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**TWISTED ALEXANDER POLYNOMIALS OF 2-BRIDGE KNOTS
FOR PARABOLIC REPRESENTATIONS**

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TWISTED ALEXANDER POLYNOMIALS OF 2-BRIDGE KNOTS FOR PARABOLIC REPRESENTATIONS

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In this paper we show that the twisted Alexander polynomial associated to a parabolic representation determines fiberedness and genus of a wide class of 2-bridge knots. As a corollary we give an affirmative answer to a conjecture of Dunfield, Friedl and Jackson for infinitely many hyperbolic knots.

1. Introduction

The twisted Alexander polynomial was introduced by Lin [2001] for knots in the 3-sphere and by Wada [1994] for finitely presentable groups. It is a generalization of the classical Alexander polynomial and gives a powerful tool in low-dimensional topology. A theory of twisted Alexander polynomials has developed rapidly over the past ten years. One of the most important aspects is the determination of fiberedness [Friedl and Vidussi 2011b] and genus (the Thurston norm) [Friedl and Vidussi 2012] of knots by the collection of the twisted Alexander polynomials corresponding to all finite-dimensional representations. For literature on other applications and related topics, we refer to the survey paper [Friedl and Vidussi 2011a].

Let K be a knot in S^3 and G_K its knot group. Namely it is the fundamental group of the complement of K in S^3 , $G_K = \pi_1(S^3 \setminus K)$. In this paper, we consider the twisted Alexander polynomial $\Delta_{K,\rho}(t) \in \mathbb{C}[t^{\pm 1}]$ associated to a parabolic representation $\rho : G_K \rightarrow \mathrm{SL}_2(\mathbb{C})$. A typical example is the holonomy representation $\rho_0 : G_K \rightarrow \mathrm{SL}_2(\mathbb{C})$ of a hyperbolic knot K , which is a lift of a discrete faithful representation $\bar{\rho}_0 : G_K \rightarrow \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$ such that $\mathbb{H}^3/\bar{\rho}_0(G_K) \cong S^3 \setminus K$ where \mathbb{H}^3 denotes the upper half space model of hyperbolic 3-space (see [Thurston 1997]). Dunfield, Friedl and Jackson [Dunfield et al. 2012] numerically computed the twisted Alexander polynomial $\mathcal{T}_K(t) = \Delta_{K,\rho_0}(t)$, which is called the hyperbolic torsion polynomial, for all hyperbolic knots of 15 or fewer crossings. Based on these huge computations, they conjectured that the hyperbolic torsion polynomial determines the knot genus and, moreover, the knot is fibered if and only if $\mathcal{T}_K(t)$ is a monic polynomial. This conjecture is nice because it would imply the fiberedness and genus of a knot is determined by the twisted Alexander polynomial associated

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to a single representation. However it is widely open except for the hyperbolic twist knots [Morifuji 2012].

The purpose of this paper is to show that the above conjecture is true for a wide class of 2-bridge knots. Since 2-bridge knots are alternating, their fiberedness and genus can be determined by the Alexander polynomial [Crowell 1959; Murasugi 1958a; 1958b]. However there seems to be no a priori reason that the same must be true for the hyperbolic torsion polynomial.

For a prime p and an integer a between 1 and $p - 1$, we say that a is a primitive root modulo p if it is a generator of the cyclic group $(\mathbb{Z}/p\mathbb{Z})^*$. Let \mathcal{P}_2 be the set of all odd primes p such that 2 is a primitive root modulo p . Note that all primes $p = 2q + 1$ such that q is a prime $\equiv 1 \pmod{4}$ are contained in \mathcal{P}_2 ; see, for example, [LeVeque 1977, Theorem 5.6].

Theorem 1.1. *Let K be the knot $J(k, 2n)$ as in Figure 1, where $k > 0$ and $n \in \mathbb{Z}$. For all hyperbolic knots K , the hyperbolic torsion polynomial $\mathcal{T}_K(t)$ determines the genus of K . Moreover for $k = 2m + 1$, $k = 2$ (twist knot), or $k = 2m$ and $|4mn - 1| \in \mathcal{P}_2$, the knot $J(k, 2n)$ is fibered if and only if $\mathcal{T}_K(t)$ is monic.*

As mentioned above, the holonomy representation ρ_0 is parabolic, so that Theorem 1.1 is an immediate corollary of the following theorem.

Theorem 1.2. *Let $\rho : G_K \rightarrow \mathrm{SL}_2(\mathbb{C})$ be a parabolic representation of $K = J(k, 2n)$. Then:*

- (1) $\Delta_{K,\rho}(t)$ determines the genus of $J(k, 2n)$.
- (2) $\Delta_{K,\rho}(t)$ determines the fiberedness of $J(k, 2n)$ if $k = 2m + 1$, $k = 2$ (twist knot), or $k = 2m$ and $|4mn - 1| \in \mathcal{P}_2$.

Remark 1.3. (1) Suppose $k = 2m$ and $n > 0$. Then $4mn - 1 \in \mathcal{P}_2$ if $4mn - 1$ is a prime and $2mn - 1$ is a prime $\equiv 1 \pmod{4}$.

- (2) It is known that the conjugacy classes of parabolic representations into $\mathrm{SL}_2(\mathbb{C})$ of the knot $J(2m, 2n)$ can be described as the zero locus of an integral polynomial in one variable. The condition $|4mn - 1| \in \mathcal{P}_2$ in Theorem 1.2 (hence Theorem 1.1) assures the irreducibility over \mathbb{Z} of this polynomial, see Section 5. We do not know whether Theorem 1.2(2) holds true for every integer m and n .

This paper is organized as follows. In Section 2, we study nonabelian representations of the knot $J(k, 2n)$ and give an explicit formula of the defining equation of the representation space. In Section 3, we investigate parabolic representations of $J(k, 2n)$. In Section 4, we quickly review the definition of the twisted Alexander polynomial and some related work on fiberedness and genus of knots. In particular, we calculate the coefficients of the highest- and lowest-degree terms of the twisted Alexander polynomial associated to a nonabelian representation of $J(k, 2n)$ and

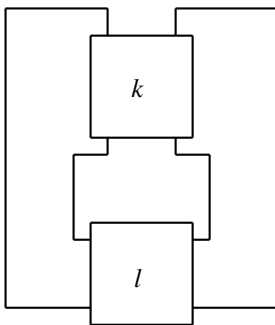


Figure 1. The knot $K = J(k, l)$. Here $k > 0$ and $l = 2n$ ($n \in \mathbb{Z}$) denote the numbers of half twists in each box. Positive numbers correspond to right-handed twists and negative numbers correspond to left-handed twists.

give the proof of [Theorem 1.2\(1\)](#). In [Section 5](#), we discuss the fibering problem and prove [Theorem 1.2\(2\)](#).

2. Non-abelian representations

Let $K = J(k, l)$ be the knot as in [Figure 1](#). Note that $J(k, l)$ is a knot if and only if kl is even, and is the trivial knot if $kl = 0$. Furthermore, $J(k, l) \cong J(l, k)$ and $J(-k, -l)$ is the mirror image of $J(k, l)$. Hence, in the following, we consider $K = J(k, 2n)$ for $k > 0$ and $|n| > 0$. When $k = 2$, $J(2, 2n)$ is the twist knot.

In this section we explicitly calculate the defining equation of the nonabelian representation space of $J(k, 2n)$.

By [\[Hoste and Shanahan 2004\]](#) the knot group of $K = J(k, 2n)$ is presented by $G_K = \langle a, b \mid w^n a = b w^n \rangle$, where

$$w = \begin{cases} (ba^{-1})^m (b^{-1}a)^m & \text{for } k = 2m, \\ (ba^{-1})^m ba(b^{-1}a)^m & \text{for } k = 2m + 1. \end{cases}$$

Let $\{S_i(z)\}_i$ be the sequence of Chebyshev polynomials, defined by $S_0(z) = 1$, $S_1(z) = z$, and $S_{i+1}(z) = zS_i(z) - S_{i-1}(z)$ for all positive integers i .

The following lemmas are standard; see, for example, [\[Tran 2013a, Lemma 2.4\]](#) and [\[Tran 2013b, Lemma 3.2\]](#).

Lemma 2.1. *One has $S_i^2(z) - zS_i(z)S_{i-1}(z) + S_{i-1}^2(z) = 1$.*

Lemma 2.2. *Suppose the sequence $\{M_i\}_i$ of 2×2 matrices satisfies the recurrence relation $M_{i+1} = zM_i - M_{i-1}$ for all integers i . Then*

(2-1) $M_i = S_{i-1}(z)M_1 - S_{i-2}(z)M_0,$

(2-2) $M_i = S_i(z)M_0 - S_{i-1}(z)M_{-1}.$

A representation $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$ is called nonabelian if $\rho(G_K)$ is a nonabelian subgroup of $\text{SL}_2(\mathbb{C})$. Taking conjugation if necessary, we can assume that ρ has the form

$$(2-3) \quad \rho(a) = A = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = B = \begin{bmatrix} s & 0 \\ 2-y & s^{-1} \end{bmatrix},$$

where $(s, y) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the matrix equation $W^n A - B W^n = 0$. Here $W = \rho(w)$. It can be easily checked that $y = \text{tr } AB^{-1}$ holds. Let $x = \text{tr } A = \text{tr } B = s + s^{-1}$.

Lemma 2.3. *One has*

$$WA - BW = \begin{bmatrix} 0 & \alpha_k(x, y) \\ (y-2)\alpha_k(x, y) & 0 \end{bmatrix},$$

where

$$\alpha_k(x, y) = \begin{cases} 1 - (y + 2 - x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y)) & \text{for } k = 2m, \\ 1 + (y + 2 - x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y)) & \text{for } k = 2m + 1. \end{cases}$$

Proof. Recall that by the Cayley–Hamilton theorem, $M^{i+1} = (\text{tr } M)M^i - M^{i-1}$ for all matrices $M \in \text{SL}_2(\mathbb{C})$ and all integers i .

If $k = 2m$ then by applying (2-1) twice, we have

$$\begin{aligned} WA &= (BA^{-1})^m (B^{-1}A)^m A \\ &= S_{m-1}^2(y)BA^{-1}B^{-1}AA - S_{m-1}(y)S_{m-2}(y)(BA^{-1}A + B^{-1}AA) + S_{m-2}^2(y)A. \end{aligned}$$

Similarly,

$$\begin{aligned} BW &= B(BA^{-1})^m (B^{-1}A)^m \\ &= S_{m-1}^2(y)BBA^{-1}B^{-1}A - S_{m-1}(y)S_{m-2}(y)(BBA^{-1} + BB^{-1}A) + S_{m-2}^2(y)B. \end{aligned}$$

Hence, by direct calculations using (2-3), we obtain

$$\begin{aligned} WA - BW &= S_{m-1}^2(y)(BA^{-1}B^{-1}AA - BBA^{-1}B^{-1}A) + S_{m-2}^2(y)(A - B) \\ &\quad - S_{m-1}(y)S_{m-2}(y)(BA^{-1}A - BBA^{-1} + B^{-1}AA - BB^{-1}A) \\ &= \begin{bmatrix} 0 & \alpha_k(x, y) \\ (y-2)\alpha_k(x, y) & 0 \end{bmatrix}, \end{aligned}$$

where

$$\alpha_k(x, y) = (s^{-2} + 1 + s^2 - y)S_{m-1}^2(y) - (s^{-2} + s^2)S_{m-1}(y)S_{m-2}(y) + S_{m-2}^2(y).$$

Since $S_{m-1}^2(y) - yS_{m-1}(y)S_{m-2}(y) + S_{m-2}^2(y) = 1$ (by Lemma 2.1) and $x = s + s^{-1}$,

$$\alpha_k(x, y) = 1 - (y + 2 - x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y)).$$

If $k = 2m + 1$ then by applying (2-2) twice, we have

$$\begin{aligned} WA &= (BA^{-1})^m BA(B^{-1}A)^m A \\ &= S_m^2(y)BAA - S_m(y)S_{m-1}(y)((BA^{-1})^{-1}BAA + BA(B^{-1}A)^{-1}A) \\ &\quad + S_{m-1}^2(y)(BA^{-1})^{-1}BA(B^{-1}A)^{-1}A \\ &= S_m^2(y)BAA - S_m(y)S_{m-1}(y)(A^3 + B^2A) + S_{m-1}^2(y)ABA. \end{aligned}$$

Similarly,

$$\begin{aligned} BW &= B(BA^{-1})^m BA(B^{-1}A)^m \\ &= S_m^2(y)BBA - S_m(y)S_{m-1}(y)(B(BA^{-1})^{-1}BA + BBA(B^{-1}A)^{-1}) \\ &\quad + S_{m-1}^2(y)B(BA^{-1})^{-1}BA(B^{-1}A)^{-1} \\ &= S_m^2(y)BBA - S_m(y)S_{m-1}(y)(BA^2 + B^3) + S_{m-1}^2(y)BAB. \end{aligned}$$

Hence, by direct calculations using (2-3), we obtain

$$\begin{aligned} WA - BW &= S_m^2(y)(BAA - BBA) + S_{m-1}^2(y)(ABA - BAB) \\ &\quad - S_m(y)S_{m-1}(y)(A^3 - BA^2 + B^2A - B^2) \\ &= \begin{bmatrix} 0 & \alpha_k(x, y) \\ (y - 2)\alpha_k(x, y) & 0 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \alpha_k(x, y) &= S_m^2(y) - (s^{-2} + s^2)S_m(y)S_{m-1}(y) + (s^{-2} + 1 + s^2 - y)S_{m-1}^2(y) \\ &= 1 + (y + 2 - x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y)). \end{aligned}$$

This completes the proof of Lemma 2.3. □

The proof of the following lemma is similar to that of Lemma 2.3.

Lemma 2.4. *One has*

$$\text{tr } W = \begin{cases} 2 + (y - 2)(y + 2 - x^2)S_{m-1}^2(y) & \text{for } k = 2m, \\ x^2 - y - (y - 2)(y + 2 - x^2)S_m(y)S_{m-1}(y) & \text{for } k = 2m + 1. \end{cases}$$

We are now ready to calculate the expression $W^n A - BW^n$ as follows. Let $\lambda = \text{tr } W$.

Proposition 2.5. *One has*

$$\begin{aligned} W^n A - BW^n &= \begin{bmatrix} 0 & S_{n-1}(\lambda)\alpha_k(x, y) - S_{n-2}(\lambda) \\ (y - 2)(S_{n-1}(\lambda)\alpha_k(x, y) - S_{n-2}(\lambda)) & 0 \end{bmatrix}. \end{aligned}$$

Proof. By applying (2-1) and Lemma 2.3, we have

$$\begin{aligned} W^n A - B W^n &= S_{n-1}(\lambda)(WA - BW) - S_{n-2}(\lambda)(A - B) \\ &= S_{n-1}(\lambda) \begin{bmatrix} 0 & \alpha_k(x, y) \\ (y-2)\alpha_k(x, y) & 0 \end{bmatrix} - S_{n-2}(\lambda) \begin{bmatrix} 0 & 1 \\ y-2 & 0 \end{bmatrix}. \end{aligned}$$

The proposition follows. □

Proposition 2.5 implies that the assignment (2-3) gives a nonabelian representation $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$ if and only if $(s, y) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the equation

$$\phi_{k,2n}(x, y) := S_{n-1}(\lambda)\alpha_k(x, y) - S_{n-2}(\lambda) = 0,$$

where $\alpha_k(x, y)$ and $\lambda = \text{tr } W$ are given by the formulas in Lemmas 2.3 and 2.4 respectively.

The polynomial $\phi_{k,2n}(x, y)$ is also known as the Riley polynomial [Riley 1984; Tkhang 1993] of $J(k, 2n)$.

3. Parabolic representations

A representation $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$ is called parabolic if the meridian μ of K is sent to a parabolic element (i.e., $\text{tr } \rho(\mu) = 2$) of $\text{SL}_2(\mathbb{C})$ and $\rho(G_K)$ is nonabelian.

Let $K = J(k, 2n)$. In this section we will show that if $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$ is a parabolic representation of the form

$$\rho(a) = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho(b) = B = \begin{bmatrix} 1 & 0 \\ 2-y & 1 \end{bmatrix},$$

where y is a real number satisfying the equation $\phi_{k,2n}(2, y) = 0$, then $y > 2$.

Lemma 3.1. *Suppose $x = 2$. Then*

$$\alpha_k^2(x, y) - \alpha_k(x, y)\lambda + 1 = \begin{cases} (y-2)^3 S_{m-1}^4(y) & \text{for } k = 2m, \\ (y-2)((y-2)S_m(y)S_{m-1}(y) + 1)^2 & \text{for } k = 2m + 1. \end{cases}$$

Proof. If $k = 2m$ then $\alpha_k(x, y) = 1 - (y + 2 - x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y))$ and $\lambda = 2 + (y - 2)(y + 2 - x^2)S_{m-1}^2(y)$ by Lemmas 2.3 and 2.4. Hence, by direct calculations using $x = 2$, we have

$$\begin{aligned} &\alpha_k^2(x, y) - \alpha_k(x, y)\lambda + 1 \\ &= (-1 - S_{m-1}^2(y) + yS_{m-1}^2(y) - yS_{m-1}(y)S_{m-2}(y) + S_{m-2}^2(y))(y-2)^2 S_{m-1}^2(y). \end{aligned}$$

Since $S_{m-1}^2(y) - yS_{m-1}(y)S_{m-2}(y) + S_{m-2}^2(y) = 1$ (by Lemma 2.1), we obtain

$$\alpha_k^2(x, y) - \alpha_k(x, y)\lambda + 1 = (y-2)^3 S_{m-1}^4(y).$$

If $k = 2m + 1$ then $\alpha_k(x, y) = 1 + (y + 2 - x^2)S_{m-1}(y) (S_m(y) - S_{m-1}(y))$ and $\lambda = x^2 - y - (y - 2)(y + 2 - x^2)S_m(y)S_{m-1}(y)$ by Lemmas 2.3 and 2.4. Hence, by direct calculations using $x = 2$, we have

$$\begin{aligned} \alpha_k^2(x, y) - \alpha_k(x, y)\lambda + 1 &= (y - 2) \left(1 + (2 - y)S_{m-1}^2(y) + (y - 2)S_{m-1}^4(y) \right. \\ &\quad \left. + 2(y - 2)S_{m-1}(y)S_m(y) + (2 - y)yS_{m-1}^3(y)S_m(y) \right. \\ &\quad \left. + (y^2 - 3y + 2)S_{m-1}^2(y)S_m^2(y) \right). \end{aligned}$$

By replacing $yS_{m-1}^3(y)S_m(y) = S_{m-1}^2(y)(S_{m-1}^2(y) + S_m^2(y) - 1)$ in this equality, we obtain

$$\alpha_k^2(x, y) - \alpha_k(x, y)\lambda + 1 = (y - 2) \left((y - 2)S_m(y)S_{m-1}(y) + 1 \right)^2,$$

as claimed. \square

Proposition 3.2. *Suppose y is a real number satisfying the equation $\phi_{k,2n}(2, y) = 0$. Then $y > 2$.*

Proof. Suppose $\phi_{k,2n}(x, y) = 0$. Then $S_{n-1}(\lambda)\alpha_k(x, y) = S_{n-2}(\lambda)$. Hence

$$\begin{aligned} 1 &= S_{n-1}^2(\lambda) - \lambda S_{n-1}(\lambda)S_{n-2}(\lambda) + S_{n-2}^2(\lambda) \\ &= (\alpha_k^2(x, y) - \alpha_k(x, y)\lambda + 1) S_{n-1}^2(\lambda). \end{aligned}$$

If we also suppose that $x = 2$ and y is a real number, then the above equality implies that $\alpha_k^2(x, y) - \alpha_k(x, y)\lambda + 1 > 0$. By Lemma 3.1, we must have $y > 2$. \square

4. Twisted Alexander polynomials

In this section we explicitly calculate the coefficients of the highest- and lowest-degree terms of the twisted Alexander polynomial associated to a nonabelian representation of $J(k, 2n)$ and give the proof of Theorem 1.2(1).

Twisted Alexander polynomials. For a knot group $G_K = \pi_1(S^3 \setminus K)$, we choose and fix a Wirtinger presentation

$$G_K = \langle a_1, \dots, a_q \mid r_1, \dots, r_{q-1} \rangle.$$

Then the abelianization homomorphism $f : G_K \rightarrow H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$ is given by $f(a_1) = \dots = f(a_q) = t$. Here we specify a generator t of $H_1(S^3 \setminus K; \mathbb{Z})$ and denote the sum in \mathbb{Z} multiplicatively. Let us consider a linear representation $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$.

The maps ρ and f naturally induce two ring homomorphisms $\tilde{\rho} : \mathbb{Z}[G_K] \rightarrow M(2, \mathbb{C})$ and $\tilde{f} : \mathbb{Z}[G_K] \rightarrow \mathbb{Z}[t^{\pm 1}]$ respectively, where $\mathbb{Z}[G_K]$ is the group ring of G_K and $M(2, \mathbb{C})$ is the matrix algebra of degree 2 over \mathbb{C} . Then $\tilde{\rho} \otimes \tilde{f}$ defines a ring homomorphism $\mathbb{Z}[G_K] \rightarrow M(2, \mathbb{C}[t^{\pm 1}])$. Let F_q denote the free group

on generators a_1, \dots, a_q and $\Phi : \mathbb{Z}[F_q] \rightarrow M(2, \mathbb{C}[t^{\pm 1}])$ the composition of the surjection $\mathbb{Z}[F_q] \rightarrow \mathbb{Z}[G_K]$ induced by the presentation of G_K and the map $\tilde{\rho} \otimes \tilde{f} : \mathbb{Z}[G_K] \rightarrow M(2, \mathbb{C}[t^{\pm 1}])$.

Consider the $(q - 1) \times q$ matrix M whose (i, j) -component is the 2×2 matrix

$$\Phi \left(\frac{\partial r_i}{\partial a_j} \right) \in M(2, \mathbb{Z}[t^{\pm 1}]),$$

where $\partial/\partial a$ denotes the free differential calculus. For $1 \leq j \leq q$, let us denote by M_j the $(q - 1) \times (q - 1)$ matrix obtained from M by removing the j -th column. We regard M_j as a $2(q - 1) \times 2(q - 1)$ matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$. Then Wada's twisted Alexander polynomial of a knot K associated to a representation $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$ is defined to be a rational function

$$\Delta_{K,\rho}(t) = \frac{\det M_j}{\det \Phi(1 - a_j)}$$

and moreover well-defined up to a factor t^{2n} ($n \in \mathbb{Z}$). It is also known that if two representations ρ, ρ' are conjugate, then $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$ holds. See [Wada 1994] and [Goda et al. 2005] for details.

Remark 4.1. Let $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$ be a nonabelian representation.

- (1) The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ associated to ρ is always a Laurent polynomial for any knot K [Kitano and Morifuji 2005].
- (2) The twisted Alexander polynomial is reciprocal; that is, $\Delta_{K,\rho}(t) = t^i \Delta_{K,\rho}(t^{-1})$ for some $i \in \mathbb{Z}$ [Hillman et al. 2010; Friedl et al. 2012].
- (3) If K is a fibered knot, then $\Delta_{K,\rho}(t)$ is a monic polynomial for every nonabelian representation ρ [Goda et al. 2005]. It is also known that the converse holds for alternating knots [Kim and Morifuji 2012, Remark 4.2].
- (4) If K is a knot of genus g , then $\deg(\Delta_{K,\rho}(t)) \leq 4g - 2$ [Friedl and Kim 2006]. Moreover if K is fibered, then the equality holds [Kitano and Morifuji 2005].

We say the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ determines the knot genus $g(K)$ if $\deg(\Delta_{K,\rho}(t)) = 4g(K) - 2$ holds. For a hyperbolic knot K , the hyperbolic torsion polynomial $\mathcal{T}_K(t)$ is defined to be $\Delta_{K,\rho_0}(t)$ for the holonomy representation $\rho_0 : G_K \rightarrow \text{SL}_2(\mathbb{C})$. We note that it is normalized so that $\mathcal{T}_K(t) = \mathcal{T}_K(t^{-1})$ holds.

Proof of part (1) of Theorem 1.2. It is known that the genus of $J(k, 2n)$, where $k > 1$ and $|n| > 0$, is 1 if k is even, and is $|n|$ if k is odd. Moreover, the genus of $J(1, 2n)$ (the $(2, 2n - 1)$ -torus knot) is $n - 1$ if $n > 0$ and is $-n$ if $n < 0$.

We first consider the case $n > 0$. Let $r = w^n a w^{-n} b^{-1}$, where w is as defined in Section 2. By direct calculations, we have

$$(4-1) \quad \frac{\partial r}{\partial a} = w^n \left(1 + (1 - a)(w^{-1} + \dots + w^{-n}) \frac{\partial w}{\partial a} \right),$$

where, for $k = 2m$,

$$\frac{\partial w}{\partial a} = - (ba^{-1} + \dots + (ba^{-1})^m) + (ba^{-1})^m (1 + b^{-1}a + \dots + (b^{-1}a)^{m-1}b^{-1}),$$

and, for $k = 2m + 1$,

$$\frac{\partial w}{\partial a} = - (ba^{-1} + \dots + (ba^{-1})^m) + (ba^{-1})^m b (1 + ab^{-1} + \dots + (ab^{-1})^m).$$

Suppose $\rho : G_K \rightarrow \text{SL}_2(\mathbb{C})$ is a nonabelian representation given by (2-3). Then the twisted Alexander polynomial of K associated to ρ is

$$\Delta_{K,\rho}(t) = \frac{\det \Phi\left(\frac{\partial r}{\partial a}\right)}{\det \Phi(1-b)} = \frac{\det \Phi\left(\frac{\partial r}{\partial a}\right)}{1-tx+t^2}.$$

The case $J(2m, 2n)$, $n > 0$. From (4-1) we have

$$\det \Phi\left(\frac{\partial r}{\partial a}\right) = |I + (I - tA)(W^{-1} + \dots + W^{-n})V|,$$

where I is the 2×2 identity matrix and

$$V = -(BA^{-1} + \dots + (BA^{-1})^m) + (BA^{-1})^m (I + B^{-1}A + \dots + (B^{-1}A)^{m-1})t^{-1}B^{-1}.$$

The next lemma follows easily.

Lemma 4.2. *The highest- and lowest-degree terms of $\det \Phi\left(\frac{\partial r}{\partial a}\right)$ are respectively*

$$|A(W^{-1} + \dots + W^{-n})(BA^{-1} + \dots + (BA^{-1})^m)|t^2$$

and

$$|(W^{-1} + \dots + W^{-n})(BA^{-1})^m(I + B^{-1}A + \dots + (B^{-1}A)^{m-1})B^{-1}|t^{-2}.$$

Let $\{T_i(z)\}_i$ be the sequence of Chebyshev polynomials defined by $T_0(z) = 2$, $T_1(z) = z$ and $T_{i+1}(z) = zT_i(z) - T_{i-1}(z)$ for all integers i . Recall that $y = \text{tr } AB^{-1}$ and $\lambda = \text{tr } W$.

Proposition 4.3. *The highest- and lowest-degree terms of $\det \Phi\left(\frac{\partial r}{\partial a}\right)$ are respectively*

$$\frac{T_n(\lambda) - 2}{\lambda - 2} \frac{T_m(y) - 2}{y - 2} t^2 \quad \text{and} \quad \frac{T_n(\lambda) - 2}{\lambda - 2} \frac{T_m(y) - 2}{y - 2} t^{-2}.$$

Proof. Let β_{\pm} be the roots of $z^2 - \lambda z + 1$ and γ_{\pm} the roots of $z^2 - yz + 1$. Lemma 4.2 implies that the highest- and lowest degree terms of $\det \Phi(\partial r/\partial a)$ are respectively

$$(1 + \beta_+ + \dots + \beta_+^{n-1})(1 + \beta_- + \dots + \beta_-^{n-1}) \times (1 + \gamma_+ + \dots + \gamma_+^{m-1})(1 + \gamma_- + \dots + \gamma_-^{m-1})t^2$$

and

$$(1 + \beta_+ + \dots + \beta_+^{n-1})(1 + \beta_- + \dots + \beta_-^{n-1}) \times (1 + \gamma_+ + \dots + \gamma_+^{m-1})(1 + \gamma_- + \dots + \gamma_-^{m-1})t^{-2}.$$

Proposition 4.3 then follows from **Lemma 4.4** below. □

Lemma 4.4. $(1 + \beta_+ + \dots + \beta_+^{n-1})(1 + \beta_- + \dots + \beta_-^{n-1}) = \frac{T_n(\lambda) - 2}{\lambda - 2} \in \mathbb{Z}[\lambda].$

Proof. The left-hand side is equal to

$$\frac{(\beta_+^n - 1)(\beta_-^n - 1)}{(\beta_+ - 1)(\beta_- - 1)} = \frac{\beta_+^n + \beta_-^n - 2}{\beta_+ + \beta_- - 2} = \frac{T_n(\lambda) - 2}{\lambda - 2}.$$

The lemma follows. □

Proposition 4.3 implies that the highest- and lowest-degree terms of the twisted Alexander polynomial

$$\Delta_{K,\rho}(t) = \frac{\det \Phi \left(\frac{\partial r}{\partial a} \right)}{1 - tx + t^2}$$

are respectively $U_{m,n}(y)t^0$ and $U_{m,n}(y)t^{-2}$, where

$$U_{m,n}(y) = \frac{T_n(\lambda) - 2}{\lambda - 2} \frac{T_m(y) - 2}{y - 2}.$$

Hence to prove **Theorem 1.2(1)** for $J(2m, 2n)$, $n > 0$, we only need to show that the coefficients of these terms are nonzero under the assumption that $\phi_K(2, y) = S_{n-1}(\lambda)\alpha_k(2, y) - S_{n-2}(\lambda) = 0$ (because the roots of this equation correspond to the parabolic representations). To this end, we show that at $x = 2$ the polynomials $\phi_K(2, y)$ and $U_{m,n}(y)$ do not have any common zero $y \in \mathbb{C}$ (in fact, if they have a common zero, the highest- and lowest-degree terms vanish at $x = 2$). It is equivalent to show that at $x = 2$ these polynomials are relatively prime in $\mathbb{C}[y]$.

Recall that $\lambda = \text{tr } W = (y - 2)(y + 2 - x^2)S_{m-1}^2(y) + 2$ and

$$\alpha_k(x, y) = 1 - (y + 2 - x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y)).$$

The next two lemmas will complete the proof of **Theorem 1.2(1)** for $J(2m, 2n)$, $n > 0$.

Lemma 4.5. *Suppose $x = 2$. Then $\text{gcd}(\phi_K(x, y), \frac{T_n(\lambda) - 2}{\lambda - 2}) = 1$ in $\mathbb{C}[y]$.*

Proof. It is equivalent to show that at $x = 2$, $\phi_K(x, y)$ and $(T_n(\lambda) - 2)/(\lambda - 2)$ do not have any common root $y \in \mathbb{C}$.

Suppose $T_n(\lambda) = 2$ and $\lambda \neq 2$ then $\beta_+^n = \beta_-^n = 1$ and $\beta_+ \neq 1$. If $\beta_+ \neq -1$ then $S_{n-1}(\lambda) = (\beta_+^n - \beta_-^n)/(\beta_+ - \beta_-) = 0$ and $S_{n-2}(\lambda) = (\beta_+^{n-1} - \beta_-^{n-1})/(\beta_+ - \beta_-) = -1$; hence $\phi_K(x, y) = 1 \neq 0$.

If $\beta_+ = -1$ (in this case n must be even) then $\lambda = -2$. It implies that $S_{n-1}(\lambda) = -n$, and $S_{n-2}(\lambda) = n - 1$. Hence $\phi_K(x, y) = -n\alpha_k(x, y) - (n - 1)$.

Suppose $\phi_K(x, y) = 0$. Then $\alpha_k(x, y) = 1/n - 1$. We have

$$(y - 2)(y + 2 - x^2)S_{m-1}^2(y) = \lambda - 2 = -4,$$

$$(y + 2 - x^2) (S_{m-1}^2(y) - S_{m-1}(y)S_{m-2}(y)) = 1 - \alpha_k(x, y) = 2 - \frac{1}{n}.$$

Thus $(y+2-x^2)S_{m-1}^2(y) = \frac{-4}{y-2}$, $(y+2-x^2)S_{m-1}(y)S_{m-2}(y) = -\left(\frac{4}{y-2} + 2 - \frac{1}{n}\right)$.

Since

$$S_{m-1}^4(y) - yS_{m-1}^2(y)(S_{m-1}(y)S_{m-2}(y)) + (S_{m-1}(y)S_{m-2}(y))^2 = S_{m-1}^2(y)$$

(by Lemma 2.1), we must have

$$\frac{16}{(y-2)^2} - \frac{4y}{y-2} \left(\frac{4}{y-2} + 2 - \frac{1}{n}\right) + \left(\frac{4}{y-2} + 2 - \frac{1}{n}\right)^2 = \frac{-4(y+2-x^2)}{y-2},$$

that is, $y = 2 + 4n^2x^2$.

If $x \in \mathbb{R}$ then $y = 2 + 4n^2x^2 \in \mathbb{R}$ and

$$-4 = (y - 2)(y + 2 - x^2)S_{m-1}^2(y) = 4n^2x^2(4 + (4n^2 - 1)x^2)S_{m-1}^2(y) \geq 0,$$

a contradiction. Hence $\phi_K(x, y) \neq 0$ when $\frac{T_n(\lambda) - 2}{\lambda - 2} = 0$ and $x \in \mathbb{R}$. The lemma follows. □

Lemma 4.6. *Suppose $x = 2$. Then $\gcd(\phi_K(x, y), \frac{T_m(y) - 2}{y - 2}) = 1$ in $\mathbb{C}[y]$.*

Proof. Suppose $T_m(y) = 2$ and $y \neq 2$ then $\gamma_+^m = \gamma_-^m = 1$ and $\gamma_+ \neq 1$. If $\gamma_+ \neq -1$ then $S_{m-1}(y) = (\gamma_+^m - \gamma_-^m)/(\gamma_+ - \gamma_-) = 0$, hence $\lambda = 2$ and $\alpha_k(x, y) = 1$. This implies that $\phi_K(x, y) = S_{n-1}(2) - S_{n-2}(2) = 1 \neq 0$.

If $\gamma_+ = -1$ (in this case m must be even) then $y = -2$. We have

$$\lambda = (y - 2)(y + 2 - x^2)S_{m-1}^2(y) + 2 = 4m^2x^2 + 2$$

and

$$\alpha_k(x, y) = 1 - (y + 2 - x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y)) = m(2m - 1)x^2 + 1.$$

This implies that

$$\phi_K(x, y) = S_{n-1}(\lambda)\alpha_k(x, y) - S_{n-2}(\lambda) = (m(2m - 1)x^2 + 1)S_{n-1}(\lambda) - S_{n-2}(\lambda).$$

If $x \in \mathbb{Z}$ then $\lambda = 4m^2x^2 + 2 \in \mathbb{Z}$ is even. This means that $\phi_K(x, y)$ is odd, since

$$\phi_K(x, y) \equiv S_{n-1}(\lambda) - S_{n-2}(\lambda) \pmod{2}.$$

Thus $\phi_K(x, y) \neq 0$ when $\frac{T_m(y) - 2}{y - 2} = 0$ and $x \in \mathbb{Z}$. The lemma follows. □

The case $J(2m + 1, 2n)$, $n > 0$. From (4-1) we have

$$\det \Phi \left(\frac{\partial r}{\partial a} \right) = |I + (I - tA)(t^{-2}W^{-1} + \dots + t^{-2n}W^{-n})V| t^{4n},$$

where

$$V = -(BA^{-1} + \dots + (BA^{-1})^m) + t(BA^{-1})^m B(I + AB^{-1} + \dots + (AB^{-1})^m).$$

We first consider the case $m = 0$ (in this case we must have $n > 1$ so that K is a nontrivial knot). Then $W = BA$ and

$$\begin{aligned} \det \Phi \left(\frac{\partial r}{\partial a} \right) &= |I + (I - tA)(t^{-2}W^{-1} + \dots + t^{-2n}W^{-n})tB| t^{4n} \\ &= |(t^{-2}W^{-1} + \dots + t^{-2n}W^{-n})tB - tA(t^{-4}W^{-2} + \dots + t^{-2n}W^{-n})tB| t^{4n}. \end{aligned}$$

This implies that the highest- and lowest-degree terms of $\det \Phi(\partial r/\partial a)$ are $|t^{-1}W^{-1}B| t^{4n} = t^{4n-2}$ and $|t^{1-2n}W^{-n}B| t^{4n} = t^2$, respectively. Hence the highest- and lowest degree terms of $\Delta_{K,\rho}(t)$ are t^{4n-4} and t^2 , respectively. Since the genus of $J(1, 2n)$, where $n > 1$, is $n - 1$, we complete the proof of Theorem 1.2(1) for $J(1, 2n)$, $n > 1$.

We now consider the case $m > 0$. In this case, we have the following.

Lemma 4.7. (1) *The highest-degree term of $\det \Phi \left(\frac{\partial r}{\partial a} \right)$ is*

$$\begin{aligned} |I - AW^{-1}(BA^{-1})^m B(I + AB^{-1} + \dots + (AB^{-1})^m)| t^{4n} \\ = |I + BA^{-1} + \dots + (BA^{-1})^{m-1}| t^{4n}. \end{aligned}$$

(2) *The lowest-degree term of $\det \Phi \left(\frac{\partial r}{\partial a} \right)$ is*

$$|-W^{-n}(BA^{-1} + \dots + (BA^{-1})^m)| t^0 = |I + BA^{-1} + \dots + (BA^{-1})^{m-1}| t^0.$$

Lemmas 4.7 and 4.4 imply the following.

Proposition 4.8. *The highest- and lowest-degree terms of $\det \Phi \left(\frac{\partial r}{\partial a} \right)$ are respectively*

$$\frac{T_m(y) - 2}{y - 2} t^{4n} \quad \text{and} \quad \frac{T_m(y) - 2}{y - 2} t^0.$$

Proposition 4.8 implies that the highest- and lowest-degree terms of $\Delta_{K,\rho}(t)$ are $(T_m(y) - 2)/(y - 2)t^{4n-2}$ and $(T_m(y) - 2)/(y - 2)t^0$, respectively. Hence to prove Theorem 1.2(1) for $J(2m + 1, 2n)$, where $m, n > 0$, we only need to show that at $x = 2$ (parabolic representation) the polynomials $\phi_K(x, y) = S_{n-1}(\lambda)\alpha_k(x, y) - S_{n-2}(\lambda)$ and $(T_m(y) - 2)/(y - 2)$ are relatively prime in $\mathbb{C}[y]$.

Recall that $\lambda = \text{tr } W = x^2 - y - (y - 2)(y + 2 - x^2)S_m(y)S_{m-1}(y)$ and

$$\alpha_k(x, y) = 1 + (y + 2 - x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y)).$$

The next lemma will complete the proof of [Theorem 1.2\(1\)](#) for $J(2m + 1, 2n)$, where $m, n > 0$.

Lemma 4.9. *Suppose $x = 2$. Then $\text{gcd}\left(\phi_K(x, y), \frac{T_m(y)-2}{y-2}\right) = 1$ in $\mathbb{C}[y]$.*

Proof. Suppose $T_m(y) = 2$ and $y \neq 2$, then $\gamma_+^m = \gamma_-^m = 1$ and $\gamma_+ \neq 1$. If $\gamma_+ \neq -1$ then $S_{m-1}(y) = 0$ and $S_m(y) = 1$; hence $\lambda = x^2 - y$ and $\alpha_k(x, y) = 1$. This implies that $\phi_K(x, y) = S_{n-1}(\lambda) - S_{n-2}(\lambda)$.

Since $\gamma_+^m = 1$, we have $y = \gamma_+ + \gamma_+^{-1} = 2 \cos(2\pi j/m)$ for some $0 < j < m$. If $\phi_K(x, y) = S_{n-1}(\lambda) - S_{n-2}(\lambda) = 0$ then $\lambda = 2 \cos((2j' - 1)\pi/(2n - 1))$ for some $1 \leq j' \leq n - 1$; see [\[Le and Tran 2012, Lemma 4.13\]](#), for example. Hence

$$x^2 = y + \lambda = 2 \left(\cos \frac{2\pi j}{m} + \cos \frac{(2j' - 1)\pi}{2n - 1} \right) < 4.$$

If $\gamma_+ = -1$ (in this case m must be even) then $y = -2$. We have

$$\lambda = -(y - 2)(y + 2 - x^2)S_m(y)S_{m-1}(y) + x^2 - y = (2m + 1)^2 x^2 + 2$$

and

$$\alpha_k(x, y) = 1 + (y + 2 - x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y)) = m(2m + 1)x^2 + 1.$$

If x is an even integer then $\lambda = (2m + 1)^2 x^2 + 2$ is an even integer and $\alpha_k(x, y) = m(2m + 1)x^2 + 1$ is an odd integer. Hence

$$\phi_K(x, y) = S_{n-1}(\lambda)\alpha_k(x, y) - S_{n-2}(\lambda) \equiv S_{n-1}(\lambda) - S_{n-2}(\lambda) \pmod{2}$$

is odd and so is nonzero.

In both cases, we obtain $\phi_K(x, y) \neq 0$ when $\frac{T_m(y)-2}{y-2} = 0$ and x is an even integer at least 2. The lemma follows. □

Next we consider the case $n < 0$. We put $l = -n$ ($l > 0$). For $r = w^n a w^{-n} b^{-1} = w^{-l} a w^l b^{-1}$, we have

$$(4-2) \quad \frac{\partial r}{\partial a} = \frac{\partial w^{-l}}{\partial a} + w^{-l} \left(1 + a \frac{\partial w^l}{\partial a} \right) = w^{-l} \left(1 - (1-a)(1+w+\dots+w^{l-1}) \frac{\partial w}{\partial a} \right).$$

The case $J(2m, 2n)$, $n < 0$. From (4-2) we have

$$\det \Phi \left(\frac{\partial r}{\partial a} \right) = |I - (I - tA)(I + W + \dots + W^{l-1})V|,$$

where

$$V = -(BA^{-1} + \dots + (BA^{-1})^m) + (BA^{-1})^m (I + B^{-1}A + \dots + (B^{-1}A)^{m-1}) t^{-1} B^{-1}.$$

Lemma 4.10. (1) *The highest-degree term of $\det \Phi\left(\frac{\partial r}{\partial a}\right)$ is*

$$|-A(I + W + \dots + W^{l-1})(BA^{-1} + \dots + (BA^{-1})^m)| t^2.$$

(2) *The lowest-degree term of $\det \Phi\left(\frac{\partial r}{\partial a}\right)$ is*

$$|-(I + W + \dots + W^{l-1})(BA^{-1})^m(I + B^{-1}A + \dots + (B^{-1}A)^{m-1})B^{-1}| t^{-2}.$$

We can apply a similar argument to that of the parallel case with $n > 0$ (page 441) to conclude that $\Delta_{K,\rho}(t)$, for ρ parabolic, determines the knot genus in this case.

The case $J(2m + 1, 2n)$, $n < 0$. From (4-2) we have

$$\begin{aligned} \det \Phi\left(\frac{\partial r}{\partial a}\right) &= |t^{-2l} W^{-l} (I - (I - tA)(I + t^2W + \dots + t^{2(l-1)}W^{l-1})V)| \\ &= |I - (I - tA)(I + t^2W + \dots + t^{2(l-1)}W^{l-1})V| t^{-4l}, \end{aligned}$$

where

$$V = -(BA^{-1} + \dots + (BA^{-1})^m) + t(BA^{-1})^m B(I + AB^{-1} + \dots + (AB^{-1})^m).$$

Lemma 4.11. (1) *The highest-degree term of $\det \Phi\left(\frac{\partial r}{\partial a}\right)$ is*

$$\begin{aligned} |AW^{l-1}(BA^{-1})^m B(I + AB^{-1} + \dots + (AB^{-1})^m)| t^0 \\ = |I + AB^{-1} + \dots + (AB^{-1})^m| t^0. \end{aligned}$$

(2) *The lowest-degree term of $\det \Phi\left(\frac{\partial r}{\partial a}\right)$ is*

$$|I + BA^{-1} + \dots + (BA^{-1})^m| t^{-4l}.$$

We can apply a similar argument to that of the parallel case with $n > 0$ (page 444) to conclude again that $\Delta_{K,\rho}(t)$, for ρ parabolic, determines the knot genus in this case.

The case analysis starting on page 441 covers all possibilities. Theorem 1.2(1) follows immediately. □

5. The fibering problem

In this section we study some properties of the parabolic representation spaces of 2-bridge knots and give the proof of Theorem 1.2(2).

Parabolic representations of 2-bridge knots. Consider the 2-bridge knot $K = \mathfrak{b}(p, q)$, where $p > q \geq 1$ are relatively prime. The knot group G_K has a presentation $G_K = \langle a, b \mid wa = bw \rangle$, where $w = a^{\varepsilon_1} b^{\varepsilon_2} \dots a^{\varepsilon_{p-2}} b^{\varepsilon_{p-1}}$ and $\varepsilon_j = (-1)^{\lfloor jq/p \rfloor}$ (see, e.g., [Burde and Zieschang 2003]).

Let $\phi_K(x, y)$ be the defining equation for the nonabelian representations into $SL_2(\mathbb{C})$ of G_K , where $x = \text{tr } \rho(a) = \text{tr } \rho(b)$ and $y = \text{tr } \rho(ab^{-1})$. Then $\phi_K(2, y)$ is the defining equation for the parabolic representations. It is known that $\phi_K(2, y) \in \mathbb{Z}[y]$ is a monic polynomial of degree $d = (p - 1)/2$; see [Riley 1984; Tkhang 1993].

We want to study the irreducibility of $\phi_K(2, y) \in \mathbb{Z}[y]$.

Lemma 5.1. *One has $\phi_K(2, y) = S_d(y) + S_{d-1}(y)$ in $\mathbb{Z}_2[y]$.*

Proof. The proof is similar to that of [Le and Tran 2012, Proposition A.2].

Suppose ρ is a parabolic representation. Let $A = \rho(a)$, $B = \rho(b)$ and $W = \rho(w)$. Taking conjugation if necessary, we can assume that

$$(5-1) \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 2 - y & 1 \end{bmatrix},$$

where $y = \text{tr } AB^{-1} \in \mathbb{C}$ satisfies the matrix equation $WA - BW = 0$.

By the Cayley–Hamilton theorem applying for matrices in $SL_2(\mathbb{C})$ we have $A + A^{-1} = \text{tr}(A)I = 2I = 0 \pmod{2}$, that is, $A^{-1} = A \pmod{2}$. Similarly, $B^{-1} = B \pmod{2}$. This implies that $W = A^{\varepsilon_1} B^{\varepsilon_2} \dots A^{\varepsilon_{2d-1}} B^{\varepsilon_{2d}} = (AB)^d \pmod{2}$. By applying (2-2), we have

$$\begin{aligned} WA + BW &= (AB)^d A + B(AB)^d \\ &= S_d(y)(A + B) + S_{d-1}(y)(B^{-1} + A^{-1}) \\ &= (S_d(y) + S_{d-1}(y))(A + B) \pmod{2}, \end{aligned}$$

where $A + B = \begin{bmatrix} 0 & 1 \\ y & 0 \end{bmatrix} \pmod{2}$. Hence $\phi_K(2, y) = S_d(y) + S_{d-1}(y)$ in $\mathbb{Z}_2[y]$. □

Recall from the Introduction that \mathcal{P}_2 is the set of all odd primes p such that 2 is a primitive root modulo p .

Lemma 5.2. *Suppose $p \in \mathcal{P}_2$. Then $S_d(y) + S_{d-1}(y) \in \mathbb{Z}_2[y]$ is irreducible.*

Proof. Let $y = u + u^{-1}$. Then

$$S_d(y) + S_{d-1}(y) = \frac{u^{d+1} + u^{-(d+1)}}{u + u^{-1}} + \frac{u^d + u^{-d}}{u + u^{-1}} = u^{-d} \frac{1 + u^{2d+1}}{1 + u}.$$

Suppose $p \in \mathcal{P}_2$. We will show that $(1 + u^p)/(1 + u) \in \mathbb{Z}_2[u]$ is irreducible. This will imply that $S_d(y) + S_{d-1}(y) \in \mathbb{Z}_2[y]$ is irreducible.

We have $(1 + u^p)/(1 + u) = u^{p-1} + \dots + u + 1$ is the p -th-cyclotomic polynomial $C_p(u) \in \mathbb{Z}_2[u]$ (since p is an odd prime). It is well known that $C_p(u) \in \mathbb{Z}_2[u]$ is irreducible if $p \in \mathcal{P}_2$; see for example, [Roman 2006, Theorem 11.2.8]. The lemma follows. □

Proposition 5.3. *Suppose $p \in \mathcal{P}_2$. Then $\phi_K(2, y) \in \mathbb{Z}[y]$ is irreducible.*

Proof. By Lemma 5.1, $\phi_K(2, y) = S_d(y) + S_{d-1}(y) \in \mathbb{Z}_2[y]$. Since $p \in \mathcal{P}_2$, the polynomial $S_d(y) + S_{d-1}(y) \in \mathbb{Z}_2[y]$ is irreducible by Lemma 5.2. This implies that $\phi_K(2, y)$ is irreducible in $\mathbb{Z}_2[y]$. Since $\phi_K(2, y) \in \mathbb{Z}[y]$ is a monic polynomial in y , it is irreducible in $\mathbb{Z}[y]$. □

Proof of part (2) of Theorem 1.2. It is known that $J(k, 2n)$ is fibered only for the trivial knot $J(k, 0)$, the trefoil knot $J(2, 2)$, the figure eight knot $J(2, -2)$, the knots $J(1, 2n)$ for any n , and the knots $J(3, 2n)$ for $n > 0$.

The case $J(2m, 2n)$, $m > 1$. We will apply Proposition 5.3 to study the fibered problem for $K = J(2m, 2n)$.

Let $p = |4mn - 1|$ then it is known that $\phi_K(2, y)$ has degree $(p - 1)/2$. By Proposition 5.3, the polynomial $\phi_K(2, y) \in \mathbb{Z}[y]$ is irreducible if $p \in \mathcal{P}_2$.

Proposition 5.4. *Suppose $m > 1$ and $p = |4mn - 1| \in \mathcal{P}_2$. Then $\Delta_{K,\rho}(t)$ is nonmonic for every parabolic representation ρ .*

Proof. We only need to consider the case $n > 0$. The case $n < 0$ is similar.

Suppose ρ is a parabolic representation, that is, $x = 2$. Since $k = 2m$, by Proposition 4.3 the coefficient of the highest-degree term of $\Delta_{K,\rho}(t)$ is $h(y) = (T_n(\lambda) - 2)/(\lambda - 2) \times (T_m(y) - 2)/(y - 2)$, an integer polynomial in y of degree $(n - 1)(2m) + (m - 1) = 2mn - (m + 1) < 2mn - 1 = (p - 1)/2$.

Since $p \in \mathcal{P}_2$, the polynomial $\phi_K(2, y) \in \mathbb{Z}[y]$ is irreducible. This implies that $\phi_K(2, y)$ does not divide $h(y) - 1$ in $\mathbb{Z}[y]$. Hence $h(y) \neq 1$ when $\phi_K(2, y) = 0$. The proposition follows. □

Twist knots $J(2, 2n)$. For $K = J(2, 2n)$ we have $\lambda = y^2 - yx^2 + 2x^2 - 2$ and $\phi_K(x, y) = -(y + 1 - x^2)S_{n-1}(\lambda) - S_{n-2}(\lambda)$. Suppose ρ is a nonabelian representation. By Proposition 4.3 the coefficient of the highest-degree term of $\Delta_{K,\rho}(t)$ is $(T_n(\lambda) - 2)/(\lambda - 2)$. We want to show that for $|n| > 1$, we have $(T_n(\lambda) - 2)/(\lambda - 2) \neq 1$ when $\phi_K(x, y) = 0$ and $x = 2$. This will imply that for any parabolic representation ρ , $\Delta_{K,\rho}(t)$ is monic if and only if $|n| = 1$.

Lemma 5.5. *If $x = 2$, then $\gcd\left(\phi_K(2, y), \frac{T_n(\lambda) - 2}{\lambda - 2} - 1\right) = 1$ in $\mathbb{C}[y]$ for $|n| > 1$.*

Proof. We only need to consider the case $n > 1$. The case $n < -1$ is similar.

Suppose $T_n(\lambda) = \lambda$ and $\lambda \neq 2$. Then $\beta_+^n + \beta_-^n = \beta_+ + \beta_-$, i.e., $\beta_+^{n-1} = 1$ or $\beta_+^{n+1} = 1$. It implies that $\lambda = -2$, or $\lambda = 2 \cos 2j\pi/(n - 1)$ for some $1 \leq j \leq n - 2$ and $j \neq (n - 1)/2$, or $\lambda = 2 \cos 2j\pi/(n + 1)$ for some $1 \leq j \leq n$ and $j \neq (n + 1)/2$.

Case 1: $\lambda = -2$ (in this case n must be odd). By similar arguments as in the proof of Lemma 4.5, we have $\phi_K(x, y) \neq 0$ if $x \in \mathbb{R}$.

Case 2: $\lambda = 2 \cos 2j\pi/(n - 1)$ for some $1 \leq j \leq n - 2$ and $j \neq (n - 1)/2$. Then $S_{n-1}(\lambda) = 1$ and $S_{n-2}(\lambda) = 0$, hence $\phi_K(x, y) = -(y + 1 - x^2)$.

Suppose $\phi_K(x, y) = 0$. Then $y = x^2 - 1$ and $\lambda = y^2 - yx^2 + 2x^2 - 2 = x^2 - 1$. This cannot occur if $x^2 - 1 \geq 2$, since $\lambda < 2$. Hence $\phi_K(x, y) \neq 0$ if $x^2 \geq 3$.

Case 3: $\lambda = 2 \cos 2j\pi/(n + 1)$ for some $1 \leq j \leq n$ and $j \neq (n + 1)/2$. Then $S_{n-1}(\lambda) = -1$ and $S_{n-2}(\lambda) = -\lambda$, hence $\phi_K(x, y) = y + 1 - x^2 + \lambda$.

Suppose $\phi_K(x, y) = 0$. Then $y = x^2 - \lambda - 1$ and $\lambda = y^2 - yx^2 + 2x^2 - 2 = \lambda^2 + \lambda(2 - x^2) + x^2 - 1$, that is, $\lambda^2 - \lambda(x^2 - 1) + x^2 - 1 = 0$. This equation does not have any real solution λ if $1 < x^2 < 5$. Hence $\phi_K(x, y) \neq 0$ if $1 < x^2 < 5$.

In all cases, $\phi_K(x, y) \neq 0$ when $(T_n(\lambda) - 2)/(\lambda - 2) = 1$ and $3 \leq x^2 < 5$. The lemma follows. □

Remark 5.6. Lemma 5.5 gives a proof of [Morifuji 2012, Theorem 1.2] that does not use the irreducibility of $\phi_{J(2,2n)}(2, y) \in \mathbb{Z}[y]$ proved in [Hoste and Shanahan 2001].

The case $J(2m + 1, 2n)$. Let $K = J(2m + 1, 2n)$. Suppose ρ is a nonabelian representation. By Proposition 4.8 and Lemma 4.11, the coefficient of the highest-degree term of $\Delta_{K,\rho}(t)$ is

$$\frac{T_m(y) - 2}{y - 2} \quad \text{if } n > 0 \quad \text{and} \quad \frac{T_{m+1}(y) - 2}{y - 2} \quad \text{if } n < 0.$$

We want to show that for $m > 1$, we have $(T_m(y) - 2)/(y - 2) \neq 1$ when $\phi_K(2, y) = 0$. This will imply that for any parabolic representation ρ , $\Delta_{K,\rho}(t)$ is monic if and only if $K = J(1, 2n)$, or $K = J(3, 2n)$ and $n > 0$.

The key point of the proof of the following lemma is to apply Proposition 3.2.

Lemma 5.7. *If $x = 2$, then $\gcd\left(\phi_K(2, y), \frac{T_m(y) - 2}{y - 2} - 1\right) = 1$ in $\mathbb{C}[y]$ for $m > 1$.*

Proof. Suppose $T_m(y) = y$ and $y \neq 2$. Then $\gamma_+^m + \gamma_-^m = \gamma_+ + \gamma_-$, i.e., $\gamma_+^{m-1} = 1$ or $\gamma_+^{m+1} = 1$. This implies that $y = -2$, or $y = 2 \cos 2j\pi/(m - 1)$ for some $1 \leq j \leq m - 2$ and $j \neq (m - 1)/2$, or $y = 2 \cos 2j\pi/(m + 1)$ for some $1 \leq j \leq m$ and $j \neq (m + 1)/2$. In all cases, $y \in \mathbb{R}$ and $y < 2$. Proposition 3.2 then implies that $\phi_K(2, y) \neq 0$. The lemma follows. □

The case analysis on the last two pages cover all possibilities, showing part (2) of Theorem 1.2. This completes the proof of the theorem. □

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