-2) $$(\xi \cdot \kappa) = 2\nu(\kappa - 1).$$

For the 3-dimensional case we have for the Ricci operator Q

$$Q = \left(\tfrac{1}{2}r - \kappa\right)I + \left(-\tfrac{1}{2}r + 3\kappa\right)\eta\otimes\xi + \mu h + \nu\phi h, \tag{3-3}$$

$$Q\phi - \phi Q = 2\nu h - 2\mu\phi h, \tag{3-4}$$

$$r = 4\kappa + 2H, \tag{3-5}$$

where r is the scalar curvature and H is the ϕ-sectional curvature. From now on, we suppose $\kappa < 1$ everywhere on M^3 and we use X, Y, U to denote arbitrary elements of $\mathcal{X}(M)$. We have

$$r = \frac{1}{\lambda}\Delta\lambda - (\xi\cdot\nu) - \frac{\|\operatorname{grad}\lambda\|^2}{\lambda^2} + 2(\kappa-\mu), \tag{3-6}$$

where Δ is the Laplace operator and for the gradient of a function f we have

$$g(\operatorname{grad} f, X) = X(f) = df(X), \tag{3-7}$$

$$(\xi\cdot r) = 2(\xi\cdot\kappa), \quad (\xi\cdot H) = -(\xi\cdot\kappa). \tag{3-8}$$

For a 3-dimensional (κ,μ)-contact metric manifold, that is, for constant κ, μ we have (see [Blair et al. 1995] and [Markellos 2009])

$$r = 2(\kappa-\mu), \tag{3-9}$$

$$\begin{aligned} R(X,Y)U = {} & \mu[g(Y,U)hX - g(X,U)hY + g(hY,U)X - g(hX,U)Y] \\ & + \nu[g(Y,U)\phi hX - g(X,U)\phi hY + g(\phi hY,U)X - g(\phi hX,U)Y] \\ & + (\kappa-H)[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\xi \\ & + (\kappa-H)[\eta(Y)\eta(U)X - \eta(X)\eta(U)Y] \\ & + H[g(Y,U)X - g(X,U)Y], \end{aligned} \tag{3-10}$$

$$\begin{aligned} (\nabla_X h)Y = & -\tfrac{1}{2(1-\kappa)}g(hX,Y)\operatorname{grad}\kappa - \tfrac{1}{2(1-\kappa)}g(hX,\phi Y)\phi(\operatorname{grad}\kappa) \\ & + [(1-\kappa)g(X,\phi Y) + g(hX,\phi Y) - \nu g(hX,Y)]\xi \\ & + \eta(Y)[(\kappa-1)\phi X + h\phi X] + \eta(X)[\mu h\phi Y + \nu hY], \end{aligned} \tag{3-11}$$

$$(\nabla_X\phi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX), \tag{3-12}$$

while $(\nabla_X\phi h)Y = (\nabla_X\phi)hY + \phi(\nabla_X h)Y$ is calculated from (3-11) and (3-12):

$$\begin{aligned} (\nabla_X\phi h)Y = & [g(X+hX,hY) + \nu g(hX,\phi Y)]\xi \\ & - \tfrac{1}{2(1-\kappa)}g(hX,Y)\phi(\operatorname{grad}\kappa) + \tfrac{1}{2(1-\kappa)}g(hX,\phi Y)\operatorname{grad}\kappa \\ & + \eta(Y)[(\kappa-1)\phi^2X + hX] + \eta(X)[\mu hY + \nu\phi hY]. \end{aligned} \tag{3-13}$$

From (3-3) and (3-5) we calculate the Ricci tensor $S(X,Y)=g(QX,Y)$:

$$S(X,Y)=(\kappa+H)g(X,Y)+(\kappa-H)\eta(X)\eta(Y)+\mu g(hX,Y) + \nu g(\phi hX,Y); \tag{3-14}$$

hence,

$$S(hX,Y)=(\kappa+H)g(hX,Y)-\mu(\kappa-1)[g(X,Y)-\eta(X)\eta(Y)] + \nu(\kappa-1)g(X,\phi Y), \tag{3-15}$$

$$S(\phi hX,Y)=(\kappa+H)g(\phi hX,Y)-\nu(\kappa-1)[g(X,Y)-\eta(X)\eta(Y)] + \mu(\kappa-1)g(\phi X,Y). \tag{3-16}$$

4. Some auxiliary results

Equation (1-5) gives for the 3-dimensional case and for all $X, Y, U \in \mathcal{X}(M)$

$$Z(X,Y)U = R(X,Y)U - \tfrac{1}{6} r R_0(X,Y)U, \tag{4-1}$$

where R_0 is given by (1-2) and hence

$$R_0(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{4-2}$$

while (1-4) for a (κ,μ,ν)-*contact metric manifold* is written in the form

$$R(X,Y)\xi = (\kappa I + \mu h + \nu\phi h)R_0(X,Y)\xi, \tag{4-3}$$

which is equivalent to

$$R(\xi,X) = R_0(\xi,(\kappa I + \mu h + \nu\phi h)X). \tag{4-4}$$

From (4-3) we get

$$R(\xi,X)\xi = \kappa(\eta(X)\xi - X) - \mu hX - \nu\phi hX. \tag{4-5}$$

Proposition 4.1. *In a (κ,μ,ν)-contact metric manifold M^3, the concircular curvature tensor Z satisfies*

$$Z(X,Y)\xi = \Big(\big(\kappa - \tfrac{1}{6}r\big)I + \mu h + \nu\phi h\Big)R_0(X,Y)\xi, \tag{4-6}$$

$$Z(\xi,X) = \big(\kappa - \tfrac{1}{6}r\big)R_0(\xi,X) + \mu R_0(\xi,hX) + \nu R_0(\xi,\phi hX). \tag{4-7}$$

Consequently, we have

$$Z(\xi,X)\xi = \big(\kappa - \tfrac{1}{6}r\big)(\eta(X)\xi - X) - \mu hX - \nu\phi hX, \tag{4-8}$$

$$\eta(Z(X,Y)\xi) = 0, \tag{4-9}$$

$$\eta(Z(\xi,X)Y) = \big(\kappa - \tfrac{1}{6}r\big)\big(g(X,Y) - \eta(X)\eta(Y)\big) + \mu g(hX,Y) + \nu g(\phi hX,Y). \tag{4-10}$$

Proof. Equations (4-1), (4-3), (4-4) lead us to conclude equations (4-6) and (4-7). Equation (4-7) implies (4-8) while (4-6) and (4-7) imply (4-9) and (4-10) respectively by virtue of (2-3). □

Proposition 4.2. *In a (κ,μ,ν)-contact metric manifold M^3 we have*

$$S(Z(\xi,X)Y,\xi)=2\kappa\left(\kappa-\tfrac{1}{6}r\right)(g(X,Y)-\eta(X)\eta(Y))+2\kappa\mu g(hX,Y)+2\kappa\nu g(\phi hX,Y), \tag{4-11}$$

$$S(Z(\xi,X)\xi,Y)=2\kappa\left(\kappa-\tfrac{1}{6}r\right)\eta(X)\eta(Y)-\left(\kappa-\tfrac{1}{6}r\right)S(X,Y)-\mu S(hX,Y)-\nu S(\phi hX,Y). \tag{4-12}$$

Proof. For a (κ,μ,ν)-contact metric manifold M^3 we obtain from (3-14)

$$S(X,\xi)=2\kappa\eta(X). \tag{4-13}$$

From (4-7), (4-10), (4-13) we get (4-11), while (4-8) and (4-13) yield (4-12). □

Proposition 4.3. *Let M^3 be a non-Sasakian (κ,μ,ν)-contact metric manifold.*

(i) *If M^3 satisfies*

$$\nu(\kappa-H)=0, \tag{4-14}$$

$$\mu(\kappa-H)=0, \tag{4-15}$$

$$\tfrac{1}{3}(\kappa-H)^2+(\kappa-1)(\mu^2+\nu^2)=0, \tag{4-16}$$

then the manifold is either flat or locally isometric to SU(2) *or* SL(2, R), *where these two Lie groups are equipped with a left invariant metric.*

(ii) *If M^3 satisfies*

$$\nu H=0, \tag{4-17}$$

$$\mu H=0, \tag{4-18}$$

$$\kappa(\kappa-H)+(\kappa-1)(\mu^2+\nu^2)=0, \tag{4-19}$$

then the manifold is a generalized (κ,μ)-contact metric manifold with $(\xi\cdot\mu)=0$.

Proof. (i) Let M be a 3-dimensional (κ,μ,ν)-contact metric manifold with $\kappa<1$ everywhere. We suppose that there is a point $p\in M$ where $\nu\neq 0$. The continuity of this function implies that there is a neighborhood $F_p\subseteq M$ of p, where $\nu\neq 0$ everywhere in F_p or by virtue of (4-14), $\kappa-H=0$. Differentiating this equation with respect to ξ and using (3-8) and (3-2) we conclude that $\kappa=1$ everywhere in F_p, which is a contradiction since $F_p\subseteq M$. Hence, $\nu=0$ everywhere in M and M is a generalized (κ,μ)-contact metric manifold.

Similarly we suppose that there is a point $p\in M$ where $\kappa-H\neq 0$. There is a neighborhood $F_p\subseteq M$ of p, where $\kappa-H\neq 0$ everywhere in F_p or by virtue of

(4-15), $\mu = 0$. Setting $\mu = \nu = 0$ in (4-16) we are led to $\frac{1}{3}(\kappa - H)^2 = 0$ which is a contradiction in F_p. Hence $\kappa - H = 0$ everywhere in M and from (4-16), $\mu = 0$. Since in a generalized (κ, μ)-contact metric manifold the constancy of one of the κ or μ implies the constancy of the other [Koufogiorgos and Tsichlias 2000], we can conclude that κ is constant in this $N(\kappa)$-contact metric manifold. From (3-4) and because $\mu = \nu = 0$ we get $Q\phi = \phi Q$; by [Blair et al. 1990, Theorem 3.3] and the main theorem of [Blair and Chen 1992] such a manifold is either Sasakian, flat, locally isometric to a left invariant metric on the Lie group SU(2) with $\kappa > 0$, or SL(2, R) with $\kappa < 0$. Finally, we can remark that the equations $\kappa - H = 0$ and (3-5) give $r = 6\kappa$, $\kappa < 1$, and hence r is constant.

(ii) We suppose that there is a point $p \in M$ where $\nu \neq 0$. Then there is a neighborhood $F_p \subseteq M$ of p, where $\nu \neq 0$ everywhere in F_p or by virtue of (4-17), $H = 0$. Differentiating this equation with respect to ξ and using (3-8) and (3-2) we conclude that $\kappa = 1$ everywhere in F_p, which is a contradiction since $F_p \subseteq M$. Hence, $\nu = 0$ everywhere in M and M is a generalized (κ, μ)-contact metric manifold.

For (4-18), we suppose that there is a point $p \in M$ where $H \neq 0$. There is a neighborhood $F_p \subseteq M$ of p, where $H \neq 0$ everywhere in F_p or by virtue of (4-18), $\mu = 0$. Since μ is constant, κ is also constant and hence from (3-5) and (3-9), $H = -\kappa - \mu$ or more explicitly $H = -\kappa$. From (4-19) and because $\mu = \nu = 0$ we get $\kappa = 0$ and obviously $H = 0$, which is a contradiction in F_p. Hence $H = 0$ everywhere in M and from (4-19), $\kappa^2 + (\kappa - 1)\mu^2 = 0$. Differentiating this equation with respect to ξ and by virtue of (3-2) and $\nu = 0$ we conclude $(\xi \cdot \mu) = 0$, while (3-5) implies $r = 4\kappa$ with $\kappa < 1$. □

Remark 4.4. The generalized (κ, μ)-contact metric manifolds in dimension 3 with $\kappa < 1$ (equivalently $\lambda \neq 0$) and $(\xi \cdot \mu) = 0$ have been studied by T. Koufogiorgos and C. Tsichlias [2008]. They proved in [2008, Theorem 4.1] that at any point of $P \in M$, precisely one of the following relations is valid: $\mu = 2(1 + \sqrt{1 - \kappa})$, or $\mu = 2(1 - \sqrt{1 - \kappa})$, while there exists a chart (U,(x,y,z)) with $P \in U \subseteq M$ such that the functions κ, μ depend only on z and the tensors fields η, ξ, ϕ, g take a suitable form. We can also add that such a manifold according to Definition 2.2 is a semi-K contact manifold.

Theorem 4.5 [Blair 2002, page 101]. *Let M^{2n+1} be a contact metric manifold satisfying the condition $R(X, Y)\xi = 0$. Then M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

5. Main results

Theorem 5.1. *A non-Sasakian Ricci flat 3-dimensional (κ, μ, ν)-contact metric manifold is flat.*

Proof. Since the manifold M is Ricci flat, from (4-13) we have

$$0 = S(\xi, \xi) = 2\kappa,$$

or $\kappa = 0$. Then, (3-2) yields $\nu = 0$, so M is a generalized (κ, μ)-contact metric manifold with $\kappa = 0$. In a generalized (κ, μ)-contact metric manifold the constancy of one of κ or μ implies the constancy of the other [Koufogiorgos and Tsichlias 2000], so μ is also constant. We set $\kappa = \nu = 0$ in (3-14) and by virtue of (3-5) and (3-9) we have

$$S(X, Y) = \mu[g(hX, Y) - g(X, Y) + \eta(X)\eta(Y)] \tag{5-1}$$

for all $X, Y \in \mathcal{X}(M)$. For any point $p \in M$ we consider a local orthonormal *h-basis* as in Lemma 3.1. In the last equation we set (i) $X = Y = e$ and (ii) $X = Y = \phi e$ and since we have a Ricci flat manifold we get respectively

$$0 = S(e, e) = \mu(\lambda - 1),$$
$$0 = S(\phi e, \phi e) = \mu(-\lambda - 1).$$

By adding these equations we see that $\mu = 0$, and Theorem 4.5 completes the proof. □

Remark 5.2. For a Sasakian 3-manifold, from Equation (3-14) with $\kappa = 1$ and $h = 0$, setting $X = Y = \xi$ yields $S(\xi, \xi) = 2$ and hence a Sasakian manifold cannot be Ricci flat.

Theorem 5.3. *A non-Sasakian Einstein* 3*-dimensional* (κ, μ, ν)*-contact metric manifold is flat.*

Proof. Since the manifold is Einstein, Equation (3-3) gives

$$\left(\tfrac{1}{2}r - \kappa\right)X + \left(-\tfrac{1}{2}r + 3\kappa\right)\eta(X)\xi + \mu hX + \nu\phi hX = aX. \tag{5-2}$$

For any point $p \in U$ as in Lemma 3.1 we consider a local orthonormal *h-basis* and we set in (5-2) $X = \xi$, $X = e$ and $X = \phi e$. We obtain respectively

$$2\kappa = a, \quad \nu = 0,$$
$$\tfrac{1}{2}r - \kappa + \lambda\mu = a, \quad \tfrac{1}{2}r - \kappa - \lambda\mu = a.$$

We have a generalized (κ, μ)-contact metric manifold with $\kappa < 1$ or equivalently $\lambda \neq 0$. From the two last equations we get $\mu = 0$ and hence κ is constant [Koufogiorgos and Tsichlias 2000]. In a 3-dimensional (κ, μ)-contact metric manifold $r = 2(\kappa - \mu)$. By substituting r in the last equation we obtain $a = 0$ or equivalently $\kappa = 0$, and Theorem 4.5 completes the proof. □

Remark 5.4. According to [Yano and Kon 1984, Proposition 3.3, page 38], a 3-dimensional Einstein manifold M is a space of constant curvature. Hence, a Sasaki–Einstein 3-manifold, since it has constant curvature, must have curvature 1.

Theorem 5.5. *If M is a 3-dimensional concircularly symmetric (κ, μ, ν)-contact metric manifold, then M is either flat or locally isometric to the sphere $S^3(1)$.*

Proof. We consider the open subsets of M:

$$U_1 = \{p \in M : \kappa = 1 \text{ in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \text{ in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is an open and dense subset of M.

In the case where $M = U_1$ the manifold is a Sasakian concircularly symmetric manifold.

Next, we assume that U_2 is not empty. Differentiating (4-1) and using (1-2), (2-1), (2-2), (2-5), (2-6), (3-7), (3-10), (3-11), (3-13), with $\kappa < 1$ everywhere, it follows that

$$\begin{aligned}
(\nabla_W Z)(X, Y)U = {} & [(W\cdot H) - \tfrac{1}{6}(W\cdot r)][g(Y,U)X - g(X,U)Y] \\
& + [(W\cdot\kappa) - (W\cdot H)][g(Y,U)\eta(X) - g(X,U)\eta(Y)]\xi \\
& + [(W\cdot\kappa) - (W\cdot H)][\eta(Y)\eta(U)X - \eta(X)\eta(U)Y] \\
& + (W\cdot\mu)[g(Y,U)hX - g(X,U)hY + g(hY,U)X - g(hX,U)Y] \\
& + (W\cdot\nu)[g(Y,U)\phi hX - g(X,U)\phi hY + g(\phi hY,U)X - g(\phi hX,U)Y] \\
& + (\kappa - H)\big\{[g(Y,U)g(W+hW,\phi X) - g(X,U)g(W+hW,\phi Y)]\xi \\
& \qquad + [\eta(Y)X - \eta(X)Y]g(W+hW,\phi U) \\
& \qquad + [g(W+hW,\phi Y)X - g(W+hW,\phi X)Y]\eta(U) \\
& \qquad - [g(Y,U)\eta(X) - g(X,U)\eta(Y)](\phi W + \phi hW)\big\} \\
& + \mu\Big[\big\{\tfrac{1}{2(\kappa-1)}g(hW,X)\operatorname{grad}\kappa + \tfrac{1}{2(\kappa-1)}g(hW,\phi X)\phi(\operatorname{grad}\kappa) \\
& \qquad + [(1-\kappa)g(W,\phi X) + g(hW,\phi X) - \nu g(hW,X)]\xi \\
& \qquad + \eta(X)[(\kappa-1)\phi W + h\phi W] + \eta(W)(\mu h\phi X + \nu hX)\big\}g(Y,U) \\
& \quad - \big\{\tfrac{1}{2(\kappa-1)}g(hW,Y)\operatorname{grad}\kappa + \tfrac{1}{2(\kappa-1)}g(hW,\phi Y)\phi(\operatorname{grad}\kappa) \\
& \qquad + [(1-\kappa)g(W,\phi Y) + g(hW,\phi Y) - \nu g(hW,Y)]\xi \\
& \qquad + \eta(Y)[(\kappa-1)\phi W + h\phi W] + \eta(W)(\mu h\phi Y + \nu hY)\big\}g(X,U) \\
& \quad + \big\{\tfrac{1}{2(\kappa-1)}g(hW,Y)(U\cdot\kappa) - \tfrac{1}{2(\kappa-1)}g(hW,\phi Y)(\phi U\cdot\kappa) \\
& \qquad + [(1-\kappa)g(W,\phi Y) + g(hW,\phi Y) - \nu g(hW,Y)]\eta(U) \\
& \qquad + \eta(Y)g((\kappa-1)\phi W + h\phi W, U) + \eta(W)g(\mu h\phi Y + \nu hY, U)\big\}X
\end{aligned}$$

$$
\begin{aligned}
&- \Big\{ \tfrac{1}{2(\kappa-1)} g(hW, X)(U \cdot \kappa) - \tfrac{1}{2(\kappa-1)} g(hW, \phi X)(\phi U \cdot \kappa) \\
&\qquad + [(1-\kappa) g(W, \phi X) + g(hW, \phi X) - \nu g(hW, X)]\eta(U) \\
&\qquad + \eta(X) g((\kappa-1)\phi W + h\phi W, U) + \eta(W) g(\mu h\phi X + \nu hX, U) \Big\} Y \Big] \\
&+ \nu \Big[\Big\{ \tfrac{1}{2(\kappa-1)} g(hW, X)\phi(\operatorname{grad}\kappa) - \tfrac{1}{2(\kappa-1)} g(hW, \phi X)\operatorname{grad}\kappa \\
&\qquad + [g(W + hW, hX) + \nu g(hW, \phi X)]\xi \\
&\qquad + \eta(X)[(\kappa-1)\phi^2 W + hW] + \eta(W)[\mu hX + \nu\phi hX] \Big\} g(Y, U) \\
&- \Big\{ \tfrac{1}{2(\kappa-1)} g(hW, Y)\phi(\operatorname{grad}\kappa) - \tfrac{1}{2(\kappa-1)} g(hW, \phi Y)\operatorname{grad}\kappa \\
&\qquad + [g(W + hW, hY) + \nu g(hW, \phi Y)]\xi \\
&\qquad + \eta(Y)[(\kappa-1)\phi^2 W + hW] + \eta(W)[\mu hY + \nu\phi hY] \Big\} g(X, U) \\
&+ \Big\{ \tfrac{-1}{2(\kappa-1)} g(hW, Y)(\phi U \cdot \kappa) - \tfrac{1}{2(\kappa-1)} g(hW, \phi Y)(U \cdot \kappa) \\
&\qquad + [g(W + hW, hY) + \nu g(hW, \phi Y)]\eta(U) \\
&\qquad + \eta(Y) g((\kappa-1)\phi^2 W + hW, U) + \eta(W) g(\mu hY + \nu\phi hY, U) \Big\} X \\
&- \Big\{ \tfrac{-1}{2(\kappa-1)} g(hW, X)(\phi U \cdot \kappa) - \tfrac{1}{2(\kappa-1)} g(hW, \phi X)(U \cdot \kappa) \\
&\qquad + [g(W + hW, hX) + \nu g(hW, \phi X)]\eta(U) \\
&\qquad + \eta(X) g((\kappa-1)\phi^2 W + hW, U) + \eta(W) g(\mu hX + \nu\phi hX, U) \Big\} Y \Big].
\end{aligned}
$$

In this equation, we set $W = \xi$ and by virtue of (2-1), (2-3), (3-8) we obtain

$$
\begin{aligned}
(\nabla_\xi Z)(X, Y)U &= 2(\xi \cdot \kappa)[g(Y, U)\eta(X) - g(X, U)\eta(Y)]\xi \\
&- \tfrac{4}{3}(\xi \cdot \kappa)[g(Y, U)X - g(X, U)Y] \\
&+ (\xi \cdot \mu)[g(Y, U)hX - g(X, U)hY + g(hY, U)X - g(hX, U)Y] \\
&+ (\xi \cdot \nu)[g(Y, U)\phi hX - g(X, U)\phi hY + g(\phi hY, U)X - g(\phi hX, U)Y] \\
&+ \mu \big\{ g(Y, U)(\mu h\phi X + \nu hX) - g(X, U)(\mu h\phi Y + \nu hY) \\
&\qquad + g(\mu h\phi Y + \nu hY, U)X - g(\mu h\phi X + \nu hX, U)Y \big\} \\
&+ \nu \big\{ g(Y, U)(\mu hX + \nu\phi hX) - g(X, U)(\mu hY + \nu\phi hY) \\
&\qquad + g(\mu hY + \nu\phi hY, U)X - g(\mu hX + \nu\phi hX, U)Y \big\}.
\end{aligned} \tag{5-3}
$$

For any point $p \in U_2$ we consider a local orthonormal *h-basis* as in Lemma 3.1. We set in (5-3): $X = U = e$, $Y = \phi e$ which yields

$$
(\nabla_\xi Z)(e, \phi e)e = \tfrac{4}{3}(\xi \cdot \kappa)\phi e.
$$

Since the manifold is concircularly symmetric we conclude that

$$
(\xi \cdot \kappa) = 0,
$$

or equivalently, by virtue of (3-2), $\nu = 0$. We set in (5-3): $X = e$, $Y = U = \xi$ and $\nu = 0$, and get

$$(\nabla_\xi Z)(e, \xi)\xi = \lambda[(\xi \cdot \mu)e - \mu^2 \phi e].$$

The manifold is concircularly symmetric and hence $\mu = 0$. The constancy of μ implies the constancy of κ [Koufogiorgos and Tsichlias 2000] and finally [Blair et al. 2005, Theorem 5.2] completes the proof. □

Theorem 5.6. *Let M a 3-dimensional (κ, μ, ν)-contact metric manifold. If the concircular curvature tensor Z satisfies the condition $Z(\xi, X) \cdot Z = 0$, then M is either Sasakian ($\kappa = 1$), flat or locally isometric to either* SU(2) *or* SL(2, *R*), *where these two Lie groups are equipped with a left invariant metric and they have constant scalar curvature $r = 6\kappa$ ($\kappa < 1$).*

Proof. We consider the open subsets of M:

$$U_1 = \{p \in M : \kappa = 1 \text{ in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \text{ in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is open and dense subset of M.

In the case where $M = U_1$ the manifold is Sasakian and then according to [Blair et al. 2005, Theorem 4.1], it has constant curvature 1.

Next, we assume that U_2 is not empty. Note that the condition $Z(\xi, X) \cdot Z = 0$ implies $(Z(\xi, U) \cdot Z)(X, Y)\xi = 0$ or more explicitly

$$Z(\xi, U)Z(X, Y)\xi - Z(Z(\xi, U)X, Y)\xi - Z(X, Z(\xi, U)Y)\xi - Z(X, Y)Z(\xi, U)\xi = 0$$

which by virtue of (1-1), (1-4), (2-3), (4-1), (4-6), (4-7), (4-8), (4-9), (4-10) yields

$$\begin{aligned}
0 =\, &\mu\left(\kappa - \tfrac{1}{6}r\right)[\eta(Y)g(hU, X) - \eta(X)g(hU, Y)]\xi \\
&+ \mu^2[\eta(Y)g(hU, hX) - \eta(X)g(hU, hY)]\xi \\
&+ \nu\left(\kappa - \tfrac{1}{6}r\right)[\eta(Y)g(\phi hU, X) - \eta(X)g(\phi hU, Y)]\xi \\
&+ \nu^2[\eta(Y)g(\phi hU, \phi hX) - \eta(X)g(\phi hU, \phi hY)]\xi \\
&+ \left(\kappa - \tfrac{1}{6}r\right)^2 g(U, X)Y + \mu\left(\kappa - \tfrac{1}{6}r\right)g(hU, X)Y + \nu\left(\kappa - \tfrac{1}{6}r\right)g(\phi hU, X)Y \\
&+ \mu\left(\kappa - \tfrac{1}{6}r\right)g(U, X)hY + \mu^2 g(hU, X)hY + \mu\nu g(\phi hU, X)hY \\
&+ \nu\left(\kappa - \tfrac{1}{6}r\right)g(U, X)\phi hY + \mu\nu g(hU, X)\phi hY + \nu^2 g(\phi hU, X)\phi hY \\
&- \left(\kappa - \tfrac{1}{6}r\right)^2 g(U, Y)X - \mu\left(\kappa - \tfrac{1}{6}r\right)g(hU, Y)X - \nu\left(\kappa - \tfrac{1}{6}r\right)g(\phi hU, Y)X \\
&- \mu\left(\kappa - \tfrac{1}{6}r\right)g(U, Y)hX - \mu^2 g(hU, Y)hX - \mu\nu g(\phi hU, Y)hX \\
&- \nu\left(\kappa - \tfrac{1}{6}r\right)g(U, Y)\phi hX - \mu\nu g(hU, Y)\phi hX - \nu^2 g(\phi hU, Y)\phi hX \\
&+ \left(\kappa - \tfrac{1}{6}r\right)Z(X, Y)U + \mu Z(X, Y)hU + \nu Z(X, Y)\phi hU.
\end{aligned} \tag{5-4}$$

For any point $p \in U_2$ we consider a local orthonormal *h-basis* as in Lemma 3.1. In (5-4) we set $X = U = e$, $Y = \phi e$, and by virtue of (2-3), (2-4) we obtain

$$[(\kappa - \tfrac{1}{6}r)^2 - \lambda^2(\mu^2 + \nu^2)]\phi e + (\kappa - \tfrac{1}{6}r)Z(e, \phi e)e + \mu Z(e, \phi e)he \\ + \nu Z(e, \phi e)\phi he = 0. \tag{5-5}$$

Equation (4-1) by virtue of (1-2), (2-4) and (3-10) yields

$$\begin{aligned} Z(e, \phi e)e &= (-H + \tfrac{1}{6}r)\phi e, \\ Z(e, \phi e)he &= \lambda(-H + \tfrac{1}{6}r)\phi e, \\ Z(e, \phi e)\phi he &= \lambda(H - \tfrac{1}{6}r)e. \end{aligned} \tag{5-6}$$

Substituting (5-6) in (5-5) we obtain

$$\nu\lambda(H - \tfrac{1}{6}r)e + [(\kappa - \tfrac{1}{6}r)(\kappa - H) - \lambda^2(\mu^2 + \nu^2) - \lambda\mu(H - \tfrac{1}{6}r)]\phi e = 0,$$

and hence

$$\nu\lambda(H - \tfrac{1}{6}r) = 0, \tag{5-7}$$

$$(\kappa - \tfrac{1}{6}r)(\kappa - H) - \lambda^2(\mu^2 + \nu^2) - \lambda\mu(H - \tfrac{1}{6}r) = 0. \tag{5-8}$$

In (5-4) we set $X = e$, $Y = U = \phi e$, and by virtue of (2-3), (2-4) we obtain

$$[-(\kappa - \tfrac{1}{6}r)^2 + \lambda^2(\mu^2 + \nu^2)]e + (\kappa - \tfrac{1}{6}r)Z(e, \phi e)\phi e \\ + \mu Z(e, \phi e)h\phi e + \nu Z(e, \phi e)\phi h\phi e = 0. \tag{5-9}$$

Equation (4-1) by virtue of (1-2), (2-4) and (3-10) yields

$$\begin{aligned} Z(e, \phi e)\phi e &= (H - \tfrac{1}{6}r)e, \\ Z(e, \phi e)h\phi e &= \lambda(-H + \tfrac{1}{6}r)e, \\ Z(e, \phi e)\phi h\phi e &= \lambda(-H + \tfrac{1}{6}r)e. \end{aligned} \tag{5-10}$$

Substituting the equations (5-10) in (5-9) we obtain

$$[-(\kappa - \tfrac{1}{6}r)(\kappa - H) + \lambda^2(\mu^2 + \nu^2) - \lambda\mu(H - \tfrac{1}{6}r)]e - \nu\lambda(H - \tfrac{1}{6}r)\phi e = 0,$$

and hence, in addition to (5-7), we get

$$-(\kappa - \tfrac{1}{6}r)(\kappa - H) + \lambda^2(\mu^2 + \nu^2) - \lambda\mu(H - \tfrac{1}{6}r) = 0. \tag{5-11}$$

Since we work in U_2 where $\kappa \neq 1$ (more precisely $\kappa < 1$) or equivalently $\lambda \neq 0$, the equations (5-7), (5-8) and (5-11) by virtue of (3-5) yield the equations (4-14), (4-15) and (4-16). Finally Proposition 4.3 completes the proof. □

Corollary 5.7. *Let M be a 3-dimensional (κ, μ, ν)-contact metric manifold. If the concircular curvature tensor Z satisfies the condition $Z(\xi, X) \cdot Z = 0$, then M is a pseudosymmetric manifold, in the sense of Deszcz, of constant type.*

Proof. From [Blair et al. 1990, Proposition 3.2] this manifold is an η-Einstein and then [Cho and Inoguchi 2005, Proposition 1.2] completes the proof. □

Theorem 5.8. *Let M be a 3-dimensional (κ, μ, ν)-contact metric manifold. If the concircular curvature tensor Z satisfies the condition $Z(\xi, X) \cdot S = 0$, then M is either Sasakian ($\kappa = 1$), flat or locally isometric to either $\mathrm{SU}(2)$ or $\mathrm{SL}(2, R)$, where these two Lie groups are equipped with a left invariant metric and they have constant scalar curvature $r = 6\kappa$ ($\kappa < 1$).*

Proof. We consider the open subsets of M:

$$U_1 = \{p \in M : \kappa = 1 \text{ in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \text{ in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is an open and dense subset of M.

In the case where $M = U_1$, the manifold is Sasakian and according to [Tripathi and Kim 2004, Theorem 1.4], it has constant curvature 1.

Next, we assume that U_2 is not empty; we work in U_2 where $\kappa < 1$ everywhere. The condition $Z(\xi, X) \cdot S = 0$ or equivalently

$$0 = (Z(\xi, X) \cdot S)(Y, W) = Z(\xi, X) \cdot S(Y, W) - S(Z(\xi, X)Y, W) - S(Y, Z(\xi, X)W)$$

implies

$$S(Z(\xi, X)Y, W) + S(Y, Z(\xi, X)W) = 0 \tag{5-12}$$

which in view of (4-11) and (4-12) yields

$$\left(\kappa - \tfrac{1}{6}r\right)[S(X, Y) - 2\kappa g(X, Y)] + \mu[S(hX, Y) - 2\kappa g(hX, Y)] + \nu[S(\phi hX, Y) - 2\kappa g(\phi hX, Y)] = 0. \tag{5-13}$$

For any point $p \in U_2$ we consider an h-basis. In (5-13) setting (i) $X = Y = e$, (ii) $X = Y = \phi e$ and (iii) $X = e$ and $Y = \phi e$, and by virtue of (3-14), (3-15) and (3-16), we obtain respectively

$$\left(\kappa - \tfrac{1}{6}r\right)(H - \kappa + \lambda\mu) + \mu(\lambda H - \lambda\kappa - \mu\kappa + \mu) - \nu^2(\kappa - 1) = 0, \tag{5-14}$$
$$\left(\kappa - \tfrac{1}{6}r\right)(H - \kappa - \lambda\mu) + \mu(-\lambda H + \lambda\kappa - \mu\kappa + \mu) - \nu^2(\kappa - 1) = 0, \tag{5-15}$$

and (4-14). By virtue of (3-5) and by subtracting (5-15) from (5-14) we obtain (4-15), while by adding equations (5-14) and (5-15) we get (4-16). Proposition 4.3 completes the proof. □

Corollary 5.9. *Let M be a 3-dimensional (κ, μ, ν)-contact metric manifold. If the concircular curvature tensor Z satisfies the condition $Z(\xi, X) \cdot S = 0$, then M is a pseudosymmetric manifold, in the sense of Deszcz, of constant type.*

Proof. From [Blair et al. 1990, Proposition 3.2] this manifold is an η-Einstein and then [Cho and Inoguchi 2005, Proposition 1.2] completes the proof. □

Theorem 5.10. *Let $M^3(\eta, \xi, \phi, g)$ be a 3-dimensional (κ, μ, ν)-contact metric manifold satisfying the condition $R(\xi, X) \cdot Z = 0$. Then, there are at most two open subsets of M^3 for which their union is an open and dense subset of M^3, and each of them as an open submanifold of M^3 is either (a) a Sasakian manifold or (b) a semi-K generalized (κ, μ)-contact metric manifold with $(\xi \cdot \mu) = 0$ and $r = 4\kappa$.*

Proof. We consider the open subsets of M:

$$U_1 = \{p \in M : \kappa = 1 \text{ in a neighborhood of } p\},$$
$$U_2 = \{p \in M : \kappa \neq 1 \text{ in a neighborhood of } p\},$$

where $U_1 \cup U_2$ is open and dense in M.

In the case where $M = U_1$, the manifold is Sasakian and according to [Blair et al. 2005, Theorem 4.3], it has constant curvature 1.

Next, we assume that U_2 is not empty. Firstly, we remark that the condition $R(\xi, X) \cdot Z = 0$ implies $(R(\xi, U) \cdot Z)(X, Y)\xi = 0$ or more explicitly

$$R(\xi, U)Z(X, Y)\xi - Z(R(\xi, U)X, Y)\xi - Z(X, R(\xi, U)Y)\xi - Z(X, Y)R(\xi, U)\xi = 0$$

which by virtue of (1-1), (1-4), (2-3), (3-10), (4-1), (4-9) yields

$$\begin{aligned}
(5\text{-}16)\quad 0 = {} & \mu\kappa[\eta(Y)g(U, hX) - \eta(X)g(U, hY)]\xi \\
& + \nu\kappa[\eta(Y)g(U, \phi hX) - \eta(X)g(U, \phi hY)]\xi \\
& + \mu^2[\eta(Y)g(hU, hX) - \eta(X)g(hU, hY)]\xi \\
& + \nu^2[\eta(Y)g(\phi hU, \phi hX) - \eta(X)g(\phi hU, \phi hY)]\xi \\
& + \kappa\big(\kappa - \tfrac{1}{6}r\big)g(U, X)Y + \kappa\mu g(U, X)hY + \kappa\nu g(U, X)\phi hY \\
& - \kappa\big(\kappa - \tfrac{1}{6}r\big)g(U, Y)X - \kappa\mu g(U, Y)hX - \kappa\nu g(U, Y)\phi hX \\
& + \mu\big(\kappa - \tfrac{1}{6}r\big)g(hU, X)Y + \mu^2 g(hU, X)hY + \mu\nu g(hU, X)\phi hY \\
& - \mu\big(\kappa - \tfrac{1}{6}r\big)g(hU, Y)X - \mu^2 g(hU, Y)hX - \mu\nu g(hU, Y)\phi hX \\
& + \nu\big(\kappa - \tfrac{1}{6}r\big)g(\phi hU, X)Y + \mu\nu g(\phi hU, X)hY + \nu^2 g(\phi hU, X)\phi hY \\
& - \nu\big(\kappa - \tfrac{1}{6}r\big)g(\phi hU, Y)X - \mu\nu g(\phi hU, Y)hX - \nu^2 g(\phi hU, Y)\phi hX \\
& + \kappa Z(X, Y)U + \mu Z(X, Y)hU + \nu Z(X, Y)\phi hU.
\end{aligned}$$

For any point $p \in U_2$ we consider a local orthonormal *h-basis* as in Lemma 3.1. In (5-16) we set $X = U = e$, $Y = \phi e$ and by virtue of (2-3), (2-4) we obtain

$$\tfrac{1}{6}r\nu\lambda e + \big[\kappa^2 - \tfrac{1}{6}r\kappa - \lambda^2(\mu^2 + \nu^2) - \tfrac{1}{6}r\lambda\mu\big]\phi e + \kappa Z(e, \phi e)e + \mu Z(e, \phi e)he + \nu Z(e, \phi e)\phi he = 0,$$

which by (5-6) gives

$$\nu\lambda H e + [\kappa(\kappa - H) - \lambda^2(\mu^2+\nu^2) - \lambda\mu H]\phi e = 0,$$

and hence

$$\nu\lambda H = 0, \tag{5-17}$$

$$\kappa(\kappa - H) - \lambda^2(\mu^2+\nu^2) - \lambda\mu H = 0. \tag{5-18}$$

In (5-16) we set $X = e$, $Y = U = \phi e$, and by virtue of (2-3), (2-4) we obtain

$$\big[-\kappa^2 + \tfrac{1}{6} r\kappa + \lambda^2(\mu^2+\nu^2) - \tfrac{1}{6} r\lambda\mu\big]e - \tfrac{1}{6} r\lambda\nu\phi e + \kappa Z(e,\phi e)\phi e \\ + \mu Z(e,\phi e)h\phi e + \nu Z(e,\phi e)\phi h\phi e = 0$$

which by virtue of (5-10) yields

$$[-\kappa(\kappa - H) + \lambda^2(\mu^2+\nu^2) - \lambda\mu H]e - \nu\lambda H\phi e = 0,$$

and hence, in addition from (5-17), we get

$$-\kappa(\kappa - H) + \lambda^2(\mu^2+\nu^2) - \lambda\mu H = 0. \tag{5-19}$$

Since we work in U_2 where $\kappa < 1$ or equivalently $\lambda \neq 0$, the equations (5-17), (5-18) and (5-19) yield the equations (4-17), (4-18) and (4-19) and hence Proposition 4.3 completes the proof. Our open submanifold U_2 is a generalized (κ, μ)-contact metric 3-manifold with $(\xi \cdot \mu) = 0$ and according to Remark 4.4 this submanifold is a semi-K contact manifold.

We have proved:

(a) If $M = U_1$ then M is Sasakian with $\kappa = 1$.

(b) If $M = U_2$ then M is a semi-K generalized (κ, μ)-contact metric manifold with $\kappa < 1$, $(\xi \cdot \mu) = 0$ and $r = 4\kappa$.

(c) If $U_1 \neq \varnothing$ and $U_2 \neq \varnothing$, the union $U_1 \cup U_2$ is open and dense in M; also, $\kappa = 1$ in U_1 and $\kappa < 1$ in U_2. The function κ is continuous in U_1 and in U_2. □

Remark 5.11. According to Proposition 4.3 and [Blair 2002, Theorem 7.5, p. 101]. U_2 becomes flat when $\mu = 0$ since Equation (4-19) yields $\kappa = 0$.

Acknowledgements

The authors would like to express their sincere thanks to Professor Ph. J. Xenos for his invaluable help and the given information. They also thank the referee for all comments and corrections which improved this article.

References

[Blair 2002] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, Birkhäuser, Boston, 2002. MR 2002m:53120 Zbl 1011.53001

[Blair and Chen 1992] D. E. Blair and H. Chen, "A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, II", *Bull. Inst. Math. Acad. Sinica* **20**:4 (1992), 379–383. MR 94b:53062 Zbl 0767.53023

[Blair et al. 1990] D. E. Blair, T. Koufogiorgos, and R. Sharma, "A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$", *Kodai Math. J.* **13**:3 (1990), 391–401. MR 91j:53015 Zbl 0716.53041

[Blair et al. 1995] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, "Contact metric manifolds satisfying a nullity condition", *Israel J. Math.* **91**:1-3 (1995), 189–214. MR 96f:53037 Zbl 0837.53038

[Blair et al. 2005] D. E. Blair, J.-S. Kim, and M. M. Tripathi, "On the concircular curvature tensor of a contact metric manifold", *J. Korean Math. Soc.* **42**:5 (2005), 883–892. MR 2006g:53120 Zbl 1084.53039

[Boeckx 2000] E. Boeckx, "A full classification of contact metric (k, μ)-spaces", *Illinois J. Math.* **44**:1 (2000), 212–219. MR 2001b:53099 Zbl 0969.53019

[Cho and Inoguchi 2005] J. T. Cho and J.-I. Inoguchi, "Pseudo-symmetric contact 3-manifolds", *J. Korean Math. Soc.* **42**:5 (2005), 913–932. MR 2006c:53085 Zbl 1081.53018

[Deszcz 1992] R. Deszcz, "On pseudosymmetric spaces", *Bull. Soc. Math. Belg. Sér. A* **44**:1 (1992), 1–34. MR 96c:53068 Zbl 0808.53012

[Gouli-Andreou and Xenos 1998] F. Gouli-Andreou and Ph. J. Xenos, "On 3-dimensional contact metric manifolds with $\nabla_\xi \tau = 0$", *J. Geom.* **62**:1-2 (1998), 154–165. MR 99e:53035 Zbl 0905.53024

[Gouli-Andreou et al. 2012] F. Gouli-Andreou, E. Moutafi, and Ph. J. Xenos, "On 3-dimensional (κ, μ, ν)-contact metrics", *Differ. Geom. Dyn. Syst.* **14** (2012), 55–68. MR 2914642 Zbl 1246.53106

[Koufogiorgos and Tsichlias 2000] T. Koufogiorgos and C. Tsichlias, "On the existence of a new class of contact metric manifolds", *Canad. Math. Bull.* **43**:4 (2000), 440–447. MR 2001h:53113 Zbl 0978.53086

[Koufogiorgos and Tsichlias 2008] T. Koufogiorgos and C. Tsichlias, "Generalized (κ, μ)-contact metric manifolds with $\xi\mu = 0$", *Tokyo Journal of Mathematics* **31**:1 (2008), 39–57. MR 2009d:53114 Zbl 1158.53020

[Koufogiorgos et al. 2008] T. Koufogiorgos, M. Markellos, and V. J. Papantoniou, "The harmonicity of the Reeb vector field on contact metric 3-manifolds", *Pacific J. Math.* **234**:2 (2008), 325–344. MR 2008m:53187 Zbl 1154.53052

[Markellos 2009] M. Markellos, *Study of specific categories of Riemannian contact manifolds*, thesis, University of Patra, 2009, Available at http://phdtheses.ekt.gr/eadd/handle/10442/18245. In Greek.

[Okumura 1962] M. Okumura, "Some remarks on space with a certain contact structure", *Tôhoku Math. J.* (2) **14** (1962), 135–145. MR 26 #708 Zbl 0119.37701

[Olszak 1979] Z. Olszak, "On contact metric manifolds", *Tôhoku Math. J.* (2) **31**:2 (1979), 247–253. MR 81b:53032 Zbl 0397.53026

[Perrone 1992] D. Perrone, "Contact Riemannian manifolds satisfying $R(X, \xi) \cdot R = 0$", *Yokohama Math. J.* **39**:2 (1992), 141–149. MR 92m:53053 Zbl 0777.53046

[Perrone 2003] D. Perrone, "Harmonic characteristic vector fields on contact metric three-manifolds", *Bull. Austral. Math. Soc.* **67**:2 (2003), 305–315. MR 2004c:53097 Zbl 1034.53050

[Perrone 2004] D. Perrone, "Contact metric manifolds whose characteristic vector field is a harmonic vector field", *Differential Geom. Appl.* **20**:3 (2004), 367–378. MR 2005a:53134 Zbl 1061.53028

[Tanno 1969] S. Tanno, "Isometric immersions of Sasakian manifolds in spheres", *Kōdai Math. Sem. Rep.* **21** (1969), 448–458. MR 41 #6120 Zbl 0196.25501

[Tripathi 2006] M. M. Tripathi, "Classification of ξ-Ricci-semisymmetric (κ, μ)-manifolds", *Balkan J. Geom. Appl.* **11**:2 (2006), 134–142. MR 2007c:53059 Zbl 1109.53049

[Tripathi and Kim 2004] M. M. Tripathi and J.-S. Kim, "On the concircular curvature tensor of a (κ, μ)-manifold", *Balkan J. Geom. Appl.* **9**:1 (2004), 104–114. MR 2006k:53138 Zbl 1073.53064

[Vanhecke 1977] L. Vanhecke, "Curvature tensors", *J. Korean Math. Soc.* **14**:1 (1977), 143–151. MR 57 #13749 Zbl 0367.53006

[Yano 1940a] K. Yano, "Concircular geometry, I: Concircular transformations", *Proc. Imp. Acad. Tokyo* **16** (1940), 195–200. MR 2,165a Zbl 0024.08102

[Yano 1940b] K. Yano, "Concircular geometry, II: Integrability conditions of $\rho_{\mu\nu} = \phi g_{\mu\nu}$", *Proc. Imp. Acad. Tokyo* **16** (1940), 354–360. MR 2,165b Zbl 0024.18403

[Yano 1940c] K. Yano, "Concircular geometry, III: Theory of curves", *Proc. Imp. Acad. Tokyo* **16** (1940), 442–448. MR 2,303d Zbl 0025.08503

[Yano 1940d] K. Yano, "Concircular geometry, IV: Theory of subspaces", *Proc. Imp. Acad. Tokyo* **16** (1940), 505–511. MR 2,303e Zbl 0025.08504

[Yano 1942] K. Yano, "Concircular geometry, V: Einstein spaces", *Proc. Imp. Acad. Tokyo* **18** (1942), 446–451. MR 7,330f Zbl 0060.38601

[Yano and Bochner 1953] K. Yano and S. Bochner, *Curvature and Betti numbers*, Annals of Mathematics Studies **32**, Princeton University Press, Princeton, NJ, 1953. MR 15,989f Zbl 0051.39402

[Yano and Kon 1984] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics **3**, World Scientific, Singapore, 1984. MR 86g:53001 Zbl 0557.53001

Received November 15, 2012. Revised December 19, 2012.

FLORENCE GOULI-ANDREOU
DEPARTMENT OF MATHEMATICS
ARISTOTLE UNIVERSITY OF THESSALONIKI
54124 THESSALONIKI
GREECE

fgouli@math.auth.gr

EVAGGELIA MOUTAFI
57007 ADENDRON
GREECE

moutafi@sch.gr

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US $410/year for the electronic version, and $535/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 269 No. 1 May 2014