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RESTRICTED SUCCESSIVE MINIMA

MARTIN HENK AND CARSTEN THIEL

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We give bounds on the successive minima of an o -symmetric convex body under the restriction that the lattice points realizing the successive minima are not contained in a collection of forbidden sublattices. Our investigations extend former results to forbidden full-dimensional lattices, to all successive minima and complement former results in the lower-dimensional case.

1. Introduction

Let \mathcal{K}_o^n be the set of all o -symmetric convex bodies in \mathbb{R}^n with nonempty interior, i.e., $K \in \mathcal{K}_o^n$ is an n -dimensional compact convex set satisfying $K = -K$. The volume, i.e., the n -dimensional Lebesgue measure, of a subset $X \subset \mathbb{R}^n$ is denoted by $\text{vol } X$. By a lattice $\Lambda \subset \mathbb{R}^n$ we understand a free \mathbb{Z} -module of rank $\text{rg } \Lambda \leq n$. The set of all lattices is denoted by \mathcal{L}^n , and $\det \Lambda$ denotes the determinant of $\Lambda \in \mathcal{L}^n$, that is the $(\text{rg } \Lambda)$ -dimensional volume of a fundamental cell of Λ .

For $K \in \mathcal{K}_o^n$ and $\Lambda \in \mathcal{L}^n$, Minkowski introduced the i -th successive minimum $\lambda_i(K, \Lambda)$, $1 \leq i \leq \text{rg } \Lambda$, as the smallest positive number λ such that λK contains at least i linearly independent lattice points of Λ , i.e.,

$$\lambda_i(K, \Lambda) = \min\{\lambda \in \mathbb{R}_{\geq 0} : \dim(\lambda K \cap \Lambda) \geq i\}, \quad 1 \leq i \leq \text{rg } \Lambda.$$

Minkowski's first fundamental theorem (see, e.g., [Gruber 2007, Sections 22–23]) on successive minima establishes an upper bound on the first successive minimum in terms of the volume of a convex body. More precisely, for $K \in \mathcal{K}_o^n$ and $\Lambda \in \mathcal{L}^n$ with $\text{rg } \Lambda = r$, it may be formulated as

$$(1-1) \quad \lambda_1(K, \Lambda)^r \text{vol}_r(K \cap \text{lin } \Lambda) \leq 2^r \det \Lambda,$$

where $\text{vol}_r(\cdot)$ denotes the r -dimensional volume, here with respect to the subspace $\text{lin } \Lambda$, the linear hull of Λ . In the case $r = n$ we just write $\text{vol}(\cdot)$. One of the many successful applications of this inequality is related to ‘‘Siegel’s lemma’’, a catch-all term for results bounding the norm of a nontrivial lattice point lying in a linear subspace given as $\ker A$ where $A \in \mathbb{Z}^{m \times n}$ is an integral matrix of rank m . For instance, with respect to the maximum norm $|\cdot|_\infty$, it was shown in [Bombieri

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and Vaaler 1983] (see also [Ball and Pajor 1990]) that there exists a $z \in \ker A \setminus \{0\}$ such that

$$|z|_\infty \leq \sqrt{\det(AA^\top)^{\frac{1}{n-m}}}.$$

In fact, this follows by (1-1), where $K = [-1, 1]^n$ is the cube of edge length 2, $\Lambda = \ker A \cap \mathbb{Z}^n$ is an $(n - m)$ -dimensional lattice of determinant at most $\sqrt{\det(AA^\top)}$, and Vaaler’s result [1979] on the minimal volume of a slice of a cube, which here gives $\text{vol}_{n-m}([-1, 1]^n \cap \text{lin } \Lambda) \geq 2^{n-m}$. For generalizations of Siegel’s lemma to number fields we refer to [Bombieri and Vaaler 1983; Fukshansky 2006a; 2006b; Gaudron 2009; Gaudron and Rémond 2012a; Vaaler 2003].

Motivated by questions in Diophantine approximation, Fukshansky [2006a] studied an inverse problem to that addressed in Siegel’s lemma, namely to bound the norm of lattice points which are not contained in the union of proper sublattices. To describe his result we need a bit more notation.

For a collection of sublattices $\Lambda_i \subset \Lambda$, $1 \leq i \leq s$, with $\bigcup_{i=1}^s \Lambda_i \neq \Lambda$ we call

$$\lambda_i\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) = \min\left\{\lambda \in \mathbb{R}_{\geq 0} : \dim\left(\lambda K \cap \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) \geq i\right\}, \quad 1 \leq i \leq \text{rg } \Lambda,$$

the i -th restricted successive minimum of K with respect to $\Lambda \setminus \bigcup_{i=1}^s \Lambda_i$. Observe that by the compactness of K and the discreteness of $\Lambda \setminus \bigcup_{i=1}^s \Lambda_i$ these minima are well-defined. Furthermore, they behave nicely with respect to dilations, as for $\mu > 0$ we have

$$(1-2) \quad \lambda_i\left(\mu K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) = \lambda_i\left(K, \frac{1}{\mu}\left(\Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right)\right) = \frac{1}{\mu} \lambda_i\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right).$$

Moreover, for a lattice $\Lambda \in \mathcal{L}^n$, $r = \text{rg } \Lambda$, and a basis (b_1, \dots, b_r) , $b_j \in \mathbb{R}^n$, of Λ , let $v(\Lambda) \in \mathbb{R}^{\binom{r}{2}}$ be the vector with entries $\det B_j$, where B_j is an $r \times r$ submatrix of (b_1, \dots, b_r) . Observe that up to the order of the coordinates the vector is independent of the given basis, and on account of the Cauchy–Binet formula the Euclidean norm of $v(\Lambda)$ is the determinant of the lattice. With this notation, Fukshansky [2006a, Theorem 1.1] proved

$$(1-3) \quad \lambda_1\left([-1, 1]^n, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) \leq \left(\frac{3}{2}\right)^{r-1} r^r \left(\sum_{i=1}^s \frac{1}{|v(\Lambda_i)|_\infty} + \sqrt{s}\right) |v(\Lambda)|_\infty + 1,$$

for proper sublattices Λ_i , $1 \leq i \leq s$, where *proper* means $\text{rg } \Lambda_i < \text{rg } \Lambda = r$. This result was generalized and improved in various ways in [Gaudron 2009] and [Gaudron and Rémond 2012a]. In particular, (1-3) has been extended to all o -symmetric bodies as well as to the adelic setting (see also [Gaudron and Rémond 2012b, Lemma 3.2] for an application). For instance, the following is a simplified version of [Gaudron 2009,

Theorem 6.1] when we assume that $\text{rg } \Lambda_i = \text{rg } \Lambda - 1 = r - 1$ (see also [Gaudron and Rémond 2012a, Theorem 2.2, Corollary 3.3]):

$$(1-4) \quad \lambda_1 \left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i \right) \leq \nu \max_{1 \leq i \leq s} \left\{ 1, \frac{\nu^{r-1} \text{vol}(K \cap \text{lin } \Lambda_i)}{\omega_r \det \Lambda_i}, \left(\frac{\nu}{\lambda_1(K, \Lambda \cap \text{lin } \Lambda_i)} \right)^{\frac{r-2}{2}} \right\},$$

where $\nu = 7r(s\omega_r \det \Lambda / \text{vol } K)^{1/r}$ and ω_r is the volume of the r -dimensional unit ball.

In our first theorem we want to complement these results on forbidden lower-dimensional lattices by a bound which takes care of the size or the structure of the individual forbidden sublattices such that the bound becomes essentially (1-1) if $\lambda_1(K, \Lambda_i) \rightarrow \infty$ for $1 \leq i \leq s$. In this case the bounds in (1-3) and (1-4) still have a dependency on s of order \sqrt{s} and $s^{1/r}$, respectively. Here we have the following result.

Theorem 1.1. *Let $K \in \mathcal{K}_o^n$, $\Lambda \in \mathcal{L}^n$, $\text{rg } \Lambda = n \geq 2$, and let $\Lambda_i \subset \Lambda$, $1 \leq i \leq s$, $\text{rg } \Lambda_i \leq n - 1$, be sublattices. Then*

$$\lambda_1 \left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i \right) < 6^{n-1} \frac{\det \Lambda}{\lambda_1(K, \Lambda)^{n-2} \text{vol } K} \left(\sum_{i=1}^s \frac{1}{\lambda_1(K, \Lambda_i)} \right) + \sqrt[n]{2^n \frac{\det \Lambda}{\text{vol } K}}.$$

Note that, if $s = 0$ or all the $\lambda_1(K, \Lambda_i)$ are very large, we get essentially (1-1).

Our second main theorem deals with forbidden full-dimensional sublattices — those for which $\text{rg } \Lambda_i = \text{rg } \Lambda$, $1 \leq i \leq s$.

Theorem 1.2. *Let $K \in \mathcal{K}_o^n$, $\Lambda \in \mathcal{L}^n$, $\text{rg } \Lambda = n \geq 2$, and let $\Lambda_i \subset \Lambda$, $1 \leq i \leq s$, $\text{rg } \Lambda_i = n$, be sublattices such that $\bigcup_{i=1}^s \Lambda_i \neq \Lambda$. Then*

$$\lambda_1 \left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i \right) < \frac{2^n \det \Lambda}{\lambda_1(K, \bar{\Lambda})^{n-1} \text{vol } K} \left(\sum_{i=1}^s \frac{\det \bar{\Lambda}}{\det \Lambda_i} - s + 1 \right) + \lambda_1(K, \bar{\Lambda}),$$

where $\bar{\Lambda} = \bigcap_{i=1}^s \Lambda_i$.

In the special case $s = 1$, since we may assume $\lambda_1(K, \Lambda_1) = \lambda_1(K, \Lambda)$, we get the following immediate consequence:

Corollary 1.3. *Let $K \in \mathcal{K}_o^n$, $\Lambda \in \mathcal{L}^n$, $\text{rg } \Lambda = n \geq 2$, and let $\Lambda_1 \subsetneq \Lambda$, $\text{rg } \Lambda_1 = n$, be a sublattice. Then*

$$\lambda_1(K, \Lambda \setminus \Lambda_1) \leq \frac{2^n \det \Lambda}{\lambda_1(K, \Lambda_1)^{n-1} \text{vol } K} + \lambda_1(K, \Lambda).$$

The following example shows that the bound in Theorem 1.2 as well as the one of the corollary above cannot be improved in general by a multiplicative factor.

Example 1.4. Let $K \in \mathcal{H}_o^2$ be the rectangle $K = [-1, 1] \times [-\alpha, \alpha]$ of edge-lengths 2 and 2α , $\alpha \leq 1$, and of volume 4α . Let $\Lambda = \mathbb{Z}^2$, and define the sublattices

$$\Lambda_1 = \{(z_1, z_2)^\top \in \mathbb{Z}^2 : z_2 \equiv 0 \pmod{2}\}, \quad \Lambda_2 = \{(z_1, z_2)^\top \in \mathbb{Z}^2 : z_1 \equiv 0 \pmod{p}\},$$

where $p > 2$ is a prime. Then $\det \Lambda = 1$, $\det \Lambda_1 = 2$, $\det \Lambda_2 = p$, and

$$\bar{\Lambda} = \Lambda_1 \cap \Lambda_2 = \{(z_1, z_2)^\top \in \mathbb{Z}^2 : z_2 \equiv 0 \pmod{2}, z_1 \equiv 0 \pmod{p}\}$$

with $\det \bar{\Lambda} = 2p$. For $\alpha \leq 2/p$ we therefore have $\lambda_1(K, \bar{\Lambda}) = p$. Regarding the set $\Lambda \setminus (\Lambda_1 \cup \Lambda_2)$, we observe that the lattice points on the axes are forbidden, but not $(1, 1)^\top$ and so $\lambda_1(K, \Lambda \setminus (\Lambda_1 \cup \Lambda_2)) = 1/\alpha$. Putting everything together, the bound in [Theorem 1.2](#) evaluates for $\alpha \leq 2/p$ to

$$\frac{1}{\alpha} = \lambda_1(\Lambda \setminus (\Lambda_1 \cup \Lambda_2)) < \frac{4}{p4\alpha}(p+1) + p = \frac{1}{\alpha} + \frac{1}{p\alpha} + p.$$

Hence for $\alpha = 2/p^2$ and $p \rightarrow \infty$ the bound cannot be improved by a multiplicative factor.

In the situation of [Corollary 1.3](#), i.e., when we consider only the forbidden lattice Λ_1 , the upper bound in the corollary evaluates to $1/\alpha + 1$, whereas, as before, $\lambda_1(K, \Lambda \setminus \Lambda_1) = 1/\alpha$.

Before beginning with the proofs of our results we would like to mention a closely related problem, namely to cover $K \cap \Lambda$, $K \in \mathcal{H}_o^n$, by a minimal number $\gamma(K)$ of lattice hyperplanes. Obviously, having a $\nu > 0$ with $\gamma(\nu K) \geq s + 1$ implies that

$$\lambda_1\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) \leq \nu$$

in the case of lower-dimensional sublattices Λ_i . For bounds on $\gamma(K)$ in terms of the successive minima and other functionals from the geometry of numbers we refer to [\[Bárány et al. 2001; Bezdek and Hausel 1994; Bezdek and Litvak 2009\]](#).

Finally, we remark that restricted successive minima have also been investigated from an algorithmic point of view. Blömer and Naewe [\[2007\]](#) studied the complexity of computing $\lambda_1(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i)$ for $s = 1$ and when K is the unit ball of an l_p -norm. They show, among other things, that some of the well-known lattice problems, like the shortest or closest lattice vector problem, are polynomial reducible to computing/approximating $\lambda_1(K, \Lambda \setminus \Lambda_1)$. Moreover, as in the case of these lattice problems an LLL-reduced basis (see [\[Grötschel et al. 1993, Chapter 5\]](#)) can be used to find in polynomial time a lattice vector b which approximates $\lambda_1(B_n, \Lambda \setminus \Lambda_1)$ up to a factor of 2^{n-1} [\[Blömer and Naewe 2007, Theorem 3.6\]](#). Here B_n is the unit ball of the Euclidean norm. Hence, [Theorem 1.1](#) implies (see [\[Grötschel et al. 1993, Theorem 5.3.13a\]](#) for a similar result in the standard setting $s = 0$):

Corollary 1.5. *Let $\Lambda \in \mathcal{L}^n$, $\text{rg } \Lambda = n \geq 2$, and let $\Lambda_1 \subset \Lambda$, $\text{rg } \Lambda_1 \leq n - 1$, be a sublattice. There exists a polynomial time algorithm for computing a vector $b \in \Lambda \setminus \Lambda_1$ of Euclidean length*

$$\|b\| < 2^{n-1} \left(6^{n-1} \frac{\det \Lambda}{\lambda_1(K, \Lambda)^{n-2} \text{vol } K} \frac{1}{\lambda_1(K, \Lambda_1)} + \sqrt[n]{2^n \frac{\det \Lambda}{\text{vol } K}} \right).$$

It seems to be a challenging problem to extend this result to more than one forbidden sublattice as well as to full-dimensional forbidden lattices.

The paper is organized as follows. The proof of [Theorem 1.1](#) will be given in the next section and full-dimensional forbidden sublattices, i.e., [Theorem 1.2](#), will be treated in [Section 3](#). In each of the sections we also present some extensions of the results above to higher successive minima, i.e., to $\lambda_i(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i)$, $i > 1$.

2. Avoiding lower-dimensional sublattices

In the course of the proof we have to estimate the number of lattice points in a centrally symmetric convex body, i.e., to bound $|K \cap \Lambda|$ from below and above. Assuming $K \in \mathcal{H}_o^n$ and $\text{rg } \Lambda = n$, we will use as a lower bound a classical result of van der Corput (see, e.g., [\[Gruber and Lekkerkerker 1987, p. 51\]](#)):

$$(2-1) \quad |K \cap \Lambda| \geq 2 \left\lfloor \frac{\text{vol } K}{2^n \det \Lambda} \right\rfloor + 1 > \frac{\text{vol } K}{2^{n-1} \det \Lambda} - 1.$$

As upper bound we will use a bound in terms of the first successive minima by Betke, Henk and Wills [\[Betke et al. 1993\]](#):

$$(2-2) \quad |K \cap \Lambda| \leq \left(\frac{2}{\lambda_1(K, \Lambda)} + 1 \right)^n.$$

Proof of [Theorem 1.1](#). By scaling K with $\lambda_1(K, \Lambda)$ we may assume without loss of generality that $\lambda_1(K, \Lambda) = 1$, i.e., K contains no nontrivial lattice point in its interior (cf. (1-2)). Let $n_i = \text{rg } \Lambda_i < n$. For $\gamma \geq 1$, since $\lambda_1(K, \Lambda_i) \geq \lambda_1(K, \Lambda) = 1$ we get, from (2-2),

$$(2-3) \quad |\gamma K \setminus \{0\} \cap \Lambda_i| \leq \left(\gamma \frac{2}{\lambda_1(K, \Lambda_i)} + 1 \right)^{n_i} - 1 < \gamma^{n-1} 3^{n-1} \frac{1}{\lambda_1(K, \Lambda_i)}.$$

Hence, for $\gamma \geq 1$, we have

$$(2-4) \quad \left| \gamma K \setminus \{0\} \cap \left(\bigcup_{i=1}^s \Lambda_i \right) \right| < \gamma^{n-1} 3^{n-1} \sum_{i=1}^s \frac{1}{\lambda_1(K, \Lambda_i)}.$$

Combining this bound with the upper bound (2-1) leads, for $\gamma \geq 1$, to the estimate

$$\begin{aligned} \left| \gamma K \setminus \{0\} \cap \Lambda \setminus \bigcup_{i=1}^s \Lambda_i \right| &> \gamma^n \frac{\text{vol } K}{2^{n-1} \det \Lambda} - 2 - \left| \gamma K \setminus \{0\} \cap \left(\bigcup_{i=1}^s \Lambda_i \right) \right| \\ &> \gamma^n \frac{\text{vol } K}{2^{n-1} \det \Lambda} - \gamma^{n-1} 3^{n-1} \left(\sum_{i=1}^s \frac{1}{\lambda_1(K, \Lambda_i)} \right) - 2 \\ &= \frac{\text{vol } K}{2^{n-1} \det \Lambda} (\gamma^n - \gamma^{n-1} \beta - \rho), \end{aligned}$$

where

$$\beta = 6^{n-1} \frac{\det \Lambda}{\text{vol } K} \left(\sum_{i=1}^s \frac{1}{\lambda_1(K, \Lambda_i)} \right), \quad \rho = 2^n \frac{\det \Lambda}{\text{vol } K}.$$

Hence, given β and ρ , we have to determine a $\gamma \geq 1$ such that $\gamma^n - \gamma^{n-1} \beta - \rho > 0$. To this end let $\bar{\gamma} = \beta + \rho^{1/n}$. Then

$$\begin{aligned} (2-5) \quad \bar{\gamma}^n - \bar{\gamma}^{n-1} \beta &= (\beta + \rho^{1/n})^n - (\beta + \rho^{1/n})^{n-1} \beta \\ &= \rho^{1/n} (\beta + \rho^{1/n})^{n-1} > \rho^{1/n} \rho^{(n-1)/n} = \rho. \end{aligned}$$

Finally, we observe that

$$\bar{\gamma} > \rho^{1/n} = \left(2^n \frac{\det \Lambda}{\text{vol } K} \right)^{1/n} \geq \lambda_1(K, \Lambda) = 1,$$

by (1-1) and our assumption. Hence, $\bar{\gamma} > 1$ and in view of (2-5) we have $\lambda_1(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i) < \bar{\gamma}$, which by the definition of $\bar{\gamma}$ yields the desired bound of the theorem with respect to our normalization $\lambda_1(K, \Lambda) = 1$. \square

Compared to the bounds in (1-3) and (1-4), our formula uses only the successive minima and not the determinants of the forbidden sublattices which reflect more the structure of a lattice. However, instead of (2-2) one can use a Blichfeldt-type bound, proved in [Henze 2013], for o -symmetric convex bodies K with $\dim(K \cap \Lambda) = n$; namely, if $L_n(x)$ is the n -th Laguerre polynomial,

$$|K \cap \Lambda| \leq \frac{n! \text{vol } K}{2^n \det \Lambda} L_n(-2),$$

This leads to a bound on $\lambda_1(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i)$ where the sum over $1/\lambda_1(K, \Lambda_i)$ is replaced by a sum over ratios of the type $\text{vol}_{\dim H}(K \cap H) / \det(\Lambda_i \cap H)$ for certain lower-dimensional planes $H \subseteq \text{lin } \Lambda_i$. In general, however, we have no control over the dimension of these hyperplanes H nor on the volume of the sections.

Theorem 1.1 can easily be extended inductively to higher restricted successive minima $\lambda_{j+1}(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i)$, $1 \leq j \leq n - 1$, by avoiding, in addition, a j -dimensional lattice containing j linearly independent lattice points corresponding to the successive minima $\lambda_i(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i)$, $1 \leq i \leq j$.

Corollary 2.1. *Under the assumptions of Theorem 1.1 we have, for $j = 1, \dots, n-1$,*

$$\lambda_{j+1}\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) < 6^{n-1} \frac{\det \Lambda}{\lambda_1(K, \Lambda)^{n-2} \operatorname{vol} K} \left(\sum_{i=1}^s \frac{1}{\lambda_1(K, \Lambda_i)} \right) + \left(\frac{3^j}{\lambda_1(K, \Lambda)^j} 2^{n-1} \frac{\det \Lambda}{\operatorname{vol} K} + \left(2^n \frac{\det \Lambda}{\operatorname{vol} K} \right)^{\frac{n-j}{n}} \right)^{\frac{1}{n-j}}.$$

Proof. Let $z_i \in \lambda_i(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i) \cap \Lambda$, $1 \leq i \leq j$, be linearly independent, and let $\bar{\Lambda} = \Lambda \cap \operatorname{lin}\{z_1, \dots, z_j\}$. Then

$$(2-6) \quad \lambda_{j+1}\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) = \lambda_1\left(K, \Lambda \setminus \left(\bigcup_{i=1}^s \Lambda_i \cup \bar{\Lambda}\right)\right),$$

and we now follow the proof of Theorem 1.1. In particular, we assume $\lambda_1(K, \Lambda) = 1$. In addition to the upper bounds on $|\gamma K \setminus \{0\} \cap \Lambda_i|$, $1 \leq i \leq s$, in (2-3), we also use for $\gamma \geq \lambda_1(K, \bar{\Lambda}) \geq \lambda_1(K, \Lambda) = 1$ the bound

$$(2-7) \quad |\gamma K \setminus \{0\} \cap \bar{\Lambda}| < \left(\frac{2\gamma}{\lambda_1(K, \bar{\Lambda})} + 1\right)^j \leq 3^j \left(\frac{\gamma}{\lambda_1(K, \bar{\Lambda})}\right)^j.$$

Combining this bound with (2-1) leads for $\gamma \geq \lambda_1(K, \bar{\Lambda})$ to

$$\begin{aligned} & \left| \gamma K \setminus \{0\} \cap \Lambda \setminus \left(\bigcup_{i=1}^s \Lambda_i \cup \bar{\Lambda}\right) \right| \\ & > \gamma^n \frac{\operatorname{vol} K}{2^{n-1} \det \Lambda} - 2 - \left| \gamma K \setminus \{0\} \cap \left(\bigcup_{i=1}^s \Lambda_i\right) \right| - |\gamma K \setminus \{0\} \cap \bar{\Lambda}| \\ & > \gamma^n \frac{\operatorname{vol} K}{2^{n-1} \det \Lambda} - 2 - \gamma^{n-1} 3^{n-1} \left(\sum_{i=1}^s \frac{1}{\lambda_1(K, \Lambda_i)}\right) - 3^j \left(\frac{\gamma}{\lambda_1(K, \bar{\Lambda})}\right)^j \\ & = \frac{\operatorname{vol} K}{2^{n-1} \det \Lambda} (\gamma^n - \gamma^{n-1} \beta - \gamma^j \alpha - \rho), \end{aligned}$$

with

$$\beta = 6^{n-1} \frac{\det \Lambda}{\operatorname{vol} K} \left(\sum_{i=1}^s \frac{1}{\lambda_1(K, \Lambda_i)}\right), \quad \alpha = \frac{3^j}{\lambda_1(K, \bar{\Lambda})^j} 2^{n-1} \frac{\det \Lambda}{\operatorname{vol} K}, \quad \rho = 2^n \frac{\det \Lambda}{\operatorname{vol} K}.$$

Now setting $\bar{\gamma} = \beta + (\alpha + \rho)^{\frac{n-j}{n}}$ we see as in the proof of Theorem 1.1 that

$$(2-8) \quad \begin{aligned} \bar{\gamma}^n - \bar{\gamma}^{n-1} \beta - \bar{\gamma}^j \alpha - \rho &= \bar{\gamma}^j (\bar{\gamma}^{n-j} - \beta \bar{\gamma}^{n-j-1} - \alpha) - \rho \\ &> \bar{\gamma}^j \rho^{(n-j)/n} - \rho > 0. \end{aligned}$$

Since $\bar{\gamma} > \beta + \rho^{1/n}$, which is, by the proof of Theorem 1.1, an upper bound on $\lambda_1(K, \bar{\Lambda})$, we also have $\bar{\gamma} > \lambda_1(K, \bar{\Lambda})$ and so we know $\lambda_{j+1}(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i) < \bar{\gamma}$,

by (2-8). By the definition of $\bar{\gamma}$ we get the required upper bound with respect to the normalization $\lambda_1(K, \Lambda) = 1$. \square

An upper bound on $\lambda_j(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i)$ of a different kind involves the so-called covering radius $\mu(K, \Lambda)$ of a convex body $K \in \mathcal{K}_o^n$ and a lattice $\Lambda \in \mathcal{L}^n$, $\text{rg } \Lambda = n$. This is the smallest positive number μ such that any translate of μK contains a lattice point:

$$\mu(K, \Lambda) = \min\{\mu > 0 : (t + \mu K) \cap \Lambda \neq \emptyset \text{ for all } t \in \mathbb{R}^n\}.$$

(see [Gruber and Lekkerkerker 1987, Chapter 2, Section 13]).

Proposition 2.2. *Under the assumptions of Theorem 1.1 we have*

$$\lambda_1\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) \leq (s + 1)\mu(K, \Lambda),$$

and hence $\lambda_j\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) \leq (s + 2)\mu(K, \Lambda)$ for $2 \leq j \leq n$.

Proof. Observe that on account of (2-6) the bound for $j \geq 2$ follows from the one for $\lambda_1(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i)$. For the proof in the case $j = 1$ let $H_i = \text{lin } \Lambda_i$, $1 \leq i \leq s$, and for brevity we write $\bar{\mu}$ instead of $\mu(K, \Lambda)$. By Ball’s solution [1991] of the affine plank problem for o -symmetric convex bodies, applied to $\bar{\mu}K$, we know that there exists a $t \in \mathbb{R}^n$ such that

$$\left(t + \frac{1}{s+1}\bar{\mu}K\right) \subset \bar{\mu}K \quad \text{and} \quad \text{int}\left(t + \frac{1}{s+1}\bar{\mu}K\right) \cap H_i = \emptyset, \quad 1 \leq i \leq s,$$

where $\text{int}(\cdot)$ denotes the interior. Thus, for any $\epsilon > 0$ the body $(s+1+\epsilon)\bar{\mu}K$ contains a translate $t_\epsilon + \bar{\mu}K$ having no points in common with H_i , $1 \leq i \leq s$. Hence, together with the definition of the covering radius, we have $(t_\epsilon + \bar{\mu}K) \cap \Lambda \setminus \bigcup_{i=1}^s \Lambda_i \neq \emptyset$ and so $\lambda_1(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i) \leq (s+1+\epsilon)\bar{\mu}$. By the arbitrariness of ϵ and the compactness of K the assertion follows. \square

For a comparable uniform bound in the much more general adelic setting and, of course, with a completely different method see [Gaudron and Rémond 2012a, Proposition 3.2].

3. Avoiding full-dimensional sublattices

If the forbidden sublattices are full-dimensional we cannot argue as in the lower-dimensional case, since now the number of forbidden lattice points in $\lambda K \cap \bigcup_{i=1}^s \Lambda_i$ grows with the same order of magnitude with respect to λ as the number of points in $\lambda K \cap \Lambda$.

The tool we use in this full-dimensional case is the torus group $\mathbb{R}^n / \bar{\Lambda}$ for a certain lattice $\bar{\Lambda}$. For a more detailed discussion we refer to [Gruber 2007, Section 26].

We recall that this quotient of abelian groups is a compact topological group and we may identify $\mathbb{R}^n/\bar{\Lambda}$ with a fundamental parallelepiped P of $\bar{\Lambda}$:

$$\mathbb{R}^n/\bar{\Lambda} \sim P = \{\rho_1 b_1 + \dots + \rho_n b_n : 0 \leq \rho_i < 1\},$$

where b_1, \dots, b_n form a basis of $\bar{\Lambda}$. Hence for $X \subset \mathbb{R}^n$, the set X modulo $\bar{\Lambda}$, $X/\bar{\Lambda}$, can be described as

$$X/\bar{\Lambda} = \{y \in P : \exists b \in \bar{\Lambda} \text{ such that } y + b \in X\} = (\bar{\Lambda} + X) \cap P,$$

and we can think of $\bar{X} \subseteq \mathbb{R}^n/\bar{\Lambda}$ as its image under inclusion into \mathbb{R}^n . In the same spirit we may identify addition \oplus in $\mathbb{R}^n/\bar{\Lambda}$ with the corresponding operation in \mathbb{R}^n , i.e., for $\bar{X}_1, \bar{X}_2 \subset \mathbb{R}^n/\bar{\Lambda}$ we have

$$\bar{X}_1 \oplus \bar{X}_2 = ((\bar{X}_1 + \bar{X}_2) + \bar{\Lambda}) \cap P.$$

As $\mathbb{R}^n/\bar{\Lambda}$ is a compact abelian group, there is a unique Haar measure $\text{vol}_T(\cdot)$ on it normalized to $\text{vol}_T(\mathbb{R}^n/\bar{\Lambda}) = \det \bar{\Lambda}$, and for a “nice” measurable set $X \subset \mathbb{R}^n$ or $\bar{X} \subset \mathbb{R}^n/\bar{\Lambda}$ we have

$$\text{vol}_T(X/\bar{\Lambda}) = \text{vol}((\bar{\Lambda} + X) \cap P) \quad \text{and} \quad \text{vol}_T(\bar{X}) = \text{vol}((\bar{\Lambda} + \bar{X}) \cap P).$$

Regarding the volume of the sum of two sets $\bar{X}_1, \bar{X}_2 \subset \mathbb{R}^n/\bar{\Lambda}$ we have the following classical sum theorem of Kneser and Macbeath [Gruber 2007, Theorem 26.1]:

$$(3-1) \quad \text{vol}_T(\bar{X}_1 \oplus \bar{X}_2) \geq \min\{\text{vol}_T(\bar{X}_1) + \text{vol}_T(\bar{X}_2), \det \bar{\Lambda}\}.$$

We also note that for an o -symmetric convex body $K \in \mathfrak{K}_o^n$ and $\lambda \geq 0$ the set $\bar{\Lambda} + \lambda K$ forms a lattice packing, i.e., for any two different lattice points $\bar{a}, \bar{b} \in \bar{\Lambda}$ the translates $\bar{a} + \lambda K$ and $\bar{b} + \lambda K$ do not overlap if and only if $\lambda \leq \lambda_1(K, \bar{\Lambda})/2$. Hence we know that, for $0 \leq \lambda \leq \lambda_1(K, \bar{\Lambda})/2$,

$$(3-2) \quad \text{vol}_T(\lambda K/\bar{\Lambda}) = \text{vol}((\lambda K + \bar{\Lambda}) \cap P) = \lambda^n \text{vol } K.$$

Furthermore, we also need a “torus version” of van der Corput’s result (2-1):

Lemma 3.1. *Let $K \in \mathfrak{K}_o^n$, $\Lambda \in \mathcal{L}^n$, $\text{rg } \Lambda = n$ and let $\bar{\Lambda} \subsetneq \Lambda$ be a sublattice with $\text{rg } \bar{\Lambda} = n$, and let $m \in \mathbb{N}$ with $m \det \Lambda < \det \bar{\Lambda}$. If $\text{vol}_T(\frac{1}{2}K/\bar{\Lambda}) \geq m \det \Lambda$ then*

$$\#(K/\bar{\Lambda} \cap \Lambda) \geq m + 1;$$

i.e., K contains at least $m + 1$ lattice points of Λ belonging to different cosets modulo $\bar{\Lambda}$.

Proof. By the compactness of K and the discreteness of lattices we may assume $\text{vol}_T(\frac{1}{2}K/\bar{\Lambda}) > m \det \Lambda$. Let P be a fundamental parallelepiped of the lattice $\bar{\Lambda}$. Then by assumption we have for the measurable set $X = (\frac{1}{2}K + \bar{\Lambda}) \cap P$ that $\text{vol } X > m \det \Lambda$. According to a result of van der Corput [Gruber and Lekkerkerker

[1987, Section 6.1, Theorem 1] we know that there exists pairwise different $x_i \in X$, $1 \leq i \leq m+1$, such that $x_i - x_j \in \Lambda$. By the o -symmetry and convexity of K we have $(X - X) = (K + \bar{\Lambda}) \cap (P - P)$, and since $(P - P) \cap \bar{\Lambda} = \{0\}$ we conclude that

$$x_i - x_j \in (K + \bar{\Lambda}) \cap \Lambda \setminus \bar{\Lambda}, \quad i \neq j.$$

Hence the m points $x_i - x_1 \in K + \bar{\Lambda}$, $i = 2, \dots, m+1$, belong to different nontrivial cosets of Λ modulo $\bar{\Lambda}$ and thus $\#(K/\bar{\Lambda} \cap \Lambda) \geq m+1$, where the additional 1 counts the origin. \square

We now state some simple facts on the intersection of full-dimensional sublattices.

Lemma 3.2. *Let $\Lambda \in \mathcal{L}^n$, $\Lambda_i \subseteq \Lambda$, $1 \leq i \leq s$, $\text{rg } \Lambda_i = \text{rg } \Lambda = n$, and let $\bar{\Lambda} = \bigcap_{i=1}^s \Lambda_i$. Then $\bar{\Lambda} \in \mathcal{L}^n$ with $\text{rg } \bar{\Lambda} = n$, and*

$$\max_{1 \leq i \leq s} \det \Lambda_i \leq \det \bar{\Lambda} \leq (\det \Lambda)^{1-s} (\det \Lambda_1) \cdots (\det \Lambda_s).$$

Moreover, with $m = \sum_{i=1}^s \det \bar{\Lambda} / \det \Lambda_i - s + 1$ we have:

- (i) The union $\bigcup_{i=1}^s \Lambda_i$ is covered by at most m cosets of Λ modulo $\bar{\Lambda}$.
- (ii) If $\det \bar{\Lambda} / \det \Lambda \geq m + 1$ then $\Lambda \neq \bigcup_{i=1}^s \Lambda_i$.

Proof. In order to show that $\bar{\Lambda}$ is a full-dimensional lattice it suffices to consider $s = 2$. Obviously, $\Lambda_1 \cap \Lambda_2$ is a discrete subgroup of Λ and it also contains n linearly independent points, e.g., $(\det \Lambda_2)a_1, \dots, (\det \Lambda_2)a_n$, where a_1, \dots, a_n is a basis of Λ_1 . Hence $\bar{\Lambda}$ is a full-dimensional lattice; see [Gruber and Lekkerkerker 1987, Section 3.2, Theorem 2]. The lower bound on $\det \bar{\Lambda}$ is clear by the inclusion $\bar{\Lambda} \subseteq \Lambda_i$, $1 \leq i \leq s$. For the upper bound we observe that two points $g, h \in \Lambda$ belong to different cosets modulo $\bar{\Lambda}$ if and only if g and h belong to different cosets of Λ modulo at least one Λ_i . There are $\det \Lambda_i / \det \Lambda$ many cosets for each i and so we get the upper bound.

For (i) we note that since Λ_i is the union of $\det \bar{\Lambda} / \det \Lambda_i$ many cosets modulo $\bar{\Lambda}$, the union is certainly covered by $\sum_{i=1}^s \det \bar{\Lambda} / \det \Lambda_i = m + s - 1$ many cosets of Λ modulo $\bar{\Lambda}$. But here we have counted the trivial coset at least s times. Part (ii) is a direct consequence of part (i). \square

Lemma 3.2(ii) implies, in particular, that the union of two strict sublattices can never be the whole lattice. This is no longer true for three sublattices, as we see in the next example, which also shows that Lemma 3.2(ii) is not an equivalence.

Example 3.3. Let $\Lambda = \mathbb{Z}^2$, and let $\Lambda_1, \dots, \Lambda_4 \subset \mathbb{Z}^2$ be the sublattices

$$\begin{aligned} \Lambda_1 &= \{(z_1, z_2)^T \in \mathbb{Z}^2 : z_2 \equiv 0 \pmod{2}\}, & \Lambda_2 &= \{(z_1, z_2)^T \in \mathbb{Z}^2 : z_1 \equiv 0 \pmod{2}\}, \\ \Lambda_3 &= \{(z_1, z_2)^T \in \mathbb{Z}^2 : z_2 \equiv 0 \pmod{3}\}, & \Lambda_4 &= \{(z_1, z_2)^T \in \mathbb{Z}^2 : z_1 \equiv z_2 \pmod{2}\}. \end{aligned}$$

Then $\Lambda_1 \cup \Lambda_2 \cup \Lambda_4 = \Lambda$ but $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \neq \Lambda$. Furthermore $\det \Lambda = 1$, $\det \Lambda_1 = \det \Lambda_2 = \det \Lambda_4 = 2$, $\det \Lambda_3 = 3$ and

$$\bar{\Lambda} = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3 = \{(z_1, z_2)^T \in \mathbb{Z}^2 : z_1 \equiv 0 \pmod{2}, z_2 \equiv 0 \pmod{6}\}$$

with $\det \bar{\Lambda} = 12$, while $\sum_{i=1}^3 \frac{\det \bar{\Lambda}}{\det \Lambda_i} - 1 = 15$.

We now come to the proof of the full-dimensional case.

Proof of Theorem 1.2. Let $\Lambda_1, \dots, \Lambda_s$ be the full-dimensional forbidden sublattices of the given lattice Λ and let $\bar{\Lambda} = \bigcap_{i=1}^s \Lambda_i$. Let

$$m = \min \left\{ \sum_{i=1}^s \frac{\det \bar{\Lambda}}{\det \Lambda_i} - s + 1, \frac{\det \bar{\Lambda}}{\det \Lambda} \right\}.$$

Claim 1. Let $\lambda > 0$ with $\text{vol}_T((\lambda \frac{1}{2} K) / \bar{\Lambda}) \geq m \det \Lambda$. Then

$$\lambda_1 \left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i \right) \leq \lambda.$$

To verify the claim, we first assume

$$m = \sum_{i=1}^s \frac{\det \bar{\Lambda}}{\det \Lambda_i} - s + 1 < \frac{\det \bar{\Lambda}}{\det \Lambda}.$$

By Lemma 3.1, λK contains $m + 1$ lattice points of Λ belonging to different cosets with respect to $\bar{\Lambda}$. By Lemma 3.2 (i), $\bigcup_{i=1}^s \Lambda_i$ is covered by at most m cosets of Λ modulo $\bar{\Lambda}$, and thus λK contains a lattice point of $\Lambda \setminus \bigcup_{i=1}^s \Lambda_i$.

Next suppose that $m = \det \bar{\Lambda} / \det \Lambda$. Then

$$\text{vol}_T((\lambda \frac{1}{2} K) / \bar{\Lambda}) = \det \bar{\Lambda} = \text{vol}_T(\mathbb{R}^n / \bar{\Lambda})$$

and, in particular, λK contains a representative of each coset of Λ modulo $\bar{\Lambda}$. By assumption there exists a coset containing a point $a \in \Lambda \setminus \bigcup_{i=1}^s \Lambda_i$, and hence all points of this coset, that is $a + \bar{\Lambda}$, lie in $\Lambda \setminus \bigcup_{i=1}^s \Lambda_i$.

This verifies the claim and it remains to compute a λ with

$$(3-3) \quad \text{vol}_T((\lambda \frac{1}{2} K) / \bar{\Lambda}) \geq m \det \Lambda.$$

To this end we set $\lambda_1 = \lambda_1(K, \bar{\Lambda})$ and we write an arbitrary $\lambda > 0$ modulo λ_1 in the form $\lambda = \lfloor \lambda / \lambda_1 \rfloor \lambda_1 + \rho \lambda_1$, with $0 \leq \rho < 1$. Hence, in view of the sum theorem of Kneser and Macbeath (3-1) and the packing property (3-2) of λ_1 with respect to $\frac{1}{2} K$, we may write

$$\begin{aligned} \text{vol}_T((\lambda \frac{1}{2} K) / \bar{\Lambda}) &= \text{vol}_T\left(\left(\left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor \frac{\lambda_1}{2} + \rho \frac{\lambda_1}{2}\right) K\right) / \bar{\Lambda} \\ &= \text{vol}_T\left(\underbrace{\left(\frac{\lambda_1}{2} K\right) / \bar{\Lambda} \oplus \dots \oplus \left(\frac{\lambda_1}{2} K\right) / \bar{\Lambda}}_{\lfloor \lambda / \lambda_1 \rfloor} \oplus \left(\frac{\rho \lambda_1}{2} K\right) / \bar{\Lambda}\right) \\ &\geq \min\left\{\left(\left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor + \rho^n\right) \left(\frac{\lambda_1}{2}\right)^n \text{vol } K, \det \bar{\Lambda}\right\}. \end{aligned}$$

Thus, (3-3) is certainly satisfied for a $\bar{\lambda}$ with

$$(3-4) \quad \left(\left\lfloor \frac{\bar{\lambda}}{\lambda_1} \right\rfloor + \rho^n\right) \left(\frac{\lambda_1}{2}\right)^n \text{vol } K = \left(\sum_{i=1}^s \frac{\det \bar{\Lambda}}{\det \Lambda_i} - s + 1\right) \det \Lambda.$$

Using that

$$(3-5) \quad \left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor + \rho^n > \frac{\lambda - \lambda_1}{\lambda_1},$$

we find

$$\lambda_1\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) \leq \bar{\lambda} < \frac{2^n \det \Lambda}{\lambda_1^{n-1} \text{vol } K} \left(\sum_{i=1}^s \frac{\det \bar{\Lambda}}{\det \Lambda_i} - s + 1\right) + \lambda_1. \quad \square$$

Remark 3.4. The bound in [Theorem 1.2](#) can be slightly improved in lower dimensions by noticing that in (3-5) we may replace $(\lambda - \lambda_1) / \lambda_1$ by $\lambda / \lambda_1 - \rho + \rho^n$. Since $\rho - \rho^n$ takes its maximum at $\rho = (1/n)^{1/(n-1)}$ we get in this way

$$\begin{aligned} \lambda_1\left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i\right) &\leq \frac{2^n \det \Lambda}{\lambda_1(K, \bar{\Lambda})^{n-1} \text{vol } K} \left(\sum_{i=1}^s \frac{\det \bar{\Lambda}}{\det \Lambda_i} - s + 1\right) + n^{-1/(n-1)} \frac{n-1}{n} \lambda_1(K, \bar{\Lambda}). \end{aligned}$$

There is a straightforward way to extend [Theorem 1.2](#) to higher successive minima which we will first present in the special case $s = 1$.

Corollary 3.5. *Under the assumptions of [Corollary 1.3](#) we have, for $1 \leq i \leq n$,*

$$\lambda_i(K, \Lambda \setminus \Lambda_1) \leq \frac{2^n \det \Lambda}{\lambda_1(K, \Lambda_1)^{n-1} \text{vol } K} + \lambda_1(K, \Lambda) + \lambda_i(K, \Lambda).$$

Proof. By [Corollary 1.3](#) it suffices to show $\lambda_i(K, \Lambda \setminus \Lambda_1) \leq \lambda_1(K, \Lambda \setminus \Lambda_1) + \lambda_i(K, \Lambda)$ for $i = 2, \dots, n$. To this end let $a \in \lambda_1(K, \Lambda \setminus \Lambda_1) K \cap \Lambda \setminus \Lambda_1$ and let b_1, \dots, b_n be linearly independent with $b_j \in \lambda_j(K, \Lambda) K \cap \Lambda, j = 1, \dots, n$. Since not both b_j and $a + b_j$ can belong to the forbidden sublattice Λ_1 we can select from each pair $b_j, a + b_j$ one contained in $\Lambda \setminus \Lambda_1, 1 \leq j \leq n$. Let these points be denoted by $\bar{b}_j, j = 1, \dots, n$. Then $a, \bar{b}_j \in (\lambda_1(K, \Lambda \setminus \Lambda_1) + \lambda_j(K, \Lambda)) K, 1 \leq j \leq n$.

Now choose k such that $a \notin \text{lin}(\{b_1, \dots, b_n\} \setminus \{b_k\})$. Then the lattice points $a, \bar{b}_1, \dots, \bar{b}_{k-1}, \bar{b}_{k+1}, \dots, \bar{b}_n$ are linearly independent and we are done. \square

For $s > 1$ the excluded substructure $\bigcup_{i=1}^s \Lambda_i$ is, in general, not a lattice anymore and so we cannot argue as above. Therefore, in this case, we choose the vectors b_j , $1 \leq j \leq n$, from the lattice $\bar{\Lambda} = \bigcap_{i=1}^s \Lambda_i$. Then for $a \in \Lambda \setminus \bigcup_{i=1}^s \Lambda_i$ we have

$$a, a + b_1, a + b_2, \dots, a + b_n \in \Lambda \setminus \bigcup_{i=1}^s \Lambda_i,$$

and analogously to the proof of [Corollary 3.5](#) we get:

Corollary 3.6. *Under the assumptions of [Theorem 1.2](#) we have, for $1 \leq i \leq n$,*

$$\begin{aligned} \lambda_i \left(K, \Lambda \setminus \bigcup_{i=1}^s \Lambda_i \right) \\ \leq \frac{2^n \det \Lambda}{\lambda_1(K, \bar{\Lambda})^{n-1} \text{vol } K} \left(\sum_{i=1}^s \frac{\det \bar{\Lambda}}{\det \Lambda_i} - s + 1 \right) + \lambda_1(K, \bar{\Lambda}) + \lambda_i(K, \bar{\Lambda}). \end{aligned}$$

Remark 3.7. It is also possible to extend lower-dimensional lattices to lattices of full rank by adjoining “sufficiently large” vectors, i.e., for each Λ_i of rank n_i choose linearly independent $z_{i,n_i+1}, \dots, z_{i,n} \in \Lambda \setminus \Lambda_i$ and consider the lattice $\bar{\Lambda}_i$ spanned by Λ_i and $z_{i,n_i+1}, \dots, z_{i,n}$. If $z_{i,j}$ are such that $\lambda_j(K, \bar{\Lambda}_i)$ is very large for $j > n_i$, one can apply the results from [Section 3](#) to the collection $\bar{\Lambda}_i$, $1 \leq i \leq s$. However, the bounds obtained in this way are in general weaker, with one exception in the case $s = 1$ for the bound on $\lambda_1(K, \Lambda \setminus \Lambda_1)$. Here we get

$$\lambda_1(K, \Lambda \setminus \Lambda_1) \leq \frac{2^n \det \Lambda}{\lambda_1(K, \Lambda_1)^{n-1} \text{vol } K} + \lambda_1(K, \Lambda)$$

for $\Lambda_1 \subsetneq \Lambda$ with $\text{rg } \Lambda_1 < n$, which improves on [Theorem 1.1](#).

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MARTIN HENK
martin.henk@ovgu.de

CARSTEN THIEL
carsten.thiel@gmail.com

FAKULTÄT FÜR MATHEMATIK
OTTO-VON-GUERICKE UNIVERSITÄT MAGDEBURG
UNIVERSITÄTSPLATZ 2
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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
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University of California
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chari@math.ucr.edu

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Department of Mathematics
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liu@math.ucla.edu

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Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

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