Pacific Journal of Mathematics

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Volume 269 No. 1 May 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 269, No. 1, 2014
dx.doi.org/10.2140/pjm.2014.269.169

ON THE SET OF MAXIMAL NILPOTENT SUPPORTS OF SUPERCUSPIDAL REPRESENTATIONS

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Let G be a quasisplit reductive group over a p-adic field k, T a maximal unramified anisotropic torus of $G(k)$, and χ a character of $T(k)$ satisfying certain conditions. Assume the residue characteristic p of k is large enough. It was shown by DeBacker and Reeder that the irreducible supercuspidal representation π_χ of $G(k)$ associated to $(T(k), \chi)$ is generic if and only if $\mathcal{B}(T, k)$ is a special vertex of $\mathcal{B}(G, k)$. We compute the set of maximal nilpotent support $\mathcal{N}_{\text{wh,max}}(\pi_\chi)$ when $\mathcal{B}(T, k)$ is not a special point in $\mathcal{B}(G, k)$.

1. Introduction

Let k be a p-adic field and ψ a nontrivial character of k. Let $\boldsymbol{G}$ be a split orthogonal or symplectic group over k, $\boldsymbol{\mathfrak{g}}$ the Lie algebra of $\boldsymbol{G}$, $G = \boldsymbol{G}(k)$, and $\mathfrak{g} = \boldsymbol{\mathfrak{g}}(k)$. Let $\mathfrak{g}_{\text{nil}}$ be the set of nilpotent elements in $\mathfrak{g}$ upon which G acts by the adjoint action. Let O be an orbit in $\mathfrak{g}_{\text{nil}}/G$, $z \in O$, and let $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ be a Lie algebra homomorphism with

$$\phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = z.$$

Identify a scalar $t \in k$ with the diagonal matrix $\operatorname{diag}(t, t^{-1}) \in \mathfrak{sl}_2(k)$. For $j \in \mathbb{Z}$, let

$$\mathfrak{g}_j = \{Y \in \mathfrak{g} \mid \operatorname{Ad} \circ \phi(t)(Y) = itY \text{ for all } t \in k\}.$$

Then $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, $z \in \mathfrak{g}_{-2}$.

Let $N_{\geq 2}$ (resp. $N_{\geq 1}$) be the unipotent subgroup of G with Lie algebra $\mathfrak{n}_{\geq 2} = \bigoplus_{j \geq 2} \mathfrak{g}_j$ (resp. $\mathfrak{n}_{\geq 1} = \bigoplus_{j \geq 1} \mathfrak{g}_j$) and $\psi_z(n) = \psi(\operatorname{tr}(z \log n))$ be a character of $N_{\geq 2}$. Let S_z be the irreducible representation of $N_{\geq 1}$ whose restriction to $N_{\geq 2}$ is a multiple of ψ_z. Let π be an irreducible representation of G; following [Mœglin and Waldspurger 1987], let $\mathcal{N}_{\text{wh}}(\pi)$ be the subset of nilpotent orbits such that $O \in \mathcal{N}_{\text{wh}}(\pi)$ if and only if $\operatorname{Hom}_{N_{\geq 1}}(\pi, S_z) \neq 0$ for any $z \in O$. Let $\mathcal{N}_{\text{wh,max}}(\pi)$ be the subset of maximal elements in $\mathcal{N}_{\text{wh}}(\pi)$ with respect to the inclusion relation of closure of orbits.

MSC2010: 22E50.
Keywords: supercuspidal representations, Bruhat–Tits building, nilpotent orbits.

On the other hand, let $\boldsymbol{T}$ be a maximal K-split anisotropic torus of $\boldsymbol{G}$; here, K is the maximal unramified extension of k. Then $T = \boldsymbol{T}(k)$ is a maximal unramified anisotropic torus of G. Let χ be a character of T satisfying certain conditions described in [Adler 1998] or [Reeder 2008]. There is a supercuspidal irreducible representation π_χ of G associated to (T, χ). Identify $\mathcal{B}(\boldsymbol{T}, k)$ as a point in $\mathcal{B}(\boldsymbol{G}, k)$. In [DeBacker and Reeder 2010], it was shown that π_χ is generic (that is, $\mathcal{N}_{\mathrm{wh}}(\pi_\chi)$ contains a regular nilpotent orbit) if and only if $\mathcal{B}(T, k)$ is a special point in $\mathcal{B}(G, k)$. In [Barbasch and Moy 1997], it was shown that if χ is of depth zero, the character of π_χ can be expanded as linear combination of orbital integrals over elements in $N_{\mathrm{wh}}(\pi_\chi)$.

For those (T, χ) with $\mathcal{B}(\boldsymbol{T}, k)$ nonspecial (that is, when $\mathrm{rank}(G)$ is large enough for $\mathcal{B}(G)$ to contain nonspecial vertices), we show in Theorem 3.2 that if χ is of *positive* depth, there is one element in $\mathcal{N}_{\mathrm{wh,max}}(\pi_\chi)$ which is related to $\mathcal{B}(\boldsymbol{T}, k)$. Note that in this case the supercuspidal representation π_χ is of *positive integral depth*. We also apply this theorem to irreducible representations in Π'_φ, the L-packet of φ, where φ is the Langlands parameter of π_χ.

This article is organized as follows: in Section 2, preliminary notation are recalled, including vertices in Bruhat–Tits building, L-packet of positive-depth supercuspidal representations [Reeder 2008], classification of maximal unramified anisotropic tori [DeBacker 2006], and classification of rational nilpotent orbits [Waldspurger 2001]. We also show by example in the Appendix how to choose a particular element from a rational nilpotent orbit. The main theorems are stated and proved in Section 3.

2. Preliminary

2A. ***Notation.*** Let k be a nonarchimedean local field of characteristic 0 with residue field $\mathfrak{f}$, and let p be the characteristic of $\mathfrak{f}$. Let $\mathfrak{O}$ be the ring of integers of k and $\mathfrak{P}$ the maximal ideal of $\mathfrak{O}$. Let K be the maximal unramified field extension of k and $\mathfrak{F}$ the residue field of K. Let ν be the normalized valuation of k and ν_K the extension of ν to K. Let ψ be an additive character of k with conductor $\mathfrak{P}$, and denote the character of $\mathfrak{f} = \mathfrak{O}/\mathfrak{P}$ derived from ψ by ψ also.

Throughout this paper, assume p is large enough that p is a good prime in the sense in [Carter 1972].

Let W be a finite-dimensional vector space over k, $\langle\cdot,\cdot\rangle$ a nondegenerate bilinear form on W, and $d = \dim_k(W)$. Assume that

$$\langle v, w\rangle = \epsilon_W \langle w, v\rangle \quad \text{for all } v, w \in W,$$

with $\epsilon_W = \pm 1$. Let $\boldsymbol{G}$ be the reductive group defined over k with

$$\boldsymbol{G} = \begin{cases} \mathbf{SO}(W) & \text{if } \epsilon_W = 1, \\ \mathbf{Sp}(W) & \text{if } \epsilon_W = -1. \end{cases}$$

Throughout this paper, assume that W has a k-basis $\{e_1, \dots, e_d\}$ satisfying

$$\langle e_j, e_k \rangle = \begin{cases} 0 & \text{if } j + k \neq d + 1, \\ 1 & \text{if } j + k = d + 1, j \leq k. \end{cases}$$

Then $\boldsymbol{G}$ is a connected split reductive group over k with finite center. Where no confusion will result, denote $\boldsymbol{G}$ by $\mathbf{SO}(d), \mathbf{Sp}(d)$ for $\epsilon_W = 1, -1$, respectively.

Let $J_W = (a_{i,j})$ be the matrix of degree d such that ${}^tJ_W = \epsilon_W J_W$ and

$$a_{j,k} = \delta_{j,d+1-k} \quad \text{for } j \leq k.$$

Let $\bar{k}$ be the algebraic closure of k and $R \subset \bar{k}$ a commutative k-algebra. Then $\boldsymbol{G}(R)$, the set of R-rational points of $\boldsymbol{G}$, is identified with the set of R-valued matrices g of degree d satisfying

$${}^t g J_W g = J_W, \quad \det(g) = 1.$$

Let $\boldsymbol{\mathfrak{g}}$ be the Lie algebra of $\boldsymbol{G}$; then $\boldsymbol{\mathfrak{g}}(R)$ is identified with the set of R-valued matrices g of degree d satisfying

$${}^t g J_W + J_W g = 0.$$

2B. ***Vertices of Bruhat–Tits building of G.*** Let $G = \boldsymbol{G}(k)$ and $\mathfrak{g} = \boldsymbol{\mathfrak{g}}(k)$. Let $\mathscr{B}(G) = \mathscr{B}(\boldsymbol{G}, k)$ be the Bruhat–Tits building of G. For $x \in \mathscr{B}(G)$, let G_x be the parahoric subgroup attached to x and $G_{x,+}$ the prounipotent radical of G_x. Let G_x be the connected reductive group defined over $\mathfrak{f}$ such that $G_x/G_{x,+}$ is the group of $\mathfrak{f}$-rational points of G_x. If F is a G-facet of $\mathscr{B}(G)$ and $x \in F$, let $G_F = G_x$, $G_{F,0+} = F_{x,0+}$, and $\mathsf{G}_F = \mathsf{G}_x$.

Let $\boldsymbol{S}$ be the maximal k-split torus of $\boldsymbol{G}$ containing all diagonal matrices in $\boldsymbol{G}$, $\boldsymbol{B}$ the Borel subgroup of $\boldsymbol{G}$ containing all upper triangular matrices in $\boldsymbol{G}$, $S = \boldsymbol{S}(k)$, and $B = \boldsymbol{B}(k)$. Let Φ be the set of roots of G with respect to S, Φ^+ the set of positive roots of G with respect to B, and $\Delta \subset \Phi^+$ the subset of simple roots of Φ^+. Let $\boldsymbol{\mathfrak{s}}$ be the Lie algebra of $\boldsymbol{S}$; then $\mathfrak{s} = \boldsymbol{\mathfrak{s}}(k)$ consists of all diagonal matrices in $\mathfrak{g}$. By taking differentials, roots in Φ are identified with linear functions on $\mathfrak{s}$.

Identify $\mathfrak{s}$ with k^n by the following isomorphism:

$$s = \mathrm{diag}(c_1, \dots, c_d) \in \mathfrak{s} \mapsto (c_1, \dots, c_n) \in k^n;$$

here, $n = [d/2]$. For $i = 1, \dots, n$, the i-th coordinate function e_i on k^n is identified with a linear function on $\mathfrak{s}$, still denoted by e_i. Let γ, α_i $(i = 1, \dots, n)$ be positive roots as follows:

$$\begin{array}{llll} & \alpha_i = e_i - e_{i+1}, & i = 1, \dots, n; & \\ & \alpha_n = e_n, & \gamma = e_1 + e_2, & \text{if } \boldsymbol{G} = \mathbf{SO}(2n+1); \\ \text{or} & \alpha_n = e_{n-1} + e_n, & \gamma = e_1 + e_2, & \text{if } \boldsymbol{G} = \mathbf{SO}(2n); \\ \text{or} & \alpha_n = 2e_n, & \gamma = 2e_1, & \text{if } \boldsymbol{G} = \mathbf{Sp}(2n). \end{array}$$

Then $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and γ is the highest root in Φ with respect Δ.

Let Φ_{af} be the set of affine roots of G with respect to S. As a subset of affine functions on $\mathfrak{s}$,

$$\Phi_{\mathrm{af}} = \{\alpha + m \mid \alpha \in \Phi, m \in \mathbb{Z}\}.$$

Let $\alpha_0 = 1 - \gamma \in \Phi_{\mathrm{af}}$ and $\Sigma = \Delta \cup \{\alpha_0\}$. Then every affine root is an integral combination of elements in Σ.

Let $X^*(S)$ be the character group of S, $X_*(S)$ the dual group of $X^*(S)$, and

$$\mathfrak{a} := X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $A = A(S)$ be the underlying affine space of $\mathfrak{a}$. Then A is an apartment in $\mathcal{B}(G)$. By fixing a hyperspecial point $o \in A$, one can identify A with $\mathfrak{a}$ and elements in Φ_{af} with affine functions on $\mathfrak{a}$.

Let C be the fundamental chamber of A defined by

$$C = \{z \in A \mid 0 < \alpha(z) < 1 \text{ for all } \alpha \in \Sigma\}.$$

For $\alpha \in \Phi_{\mathrm{af}}$, let $H_\alpha = \{z \in A \mid \alpha(z) = 0\}$. Then the H_α $(\alpha \in \Sigma)$ are walls of $\overline{C}$. For $0 \le i \le n$, let $y_i \in \overline{C}$, such that $\{y_i\} = \bigcap_{\substack{\alpha \in \Sigma \\ \alpha \neq \alpha_i}} H_{\alpha_j}$. Then the y_i $(i = 0, \dots, n)$ are vertices of $\overline{C}$. Let

$$I_{\mathrm{nsp}} = \begin{cases} \{2, \dots, n\} & \text{if } \boldsymbol{G} = \mathbf{SO}(2n+1), \\ \{2, \dots, n-2\} & \text{if } \boldsymbol{G} = \mathbf{SO}(2n), \\ \{1, \dots, n-1\} & \text{if } \boldsymbol{G} = \mathbf{Sp}(2n). \end{cases} \tag{1}$$

Then y_i is not a special vertex (see [Tits 1979]) for all $i \in I_{\mathrm{nsp}}$, and

$$\mathsf{G}_{y_i}(\mathfrak{f}) \simeq \begin{cases} \mathrm{SO}(2i, \mathfrak{f}) \times \mathrm{SO}(2n-2i+1, \mathfrak{f}) & \text{if } \boldsymbol{G} = \mathbf{SO}(2n+1), \\ \mathrm{SO}(2i, \mathfrak{f}) \times \mathrm{SO}(2n-2i, \mathfrak{f}) & \text{if } \boldsymbol{G} = \mathbf{SO}(2n), \\ \mathrm{Sp}(2i, \mathfrak{f}) \times \mathrm{Sp}(2n-2i, \mathfrak{f}) & \text{if } \boldsymbol{G} = \mathbf{Sp}(2n). \end{cases}$$

2C. ***On the stable conjugacy classes of maximal tori.*** If $\boldsymbol{T}$ is a maximal K-split k-torus of $\boldsymbol{G}$ defined over k, then $T = \boldsymbol{T}(k)$ is a maximal unramified torus of G [DeBacker 2006]. In this case, let $\mathcal{B}(T) = \mathcal{B}(\boldsymbol{T}, k)$. By [Adler 1998], choose a $\mathrm{Gal}(K/k)$-equivariant embedding of $\mathcal{B}(\boldsymbol{T}, K)$ into $\mathcal{B}(\boldsymbol{G}, K)$; then $\mathcal{B}(T)$ is identified with a subset of $\mathcal{B}(G)$:

$$\mathcal{B}(T) = \mathcal{B}(\boldsymbol{T}, K)^{\Gamma} \subset \mathcal{B}(\boldsymbol{G}, K)^{\Gamma} = \mathcal{B}(G).$$

DeBacker [2006] defines a set I^m and an equivalence relation "$\sim$" on I^m, so that there is a one-to-one and onto correspondence between $I^m/\sim$ and the set of G-conjugacy classes of unramified maximal tori in G. Elements in I^m are of the form (F, T), where F is an arbitrary G-facet in $\mathcal{B}(G)$ and T is a maximal minisotropic $\mathfrak{f}$-torus in G_F. Let $C(F, \mathsf{T})$ be the G-conjugacy class of maximal unramified tori in G corresponding to the equivalence class in I^m containing (F, T).

Let $o \in \mathscr{B}(G)$ be one of the special points chosen in Section 2B, to which we associate a conjugacy class of a maximal anisotropic $\mathfrak{f}$-torus in G_o and a conjugacy class in $\boldsymbol{W}(\mathsf{G}_o)$ (see [DeBacker 2006; Carter 1985]). Here $\boldsymbol{W}(\mathsf{G}_o)$ is the Weyl group of G_o. Let T_o (resp. w_o) be a representative of the conjugacy class of a maximal anisotropic $\mathfrak{f}$ torus (resp. the $\boldsymbol{W}(\mathsf{G}_o)$-conjugacy class). Then $(\{o\}, \mathsf{T}_o) \in I^m$. Take $T = \boldsymbol{T}(k) \in C(\{o\}, \mathsf{T}_o)$; then T is a maximal unramified anisotropic k-torus in G (see [DeBacker 2006]).

Let $\mathscr{S}(\mathsf{T}_o)$ be the subset of I^m consisting of elements (F, T) such that if $\boldsymbol{W}(\mathsf{G}_F)$ is identified with a subgroup of $\boldsymbol{W}(\mathsf{G}_o)$, then ${}^{\boldsymbol{W}(\mathsf{G}_F)}w_F \cap {}^{\boldsymbol{W}(\mathsf{G}_o)}w_o \neq \varnothing$, where w_F is a representative of the $\boldsymbol{W}(\mathsf{G}_F)$-conjugacy class corresponding to T. Then $\mathscr{S}(\mathsf{T}_o)$ depends only on the conjugacy class of w_o in $\boldsymbol{W}(\mathsf{G}_o)$. In fact, $\mathscr{S}(\mathsf{T}_o)$ is the set of G-conjugacy classes of maximal unramified anisotropic tori in the stable conjugacy class of T in G, which is the stable conjugacy class of maximal unramified tori in G corresponding to w_o [ibid., Corollary 4.3.2]. Let "$\sim$" be the equivalence relation on $\mathscr{S}(\mathsf{T}_o)$ inherited from I^m.

We briefly recall the classification of conjugacy classes in $\boldsymbol{W}(\mathsf{G}_o)$. Since G_o is split special orthogonal group or symplectic group over $\mathfrak{f}$,

$$\boldsymbol{W}(\mathsf{G}_o) \simeq \begin{cases} S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n & \text{if } \mathsf{G}_o = \mathrm{SO}(2n+1) \text{ or } \mathrm{Sp}(2n), \\ S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^{n-1} & \text{if } \mathsf{G}_o = \mathrm{SO}(2n), n \geq 2. \end{cases}$$

Here S_n is the n-th symmetric group. Conjugacy classes in $\boldsymbol{W}(\mathsf{G}_o)$ are parametrized by the set of pairs of partitions $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ with $S(\boldsymbol{\lambda}) + S(\boldsymbol{\mu}) = n$; moreover, if $\mathsf{G}_o = \mathrm{SO}(2n)$, $c(\mu)$ is even [Carter 1972, Propositions 24, 25]. Here, terminology in [Waldspurger 2001] is used: for a partition $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n, \dots)$,

$$S(\boldsymbol{\lambda}) = \sum_{i=1}^{\infty} \lambda_i, \quad c(\boldsymbol{\lambda}) = |\{i \geq 1 \mid \lambda_i \neq 0\}|.$$

In particular, conjugacy classes of anisotropic maximal tori in $\mathsf{G}_o(\mathfrak{f})$ are parametrized by the subset consisting of $(\varnothing, \boldsymbol{\mu})$, with $S(\boldsymbol{\mu}) = n$; if $\mathsf{G}_o = \mathrm{SO}(2n)$, $c(\boldsymbol{\mu})$ is even.

Assume $(\varnothing, \boldsymbol{\mu})$ corresponds to the conjugacy class of w_o in $W(\mathsf{G}_o)$, and write

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_s), \quad \mu_1 \geq \cdots \geq \mu_s \geq 1,$$

so that $S(\boldsymbol{\mu}) = n$, and s is even if $G = \mathrm{SO}(2n)$. Let

$$\mathscr{S}(\boldsymbol{\mu}) = \{\boldsymbol{\mu}' = (\mu_{j_1}, \dots, \mu_{j_{s-2m}}) \mid \text{for some } 1 \leq j_1 < j_2 < \cdots < j_{s-2m}, 0 \leq 2m \leq s\}, \quad \text{if } G = \mathrm{SO}(2n+1) \text{ or } \mathrm{SO}(2n);$$

$$\mathscr{S}(\boldsymbol{\mu}) = \{\boldsymbol{\mu}' = (\mu_{j_1}, \dots, \mu_{j_{s-m}}) \mid \text{for some } 1 \leq j_1 < j_2 < \cdots < j_{s-m}, 0 \leq m \leq s\}, \quad \text{if } G = \mathrm{Sp}(2n);$$

For $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, define

$$i := i_{\boldsymbol{\mu}'} := i(\boldsymbol{\mu}') := S(\boldsymbol{\mu}) - S(\boldsymbol{\mu}').$$

Then ${}^{\boldsymbol{W}(\mathsf{G}_o)}w_o \cap \boldsymbol{W}(\mathsf{G}_{y_i}) \neq \varnothing$. Here ${}^{\boldsymbol{W}(\mathsf{G}_o)}w_o$ is the conjugacy class of w_o and $\boldsymbol{W}(\mathsf{G}_{y_i})$ is the Weyl group of G_{y_i} identified as a subgroup of $\boldsymbol{W}(\mathsf{G}_o)$. By [DeBacker 2006, Corollary 4.3.2], there is a maximal anisotropic torus $\mathsf{T}_{\boldsymbol{\mu}'}$ in $\mathsf{G}_{y_i}(\mathfrak{f})$ that is $\mathsf{G}_o(\mathfrak{f})$-conjugate to T_o. Hence $(\{y_{i(\boldsymbol{\mu}')}\}, \mathsf{T}_{\boldsymbol{\mu}'}) \in \mathcal{S}(\mathsf{T}_o)$.

Take $T_{\boldsymbol{\mu}'} \in C(\{y_{i(\boldsymbol{\mu}')}\}, \mathsf{T}_{\boldsymbol{\mu}'})$; then $T_{\boldsymbol{\mu}'}$ is a maximal unramified anisotropic torus in G stably conjugate to T and $\mathcal{B}(T_{\boldsymbol{\mu}'}) = \{y_{i_{\boldsymbol{\mu}'}}\}$. In particular, $\boldsymbol{\mu} \in \mathcal{S}(\boldsymbol{\mu})$. Take $T_{\boldsymbol{\mu}} = T$. Conversely, all G-conjugacy classes in the stable conjugacy class of T have a representative of this form.

Lemma 2.1. *The set* $\{(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'}) \mid \boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})\}$ *is a complete set of representatives of* $\mathcal{S}(\mathsf{T}_o)/\sim$.

Proof. It remains to show that the pairs $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ are not equivalent to one another, for $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$. If $i_{\boldsymbol{\mu}'} = i_{\boldsymbol{\mu}''}$ for distinct $\boldsymbol{\mu}', \boldsymbol{\mu}'' \in \mathcal{S}(\boldsymbol{\mu})$, then by the choice of $\mathsf{T}_{\boldsymbol{\mu}'}$ and $\mathsf{T}_{\boldsymbol{\mu}''}$, $\mathsf{T}_{\boldsymbol{\mu}'}$ is not conjugate to $\mathsf{T}_{\boldsymbol{\mu}''}$ in $\mathsf{G}_{y_{i_{\boldsymbol{\mu}'}}}$; therefore $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ is not equivalent to $(\{y_{i_{\boldsymbol{\mu}''}}\}, \mathsf{T}_{\boldsymbol{\mu}''})$.

If $i_{\boldsymbol{\mu}'} \neq i_{\boldsymbol{\mu}''}$ for $\boldsymbol{\mu}', \boldsymbol{\mu}'' \in \mathcal{S}(\boldsymbol{\mu})$, we will show $y_{i_{\boldsymbol{\mu}'}}$ is not associated to $y_{i_{\boldsymbol{\mu}''}}$. As a consequence, $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ is not equivalent to $(\{y_{i_{\boldsymbol{\mu}''}}\}, \mathsf{T}_{\boldsymbol{\mu}''})$.

The case for $\boldsymbol{G} = \mathbf{Sp}(2n)$ is trivial, since the vertices $y_0, y_1, \dots, y_n$ of $\overline{C}$ are not associated to each other.

If $\boldsymbol{G} = \mathbf{SO}(2n+1)$, among all vertices $y_0, y_1, \dots, y_n$ of $\overline{C}$, y_0 is associated to y_1, and $y_0, y_2, \dots, y_n$ are not associated to each other. For $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, if $i_{\boldsymbol{\mu}'} \neq 0$, then $i_{\boldsymbol{\mu}'} \geq 2$. As a result, $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ is not equivalent to $(\{y_{i_{\boldsymbol{\mu}''}}\}, \mathsf{T}_{\boldsymbol{\mu}''})$.

If $\boldsymbol{G} = \mathbf{SO}(2n)$, among all vertices $y_0, y_1, \dots, y_n$, y_0 is associated to y_1, y_{n-1} is associated to y_n, and $y_0, y_2, \dots, y_{n-2}, y_n$ are not associated to each other. For $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, if $i_{\boldsymbol{\mu}'} \neq 0$, then $i_{\boldsymbol{\mu}'} \neq 1, i_{\boldsymbol{\mu}'} \neq n-1$. Then $(\{y_{i_{\boldsymbol{\mu}'}}\}, \mathsf{T}_{\boldsymbol{\mu}'})$ is not equivalent to $(\{y_{i_{\boldsymbol{\mu}''}}\}, \mathsf{T}_{\boldsymbol{\mu}''})$. □

2D. ***L-packet.*** Keep the notation of the previous subsection. Let $\mathfrak{t}_{\boldsymbol{\mu}}$ (resp. $\mathfrak{t}_{\boldsymbol{\mu}}(K)$) be the Lie algebra of $T_{\boldsymbol{\mu}}$ (resp. $\boldsymbol{T}_{\boldsymbol{\mu}}(K)$). For $s \in \mathbb{Z}$, let $\mathfrak{t}_{\boldsymbol{\mu},s}$ (resp. $T_{\boldsymbol{\mu},s}$) be the s-th filtration of $\mathfrak{t}_{\boldsymbol{\mu}}$ (resp. $T_{\boldsymbol{\mu}}$) [Adler 1998]. Let r be a positive integer, $X_{\boldsymbol{\mu}}$ a good element in $\mathfrak{t}_{\boldsymbol{\mu},-r}$ (i.e., $X_{\boldsymbol{\mu}} \in \mathfrak{t}_{-r}$), and for every root α of $\boldsymbol{T}_{\boldsymbol{\mu}}(K)$ in $\boldsymbol{G}(K)$, assume $d\alpha(X_{\boldsymbol{\mu}}) \neq 0$. Let $\chi_{\boldsymbol{\mu}}$ be a character of $T_{\boldsymbol{\mu}}$ satisfying $\chi_{\boldsymbol{\mu}}|_{T_{\boldsymbol{\mu},r+1}} = 1$,

$$\chi_{\boldsymbol{\mu}}(\exp_o(Y)) = \psi(\operatorname{tr}(X_{\boldsymbol{\mu}} Y)) \quad \text{for all } Y \in \mathfrak{t}_{\boldsymbol{\mu},r}.$$

Here $\exp_o$ is the mock exponential map defined in [Adler 1998].

Let $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}}$ be the supercuspidal representation constructed by using $\chi_{\boldsymbol{\mu}}$ and $X_{\boldsymbol{\mu}}$, $\varphi : \mathcal{W}_k \to {}^L G$ be the L-parameter of $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}}$ (see [Adler 1998; Reeder 2008]), where

$\mathcal{W}_k$ is the Weil group of k. For $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, let $g \in \boldsymbol{G}(K)_o$ be an element such that $T_{\boldsymbol{\mu}'}(k) = {}^g T_{\boldsymbol{\mu}}(k)$; then $X_{\boldsymbol{\mu}'} = {}^g X_{\boldsymbol{\mu}}$ is a good element in $\mathfrak{t}_{\boldsymbol{\mu}',-r}$. Define a depth r character $\chi_{\boldsymbol{\mu}'}$ of $T_{\boldsymbol{\mu}'}$ by $\chi_{\boldsymbol{\mu}'} := {}^g\chi_{\boldsymbol{\mu}'}$; then,

$$\chi_{\boldsymbol{\mu}'}(\exp_{y_{i(\boldsymbol{\mu}')}}(Y)) = \psi(\operatorname{tr} X_{\boldsymbol{\mu}'}Y) \quad \text{for all } Y \in \mathfrak{t}_{\boldsymbol{\mu}',r}.$$

Let $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}$ be the supercuspidal representation of G constructed by using $\chi_{\boldsymbol{\mu}'}$ and $X_{\boldsymbol{\mu}'}$. Then:

Theorem 2.2 [Reeder 2008]. *The set $\Pi'(\varphi) = \{\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'} \mid \boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})\}$ is the L-packet associated to φ.*

The main result of this paper concerns nilpotent orbits supporting representations in $\Pi'(\varphi)$. Prior to the statement of the main theorems, we recall the classification of k-rational nilpotent orbits in $\mathfrak{g}$ [Waldspurger 2001, §I.6] and define a partition $\boldsymbol{\lambda}^i$ for every $i \in I_{\text{nsp}}$.

2E. ***Nilpotent orbits.*** Let $\boldsymbol{\lambda} = (\lambda_i)_{i\in\mathbb{N}}$ be a sequence of nonnegative integers such that $\lambda_j = 0$ for j sufficiently large. Define

$$S(\boldsymbol{\lambda}) = \sum_{j\geq 1} \lambda_j, \quad c(\boldsymbol{\lambda}) = |\{j \geq 1 \mid \lambda_j \neq 0\}|, \quad c_i(\boldsymbol{\lambda}) = |\{j \mid \lambda_j = i\}| \text{ for all } i \in \mathbb{N}.$$

If $\lambda_1 \geq \lambda_2 \geq \cdots$, $\boldsymbol{\lambda}$ is called a partition. Let $\mathcal{P}$ be the set of all partitions and $\mathcal{P}(n)$ the subset of all $\boldsymbol{\lambda} \in \mathcal{P}$ such that $S(\boldsymbol{\lambda}) = n$. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}$, let $\boldsymbol{\lambda} \cup \boldsymbol{\mu}$ be the unique partition such that $c_i(\boldsymbol{\lambda} \cup \boldsymbol{\mu}) = c_i(\boldsymbol{\lambda}) + c_i(\boldsymbol{\mu})$ for all $i \in \mathbb{N}$.

Let W be the vector space defined in Section 2A and $d = \dim_k W$. If $\epsilon_W = 1$, let $\mathcal{P}(W)$ be the set of partitions $\boldsymbol{\lambda} \in \mathcal{P}(d)$ such that c_i is even for all even i. If $\epsilon_W = -1$, let $\mathcal{P}(W)$ be the set of partitions $\boldsymbol{\lambda} \in \mathcal{P}(d)$ so that c_i is even for all odd i. Let $\operatorname{Nil}_I(W)$ be the set of $(\boldsymbol{\lambda}, (q_i))$ with $\boldsymbol{\lambda} \in \mathcal{P}(W)$, and let q_i, $i \in \mathbb{N}$, be quadratic forms satisfying these conditions:

- If $\epsilon_W = 1$, q_i is a nondegenerate quadratic form on k^{c_i} for i odd, $q_i = 0$ for i even, moreover the quadratic form $\bigoplus_{i\in\mathbb{N}} q_i$ has the same anisotropic kernel as q_W; here, q_W is the quadratic form on W defined by $q_W(v) = \langle v, v\rangle$.
- If $\epsilon_W = -1$, q_i is a nondegenerate quadratic form on k^{c_i} for i even, $q_i = 0$ for i odd.

Definition 2.3. $(\boldsymbol{\lambda}, (q_i)) \in \operatorname{Nil}_I(W)$ is called exceptional if $\epsilon_W = 1$, $4 \mid d$, and λ_i is even for all $i \in \mathbb{N}$. In this case, $q_i = 0$ for all $i \in \mathbb{N}$.

Definition 2.4.

- If $\epsilon_W = -1$, let $\operatorname{Nil}(W) = \operatorname{Nil}_I(W)$;
- If $\epsilon_W = 1$, $4 \nmid d$, let $\operatorname{Nil}(W) = \operatorname{Nil}_I(W)$;
- If $\epsilon_W = 1$, $4 \mid d$, let $\operatorname{Nil}(W)$ be the set consisting all nonexceptional $(\boldsymbol{\lambda}, (q_i)) \in \operatorname{Nil}_I(W)$ and $(\boldsymbol{\lambda}, (q_i), \varepsilon)$ with $(\boldsymbol{\lambda}, (q_i))$ exceptional, $\varepsilon = \pm 1$.

By [Waldspurger 2001], there is a bijective correspondence between $\mathrm{Nil}(W)$ and $\mathfrak{g}_{\mathrm{nil}}/G$, the set of k-rational nilpotent orbits. Define a partial order on $\mathcal{P}(n)$: for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}(n)$, $\boldsymbol{\lambda} \geq \boldsymbol{\mu}$ if and only if for all $j \geq 1$, $\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$.

Definition 2.5. Define a partial order on the set of nilpotent orbits in $\mathfrak{g}$: $O_1 \geq O_2$ if and only if $\overline{O}_1 \supset \overline{O}_2$. Here the closure is taken with respect to the usual topology in $\mathfrak{g}$.

Lemma 2.6. *Let O_1, O_2 be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}, (q_i))$ or $(\boldsymbol{\lambda}, \varnothing, \varepsilon)$ and $(\boldsymbol{\mu}, (q_i'))$ or $(\boldsymbol{\mu}, \varnothing, \varepsilon')$ respectively. If $O_1 > O_2$, then $\boldsymbol{\lambda} > \boldsymbol{\mu}$.*

Proof. The proof is similar to that of Theorem 6.2.5 of [Collingwood and McGovern 1993]. Take arbitrary $X \in O_1$, $Y \in O_2$, with O_1, O_2 corresponding to $(\boldsymbol{\lambda}, (q_i))$ or $(\boldsymbol{\lambda}, \varnothing, \varepsilon)$ and $(\boldsymbol{\mu}, (q_i'))$ or $(\boldsymbol{\mu}, \varnothing, \varepsilon')$ respectively. If $O_1 > O_2$, then $\overline{O}_1 \supsetneq \overline{O}_2$,

$$\mathrm{rank}(X^k) > \mathrm{rank}(Y^k) \quad \text{for all } k \geq 1,$$

since the condition that rank of a matrix be strictly less than a fixed number is a closed condition for the usual topology. Now $\boldsymbol{\lambda} > \boldsymbol{\mu}$ by of [ibid., Lemma 6.2.2], ☐

Example 2.7. Regular nilpotent orbits in $\mathfrak{g}_{\mathrm{nil}}$ are those corresponding to:

- $([2n+1], q_{2n+1})$, if $\epsilon_W = 1$, $d = 2n+1$. Here q_{2n+1} is the nondegenerate quadratic form on k defined by $q_{2n+1}(x) = x^2$.
- $([2n-1, 1], (q_{2n-1}, q_1))$, if $\epsilon_W = 1$, $d = 2n$. Here q_{2n-1}, q_1 are nondegenerate quadratic forms on k such that $q_{2n-1} \oplus q_1 \simeq q'$, where q' is the quadratic form on k^2 defined by $q'(x, y) = 2xy$ for all $x, y \in k$.
- $([2n], q_{2n})$, if $\epsilon_W = -1$, $d = 2n$. Here q_{2n} is a nondegenerate quadratic form on k.

Let I_{nsp} be the set defined in (1). For $i \in I_{\mathrm{nsp}}$, let $\boldsymbol{\lambda}^i = \boldsymbol{\mu}' \cup \boldsymbol{\mu}''$ with

$$\begin{aligned}
&\boldsymbol{\mu}' = [2i-1, 1], && \boldsymbol{\mu}'' = [2n-2i+1], && \text{if } \epsilon_W = 1, d = 2n+1;\\
&\boldsymbol{\mu}' = [2i-1, 1], && \boldsymbol{\mu}'' = [2n-2i-1, 1], && \text{if } \epsilon_W = 1, d = 2n;\\
&\boldsymbol{\mu}' = [2i], && \boldsymbol{\mu}'' = [2n-2i], && \text{if } \epsilon_W = -1, d = 2n.
\end{aligned}$$

For $i \notin I_{\mathrm{nsp}}$, let

$$\boldsymbol{\lambda}^i = \begin{cases} [d] & \text{if } \epsilon_W = 1,\ d = 2n+1,\\ [d-1, 1] & \text{if } \epsilon_W = 1,\ d = 2n,\\ [d] & \text{if } \epsilon_W = -1,\ d = 2n. \end{cases}$$

Lemma 2.8. *Let $i \in I_{\mathrm{nsp}}$. Let O', O^i be nilpotent orbits in $\mathfrak{g}_{\mathrm{nil}}$ corresponding to $(\boldsymbol{\lambda}', (q_j'))$ or $(\boldsymbol{\lambda}', \varnothing, \varepsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$. Assume $O' > O^i$. Then:*

- *If* $\boldsymbol{G} = \mathbf{SO}(2n+1)$, *then* $\boldsymbol{\lambda}' = [2n+1]$ *or* $[m, 2n-m, 1]$ *for some odd* $m > \max(2i-1, 2n-2i+1)$.
- *If* $\boldsymbol{G} = \mathbf{SO}(2n)$ *and* $i \neq n/2$, *then* $\boldsymbol{\lambda}' = [m, 2n-m]$ *for some odd* $m \geq \max(2i-1, 2n-2i-1)$, *or* $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ *for some odd* $m > \max(2i-1, 2n-2i-1)$.
- *If* $\boldsymbol{G} = \mathbf{SO}(2n)$ *and* $i = n/2$, *then* $\boldsymbol{\lambda}' = [n^2]$, *or*

$$\boldsymbol{\lambda}' = [m, 2n-m] \text{ or } [m, 2n-m-2, 1^2]$$

for some odd $m > \max(2i-1, 2n-2i-1)$.
- *If* $\boldsymbol{G} = \mathbf{Sp}(2n)$, *then* $\boldsymbol{\lambda}' = [m, 2n-m]$ *for some even* $m > \max(2i, 2n-2i)$.

Proof. Assume $\boldsymbol{\lambda}' = [\lambda_1', \lambda_2', \dots] \in \mathcal{P}(W)$, with $\lambda_1' \geq \lambda_2' \geq \cdots$. By Lemma 2.6, if $O' > O^i$, then $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$.

Assume $\boldsymbol{G} = \mathbf{SO}(2n+1)$, $\boldsymbol{\lambda}^i = [2i-1, 1] \cup [2n-2i+1]$. First, assume $2i-1 \geq 2n-2i+1$, $\boldsymbol{\lambda}^i = [2i-1, 2n-2i+1, 1]$.

By definition, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$ if and only if $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$ and

$$\lambda_1' \geq 2i-1, \quad \lambda_1' + \lambda_2' \geq 2n, \quad \lambda_1' + \lambda_2' + \lambda_3' = 2n+1.$$

Then $\lambda_3' = 0$ or $\lambda_3' = 1$. If $\lambda_3' = 0$, $\lambda_2' = 0$, then $\boldsymbol{\lambda}' = [2n+1] > \boldsymbol{\lambda}^i$. If $\lambda_3' = 0$, $\lambda_2' \neq 0$, then $\boldsymbol{\lambda}' = [\lambda_1', 2n+1-\lambda_1'] \notin \mathcal{P}(W)$, which contradicts the assumption $\boldsymbol{\lambda}' \in \mathcal{P}(W)$.

If $\lambda_3' = 1$, $\boldsymbol{\lambda}' = [m, 2n-m, 1]$ for some $m \geq 2i-1$. If $m = 2i-1$, then $\boldsymbol{\lambda}' = \lambda^i$, which contradicts the assumption $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$. Hence $m > 2i-1$. If m is even, then $c_m(\boldsymbol{\lambda}')$ is even and $2n-m = m$; hence $m = n$, and $\boldsymbol{\lambda}' = [n^2, 1]$. On the other hand, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$, $2i-1 = 2n-2i+1 = n = m$, which contradicts $m > 2i-1$. In conclusion, $\boldsymbol{\lambda}' = [m, 2n-m, 1]$ for some odd $m > 2i-1$.

Similarly, if $2n-2i-1 \geq 2i-1$, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i = [2n-2i-1, 2i-1, 1]$, then $\boldsymbol{\lambda}' = [m, 2n-m, 1]$ for some odd $m > 2n-2i+1$. This concludes the proof for $\boldsymbol{G} = \mathbf{SO}(2n+1)$.

Assume $\boldsymbol{G} = \mathbf{SO}(2n)$, $\boldsymbol{\lambda}^i = [2i-1, 1] \cup [2n-2i-1, 1]$. First, assume $2i-1 > 2n-2i-1$, $\boldsymbol{\lambda}^i = [2i-1, 2n-2i-1, 1^2]$.

By definition, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$ if and only if $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$ and

$$\lambda_1' \geq 2i-1, \quad \lambda_1' + \lambda_2' \geq 2n-2, \quad \lambda_1' + \lambda_2' + \lambda_3' \geq 2n-1, \quad \lambda_1' + \lambda_2' + \lambda_3' + \lambda_4' = 2n.$$

Then $\lambda_4' = 0$ or $\lambda_4' = 1$. Assume $\lambda_4' = 0$; then, $\lambda_3' = 0$ or $\lambda_3' = 1$. If $\lambda_3' = 1$, $\lambda_4' = 0$, then λ_1' and λ_2' have different parity, so $\boldsymbol{\lambda}' \notin \mathcal{P}(W)$. If $\lambda_3' = \lambda_4' = 0$, then $\boldsymbol{\lambda}' = [m, 2n-m]$ with $m \geq 2i-1$. If m is even, then $c_m(\boldsymbol{\lambda}')$ is even, $m = 2n-m = n$. Hence $m = n > 2i-1 > 2n-2i-1$, which has no solution since the second inequality requires $2i-1 > n-1$. In conclusion, if $\lambda_4' = 0$, then $\boldsymbol{\lambda}' = [m, 2n-m]$ for some odd $m \geq 2i-1$.

If $\lambda'_4 = 1$, then $\lambda'_3 = 1$, $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some $m \geq 2i-1$. If $m = 2i-1$, then $\boldsymbol{\lambda}' = \boldsymbol{\lambda}^i$ which contradicts the assumption $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$. Hence $m > 2i-1$. If m is even, then $c_m(\boldsymbol{\lambda}')$ is even, $m = 2n-m-2 = n-1$. Hence $m = n-1 > 2i-1 > 2n-2i-1$, which has no solution since the second inequality requires $2i-1 > n-1$. In conclusion, if $\lambda'_4 = 1$, then $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some odd $m > 2i-1$.

Similarly, if $2n-2i-1 > 2i-1$, then $\boldsymbol{\lambda}' = [m, 2n-m]$ for some odd $m \geq \max(2i-1, 2n-2i-1)$, or $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some odd $m > \max(2i-1, 2n-2i-1)$.

Assume now $2i-1 = 2n-2i-1$. Then n is even, $i = n/2$, and $\boldsymbol{\lambda}^i = [(n-1)^2, 1^2]$. Assume $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$, $\lambda \in \mathcal{P}(W)$. Then

$$\lambda'_1 \geq n-1, \quad \lambda'_1+\lambda'_2 \geq 2n-2, \quad \lambda'_1+\lambda'_2+\lambda'_3 \geq 2n-1, \quad \lambda'_1+\lambda'_2+\lambda'_3+\lambda'_4 = 2n.$$

If $\lambda'_1 = n-1$, then $\lambda'_2 = n-1$, $\boldsymbol{\lambda}' = [(n-1)^2, 1^2] = \boldsymbol{\lambda}^i$, contradicting the assumption $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$. Hence $\lambda'_1 \geq n$. If λ'_1 is even, then $c_{\lambda'_1}$ is even, $\lambda'_1 = \lambda'_2 = n$, and $\boldsymbol{\lambda} = [n^2]$. If $m = \lambda'_1 > n$ is odd, then $m > \max(2i-1, 2n-2i-1) = n-1$ and $\boldsymbol{\lambda}' = [m, 2n-m]$ or $[m, 2n-m-2, 1^2]$. This concludes the proof for $\boldsymbol{G} = \mathbf{SO}(2n)$.

Assume $\boldsymbol{G} = \mathbf{Sp}(2n)$. Without loss of generality, assume $2i \geq 2n-2i$; i.e., $i \geq n/2$. Then $\boldsymbol{\lambda}^i = [2i, 2n-2i]$. By definition, $\boldsymbol{\lambda}' > \boldsymbol{\lambda}^i$ if and only if $\lambda' \neq \boldsymbol{\lambda}^i$ and

$$\lambda'_1 \geq 2i, \qquad \lambda'_1 + \lambda'_2 = 2n.$$

Hence $\boldsymbol{\lambda} = [\lambda'_1, 2n-\lambda'_1]$. If $\lambda'_1 = 2i$, then $\lambda'_2 = 2n-2i$, $\boldsymbol{\lambda}' = \boldsymbol{\lambda}^i$, which contradicts the assumption $\boldsymbol{\lambda}' \neq \boldsymbol{\lambda}^i$. Hence $\lambda'_1 > 2i \geq n$. If λ'_1 is odd, then $c_{\lambda'_1}\boldsymbol{\lambda}'$ is even, $\lambda'_1 = \lambda'_2 = n$, which contradicts $\lambda'_1 > n$. As a result, $\boldsymbol{\lambda}' = [m, 2n-m]$ with $m = \lambda'_1 > 2i$ even. This concludes the proof for $\boldsymbol{G} = \mathbf{Sp}(2n)$. □

2F. *Nilpotent support.* Let O' be a rational nilpotent orbit in $\mathfrak{g}/G$ and fix an element $z \in O'$. Let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$; i.e., let there be a Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ such that

$$z = \phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right), \quad h = \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \quad z' = \phi\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

For $i \in \mathbb{Z}$, let $\mathfrak{g}_i = \{Z \in \mathfrak{g} \mid \mathrm{Ad}(h)(Z) = iZ\}$. Then $z \in \mathfrak{g}_{-2}$ and $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$.

Define nilpotent subalgebras $\mathfrak{n}'_{\geq 1}, \mathfrak{n}'_{\geq 2}$ of $\mathfrak{g}$ and unipotent subgroups $N'_{\geq 1}, N'_{\geq 2}$ of G as follows:

$$\mathfrak{n}'_{\geq 1} = \bigoplus_{i \geq 1} \mathfrak{g}_i, \quad N'_{\geq 1} = \exp(\mathfrak{n}'_{\geq 1}),$$
$$\mathfrak{n}'_{\geq 2} = \bigoplus_{i \geq 2} \mathfrak{g}_i, \quad N'_{\geq 2} = \exp(\mathfrak{n}'_{\geq 2}). \tag{2}$$

Let ψ_z be the character of $N'_{\geq 2}$ defined by

$$\psi_z(Z) = \psi \circ \mathrm{tr}(z \cdot \log Z) \quad (Z \in N'_{\geq 2}). \tag{3}$$

Then $\mathrm{Ker}(\psi_z)$ is a subgroup of $N'_{\geq 2}$. If $\mathfrak{n}'_{\geq 1} = \mathfrak{n}'_{\geq 2}$, so $N'_{\geq 1} = N'_{\geq 2}$, let S_z be the character ψ_z of $N'_{\geq 1}$. If $\mathfrak{n}'_{\geq 1} \neq \mathfrak{n}'_{\geq 2}$, then $\mathfrak{g}_1 \neq 0$ and $N'_{\geq 1}/\mathrm{Ker}(\psi_z)$ is isomorphic to a Heisenberg group over $\mathfrak{f}$ with center $N'_{\geq 2}/\mathrm{Ker}(\psi_z)$. In this case, let S_z be the irreducible representation of $N'_{\geq 1}$ whose restriction to $N'_{\geq 2}$ is a multiple of ψ_z.

Definition 2.9. Keep the notation above. Following [Mœglin and Waldspurger 1987], denote by $\mathcal{N}_{\mathrm{wh}}(\pi)$ the set of all nilpotent orbits O' in $\mathfrak{g}/G$ such that, for some smooth irreducible representation π of G, we have $\mathrm{Hom}_{N'_{\geq 1}}(\pi, S_z) \neq 0$. Let $\mathcal{N}_{\mathrm{wh,max}}(\pi)$ be the subset of maximal elements in $\mathcal{N}_{\mathrm{wh}}(\pi)$ with respect to the inclusion relation of closure of orbits.

3. Main theorems

The main results of this paper are the following theorems, whose proofs are given starting on page 185 and page 192, respectively.

Theorem 3.1. *Let $\pi \in \Pi'(\varphi)$. Assume $\pi = \pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}$ for some $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, $i = i_{\boldsymbol{\mu}'}$. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. Take arbitrary $z \in O'$. Then*

$$\mathrm{Hom}_{N'_{\geq 1}}(\pi, S_z) = 0.$$

Theorem 3.2. *Let $\pi \in \Pi'(\varphi)$. Assume $\pi = \pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}$ for some $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu}), i = i_{\boldsymbol{\mu}'}$. Then there is a nilpotent orbit O^i corresponding to $(\boldsymbol{\lambda}^i, (q_j))$ such that $O^i \in \mathcal{N}_{\mathrm{wh,max}}(\pi)$.*

If $i \notin I_{\mathrm{nsp}}$, then y_i is special. In this case, Theorem 3.1 is void and Theorem 3.2 is proved in [DeBacker and Reeder 2010].

The subset $\boldsymbol{\Gamma_z}$ of $\boldsymbol{\Phi^+}$. Assume now $i \in I_{\mathrm{nsp}}$; that is, $\mathrm{rank}(G)$ is large enough for I_{nsp} to be nonempty. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. In this subsection, we will choose a particular element $z \in O'$ such that

$$N'_{\geq 2} \subset B, \quad N'_{\geq 4} \subset B. \tag{4}$$

Here B is the Borel subgroup consisting of upper triangular matrices in G and $N'_{\geq j}$ is the object defined in Section 2F for any $\mathfrak{sl}_2$ triple $\{z, h, z'\}$ attached to z in $\mathfrak{g}$. Let $\Gamma'_z \subset \Phi^+$ be the subset of positive roots such that $\alpha \in \Gamma'_z$ if and only if the root space $\mathfrak{u}_\alpha \subset \mathfrak{n}'_{\geq 4}$, and let

$$\Gamma_z := \Phi^+ \backslash \Gamma'_z. \tag{5}$$

The following notation is used frequently: let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_s)$ be a sequence of positive integers such that $d = \sum_{j=1}^{s} \nu_j$. Then every matrix $a \in \mathfrak{gl}(d,k)$ can be written in blocks $a = (a_{j,\ell})_{j,\ell \le s}$, with $a_{jj} \in \mathfrak{gl}(\nu_j, k)$. Let A_j be an arbitrary $\nu_{j+1} \times \nu_j$ matrix for $1 \le j \le s-1$, and let $z(\boldsymbol{\nu}; A_1, \dots, A_{s-1}) = (z_{j,\ell})_{j,\ell \le s}$ be the nilpotent element in $\mathfrak{gl}(d,k)$ such that

$$z_{j,\ell} = \begin{cases} A_\ell & j = \ell + 1, \\ 0_{\nu_j \times \nu_\ell} & j \ne \ell + 1. \end{cases}$$

Assume $\boldsymbol{G} = \mathbf{SO}(2n+1)$. By Lemma 2.8, $\boldsymbol{\lambda}' = [2n+1]$ or $[m, 2n-m, 1]$ with m odd and $m > \max(2i-1, 2n-2i+1)$.

First, assume $\boldsymbol{\lambda}' = [2n+1]$, $q'_{2n+1} = q_{2n+1}$ as in Example 2.7. Let

$$z = z(\boldsymbol{\nu}; 1, \dots, 1, -1, \dots, -1), \tag{6}$$

with $\boldsymbol{\nu} = (1^{2n+1})$ a regular nilpotent element in $\mathfrak{g}$. Let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and $\mathfrak{g}_j, \mathfrak{n}'_{\ge j}, N'_{\ge j}$ the objects defined in Section 2F. Then, we naturally have

$$N'_{\ge 2} = \{n = (n_{j,\ell})_{j,\ell \le 2n+1} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \ge \ell\} \subset B,$$
$$N'_{\ge 4} = \{n = (n_{j,\ell})_{j,\ell \le 2n+1} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \ge \ell - 1\} \subset B.$$

Let Γ_z be the subset of Φ^+ defined in (5); then,

$$\Gamma_z = \{\alpha_j \mid j = 1, \dots, n\}. \tag{7}$$

Second, assume $m = 2n-1$. Then $\boldsymbol{\lambda}' = [2n-1, 1^2]$, q'_{2n-1} is a nondegenerate quadratic form on k, identified with a nonzero element in $k^\times$, and q'_1 is a nondegenerate quadratic form on k^2, such that $q'_{2n-1} \oplus q'_1$ is isometric to the quadratic form on k^3

$$(u, v, w) \mapsto 2uw + v^2 \quad (u, v, w \in k).$$

Let

$$z = z(\boldsymbol{\nu}; 1, 1, \dots, 1, A^*, A, -1, \dots, -1), \tag{8}$$

with $\boldsymbol{\nu} = (1^{n-1}, 3, 1^{n-1})$,

$$A^* = (a_m, b_m, c_m)^t, \quad A = -(c_m, b_m, a_m),$$

such that $AA^* = -q'_{2n-1}$. Then $z \in O'$, as shown in the Appendix.

Let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and $\mathfrak{g}_j, \mathfrak{n}'_{\ge j}, N'_{\ge j}$ the objects defined in Section 2F. Let $s = s(\boldsymbol{\nu}) = 2n-1 = m$. It is shown in the Appendix that

$$N'_{\ge 2} = \{n = (n_{j,\ell})_{j,\ell \le s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \ge \ell\},$$
$$N'_{\ge 4} = \{n = (n_{j,\ell})_{j,\ell \le s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \ge \ell - 1\};$$

that is, (4) is satisfied. Let Γ_z be the subset of Φ^+ defined in (5); then,

$$\Gamma_z = \{\alpha_j \mid j = 1, \dots, n-2\} \cup \{e_{n-1} \pm e_n\} \cup \{e_{n-1}, e_n\}. \tag{9}$$

Here the α_j $(j = 0, 1, \dots n)$ are simple roots defined in Section 2B.

Third, assume $m < 2n-1$. Then $\boldsymbol{\lambda}' = [m, 2n-m, 1]$, and q'_m, q'_{2n-m}, q'_1 are nondegenerate quadratic forms on k such that $q'_m \oplus q'_{2n-m} \oplus q'_1$ is isometric to quadratic form $(u, v, w) \mapsto 2uw + v^2$ $(u, v, w \in k)$. Let

$$z = z(\boldsymbol{\nu}; 1 \dots, 1, a^*, 1_2, \dots, 1_2, A^*, A, -1_2, \dots, -1_2, a, -1, \dots, -1), \tag{10}$$

with $\boldsymbol{\nu} = (1^{m-n}, 2^{n-(m+1)/2}, 3, 2^{n-(m+1)/2}, 1^{m-n})$, $a^* = (1, 0)^t$, $a = -(0, 1)$,

$$A^* = \begin{pmatrix} a_m & a_{2n-m} \\ b_m & b_{2n-m} \\ c_m & c_{2n-m} \end{pmatrix}, \quad A = -\begin{pmatrix} c_{2n-m} & b_{2n-m} & a_{2n-m} \\ c_m & b_m & a_m \end{pmatrix},$$

such that

$$AA^* = -\begin{pmatrix} 0 & q'_{2n-m} \\ q'_m & 0 \end{pmatrix}.$$

Working as in the Appendix, given $z \in O'$, let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and let $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ be the objects defined in Section 2F. Let $s = s(\nu) = m$; then, (4) is satisfied:

$$N'_{\geq 2} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B,$$
$$N'_{\geq 4} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5); then,

$$\begin{aligned} \Gamma_z = &\{\alpha_j \mid j = 1, \dots, m-n\} \cup \{e_{m-n} - e_{m-n+2}\} \\ &\cup \{\alpha_{m-n+2j-1} \mid j = 1, \dots, n-(m+1)/2\} \\ &\cup \bigcup_{j=1}^{n-\frac{m+3}{2}} \{e_{m-n+2j-1} - e_{m-n+2j+1}, e_{m-n+2j-1} - e_{m-n+2j+2}\} \\ &\cup \bigcup_{j=1}^{n-\frac{m+3}{2}} \{e_{m-n+2j} - e_{m-n+2j+1}, e_{m-n+2j} - e_{m-n+2j+2}\} \\ &\cup \{e_{n-2} \pm e_n\} \cup \{e_{n-1} \pm e_n\} \cup \{e_{n-2}, e_{n-1}, e_n\}. \end{aligned} \tag{11}$$

Assume $\boldsymbol{G} = \mathbf{SO}(2n)$. By Lemma 2.8, $\boldsymbol{\lambda}'$ is one of $[n^2]$, $[m, 2n-m]$, or $[m, 2n-m-2, 1^2]$ for some odd $m \geq \max(2i-1, 2n-2i-1)$.

First, assume $m = 2n-3$ and $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2] = [2n-3, 1^3]$. Then q'_{2n-3} and q'_1 are nondegenerate quadratic forms on k and k^3, respectively, such that $q'_{2n-3} \oplus q'_1$ is isometric to the quadratic form on k^4 defined by $(u, v, w, x) = 2ux + 2vw$ $(u, v, w, x \in k)$. Let $\boldsymbol{\nu} = (1^{n-2}, 4, 1^{n-2})$, $s = s(\boldsymbol{\nu}) = 2n-3 = m$, and $z = z(\boldsymbol{\nu}; 1, \dots, 1, A^*, A, -1, \dots, -1)$, with

$$A^* = (a_{2n-3}, b_{2n-3}, c_{2n-3}, d_{2n-3})^t, \quad A = -(d_{2n-3}, c_{2n-3}, b_{2n-3}, a_{2n-3})$$

satisfying $AA^* = -q'_{2n-3}$. Similar to that in the Appendix, $z \in O'$. Let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ the objects defined in Section 2F. Then

$$N'_{\geq 2} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B,$$
$$N'_{\geq 4} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5); then,

(12) $\Gamma_z = \{\alpha_j \mid j = 1, \dots, n-3\} \cup \{e_{n-2} \pm e_{n-1}\} \cup \{e_{n-2} \pm e_n\} \cup \{e_{n-1} \pm e_n\}$.

Second, assume $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some odd $m < 2n-3$, $m > \max(2i-1, 2n-2i-1)$. Since $m > 2n-m-2 > 1$, q'_m, q'_{2n-m-2} are quadratic forms on k and q'_1 is a quadratic form on k^2 such that $q'_m \oplus q'_{2n-m-2} \oplus q'_1$ is isometric to the quadratic form on k^4 defined by

$$(u, v, w, x) = 2ux + 2vw \quad (u, v, w, x \in k).$$

Let $\boldsymbol{\nu} = (1^{m-n+1}, 2^{n-\frac{m+3}{2}}, 4, 2^{n-\frac{m+3}{2}}, 1^{m-n+1})$, $s = s(\nu) = m$, and

$$z = z(\boldsymbol{\nu}; 1, \dots, 1, a^*, 1_2, \dots, 1_2, A^*, A, -1_2, \dots, -1_2, a, -1, \dots, -1),$$

with $a^* = (1, 0)^t, a = -(0, 1)$,

$$A^* = \begin{pmatrix} a_m & a_{2n-m-2} \\ b_m & b_{2n-m-2} \\ c_m & c_{2n-m-2} \\ d_m & c_{2n-m-2} \end{pmatrix}, A = -\begin{pmatrix} d_{2n-m-2} & c_{2n-m-2} & b_{2n-m-2} & a_{2n-m-2} \\ d_m & c_m & b_m & a_m \end{pmatrix},$$

such that

$$AA^* = -\begin{pmatrix} 0 & q'_{2n-m-2} \\ q'_m & 0 \end{pmatrix}.$$

Working as in the Appendix, given $z \in O'$, let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and let $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ be the objects defined in Section 2F. Then

$$N'_{\geq 2} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B,$$
$$N'_{\geq 4} = \{n = (n_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid n_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5); then,

$$\begin{aligned}(13)\quad \Gamma_z =&\{\alpha_j \mid j=1,\dots,m-n+1\}\cup\{e_{m-n+1}-e_{m-n+3}\}\\ &\cup\{\alpha_{m-n+1+2j-1}\mid j=1,\dots,n-(m+3)/2\}\\ &\cup\bigcup_{j=1}^{n-\frac{m+5}{2}}\{e_{m-n+1+2j-1}-e_{m-n+1+2j+1},e_{m-n+1+2j-1}-e_{m-n+1+2j+2}\}\\ &\cup\bigcup_{j=1}^{n-\frac{m+5}{2}}\{e_{m-n+1+2j}-e_{m-n+1+2j+1},e_{m-n+1+2j}-e_{m-n+1+2j+2}\}\\ &\cup\{e_{n-3}\pm e_{n-1},e_{n-3}\pm e_n\}\cup\{e_{n-2}\pm e_{n-1},e_{n-2}\pm e_n\}\\ &\cup\{e_{n-1}\pm e_n\}.\end{aligned}$$

Third, assume $\boldsymbol{\lambda}'=[m,2n-m]$ for some odd $m\geq n$. If $m>n$, then q'_m, q'_{2n-m} are quadratic forms on k such that $q'_m\oplus q'_{2n-m}$ is isometric to the quadratic form on k^2 defined by $(u,w)\mapsto 2uw$. If $m=n$ is odd, then $\boldsymbol{\lambda}'=[n^2]$, and q'_n is the quadratic form on k^2 isometric to the quadratic form on k^2 defined by $(u,w)\mapsto 2uw$.

Let $\boldsymbol{\nu}=(1^{m-n},2^{2n-m},1^{m-n})$, $s=s(\boldsymbol{\nu})=m$, and

$$z=\begin{cases} z(\boldsymbol{\nu};1_2,\dots,1_2,A^*,A,-1_2,\dots,-1_2), & m=n,\\ z(\boldsymbol{\nu};1,\dots,1,a^*,1_2,\dots,1_2,A^*,A,-1_2,\dots,-1_2,a,-1,\dots,-1), & m>n,\end{cases}$$

with $a^*=(1,0)^t, a=-(0,1)$,

$$A^*=\begin{pmatrix}a_m & a_{2n-m}\\ b_m & b_{2n-m}\end{pmatrix},\quad A=-\begin{pmatrix}b_{2n-m} & a_{2n-m}\\ b_m & a_m\end{pmatrix},$$

satisfying

$$AA^*=-\begin{cases}\begin{pmatrix}0 & q'_{2n-m}\\ q'_m & 0\end{pmatrix} & \text{if } m>n,\\ -\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} & \text{if } m=n.\end{cases}$$

Working as in the Appendix, given $z\in O'$, let $\{z,h,z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and let $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ be the objects defined in Section 2F. Then

$$\begin{aligned}N'_{\geq 2}&=\{n=(n_{j,\ell})_{j,\ell\leq s}\in\mathfrak{g}\mid n_{j,\ell}=0_{\nu_j\times\nu_\ell}\text{ if } j\geq \ell\}\subset B,\\ N'_{\geq 4}&=\{n=(n_{j,\ell})_{j,\ell\leq s}\in\mathfrak{g}\mid n_{j,\ell}=0_{\nu_j\times\nu_\ell}\text{ if } j\geq \ell-1\}\subset B.\end{aligned}$$

Let $\Gamma_z\subset\Phi^+$ be the subset of positive roots defined in (5); then,

$$
\begin{aligned}
(14)\qquad \Gamma_z =&\{\alpha_j \mid j=1,\dots,m-n\}\cup\{e_{m-n}-e_{m-n+2}\}\\
&\cup\{\alpha_{m-n+2j-1}\mid j=1,\dots,n-(m+1)/2\}\\
&\cup\bigcup_{j=1}^{n-\frac{m+3}{2}}\{e_{m-n+2j-1}-e_{m-n+2j+1}, e_{m-n+2j-1}-e_{m-n+2j+2}\}\\
&\cup\bigcup_{j=1}^{n-\frac{m+3}{2}}\{e_{m-n+2j}-e_{m-n+2j+1}, e_{m-n+1+2j}-e_{m-n+2j+2}\}\\
&\cup\{e_{n-1}\pm e_n\}\cup\{e_{n-2}\pm e_n\}.
\end{aligned}
$$

Fourth, assume n is even and $\boldsymbol{\lambda}'=[n^2]$. Let $\boldsymbol{\nu}=(2^n)$,

$$
(15)\qquad z=z(\nu;1_2,\dots,1_2,A,-1_2,\dots,-1_2),
$$

with $A=\operatorname{diag}(1,-1)$. Working as in the Appendix, take $z_\varepsilon\in O'_\varepsilon$, where O'_ε is the nilpotent orbit corresponding to $(\boldsymbol{\lambda}',\varnothing,\varepsilon)$ for some $\varepsilon=1$ or -1. Let $\{z_\varepsilon,h_\varepsilon,z_\varepsilon\}$ be an $\mathfrak{sl}_2$ triple attached to z_ε in $\mathfrak{g}$, and $\mathfrak{g}_j$, $\mathfrak{n}'_{\geq j}$, $N'_{\geq j}$ the objects defined in Section 2F. Then

$$
\begin{aligned}
N'_{\geq 2}&=\{u=(u_{j,\ell})_{j,\ell\leq n}\in\mathfrak{g}\mid u_{j,\ell}=0_{\nu_j\times\nu_\ell}\text{ if } j\geq \ell\}\subset B,\\
N'_{\geq 4}&=\{u=(u_{j,\ell})_{j,\ell\leq n}\in\mathfrak{g}\mid u_{j,\ell}=0_{\nu_j\times\nu_\ell}\text{ if } j\geq \ell-1\}\subset B.
\end{aligned}
$$

Let $\Gamma_{z_\varepsilon}\subset\Phi^+$ be the subset of positive roots defined in (5) for z_ε; then,

$$
\begin{aligned}
(16)\qquad \Gamma_{z_\epsilon}=&\{\alpha_{2j-1}\mid j=1,\dots,n/2-1\}\cup\{e_{n-1}\pm e_n\}\\
&\cup\bigcup_{j=1}^{\frac{n}{2}-1}\{e_{2j-1}-e_{2j+1}, e_{2j-1}-e_{2j+2}, e_{2j}-e_{2j+1}, e_{2j}-e_{2j+2}\}.
\end{aligned}
$$

Let $w_0=(a_{\ell,\ell'})_{2n\times 2n}$ be the element in $\mathbf{O}(2n)$ satisfying

$$
\begin{cases}
a_{n,n+1}=a_{n+1,n}=a_{j,j}=1 & \text{if } 1\leq j\leq 2n,\ j\neq n,\ j\neq n+1,\\
a_{\ell,\ell'}=0 & \text{otherwise.}
\end{cases}
$$

Let $z_{-\varepsilon}=w_0z_\varepsilon w_0^{-1}$; then $z_{-\varepsilon}\in O'_{-\epsilon}$, where $O'_{-\epsilon}$ is the nilpotent orbit corresponding to $(\boldsymbol{\lambda}',\phi,-\varepsilon)$. Let $\{z_{-\varepsilon},h_{-\varepsilon},z_{-\varepsilon}\}$ be an $\mathfrak{sl}_2$ triple attached to $z_{-\varepsilon}$ in $\mathfrak{g}$ and $\mathfrak{g}''_j$, $\mathfrak{n}''_{\geq j}$, $N''_{\geq j}$ the objects defined in Section 2F. Then

$$
N''_{\geq 2}=w_0N'_{\geq 2}w_0^{-1}\subset B,\quad N''_{\geq 4}=w_0N'_{\geq 4}w_0^{-1}\subset B.
$$

Let $\Gamma_{z_{-\varepsilon}}\subset\Phi^+$ be the subset of positive roots defined in (5) for $z_{-\varepsilon}$, then

$$
(17)\qquad \Gamma_{z_{-\varepsilon}}=\{e_{n-3}+e_n, e_{n-2}+e_n\}\cup\Gamma_{z_\varepsilon}\backslash\{e_{n-3}-e_n, e_{n-2}-e_n\}.
$$

Assume $\boldsymbol{G} = \mathbf{Sp}(2n)$. By Lemma 2.8, $\boldsymbol{\lambda}' = [m, 2n-m]$ for some even $m > \max(2i, 2n-2i)$. Then $m > 2n-m$, and q'_m, q'_{2n-m} are nondegenerate quadratic forms on k. Let $\boldsymbol{\nu} = (1^{m-n}, 2^{2n-m}, 1^{m-n})$, $s = s(\boldsymbol{\nu}) = m$, and

$$z = z(\boldsymbol{\nu}; 1, \ldots, 1, a^*, 1_2, \ldots, 1_2, A, -1_2, \ldots, -1_2, a, -1, \ldots, -1),$$

with $a^* = (1,0)^t, a = -(0,1)$, $A = \left(\begin{smallmatrix} b & a \\ a & c \end{smallmatrix}\right)$, such that $q'_m \oplus q'_{2n-m}$ is isometric to the quadratic form given by the symmetric matrix A.

Working as in the Appendix, given $z \in O'$, let $\{z, h, z'\}$ be an $\mathfrak{sl}_2$ triple attached to z in $\mathfrak{g}$ and let $\mathfrak{g}_j, \mathfrak{n}'_{\geq j}, N'_{\geq j}$ be the objects defined in Section 2F. Then

$$N'_{\geq 2} = \{u = (u_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid u_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell\} \subset B,$$
$$N'_{\geq 4} = \{u = (u_{j,\ell})_{j,\ell \leq s} \in \mathfrak{g} \mid u_{j,\ell} = 0_{\nu_j \times \nu_\ell} \text{ if } j \geq \ell - 1\} \subset B.$$

Let $\Gamma_z \subset \Phi^+$ be the subset of positive roots defined in (5) for z; then,

$$\begin{aligned} \Gamma_z =& \{\alpha_j \mid j = 1, \ldots, m-n\} \cup \{e_{m-n} - e_{m-n+2}\} \\ & \cup \{\alpha_{m-n+2j-1} \mid j = 1, \ldots, n-(m)/2\} \\ & \cup \bigcup_{j=1}^{n-\frac{m}{2}-1} \{e_{m-n+2j-1} - e_{m-n+2j+1}, e_{m-n+2j-1} - e_{m-n+2j+2}\} \\ & \cup \bigcup_{j=1}^{n-\frac{m}{2}-1} \{e_{m-n+2j} - e_{m-n+2j+1}, e_{m-n+1+2j} - e_{m-n+2j+2}\} \\ & \cup \{e_{n-1} + e_n, 2e_{n-1}, 2e_n\}. \end{aligned} \tag{18}$$

Proof of Theorem 3.1. We keep the notation used so far in this section and in Section 2B. For $i \in I_{\text{nsp}}$, let

$$\Sigma_i = \{\alpha_j \mid j = 1, \ldots, n, \, j \neq i\} \cup \{-\gamma\},$$

which is a set of simple roots of a root subsystem of Φ. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. Let $z \in O', \Gamma_z \subset \Phi^+$ be as defined (6), (8), (10), (15), and set $\Gamma'_z = \Phi^+ \backslash \Gamma_z$.

Lemma 3.3. *Let w be a Weyl element of G such that $w^{-1}(\Sigma_i) \subset \Phi^+$. Then $w^{-1}(\Sigma_i) \cap \Gamma'_z \neq \varnothing$.*

Proof. First assume $\boldsymbol{G} = \mathbf{SO}(2n+1)$. Then $-\gamma = -e_1 - e_2, \alpha_j = e_j - e_{j+1}$ for $j = 1, \ldots, n-1$, and $\alpha_n = e_n$. Let w be a Weyl element of G such that $w^{-1}(\Sigma_i) \subset \Phi^+$; then, there is a permutation σ of $\{1, 2, \ldots, n\}$ satisfying $\sigma(1) > \sigma(2) > \cdots > \sigma(i)$,

$\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$, such that

$$w^{-1}(e_j) = \begin{cases} \pm e_{\sigma(1)} & \text{if } j = 1, \\ -e_{\sigma(j)} & \text{if } 2 \le j \le i, \\ e_{\sigma(j)} & \text{if } i+1 \le j \le n. \end{cases} \tag{19}$$

Assume on the contrary that $w^{-1}(\Sigma_i) \cap \Gamma_z' = \varnothing$; then

$$w^{-1}(\Sigma_i) \subset \Gamma_z. \tag{20}$$

If $i = n$, then $\boldsymbol{\lambda}^i = [2n-1, 1^2]$, $\Sigma_n = \{\alpha_j \mid 1 \le j < n\} \cup \{-\gamma\}$. Then by Lemma 2.8, $\boldsymbol{\lambda}' = [2n+1]$ and $q_{2n+1}' = q_{2n+1}$, and by (7), $\Gamma_z = \{\alpha_j \mid j = 1, \dots, n\}$. If w satisfies (19) and (20), then $\sigma(j) = n+1-j$,

$$w^{-1}(e_1) = \pm e_n, \qquad w^{-1}(e_j) = -e_{n+1-j} \quad (1 < j \le n).$$

As a result, $w^{-1}(\Sigma_n) = \{\alpha_j \mid 1 \le j < n\} \cup \{e_{n-1} + e_n\} \not\subset \Gamma_z$, which contradicts (20). Hence $w^{-1}(\Sigma_n) \cap \Gamma_z' \ne \varnothing$.

If $i < n$, by Lemma 2.8, $\boldsymbol{\lambda}' = [2n+1]$ or $[m, 2n-m, 1]$ for some odd $m > \max(2i-1, 2n-2i+1)$. Let w be a Weyl element satisfying (19) and (20). Since $\pm e_1 - e_2, e_n \in \Sigma_i$, we have

$$w^{-1}(\pm e_1 - e_2) = e_{\sigma(2)} \pm e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(e_n) = e_{\sigma(n)} \in \Gamma_z. \tag{21}$$

If $\boldsymbol{\lambda}' = [2n+1]$, then $\Gamma_z = \{\alpha_j \mid 1 \le j \le n\}$ and $e_{\sigma(2)} + e_{\sigma(1)} \notin \Gamma_z$, which contradicts (21).

If $\boldsymbol{\lambda}' = [m, 2n-m, 1]$, $m = 2n-1$, then Γ_z is the set in (9). By (21), $\sigma(2) = n-1$, $\sigma(1) = n$, while $\sigma(n) = n$ or $n-1$, which is impossible since σ is a permutation.

If $\boldsymbol{\lambda}' = [m, 2n-m, 1]$, $m < 2n-1$, then Γ_z is the set in (11). By (21), $\sigma(1) = n$, $\{\sigma(2), \sigma(n)\} = \{n-2, n-1\}$. If $e_2 - e_3, e_{n-1} - e_n \in \Sigma_i$, then by (20),

$$w^{-1}(e_2 - e_3) = e_{\sigma(3)} - e_{\sigma(2)} \in \Gamma_z, \quad w^{-1}(e_{n-1} - e_n) = e_{\sigma(n-1)} - e_{\sigma(n)} \in \Gamma_z.$$

Then $\{\sigma(3), \sigma(n-1)\} = \{n-4, n-3\}$. Since $m > \max(2i-1, 2n-2i+1)$, we have

$$n - \frac{m+1}{2} < \min(n-i, i-1),$$

so the procedure can be repeated $n - \frac{m+1}{2}$ times. Then, for $\ell = 2, \dots, n - \frac{m-1}{2}$,

$$\{\sigma(\ell), \sigma(n+2-\ell)\} = \{n - 2(\ell-1), n - 2(\ell-1) + 1\}.$$

In particular, for $\ell_0 = n - \frac{m-1}{2}$ and $n + 2 - \ell_0 = \frac{m+3}{2}$,

$$\{\sigma(\ell_0), \sigma(n+2-\ell_0)\} = \Big\{\sigma(\ell_0), \sigma\Big(\frac{m+3}{2}\Big)\Big\} = \{m-n+1, m-n+2\}.$$

Since $m > 2i-1$, we have $m > 2n-2i+1$,

$$\ell_0 = n - \frac{m-1}{2} < i, \quad i+1 < \frac{m+3}{2} = n+2-\ell_0, \quad e_{\ell_0} - e_{\ell_0+1}, e_{\frac{m+1}{2}} - e_{\frac{m+3}{2}} \in \Sigma_i.$$

By (20),

$$w^{-1}(e_{\ell_0} - e_{\ell_0+1}) = e_{\sigma(\ell_0+1)} - e_{\sigma(\ell_0)} \in \Gamma_z,$$
$$w^{-1}(e_{\frac{m+1}{2}} - e_{\frac{m+3}{2}}) = e_{\sigma(\frac{m+1}{2})} - e_{\sigma(\frac{m+3}{2})} \in \Gamma_z.$$

Then $\sigma(\ell_0+1) = \sigma(\frac{1}{2}(m+1)) = m-n$, which contradicts the assumption that σ is a permutation, for $\ell_0+1 \le i$, $(m+1)/2 \ge i+1$, $\ell_0+1 \ne (m+1)/2$. Hence $w^{-1}(\Sigma_i) \cap \Gamma'_z \ne \varnothing$, concluding the proof for $\boldsymbol{G} = \mathbf{SO}(2n+1)$.

Assume now $\boldsymbol{G} = \mathbf{SO}(2n)$; then we have $-\gamma = -e_1 - e_2$, $\alpha_j = e_j - e_{j+1}$ for $j = 1, \dots, n-1$, and $\alpha_n = e_{n-1} + e_n$. Let w be a Weyl element of G such that $w^{-1}(\Sigma_i) \subset \Phi^+$; then, there is a permutation σ of $\{1, 2, \dots, n\}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ satisfying $\sigma(1) > \sigma(2) > \cdots > \sigma(i)$, $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$, $(-1)^{i-1}\varepsilon_1\varepsilon_2 = 1$, such that

$$w^{-1}(e_j) = \begin{cases} \varepsilon_1 e_{\sigma(1)} & \text{if } j = 1, \\ -e_{\sigma(j)} & \text{if } 2 \le j \le i, \\ e_{\sigma(j)} & \text{if } i+1 \le j \le n-1, \\ \varepsilon_2 e_{\sigma(n)} & \text{if } j = n. \end{cases} \tag{22}$$

Assume on the contrary that $w^{-1}(\Sigma_i) \cap \Gamma'_z = \varnothing$; then

$$w^{-1}(\Sigma_i) \subset \Gamma_z. \tag{23}$$

By Lemma 2.8, $\boldsymbol{\lambda}'$ is of the form $[m, 2n-m-2, 1^2]$ or $[m, 2n-m]$.

Assume first $m = 2n-3 > \max(2i-1, 2n-2i-1)$, $\boldsymbol{\lambda}' = [2n-3, 1^3]$; then Γ_z is the set in (12). Since $i \in I_{\mathrm{nsp}}$, I_{nsp} is nonempty and $n \ge 4$. Hence $1, 2, n-1, n$ are four distinct numbers. On the other hand, $\pm e_1 - e_2, e_{n-1} \pm e_n \in \Sigma_i$, so by (23),

$$w^{-1}(\pm e_1 - e_2) = e_{\sigma(2)} \pm \epsilon_1 e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(e_{n-1} \pm e_n) = e_{\sigma(n-1)} \pm \epsilon_2 e_{\sigma(n)} \in \Gamma_z.$$

Hence the cardinality of $\{\sigma(1), \sigma(2), \sigma(n-1), \sigma(n)\}$ is 3, which contradicts the assumption that σ is a permutation.

Second, assume $\boldsymbol{\lambda}' = [m, 2n-m-2, 1^2]$ for some odd m with $m < 2n-3$, $m > \max(2i-1, 2n-2i-1)$. Then Γ_z is the set in (13). Since $\pm e_1 - e_2, e_{n-1} \pm e_n \in \Sigma_i$, we have, by (23),

$$w^{-1}(\pm e_1 - e_2) = e_{\sigma(2)} \pm \epsilon_1 e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(e_{n-1} \pm e_n) = e_{\sigma(n-1)} \pm \epsilon_2 e_{\sigma(n)} \in \Gamma_z.$$

Then $\{\sigma(1), \sigma(n)\} = \{n-1, n\}$ and $\{\sigma(2), \sigma(n-1) = \{n-2, n-3\}$. If $e_2 - e_3$, $e_{n-2} - e_{n-1} \in \Sigma_i$, then by (23),

$$w^{-1}(e_2 - e_3) = e_{\sigma(3)} - e_{\sigma(2)} \in \Gamma_z, \quad w^{-1}(e_{n-2} - e_{n-1}) = e_{\sigma(n-2)} - e_{\sigma(n-1)} \in \Gamma_z.$$

Then $\{\sigma(3), \sigma(n-2)\} = \{n-5, n-4\}$. Since $m > 2i-1$, $m > 2n-2i-1$,

$$n - \frac{m+3}{2} < \min(i-1, n-i-1),$$

the procedure can be repeated $n - \frac{m+3}{2}$ times. Then for $\ell = 1, 2, \ldots, n - \frac{m+1}{2}$,

$$\{\sigma(\ell), \sigma(n+1-\ell)\} = \{n-2(\ell-1), n-2(\ell-1)-1\}.$$

In particular, for $\ell_0 = n - \frac{m+1}{2}$, we have $n+1-\ell_0 = \frac{m+3}{2}$,

$$\{\sigma(\ell_0), \sigma(n+1-\ell_0)\} = \left\{\sigma(\ell_0), \frac{m+3}{2}\right\} = \{m-n+3, m-n+2\}.$$

Since $m > 2i-1$, we have $m > 2n-2i-1$,

$$\ell_0 = n - \frac{m+1}{2} < i, \quad i+1 < \frac{m+3}{2} = n+1-\ell_0, \quad e_{\ell_0} - e_{\ell_0+1}, e_{\frac{m+1}{2}} - e_{\frac{m+3}{2}} \in \Sigma_i.$$

By (23),

$$w^{-1}(e_{\ell_0} - e_{\ell_0+1}) = e_{\sigma(\ell_0+1)} - e_{\sigma(\ell_0)} \in \Gamma_z,$$
$$w^{-1}(e_{\frac{m+1}{2}} - e_{\frac{m+3}{2}}) = e_{\sigma(\frac{m+1}{2})} - e_{\sigma(\frac{m+3}{2})} \in \Gamma_z.$$

Then $\sigma(\ell_0+1) = \sigma(\frac{1}{2}(m+1)) = m-n+1$, which contradicts the assumption that σ is a permutation, for $\ell_0 + 1 \le i$, $\frac{1}{2}(m+1) \ge i+1$, $\ell_0 + 1 \ne \frac{1}{2}(m+1)$.

Third, assume $\boldsymbol{\lambda}' = [m, 2n-m]$ for some odd $m \ge \max(2i-1, 2n-2i+1)$. Then Γ_z is the set in (14). Since $\pm e_1 - e_2, e_{n-1} \pm e_n \in \Sigma_i$, we have, by (23),

$$w^{-1}(\pm e_1 - e_2) = e_{\sigma(2)} \pm \epsilon_1 e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(e_{n-1} \pm e_n) = e_{\sigma(n-1)} \pm \epsilon_2 e_{\sigma(n)} \in \Gamma_z.$$

Then $\sigma(1) = \sigma(n) = n$, which contradicts the assumption that σ is a permutation.

Fourth, assume n is even and $\boldsymbol{\lambda}' = [n^2]$. Then Γ_z is either the set in (16) or the set in (17). Since $\pm e_1 - e_2, e_{n-1} \pm e_n$ belong to Σ_i, by (23),

$$w^{-1}(\pm e_1 - e_2) = e_{\sigma(2)} \pm \epsilon_1 e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(e_{n-1} \pm e_n) = e_{\sigma(n-1)} \pm \epsilon_2 e_{\sigma(n)} \in \Gamma_z.$$

Then $\sigma(1) = \sigma(n) = n$, which contradicts the assumption that σ is a permutation. Hence $w^{-1}(\Sigma_i) \cap \Gamma_z' \ne \varnothing$. This concludes the proof for $\boldsymbol{G} = \mathbf{SO}(2n)$.

Assume now $\boldsymbol{G} = \mathbf{Sp}(2n)$; then we have $-\gamma = -2e_1$, $\alpha_j = e_j - e_{j+1}$ for $j = 1, \ldots, n-1$, and $\alpha_n = 2e_n$. Since $w^{-1}(\Sigma_i) \subset \Phi^+$, there is a permutation σ of $\{1, 2, \ldots, n\}$, satisfying $\sigma(1) > \sigma(2) > \cdots > \sigma(i)$, $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$, such that

$$w^{-1}(e_j) = \begin{cases} -e_{\sigma(j)} & \text{if } 1 \le j \le i, \\ e_{\sigma(j)} & \text{if } i+1 \le j \le n. \end{cases} \tag{24}$$

By Lemma 2.8, $\boldsymbol{\lambda}' = [m, 2n-m]$ for some even $m > \max(2i, 2n-2i)$. Then Γ_z

is the set in (18). Assume on the contrary that $w^{-1}(\Sigma_i) \cap \Gamma'_z = \varnothing$; then

$$w^{-1}(\Sigma_i) \subset \Gamma_z.$$

Since $-2e_1, 2e_n \in \Sigma_i$, we have

$$w^{-1}(-2e_1) = 2e_{\sigma(1)} \in \Gamma_z, \quad w^{-1}(2e_n) = 2e_{\sigma(n)} \in \Gamma_z.$$

Then $\{\sigma(1), \sigma(n)\} = \{n-1, n\}$. If $e_1 - e_2, e_{n-1} - e_n \in \Sigma_i$,

$$w^{-1}(e_1 - e_2) = e_{\sigma(2)} - e_{\sigma(1)} \in \Gamma_{O'}, \quad w^{-1}(e_{n-1} - e_n) = e_{\sigma(n-1)} - e_{\sigma(n)} \in \Gamma_{O'}.$$

Then $\{\sigma(2), \sigma(n-1)\} = \{n-3, n-2\}$. Since $m > 2i$ and $m > 2n-2i$, we have

$$n - \frac{m}{2} < \max(i, n-i),$$

the above procedure can be repeated $n - \frac{m}{2}$ times. Then for $\ell = 1, 2, \ldots, n - \frac{m}{2}$,

$$\{\sigma(\ell), \sigma(n+1-\ell)\} = \{n - 2(\ell-1), n - 2(\ell-1) - 1\}.$$

In particular, for $\ell_0 = n - \frac{m}{2}$ and $n + 1 - \ell_0 = \frac{m}{2} + 1$, we have

$$\{\sigma(\ell_0), \sigma(n+1-\ell_0)\} = \left\{\sigma(\ell_0), \sigma\left(\frac{m}{2} + 1\right)\right\} = \{m-n+1, m-n+2\}.$$

Since $m > 2i$, $m > 2n - 2i$,

$$\ell_0 = n - \frac{m}{2} < i, \quad i + 1 < \frac{m}{2} + 1 = n + 1 - \ell_0, \quad e_{\ell_0} - e_{\ell_0+1}, e_{\frac{m}{2}} - e_{\frac{m}{2}+1} \in \Sigma_i.$$

By assumption,

$$\begin{aligned} w^{-1}(e_{\ell_0} - e_{\ell_0+1}) &= e_{\sigma(\ell_0+1)} - e_{\sigma(\ell_0)} \in \Gamma_{O'}, \\ w^{-1}(e_{\frac{m}{2}} - e_{\frac{m}{2}+1}) &= e_{\sigma(\frac{m}{2})} - e_{\sigma(\frac{m}{2}+1)} \in \Gamma_{O'}. \end{aligned}$$

Then $\sigma(\ell_0 + 1) = \sigma(m/2) = m - n$. But $i \geq \ell_0 + 1 \neq m/2 > i$, which contradicts the assumption that σ is a permutation. Hence $w^{-1}(\Sigma_i) \subset \Phi^+$. This conclude the proof for $\boldsymbol{G} = \mathbf{Sp}(2n)$. □

Let $A = A(S)$ be the apartment of $\mathcal{B}(G)$ defined by the maximal split torus S of G; see Section 2B. Let r be a positive integer. $F \subset A$ is called an r-facet if F is connected and there is a finite subset Φ_F of Φ_{af} such that

$$\psi(x) = r \quad \text{for all } x \in F,\ \psi \in \Phi_F.$$

Here Φ_{af} is the set of affine roots associated to S. For more details on r-facets, see [DeBacker 2002]. Since r is integer, the r-facet is in fact the usual facet.

Lemma 3.4. *For $i \in I_{\mathrm{nsp}}$, let w be a Weyl element satisfying $w^{-1}(\Sigma_i) \subset \Phi^+$. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. Let $z \in O'$ be the nilpotent element in (6), (8), (10), (15), and let $r > 0$ a positive integer. Then there is an r-facet F such that $y_i \in \partial F$ and*

$$(wN'_{\geq 4}w^{-1} \cap G_{y_i,r})G_{y_i,r+} \supset G_{F,r+}.$$

Here y_i is the vertex of the fundamental chamber C defined in Section 2B and $N'_{\geq j}$ is the object defined in Section 2F for any $\mathfrak{sl}_2$ triple $\{z, h, z'\}$ attached to z in $\mathfrak{g}$.

Proof. Let $\Gamma_z \subset \Phi^+$ be the set defined in (7), (9), (11), (13), and set $\Gamma'_z = \Phi^+ \backslash \Gamma_z$. By Lemma 3.3, $w^{-1}(\Sigma_i) \cap \Gamma'_z \neq \varnothing$. Take $\beta \in \Sigma_i$ such that $w^{-1}(\beta) \in \Gamma'_z$ and, let x_β be an arbitrary point in the apartment $\mathscr{A}$ such that $0 < \beta(x_\beta) < \frac{1}{2}$ and $\alpha(x_\beta) = 0$ for all $\alpha \in \Sigma_i$ distinct from β. Let F be the smallest r-facet containing x_β. Then $y_i \in \partial F$ and F satisfies the requirement of the lemma.

In fact, let Φ_i be the root subsystem generated by Σ_i and Φ_i^+ the subset of positive roots of Φ_i generated by Σ_i. Then by definition

$$\mathfrak{g}_{F,r+} := \mathfrak{g}_{x_\beta,r+} = \Bigg(\prod_{\substack{\delta \in \Phi_i \\ \delta(x_\beta) > \delta(y_i)}} u_{\delta,r} \Bigg) + \mathfrak{g}_{y_i,r+} \subset \mathfrak{g}_{y_i,r}.$$

Note that the following sets are the same:

$$\begin{aligned}\{\delta \in \Phi_i \mid \delta(x_\beta) > \delta(y_i)\} &= \{\delta \in \Phi_i^+ \mid \delta - \beta \in \Phi_i^+\} \\ &= \{\delta \in \Phi_i^+ \mid \delta \in \beta + \Phi_i^+\} \\ &= \{w(\alpha) \in \Phi_i^+ \mid \alpha \in w^{-1}(\beta) + w^{-1}(\Phi_i^+)\}.\end{aligned}$$

By Lemma 3.3, $w^{-1}(\beta) \in \Gamma'_z$; that is, the root space $\mathfrak{u}_{w^{-1}(\beta)} \subset \mathfrak{n}'_{\geq 4}$. On the other hand, since $w^{-1}(\Sigma_i) \subset \Phi^+$, $w^{-1}(\Phi_i^+) \subset \Phi^+$. For all $\delta \in \Phi^+$, $\mathfrak{u}_\delta \in \mathfrak{n}'_{\geq 0}$ (see Appendix), so $u_\alpha \subset \mathfrak{n}'_{\geq 4}$ for all $\alpha \in \Phi^+ \cap (w^{-1}(\beta) + w^{-1}(\Sigma_i))$.

Hence $\mathfrak{g}_{F,r+} \subset w\mathfrak{n}'_{\geq 4}w^{-1} \cap \mathfrak{g}_{y_i,r} + \mathfrak{g}_{y_i,r+}$, and thus

$$(wN'_{\geq 4}w^{-1} \cap G_{y_i,r})G_{y_i,r+} \supset G_{F,r+}. \qquad \square$$

Proposition 3.5. *Let $\pi = \pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'} \in \Pi'(\varphi)$ be an irreducible representation defined in Section 2D such that $i = i(\boldsymbol{\mu}') \in I_{\mathrm{nsp}}$. Let O', O^i be nilpotent orbits in $\mathfrak{g}$ corresponding to $(\boldsymbol{\lambda}', (q'_j))$ or $(\boldsymbol{\lambda}', \phi, \epsilon)$ and $(\boldsymbol{\lambda}^i, (q_j))$ respectively, with $O' > O^i$. Let $z \in O'$ be the nilpotent element in (6), (8), (10), (15), and let $N'_{\geq j}$ be the object defined in Section 2F for any $\mathfrak{sl}_2$ triple $\{z, h, z'\}$ attached to z in $\mathfrak{g}$.*

Let $N' = N'_{\geq 2}$ and ψ_z the character of N' defined in (3). Let v be a representative of a double coset in $G_{y_i} \backslash G / N'$ and ψ_z^v the character of $vN'v^{-1} \cap G_{y_i}$ defined as *follows*: *for all* $x \in vN'v^{-1} \cap G_{y_i}$,

$$\psi_z^v(x) := \psi_z(v^{-1}xv). \tag{25}$$

Let $r > 0$ *be a positive integer. Then there is an* r*-facet* F *such that* $y_i \in \partial F$ *and*

$$(vN'v^{-1} \cap G_{y_i,r})G_{y_i,r+}/G_{y_i,r+} \supset G_{F,r+}/G_{y_i,r+}, \quad \psi_z^v|_{G_{F,r+}} = 1.$$

Proof. Let S, B be the split torus and the Borel subgroup of G defined in Section 2B and U the unipotent subgroup of B. Let v be a representative of $G_{y_i}\backslash G/N'$; then,

$$v = w \cdot a \cdot u$$

for some Weyl element w of G such that $w^{-1}(\Sigma_i) \subset \Phi^+$, $a \in S$, and $u \in U/N'$, where Σ_i is the set defined in Lemma 3.3 (see [Reeder 1997]).

Note that a, u normalize N', and let $\psi' = \psi_z^{au}$, the character of N' defined in (25) with v replaced by au. By Lemma 3.4, there is an r-facet F with $y_i \in \partial F$ such that

$$(vN'v^{-1} \cap G_{y_i,r})G_{y_i,r+} \supset (wN'_{\geq 4}w^{-1} \cap G_{y_i,r})G_{y_i,r+} \supset G_{F,r+}.$$

For all $x \in G_{F,r+}$,

$$v^{-1}xv \in (au)^{-1}w^{-1}[wN'_{\geq 4}w^{-1}]wau \subset (au)^{-1}N'_{\geq 4}au = N_{\geq 4}.$$

By the definition of ψ_z, $\psi_z^v(x) = \psi_z(v^{-1}xv) = 1$. □

We can now conclude the proof of Theorem 3.1. By the discreteness criterion in [DeBacker and Reeder 2010, Lemma 2.4],

$$\chi(\pi) := \{x \in \mathcal{B}(G) \mid V_\pi^{G_{x,r+}} \neq 0\} = G.y_i,$$

and the $G_{y_i,r}/G_{y_i,r+}$-module $V_\pi^{G_{y_i,r+}}$ is cuspidal; i.e., for any r-facet F with $y_i \in \partial F$,

$$(V_\pi^{G_{y_i,r+}})^{\mathsf{L}_F} = 0. \tag{26}$$

Here $\mathsf{L}^F = G_{F,r+}/G_{y_i,r+}$ and V_π is the representation space of π.

Assume on the contrary $\mathrm{Hom}_{N'}(\pi, \psi_z) \neq 0$. By the construction of π in [Adler 1998], $\pi = c - \mathrm{Ind}_{G_{y_i}}^G(\Xi)$ for some irreducible representation Ξ of G_{y_i}. Let V_Ξ be the space of Ξ. Then

$$\mathrm{Hom}_{N'}(\pi, \psi_z) = \prod_{v \in G_{y_i}\backslash G/N'} \mathrm{Hom}_{vN'v^{-1}\cap G_{y_i}}(\Xi, \psi_z^v),$$

and there is some $v \in G_{y_i}\backslash G/N'$ such that $\mathrm{Hom}_{vN'v^{-1}\cap G_{y_i}}(\Xi, \psi_z^v) \neq 0$. Then

$$\mathrm{Hom}_{vN'v^{-1}\cap G_{y_i,r}}(\Xi, \psi_z^v) \neq 0.$$

Applying Proposition 3.5, there is an r-facet F such that $y_i \in \partial F$ and $V_{\Xi}^{G_{F,r+}} \neq 0$. Then $V_{\pi}^{G_{F,r+}} \neq 0$, which contradicts the discreteness criterion (26). $\square$

Proof of Theorem 3.2. Let $\bar{\mathfrak{f}}$ be the algebraic closure of $\mathfrak{f}$. Assume the characteristic p of $\mathfrak{f}$ is large enough that p is a good prime in the sense of [Carter 1972].

Keep the notation of Proposition 3.5. Then $i = i(\boldsymbol{\mu}') \in I_{\text{nsp}}$ and $G_{y_i,r}/G_{y_i,r+} = \mathfrak{g}_1(\mathfrak{f}) \times \mathfrak{g}_2(\mathfrak{f})$, with $\mathfrak{g}_1 = \mathfrak{so}(2i, f)$ or $\mathfrak{sp}(2i, f)$ (see Section 2B). Let $\bar{\xi}_j \in \mathfrak{g}_j(\mathfrak{f})$ $(j = 1, 2)$ be regular nilpotent elements and $\{\bar{\xi}_j, \bar{h}_j, \bar{\xi}'_j\}$ an $\mathfrak{sl}_2$ triple in $\mathfrak{g}_j(\mathfrak{f})$ attached to $\bar{\xi}_j$. Let

$$\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2), \quad \bar{h} = (\bar{h}_1, \bar{h}_2), \quad \bar{\xi}' = (\bar{\xi}_1, \bar{\xi}_2).$$

Then $(\bar{\xi}, \bar{h}, \bar{\xi}')$ is an $\mathfrak{sl}_2$ triple in $\mathfrak{g}_1(\mathfrak{f}) \times \mathfrak{g}_2(\mathfrak{f})$.

Recall that if $\boldsymbol{\mu}' \in \mathcal{S}(\boldsymbol{\mu})$, $i = i_{\boldsymbol{\mu}'} \in I_{\text{nsp}}$, then $\mathsf{T} := \mathsf{T}_{\boldsymbol{\mu}'} = \mathsf{T}_1 \times \mathsf{T}_2$ is a maximal anisotropic torus in G_{y_i}. Let $T := T_{\boldsymbol{\mu}'}$ be the maximal anisotropic unramified torus in G associated to $(y_i, \mathsf{T}_{\boldsymbol{\mu}'})$ in Section 2C. Let $X = X_{\boldsymbol{\mu}'} \in \mathfrak{t} = \text{Lie}(T)$ be the good element of depth $-r$ defining $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}$, whose image under the natural projection

$$\mathfrak{g}_{y_i,-r} \to \mathfrak{g}_{y_i,-r}/\mathfrak{g}_{y_i,-r+} \simeq \mathfrak{g}_1 \times \mathfrak{g}_2.$$

is denoted by $\bar{X} = (\bar{X}_1, \bar{X}_2)$. Since X is a good element in $\mathfrak{t}$ with $C_G(X) = T$, $\bar{X}_j$ is a regular semisimple element in $\text{Lie}(\mathsf{T}_j)(\mathfrak{f})$ for $j = 1, 2$.

Let $O_{\bar{X}_j}$ be the orbit of $\bar{X}_j$ in $\mathfrak{g}_j(\bar{\mathfrak{f}})/\mathsf{G}_j(\bar{\mathfrak{f}})$. By [Slodowy 1980, §7.4, Corollary 2], the Slodowy slice

$$\bar{V}_j := \bar{\xi}_j + C_{\mathfrak{g}_j(\bar{\mathfrak{f}})}(\bar{\xi}'_j) \tag{27}$$

intersects $O_{\bar{X}_j}$ at a unique $\mathfrak{f}$-rational point $\bar{X}'_j \in \mathfrak{g}_j(\mathfrak{f})$.

Since X is good, $C_{\mathsf{G}_j(\bar{\mathfrak{f}})}(\bar{X}_j)$ is connected [Carter 1985, Theorem 3.5.3]. Then there is a $\bar{g}_j \in \mathsf{G}_j(\mathfrak{f})$ such that $\text{Ad}(\bar{g}_j)(X_j) = \bar{X}'_j$ [Digne and Michel 1991, §3.25]. Moreover $\mathsf{T}'_j = C_{\mathsf{G}_j}(\bar{X}'_j) = \text{Ad}(\bar{g}_j)(\mathsf{T}_j)$ is a maximal anisotropic torus of $\mathsf{G}_j(\mathfrak{f})$, with $\mathsf{G}_j(\mathfrak{f})$-conjugate to T_j. Let $\bar{g} = (\bar{g}_1, \bar{g}_2) \in \mathsf{G}(\mathfrak{f})$; then, $\text{Ad}(\bar{g})(\mathsf{T}_1 \times \mathsf{T}_2) = \mathsf{T}' := \mathsf{T}'_1 \times \mathsf{T}'_2$.

Let $g \in G_{y_i,0} - g_{y_i,0+}$ such that g projects to $\bar{g}$, $T' := \text{Ad}(g)(T)$, and $X' := \text{Ad}(g)(X) \in \mathfrak{t}'$. Then T' is the maximal unramified torus in G, associated to (y_i, T'), X' is a good element in $\mathfrak{g}_{y_i,-r} \backslash \mathfrak{g}_{y_i,-r+}$, whose image under the natural projection in G_{y_i} is $\bar{X}' = (\bar{X}'_1, \bar{X}'_2)$. Note that $\bar{X}' \in \bar{V}_1(\mathfrak{f}) \times \bar{V}_2(\mathfrak{f})$, where

$$\bar{V}_1(\mathfrak{f}) = \bar{\xi}_1 + C_{\mathfrak{g}_1(\mathfrak{f})}(\bar{\xi}'_1), \quad \bar{V}_2(\mathfrak{f}) = \bar{\xi}_2 + C_{\mathfrak{g}_2(\mathfrak{f})}(\bar{\xi}'_2)$$

are sets of $\mathfrak{f}$-rational points of $\bar{V}_1, \bar{V}_2$ respectively. Without loss of generality, assume $X = X'$. Then the natural image $\bar{X}$ of X in $\mathfrak{g}_{y_i,-r}/\mathfrak{g}_{y_i,-r+}$ belongs to $\bar{V}_1(\mathfrak{f}) \times \bar{V}_2(\mathfrak{f})$.

By [DeBacker 2002, Corollary 4.3.2], let $(\xi, h, \xi') \in \mathfrak{g}_{y_i,-r} \times \mathfrak{g}_{y_i,0} \times \mathfrak{g}_{y_i,r}$ be an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$ such that $\{\xi, h, \xi'\}$ lifts $\{\bar{\xi}, \bar{h}, \bar{\xi}'\}$ respectively and $O' = \mathrm{Ad}(G)(\xi)$ the nilpotent orbit of ξ in $\mathfrak{g}$. By the choice of $\{\xi, h, \xi'\}$, $O' = O^i$ is a nilpotent orbit corresponding to $(\boldsymbol{\lambda}^i, (q_j))$. Let $N'_{\geq j}$ be the object defined in Section 2F for the triple $\{\xi, h, \xi'\}$ attached to ξ in $\mathfrak{g}$.

We can now conclude the proof of Theorem 3.2. Let $N' = N'_{\geq 2}$ and let S_ξ be the character ψ_ξ of N':

$$S_\xi(\exp Y) = \psi \circ \mathrm{tr}(\xi Y), \quad Y \in \mathrm{Lie}(N').$$

On the other hand, by the construction in [Adler 1998], $\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'} = c - \mathrm{Ind}_{G_{y_i}}^{G(k)}(\Xi)$, while $\Xi = \mathrm{Ind}_{TJ}^{G_{y_i}}(\sigma_\chi)$. Here

$$J = \exp_{y_i}(\mathfrak{J}), \qquad \mathfrak{J} = \mathfrak{t}_{y_i,r} + \mathfrak{t}^{\perp}_{y_i,\frac{r}{2}},$$
$$J^+ = \exp_{y_i}(\mathfrak{J}^+), \quad \mathfrak{J}^+ = \mathfrak{t}_{y_i,r} + \mathfrak{t}^{\perp}_{y_i,\frac{r}{2}+},$$

with $\mathfrak{t}^\perp$ the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to the killing form. Here TJ and TJ^+ are subgroups of G, since T normalizes J and J^+, and σ_χ is the irreducible representation of TJ such that $\sigma_\chi|_{TJ^+}$ is a multiple of χ, where χ is the character of TJ^+ extending $\chi_{\boldsymbol{\mu}'}$ on T, such that

$$\chi(\exp_{y_i} Y) = \psi(\mathrm{tr}(X \cdot Y)) \quad \text{for all } Y \in \mathfrak{J}^+.$$

Note that T is anisotropic and $N' \cap TJ = N' \cap J \supset N' \cap J^+$, while $N' \cap J / N' \cap J^+$ is an isotropic subspace over $\mathfrak{f}$ with respect to the nondegenerate symplectic form defined on J/J^+ by $(n, n') \mapsto \psi_\xi([\log n, \log n'])$. On the other hand, since $\bar{X} \in \bar{V}_1(\mathfrak{f}) \times \bar{V}_2(\mathfrak{f})$, $\chi|_{J^+ \cap N'} = \psi_\xi|_{J^+ \cap N'}$. By the definition of σ_χ,

$$\mathrm{Hom}_{N' \cap TJ}(\sigma_\chi, \psi_\xi) = \mathrm{Hom}_{N' \cap J}(\sigma_\chi, \psi_\xi) \neq 0.$$

Apply Lemma 3.6 below with G_1 replaced by G_{y_i}, G_2 by $N' \cap G_{y_i}$, and H_1 by TJ; then,

$$\mathrm{Hom}_{N' \cap G_{y_i}}(\Xi, \psi_\xi) \neq 0. \tag{28}$$

Since $\mathrm{Hom}_{N'}(\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}, S_\xi) = \prod_{v \in G_{y_i} \backslash G / N'} \mathrm{Hom}_{vN'v^{-1} \cap G_{y_i}}(\Xi, \psi_\xi^v)$, by (28),

$$\mathrm{Hom}_{N'}(\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'}, \psi_\xi) \neq 0.$$

Hence $O' \in \mathcal{N}_{\mathrm{wh}}(\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'})$. Combining with Theorem 3.1, $O' \in \mathcal{N}_{\mathrm{wh,max}}(\pi_{\chi_{\boldsymbol{\mu}};\boldsymbol{\mu}'})$. □

Lemma 3.6. *Let G_1 be a compact subgroup, and H_1, G_2 open compact subgroups of G_1. Let (σ, V_σ) (resp. (ξ, V_ξ)) be a smooth representation of H_1 (resp. G_2). If* $\mathrm{Hom}_{H_1 \cap G_2}(\sigma, \xi) \neq 0$, *then* $\mathrm{Hom}_{G_2}(\mathrm{Ind}_{H_1}^{G_1} \sigma, \xi) \neq 0$.

Proof. The proof is similar to that of Proposition 2.1 in [Arthur 2008]. Consider a nonzero $A \in \mathrm{Hom}_{H_1 \cap G_2}(\sigma, \xi)$, and define $J_A \in \mathrm{Hom}_{G_2}(\mathrm{Ind}_{H_1}^{G_1} \sigma, \xi)$ as follows: for arbitrary $\phi \in \mathrm{Ind}_{H_1}^{G_1} \sigma$,

$$J_A \phi = \sum_{H_1 \cap G_2 \backslash G_2} \xi(g')^{-1} A(\phi(g')) \in V_\xi.$$

For all $g \in G_2$,

$$\begin{aligned} J_A(\mathrm{Ind}\sigma)(g)\phi &= \sum_{H_1 \cap G_2 \backslash G_2} \xi(g')^{-1} A(\mathrm{Ind}\sigma(g)\phi)(g') \\ &= \sum_{H_1 \cap G_2 \backslash G_2} \xi(g')^{-1} A\phi(g'g) \\ &= \xi(g) J_A \phi. \end{aligned}$$

Take some $v \in V_\sigma$ such that $Av \neq 0$. Define $\phi_v(g) = \sigma(h)v$ if $g = h \in H_1$ and $\phi_v(g) = 0$ if $g \notin H_1$. Then $\phi_v \in \mathrm{Ind}_{H_1}^{G_1} \sigma$, and $J_A \phi_v = Av \neq 0$, so J_A is a nonzero element in $\mathrm{Hom}_{G_2}(\mathrm{Ind}_{H_1}^{G_1} \sigma, \xi)$. □

Appendix: Rational nilpotent orbits

In this section, we show by example how to choose a particular element from a rational nilpotent orbit parametrized by $(\boldsymbol{\lambda}, (q_j))$.

Let W be a $(2n+1)$-dimensional symmetric k-space as defined in Section 2A, with bilinear form q_W. Let z be a nonzero nilpotent element in $\mathfrak{g} = \mathfrak{so}(W) \subset \mathfrak{gl}(W)$, and set $G = \mathbf{SO}(k, W)$. Let $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ be a Lie algebra homomorphism with

$$\phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = z.$$

Identify a scalar $t \in k$ with the diagonal matrix $\mathrm{diag}(t, t^{-1}) \in \mathfrak{sl}_2(k)$. As in [Mœglin 1996], for $i \in \mathbb{Z}$, let

$$\begin{aligned} \mathfrak{g}(i) &= \{Y \in \mathfrak{g} \mid \mathrm{Ad} \circ \phi(t)(Y) = itY \text{ for all } t \in k\}, \\ W(i) &= \{v \in W \mid \phi(t)(v) = itv \text{ for all } t \in k\}. \end{aligned}$$

Then $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$, $W = \bigoplus_{i \in \mathbb{Z}} W(i)$.

Assume the orbit $O = \mathrm{Ad}(G)(z)$ of z is parametrized by $(\boldsymbol{\lambda}, (q_i))$ with $\boldsymbol{\lambda} = [m, 2n-m, 1]$, where $m > n$ is an odd number. For $i = 1, \ldots, 2n+1$, let

$$W_i = \mathrm{Ker}(z^i)/(\mathrm{Ker}(z^{i-1}) + z\,\mathrm{Ker}(z^{i+1})). \tag{29}$$

Then by [Waldspurger 2001, §I.6], $\dim W_i = c_i(\boldsymbol{\lambda})$ and q_i is the nondegenerate quadratic form on W_i defined by

$$q_i(\bar{v}, \bar{v}') = (-1)^{[\frac{i-1}{2}]} q_W(z^{i-1}v, v') \quad (\bar{v}, \bar{v}' \in W_i), \tag{30}$$

where v (resp. v') is an inverse image of $\bar{v}$ (resp. $\bar{v}'$) in $\mathrm{Ker}(z^i)$.

Assume $m = 2n-1$; in this case $\boldsymbol{\lambda} = [2n-1, 1^2]$, $c_1(\boldsymbol{\lambda}) = 2$, $c_{2n-1}(\boldsymbol{\lambda}) = 1$. Then $\dim W_1 = 2$ and $\dim W_m = 1$. By (29), let $v_1, v_1' \in \mathrm{Ker}\, z$, $v_m \in \mathrm{Ker}\, z^m$ such that

$$\mathrm{Ker}\, z = z\,\mathrm{Ker}\, z^2 \oplus kv_1 \oplus kv_1',$$
$$\mathrm{Ker}\, z^m = (\mathrm{Ker}\, z^{m-1} + z\,\mathrm{Ker}\, z^{m+1}) \oplus kv_m.$$

Let $\bar{v}_1, \bar{v}_1'$ be the natural images of v_1, v_1' in W_1 and $\bar{v}_m$ that of v_m in W_m. Without loss of generality, assume $\bar{v}_1, \bar{v}_1'$ are orthogonal to each other under q_1; then $q_1 = \langle q_1(\bar{v}_1, \bar{v}_1), q_1(\bar{v}_1', \bar{v}_1')\rangle$,

$$q_m = \langle q_m(\bar{v}_m, \bar{v}_m)\rangle = (-1)^{\frac{m-1}{2}} q_W(z^{m-1}v_m, v_m). \tag{31}$$

In the following, identify q_m with $q_m(\bar{v}_m, \bar{v}_m)$.

Through $\phi : \mathfrak{sl}_2 \to \mathfrak{so}(W) \subset \mathfrak{gl}(W)$, W is a representation space of $\mathfrak{sl}_2$. In fact, since O_X corresponds to $(\boldsymbol{\lambda}, (q_i))$, $W \simeq V_m \oplus V_1 \oplus V_1$, where V_j is the irreducible representation of $\mathfrak{sl}_2$ of dimension j. By the representation theory of $\mathfrak{sl}_2$, $v_1, v_1' \in W(0)$ and $v_m \in W(m-1)$. Modifying by elements in $z\,\mathrm{Ker}\, z^2$, we can assume further that the subspace generated by v_1, v_1' is $V_1 \oplus V_1$.

Then ${}_{0\neq} z^{\ell}(v_m) \in W(m-1-2\ell)$ for all $\ell = 1, \dots, m-1$, and

$$\begin{aligned}
W(m-1) &= kv_m,\\
W(m-3) &= kzv_m,\\
&\ \vdots\\
W(2) &= kz^{n-2}v_m,\\
W(0) &= kz^{n-1}v_m \oplus kv_1 \oplus kv_1',\\
W(-2) &= kz^n v_m,\\
&\ \vdots\\
W(-(m-1)) &= kz^{m-1}v_m.
\end{aligned}$$

For $j = 1, \dots, m$, let $F_j = \bigoplus_{\ell \leq -(m-1)+2(j-1)} W(\ell)$ be a subspace of W. Then

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m = W,$$

and $zF_j = F_{j-1}$ for $j = 1, \dots, m$. Take a basis of W such that

$$\begin{aligned} e_1 &= v_m, & e_{-1} &= (-1)^{\frac{m-1}{2}} q_m^{-1} z^{m-1} v_m, \\ e_2 &= z v_m, & e_{-2} &= (-1)^{\frac{m-3}{2}} q_m^{-1} z^{m-2} v_m, \\ &\vdots & &\vdots \\ e_{n-1} &= z^{n-2} v_m, & e_{-(n-1)} &= (-1) q_m^{-1} z^{n+1} v_m. \end{aligned}$$

By (30), $q_W(e_i, e_j) = 0$ unless $i + j = 0$, and $q_W(e_i, e_{-i}) = 1$. Note that $W(0)$ has orthogonal decomposition

$$W(0) = k z^{n-1} v_m \oplus k v_1 \oplus k v_1'$$

under $q_W|_{W(0)}$. By (30), $q_W(z^{n-1} v_m, z^{n-1} v_m) = q_m(\bar{v}_m, \bar{v}_m)$, $q_W(v_1, v_1) = q_1(\bar{v}_1, \bar{v}_1)$, and $q_W(v_1', v_1') = q_1(\bar{v}_1', \bar{v}_1')$. By (31),

$$\begin{aligned} q_W|_{W(0)} &= \langle q_W(z^{\frac{m-1}{2}} v_m, z^{\frac{m-1}{2}} v_m), q_W(v_1, v_1), q_W(v_1', v_1') \rangle \\ &= \langle q_m(\bar{v}_m, \bar{v}_m), q_1(\bar{v}_1, \bar{v}_1), q_1(\bar{v}_1', \bar{v}_1') \rangle \\ &= q_m \oplus q_1. \end{aligned}$$

Because $q_1 \oplus q_m$ has the same anisotropic kernel as W, let e_n, e_0, e_{-n} be a basis of $W(0)$ such that

$$q_W(e_n, e_{-n}) = 1, \quad q_W(e_0, e_0) = 1, \quad q_W(e_n, e_0) = q_W(e_{-n}, e_0) = 0.$$

Then $e_1, e_2, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}$ is a basis of W, under which the matrix of q_W is J_W (defined in Section 2A), and the matrix of z is the lower triangular block matrix

$$\begin{pmatrix} 0 & & & & & & & & \\ 1 & 0 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & 1 & 0 & & & & & \\ & & & A^* & 0_3 & & & & \\ & & & & A & 0 & & & \\ & & & & & -1 & 0 & & \\ & & & & & & \ddots & \ddots & \end{pmatrix},$$

with

$$A^* = \begin{pmatrix} a_m \\ b_m \\ c_m \end{pmatrix}, \quad A = -\begin{pmatrix} c_m & b_m & a_m \end{pmatrix},$$

where (a_m, b_m, c_m) are the coordinates of $z^{n-1} v_m$ in the basis $\{e_n, e_0, e_{-n}\}$. Note that in this case, $AA^* = -q_m$.

Acknowledgments

The author would like to thank the referee for carefully reading and giving good advice on organizing and revising this paper. This paper was started and completed during a visit to University of Toronto in 2011. The author also thanks Professor Murnaghan for her hospitality and significant discussions during the visit.

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Received October 25, 2012. Revised March 23, 2014.

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Volume 269 No. 1 May 2014

The asymptotic behavior of Palais–Smale sequences on manifolds with boundary 1
SÉRGIO ALMARAZ

The cup subalgebra of a II_1 factor given by a subfactor planar algebra is maximal amenable 19
ARNAUD BROTHIER

Representation theory of type B and C standard Levi W-algebras 31
JONATHAN BROWN and SIMON M. GOODWIN

New invariants for complex manifolds and rational singularities 73
RONG DU and YUN GAO

Homogeneity groups of ends of open 3-manifolds 99
DENNIS J. GARITY and DUŠAN REPOVŠ

On the concircular curvature of a (κ, μ, ν)-manifold 113
FLORENCE GOULI-ANDREOU and EVAGGELIA MOUTAFI

Genuses of cluster quivers of finite mutation type 133
FANG LI, JICHUN LIU and YICHAO YANG

Taut foliations in knot complements 149
TAO LI and RACHEL ROBERTS

On the set of maximal nilpotent supports of supercuspidal representations 169
QIN YUJUN

The natural filtrations of finite-dimensional modular Lie superalgebras of Witt and Hamiltonian type 199
KELI ZHENG, YONGZHENG ZHANG and WEI SONG

Free Brownian motion and free convolution semigroups: multiplicative case 219
PING ZHONG