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NOTES ON THE EXTENSION OF THE MEAN CURVATURE FLOW

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NOTES ON THE EXTENSION OF THE MEAN CURVATURE FLOW

YAN LENG, ENTAO ZHAO AND HAORAN ZHAO

In this paper, we present several new curvature conditions that assure the extension of the mean curvature flow on a finite time interval, which improve some known extension theorems.

1. Introduction

Let $F_0 : M^n \to N^{n+d}$ be a smooth isometric immersion from an n-dimensional closed (compact and without boundary) Riemannian manifold M to an $(n+d)$-dimensional Riemannian manifold N. Consider a one-parameter family of smooth isometric immersions $F : M \times [0, T) \to N$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} F(x,t) = H(x,t), \\ \quad F(x,0) = F_0(x), \end{cases} \tag{1-1}$$

where $H(x,t)$ is the mean curvature vector of $F_t(M)$ and $F_t(x) = F(x,t)$. Set $M_t = F_t(M)$. We call $F : M \times [0, T) \to N$ the mean curvature flow with initial value $F_0 : M \to N$.

The mean curvature flow is a (degenerate) quasilinear parabolic evolution equation, and one can obtain the short-time existence either by the Nash–Moser implicit function theorem or by the DeTurck trick to modify the mean curvature flow equation to a strongly parabolic equation. Without any special assumption on M_0, the mean curvature flow (1-1) will in general develop singularities in finite time, characterized by blowing up of the second fundamental form A.

Theorem 1.1 [Huisken 1984; 1986; Wang 2001]. *Suppose $T < \infty$ is the first singular time for a closed mean curvature flow in a Riemannian manifold with bounded geometry. Then we have*

$$\lim_{t \to T} \sup_{M_t} |A| = \infty.$$

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From Theorem 1.1, we see that if $\sup_{M_t\times[0,T)}|A|$ is bounded, then the mean curvature flow can be extended past the time T. Recently, Le and Šešum [2011] and Liu, Xu, Ye and Zhao [Liu et al. 2011; Xu et al. 2011a; 2011b] obtained some integral conditions to extend the mean curvature flow. Define a $(0, 2)$-tensor B on M in a local orthonormal frame field by $B_{ij} = \langle H, h_{ij}\rangle$. Cooper obtained the following characterization of the singular time.

Theorem 1.2 [Cooper 2011]. *Suppose $T < \infty$ is the first singular time for a closed mean curvature flow in a Riemannian manifold with bounded geometry. Then we have*

$$\lim_{t\to T}\sup_{M_t}|B| = \infty.$$

Similarly, we see from Theorem 1.2 that if $\sup_{M_t\times[0,T)}|B|$ is bounded, then the mean curvature flow can be extended past the time T.

In the present paper, we make an improvement of Theorems 1.1 and 1.2 by considering the integral of $|B|$ on the time interval. More precisely, we prove the following theorem.

Theorem 1.3. *Let $F_t : M^n \to N^{n+d}$ be the mean curvature flow solution of closed submanifolds on a finite time interval $[0, T)$ and assume N has bounded geometry. If the function $f(x) := \int_0^T |B|(x,t)\,dt$ is continuous on M, then the mean curvature flow can be extended past the time T.*

By the dominated convergence theorem and Theorem 1.3, we obtain the following result, which recovers Theorems 1.1 and 1.2.

Theorem 1.4. *Suppose $T < \infty$ is the first singular time for a closed mean curvature flow in a Riemannian manifold with bounded geometry. Then we have*

$$\int_0^T \sup_{M_t}|B|(t)\,dt = \infty.$$

Analogous extension theorems for the Ricci flow have been proved recently [Wang 2012; He 2014]. Some general regularity results have been obtained by Cheeger, Haslhofer and Naber [Cheeger et al. 2013] and Ecker [2013], among others. To prove our theorems we combine the ideas in [Cooper 2011] and [He 2014]. First, by a suitable blow-up argument we get a minimal submanifold in Euclidean space. Second, we prove that the volume of geodesic balls in this minimal submanifold is less than the volume of geodesic balls with same radius. By the expansion for the volume of geodesic balls, we see that this minimal submanifold is in fact totally geodesic at the base point.

2. Preliminaries

Let M^n be an n-dimensional submanifold isometrically immersed in an $(n+d)$-dimensional Riemannian manifold N^{n+d}. Let A and H be the second fundamental form and the mean curvature vector of M in N, respectively. Define a $(0,2)$-tensor B from A and H by $B = \langle A, H\rangle$. Choose a local orthonormal frame field $\{e_A\}_{A=1}^{n+d}$ in N^{n+d} such that each e_i, $i = 1, \dots, n$, is tangent to M and let $\{\omega_A\}_{A=1}^{n+d}$ be the dual frame field of $\{e_A\}_{A=1}^{n+d}$. Then A, H and B can be written as

$$A = \sum_{i,j=1}^{n} \sum_{\alpha=n+1}^{n+d} h_{ij}^{\alpha}\, \omega_i \otimes \omega_j \otimes e_\alpha = \sum_{i,j} h_{ij}\, \omega_i \otimes \omega_j,\ h_{ij} = \sum_{\alpha=n+1}^{n+d} h_{ij}^{\alpha}\, e_\alpha,$$

$$H = \sum_{\alpha} H^\alpha e_\alpha,\ H^\alpha = \sum_{i=1}^{n} h_{ii}^{\alpha},$$

$$B = \sum_{i,j=1}^{n} B_{ij}\, \omega_i \otimes \omega_j,\ B_{ij} = \sum_{\alpha=n+1}^{n+d} H^\alpha h_{ij}^{\alpha}.$$

Let $F : M \times [0, T) \to N$ be a mean curvature flow solution with initial immersion $F_0 : M \to N$. Denote by $g(t)$ and $d\mu(t)$ the induced metric and the volume form on M. Under the mean curvature flow, $g(t)$ and $d\mu(t)$ satisfy the following evolution equations.

$$\frac{\partial}{\partial t} g(t) = -2B(t), \tag{2-1}$$

$$\frac{\partial}{\partial t} d\mu(t) = -|H|^2 d\mu(t). \tag{2-2}$$

3. Proof of Theorem 1.3

Now we give the proof of Theorem 1.3.

Proof. We argue by contradiction. Suppose that T is the maximal existence time. Then by Theorem 1.1 we see that $\lim_{t\to T} \sup_{M_t} |A| = \infty$. Choose a sequence of points $(O_i, t_i) \in M \times [0, T)$, $i = 1, 2, \dots,$ such that $\lim_{i\to\infty} t_i = T$ and

$$|A|^2(O_i, t_i) = \max_{(x,t)\in M\times[0,t_i]} |A|^2(x, t) \to \infty \quad \text{as } i \to \infty.$$

Set $Q_i = |A|^2(O_i, t_i)$ and we suppose $Q_i \geq 1$ and $Q_i t_i \geq 1$. Denote by h the Riemannian metric on N. We consider the rescaled flows for $t \in [0, 1]$

$$F_i(t) = F\left(\frac{t-1}{Q_i} + t_i\right) : (M, g_i(t)) \longrightarrow (N, Q_i h),$$

where $g_i(t) = F_i(t)^*(Q_i h)$ is the induced metric on M. Then for each i, F_i is also a solution of the mean curvature flow on time interval [0, 1]. Denote by M_i the manifold M with metric $g_i(t)$. It follows from [Chen and He 2010] that there is a subsequence of $\{(M_i, g_i(t), O_i) : i = 1, 2, \dots\}$ which converges to a Riemannian manifold $(M_\infty, g_\infty(t), O_\infty)$, and the corresponding subsequence of immersions $F_i(t)$ converges to an immersion $F_\infty(t) : M_\infty \to \mathbb{R}^{n+d}$, $t \in [0, 1]$. Note that F_∞ is also a solution of the mean curvature flow on time interval [0, 1].

We first show that for any $t \in [0, 1]$, M_∞ is a minimal submanifold in $\mathbb{R}^{n+d}$. Let $B_\infty(\cdot, t)$ be the (0, 2)-tensor for $F_\infty(t)$. In fact, we prove the following:

Lemma 3.1. $B_\infty(t) = 0$ *for* $t \in [0, 1]$.

Proof. By the continuity assumption on

$$f(x) := \int_0^T |B|(x, t)\, dt$$

and the compactness of M, we can use elementary arguments to prove that

$$\lim_{t \to T} \int_t^T |B|(x, t)\, dt = 0.$$

First, we have

$$\begin{aligned} g_i(t) &= F_i(t)^*(Q_i h) = F\left(\frac{t-1}{Q_i} + t_i\right)^*(Q_i h) \\ &= Q_i F\left(\frac{t-1}{Q_i} + t_i\right)^*(h) = Q_i g\left(\frac{t-1}{Q_i} + t_i\right). \end{aligned}$$

Denote by $A_i(\cdot, t)$, $H_i(\cdot, t)$ and $B_i(\cdot, t)$ the second fundamental form, the mean curvature and the (0, 2)-tensor of $F_i(t)$, respectively. It is easy to see from the definition of second fundamental form that

$$A_i(\cdot, t) = A\left(\cdot, \frac{t-1}{Q_i} + t_i\right).$$

Since the mean curvature is the trace of the second fundamental form, we have

$$H_i(\cdot, t) = Q_i^{-1} H\left(\cdot, \frac{t-1}{Q_i} + t_i\right).$$

So for the (0, 2)-tensor we have

$$\begin{aligned} B_i(\cdot, t) &= \langle A_i(\cdot, t), H_i(\cdot, t)\rangle_{Q_i h} \\ &= \left\langle A\left(\cdot, \frac{t-1}{Q_i} + t_i\right), H\left(\cdot, \frac{t-1}{Q_i} + t_i\right)\right\rangle_h = B\left(\cdot, \frac{t-1}{Q_i} + t_i\right). \end{aligned}$$

From this we see that

$$
\begin{aligned}
|B_i(\cdot,t)|^2_{g_i(t)} &= \langle B_i(\cdot,t), B_i(\cdot,t)\rangle_{g_i(t)\otimes g_i(t)} \\
&= Q_i^{-2}\Big\langle B\Big(\cdot,\frac{t-1}{Q_i}+t_i\Big), B\Big(\cdot,\frac{t-1}{Q_i}+t_i\Big)\Big\rangle_{g((t-1)/Q_i+t_i)\otimes g((t-1)/Q_i+t_i)} \\
&= Q_i^{-2}\Big|B\Big(\cdot,\frac{t-1}{Q_i}+t_i\Big)\Big|^2_{g((t-1)/Q_i+t_i)}.
\end{aligned}
$$

For any $y \in M_\infty$, there are $y_i \in M$, $i = 1, 2, \ldots$, such that $\lim_{i\to\infty} y_i = y$.

$$
\begin{aligned}
\int_0^1 |B|_{g_\infty(t)}(y,t)\,dt &= \lim_{i\to\infty}\int_0^1 |B_i|_{g_i(t)}(y_i,t)\,dt \\
&= \lim_{i\to\infty} Q_i^{-1}\int_0^1 |B|_{g((t-1)/Q_i+t_i)}\Big(y_i, \frac{t-1}{Q_i}+t_i\Big)\,dt \\
&= \lim_{i\to\infty}\int_{t_i-Q_i^{-1}}^{t_i} |B|_{g(s)}(y_i,s)\,ds \\
&= 0.
\end{aligned}
$$

Hence we have $B_\infty(t) = 0$ for each $t \in [0, 1]$. □

Lemma 3.2. *The induced metrics $g(t)$ on M are uniformly equivalent and converge pointwise as $t \to T$ to a continuous positive-definite metric $g(T)$.*

Proof. Under the assumption that $f(x) = \int_0^T |B|(x,t)\,dt$ is continuous, we see that $f(x)$ is bounded and for any $0 \le \tau \le \theta < T$

$$
\lim_{\tau\to\theta}\int_\tau^\theta |B|(x,t)\,dt = 0
$$

uniformly. Since $g(t)$ satisfies (2-1), we can carry out the same argument as in [Hamilton 1982] to prove the lemma. □

Let $\boldsymbol{B}_{g_\infty(1)}(O_\infty, r)$ be the geodesic ball of radius r centered at $O_\infty \in M_\infty$ with respect to the metric $g_\infty(1)$, and $\mathrm{Vol}_{g_\infty(1)}(\boldsymbol{B}_{g_\infty(1)}(O_\infty, r))$ be the volume of $\boldsymbol{B}_{g_\infty(1)}(O_\infty, r)$. Denote by ω_n the volume of the unit ball in $\mathbb{R}^n$.

Lemma 3.3. $$\mathrm{Vol}_{g_\infty(1)}(\boldsymbol{B}_{g_\infty(1)}(O_\infty, r)) \le \omega_n r^n.$$

Proof. Let $\boldsymbol{B}_{g_i(1)}(O_i, r)$ be the geodesic ball with radius r centered at $O_i \in M_i$ with respect to $g_i(t)$ and $\mathrm{Vol}_{g_i(1)}(\boldsymbol{B}_{g_i(1)}(O_i, r))$ the volume of $\boldsymbol{B}_{g_i(1)}(O_i, r)$. It is easy to see that

$$
\boldsymbol{B}_{g_i(1)}(O_i, r) = \boldsymbol{B}_{g(t_i)}(O_i, Q_i^{-1/2} r).
$$

Hence

$$\begin{aligned}\frac{\mathrm{Vol}_{g_\infty(1)}(\boldsymbol{B}_{g_\infty(1)}(O_\infty, r))}{r^n} &= \lim_{i\to\infty} \frac{\mathrm{Vol}_{g_i(1)}(\boldsymbol{B}_{g_i(1)}(O_i, r))}{r^n} \\ &= \lim_{i\to\infty} \frac{\mathrm{Vol}_{g_i(1)}(\boldsymbol{B}_{g(t_i)}(O_i, Q_i^{-1/2} r))}{r^n} \\ &= \lim_{i\to\infty} \frac{Q_i^{n/2}\mathrm{Vol}_{g(t_i)}(\boldsymbol{B}_{g(t_i)}(O_i, Q_i^{-1/2} r))}{r^n} \\ &= \lim_{i\to\infty} \frac{\mathrm{Vol}_{g(t_i)}(\boldsymbol{B}_{g(t_i)}(O_i, Q_i^{-1/2} r))}{(Q_i^{-1/2} r)^n}.\end{aligned}$$

From Lemma 3.2, we see that for any $\varepsilon > 0$, there is a positive constant δ such that if $t \geq t_0 > T - \delta$, then $(1-\varepsilon)g(t_0) \leq g(t) \leq (1+\varepsilon)g(t_0)$. We may pick t_is such that $t_i \geq t_0 \geq T - \delta$. From a lemma in [Cooper 2011; Glickenstein 2003] we see that

$$\begin{aligned}\lim_{i\to\infty} &\frac{\mathrm{Vol}_{g(t_i)}(\boldsymbol{B}_{g(t_i)}(O_i, Q_i^{-1/2} r))}{(Q_i^{-1/2} r)^n} \\ &\leq \lim_{i\to\infty} \frac{\mathrm{Vol}_{g(t_i)}(\boldsymbol{B}_{g(t_0)}(O_i, ((1-\varepsilon)Q_i)^{-1/2} r))}{(Q_i^{-1/2} r)^n} \\ &\leq \lim_{i\to\infty} \frac{\mathrm{Vol}_{g(t_0)}(\boldsymbol{B}_{g(t_0)}(O_i, ((1-\varepsilon)Q_i)^{-1/2} r))}{(Q_i^{-1/2} r)^n} \\ &= (1-\varepsilon)^{-n/2} \lim_{i\to\infty} \frac{\mathrm{Vol}_{g(t_0)}(\boldsymbol{B}_{g(t_0)}(O_i, ((1-\varepsilon)Q_i)^{-1/2} r))}{(((1-\varepsilon)Q_i)^{-1/2} r)^n}.\end{aligned}$$

Since $Q_i \to \infty$ as $i \to \infty$, we have

$$\lim_{i\to\infty} \frac{\mathrm{Vol}_{g(t_0)}(\boldsymbol{B}_{g(t_0)}(O_i, ((1-\varepsilon)Q_i)^{-1/2} r))}{(((1-\varepsilon)Q_i)^{-1/2} r)^n} = \omega_n.$$

Since ε is arbitrary, we see that

$$\frac{\mathrm{Vol}_{g_\infty(1)}(\boldsymbol{B}_{g_\infty(1)}(O_\infty, r))}{r^n} \leq \omega_n. \qquad \square$$

We continue the proof of Theorem 1.3. From the expansion formula for the volume of small balls (see Theorem 3.98 of [Gallot et al. 1987]) we have

$$\frac{\mathrm{Vol}_{g_\infty(1)}(\boldsymbol{B}_{g_\infty(1)}(O_\infty, r))}{\omega_n r^n} = 1 - \frac{R(O_\infty)}{6(n+2)} r^2 + o(r^2).$$

Here $R(O_\infty)$ is the scalar curvature at $O_\infty \in (M_\infty, g_\infty(1))$. From Lemma 3.3 we see that

$$R(O_\infty) \geq 0.$$

This combined with Lemma 3.1 implies that

$$|A|_\infty(O_\infty, 1) = 0.$$

However, it is seen from the point selecting process that

$$|A|_\infty(O_\infty, 1) = 1.$$

This is a contradiction, which completes the proof of Theorem 1.3. □

Theorem 3.4. *Let $F_t : M^n \to N^{n+d}$ be the mean curvature flow solution of closed submanifolds on a finite time interval* $[0, T)$ *and assume N has bounded geometry. Suppose $T < \infty$ is the first singular time. If the function $\int_0^T |A|(x, t)\, dt < +\infty$ is continuous on M, then we have*

$$\lim_{t\to T} \sup_{M_t} |H| = \infty.$$

Proof. We suppose $|H| \le C$ uniformly for all the existence time. Then

$$\left|\frac{\partial g}{\partial t}\right| = 2|B| \le 2|H||A| \le 2C|A|.$$

By the dominated convergence theorem, we know that $\int_0^T |A|(x, t)\, dt$ is continuous in x. Then by a similar argument as in the proof of Theorem 1.3, we get the conclusion. □

Remark 3.5. Theorem 3.4 recovers [Cooper 2011, Theorem 5.1].

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