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RADIAL SOLUTIONS OF NON-ARCHIMEDEAN PSEUDODIFFERENTIAL EQUATIONS

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We consider a class of equations with the fractional differentiation operator D^α, $\alpha > 0$, for complex-valued functions $x \mapsto f(|x|_K)$ on a non-Archimedean local field K depending only on the absolute value $|\cdot|_K$. We introduce a right inverse I^α to D^α, such that the change of an unknown function $u = I^\alpha v$ reduces the Cauchy problem for an equation with D^α (for radial functions) to an integral equation whose properties resemble those of classical Volterra equations. This contrasts much more complicated behavior of D^α on other classes of functions.

1. Introduction

Pseudodifferential equations for complex-valued functions defined on a non-Archimedean local field are among the central objects of contemporary harmonic analysis and mathematical physics; see the monographs [Vladimirov et al. 1994; Kochubei 2001; Albeverio et al. 2010], and the survey [Zúñiga-Galindo 2011].

The simplest example is the fractional differentiation operator D^α, $\alpha > 0$, on the field $\mathbb{Q}_p$ of p-adic numbers (here p is a prime number). It can be defined as a pseudodifferential operator with the symbol $|\xi|_p^\alpha$ where $|\cdot|_p$ is the p-adic absolute value or, equivalently, as an appropriate convolution operator.

Already in this case, as it was first shown by Vladimirov (see [Vladimirov et al. 1994]), properties of the p-adic pseudodifferential operator are much more complicated than those of its classical counterpart. It suffices to say that, as an operator on $L_2(\mathbb{Q}_p)$, it has a point spectrum of infinite multiplicity. Considering a simple "formal" evolution equation with the operator D^α in the p-adic time variable t, Vladimirov [2003] noticed that such an equation does not possess a fundamental solution.

At the same time, it was found in [Kochubei 2008] that some of the evolution equations of the above kind behave reasonably, if one considers only solutions

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depending on $|t|_p$. This observation has led to the concept of a non-Archimedean wave equation possessing various properties resembling those of classical hyperbolic equations, up to the Huygens principle.

In this paper we consider the Cauchy problem for a class of equations like

$$D^\alpha u + a(|x|_p)u = f(|x|_p), \quad x \in \mathbb{Q}_p, \tag{1-1}$$

assuming that a solution is looked for in the class of radial functions, $u = u(|x|_p)$; the precise definition of D^α and assumptions on a, f are given below. This Cauchy problem is reduced to an integral equation resembling classical Volterra equations. It turns out that (1-1) and its generalizations considered on radial functions constitute p-adic counterparts of ordinary differential equations.

2. Preliminaries

2.1. *Local fields.* Let K be a non-Archimedean local field, that is, a nondiscrete totally disconnected locally compact topological field. It is well known that K is isomorphic either to a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers (if K has characteristic 0), or to the field of formal Laurent series with coefficients from a finite field, if K has a positive characteristic. For a summary of main notions and results regarding local fields see, for example, [Kochubei 2001].

Any local field K is endowed with an absolute value $|\cdot|_K$, such that $|x|_K = 0$ if and only if $x = 0$, $|xy|_K = |x|_K \cdot |y|_K$, $|x + y|_K \le \max(|x|_K, |y|_K)$. Set

$$O = \{x \in K : |x|_K \le 1\}, \quad P = \{x \in K : |x|_K < 1\}.$$

Then O is a subring of K, and P is an ideal in O containing such an element β that $P = \beta O$. The quotient ring O/P is actually a finite field; denote by q its cardinality. We will always assume that the absolute value is normalized, that is $|\beta|_K = q^{-1}$. The normalized absolute value takes the values q^N, $N \in \mathbb{Z}$. Note that for $K = \mathbb{Q}_p$ we have $\beta = p$ and $q = p$; the p-adic absolute value is normalized.

Denote by $S \subset O$ a complete system of representatives of the residue classes from O/P. Any nonzero element $x \in K$ admits the canonical representation in the form of the convergent series

$$x = \beta^{-n}(x_0 + x_1\beta + x_2\beta^2 + \cdots) \tag{2-1}$$

where $n \in \mathbb{Z}$, $|x|_K = q^n$, $x_j \in S$, $x_0 \notin P$. For $K = \mathbb{Q}_p$, one may take $S = \{0, 1, \ldots, p-1\}$.

The additive group of any local field is self-dual; that is, if χ is a fixed nonconstant complex-valued additive character of K, then any other additive character can be written as $\chi_a(x) = \chi(ax)$, $x \in K$, for some $a \in K$. Below we assume that χ is a rank zero character, that is $\chi(x) \equiv 1$ for $x \in O$, while there exists such an element $x_0 \in K$ that $|x_0|_K = q$ and $\chi(x_0) \neq 1$.

This duality is used in the definition of the Fourier transform over K. Denoting by dx the Haar measure on the additive group of K (normalized in such a way that the measure of O equals 1) we write

$$\tilde{f}(\xi) = \int_K \chi(x\xi) f(x)\, dx, \quad \xi \in K,$$

where f is a complex-valued function from $L_1(K)$. As usual, the Fourier transform $\mathcal{F}$ can be extended from $L_1(K) \cap L_2(K)$ to a unitary operator on $L_2(K)$. If $\mathcal{F} f = \tilde{f} \in L_1(K)$, we have the inversion formula

$$f(x) = \int_K \chi(-x\xi) \tilde{f}(\xi)\, d\xi.$$

2.2. ***Integration formulas.*** As in real analysis, there are many well known formulas for integrals of complex-valued functions defined on subsets of a local field. There exist even tables of such integrals [Vladimirov 2003]. Note that formulas for integrals on $\mathbb{Q}_p$ and its subsets, as a rule, carry over to the general case, if one substitutes the normalized absolute value for $|\cdot|_p$ and q for p.

Here we collect some formulas used in this work. Let $n \in \mathbb{Z}$, $\alpha > 0$.

$$\int\limits_{|x|_K \le q^n} |x|_K^{\alpha-1}\, dx = \frac{1-q^{-1}}{1-q^{-\alpha}} q^{\alpha n}, \tag{2-2}$$

$$\int\limits_{|x|_K = q^n} |x-a|_K^{\alpha-1}\, dx = \frac{q-2+q^{-\alpha}}{q(1-q^{-\alpha})} |a|_K^{\alpha}, \quad |a|_K = q^n. \tag{2-3}$$

$$\int\limits_{|x|_K \le q^n} \log|x|_K\, dx = \Big(n - \frac{1}{q-1}\Big) q^n \log q. \tag{2-4}$$

$$\int\limits_{|x|_K = q^n} \log|x-a|_K\, dx = \Big[\Big(1-\frac{1}{q}\Big)\log|a|_K - \frac{\log q}{q-1}\Big]|a|_K, \quad |a|_K = q^n. \tag{2-5}$$

$$\int\limits_{|x|_K \le q^n} dx = q^n; \quad \int\limits_{|x|_K = q^n} dx = \Big(1-\frac{1}{q}\Big) q^n. \tag{2-6}$$

$$\int\limits_{\substack{|x|_K = q^n \\ x_0 = k_0}} dx = q^{n-1}, \quad 0 \neq k_0 \in S \tag{2-7}$$

(the restriction $x_0 = k_0$ is in the sense of the canonical representation (2-1)).

$$\int\limits_{\substack{|x|_K=q^n\\ x_0\neq k_0}} dx = \Big(1-\frac{2}{q}\Big)q^n. \tag{2-8}$$

2.3. ***Test functions and distributions.*** A function $f : K \to \mathbb{C}$ is said to be locally constant, if there exists such an integer l that for any $x \in K$

$$f(x+x') = f(x), \quad \text{whenever } |x'| \le q^{-l}.$$

The smallest number l with this property is called the exponent of local constancy of the function f.

Let $\mathfrak{D}(K)$ be the set of all locally constant functions with compact supports; it is a vector space over $\mathbb{C}$ with the topology of double inductive limit

$$\mathfrak{D}(K) = \varinjlim_{N\to\infty} \varinjlim_{l\to\infty} \mathfrak{D}_N^l$$

where $\mathfrak{D}_N^l$ is the finite dimensional space of functions supported in the ball $B_N = \{x \in K : |x| \le q^N\}$ and having the exponents of local constancy $\le l$. The strong conjugate space $\mathfrak{D}'(K)$ is called the space of Bruhat–Schwartz distributions.

The Fourier transform preserves the space $\mathfrak{D}(K)$. Therefore the Fourier transform of a distribution defined by duality acts continuously on $\mathfrak{D}'(K)$. As in the case of $\mathbb{R}^n$, there exists a well developed theory of distributions over local fields; it includes such topics as convolution, direct product, homogeneous distributions etc (see [Vladimirov et al. 1994; Kochubei 2001; Albeverio et al. 2010]). In connection with homogeneous distributions, it is useful to introduce the subspaces of $\mathfrak{D}(K)$:

$$\Psi(K) = \{\psi \in \mathfrak{D}(K) : \psi(0) = 0\},$$
$$\Phi(K) = \Big\{\varphi \in \mathfrak{D}(K) : \int_K \varphi(x)\,dx = 0\Big\}.$$

The Fourier transform $\mathcal{F}$ is a linear isomorphism from $\Psi(K)$ onto $\Phi(K)$, thus also from $\Phi'(K)$ onto $\Psi'(K)$. The spaces $\Phi(K)$ and $\Phi'(K)$ are called the Lizorkin spaces (of the second kind) of test functions and distributions respectively; see [Albeverio et al. 2010]. Note that two distributions differing by a constant summand coincide as elements of $\Phi'(K)$.

3. Fractional differentiation and integration operators

3.1. ***Riesz kernels and operators generated by them.*** On a test function $\varphi \in \mathfrak{D}(K)$, the fractional differentiation operator D^α, $\alpha > 0$, is defined as

$$(D^\alpha\varphi)(x) = \mathcal{F}^{-1}\big[|\xi|_K^\alpha\big(\mathcal{F}(\varphi)\big)(\xi)\big](x). \tag{3-1}$$

However D^α does not act on the space $\mathcal{D}(K)$, since the function $\xi \mapsto |\xi|_K^\alpha$ is not locally constant. On the other hand, $D^\alpha : \Phi(K) \to \Phi(K)$ and $D^\alpha : \Phi'(K) \to \Phi'(K)$; see [Albeverio et al. 2010], and that was a motivation to introduce these spaces.

The operator D^α can also be represented as a hypersingular integral operator:

$$(D^\alpha \varphi)(x) = \frac{1-q^\alpha}{1-q^{-\alpha-1}} \int_K |y|_K^{-\alpha-1} [\varphi(x-y) - \varphi(x)]\, dy \tag{3-2}$$

[Vladimirov et al. 1994; Kochubei 2001]. In contrast to (3-1), the expression in the right of (3-2) makes sense for wider classes of functions. Below we study this in detail for the case of radial functions.

The expression in (3-2) is in fact the convolution $f_{-\alpha} * \varphi$, where the Riesz kernel f_s, for complex $s \notin 1 + \frac{2\pi i}{\log q}\mathbb{Z}$, is defined first for $\operatorname{Re} s > 0$ as

$$f_s(x) = \frac{|x|_K^{s-1}}{\Gamma_K(s)}, \quad \Gamma_K(s) = \frac{1-q^{s-1}}{1-q^{-s}},$$

and then extended meromorphically to the remaining nonzero values of s as a distribution from $\mathcal{D}'(K)$:

$$\langle f_s, \varphi \rangle = \frac{1-q^{-1}}{1-q^{s-1}} \varphi(0) + \frac{1-q^{-s}}{1-q^{s-1}} \Bigg[\int_{|x|_K > 1} \varphi(x) \frac{dx}{|x|_K^{1-s}} + \int_{|x|_K \le 1} (\varphi(x) - \varphi(0)) \frac{dx}{|x|_K^{1-s}} \Bigg],$$

For $s = 0$, we set $f_0(x) = \delta(x)$. For $s \in 1 + \frac{2\pi i}{\log q}\mathbb{Z}$, we define

$$f_s(x) = \frac{1-q}{\log q} \log |x|_K.$$

It is well known that $f_s * f_t = f_{s+t}$ in the sense of distributions from $\mathcal{D}'(K)$, so long as $s, t, s+t \notin 1 + \frac{2\pi i}{\log q}\mathbb{Z}$. If these kernels are considered as distributions from $\Phi'(K)$, then $f_s * f_t = f_{s+t}$ for all $s, t \in \mathbb{C}$ [Albeverio et al. 2010]. In view of this identity, it is natural to define the operator $D^{-\alpha}$, $\alpha > 0$, setting

$$(D^{-\alpha}\varphi)(x) = (f_\alpha * \varphi)(x) = \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int_K |x-y|_K^{\alpha-1} \varphi(y)\, dy, \qquad \varphi \in \mathcal{D}(K),\ \alpha \neq 1, \tag{3-3}$$

and

$$(D^{-1}\varphi)(x) = \frac{1-q}{q \log q} \int_K \log |x-y|_K \varphi(y)\, dy. \tag{3-4}$$

Then $D^\alpha D^{-\alpha} = I$ on $\mathcal{D}(K)$, if $\alpha \neq 1$. This remains valid on $\Phi(K)$ also for $\alpha = 1$.

The notions and results above are well known; see [Vladimirov et al. 1994; Albeverio et al. 2010]. We now come to new phenomena, considering the case of radial functions.

3.2. *Operators on radial functions.* Let u be a radial function, that is $u = u(|x|_K)$, $x \in K$. (In order to make the notation concise, we identify the function $x \mapsto u(|x|_K)$ on K with the function $|x|_K \mapsto u(|x|_K)$ on $q^{\mathbb{Z}}$. This abuse of notation will not lead to confusion.)

Let us find an explicit expression for $D^\alpha u$, $\alpha > 0$. Below we write $d_\alpha = (1-q^\alpha)/(1-q^{-\alpha-1})$. For $x \in K$, we denote by x_0 the element from $S \subset O$ appearing in the representation (2-1).

Lemma 1. *If a function $u = u(|x|_K)$ is such that*

$$\sum_{k=-\infty}^{m} q^k \left|u(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty} q^{-\alpha l} \left|u(q^l)\right| < \infty, \tag{3-5}$$

for some $m \in \mathbb{Z}$, then for each $n \in \mathbb{Z}$ the expression in the right-hand side of (3-2) with $\varphi(x) = u(|x|_K)$ exists for $|x|_K = q^n$, depends only on $|x|_K$, and

$$\begin{aligned}(D^\alpha u)(q^n) = d_\alpha\Bigl(1-\frac{1}{q}\Bigr)q^{-(\alpha+1)n} \sum_{k=-\infty}^{n-1} q^k u(q^k) + q^{-\alpha n-1}\frac{q^\alpha+q-2}{1-q^{-\alpha-1}}u(q^n)\\ + d_\alpha\Bigl(1-\frac{1}{q}\Bigr)\sum_{l=n+1}^{\infty} q^{-\alpha l}u(q^l).\end{aligned} \tag{3-6}$$

Proof. We find, using the ultrametric properties of the absolute value, that

$$(D^\alpha u)(x) = d_\alpha \int_{|y|_K \ge |x|_K} |y|_K^{-\alpha-1}[u(|x-y|_K) - u(|x|_K)]\,dy.$$

If $|y|_K = |x|_K$ and $y_0 \neq x_0$, the integrand vanishes. Therefore, by (2-6),

$$\begin{aligned}(D^\alpha u)(x) &= d_\alpha \sum_{k=-\infty}^{n-1} \int_{|y-x|_K = q^k} |x|_K^{-\alpha-1}[u(q^k) - u(q^n)]\,dy\\ &\quad + d_\alpha \sum_{l=n+1}^{\infty} \int_{|y|_K = q^l} q^{-l(\alpha+1)}[u(q^k) - u(q^n)]\,dy\\ &= d_\alpha\Bigl(1-\frac{1}{q}\Bigr)q^{-(\alpha+1)n}\sum_{k=-\infty}^{n-1} q^k[u(q^k)-u(q^n)]\\ &\quad + d_\alpha\Bigl(1-\frac{1}{q}\Bigr)\sum_{l=n+1}^{\infty} q^{-\alpha l}[u(q^l)-u(q^n)].\end{aligned}$$

It is clear from this expression that $(D^\alpha u)(x)$, $|x|_K = q^n$, depends only on $|x|_K$. After elementary transformations we get (3-6). $\square$

Definition. We say that the action $D^\alpha u$, $\alpha > 0$, on a radial function u is *defined in the strong sense* if the function u satisfies (3-5), so that $D^\alpha u(|x|_K)$, $|x|_K \neq 0$, is given by (3-6), and there exists the limit

$$D^\alpha u(0) \overset{\text{def}}{=} \lim_{x \to 0} D^\alpha u(|x|_K).$$

It is evident from (3-2) that D^α annihilates constant functions (recall that in $\Phi'(K)$ they are equivalent to zero). Therefore $D^{-\alpha}$ is not the only possible choice of the right inverse to D^α. In particular, we will use

$$(I^\alpha \varphi)(x) = (D^{-\alpha}\varphi)(x) - (D^{-\alpha}\varphi)(0). \tag{3-7}$$

This is defined initially for $\varphi \in \mathfrak{D}(K)$. It is seen from (3-3), (3-4), and the ultrametric property of the absolute value that

$$(I^\alpha \varphi)(x) = \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int_{|y|_K \le |x|_K} \big(|x-y|_K^{\alpha-1} - |y|_K^{\alpha-1}\big)\varphi(y)\,dy, \quad \alpha \neq 1, \tag{3-8}$$

and

$$(I^1 \varphi)(x) = \frac{1-q}{q \log q} \int_{|y|_K \le |x|_K} \big(\log|x-y|_K - \log|y|_K\big)\varphi(y)\,dy. \tag{3-9}$$

In contrast to (3-3) and (3-4), in (3-8) and (3-9) the integrals are taken, for each fixed $x \in K$, over bounded sets.

Let us calculate $I^\alpha u$ for a radial function $u = u(|x|_K)$. Obviously, $(I^\alpha u)(0) = 0$ whenever I^α is defined.

Lemma 2. *Suppose that, for some $m \in \mathbb{Z}$,*

$$\sum_{k=-\infty}^{m} \max\big(q^k, q^{\alpha k}\big)\big|u(q^k)\big| < \infty \quad \text{if } \alpha \neq 1,$$

and

$$\sum_{k=-\infty}^{m} |k| q^k \big|u(q^k)\big| < \infty \qquad \text{if } \alpha = 1.$$

Then $I^\alpha u$ exists, it is a radial function, and for any $x \neq 0$, we have

$$(I^\alpha u)(|x|_K) = q^{-\alpha}|x|_K^\alpha u(|x|_K) + \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int_{|y|_K < |x|_K} \big(|x|_K^{\alpha-1} - |y|_K^{\alpha-1}\big) u(|y|_K)\,dy \tag{3-10}$$

if $\alpha \neq 1$, and

(3-11) $$(I^1u)(|x|_K) = q^{-1}|x|_K u(|x|_K) + \frac{1-q}{q\log q}\int_{|y|_K<|x|_K} \bigl(\log|x|_K - \log|y|_K\bigr) u(|y|_K)\,dy.$$

Proof. It is sufficient to compute the integrals over the set $\{y \in K : |y|_K = |x|_K\}$, and that is done using the integration formulas (2-3) and (2-5). □

It follows from Lemma 2 that the function $I^\alpha u$ is continuous if, for example, u is bounded near the origin (see an estimate of the integral $I_{\alpha,0}$ in the proof of Theorem 1 below). If $|u(|x|_K)| \le C|x|_K^{-\varepsilon}$, as $|x|_K \ge 1$, then $|(I^\alpha u)(|x|_K)| \le C|x|_K^{\alpha-\varepsilon}$, as $|x|_K \ge 1$. Here and below we denote by C various (possibly different) positive constants.

It is easy to transform (3-10) and (3-11) further obtaining series involving $u(q^n)$.

Obviously, $D^\alpha I^\alpha = I$ on $\mathfrak{D}(K)$, if $\alpha \ne 1$, and on $\Phi(K)$, if $\alpha = 1$. Since by Lemma 1 and Lemma 2, the operators are defined in a straightforward sense for wider classes of functions, it is natural to look for conditions sufficient for this identity.

Lemma 3. *Suppose that for some $m \in \mathbb{Z}$,*

$$\sum_{k=-\infty}^{m} \max(q^k, q^{\alpha k})\bigl|v(q^k)\bigr| < \infty \quad \textit{and} \quad \sum_{l=m}^{\infty}\bigl|v(q^l)\bigr| < \infty \quad \textit{if } \alpha \ne 1,$$

$$\sum_{k=-\infty}^{m} |k|q^k\,\bigl|v(q^k)\bigr| < \infty \quad \textit{and} \quad \sum_{l=m}^{\infty} l\bigl|v(q^l)\bigr| < \infty \quad \textit{if } \alpha = 1.$$

Then there exists $(D^\alpha I^\alpha v)(|x|_K) = v(|x|_K)$ for any $x \ne 0$.

The proof consists of tedious but quite elementary calculations based on the integration formulas (2-2)–(2-8). A relatively nontrivial tool is the sum formula for the arithmetic-geometric progression (from [Gradshteyn and Ryzhik 1996, Formula 0.113]).

Using Lemma 3, we can consider the simplest Cauchy problem

$$D^\alpha u(|x|_K) = f(|x|_K), \quad u(0) = 0,$$

where f is a continuous function, such that

$$\sum_{l=m}^{\infty}\bigl|f(q^l)\bigr| < \infty, \text{ if } \alpha \ne 1, \quad \text{or} \quad \sum_{l=m}^{\infty} l\bigl|f(q^l)\bigr| < \infty, \text{ if } \alpha = 1.$$

The unique strong solution is $u = I^\alpha f$; the uniqueness follows from the fact that the equality $D^\alpha u = 0$ (in the sense of $\mathfrak{D}'(K)$) implies the equality $u = \text{const}$; see [Vladimirov et al. 1994] or [Kochubei 2001]. Therefore on radial functions, the operators D^α and I^α behave like the Caputo–Dzhrbashyan fractional derivative and the Riemann–Liouville fractional integral of real analysis (see, for example, [Kilbas et al. 2006]). However the next example illustrates a different behavior of the "fractional integral" in the non-Archimedean case.

Example. Let $f(|x|_K) \equiv 1$, $x \in K$. Then $(I^\alpha f)(|x|_K) \equiv 0$.

Proof. Let $|x|_K = q^n$. If $\alpha \neq 1$, then by (3-10), (2-2), and (2-6),

$$\begin{aligned}(I^\alpha f)(|x|_K) &= q^{-\alpha}|x|_K^\alpha + \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int_{|y|_K \le q^{n-1}} \bigl(|x|_K^{\alpha-1} - |y|_K^{\alpha-1}\bigr)\,dy \\ &= q^{-\alpha}|x|_K^\alpha + \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \left[q^{n-1}|x|_K^{\alpha-1} - \frac{1-q^{-1}}{1-q^{-\alpha}} q^{\alpha(n-1)}\right] \\ &= q^{-\alpha}|x|_K^\alpha + \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} |x|_K^\alpha \frac{q^{-1}-q^{-\alpha}}{1-q^{-\alpha}} = 0.\end{aligned}$$

If $\alpha = 1$, then by (3-11), (2-4), and (2-6),

$$\begin{aligned}(I^1 f)(|x|_K) &= q^{-1}|x|_K + \frac{1-q}{q\log q} \int_{|y|_K \le q^{n-1}} \bigl(\log|x|_K - \log|y|_K\bigr)\,dy \\ &= q^{-1}|x|_K + \frac{1-q}{q\log q}\left[q^{n-1}\log|x|_K - \Bigl(n-1-\frac{1}{q-1}\Bigr)q^{n-1}\log q\right] \\ &= |x|_K\left(q^{-1} + \frac{1-q}{q\log q}\Bigl(1+\frac{1}{q-1}\Bigr)q^{-1}\log q\right) = 0. \qquad \square\end{aligned}$$

Of course, these identities in the weaker sense of distributions from $\Phi'(K)$ are trivial, since the constant functions are identified with zero, I^α with $D^{-\alpha}$, and $D^\alpha D^{-\alpha} = I$.

On the other hand, the example shows that the condition of decay at infinity in Lemma 3 cannot be dropped.

4. Fractional differential equations

4.1. ***The Cauchy problem and an integral equation.*** In the class of radial functions $u = u(|x|_K)$, we consider the Cauchy problem

$$D^\alpha u + a(|x|_K)u = f(|x|_K), \quad x \in K, \tag{4-1}$$

$$u(0) = 0, \tag{4-2}$$

where a and f are continuous functions, that is, they have finite limits $a(0)$ and $f(0)$, as $x \to 0$.

Looking for a solution of the form $u = I^\alpha v$, where v is a radial function, we obtain formally an integral equation

$$v(|x|_K) + a(|x|_K)(I^\alpha v)(|x|_K) = f(|x|_K), \quad x \in K. \tag{4-3}$$

Let us study its solvability. Later we investigate, in what sense a solution of (4-3) corresponds to a solution of the Cauchy problem (4-1)–(4-2).

It follows from (4-3) that $v(0) = f(0)$. Suppose first that $\alpha \neq 1$. By Lemma 2, (4-3) can be written in the form

$$\text{(4-3}'\text{)}\quad \bigl[1 + q^{-\alpha} a(|x|_K)|x|_K^{\alpha}\bigr] v(|x|_K) + c_\alpha a(|x|_K) \int_{|y|_K < |x|_K} \bigl(|x|_K^{\alpha-1} - |y|_K^{\alpha-1}\bigr) v(|y|_K)\, dy = f(|x|_K), \quad x \neq 0,$$

where $c_\alpha = (1 - q^{-\alpha})/(1 - q^{\alpha-1})$.

Since a is continuous, there exists such $N \in \mathbb{Z}$ that

$$q^{-\alpha} a(|x|_K)|x|_K^{\alpha} < 1 \quad \text{for } |x|_K \le q^N.$$

On the ball $B_N = \{x \in K : |x|_K \le q^N\}$, the equation takes the form

$$\text{(4-4)}\qquad v(|x|_K) + \int_{|y|_K < |x|_K} k_\alpha(x, y) v(|y|_K)\, dy = F(|x|_K)$$

where

$$k_\alpha(x, y) = \begin{cases} \bigl[1 + q^{-\alpha} a(|x|_K)|x|_K^{\alpha}\bigr]^{-1} c_\alpha a(|x|_K)\bigl(|x|_K^{\alpha-1} - |y|_K^{\alpha-1}\bigr) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$F(|x|_K) = \bigl[1 + q^{-\alpha} a(|x|_K)|x|_K^{\alpha}\bigr]^{-1} f(|x|_K).$$

If we construct a solution of (4-4) on B_N, and if

$$\text{(4-5)}\qquad a(|x|_K) \neq -q^{\alpha m} \quad \text{for any } x \in K,\ m \in \mathbb{Z},$$

we will be able to construct a solution of (4-4), thus a solution of (4-3), successively for all $x \in K$.

If $\alpha = 1$, we use (3-11) and obtain in a similar way the equation (4-4) with

$$k_1(x,y) = \begin{cases} \dfrac{1-q}{q \log q}\bigl[1+q^{-1} a(|x|_K)|x|_K\bigr]^{-1} a(|x|_K)\bigl(\log|x|_K - \log|y|_K\bigr) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$F(|x|_K) = \bigl[1 + q^{-1} a(|x|_K)|x|_K\bigr]^{-1} f(|x|_K).$$

It is convenient to extend k_α (including the case $\alpha = 1$) by zero onto $B_N \times B_N$.

Theorem 1. *For each $\alpha > 0$, the integral equation (4-4) has a unique continuous solution on B_N.*

Proof. Let us consider the integral operator $\mathcal{K}$ appearing in (4-4) as an operator on the Banach space $C(B_N)$ of complex-valued continuous functions on B_N. By the theory of integral operators developed in sufficient generality in [Edwards 1965] (see Proposition 9.5.17), to prove that $\mathcal{K}$ is a compact operator, it suffices to check that, for any $x_0 \in B_N$,

$$\lim_{x\to x_0} \int_{B_N} |k_\alpha(x,y) - k_\alpha(x_0,y)|\,dy = 0. \tag{4-6}$$

The relation (4-6) is obvious for $x_0 \neq 0$, and also for $\alpha > 1$. For $x_0 = 0$, we have $k_\alpha(0,y) = 0$, and for $0 < \alpha < 1$, $|x|_K = q^n$, $n \geq N$, we get by (2-2) and (2-6) that

$$\begin{aligned}\int_{B_N} |k_\alpha(x,y)|\,dy &= \text{const} \int_{|y|_K \leq q^{n-1}} \left(|y|_K^{\alpha-1} - q^{n(\alpha-1)}\right) dy \\ &= \text{const}\left(\frac{1-q^{-1}}{1-q^{-\alpha}} q^{\alpha(n-1)} - q^{n(\alpha-1)}q^{n-1}\right) = \text{const}\,|x|_K^\alpha,\end{aligned}$$

which tends to 0 as $|x|_K \to 0$. For $\alpha = 1$, we use (2-4) and (2-6) to obtain that

$$\begin{aligned}\int_{B_N} |k_1(x,y)|\,dy &= \text{const} \int_{|y|_K \leq q^{n-1}} \left(\log q^n - \log |y|_K\right) dy \\ &= \text{const}\left(nq^{n-1}\log q - \left(n-1-\frac{1}{q-1}\right)q^{n-1}\log q\right) \\ &= \text{const}\,\frac{q}{q-1}q^{n-1}\log q = \text{const}\,\frac{\log q}{q-1}|x|_K,\end{aligned}$$

and this again tends to 0 as $|x|_K \to 0$.

Thus, $\mathcal{K}$ is compact, and by the Fredholm alternative [Edwards 1965, 9.10.3], our theorem will be proved if we show that $\mathcal{K}$ has no nonzero eigenvalues.

Suppose that $\mathcal{K}w = \lambda w$, $\lambda \neq 0$, for some $w \in C(B_N)$. We have $|w(y)| \leq C$,

$$|k_\alpha(x,y)| \leq M\left||x|_K^{\alpha-1} - |y|_K^{\alpha-1}\right|,$$

if $\alpha \neq 1$, and

$$|k_1(x,y)| \leq M(\log |x|_K - \log |y|_K),$$

if $\alpha = 1$, $|y|_K < |x|_K$.

In subsequent iterations we will deal with the integrals

$$I_{\alpha,m} = \int_{|y|_K < |x|_K} \left||x|_K^{\alpha-1} - |y|_K^{\alpha-1}\right| |y|_K^{\alpha m}\,dy, \quad \alpha \neq 1,$$

$$I_{1,m} = \int_{|y|_K < |x|_K} \left(\log |x|_K - \log |y|_K\right)|y|_K^m\,dy.$$

If $\alpha > 1$, we find denoting $|x|_K = q^n$ and using (2-2) that

$$I_{\alpha,m} = |x|_K^{\alpha-1} \int_{|y|_K \le q^{n-1}} |y|_K^{\alpha m}\, dy - \int_{|y|_K \le q^{n-1}} |y|_K^{\alpha(m+1)-1}\, dy = d_{\alpha,m} |x|_K^{\alpha(m+1)}$$

where, for all $m = 0, 1, 2, \ldots$

$$\begin{aligned} d_{\alpha,m} &= \frac{1-q^{-1}}{1-q^{-\alpha m-1}} q^{-\alpha m-1} - \frac{1-q^{-1}}{1-q^{-\alpha m-\alpha}} q^{-\alpha m-\alpha} \\ &= (1-q^{-1}) \frac{q^{\alpha-1}-1}{(1-q^{-\alpha m-1})(q^{\alpha m+\alpha}-1)} \le A q^{-\alpha m} \end{aligned}$$

for some $A > 0$. A similar result,

$$I_{\alpha,m} = d_{\alpha,m} |x|_K^{\alpha(m+1)}, \quad d_{\alpha,m} \le A q^{-\alpha m}, \quad m = 0, 1, 2, \ldots \tag{4-7}$$

is obtained for $0 < \alpha < 1$, so that (4-7) holds for any $\alpha \ne 1$. If $\alpha = 1$, then the integral $I_{1,m}$ is evaluated as follows. We have

$$\begin{aligned} I_{1,m} &= \sum_{k=-\infty}^{n-1} \int_{|y|_K = q^k} \big(\log |x|_K - \log |y|_K\big) |y|_K^m\, dy \\ &= \Big(1 - \frac{1}{q}\Big) \log q \sum_{k=-\infty}^{n-1} (n-k) q^{k(m+1)} \\ &= \Big(1 - \frac{1}{q}\Big) \log q \sum_{\nu=1}^{\infty} \nu q^{(n-\nu)(m+1)} = d_{1,m} |x|_K^{\alpha(m+1)} \end{aligned}$$

where

$$d_{1,m} = \Big(1 - \frac{1}{q}\Big) \log q \sum_{\nu=1}^{\infty} \nu q^{-\nu(m+1)} = \Big(1 - \frac{1}{q}\Big) \log q \frac{q^{-m-1}}{(1-q^{-m-1})^2} \le A q^{-m}$$

(we have used [Gradshteyn and Ryzhik 1996, Identity 0.231.2]). Thus, we have proved (4-7) also for $\alpha = 1$.

Let us return to a function w satisfying the relation $\mathcal{H} w = \lambda w$, $\lambda \ne 0$. Using (4-7) (separately for $\alpha \ne 1$ and $\alpha = 1$) and iterating we find by induction that

$$|w(x)| \le C (M |\lambda|^{-1} A)^m \Big(\prod_{j=0}^{m} q^{-\alpha j} \Big) |x|_K^{\alpha m}, \quad m = 0, 1, 2, \ldots, x \in B_N. \tag{4-8}$$

Since $\prod_{j=0}^{m} q^{-\alpha j} = q^{-\frac{\alpha}{2} m(m+1)}$, it follows from (4-8) that $w(x) \equiv 0$. □

4.2. ***Strong solutions.*** Below we assume that the inequality (4-5) is satisfied. Then, as we have mentioned, the solution v of (4-4) is automatically extended in a unique way from B_N onto K. The extended function v satisfies (4-3). Therefore the function $u = I^\alpha v$ satisfies (4-1) in the sense of distributions from Φ'. The initial condition (4-2) is satisfied automatically.

Let us find additional conditions on a and f, under which this construction gives a strong solution of the Cauchy problem (4-1)–(4-2). Note that, by Lemma 3 and Theorem 1, a strong solution is unique in the class of functions $u = I^\alpha v$ where v is a continuous radial function, such that $\sum_{l=m}^{\infty} |v(q^l)| < \infty$ for some $m \in \mathbb{Z}$.

Theorem 2. *Suppose that*

$$|a(|x|_K)| \le C|x|_K^{-\alpha-\varepsilon}, \quad |f(|x|_K)| \le C|x|_K^{-\varepsilon}, \quad \varepsilon > 0, C > 0, \tag{4-9}$$

as $|x|_K > 1$. *Then* $u = I^\alpha v$ *is a strong solution of the Cauchy problem* (4-1)–(4-2).

Proof. Let $v(|x|_K)$ be the solution of (4-3′) constructed above for all $x \in K$ (for $x = 0$, the integral in the right-hand side is assumed equal to zero). For $|x|_K \le q^N$ the existence of a solution v was obtained from the theory of compact operators; for larger values of $|x|_K$ we use successively (4-3′) itself. Denote

$$V_m = \sup_{|x|_K \le q^m} |v(q^m)|.$$

The sequence $\{V_m\}$ is nondecreasing.

As we assumed in Theorem 1 only the continuity of the coefficient a, we took N in such a way that the neighborhood $B_N = \{x : |x|_K \le q^N\}$ was sufficiently small. Here we assume (4-5), so that we can take any fixed integer N and obtain a solution v on B_N.

Consider the case where $\alpha \ne 1$. It follows from (4-5) and (4-9) that

$$\left|[1 + q^{-\alpha} a(|x|_K)|x|_K^\alpha]^{-1}\right| \le H$$

where $H > 0$ does not depend on $x \in K$. If $m \ge N$, then we find from (4-3′) and the above estimate for $I_{\alpha,0}$ that

$$|v(q^m)| \le c_\alpha d_{\alpha,0} H a(q^m) q^{\alpha m} V_{m-1} + H|f(g^m)|. \tag{4-10}$$

Let us choose $m_1 \ge N$ so big that

$$c_\alpha d_{\alpha,0} H a(q^m) q^{\alpha m} \le \tfrac{1}{2}, \quad H|f(g^m)| \le \tfrac{1}{2} V_{N-1},$$

as $m \ge m_1$ (that is possible due to (4-9)). Then it follows from (4-10) that $V_m \le V_{m-1}$, as $m \ge m_1$, hence that the function v is bounded on K.

Now we get from (4-3′) and the assumptions (4-9) that

$$\left|v(|x|_K)\right| \le C|x|_K^{-\varepsilon}, \quad |x|_K \ge 1, \tag{4-11}$$

$C > 0$. A similar reasoning works for $\alpha = 1$.

Taking into account the estimate (4-11) we find from Lemma 3 that

$$(D^\alpha I^\alpha v)(|x|_K) = v(|x|_K), \quad x \neq 0.$$

Therefore the function $u = I^\alpha v$ satisfies (4-1) for all $x \neq 0$. Since a, f, and u are continuous, the equation is satisfied in the strong sense. □

4.3. *Generalizations.* Instead of (4-2), one can consider an inhomogeneous initial condition $u(0) = u_0$, $u_0 \in \mathbb{C}$. Looking for a solution in the form $u = u_0 + I^\alpha v$, $v = v(|x|_K)$, we obtain the integral equation

$$v(|x|_K) + a(|x|_K)(I^\alpha v)(|x|_K) = f(|x|_K) - a(|x|_K)u_0,$$

which can be studied under the same assumptions.

All the above results carry over to the case of a matrix-valued coefficient $a(|x|_K)$ and vector-valued solutions. In this case, to obtain a strong solution, it is sufficient to demand that the spectrum of each matrix $a(|x|_K)$, $x \in K$, does not intersect the set $\{-q^N : N \in \mathbb{Z}\}$.

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