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THE NATURAL FILTRATIONS OF FINITE-DIMENSIONAL MODULAR LIE SUPERALGEBRAS OF WITT AND HAMILTONIAN TYPE

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We study the natural filtrations of the finite-dimensional modular Lie superalgebras $W(n, m)$ and $H(n, m)$. In particular, the natural filtrations which are invariant relative to the automorphisms of the Lie superalgebras are employed in order to characterize the Lie superalgebras themselves.

1. Introduction

In mathematics, a Lie superalgebra is a generalization of a Lie algebra including a $\mathbb{Z}_2$-grading. Lie superalgebras are also important in theoretical physics where they are used to describe the mathematics of supersymmetry [Varadarajan 2004]. Although many structural features of Lie superalgebras over fields of characteristic zero (see [Kac 1977; Scheunert 1979]) are well understood, there seem to be very few general results on modular Lie superalgebras. In particular, the classification problem is still open for the finite-dimensional simple Lie superalgebras over fields of positive characteristic (see [Bouarroudj and Leites 2006; Zhang 1997] for example). The treatment of modular Lie superalgebras necessitates different techniques which are set forth in [Kochetkov and Leites 1992; Petrogradski 1992]. Elduque [2007] obtained two new simple modular Lie superalgebras. These Lie superalgebras share the property that their even parts are orthogonal Lie algebras and the odd parts are their spin modules. In [Zhao 2010] modular representations of basic classical Lie superalgebras were studied. The Lie superalgebras of Cartan type play an extremely important role in the study of modular Lie superalgebras. Recent works on them can be found in [Chen and Liu 2011; Yuan et al. 2011; Zhang and Fu 2002].

It is well known that filtration techniques are of great importance in the structure and the classification theories of Lie (super)algebras (see [Block and Wilson 1988; Strade 1993; Kac 1977; Scheunert 1979]). We know that the simple Lie

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(super)algebras of Cartan type possess various natural filtration structures. For the filtration structures, the invariance may be used to make an insight for the intrinsic properties and the automorphism groups of those Lie (super)algebras. The natural filtrations of finite-dimensional modular Lie algebras of Cartan type were proved to be invariant in [Kac 1974; Kostrikin and Shafarevich 1969]. The finite-dimensional simple modular Lie superalgebras W, S, and H of Cartan type were defined in [Zhang 1997] and their natural filtrations were investigated in [Zhang and Fu 2002; Zhang and Nan 1998]. Recently, the natural filtrations of odd Hamiltonian superalgebras and special odd Hamiltonian superalgebras of formal vector fields were investigated in [Ren et al. 2012].

The finite-dimensional modular Lie superalgebras $W(n, m)$ and $H(n, m)$ were first introduced in [Awuti and Zhang 2008] and [Ren et al. 2011], respectively. In these papers, their derivation superalgebras were also determined. The starting point of our studies is the investigation of the ad-nilpotent elements of $W(n, m)$. Then the natural filtration of $W(n, m)$ is proved to be invariant by the determined ad-nilpotent elements. In the case of $H(n, m)$, the invariance of the natural filtration is studied by the methods of minimal dimension of image spaces and the derivation superalgebras. In view of the above invariance of the natural filtrations we describe the intrinsic properties of these modular Lie superalgebras.

This paper is arranged as follows. A brief summary of the relevant concepts and notations in finite-dimensional modular Lie superalgebras $W(n, m)$ and $H(n, m)$ is presented in Section 2. In Section 3, by using the ad-nilpotent elements of the Lie superalgebras $W(n, m)$, we show that the natural filtration of $W(n, m)$ is invariant under their automorphisms. In Section 4, the intrinsic properties with respect to the natural filtration of finite-dimensional modular Lie superalgebras $H(n, m)$ are investigated. Besides, the isomorphic relation between $H(n, m)$ and $H(n', m')$ is also proved by the method of the natural filtration.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ denotes an algebraic closed field of characteristic $p > 2$, n is an integer greater than 1. Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of integers, positive integers and nonnegative integers. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the residue class ring of integers modulo 2.

Let $\Lambda(n)$ be the Grassmann algebra over $\mathbb{F}$ in n variables $x_1, x_2, \ldots, x_n$. Set $\mathbb{B}_k = \{\langle i_1, i_2, \ldots, i_k\rangle \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}$ and $\mathbb{B}(n) = \bigcup_{k=0}^{n} B_k$, where $B_0 = \varnothing$. For $u = \langle i_1, i_2, \ldots, i_k\rangle \in \mathbb{B}(k)$, set $|u| = k$, $\{u\} = \{i_1, i_2, \ldots, i_k\}$, and $x^u = x_{i_1} x_{i_2} \cdots x_{i_k}$ ($|\varnothing| = 0$, $x^{\varnothing} = 1$). Then $\{x^u | u \in \mathbb{B}(n)\}$ is an $\mathbb{F}$-basis of $\Lambda(n)$.

Let Π denote the prime field of $\mathbb{F}$, that is, $\Pi = \{0, 1, \ldots, p-1\}$. Suppose that $\{z_1, z_2, \ldots, z_m\}$ is a Π-linearly independent finite subset of $\mathbb{F}$. Let $G =$

$\left\{\sum_{i=1}^{m} \lambda_i z_i \mid \lambda_i \in \Pi\right\}$. Then G is an additive subgroup of $\mathbb{F}$. Let $\mathbb{F}[y_1, y_2, \ldots, y_m]$ be the truncated polynomial algebra satisfying $y_i^p = 1$ for all $i = 1, 2, \ldots, m$. For every element $\lambda = \sum_{i=1}^{m} \lambda_i z_i \in G$, define $y^\lambda = y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_m^{\lambda_m}$. Then $y^\lambda y^\eta = y^{\lambda+\eta}$ for all $\lambda, \eta \in G$. Let $\mathbb{T}(m)$ denote $\mathbb{F}[y_1, y_2, \ldots, y_m]$. Then $\mathbb{T}(m) = \left\{\sum_{\lambda \in G} a_\lambda y^\lambda \mid a_\lambda \in \mathbb{F}\right\}$. We denote the tensor product by $\mathcal{U} = \Lambda(n) \otimes \mathbb{T}(m)$. Then $\mathcal{U}$ is an associative superalgebra with $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathbb{T}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$, that is, $\mathcal{U} = \mathcal{U}_{\bar{0}} \oplus \mathcal{U}_{\bar{1}}$, where $\mathcal{U}_{\bar{0}} = \Lambda(n)_{\bar{0}} \otimes \mathbb{T}(m)$ and $\mathcal{U}_{\bar{1}} = \Lambda(n)_{\bar{1}} \otimes \mathbb{T}(m)$.

For $f \in \Lambda(n)$ and $\alpha \in \mathbb{T}(m)$, we abbreviate $f \otimes \alpha$ as $f\alpha$. Then the elements $x^u y^\lambda$ with $u \in \mathbb{B}(n)$ and $\lambda \in G$ form an $\mathbb{F}$-basis of $\mathcal{U}$. It is easy to see that $\mathcal{U} = \bigoplus_{i=0}^{n} \mathcal{U}_i$ is a $\mathbb{Z}$-graded superalgebra, where $\mathcal{U}_i = \operatorname{span}_{\mathbb{F}}\{x^u y^\lambda \mid u \in \mathbb{B}(n), |u| = i, \lambda \in G\}$. In particular, $\mathcal{U}_0 = \mathbb{T}(m)$ and $\mathcal{U}_n = \operatorname{span}_{\mathbb{F}}\{x^\pi y^\lambda \mid \lambda \in G\}$, where $\pi := \langle 1, 2, \ldots, n\rangle \in \mathbb{B}(n)$.

In this paper, if $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a superalgebra (or $\mathbb{Z}_2$-graded linear space), let $\operatorname{hg}(A) = A_{\bar{0}} \cup A_{\bar{1}}$; that is, $\operatorname{hg}(A)$ is the set of all $\mathbb{Z}_2$-homogeneous elements of A. If $\deg x$ occurs in some expression, we regard x as a $\mathbb{Z}_2$-homogeneous element and $\deg x$ as the $\mathbb{Z}_2$-degree of x. Let $A = \bigoplus_{i=-r}^{n} A_i$ be a $\mathbb{Z}$-graded superalgebra. If $x \in A_i$, we call x a $\mathbb{Z}$-homogeneous element, i the $\mathbb{Z}$-degree of x and set $\operatorname{zd}(x) = i$. If $y \in A$, let $\mu(y)$ denote the nonzero $\mathbb{Z}$-homogeneous part of y with the least $\mathbb{Z}$-degree.

Let $\operatorname{pl}(A) = \operatorname{pl}_{\bar{0}}(A) \oplus \operatorname{pl}_{\bar{1}}(A)$ denote the general linear Lie superalgebra of the $\mathbb{Z}_2$-graded space A. For $\varphi \in \operatorname{pl}_\theta(A)$ with $\theta \in \mathbb{Z}_2$, if

$$\varphi(xy) = \varphi(x)y + (-1)^{\theta \deg x} x\varphi(y)$$

for all $x \in \operatorname{hg}(A)$ and $y \in A$, then φ is called a derivation of A with $\mathbb{Z}_2$-degree θ. Let $\operatorname{Der}_\theta A$ denote the set of all derivations of A with $\mathbb{Z}_2$-degree θ. Then $\operatorname{Der} A = \operatorname{Der}_{\bar{0}} A \oplus \operatorname{Der}_{\bar{1}} A$ is a subalgebra of $\operatorname{pl}(A)$ (see [Scheunert 1979]), which is called the derivation superalgebra of A.

Set $\mathrm{Y} = \{1, 2, \ldots, n\}$. Given $i \in \mathrm{Y}$, let $\partial/\partial x_i$ be the partial derivative on $\Lambda(n)$ with respect to x_i. For $i \in Y$, let D_i be the linear transformation on $\mathcal{U}$ such that $D_i(x^u y^\lambda) = (\partial x^u/\partial x_i) y^\lambda$ for all $u \in \mathbb{B}(n)$ and $\lambda \in G$. Then $D_i \in \operatorname{Der}_{\bar{1}} \mathcal{U}$ for all $i \in \mathrm{Y}$ since $\partial/\partial x_i \in \operatorname{Der}_{\bar{1}}(\Lambda(n))$.

Suppose that $u \in \mathbb{B}_k \subseteq \mathbb{B}(n)$ and $i \in \mathrm{Y}$. When $i \in \{u\}$, we denote the uniquely determined element of $\mathbb{B}_{k-1}$ satisfying $\{u - \langle i\rangle\} = \{u\} \setminus \{i\}$ by $u - \langle i\rangle$, and denote the number of integers less than i in $\{u\}$ by $\tau(u, i)$. When $i \notin \{u\}$, we set $\tau(u, i) = 0$ and $x^{u-\langle i\rangle} = 0$. Therefore, $D_i(x^u) = (-1)^{\tau(u,i)} x^{u-\langle i\rangle}$ for any $i \in \mathrm{Y}$ and $u \in \mathbb{B}(n)$.

We define $(fD)(g) = fD(g)$ for $f, g \in \operatorname{hg}(\mathcal{U})$ and $D \in \operatorname{hg}(\operatorname{Der} \mathcal{U})$. Since the multiplication of $\mathcal{U}$ is supercommutative, it follows that fD is a derivation of $\mathcal{U}$. Let

$$W(n, m) = \operatorname{span}_{\mathbb{F}}\{x^u y^\lambda D_i \mid u \in \mathbb{B}(n), \lambda \in G, i \in \mathrm{Y}\}.$$

Then $W(n,m)$ is a finite-dimensional Lie superalgebra contained in $\operatorname{Der}\mathcal{U}$. A direct computation shows that

$$[fD_i, gD_j] = fD_i(g)D_j - (-1)^{\deg fD_i \deg gD_j} gD_j(f)D_i, \tag{2-1}$$

where $f, g \in \mathrm{hg}(\mathcal{U})$ and $i, j \in \mathrm{Y}$.

Let $D_H : \mathcal{U} \to W(n,m)$ be the linear mapping such that for every $f \in \mathrm{hg}(\mathcal{U})$, $D_H(f) = \sum_{i=1}^n f_i D_i$, where $f_i = (-1)^{\deg f} D_i(f)$. It is easy to see that D_H is an even linear mapping and $D_i(f_j) = -D_j(f_i)$ for all $i, j \in \mathrm{Y}$. Let $\overline{H}(n,m) = \{D_H(f) \mid f \in \mathcal{U}\}$ and $H(n,m) = \big\{D_H(f) \mid f \in \bigoplus_{i=0}^{n-1} \mathcal{U}_i i\big\}$. Then $H(n,m)$ is a finite-dimensional Hamiltonian Lie superalgebra, with a $\mathbb{Z}$-gradation $H(n,m) = \bigoplus_{i=-1}^{n-3} H_i(n,m)$, where $H_i(n,m) = \{D_H(x^u y^\lambda) \mid u \in \mathbb{B}(n), |u| = i+2, \lambda \in G\}$. It was shown in [Ren et al. 2011] that $H(n,m)$ is a subalgebra of $W(n,m)$ and that

$$[D_H(f), D_H(g)] = D_H\biggl(\sum_{i=1}^n (-1)^{\deg f} D_i(f) D_i(g)\biggr), \tag{2-2}$$

$$[D_j, D_H(f)] = D_H(D_j(f)), \tag{2-3}$$

where $f, g \in \mathrm{hg}(\mathcal{U})$ and $j \in Y$.

Let $\Theta := T(m)^m = T(m) \times \cdots \times T(m)$. For every $\theta = (h_1(y), \ldots, h_m(y)) \in \Theta$, we define $\tilde{\theta} \colon G \to T(m)$ by $\tilde{\theta}(\lambda) = \sum_{j=1}^m \lambda_j h_j(y)$ for $\lambda = \sum_{j=1}^m \lambda_j z_j \in G$. It is easy to check that $\tilde{\theta}(\lambda + \eta) = \tilde{\theta}(\lambda) + \tilde{\theta}(\eta)$ for $\lambda, \eta \in G$. For every $\theta \in \Theta$, let $D_\theta \colon H(n,m) \to H(n,m)$ be the linear mapping given by $D_\theta D_H(x^u y^\lambda) = \tilde{\theta}(\lambda) D_H(x^u y^\lambda)$ for $x^u y^\lambda \in \mathcal{U}$. A direct verification shows that $D_\theta \in \operatorname{Der}_{\bar{0}} H$ for all $\theta \in \Theta$. Put $\Omega := \{D_\theta \mid \theta \in \Theta\}$.

3. The natural filtration of $W(n,m)$

In this section, W always denotes Lie superalgebras $W(n,m)$. Then $W = \bigoplus_{k=-1}^{n-1} W_k$ is $\mathbb{Z}$-graded, where $W_k = \mathrm{span}_{\mathbb{F}}\{x^u y^\lambda D_j \mid |u| = k+1, j \in \mathrm{Y}\}$.

Adopting the notion of [Jin 1992], the element x of Lie superalgebra L is called ad-nilpotent if $\operatorname{ad} x$ is a nilpotent linear transformation. The set of all ad-nilpotent elements of L is denoted by $\mathrm{nil}(L)$. Let $L_{(j)} = \bigoplus_{k \geq j} L_k$; then $\{L_{(j)} \mid j \geq -1\}$ is the natural filtration of L. If L is $\mathbb{Z}$-graded and finite-dimensional, then $L_{-1} \subseteq \mathrm{nil}(L)$ and $L_{(1)} \subseteq \mathrm{nil}(L)$.

Let $M_n(\mathbb{F})$ denote the set of all $n \times n$ matrices over $\mathbb{F}$. Notice that $\dim T(m) = p^m$. Without loss of generality, we may suppose that $\{y_1, \ldots, y_{p^m}\}$ is a standard $\mathbb{F}$-basis of $T(m)$. If

$$z = \sum_{i,j=1}^n \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in W_0,$$

where $a_{ijq} \in \mathbb{F}$, let

$$\rho(z) = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_{p^m} \end{pmatrix}_{np^m \times np^m}$$

where $A_q = (a_{ijq})_{n\times n} \in M_n(\mathbb{F})$.

Lemma 3.1. *Suppose that $z = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in W_0$. If z is ad-nilpotent, then $\rho(z)$ is a nilpotent matrix.*

Proof. Let Γ be the representation of W_0 with values in W_{-1}. Then $\Gamma(z) = \operatorname{ad} z$ and the matrix of $\Gamma(z)$ over the basis $\{y_1 D_1, \ldots, y_1 D_n, \ldots, y_{p^m} D_1, \ldots, y_{p^m} D_n\}$ of W_{-1} is

$$A = \begin{pmatrix} -(A_1)^t & & \\ & \ddots & \\ & & -(A_{p^m})^t \end{pmatrix}_{np^m \times np^m}$$

where $A_q = (a_{ijq})_{n\times n} \in M_n(\mathbb{F})$. Since z is ad-nilpotent, the representation $\Gamma(z)$ is a nilpotent linear transformation. This implies that A is nilpotent. Therefore, $\rho(z) = -A^t$ is a nilpotent matrix. □

Lemma 3.2. *Let $z = \sum_{i=k}^{n-1} z_i$, where $z_i \in W_i$ and $k \leq n-1$. If $z \in \operatorname{nil}(W)$ and $k \geq 0$, then $z_k \in \operatorname{nil}(W)$.*

Proof. Suppose that $z = z_k + z'$, where $z_k \in W_k$ and $z' \in \bigoplus_{i=k+1}^{n-1} W_i \subseteq W_{(k+1)}$. Since $z \in \operatorname{nil}(W)$, we may assume that $(\operatorname{ad} z)^t = 0$. Let x is a $\mathbb{Z}$-homogeneous element of W with $\mathbb{Z}$-degree i. Then $(\operatorname{ad} z)^t(x) = 0$. On the other hand,

$$(\operatorname{ad} z)^t(x) = (\operatorname{ad}(z_k + z'))^t(x) = (\operatorname{ad} z_k)^t(x) + h,$$

which implies $(\operatorname{ad} z_k)^t(x) + h = 0$. It is easy to see that $(\operatorname{ad} z_k)^t(x) \in W_{(kt+i)}$ and $h \in W_{(kt+i+1)} = \bigoplus_{j \geq kt+i+1} W_j$. Thus $(\operatorname{ad} z_k)^t(x) = 0$. Since x is an arbitrary $\mathbb{Z}$-homogeneous element of W, we have $(\operatorname{ad} z_k)^t(W) = 0$. Then $(\operatorname{ad} z_k)^t = 0$, that is, $z_k \in \operatorname{nil}(W)$. □

Suppose that E_{ij} denotes the $n \times n$ matrix whose (i, j) element is 1 and otherwise are zero. Obviously,

$$E_{ij} E_{kl} = \delta_{jk} E_{il}, \tag{3-1}$$

where δ_{jk} is the Kronecker delta.

If $z = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j \in W_0$, where $a_{ijq} \in \mathbb{F}$, then

$$\rho(z) = \sum_{i,j=1}^{n} a_{ij1} E_{ij} + \sum_{i,j=n+1}^{2n} a_{ij2} E_{ij} + \cdots + \sum_{i,j=n(p^m-1)+1}^{np^m} a_{ijp^m} E_{ij}. \tag{3-2}$$

Let $\Delta = \{z \in \operatorname{nil}(W) \mid \operatorname{ad} z(W) \subseteq \operatorname{nil}(W)\}$.

Lemma 3.3. *Suppose that $z=\sum_{i=-1}^{n-1} z_i$, where $z_i \in W_i$. If $z \in \Delta$, then $z_{-1}=z_0=0$.*

Proof. Suppose that $0 \neq z_{-1} = \sum_{i=1}^{n}\sum_{q=1}^{p^m} a_{iq} y_q D_i$, where $a_{iq} \in \mathbb{F}$. Let $a_{jq} \neq 0$ and $j, l \in \mathrm{Y}$ such that j, l are distinct. We may assume that $d=[z_{-1}, x_l x_j D_l]$. A direct calculation shows that

$$d=\Bigl[\sum_{i=1}^{n}\sum_{q=1}^{p^m} a_{iq} y_q D_i,\, x_l x_j D_l\Bigr]=\sum_{q=1}^{p^m}(a_{lq}x_j y_q D_l - a_{jq} x_l y_q D_l).$$

By (3-1) and (3-2), we have

$$\begin{aligned}(\rho(d))^t &= (-1)^t (a_{j1})^t E_{ll} + (-1)^{t-1} a_{l1}(a_{j1})^{t-1} E_{jl}\\ &\quad + (-1)^t (a_{(j+n)2})^t E_{(l+n)(l+n)} + (-1)^{t-1} a_{(l+n)2}(a_{(j+n)1})^{t-1} E_{(j+n)(l+n)}\\ &\quad + \cdots\\ &\quad + (-1)^t (a_{(j+p^m-n)p^m})^t E_{(l+p^m-n)(l+p^m-n)}\\ &\quad + (-1)^{t-1} a_{(l+p^m-n)p^m}(a_{(j+p^m-n)p^m})^{t-1} E_{(j+p^m-n)(l+p^m-n)}.\end{aligned}$$

Since $(a_{j1})^t \neq 0$, we have $(\rho(d))^t \neq 0$. So $\rho(d)$ is not a nilpotent matrix. By Lemma 3.1, it shows that $d \notin \mathrm{nil}(W)$. By Lemma 3.2, we have $[z, x_l x_j D_l] \notin \mathrm{nil}(W)$. Then $z \notin \Delta$. This contradicts $z \in \Delta$, and proves our assertion that $z_{-1}=0$.

Assume that $z_0 \neq 0$. Let $z_0=\sum_{i,j=1}^{n}\sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j$, $a_{ijq} \in \mathbb{F}$ and

$$l=\min\{i \mid a_{ij\lambda} \neq 0,\ i, j \in \mathrm{Y}\}, \tag{3-3}$$

$$t=\min\{j \mid a_{ij\lambda} \neq 0,\ i, j \in \mathrm{Y}\}. \tag{3-4}$$

(i) Suppose that $l \leq t$. Let

$$k=\max\{j \mid a_{lj\lambda} \neq 0,\ j \in \mathrm{Y}\}. \tag{3-5}$$

Then $a_{lkq} \neq 0$. It is easy to see that $t \leq k$. Since $l \leq t$, we have $l \leq k$. Therefore,

$$z_0=\sum_{j=t}^{k}\sum_{q=1}^{p^m} a_{ljq} x_l y_q D_j + \sum_{i=l+1}^{n}\sum_{j=t}^{n}\sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j.$$

Assume that $l=k$. It follows from $t \leq k$ that $t \leq l$. Then $t=l$, which implies that

$$z_0=\sum_{q=1}^{p^m} a_{llq} x_l y_q D_l + \sum_{i=l+1}^{n}\sum_{j=t}^{n}\sum_{q=1}^{p^m} a_{ijq} x_i y_q D_j.$$

Therefore,

$$\begin{aligned}\rho(z_0) &= a_{ll1}E_{ll} + \sum_{i=l+1}^{n}\sum_{j=t}^{n} a_{ij1}E_{ij}\\ &\quad + a_{(l+n)(l+n)2}E_{(l+n)(l+n)} + \sum_{i=l+1+n}^{2n}\sum_{j=t+n}^{2n} a_{ij2}E_{ij}\\ &\quad + \cdots\\ &\quad + a_{(l+n(p^m-1))(l+n(p^m-1))p^m}E_{(l+n)(l+n)}\\ &\quad + \sum_{i=l+1+n(p^m-1)}^{np^m}\sum_{j=t+n(p^m-1)}^{np^m} a_{ijp^m}E_{ij}\\ &= \begin{pmatrix} \begin{smallmatrix} A_1 & \\ B_1 & C_1\end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} A_{p^m} & \\ B_{p^m} & C_{p^m}\end{smallmatrix}\end{pmatrix}_{np^m\times np^m},\end{aligned}$$

where $A_k = a_{(l+(k-1)n)(l+(k-1)n)q}E_{(l+(k-1)n)(l+(k-1)n)}$ is an $(l+(k-1)n)$-by-$(l+(k-1)n)$ matrix and $q \in \{1, \dots, p^m\}$. Since $a_{ll1} \neq 0$, we have A_1 is not a nilpotent matrix. Then $\rho(z_0)$ is not a nilpotent matrix and $z_0 \notin \mathrm{nil}(W)$. Lemma 3.2 shows that $z \notin \mathrm{nil}(W)$. This is in contradiction with $z \in \Delta$; thus $l < k$.

Suppose that $h \in \mathrm{Y}$ and $h \neq l, k$. Let $d = [z_0, x_k D_l]$. From (2-1), we obtain

$$d = \sum_{q=1}^{p^m}\Bigl(a_{lkq}x_l y_q D_l + \sum_{i=l+1}^{n} a_{ikq}x_i y_q D_l - \sum_{j=t}^{k} a_{ljq}x_k y_q D_j\Bigr).$$

Since $l < k$, $\rho(d)$ also has the form

$$\begin{pmatrix} \begin{smallmatrix} A_1 & \\ B_1 & C_1\end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} A_{p^m} & \\ B_{p^m} & C_{p^m}\end{smallmatrix}\end{pmatrix}_{np^m\times np^m}.$$

It follows from $a_{lkq} \neq 0$ that A_1 is not a nilpotent matrix. Then $\rho(d)$ is not nilpotent. So $z \notin \mathrm{nil}(W)$ and $[z, x_k D_l] \notin \mathrm{nil}(W)$. This is in contradiction with $z \in \Delta$.

(ii) Suppose that $t < l$. Let $k = \max\{i \mid a_{it\lambda} \neq 0\}$ and $d' = [z, x_t D_k]$. Imitating (i), we may prove that $\rho(d')$ is also not nilpotent. The desired result follows. □

Lemma 3.4. (i) *If $z \in W_0 \cap \mathrm{nil}(W)$ and $h \in W_{(1)}$, then $z + h \in \mathrm{nil}(W)$.*

(ii) *If i, j are distinct elements of Y, then $x_i y^\lambda D_j \in \mathrm{nil}(W)$ for all $\lambda \in G$.*

(iii) *If* i, j, k *are distinct elements of* Υ, *then* $ax_jy^\lambda D_k + bx_iy^\eta D_k \in \mathrm{nil}(W)$ *and* $x_ix_jy^\lambda D_k \in \Delta$, *where* $a, b \in \mathbb{F}$ *and* λ, η *are arbitrary elements of* G.

Proof. (i) A direct verification shows that $\{\mathrm{ad}\, z\} \cup \{\mathrm{ad}\, W_{(1)}\}$ is a weakly closed subset of nilpotent elements of $\mathrm{pl}(W)$. It was shown in [Jacobson 1962, Theorem 1 of Chapter II] that each element of $\mathrm{span}_{\mathbb{F}}(\{\mathrm{ad}\, z\} \cup \{\mathrm{ad}\, W_{(1)}\})$ is a nilpotent linear transformation of W. Then $\mathrm{ad}\, z + \mathrm{ad}\, h$ is nilpotent. So $z + h$ is ad-nilpotent.

(ii) To prove $(\mathrm{ad}\, x_iy^\lambda D_j)^p = 0$, we may assume without loss of generality that $i < j$. Set η is an arbitrary element of G. If $k \neq i$, then

$$\begin{aligned}(\mathrm{ad}\, x_iy^\lambda D_j)^2(x^uy^\eta D_k) &= [x_iy^\lambda D_j, [x_iy^\lambda D_j, x^uy^\eta D_k]]\\ &= (-1)^{\tau(u,j)}[x_iy^\lambda D_j, x_ix^{u-\langle j\rangle}y^{\lambda+\eta}D_k]\\ &= 0.\end{aligned}$$

In the case of $k = i$, we have

$$\begin{aligned}(\mathrm{ad} x_iy^\lambda D_j)^3(x^uy^\eta D_k) &= [x_iy^\lambda D_j, [x_iy^\lambda D_j, [x_iy^\lambda D_j, x^uy^\eta D_i]]]\\ &= [x_iy^\lambda D_j, [x_iy^\lambda D_j, (-1)^{\tau(u,j)}x_ix^{u-\langle j\rangle}y^\lambda D_i - x^uy^{\lambda+\eta}D_j]]\\ &= (-1)^{\tau(u,j)}[x_iy^\lambda D_j, -x_ix^{u-\langle j\rangle}y^\lambda D_j - x_ix^{u-\langle j\rangle}y^{2\lambda+\eta}D_j]\\ &= 0.\end{aligned}$$

For $p > 2$ we get $(\mathrm{ad}\, x_iy^\lambda D_j)^p(x^uy^\eta D_k) = 0$. Therefore $(\mathrm{ad}\, x_iy^\lambda D_j)^p(W) = 0$. This yields $(\mathrm{ad}\, x_iy^\lambda D_j)^p = 0$. Thus $x_iy^\lambda D_j \in \mathrm{nil}(W)$.

(iii) According to (ii) and $[x_jy^\lambda D_k, x_iy^\eta D_k] = 0$, $\{\mathrm{ad}\, x_jy^\lambda D_k, \mathrm{ad}\, x_iy^\eta D_k\}$ is a weakly closed subset of nilpotent elements of pl(W). So $ax_jy^\lambda D_k + bx_iy^\eta D_k \in \mathrm{nil}(W)$, where $a, b \in \mathbb{F}$.

Suppose that $l \in \Upsilon\backslash\{i, j, k\}$. Then $x_ix_jy^\lambda D_k \in W_{(1)} \subseteq \mathrm{nil}(W)$. Let $z = \sum_{i=-1}^{n-1} z_i$, where $z_i \in W_i$. Without loss, we may assume that $[x_ix_jy^\lambda D_k, z] = f_0 + f_1$, where $f_0 = [x_ix_jy^\lambda D_k, z_{-1}] \in W_0$ and $f_1 \in W_{(1)}$. Let $z_{-1} = \sum_{l=1}^{n}\sum_{\eta\in G} a_{l\eta}y^\eta D_l$. Then

$$f_0 = [x_ix_jy^\lambda D_k, \sum_{l=1}^{n}\sum_{\eta\in G} a_{l\eta}y^\eta D_l] = -\sum_{\eta\in G}(a_{i\eta}x_jy^{\lambda+\eta}D_k + a_{j\eta}x_iy^{\lambda+\eta}D_k).$$

It follows that $f_0 \in W_0 \cap \mathrm{nil}(W)$. Statement (i) shows that $f_0 + f_1 \in \mathrm{nil}(W)$. We finally obtain $x_ix_jy^\lambda D_k \in \Delta$ for all $\lambda \in G$. □

Let $Q = \{z \in \mathrm{nil}(W) \mid \mathrm{ad}\, z(\Delta) \subseteq \Delta\}$.

Lemma 3.5. $$Q = W_{(1)}.$$

Proof. By the definition of Δ, we have $W_{(2)} \subseteq \Delta$. Lemma 3.3 show that $\Delta \subseteq W_{(1)}$. Then $[W_{(1)}, \Delta] \subseteq [W_{(1)}, W_{(1)}] \subseteq W_{(2)} \subseteq \Delta$. Thus $W_{(1)} \subseteq Q$.

Next we will prove $Q \subseteq W_{(1)}$. Let $z \in Q$ and $z = \sum_{i=-1}^{n-1} z_i$, where $z_i \in W_i$. Assume that $z_{-1} = \sum_{l=1}^{n} \sum_{\lambda\in G} a_{l\lambda} y^{\lambda} D_l \neq 0$, $a_{l\lambda} \in \mathbb{F}$. Without loss of generality, we may suppose that $a_i \neq 0$. Let $d = x_i x_j y^{\eta} D_k$, where i, j, k are distinct elements of Y and η is an arbitrary element of G. By Lemma 3.4 (iii), we have $d \in \Delta$. Let $[z, d] = h_0 + h_1$, where $h_0 = [z_{-1}, d] \in W_0$ and $h_1 \in W_{(1)}$. Since $a_i \neq 0$, we have $h_0 = \sum_{\lambda\in G}(a_{i\lambda} x_j y^{\lambda+\eta} D_k - a_{j\lambda} x_i y^{\lambda+\eta} D_k) \neq 0$. Lemma 3.3 implies that $h_0 + h_1 \notin \Delta$. It is a contradiction to $z \in Q$. Hence $z_{-1} = 0$.

Assume that $0 \neq z_0 = \sum_{i,j=1}^{n} \sum_{q=1}^{p^m} a_{ij\lambda} x_i y_q D_j$, $a_{ijq} \in \mathbb{F}$ and suppose that l and t are as the definitions in (3-3) and (3-4). We may suppose that $l \leq t$ (the proof is similar to the case $t < l$) and let k be as definition in (3-5). Similar to the first part of the proof in Lemma 3.3, we have $l < k$. Suppose that $h \in \mathrm{Y} \setminus \{l, k, t\}$ and $d_1 = x_k x_h D_l$. Lemma 3.4 (iii) shows that $d_1 \in \Delta$. Let $[z, d_1] = g_1 + g_2$, where $g_1 = [z_0, d_1] \in W_1$ and $g_2 \in W_{(2)}$. Using (2-1), we have

$$g_1 = \sum_{q=1}^{p^m}\left(a_{lkq} x_l x_h y_q D_l - \sum_{i=l+1}^{n} a_{ihq} x_i x_k y_q D_l - \sum_{j=t}^{k} a_{ljq} x_k x_h y_q D_j\right).$$

If $h < t$, then $a_{ihq} = 0$ in the above equality, where $i \in \mathrm{Y} \setminus \{1, \dots, l-1\}$. Thus

$$[D_h, g_1] = -\sum_{q=1}^{p^m}\left(a_{lkq} x_l y_q D_l + \sum_{i=l+1}^{n} a_{ihq} x_i y_q D_l + a_{hhq} x_k y_q D_l - a_{ljq} x_k y_q D_j\right).$$

By (3-2), the matrix $\rho([D_h, g_1])$ has the form

$$\begin{pmatrix} \begin{matrix} A_1 & \\ B_1 & C_1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} A_{p^m} & \\ B_{p^m} & C_{p^m} \end{matrix} \end{pmatrix}_{np^m \times np^m}$$

as in Lemma 3.3. Since $a_{lkq} \neq 0$, A_1 is not a nilpotent matrix. This implies that $\rho([D_h, g_1])$ is not nilpotent. Hence $[D_h, g_1] \notin \mathrm{nil}(W)$. Lemma 3.2 shows that $[D_h, g_1 + g_2] \notin \mathrm{nil}(W)$, that is, $[D_h, g_1 + g_2] \notin \Delta$. It is contradict with $z \in Q$. Thus $z_0 = 0$. Therefore, $z \in W_{(1)}$ and $Q \subseteq W_{(1)}$. □

It is easy to verify that Δ and Q are invariant subspaces under the automorphisms of W. According to Lemma 3.5, $W_{(1)}$ is also invariant under the automorphisms of W. Since

$$W_{(0)} = \{x \in W \mid [x, W_{(1)}] \subseteq W_{(1)}\}, \tag{3-6}$$

$$W_{(i)} = \{x \in W_{i-1} \mid [x, W] \subseteq W_{(i-1)}\}, \quad i \geq 1, \tag{3-7}$$

we easily obtain the following theorem.

Theorem 3.6. *The natural filtration of W is invariant under automorphisms of W.*

Let $\mathfrak{W}_i = W_{(i)}/W_{(i+1)}$ for $-1 \le i \le n-1$. Then $\mathfrak{W}_i$ is a $\mathbb{Z}$-graded space. Suppose that $\mathfrak{W} := \bigoplus_{i=-1}^{n-1} \mathfrak{W}_i$; then $\mathfrak{W}$ is also a $\mathbb{Z}$-graded space. Let $x + W_{(i+1)} \in \mathfrak{W}_i$ and $y + W_{(j+1)} \in \mathfrak{W}_j$. Define

$$[x + W_{(i+1)}, y + W_{(j+1)}] := [x, y] + W_{(i+j+1)}.$$

It follows from $[\mathfrak{W}_i, \mathfrak{W}_j] \subseteq \mathfrak{W}_{i+j}$ that the operator [,] on $\mathfrak{W}$ is well-defined. There exists a linear expansion such that $\mathfrak{W}$ has a operator [,]. A direct verification shows that $\mathfrak{W}$ is a Lie superalgebra with respect to the operator [,]. The Lie superalgebras $\mathfrak{W}$ is called a Lie superalgebra induced by the natural filtration of W.

Lemma 3.7. $$\mathfrak{W} \cong W.$$

Proof. Let $\phi\colon W \to \mathfrak{W}$ is a linear map such that $\phi(x) = x + W_{(i+1)}$, where $x \in W_{(i)} \setminus W_{(i+1)}$. A direct verification shows that ϕ is a homomorphism of Lie superalgebras. Suppose that $y \in \ker\phi$. If $y \neq 0$, then there exists $i \ge -1$ such that $y \in W_{(i)} \setminus W_{(i+1)}$. Since $\phi(y) = 0$, we have $y + W_{(i+1)} = 0$. Hence $y \in W_{(i+1)}$. That shows that $y = 0$. Thus, $\ker\phi = 0$. Therefore, ϕ is a monomorphism. It follows from W is finite-dimensional that ϕ is an isomorphism. □

The definition of ϕ shows that, for $i \ge -1$

$$\begin{aligned} \phi(W_i) &= \{x + W_{(i+1)} \mid x \in W_i\} = \{x + W_{(i+1)} \mid x \in W_{(i)}\} \\ &= W_{(i)}/W_{(i+1)} = \mathfrak{W}_i. \end{aligned} \tag{3-8}$$

Suppose that m, n, m', n' are elements of $\mathbb{N}$ greater than 1. Similar to W, the Lie superalgebra $W(n', m')$ will be simply denoted by W'. According to the definitions of Δ, Q, and $\mathfrak{W}$ in W, we define Δ', Q', and $\mathfrak{W}'$ in W' using the same method.

Proposition 3.8. *Suppose that $W \cong W'$ and σ is an isomorphism from W to W'. Then $\sigma(W_{(i)}) = W'_{(i)}$ for all $i \ge -1$.*

Proof. It is clear that $\sigma(W_{(-1)}) = W'_{(-1)}$ and $\sigma(\mathrm{nil}(W)) = \mathrm{nil}(W')$. A direct verification shows that $\sigma(\Delta) = \Delta'$. Hence $\sigma(Q) = Q'$. By virtue of Lemma 3.5, we have $Q = W_{(1)}$ and $Q' = W'_{(1)}$. Thus $\sigma(W_{(1)}) = W'_{(1)}$. By (3-6) and (3-7), the desired result $\sigma(W_{(i)}) = W'_{(i)}$ for all $i \ge -1$ is obtained. □

Lemma 3.9. *Suppose that $W \cong W'$ and σ is an isomorphism from W to W'. Then σ induces an isomorphism $\tilde{\sigma}$ from $\mathfrak{W}$ to $\mathfrak{W}'$ such that $\tilde{\sigma}(\mathfrak{W}_i) = \mathfrak{W}'_i$ for all $i \ge -1$.*

Proof. Define a linear map $\tilde{\sigma} : \mathfrak{W} \to \mathfrak{W}'$ such that

$$\tilde{\sigma}(x + W_{(i+1)}) = \sigma(x) + W'_{(i+1)},$$

where $x + W_{(i+1)} \in \mathfrak{W}_i$. Because of Proposition 3.8, the definition of $\tilde{\sigma}$ is reasonable and

$$\begin{aligned}\tilde{\sigma}([x+W_{(i+1)}, y+W_{(j+1)}]) &= \sigma([x,y]) + W'_{(i+j+1)} \\ &= [\sigma(x)+W'_{(i+1)}, \sigma(y)+W'_{(j+1)}] \\ &= [\tilde{\sigma}(x+W'_{(i+1)}), \tilde{\sigma}(y+W'_{(j+1)})].\end{aligned}$$

Thus $\tilde{\sigma}$ is a homomorphism from $\mathfrak{W}$ to $\mathfrak{W}'$. Clearly, $\tilde{\sigma}(\mathfrak{W}_i) = \mathfrak{W}'_i$ for all $i \geq -1$. It shows that $\tilde{\sigma}$ is an epimorphism.

Suppose that $y \in \ker\tilde{\sigma}$; then $y \in \mathfrak{W}$. So we may suppose that $y = \sum_{i=-1}^{n-1} y_i$ and $y_i \in \mathfrak{W}_i$. Since $\mathfrak{W}_i = W_{(i)}/W_{(i+1)}$, let $y_i = z_i + W_{(i+1)}$, where $z_i \in W_{(i)}$. Hence $\tilde{\sigma}(y_i) = \sigma(z_i) + W'_{(i+1)}$. It follows from $\tilde{\sigma}(y) = 0$ that $\sum_{i=-1}^{n-1} \tilde{\sigma}(y_i) = 0$. Thus $\tilde{\sigma}(y_i) = 0$, that is, $\sigma(z_i) + W'_{(i+1)} = 0$. This shows $\sigma(z_i) \in W'_{(i+1)}$. By Proposition 3.8, we have $z_i \in \sigma^{-1}(W'_{(i+1)}) = W_{(i+1)}$. Then $y_i = z_i + W_{(i+1)} = 0$ for $-1 \leq i \leq n-1$. Therefore, $y = 0$ and $\ker\tilde{\sigma} = 0$. Consequently, $\tilde{\sigma}$ is an isomorphism induced by σ such that $\tilde{\sigma}(\mathfrak{W}_i) = \mathfrak{W}'_i$ for all $i \geq -1$. □

Theorem 3.10. *$W \cong W'$ if and only if $m = m'$ and $n = n'$.*

Proof. Since the sufficiency is obvious, it suffices to prove the necessity. Suppose that $\phi\colon W \to \mathfrak{W}$ is the isomorphism given in the proof of Lemma 3.7. Similarly, there also exists the $\phi'\colon W' \to \mathfrak{W}'$. According to (3-8) and Lemma 3.9, we have

$$\phi(W_i) = \mathfrak{W}_i, \quad \phi'(W'_i) = \mathfrak{W}'_i, \quad \tilde{\sigma}(\mathfrak{W}_i) = \mathfrak{W}'_i$$

for $-1 \leq i \leq n-1$. Let $\psi = (\phi')^{-1}\tilde{\sigma}\phi$. Then

$$\psi(W_i) = (\phi')^{-1}\tilde{\sigma}\phi(W_i) = (\phi')^{-1}\tilde{\sigma}(\mathfrak{W}_i) = (\phi')^{-1}(\mathfrak{W}'_i) = W_i.$$

In particular, $\psi(W_{-1}) = W'_{-1}$ and $\psi(W_0) = W'_0$. Since $\dim W_{-1} = \dim W'_{-1}$, we get $np^m = n'p^{m'}$. By virtue of the definition of W_i, we have

$$W_0 = \mathrm{span}_{\mathbb{F}}\{x_i y^{\lambda} D_j \in W \mid i, j \in \mathrm{Y}, \lambda \in G\}.$$

Thus $\dim W_0 = n^2 p^m$. By the same method used in W_0, we may obtain $\dim W'_0 = n'^2 p^{m'}$. According to $\dim W_0 = \dim W'_0$ and $np^m = n'p^{m'}$, we have $n = n'$ and $m = m'$. In conclusion, the proof is completed. □

4. The natural filtration of $H(n, m)$

In this section we will investigate the question of the natural filtration of the Lie superalgebras $H(n, m)$. For convenience, $H(n, m)$, $\overline{H}(n, m)$ and $H_i(n, m)$ will be simply denoted by H, $\overline{H}$ and H_i.

Let $H_{(j)} = \bigoplus_{i \geq j} H_i$. Then

$$H = H_{(-1)} \supseteq H_{(0)} \supseteq H_{(1)} \supseteq \cdots \supseteq H_{(n-3)} \supseteq H_{(n-2)} = 0$$

is a descending filtration of H, which is called the natural filtration of H. We also call $\{H_{(k)} \mid k \in \mathbb{Z}\}$ a filtration of H for short, where $H_{(k)} = H$ if $k \leq -1$ and $H_{(k)} = 0$ if $k \geq n-2$.

Lemma 4.1. *Let $f_i = g_i + h_i$, where $f_i, g_i, h_i \in \mathcal{U}$ and $i = 1, \ldots, k$. If the set $\{g_i \mid i = 1, \ldots, k\}$ is linearly independent and*

$$\operatorname{span}_{\mathbb{F}}\{g_i \mid i = 1, \ldots, k\} \cap \operatorname{span}_{\mathbb{F}}\{h_i \mid i = 1, \ldots, k\} = 0,$$

then $\{f_i \mid i = 1, \ldots, k\}$ is linearly independent.

Proof. If $\sum_{i=1}^{k} a_i f_i = 0$, $a_i \in \mathbb{F}$, then $\sum_{i=1}^{k} a_i g_i = -\sum_{i=1}^{k} a_i h_i$. This shows that

$$\sum_{i=1}^{k} a_i g_i \in \operatorname{span}_{\mathbb{F}}\{g_i \mid i = 1, \ldots, k\} \cap \operatorname{span}_{\mathbb{F}}\{h_i \mid i = 1, \ldots, k\} = 0.$$

Since $\{g_i \mid i = 1, \ldots, k\}$ is linearly independent, we obtain $a_i = 0$, $i = 1, \ldots, k$. $\square$

Lemma 4.2. *If $h_1, h_2, \ldots, h_k \in H \setminus \{0\}$. If $\{h_i \mid i = 1, \ldots, k\}$ is linearly dependent, then so is $\{\mu(h_i) \mid i = 1, \ldots, k\}$.*

Proof. Since $\{h_i \mid i = 1, \ldots, k\}$ is linearly dependent, there exist $a_1, \ldots, a_k \in \mathbb{F}$ such that $\sum_{i=1}^{k} a_i h_i = 0$ and some a_i is not zero. We may suppose that $a_1, \ldots, a_s \neq 0$ and $a_{s+1} = \cdots = a_k = 0$, where $1 \leq s \leq k$. Let

$$\varepsilon = \min\{\operatorname{zd}(\mu(h_i)) \mid i = 1, \ldots, s\}.$$

Without loss of generality, we may suppose that $\operatorname{zd}(\mu(h_i)) = \varepsilon$ for $i = 1, \ldots, t$ and $\operatorname{zd}(\mu(h_i)) > \varepsilon$ for $i = t+1, \ldots, s$. It follows from $\sum_{i=1}^{k} a_i h_i = 0$ that $\sum_{i=1}^{k} a_i \mu(h_i) = 0$. Since $a_1, \ldots, a_t \neq 0$, we obtain that $\{\mu(h_i) \mid i = 1, \ldots, t\}$ is linearly dependent. Hence so is $\{\mu(h_i) \mid i = 1, \ldots, k\}$. $\square$

Lemma 4.3. *Let $g_1, g_2, \ldots, g_k \in \mathcal{U}$. If $\operatorname{zd}(\mu(g_i)) \geq 1$, $i = 1, \ldots, k$, then $\{g_i \mid i = 1, \ldots, k\}$ is linearly dependent if and only if $\{D_H(g_i) \mid i = 1, \ldots, k\}$ is.*

Proof. If $\{g_i \mid i = 1, \ldots, k\}$ is linearly dependent, there exist $a_1, \ldots, a_k \in \mathbb{F}$, not all zero, such that $\sum_{i=1}^{k} a_i g_i = 0$. Clearly, $D_H\big(\sum_{i=1}^{k} a_i g_i\big) = \sum_{i=1}^{k} a_i D_H(g_i) = 0$. Hence $\{D_H(g_i) \mid i = 1, \ldots, k\}$ is linearly dependent.

Conversely, we consider the sufficiency. Without loss of generality, we may suppose that $g = x^u y^\lambda$ for $u \in \mathbb{B}(n)$ and $\lambda \in G$ such that $D_H(g) = 0$. Then

$$D_H(x^u y^\lambda) = \sum_{i=1}^{n} (-1)^{|u|} D_i(x^u) y^\lambda D_i.$$

Hence $D_i(x^u) = 0$, which shows that $|u| = 0$. Thus $\ker(D_H) = \mathbb{T}(m)$. Since the set $\{D_H(g_i) \mid i = 1, \ldots, k\}$ is linearly dependent, there exist $a_1, \ldots, a_k \in \mathbb{F}$, not all zero, such that $\sum_{i=1}^{k} a_i D_H(g_i) = 0$. Then $D_H\left(\sum_{i=1}^{k} a_i g_i\right) = 0$. Hence $\sum_{i=1}^{k} a_i g_i \in \mathbb{T}(m)$. Notice that $\mathrm{zd}(\mu(g_i)) \geq 1$, $i = 1, \ldots, k$; thus $\sum_{i=1}^{k} a_i g_i = 0$, showing that $\{g_i \mid i = 1, \ldots, k\}$ is linearly dependent. $\square$

For a superderivation D of a Lie superalgebra L. Set $I(D) = \dim(\mathrm{Im}(D))$. If T is a subset of superderivations of L, we define $I(T) = \min\{I(D) \mid 0 \neq D \in T\}$.

Theorem 4.4. *Suppose that* $T = \mathrm{ad}(\mathrm{hg}(\overline{H}))|_H$, *then* $I(T) \geq np^m$. *Besides,* $I(\mathrm{ad}\, D_H(g)) = np^m$ *if and only if* $0 \neq D_H(g) \in \mathrm{span}_{\mathbb{F}}\{D_H(x^\pi y^\lambda) \mid \lambda \in G\}$, *where* $D_H(g) \in \mathrm{hg}(\overline{H})$.

Proof. For any $h \in \mathrm{hg}(\overline{H})$ we write $\mathrm{ad}\, h|_H$ simply as $\mathrm{ad}\, h$. A direct calculation shows that

$$[D_H(x^\pi y^\lambda), D_H(x^v y^\eta)] = D_H\left(\sum_{i=1}^{n} (-1)^n D_i(x^\pi y^\lambda) D_i(x^v y^\eta)\right)$$

for $v \in \mathbb{B}(n)$ and $\lambda, \eta \in G$.

In the case of $|v| \geq 2$ we have

$$D_i(x^v y^\eta) = (-1)^{\tau(v,i)} x^{v-\langle i \rangle} y^\eta, \quad D_i(x^\pi y^\lambda) = (-1)^{\tau(\pi,i)} x^{\pi-\langle i \rangle} y^\lambda.$$

Clearly, $\{v - \langle i \rangle\} \in \{\pi - \langle i \rangle\}$. Then $[D_H(x^\pi y^\lambda), D_H(x^v y^\eta)] = 0$ in this case.

In the case of $|v| = 1$ we may suppose that $x^v y^\eta = x_i y^\eta$ for some $i \in \mathrm{Y}$. Then

$$\begin{aligned}[D_H(x^\pi y^\lambda), D_H(x_i y^\eta)] &= D_H\left(\sum_{j=1}^{n} (-1)^n D_j(x^\pi y^\lambda) D_j(x_i y^\eta)\right) \\ &= D_H\big((-1)^{n+\tau(\pi,i)} x^{\pi-\langle i \rangle} y^{\lambda+\eta}\big).\end{aligned}$$

Since $\{x^{\pi-\langle i \rangle} y^{\lambda+\eta} \mid i \in \mathrm{Y}, \lambda, \eta \in G\}$ is a linearly independent set, Lemma 4.3 shows that $\{[D_H(x^\pi y^\lambda), D_H(x_i y^\eta)] \mid i \in \mathrm{Y}, \lambda, \eta \in G\}$ is linearly independent. Thus $I(\mathrm{ad}\, D_H(g)) = np^m$.

Next we will consider the converse inclusion. Assume that $D_H(g_0) \in \mathrm{hg}(\overline{H})$ and $D_H(g_0) \notin \mathrm{span}_{\mathbb{F}}\{D_H(x^\pi y^\lambda) \mid \lambda \in G\}$. We want to prove that $I(\mathrm{ad}\, D_H(g_0)) > np^m$. Suppose that $\mu(D_H(g_0)) = D_H(g)$. By Lemma 4.2, it suffices to prove that $I(\mathrm{ad}\, D_H(g)) > np^m$.

Let $g = x^u y^\lambda$, where $u \in \mathbb{B}(n)$ and $\lambda \in G$. Then $1 \leq |u| < n$. There exist $v \in \mathbb{B}(n)$ and $\eta \in G$ such that $D_H(x^v y^\eta) \in H$. Then

$$[D_H(x^u y^\lambda), D_H(x^v y^\eta)] = D_H\left(\sum_{i=1}^{n} (-1)^{|u|} D_i(x^u y^\lambda) D_i(x^v y^\eta)\right).$$

(1) Suppose that $|u| = 1$ and $x^u y^\lambda = x_i y^\lambda$ for some $i \in \mathrm{Y}$, then

$$[D_H(x_i y^\lambda), D_H(x^v y^\eta)] = -(-1)^{\tau(v,i)} D_H\left(x^{v-\langle i \rangle} y^{\lambda+\eta}\right).$$

Considering $x^{v-\langle i\rangle}$, the following statements hold:

$$\begin{cases} D_H(x^{v-\langle i\rangle}y^{\lambda+\eta})=0 & \text{if } |v|=0 \text{ or } 1,\\ \dim\{x^{v-\langle i\rangle}\}=\mathrm{C}_{n-1}^{1} & \text{if } |v|=2,\\ \dim\{x^{v-\langle i\rangle}\}=\mathrm{C}_{n-1}^{2} & \text{if } |v|=3,\\ \qquad\vdots & \\ \dim\{x^{v-\langle i\rangle}\}=\mathrm{C}_{n-1}^{n-1} & \text{if } |v|=n.\end{cases}$$

Therefore,

$$\dim\{x^{v-\langle i\rangle}y^{\lambda+\eta}\}=(\mathrm{C}_{n-1}^{1}+\mathrm{C}_{n-1}^{2}+\cdots+\mathrm{C}_{n-1}^{n-1})p^m=(2^{n-1}-1)p^m>np^m.$$

(2) If $1<|u|<n$, then we suppose that $|u|=l$.

For $|v|=2$ we may suppose that $x^v y^\eta=x_jx_ky^\eta$, where j,k are distinct elements of Υ and η is an arbitrary element of G. A direct calculation shows that $[D_H(x^uy^\lambda),D_H(x_jx_ky^\eta)]$ equals

$$D_H\left((-1)^{|u|}((-1)^{\tau(u,j)}x^{u-\langle j\rangle}x_k-(-1)^{\tau(u,k)}x^{u-\langle k\rangle}x_j)y^{\lambda+\eta}\right).$$

Consider $\Upsilon=(-1)^{\tau(u,j)}x^{u-\langle j\rangle}x_k-(-1)^{\tau(u,k)}x^{u-\langle k\rangle}x_j$. Then the following statements hold:

$$\begin{cases}\Upsilon=0 & \text{if } j,k\in u \text{ or } j,k\notin u,\\ \Upsilon=(-1)^{\tau(u,j)}x^{(u\setminus\{j\})\cup\{k\}} & \text{if } j\in u \text{ and } k\notin u,\\ \Upsilon=(-1)^{\tau(u,k)}x^{(u\setminus\{k\})\cup\{j\}} & \text{if } k\in u \text{ and } j\notin u.\end{cases}$$

Thus $\dim\Upsilon=l(n-l)$.

For $|v|=1$ we may suppose that $x^vy^\eta=x_iy^\eta$ for some $i\in\Upsilon$. Then

$$[D_H(x_iy^\lambda),D_H(x_iy^\eta)]=(-1)^{|u|+\tau(u,i)}D_H(x^{u-\langle i\rangle}y^{\lambda+\eta}),\quad i\in\Upsilon.$$

Hence $\dim(x^{u-\langle i\rangle})=|u|=l$. It is clear to see that $l(n-l)+l>n$. Therefore, $I(\operatorname{ad}D_H(g))\geq(l(n-l)+l)p^m>np^m$. □

Theorem 4.5. $I(\mathrm{hg}(\operatorname{Der}H))=np^m$. *Moreover, for* $D\in\mathrm{hg}(\operatorname{Der}(H))$, *we have* $I(D)=np^m$ *if and only if* D *is nonzero and lies in* $\operatorname{span}_{\mathbb{F}}\{\operatorname{ad}D_H(x^\pi y^\lambda)\mid\lambda\in G\}$.

Proof. By virtue of Theorem 4.4, we have $I(\operatorname{ad}D_H(x^\pi y^\lambda))=np^m$. By [Ren et al. 2011, Proposition 3.7], we obtain

$$\operatorname{Der}H=\operatorname{ad}(\overline{H}+\mathbb{F}y^\lambda h)\oplus\Omega,$$

where $h=\sum_{i=1}^n x_iD_i$ and $\lambda\in G$. Hence $I(\mathrm{hg}(\operatorname{Der}H))\leq np^m$. Let $D\in\mathrm{hg}(\operatorname{Der}H)$ and $I(D)\leq np^m$. Without loss of generality, we may suppose that

$$D=\operatorname{ad}D_H(g)+a\operatorname{ad}y^\lambda h+\sum_{\theta\in\Theta}b_\theta D_\theta,$$

where $a, b_\theta \in \mathbb{F}$ and $D_H(g) \in \mathrm{hg}(\overline{H})$. Then

$$
\begin{aligned}
&D(D_H(x^u y^\eta)) \\
&\quad = [D_H(g), D_H(x^u y^\eta)] + a\Big[\sum_{i=1}^{n} x_i y^\lambda D_i, D_H(x^u y^\eta)\Big] + \sum_{\theta\in\Theta} b_\theta D_\theta(D_H(x^u y^\eta))
\end{aligned}
$$

for all $u \in \mathbb{B}(n)$ and $\eta \in G$.

Next we will prove that a and b_θ are all zero for all $\theta \in \Theta$.

First of all we consider the coefficient a. A direct calculation shows that

$$
\begin{aligned}
a\Big[\sum_{i=1}^{n} x_i y^\lambda D_i, D_H(x^u y^\eta)\Big] &= \sum_{i,j=1}^{n} (-1)^{|u|} a[x_i y^\lambda D_i, D_j(x^u) y^\eta D_j] \\
&= \sum_{i,j=1}^{n} (-1)^{|u|} a[x_i y^\lambda D_i, (-1)^{\tau(u,j)} x^{u-\langle j\rangle} y^\eta D_j] \\
&= \begin{cases} -\sum_{j=1}^{n} (-1)^{|u|+\tau(u,j)} a x^{u-\langle j\rangle} y^{\lambda+\eta} D_j & \text{if } i = j, \\ (n-1)\sum_{j=1}^{n} (-1)^{|u|+\tau(u,j)} a x^{u-\langle j\rangle} y^{\lambda+\eta} D_j & \text{if } i \neq j. \end{cases}
\end{aligned}
$$

Using the similar discussion in Theorem 4.4, we obtain

$$\dim\big(\mathrm{span}_{\mathbb{F}}\{x^{u-\langle j\rangle} y^{\lambda+\eta} D_j\}\big) > np^m$$

for given $j \in u$. Since $n > 1$, we have $a = 0$.

Secondly, the other coefficient b_θ will be considered. For any $u \in \mathbb{B}(n)$ and $\eta \in G$, we have

$$
\begin{aligned}
b_\theta D_\theta(D_H(x^u y^\eta)) &= \tilde{\theta}(\eta) D_H(b_\theta x^u y^\eta) \\
&= \sum_{i=1}^{n} (-1)^{|u|} b_\theta D_i(x^u) D_i \tilde{\theta}(\eta) y^\eta \\
&= b_\theta \sum_{i=1}^{n} (-1)^{|u|+\tau(u,i)} x^{u-\langle i\rangle} D_i \tilde{\theta}(\eta) y^\eta.
\end{aligned}
$$

By the equality above and the similar discussion in Theorem 4.4, we have

$$\dim(\mathrm{span}_{\mathbb{F}}\{x^{u-\langle i\rangle} \tilde{\eta}(\mu) y^\eta D_j\}) > np^m.$$

Hence $b_\theta = 0$ for all $\theta \in \Theta$. Therefore, $D = \mathrm{ad}\, D_H(g)$. It follows from Theorem 4.4 that $I(\mathrm{hg}(\mathrm{Der}\, H)) = np^m$. In particular, $I(D) = np^m$ if and only if

$$0 \neq D \in \mathrm{span}_{\mathbb{F}}\{\mathrm{ad}\, D_H(x^\pi y^\lambda) \mid \lambda \in G\}. \qquad \square$$

We adopt the notations n', m' in Section 3 and let $H' = H(n', m')$ and $G' = \left\{\sum_{i=1}^{m'} \lambda_i z_i \mid \lambda_i \in \Pi, i = 1, \dots, m'\right\}$.

Proposition 4.6. *Let*

$$R = \operatorname{span}_{\mathbb{F}}\{D_H(x^u y^\lambda) \in H \mid u \in \mathbb{B}(n), |u| \geq 2, \lambda \in G\},$$
$$R' = \operatorname{span}_{\mathbb{F}}\{D_{H'}(x^u y^\lambda) \in H' \mid u \in \mathbb{B}(n'), |u| \geq 2, \lambda \in G'\}.$$

If σ is an isomorphism from H to H', then $\sigma(R) = R'$.

Proof. It is easy to see that the map $\xi \colon D \to \sigma D \sigma^{-1}$ is a bijection. Then ξ is an isomorphism from $\operatorname{Der} H$ to $\operatorname{Der} H'$. Thus $I(\operatorname{hg}(\operatorname{Der} H)) = I(\operatorname{hg}(\operatorname{Der} H'))$. According to Theorem 4.5, we have

$$\sigma(\operatorname{span}_{\mathbb{F}}\{\operatorname{ad} D_H(x^\pi y^\lambda)\})\sigma^{-1} = \operatorname{span}_{\mathbb{F}}\{\operatorname{ad} D_H(x^{\pi'} y^{\lambda'})\},$$

where $\pi' = \{1, \dots, n'\} \in \mathbb{B}(n')$, $\lambda \in G$, and $\lambda' \in G'$. Note that

$$[D_H(x^\pi y^\lambda), D_H(x^u y^\eta)] = D_H\left(\sum_{i=1}^{n} (-1)^n D_i(x^\pi y^\lambda) D_i(x^u y^\eta)\right).$$

for $u \in \mathbb{B}(n)$ and $\lambda, \eta \in G$. If $|u| \geq 2$, then $D_i(x^u y^\eta) = (-1)^{\tau(u,i)} x^{u - \langle i \rangle} y^\eta$ and $D_i(x^\pi y^\lambda) = (-1)^{\tau(\pi,i)} x^{\pi - \langle i \rangle} y^\lambda$. Since $\{u - \langle i \rangle\} \in \{\pi - \langle i \rangle\}$, we have

$$[D_H(x^\pi y^\lambda), D_H(x^u y^\eta)] = 0.$$

Hence

$$R = \{h \in H \mid (\operatorname{span}_{\mathbb{F}}\{\operatorname{ad} D_H(x^\pi y^\lambda)\})(h) = 0\}.$$

Similarly, $R' = \{h \in H' \mid (\operatorname{span}_{\mathbb{F}}\{\operatorname{ad} D_{H'}(x^{\pi'} y^{\lambda'})\})(h) = 0\}$. Then

$$\begin{aligned}
(\operatorname{span}_{\mathbb{F}}\{\operatorname{ad} D_{H'}(x^{\pi'} y^{\lambda'})\})(\sigma(R)) &= \sigma(\operatorname{span}_{\mathbb{F}}\{\operatorname{ad} D_H(x^\pi y^\lambda)\})\sigma^{-1}(\sigma(R)) \\
&= \sigma(\operatorname{span}_{\mathbb{F}}\{\operatorname{ad} D_H(x^\pi y^\lambda)\})(R) \\
&= \sigma(\operatorname{span}_{\mathbb{F}}\{\operatorname{ad} D_H(x^\pi y^\lambda)\})(R) \\
&= \sigma(0) \\
&= 0.
\end{aligned}$$

Thus $\sigma(R) \subseteq R'$. By the same method above, we have $\sigma^{-1}(R') \subseteq R$. Hence $R' \subseteq \sigma(R)$. In conclusion, $\sigma(R) = R'$. $\square$

Lemma 4.7. *Let $H = H_{(-1)} \supseteq H_{(0)} \supseteq \cdots \supseteq H_{(n-3)} \supseteq H_{(n-2)} = 0$ be the natural filtration of H. Then*

$$H_{(0)} = R, \quad H_{(i)} = \{h \in H_{(i-1)} \mid [h, H] \subseteq H_{(i-1)}\} \textit{ for } i \geq 1.$$

Similarly, for the natural filtration of H',

$$H'_{(0)} = R', \quad H'_{(i)} = \{h \in H'_{(i-1)} \mid [h, H'] \subseteq H'_{(i-1)}\} \text{ for } i \geq 1.$$

Proof. Suppose that $M = \{h \in H_{(i-1)} \mid [h, H] \subseteq H_{(i-1)}\}$. Note that $H_{(i)} \subseteq H_{(i-1)}$ and $[H_{(i)}, H] = [H_{(i)}, H_{(-1)}] \subseteq H_{(i-1)}$. Then the inclusion relations show that $H_{(i)} \subseteq M$.

Conversely, if $h \in M$, then $h \in H_{(i-1)}$. So we may suppose that $h = \sum_{j=i-1}^{n-3} h_j$, where $h_j \in H_j$. Let $h_{i-1} = \sum_k a_k D(x^{u_k} y^{\lambda_k})$, where $a_k \in \mathbb{F}$, $u_k \in \mathbb{B}(n)$, $|u_k| = i - 1 + 2 = i + 1 \geq 2$, and $\lambda_k \in G$.

If $h_{i-1} = 0$, then $h \in H_{(i)}$. Therefore, the desired result follows in this case.

If $h_{i-1} \neq 0$, then it follows from $h \in M$ that $[h, H_{-1}] \subseteq H_{(i-1)}$. Hence $[h_{i-1}, H_{-1}] = 0$, that is,

$$\left[\sum_k a_k D(x^{u_k} y^{\lambda_k}), D_H(x_i y^{\eta})\right] = 0$$

for all $i \in \mathrm{Y}$ and $\eta \in G$. As $|u_k| \geq 2$, there exists $i \in \mathrm{Y}$ such that

$$D_H((-1)^{|u_k|} D_i(x^{u_k} y^{\lambda_k})) \neq 0.$$

Hence $a_k = 0$ which is in contradiction with $h_{i-1} \neq 0$.

The considerations above show that $M \subseteq H_{(i)}$. Therefore,

$$H_{(i)} = \{h \in H_{(i-1)} \mid [h, H] \subseteq H_{(i-1)}\} \quad \text{for } i \geq 1.$$

Similarly, $H'_{(i)} = \{h \in H'_{(i-1)} \mid [h, H'] \subseteq H'_{(i-1)}\}$ for $i \geq 1$. □

Proposition 4.8. *Suppose that* $H \cong H'$ *and* σ *is an isomorphism from* H *to* H', *then* $\sigma(H_{(i)}) = H'_{(i)}$ *for all* $i \geq -1$.

Proof. If $i = 0$, then $H_{(0)} = R$ and $H'_{(0)} = R'$. Proposition 4.6 shows that $\sigma(H_{(0)}) = H'_{(0)}$.

If $i = -1$, then $H_{(-1)} = H$ and $H'_{(-1)} = H'$. Hence $\sigma(H_{(-1)}) = H'_{(-1)}$.

Next we use induction on i. Assume that $\sigma(H_{(i)}) = H'_{(i)}$ for $i \geq 1$. By Lemma 4.7, for $h \in H_{(i+1)}$, we have $h \in H_{(i)}$ as well as $[h, H] \subseteq H_{(i)}$. Since $h \in H_{(i)}$, the induction hypothesis yields $\sigma(h) \in H'_{(i)}$. Then

$$\sigma([h, H]) = [\sigma(h), \sigma(H)] \subseteq [H'_{(i)}, H'] \subseteq H'_{(i)}.$$

By Lemma 4.7, we have $\sigma(h) \in H'_{(i+1)}$. This implies that $\sigma(H_{(i+1)}) \subseteq H'_{(i+1)}$.

For any $h' \in H'_{(i+1)}$, we want to prove that $h' \in \sigma(H_{(i+1)})$. The fact $h' \in H'_{(i)} = \sigma(H_{(i)})$ ensures that there exists $h \in H_{(i)}$ such that $\sigma(h) = h'$. It is easy to see that $[h', H'] \subseteq H'_{(i)} = \sigma(H_{(i)})$. Since $[h', H'] = [\sigma(h), \sigma(H)] = \sigma[h, H]$, we have $[h, H] \in H_{(i)}$. Then $h \in H_{(i+1)}$, that is, $h' \in \sigma(H_{(i+1)})$. Consequently, $\sigma(H_{(i)}) = H'_{(i)}$ for all $i \geq -1$. □

Theorem 4.9. *The natural filtration of H is invariant under the automorphisms of H.*

Proof. It is a direct conclusion of Proposition 4.8. $\square$

Imitating the definition of $\mathfrak{W}_i$ in W, we let $\mathfrak{H}_i = H_{(i)}/H_{(i+1)}$ for $-1 \leq i \leq n-3$. Suppose that $\mathfrak{H} := \bigoplus_{i=-1}^{n-3} \mathfrak{H}_i$, then $\mathfrak{H}$ is a $\mathbb{Z}$-graded space. Let $x + H_{(i+1)} \in \mathfrak{H}_i$ and $y + H_{(j+1)} \in \mathfrak{H}_j$. We define

$$[x + H_{(i+1)}, y + H_{(j+1)}] := [x, y] + H_{(i+j+1)}.$$

It is easy to see that the operator [,] on $\mathfrak{H}$ is well-defined. There exists a linear expansion such that $\mathfrak{H}$ has a operator [,]. A direct verification shows that $\mathfrak{H}$ is a Lie superalgebra with respect to the operator [,]. The Lie superalgebras $\mathfrak{H}$ is called a Lie superalgebra induced by the natural filtration of H.

By the similar methods used to prove Propositions 3.7 and 3.9, the following lemmas are easy to obtain.

Lemma 4.10. $\mathfrak{H} \cong H$.

Lemma 4.11. *Suppose that $H \cong H'$ and σ is an isomorphism from H to H', then σ induces an isomorphism $\tilde{\sigma}$ from $\mathfrak{H}$ to $\mathfrak{H}'$ such that $\tilde{\sigma}(\mathfrak{H}_i) = \mathfrak{H}'_i$ for all $i \geq -1$.*

Theorem 4.12. *$H \cong H'$ if and only if $m = m'$ and $n = n'$.*

Proof. Since the sufficiency is obvious, it suffices to prove the necessity. Using the similar methods in the proof of Theorem 3.10, we have $\dim H_{-1} = \dim H'_{-1}$ and $\dim H_0 = \dim H'_0$. It follows from $W_{-1} = H_{-1}$ that $np^m = n'p^{m'}$. By virtue of the definition of H_i, we have

$$H_0 = \operatorname{span}_{\mathbb{F}}\{D_H(x_i x_j y^\lambda) \in H \mid i, j \in \mathrm{Y}, \lambda \in G\}.$$

Thus $\dim H_0 = \mathrm{C}_n^2 p^m = \frac{1}{2} n(n-1)p^m$. Similarly, $\dim H'_0 = \frac{1}{2} n'(n'-1)p^{m'}$. According to $\dim H_0 = \dim H'_0$ and $np^m = n'p^{m'}$, we have $n = n'$ and $m = m'$. Consequently, the desired result follows. $\square$

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References

[Awuti and Zhang 2008] S. Awuti and Y. Z. Zhang, "Modular Lie superalgebra $\overline{W}(n, m)$", *Dongbei Shida Xuebao* **40**:2 (2008), 7–11. MR 2009k:17035 Zbl 1171.17302

[Block and Wilson 1988] R. E. Block and R. L. Wilson, "Classification of the restricted simple Lie algebras", *J. Algebra* **114**:1 (1988), 115–259. MR 89e:17014 Zbl 0644.17008

[Bouarroudj and Leites 2006] S. Bouarroudj and D. Leites, "Simple Lie superalgebras and nonintegrable distributions in characteristic p", *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **331**:14 (2006), 15–29. In Russian; translated in *J. Math. Sci.* **141**:4 (2007), 1390–1398. MR 2007d:17025 Zbl 1104.17011

[Chen and Liu 2011] Y. Chen and W. Liu, "Finite-dimensional odd contact superalgebras over a field of prime characteristic", *J. Lie Theory* **21**:3 (2011), 729–754. MR 2012h:17024 Zbl 1239.17015

[Elduque 2007] A. Elduque, "Some new simple modular Lie superalgebras", *Pacific J. Math.* **231**:2 (2007), 337–359. MR 2009d:17027 Zbl 1207.17023

[Jacobson 1962] N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Applied Mathematics **10**, Interscience, New York, 1962. MR 26 #1345 Zbl 0121.27504

[Jin 1992] N. Jin, "Ad-nilpotent elements, quasi-nilpotent elements and invariant filtrations of infinite-dimensional Lie algebras of Cartan type", *Sci. China Ser. A* **35**:10 (1992), 1191–1200. MR 94j:17027 Zbl 0772.17007

[Kac 1974] V. G. Kac, "A description of the filtered Lie algebras with which graded Lie algebras of Cartan type are associated", *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 800–834. In Russian; translated in *Math. USSR Izv* **8**:4 (1974), 801–835. MR 51 #5685 Zbl 0317.17002

[Kac 1977] V. G. Kac, "Lie superalgebras", *Advances in Math.* **26**:1 (1977), 8–96. MR 58 #5803 Zbl 0366.17012

[Kochetkov and Leites 1992] Y. Kochetkov and D. Leites, "Simple Lie algebras in characteristic 2 recovered from superalgebras and on the notion of a simple finite group", pp. 59–67 in *Proceedings of the International Conference on Algebra, II* (Novosibirsk, 1989), edited by L. A. Bokut et al., Contemp. Math. **131**, Amer. Math. Soc., Providence, RI, 1992. MR 93g:17035 Zbl 0765.17006

[Kostrikin and Shafarevich 1969] A. I. Kostrikin and I. R. Shafarevich, "Graded Lie algebras of finite characteristic", *Izv. Akad. Nauk SSSR Ser. Mat.* **33** (1969), 251–322. In Russian; translated in *Math. USSR Izv* **3**:2 (1969), 237–304. MR 40 #5680 Zbl 0211.05304

[Petrogradski 1992] V. M. Petrogradski, "Identities in the enveloping algebras for modular Lie superalgebras", *J. Algebra* **145**:1 (1992), 1–21. MR 92m:17013 Zbl 0752.17001

[Ren et al. 2011] L. Ren, Q. Mu, and Y. Zhang, "A class of finite-dimensional Lie superalgebra of Hamiltonian type", *Algebra Colloq.* **18**:2 (2011), 347–360. MR 2012g:17025 Zbl 1219.17015

[Ren et al. 2012] L. Ren, Q. Mu, and Y. Zhang, "Odd Hamiltonian superalgebras and special odd Hamiltonian superalgebras of formal vector fields", *Pacific J. Math.* **257**:2 (2012), 471–489. MR 2972474 Zbl 06080829

[Scheunert 1979] M. Scheunert, *The theory of Lie superalgebras: An introduction*, Lecture Notes in Mathematics **716**, Springer, Berlin, 1979. MR 80i:17005 Zbl 0407.17001

[Strade 1993] H. Strade, "The classification of the simple modular Lie algebras, IV: Determining the associated graded algebra", *Ann. of Math.* (2) **138**:1 (1993), 1–59. MR 94k:17039 Zbl 0790.17011

[Varadarajan 2004] V. S. Varadarajan, *Supersymmetry for mathematicians: An introduction*, Courant Lecture Notes in Mathematics **11**, Amer. Math. Soc., Providence, RI, 2004. MR 2005g:58011 Zbl 1142.58009

[Yuan et al. 2011] J. Yuan, Y. Chen, and W. Liu, "Automorphisms and p-characters of restricted Lie superalgebras of Cartan type", *Algebra Colloq.* **18**:3 (2011), 397–410. MR 2012g:17027 Zbl 1231.17010

[Zhang 1997] Y. Zhang, "Finite-dimensional Lie superalgebras of Cartan type over fields of prime characteristic", *Chinese Sci. Bull.* **42**:9 (1997), 720–724. MR 98h:17023 Zbl 0886.17022

[Zhang and Fu 2002] Y. Zhang and H. Fu, "Finite-dimensional Hamiltonian Lie superalgebra", *Comm. Algebra* **30**:6 (2002), 2651–2673. MR 2003h:17040 Zbl 1021.17017

[Zhang and Nan 1998] Y. Zhang and J. Nan, "Finite-dimensional Lie superalgebras $W(m, n, \mathbf{t})$ and $S(m, n, \mathbf{t})$ of Cartan type", *Adv. Math. (China)* **27**:3 (1998), 240–246. MR 99h:17012 Zbl 1054.17502

[Zhao 2010] L. Zhao, "Representations of Lie superalgebras in prime characteristic, III", *Pacific J. Math.* **248**:2 (2010), 493–510. MR 2012d:17037 Zbl 1225.17022

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KELI ZHENG
SCHOOL OF MATHEMATICS AND STATISTICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN, 130024
CHINA
Current address:
DEPARTMENT OF MATHEMATICS
NORTHEAST FORESTRY UNIVERSITY
HARBIN 150040
CHINA
zhengkl561@nenu.edu.cn

YONGZHENG ZHANG
SCHOOL OF MATHEMATICS AND STATISTICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN, 130024
CHINA
zhyz@nenu.edu.cn

WEI SONG
DONGGUAN MIDDLE SCHOOL – SSL SCHOOL
DONGGUAN, 523808
CHINA
songw207@163.com

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