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Let $X_0^D(N)$, where $(D, N) = 1$, denote the Shimura curve associated to an Eichler order of level N, in an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant D. Let $W_{D,N}$ be the group of all Atkin–Lehner involutions on $X_0^D(N)$ and W_D the subgroup consisting of Atkin–Lehner involutions w_m with $m \mid D$. In this paper, we will determine Schwarzian differential equations associated to Shimura curves $X_0^D(N)/W_D$ of genus zero in the cases where there exists a squarefree integer $M > 1$ such that $X_0^D(M)/W_D$ is of genus zero.

1. Introduction

Let B be an indefinite quaternion algebra of discriminant D over $\mathbb{Q}$. For an Eichler order $\mathbb{O}$ of level N, $(D, N) = 1$, in B, we let $X_0^D(N)$ denote the Shimura curve associated to $\mathbb{O}$. For each divisor m of DN with $(m, DN/m) = 1$, we let w_m denote the Atkin–Lehner involution on $X_0^D(N)$ and $W_{D,N}$ be the group of all Atkin–Lehner involutions. We also let the subgroup of $W_{D,N}$ consisting of w_m, $m \mid D$, be denoted by W_D. (We refer the reader to [Alsina and Bayer 2004; Elkies 1998] for general definitions and properties of Shimura curves.)

The notion of Shimura curves generalizes that of classical modular curves, which correspond to the case $B = M(2, \mathbb{Q})$ with $D = 1$. Many properties and theories about classical modular curves can be extended to the case of Shimura curves. However, because of the lack of cusps in the case $D \neq 1$, there have been very few explicit methods for general Shimura curves. One of the few methods uses differential equations satisfied by automorphic forms and automorphic functions. (See [Bayer and Travesa 2007; Elkies 1998; Yang 2013b; 2004].) The idea is that even though it is difficult to explicitly construct automorphic functions that can be put into practical use, the Schwarzian differential equations associated to automorphic functions in the case of Shimura curves of genus zero can often be determined using analytic information about the automorphic functions and coverings between Shimura curves. (See Section 2 for the definition and properties of Schwarzian differential equations.)

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Then one can use the solutions of the Schwarzian differential equations in place of automorphic forms to study properties of automorphic forms. For example, Yang [2013b] devised a method to determine Hecke eigenforms in the spaces of automorphic forms, expressed in terms of solutions of Schwarzian differential equations. In [Tu and Yang 2013], we obtained several new algebraic transformations of ${}_2F_1$-hypergeometric functions by interpreting identities among hypergeometric functions as identities among automorphic forms on different Shimura curves.

In view of the significance of Schwarzian differential equations, it is important to determine the Schwarzian differential equation for each of the Shimura curves $X_0^D(N)/G$, $G < W_{D,N}$, of genus zero. Elkies [1998] worked out the Schwarzian equations on $X_0^{10}(1)/W_{10}$, $X_0^{14}(1)/W_{14}$, and $X_0^{15}(1)/W_{15}$. Bayer and Travesa [2007] computed all the Schwarzian differential equations for the Shimura curves $X_0^6(1)/G$ with $G < W_6$. Yang [2013b] also gave Schwarzian differential equations on $X_0^6(1)/W_6$ and $X_0^{10}(1)/W_{10}$ from the properties of the automorphic derivatives. (See Section 2.)

In this paper, we will consider the cases $X_0^D(N)/W_D$ when there exists an integer $M > 1$ such that $X_0^D(M)/W_D$ has genus zero. The reason for this restriction is that we need additional information from coverings between Shimura curves of genus zero in order to completely determine the differential equations. (Note that in [Yang 2013b], a covering between Shimura curves of different levels is also needed in order to compute Hecke operators.) In the process, we also need to work out equations for some Shimura curves of genus one and hyperelliptic Shimura curves, which are useful in determining the covering maps between Shimura curves. As a byproduct of our computation of coverings $X_0^D(N)/W_D \to X_0^D(1)/W_D$, we can also determine the values of Hauptmoduln at several CM-points.

A possible future work related to Schwarzian differential equations is Ramanujan-type series for Shimura curves. A typical example of Ramanujan-type identities for the classical modular curves is

$$\sum_{n=0}^{\infty} \frac{(6n+1)(1/2)_n^3}{(n!)^3}\left(\frac{1}{4}\right)^n = \frac{4}{\pi},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. Yang [2013a] gave several Ramanujan-type formulae for the Shimura curve $X_0^6(1)/W_6$. He conjectured that the general Ramanujan-type identities for Shimura curves are

$$\sum_{n=0}^{\infty}(R_1 n + R_2)A_n t_0^n = R_3 \frac{\pi}{\Omega_d^2},$$

where $R_1, R_2, R_3 \in \overline{\mathbb{Q}}$, $\sum_0^{\infty} A_n t^n$ is the expansion of a meromorphic automorphic form of weight 2 with respect to a Hauptmodul t of a Shimura curve of genus zero such that t takes value 0 at a CM-point of discriminant d, and t_0 is the value of t at some CM-point of discriminant $d' \neq d$. The number Ω_d is the period of an elliptic curve E over $\overline{\mathbb{Q}}$ with CM by an imaginary quadratic number field of discriminant d. In the same article, he also gave some numerical results of p-adic analogues of these Ramanujan-type identities. It is natural to expect that those p-adic identities should be related to the p-adic periods of elliptic curves with CM. In this paper, in support of his conjecture, we will numerically obtain Ramanujan-type identities for $X_0^{14}(1)/W_{14}$ using our Schwarzian differential equation. However, we are not able to give a rigorous proof at present.

The rest of the paper is organized as follows. In Section 2, we will review the definition of properties of Schwarzian differential equations. In Section 3, we determine all Shimura curves $X_0^D(N)/W_D$ of genus 0, $N > 1$. In Section 4, we will find explicit coverings of $X_0^D(N)/W_D \to X_0^D(1)/W_D$. The equations for Shimura curves and the methods to obtain them given in [González and Rotger 2004; 2006; Molina 2012] are important here. The explicit coverings will be used later. In Section 5, we will work out Schwarzian differential equations and examples for Ramanujan-type identities from the Shimura curve $X_0^{14}(1)/W_{14}$.

From now on, for simplicity of statements, all Shimura curves mentioned below are assumed not to be classical modular curves.

2. Schwarzian differential equations

Let $t(\tau)$ be a nonconstant automorphic function on a Shimura curve X. It is straightforward to verify that $t'(\tau)$ is a meromorphic automorphic form of weight 2 on X and that the *Schwarzian derivative*

$$\{t, \tau\} := \frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\left(\frac{t''(\tau)}{t'(\tau)}\right)^2$$

is a meromorphic automorphic form of weight 4 on X. Thus, the ratio of $\{t, \tau\}$ and $t'(\tau)^2$ is an automorphic function on X. In particular, if X has genus zero and $t(\tau)$ is a Hauptmodul, that is, if t generates the field of automorphic functions on X, then

$$Q(t) := -\frac{\{t, \tau\}}{2t'(\tau)^2}$$

is a rational function of t. In [Bayer and Travesa 2007], given a thrice-differentiable function f of z, the function

$$D(f, z) := -\frac{\{f, z\}}{2f'(z)^2}$$

is called the *automorphic derivative* associated to f.

Now the relation $2Q(t)t'(\tau)^2 + \{t, \tau\} = 0$ can also be written as

$$\frac{d^2}{dt(\tau)^2} t'(\tau)^{1/2} + Q(t)t'(\tau)^{1/2} = 0.$$

In other words, if we consider $t'(\tau)^{1/2}$ as a function of t, then $t'(\tau)^{1/2}$ is a solution of the differential equation

$$(\dagger) \qquad \frac{d^2}{dt^2} f + Q(t) f = 0.$$

Definition 1. The differential equation (†) is called the *Schwarzian differential equation* associated to $t(\tau)$.

The significance of Schwarzian differential equations can be seen from the following result.

Proposition 2 [Yang 2013b]. *Assume that a Shimura curve X has genus zero with elliptic points $\tau_1, \dots, \tau_r$ of orders $e_1, \dots, e_r$, respectively. Let $t(\tau)$ be a Hauptmodul of X and set $a_i = t(\tau_i)$, $i = 1, \dots, r$. For a positive even integer $k \geq 4$, let*

$$d_k = \dim S_k(X) = 1 - k + \sum_{j=1}^{r} \left\lfloor \frac{k}{2}\left(1 - \frac{1}{e_j}\right) \right\rfloor,$$

$S_k(X)$ being the space of automorphic forms of weight k on X. A basis for $S_k(X)$ is

$$t'(\tau)^{k/2} t(\tau)^j \prod_{\substack{j=1 \\ a_j \neq \infty}}^{r} \big(t(\tau) - a_j\big)^{-\lfloor k(1-1/e_j)/2 \rfloor}, \qquad j = 0, \dots, d_k - 1.$$

In other words, if we can determine the Schwarzian differential equation associated to a Hauptmodul on a Shimura curve, then we can express automorphic forms of any even weight k on this Shimura curve in terms of solutions of the differential equation. This provides a concrete space that we can use to study properties of automorphic forms. For example, Yang [2013b] demonstrated how to compute Hecke operators on these spaces.

Now the upshot is that it is often possible to determine a Schwarzian differential equation without constructing a Hauptmodul first. This is especially true when a Shimura curve of genus zero has three elliptic points. This is due to the well-known fact that a second-order Fuchsian differential equation with precisely three singularities is uniquely determined its local exponents at the three points. For general Shimura curves, the following properties of Schwarzian differential equations and automorphic derivatives are very useful in determining the differential equations.

Proposition 3. *Assume that $X(\mathbb{O})$ has genus zero with elliptic points $\tau_1, \dots, \tau_r$ of order $e_1, \dots, e_r$, respectively. Let $t(\tau)$ be a Hauptmodul of $X(\mathbb{O})$ and set $a_i = t(\tau_i)$, $i = 1, \dots, r$. Then the automorphic derivative $Q(t) = D(t, \tau)$ is equal to*

$$Q(t) = \frac{1}{4} \sum_{\substack{j=1 \\ a_j \neq \infty}}^{r} \frac{1 - 1/e_j^2}{(t - a_j)^2} + \sum_{\substack{j=1 \\ a_j \neq \infty}}^{r} \frac{B_j}{t - a_j}$$

for some constants B_j. Moreover, if $a_j \neq \infty$ for all j, then the constants B_j satisfy

$$\sum_{j=1}^{r} B_j = \sum_{j=1}^{r} \bigl(a_j B_j + \tfrac{1}{4}(1 - 1/e_j^2)\bigr) = \sum_{j=1}^{r} \bigl(a_j^2 B_j + \tfrac{1}{2} a_j (1 - 1/e_j^2)\bigr) = 0.$$

Also, if $a_r = \infty$, then the B_j satisfy

$$\sum_{j=1}^{r-1} B_j = 0, \quad \sum_{j=1}^{r-1} \bigl(a_j B_j + \tfrac{1}{4}(1 - 1/e_j^2)\bigr) = \tfrac{1}{4}\bigl(1 - 1/e_r^2\bigr).$$

Proposition 4 [Yang 2013b]. *Automorphic derivatives have the following properties.*

(1) $D((az+b)/(cz+d), z) = 0$ *for all* $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{C})$.

(2) $D(g \circ f, z) = D(g, f(z)) + D(f, z)/(dg/df)^2$.

Proposition 5. *Let $t(\tau)$ be a Hauptmodul for a Shimura curve X of genus 0. Let $R(x) \in \mathbb{C}(x)$ be the rational function such that the automorphic derivative $Q(t) = D(t, \tau)$ is equal to $R(z)$. Assume that γ is an element of $\mathrm{SL}(2, \mathbb{R})$ normalizing the order $\mathbb{O}$ associated to X and let σ be the automorphism of X induced by γ. If $\sigma : t \mapsto (at + b)/(ct + d)$, then $R(x)$ satisfies*

$$\frac{(ad - bc)^2}{(cx + d)^4} R\left(\frac{ax + b}{cx + d}\right) = R(x).$$

Proof. We shall compute $D(t(\gamma\tau), \tau)$ in two ways. By Proposition 4, we have

$$D(t(\gamma\tau), \tau) = D\left(\frac{at(\tau) + b}{ct(\tau) + d}, t(\tau)\right) + \frac{D(t(\tau), \tau)}{(dt(\gamma\tau)/dt(\tau))^2} = 0 + \frac{(ct + d)^4 R(t)}{(ad - bc)^2}.$$

On the other hand, by the same proposition, we also have

$$D(t(\gamma\tau), \tau) = D(t(\gamma\tau), \gamma\tau) + \frac{D(\gamma\tau, \tau)}{(dt(\gamma\tau)/d\gamma\tau)^2} = R(t(\gamma\tau)) = R\left(\frac{at + b}{ct + d}\right).$$

Comparing the two expressions, we get the formula. □

3. Shimura curves of genus zero

In this section, we will determine all pairs of integers (D, N), $D, N > 1$, such that $X_0^D(N)/W_D$ has genus 0. As explained in the introduction, we will need explicit coverings $X_0^D(N)/W_D \to X_0^D(1)/W_D$ in order to determine Schwarzian differential equations.

To describe the genus formula for $X_0^D(N)/W_D$, we need to recall the definition of CM-points first. Let B be a quaternion algebra of discriminant D over $\mathbb{Q}$ and $\mathcal{O}$ an Eichler order of level N in B. Fix an embedding ι of B into $M(2, \mathbb{R})$. Let K be an imaginary quadratic field and R an order of discriminant $d_R = f^2 d_K$ in K. Following Eichler, we say an embedding $\phi : R \to \mathcal{O}$ is *optimal* if $\phi(K) \cap \mathcal{O} = \phi(R)$. Now the action of the set $\iota \circ \phi(R \backslash \{0\}) \subset \mathrm{GL}^+(2, \mathbb{R})$ on the upper half-plane $\mathbb{H}$ fixes precisely one point τ_ϕ. Such a point is called a *CM-point* (point with complex multiplication) of discriminant d_R. We denote the set of CM-points of discriminant d_R, up to $\mathcal{O}_1^*$-equivalence, by $\mathrm{CM}(d_R)$.

Lemma 6 [Ogg 1983]. *Assume that m is a squarefree divisor of DN such that $(m, DN/m) = 1$. Then the set of the fixed points of an Atkin–Lehner involution w_m, $m > 1$, on $X_0^D(N)$ is*

$$\begin{cases} \mathrm{CM}(-4) \cup \mathrm{CM}(-8) & \text{if } m = 2, \\ \mathrm{CM}(-m) \cup \mathrm{CM}(-4m) & \text{if } m \equiv 3 \bmod 4, \\ \mathrm{CM}(-4m) & \text{otherwise}. \end{cases}$$

We remark that in the case m is not squarefree, the fixed points of w_m will still be CM-points, but the description is complicated. (In general, they will be a proper subset of $\bigcup_{f^2 | 4m} \mathrm{CM}(-4m/f^2)$.)

From this lemma, it is easy to determine the number of elliptic points on $X_0^D(N)/G$ for any subgroup G of $W_{D,N}$ such that m is squarefree for any w_m in G.

Lemma 7. *Let G be a nontrivial subgroup of the group $W_{D,N}$ of Atkin–Lehner involutions on $X_0^D(N)$ such that m is squarefree for any $w_m \in G$. Then the only possible orders of elliptic points on $X_0^D(N)/G$ are* 2, 3, 4, *and* 6.

(1) *If $w_2 \in G$, then the number of elliptic points of order* 2 *on $X_0^D(N)/G$ is*

$$\frac{2}{|G|} \begin{cases} \sum_{\substack{w_m \in G \\ m \neq 1}} (\#\mathrm{CM}(-4m) + \#\mathrm{CM}(-m)) - \#\mathrm{CM}(-3) & \text{if } w_3 \in G, \\ \sum_{\substack{w_m \in G \\ m \neq 1}} (\#\mathrm{CM}(-4m) + \#\mathrm{CM}(-m)) & \text{if } w_3 \notin G. \end{cases}$$

If $w_2 \notin G$, *then the number is* $(\#\mathrm{CM}(-4)+2A)/|G|$, *where* A *is*

$$\begin{cases} \sum_{\substack{w_m\in G\\ m\neq 1}} (\#\mathrm{CM}(-4m)+\#\mathrm{CM}(-m)) - \#\mathrm{CM}(-3) & \text{if } w_3\in G,\\ \sum_{\substack{w_m\in G\\ m\neq 1}} (\#\mathrm{CM}(-4m)+\#\mathrm{CM}(-m)) & \text{if } w_3\notin G.\end{cases}$$

(*If* $-m$ *is not a discriminant, we simply set* $\#\mathrm{CM}(-m)=0$.)

(2) *If* $w_3 \in G$, *then there are no elliptic points of order* 3 *on* $X_0^D(N)/G$. *If* $w_3 \notin G$, *then the number of elliptic points of order* 3 *is* $\#\mathrm{CM}(-3)/|G|$.

(3) *If* $w_2 \notin G$, *then there are no elliptic points of order* 4 *on* $X_0^D(N)/G$. *If* $w_2 \in G$, *then the number of elliptic points of order* 4 *is* $2\,\#\mathrm{CM}(-4)/|G|$.

(4) *If* $w_3 \notin G$, *then there are no elliptic points of order* 6 *on* $X_0^D(N)/G$. *If* $w_3 \in G$, *then the number of elliptic points of order* 6 *is* $2\,\#\mathrm{CM}(-3)/|G|$.

Proof. The fact that only 2, 3, 4, and 6 can be the orders of elliptic points on $X_0^D(N)/G$ is well-known.

Let $w_m \in G$. By Lemma 6, the fixed points of w_m consist of CM(-4), CM($-m$), or CM($-4m$), depending on m. If $m \neq 1, 3$, then points in CM($-4m$) or CM($-m$) are fixed only by w_m and every other Atkin–Lehner involution other than w_1 permutes them. Thus, there are totally $|G|/2$ points in CM($-4m$) or CM($-m$) that are mapped to the same point in the covering $X_0^D(N) \to X_0^D(N)/G$. For points in CM(-4), which constitute elliptic points of order 2 on $X_0^D(N)$, they are also fixed by w_2. Thus, if $w_2 \in G$, then there are $2\,\#\mathrm{CM}(-4)/|G|$ elliptic points of order 4 on $X_0^D(N)/G$. If $w_2 \notin G$, points in CM(-4) contribute another $\#\mathrm{CM}(-4)/|G|$ elliptic points of order 2 on $X_0^D(N)/G$. For points in CM(-3), which are elliptic points of order 3 on $X_0^D(N)$, they are also fixed by w_3. If $w_3 \in G$, then they become elliptic points of order 6 on $X_0^D(N)/G$ and there are $2\,\#\mathrm{CM}(-3)/|G|$ such points. If $w_3 \notin G$, then they remain elliptic points of order 3. There are $\#\mathrm{CM}(-3)/|G|$ such points. Summarizing, we get the lemma. □

In view of these lemmas, a formula for the genus of $X_0^D(N)/G$, $G < W_{D,N}$, will involve the numbers of CM-points of certain discriminants. The general formula for the number of CM-points of an arbitrary discriminant is complicated to state. (See [Alsina and Bayer 2004; Ogg 1983].) For the goal of this section, we only need to know the number of CM-points of discriminant -3, d_K, or $4d_K$ in the case $d_K \equiv 1 \bmod 4$, for $K=\mathbb{Q}(\sqrt{-m})$ with $m\,|\,D$.

Lemma 8 [Ogg 1983]. *For* $m\,|\,D$ *or* $m=3$, *let* d_K *denote the discriminant of the field* $K=\mathbb{Q}(\sqrt{-m})$. *We have*

$$\#\mathrm{CM}(d_K) = h(d_K)\begin{cases} 0 & \text{if } p^2\,|\,N \text{ for some } p\,|\,d_K,\\ \prod_{p|D}\left(1-\left(\frac{d_K}{p}\right)\right)\prod_{p|N}\left(1+\left(\frac{d_K}{p}\right)\right) & \text{otherwise.}\end{cases}$$

Also, for $m \mid D$ *with* $m \equiv 3 \bmod 4$, *we have*

$$\#\mathrm{CM}(4d_K) = \delta h(4d_K) \begin{cases} 0 & \text{if } 2 \mid D, \\ \prod_{p \mid D} \bigl(1 - \bigl(\frac{4d_K}{p}\bigr)\bigr) \prod_{p \mid N} \bigl(1 + \bigl(\frac{4d_K}{p}\bigr)\bigr) & \text{if } 2 \nmid D, \end{cases}$$

where, when $m \equiv 7 \bmod 8$,

$$\delta = \begin{cases} 6 & \text{if } 8 \mid N, \\ 4 & \text{if } 4 \mid N, \\ 2 & \text{if } 2 \mid N, \\ 1 & \text{if } 2 \nmid N, \end{cases}$$

and when $m \equiv 3 \bmod 8$,

$$\delta = \begin{cases} 0 & \text{if } 8 \mid N, \\ 2 & \text{if } 2 \mid N \text{ or } 4 \mid N, \\ 1 & \text{if } 2 \nmid N. \end{cases}$$

Here $h(d)$ *is the class number of the imaginary quadratic order of discriminant* d.

Proof. These formulas are just special cases of Theorems 1 and 2 of [Ogg 1983]. □

Lemma 9. *The complete list of integers* (D, N) *with* $D, N > 1$ *such that the Shimura curve* $X_0^D(N)/W_D$ *has genus zero, is*

$$(6, 5),\ (6, 7),\ (6, 13),\ (10, 3),\ (10, 7),\ (14, 3),\ (14, 5),$$
$$(15, 2),\ (15, 4),\ (21, 2),\ (26, 3),\ (35, 2),\ (39, 2).$$

Proof. Let Γ be a congruence Fuchsian subgroup of $\mathrm{SL}(2, \mathbb{R})$. (See [Katok 1992] for the definition of a congruence Fuchsian subgroup; the groups considered here are all congruence Fuchsian subgroups.) A famous result of Selberg [1965] stated that if Γ is a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$, then the first eigenvalue λ_1 of the Laplace operator on the space of square-integrable function on $\Gamma\backslash\mathbb{H}$ is not less than $3/16$. By combining this result with the Jacquet–Langlands correspondence, Vignéras [1983] showed that the same inequality also holds for congruence Fuchsian subgroups coming from indefinite quaternion algebras over $\mathbb{Q}$ of discriminant not equal to 1.

On the other hand, Zograf [1991] showed that if the area $A(\Gamma\backslash\mathbb{H})$ is at least $16(g(\Gamma)+1)$, then $\lambda_1 < 4(g(\Gamma)+1)/A(\Gamma\backslash\mathbb{H})$. Here $g(\Gamma)$ denotes the genus of Γ and the area is normalized such that $A(\mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}) = 1/6$. Combining Selberg's inequality and Zograf's result, one sees that if a congruence Fuchsian subgroup has genus 0, then the area must be less than $64/3$.

Now recall from [Shimizu 1965] that the area of $X_0^D(N)$ is given by

$$\frac{DN}{6} \prod_{p \mid D} \Bigl(1 - \frac{1}{p}\Bigr) \prod_{p \mid N} \Bigl(1 + \frac{1}{p}\Bigr).$$

This immediately shows that if the number of prime factors of D is at least 6, then the genus of $X_0^D(N)/W_D$ cannot be 0 for any $N \geq 2$. Also, if $D = pq$ is a product of two primes such that $(p-1)(q-1) > 512/3$, then $X_0^D(N)/W_D$ must have a positive genus for any $N \geq 2$. A similar bounds exists for the case D has 4 prime factors. This leaves finitely many cases to check.

Now recall that the genus of a Shimura curve X is given by

$$g(X) = 1 + \frac{A(X)}{2} - \frac{1}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{e_i}\right),$$

where the sum runs through all elliptic points with e_i being their respective orders. For $X = X_0^D(N)/W_D$, by Lemma 7, we have

$$\begin{aligned} g(X) = 1 + \frac{A(X)}{2} - \frac{1}{4}\sum_{\substack{m \mid D \\ m \neq 1,3}} \frac{1}{2^{r-1}}(\#\mathrm{CM}(-4m) + \#\mathrm{CM}(-m)) \\ - \begin{cases} \frac{1}{4\cdot 2^r}\,\#\mathrm{CM}(-4) & \text{if } 2\nmid D, \\ \frac{3}{8\cdot 2^{r-1}}\,\#\mathrm{CM}(-4) & \text{if } 2\mid D \end{cases} \\ - \begin{cases} \frac{1}{3\cdot 2^r}\,\#\mathrm{CM}(-3) & \text{if } 3\nmid D, \\ \left(\frac{1}{4\cdot 2^{r-1}}\,\#\mathrm{CM}(-12) + \frac{5}{12\cdot 2^{r-1}}\,\#\mathrm{CM}(-3)\right) & \text{if } 3\mid D, \end{cases} \end{aligned}$$

where r is the number of prime divisors of D. (Of course, if d is not a discriminant, then we simply let $\mathrm{CM}(d)$ be the empty set.)

Using the Selberg–Zograf bound, the genus formula in the paragraph above and Lemma 8, we check case by case that the pairs of integers given in the lemma are the only cases where $X_0^D(N)/W_D$, $N > 1$, has genus zero. □

We now tabulate all Shimura curves $X_0^D(M)/W_D$ of genus 0 for integers D that appear in the lemma. We will also give a description of their elliptic points. We wish to determine the Schwarzian differential equations for these curves. Here v_j denotes the number of elliptic points of order j on $X_0^D(M)/W_D$. Here we also let $\mathrm{CM}(-m)$ denote the set of points on $X_0^D(N)/W_D$ that are the image of CM-points of discriminants $-m$ under the covering $X_0^D(N) \to X_0^D(N)/W_D$. The number n in $\mathrm{CM}(-m)^{\times n}$ means the number of elements in $\mathrm{CM}(-m)$ is n. If $n = 1$, we omit this annotation.

4. Coverings of Shimura curves

The goal of this section is to obtain explicit coverings of $X_0^D(N)/W_D \to X_0^D(1)/W_D$ for pairs of D and N given in Lemma 9. That is, we wish to find a Hauptmodul t_1

D, N	v_2, v_3, v_4, v_6	elliptic points
6, 1	1, 0, 1, 1	CM(−3), CM(−4), CM(−24)
6, 5	2, 0, 2, 0	$\mathrm{CM}(-4)^{\times 2}$, $\mathrm{CM}(-24)^{\times 2}$
6, 7	2, 0, 0, 2	$\mathrm{CM}(-3)^{\times 2}$, $\mathrm{CM}(-24)^{\times 2}$
6, 13	0, 0, 2, 2	$\mathrm{CM}(-3)^{\times 2}$, $\mathrm{CM}(-4)^{\times 2}$
10, 1	3, 1, 0, 0	CM(−3), CM(−8), CM(−20), CM(−40)
10, 3	4, 1, 0, 0	CM(−3), $\mathrm{CM}(-8)^{\times 2}$, $\mathrm{CM}(-20)^{\times 2}$
10, 7	4, 2, 0, 0	$\mathrm{CM}(-3)^{\times 2}$, $\mathrm{CM}(-20)^{\times 2}$, $\mathrm{CM}(-40)^{\times 2}$
14, 1	3, 0, 1, 0	CM(−4), CM(−8), $\mathrm{CM}(-56)^{\times 2}$
14, 3	6, 0, 0, 0	$\mathrm{CM}(-8)^{\times 2}$, $\mathrm{CM}(-56)^{\times 4}$
14, 5	4, 0, 2, 0	$\mathrm{CM}(-4)^{\times 2}$, $\mathrm{CM}(-56)^{\times 4}$
15, 1	3, 0, 0, 1	CM(−3), CM(−12), CM(−15), CM(−60)
15, 2	6, 0, 0, 0	$\mathrm{CM}(-12)^{\times 2}$, $\mathrm{CM}(-15)^{\times 2}$, $\mathrm{CM}(-60)^{\times 2}$
15, 4	8, 0, 0, 0	$\mathrm{CM}(-12)^{\times 2}$, $\mathrm{CM}(-15)^{\times 2}$, $\mathrm{CM}(-60)^{\times 4}$
21, 1	5, 0, 0, 0	CM(−4), CM(−7), CM(−28), $\mathrm{CM}(-84)^{\times 2}$
21, 2	7, 0, 0, 0	CM(−4), $\mathrm{CM}(-7)^{\times 2}$, $\mathrm{CM}(-28)^{\times 2}$, $\mathrm{CM}(-84)^{\times 2}$
26, 1	5, 0, 0, 0	CM(−8), CM(−52), $\mathrm{CM}(-104)^{\times 3}$
26, 3	8, 0, 0, 0	$\mathrm{CM}(-8)^{\times 2}$, $\mathrm{CM}(-104)^{\times 6}$
35, 1	6, 0, 0, 0	CM(−7), CM(−28), CM(−35), $\mathrm{CM}(-140)^{\times 3}$
35, 2	10, 0, 0, 0	$\mathrm{CM}(-7)^{\times 2}$, $\mathrm{CM}(-28)^{\times 2}$, $\mathrm{CM}(-140)^{\times 6}$
39, 1	6, 0, 0, 0	$\mathrm{CM}(-52)^{\times 2}$, $\mathrm{CM}(-39)^{\times 2}$, $\mathrm{CM}(-156)^{\times 2}$
39, 2	10, 0, 0, 0	$\mathrm{CM}(-52)^{\times 2}$, $\mathrm{CM}(-39)^{\times 4}$, $\mathrm{CM}(-156)^{\times 4}$

Table 1. All Shimura curves $X_0^D(M)/W_D$ of genus 0 for integers D appearing in Lemma 9.

of $X_0^D(1)/W_D$, a Hauptmodul t_N of $X_0^D(N)/W_D$, and the relation between them. Of course, there are infinitely many choices for t_1 and t_N. For $X_0^D(N)/W_D$, we will choose t_N such that the Atkin–Lehner involution w_N acts by $w_N : t_N \mapsto -t_N$. This will make the determination of Schwarzian differential equation simpler.

Case D = 6. In the case $D = 6$, all the coverings $X_0^6(N)/W_6 \to X_0^6(1)/W_6$, $N = 5, 7, 13$, are already given in [Elkies 1998]. Here we just modify the t_N in [Elkies 1998] such that the new t_N satisfies $w_N : t_N \mapsto -t_N$.

Lemma 10 [Elkies 1998]. (1) *There is a Hauptmodul t_1 for $X_0^6(1)/W_6$ that takes values* 0, 1, *and* ∞ *at the CM-points of discriminants* -24, -4, *and* -3, *respectively.*

(2) *There is a Hauptmodul* $t = t_5$ *for* $X_0^6(5)/W_6$ *that takes values* $\pm i/8$ *and* $\pm\sqrt{-6}/3$ *at the CM-points of discriminants* -4 *and* -24, *respectively. The relation between* t_1 *and* t *is*

$$t_1 = \frac{(2+3t^2)(34-117t+1824t^2)^2}{125(1+6t)^6} = 1 + \frac{27(1+64t^2)(3-7t)^4}{125(1+6t)^6}.$$

The Atkin–Lehner involution w_5 *acts by* $w_5 : t \mapsto -t$.

(3) *There is a Hauptmodul* $t = t_7$ *for* $X_0^6(7)/W_6$ *that takes values* $\pm\sqrt{-3}/9$ *and* $\pm\sqrt{-6}/8$ *at the CM-points of discriminants* -3 *and* -24, *respectively. The relation between* t_1 *and* t *is*

$$t_1 = -\frac{(3+32t^2)(78-396t+1963t^2-12312t^3)^2}{4(1+27t^2)(3+10t)^6}.$$

The Atkin–Lehner involution w_7 *acts by* $w_7 : t \mapsto -t$.

(4) *There is a Hauptmodul* $t = t_{13}$ *for* $X_0^6(13)/W_6$ *that takes values* $\pm 4\sqrt{-3}/9$ *and* $\pm 3i/4$ *at the CM-points of discriminants* -3 *and* -4, *respectively. The relation between* t_1 *and* t *is*

$$t_1 = 1 - \frac{27(9+16t^2)(144-98t+246t^2-161t^3)^4}{16(16+27t^2)(30+3t+55t^2)^6}.$$

The Atkin–Lehner involution w_{13} *acts by* $w_{13} : t \mapsto -t$.

Proof. Elkies [1998] showed that explicit coverings of $X_0^6(N)/W_6 \to X_0^6(1)/W_6$, $N = 5, 7, 13$, are given by

$$t_1 = 1 + 135s^4 + 324s^5 + 540s^6, \qquad w_5 : s \mapsto \frac{42-55s}{55+300s},$$

$$t_1 = -\frac{(4s^2+4s+25)(2s^3-3s^2+12s-2)^2}{108(7s^2-8s+37)}, \quad w_7 : s \mapsto \frac{116-9s}{9+20s},$$

and

$$t_1 = \frac{(s^7-50s^6+63s^5-5040s^4+783s^3-168426s^2-6831s-1864404)^2}{4(7s^2+2s+247)(s^2+39)^6}$$

with

$$w_{13} : s \mapsto \frac{5s+72}{2s-5},$$

respectively. Choosing t such that

$$s = \frac{7t-3}{30t+5}, \quad s = \frac{-29t+6}{10t+3}, \quad s = \frac{-8t+9}{2t+1},$$

respectively, we get the lemma. □

Case D = 10. The covering $X_0^{10}(3)/W_{10} \to X_0^{10}(1)/W_{10}$ has also been given in [Elkies 1998]. Here we mainly work on the case $N = 7$.

Lemma 11. (1) *There is a Hauptmodul t_1 for $X_0^{10}(1)/W_{10}$ that takes values* 0, ∞, 2, *and* 27 *at the CM-points of discriminants* -3, -8, -20, *and* -40, *respectively.*

(2) *There is a Hauptmodul $t = t_3$ for $X_0^{10}(3)/W_{10}$ that takes values* 0, $\pm 1/4\sqrt{-2}$, $\pm 1/\sqrt{-5}$ *at the CM-points of discriminants* -3, -8, *and* -20, *respectively. The relation between t_1 and t is*

$$t_1 = \frac{108t(1-2t)^3}{(1+32t^2)(1+7t)^2} = 2 - \frac{2(1+5t^2)(1-20t)^2}{(1+32t^2)(1+7t)^2}.$$

The Atkin–Lehner involution w_3 acts by $w_3 : t \mapsto -t$.

(3) *There is a Hauptmodul $t = t_7$ for $X_0^{10}(7)/W_{10}$ that takes values* $\pm 1/3\sqrt{-3}$, $\pm 1/2\sqrt{-5}$, *and* $\pm\sqrt{-10}/16$ *at the CM-points of discriminants* -3, -20, *and* -40, *respectively. The relation between t_1 and t is*

$$t_1 = \frac{8(1+27t^2)(2-3t+44t^2)^3}{7(1+4t+55t^2+102t^3+736t^4)^2}.$$

The Atkin–Lehner involution w_7 acts by $w_7 : t \mapsto -t$.

Proof. In [Elkies 1998], it is shown that an explicit covering $X_0^{10}(3)/W_{10} \to X_0^{10}(1)/W_{10}$ is given by

$$t_1 = \frac{216(s-1)^3}{(s+1)^2(9s^2-10s+17)}$$

with $w_3 : s \mapsto 10/9 - s$. Let t be the Hauptmodul of $X_0^{10}(1)/W_{10}$ with

$$s = \frac{2}{9t} + \frac{5}{9}.$$

Then the relation of t_1 and t and the action of w_3 are given as in the lemma.

We next consider the case $N = 7$. According to Theorem 3.4 of [González and Rotger 2006], an equation for $X_0^{10}(7)$ is given by

$$y^2 = -27x^4 - 40x^3 + 6x^2 + 40x - 27. \tag{1}$$

The actions of the Atkin–Lehner involutions on this model of $X_0^{10}(7)$ are given by

$$w_{70} : (x, y) \mapsto (x, -y), \qquad w_5 : (x, y) \mapsto \left(-\frac{1}{x}, -\frac{y}{x^2}\right),$$

and

$$w_{10} : (x, y) \mapsto \left(\frac{2x+1}{x-2}, \frac{5y}{(x-2)^2}\right).$$

Since CM(-20) are fixed points under the action of w_5, their coordinates on (1) are $(i, \pm 2\sqrt{5}(1+2i))$ and $(-i, \pm 2\sqrt{5}(1-2i))$. Likewise, we find that CM(-40) have coordinates $(2+\sqrt{5}, \pm 8\sqrt{-10}(2+\sqrt{5}))$ and $(2-\sqrt{5}, \pm 8\sqrt{-10}(2-\sqrt{5}))$. Furthermore, from the method of [González and Rotger 2006], we know that the two points at infinity are CM-points of discriminant -3. Thus, the coordinates of CM(-3) are ∞, $(0, \pm 3\sqrt{-3})$, $(2, \pm 15\sqrt{-3})$, and $(-1/2, \pm 15\sqrt{-3}/4)$.

From (1), we can obtain an equation $w^2+27z^2+40z+20=0$ for $X_0^{10}(7)/\langle w_{10}\rangle$, where the covering $X_0^{10}(7) \to X_0^{10}(7)/\langle w_{10}\rangle$ is given by

$$(x, y) \mapsto (w, z) = \left(\frac{y}{x-2}, \frac{x^2+1}{x-2}\right).$$

In this equation for $X_0^{(10)}(7)/\langle w_{10}\rangle$, the actions of the Atkin–Lehner involutions are given by

$$w_{70} = w_7 : (w, z) \mapsto (-w, z), \quad w_2 = w_5 : (w, z) \mapsto \left(\frac{w}{2z+1}, \frac{-z}{2z+1}\right).$$

The coordinates of CM(-3) are the two points at ∞ and $(\pm 3\sqrt{-3}/2, -1/2)$. Also, the coordinates of CM(-20) are $(\pm 2\sqrt{-5}, 0)$, and the coordinates of CM(-40) are $(\pm 8\sqrt{-2}(2+\sqrt{5}), 4+2\sqrt{5})$ and $(\pm 8\sqrt{-2}(2-\sqrt{5}), 4-2\sqrt{5})$.

Now set $t = (z+1)/w$. We can check that t is invariant under w_2 and that $(w, z) \mapsto t = (z+1)/w$ is 2-to-1. Thus, t is a Hauptmodul of $X_0^{10}(7)/W_{10}$. The coordinates of the CM-points of discriminants -3, -20, and -40 are $\pm 1/3\sqrt{-3}$, $\pm 1/2\sqrt{-5}$, and $\pm\sqrt{-10}/16$, respectively. It follows that the relation between t_1 and t is

$$t_1 = \frac{A(1+27t^2)(1+a_1t+a_2t^2)^3}{(1+b_1t+b_2t^2+b_3t^3+b_4t^4)^2}$$

with

$$\begin{aligned} A(1+27t^2)(1+a_1t+a_2t^2)^3 - 2(1+b_1t+b_2t^2+b_3t^3+b_4t^4)^2 \\ = B(1+20t^2)(1+c_1t+c_2t^2+c_3t^3)^2, \\ A(1+27t^2)(1+a_1t+a_2t^2)^3 - 27(1+b_1t+b_2t^2+b_3t^3+b_4t^4)^2 \\ = C\big(1+\tfrac{128}{5}t^2\big)(1+d_1t+d_2t^2+d_3t^3)^2 \end{aligned}$$

for some constants A, B, C, a_j, b_j, c_j, and d_j. Comparing the coefficients, we get

$$t_1 = \frac{8(1+27t^2)(2-3t+44t^2)^3}{7(1+4t+55t^2+102t^3+736t^4)^2}$$

(or the same expression with t replaced by $-t$). This proves the lemma. □

***Case* $D = 14$.** The case $D = 14$ is also worked out in [Elkies 1998]. Here we only need to make a change of variable so that w_N acts by $w_N : t_N \to -t_N$.

Lemma 12 [Elkies 1998]. (1) *There is a Hauptmodul t_1 for $X_0^{14}(1)/W_{14}$ that takes values* ∞, 0, *and* $(-13 \pm 7\sqrt{-7})/32$ *at CM-points of discriminants* -4, -8, *and* -56, *respectively.*

(2) *There is a Hauptmodul $t = t_3$ for $X_0^{14}(3)/W_{14}$ that takes values $\pm 1/\sqrt{-2}$ and $(\pm 9\sqrt{-7} \pm 4\sqrt{-14})/49$ at CM-points of discriminants -8 and -56, respectively. The relation between t_1 and t is*

$$t_1 = \frac{4(1+2t^2)(1-5t)^2}{9(1+t)^4}.$$

The Atkin–Lehner involution w_3 acts by $w_3 : t \mapsto -t$.

(3) *There is a Hauptmodul $t = t_5$ for $X_0^{14}(5)/W_{14}$ that takes values $\pm i/4$ and $(\pm 5\sqrt{-7} \pm 4\sqrt{-14})/7$ at CM-points of discriminants -4 and -56, respectively. The relation between t_1 and t is*

$$t_1 = -\frac{5(1-t+17t^2-13t^3)^2}{(1+16t^2)(1+3t)^4}.$$

The Atkin–Lehner involution w_5 acts by $w_5 : t \mapsto -t$.

Proof. In [Elkies 1998], it is shown that explicit coverings $X_0^{14}(N)/W_{14} \to X_0^{14}(1)/W_{14}$ can be given by

$$t_1 = \tfrac{1}{27}(s^4 + 2s^3 + 9s^2), \quad w_3 : s \mapsto \frac{5-2s}{2+s}$$

and

$$t_1 = -\frac{(256s^3 + 224s^2 + 232s + 217)^2}{50000(s^2+1)}, \quad w_5 : s \mapsto \frac{24-7s}{7+24s},$$

respectively. Choosing t with

$$s = \frac{1-5t}{1+t}, \quad s = \frac{3-16t}{4+12t},$$

respectively, we get the lemma. □

***Case* $D = 15$.** An explicit covering $X_0^{15}(2)/W_{15} \to X_0^{15}(1)/W_{15}$ is given in [Elkies 1998]. Here we only need make a change of variable so w_N acts by $w_N : t_N \to -t_N$.

Lemma 13. (1) *There is a Hauptmodul for $X_0^{15}(1)/W_{15}$ that takes values* ∞, 0, 81, *and* 1 *at CM-points of discriminants* -3, -12, -15, *and* -60, *respectively.*

(2) *There is a Hauptmodul* t_2 *for* $X_0^{15}(2)/W_{15}$ *that takes values* ± 1, $\pm\sqrt{-15}/3$, *and* $\pm 1/5$ *at CM-points of discriminant* -12, -15, *and* -60, *respectively. The relation between* t_1 *and* t_2 *is*

$$t_1 = \frac{27(1-t_2)(1-3t_2)^2}{2(1+t_2)^3} = 1 + \frac{(1-5t_2)(5-7t_2)^2}{2(1+t_2)^3} = 81 - \frac{27(1+5t_2)(5+3t_2^2)}{2(1+t_2)^3}.$$

The Atkin–Lehner involution w_2 *acts by* $w_2 : t_2 \mapsto -t_2$.

(3) *There is a Hauptmodul* t_4 *for* $X_0^{15}(4)/W_{15}$ *that takes values* $\pm 1/\sqrt{-3}$, $\pm\sqrt{-15}/5$, *and* $(\pm 1 \pm \sqrt{-15})/8$ *at CM-points of discriminants* -12, -15, *and* -60, *respectively. The relation between* t_4 *and* t_2 *is*

$$t_2 = \frac{5t_4^2 + 2t_4 + 1}{7t_4^2 - 2t_4 + 3}.$$

Proof. In [Elkies 1998], an explicit covering $X_0^{15}(2)/W_{15} \to X_0^{15}(1)/W_{15}$ is given by

$$t_1 = \tfrac{1}{4}s(s-3)^2, \quad w_2 : s \mapsto \frac{36}{s}.$$

Choosing a Hauptmodul t for $X_0^{15}(2)/W_{15}$ with

$$s = \frac{6-6t}{1+t},$$

we establish the claim about $X_0^{15}(2)/W_{15}$.

For the covering map $X_0^{15}(4)/W_{15} \to X_0^{15}(2)/W_{15}$, it is clear that one of the CM-points of discriminant -12 on $X_0^{15}(2)/W_{15}$ becomes two CM-points of discriminant -12 on $X_0^{15}(4)/W_{15}$, and the other is ramified. To determine the ramification data of this covering completely, we need to consider the optimal embeddings of the quadratic orders of the field $\mathbb{Q}(\sqrt{-15})$ into the Eichler order of level 2 and the Eichler order of level 4 in the quaternion algebra B over $\mathbb{Q}$ with discriminant 15 at the finite place $p = 2$.

Let $R_1 = \mathbb{Z} + \mathbb{Z}\alpha$, $p_1(x) = x^2 + x + 4$ be the irreducible polynomial of α over $\mathbb{Q}$, and $R_2 = \mathbb{Z} + \mathbb{Z}\beta$, $p_2(x) = x^2 + 15$ be the irreducible polynomial of β over $\mathbb{Q}$. Up to conjugation, we may assume that in the localization $M(2, \mathbb{Q}_2)$ of B at the finite place 2, the Eichler orders $\mathbb{O}_2$, $\mathbb{O}_4$ of level 2 and 4 are

$$\mathbb{O}_2 = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 2\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}, \quad \mathbb{O}_4 = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 4\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix},$$

respectively. Then the inequivalent optimal embeddings of R_1 into $\mathbb{O}_2$ can be given by sending α to

$$A_{-15,1} = \begin{pmatrix} 0 & -1 \\ 4 & -1 \end{pmatrix} \quad \text{and} \quad A_{-15,2} = \begin{pmatrix} -1 & -1 \\ 4 & 0 \end{pmatrix};$$

the inequivalent optimal embeddings of R_2 into $\mathcal{O}_2$ can be given by sending β to

$$A_{-60,1}=\begin{pmatrix}1&-1\\16&-1\end{pmatrix}\quad\text{and}\quad A_{-60,2}=\begin{pmatrix}1&-8\\2&-1\end{pmatrix}.$$

The inequivalent optimal embeddings of R_1 and R_2 into $\mathcal{O}_4$ are given by

$$B_{-15,1}=\begin{pmatrix}0&-1\\4&-1\end{pmatrix}\quad\text{and}\quad B_{-15,2}=\begin{pmatrix}-1&-1\\4&0\end{pmatrix},$$

and

$$B_{-60,1}=\begin{pmatrix}1&-1\\16&-1\end{pmatrix}\quad\text{and}\quad B_{-60,2}=\begin{pmatrix}-1&-1\\16&1\end{pmatrix},$$
$$B_{-60,3}=\begin{pmatrix}1&-4\\4&-1\end{pmatrix}\quad\text{and}\quad B_{-60,4}=\begin{pmatrix}-1&-4\\4&1\end{pmatrix},$$

respectively. Furthermore, we can check the embeddings sending β to $B_{-60,3}$, $B_{-60,4}$ give optimal embeddings of R_1 into $\mathcal{O}_2$, and the matrices $B_{-60,1}$, $B_{-60,2}$, and $A_{-60,1}$ are conjugate to each other in $\mathcal{O}_2$.

According this information, we can conclude that each CM-point of discriminant -15 on $X_0^{15}(2)/W_{15}$ becomes one CM-point of discriminant -15 and one CM-point of discriminant -60 on $X_0^{15}(4)/W_{15}$. One of the CM-points of discriminant -60 on $X_0^{15}(2)/W_{15}$ becomes two CM-points of discriminant -60 on $X_0^{15}(4)/W_{15}$, and the other CM-points of discriminant -60 on $X_0^{15}(2)/W_{15}$ is ramified.

We now suppose that the covering $X_0^{15}(4)/W_{15}\to X_0^{15}(2)/W_{15}$ is given by

$$t_2=\frac{a_2t^2+a_1t+a_3}{t^2+b_1t+b_2},$$

where $t=t_4$ is a Hauptmodul for $X_0^{15}(4)/W_{15}$. Since the Atkin–Lehner involution w_2 switches the two CM-points of discriminant -12 on $X_0^{15}(2)/W_{15}$, we may assume that the CM-point of discriminant -12 having coordinate 1 is a ramified point. According to the ramification data and the fields of definition of these CM-points, without loss the generality, we may assume that t has repeated roots 1 when $t_2=1$, and assume that the CM-points of discriminant -12 of $X_0^{15}(4)/W_{15}$ that lie above the unramified CM-point of discriminant -12 of $X_0^{15}(2)/W_{15}$ are $\pm 1/\sqrt{-3}$. Therefore, we have

$$t_2=\frac{(2a-3)t^2+(3a-1)t+1-2a}{t^2+(1-3a)t+a},$$

for some constant a. From the information of the CM-points of discriminant -60,

$$t_2^2-1=\frac{(t-c)^2(t^2+c_1t+c_2)}{(t^2+(1-3a)t+a)^2},$$

and the roots of $t^2+c_1t+c_2$ are in the field $\mathbb{Q}(\sqrt{-3}, \sqrt{5})$, we can deduce that

$$t_2 = \frac{5t^2+2t+1}{7t^2-2t+3}.$$

We get the lemma. □

***Case D* = 21.** We will need an equation for some Atkin–Lehner quotient of $X_0^{21}(2)$ in order to determine the coordinates of elliptic points on $X_0^{21}(2)$.

Lemma 14. *An equation for* $X_0^{21}(2)/\langle w_{21}\rangle$ *is* $y^2=(x+12)(x^2-7x+28)$. *Moreover, the action of the Atkin–Lehner involution* $w_3 = w_7$ *on this curve is given by the map* $(x, y) \mapsto (x, -y)$. *Also, the two rational points* ∞ *and* $(-12, 0)$ *are the CM-points of discriminant* -28, *and the other two* 2*-torsion points* $(7 \pm 3\sqrt{-7})/2, 0)$ *are the CM-points of discriminant* -7.

Proof. We follow the methods of [González and Rotger 2006]. The Shimura curve $X_0^{21}(2)/\langle w_{21}\rangle$ has genus 1. By Lemma 5.10 of that paper, the two CM-points of discriminant -28 are $\mathbb{Q}$-rational points on this curve. Thus, $X_0^{21}(2)/\langle w_{21}\rangle$ is an elliptic curve over $\mathbb{Q}$. Now in the space $S_2(\Gamma_0(42))^{21\text{-new}}$ the unique Hecke eigenform with $+$-eigenvalue for w_{21} is coming from the newform space of $S_2(\Gamma_0(42))$. Therefore, the elliptic curve $X_0^{21}(2)/\langle w_{21}\rangle$ has conductor 42. Using the Cherednik–Drinfeld theory of p-adic uniformization of Shimura curves, we find that the types of singular fibers at primes of bad reduction of $X_0^{21}(2)/\langle w_{21}\rangle$ agree with those of the elliptic curve 42A1, in Cremona's notation. The global minimal model of the elliptic curve 42A1 is $y^2+xy+y=x^3+x^2-4x+5$. With a simple change of variables, we write it as $y^2=(x+12)(x^2-7x+28)$.

Now the covering $X_0^{21}(2)/\langle w_{21}\rangle \to X_0^{21}(2)/W_{21}$ is ramified at the two CM-points of discriminant -7 and the two CM-points of discriminant -28. If we let one of the CM-points of discriminant -28 be the point at infinity, then an equation for $X_0^{21}(2)/\langle w_{21}\rangle$ is of the form $y^2 = f(x)$ for some polynomial $f(x) = x^3+\cdots$ of degree 3 in $\mathbb{Q}[x]$ with the Atkin–Lehner involution w_3 acting by $(x, y) \mapsto (x, -y)$. Up to a transformation of the form $x \mapsto ax+b$, this polynomial $f(x)$ must be the polynomial $(x+12)(x^2-7x+28)$. This proves the lemma. □

Remark 15. According to Cremona's table of elliptic curves [1997], the elliptic curve 42A1 has 8 rational points. Thus, $X_0^{21}(2)/\langle w_{21}\rangle$ also has 8 $\mathbb{Q}$-rational points. Two of them are the CM-points of discriminant -28 mentioned above. The rest of $\mathbb{Q}$-rational points consist of two CM-points of discriminant -4 and four CM-points of discriminant -16.

Lemma 16. *There is a Hauptmodul* t_1 *for* $X_0^{21}(1)/W_{21}$ *that takes values* 49, 0, ∞, *and* $(47 \pm 8\sqrt{-3})/7$ *at CM-points of discriminants* -4, -7, -28, *and* -84, *respectively.*

Also, there is a Hauptmodul $t = t_2$ *for* $X_0^{21}(2)/W_{21}$ *that takes values* 0, $\pm 1/3\sqrt{-7}$, ± 1, *and* $\pm 1/3\sqrt{-3}$ *at CM-points of discriminants* -4, -7, -28, *and* -84, *respectively. The relation between* t_1 *and* t *is*

$$t_1 = \frac{49(1+t)(1+63t^2)}{(1-t)(1-15t)^2} = 49 + \frac{1568t(1-3t)^2}{(1-t)(1-15t)^2}.$$

The Atkin–Lehner involution w_2 *acts by* $w_2 : t \mapsto -t$.

Proof. According to [González and Rotger 2006], an equation for $X_0^{21}(1)$ is given by $y^2 = -7x^4 + 94x^2 - 343$ with the actions of the Atkin–Lehner involutions given by

$$w_3 : (x, y) \mapsto (-x, -y), \quad w_7 : (x, y) \mapsto (-x, y), \quad w_{21} : (x, y) \mapsto (x, -y).$$

The Atkin–Lehner involution w_7 fixes the two points at ∞ and $(0, \pm 7\sqrt{-7})$. Since the equation has a symmetry $(x, y) \mapsto (7/x, 7y/x^2)$, we might as well assume that the two points $(0, \pm 7\sqrt{-7})$ are the CM-points of discriminant -7 and the two points at infinity are the CM-points of discriminant -28. Moreover, the four points with $y = 0$ correspond to the four CM-points of discriminant -84.

Since w_3 acts by $(x, y) \to (-x, -y)$, an equation for $X_0^{21}(1)/\langle w_3\rangle$ is $y^2 = -7x^3 + 94x^2 - 343x$, where the covering $X_0^{21}(1) \to X_0^{21}(1)/\langle w_3\rangle$ is given by $(x, y) \mapsto (x^2, xy)$. Then $t_1 = x$ generates the function field of X_0^{21}/W_{21}. The values of t_1 at the CM-points of discriminants -7, -28, and -84 are 0, ∞, and $(47 \pm 8\sqrt{-3})/7$, respectively. The value of t_1 at the CM-point of discriminant -4 will be determined later.

By Lemma 14, an equation $X_0^{21}(2)/\langle w_{21}\rangle$ is $y^2 = (x+12)(x^2 - 7x + 28)$ with the Atkin–Lehner involution $w_3 = w_7$ acting by $(x, y) \to (x, -y)$. Thus, $s = x$ generates the function field of $X_0^{21}(2)/W_{21}$. According to the lemma, the values of s at the CM-points of discriminant -7 are $(7 \pm 3\sqrt{-7})/2$ and those at CM-points of discriminant -28 are -12 and ∞. The Atkin–Lehner involution w_2 switches the two CM-points of discriminant -28. It also switches the two CM-points of discriminant -7. (Note that in general, w_2 can send a CM-point of discriminant $-d$ on $X_0^D(N)/G$ to a CM-point of discriminant $-4d$ and vice versa. Here because w_2 is defined over $\mathbb{Q}$, it must send a $\mathbb{Q}$-rational point to another $\mathbb{Q}$-rational point.) This information suffices to determine w_2 in terms of s. We find

$$w_2 : s \mapsto \frac{-12s + 112}{s + 12}.$$

Choosing a new Hauptmodul

$$t = \frac{4 - s}{28 + s},$$

we have $w_2 : t \mapsto -t$. The new coordinates of CM-points of discriminants -7 and -28 are $\pm 1/3\sqrt{-7}$ and ± 1, respectively. Also, since w_2 fixes the unique CM-point of discriminant -4, we find that the CM-point of discriminant -4 has coordinate 0. We now determine the relation between t_1 and t.

Replacing t by $-t$ if necessary, we may assume that the CM-point of discriminant -28 of $X_0^{21}(2)/W_{21}$ that lies above the CM-point of discriminant -7 of $X_0^{21}(1)/W_{21}$ is -1. Then

$$t_1 = \frac{A(1+t)(1+63t^2)}{(1-t)(1-at)^2}$$

for some constants A and a. Since $X_0^{21}(2)/W_{21} \to X_0^{21}(1)/W_{21}$ is also ramified at the CM-points of discriminant -84, the discriminant of the polynomial

$$A(1+t)(1+63t^2) - B(1-t)(1-at)^2$$

in t must be divisible by the polynomial $7B^2 - 94B + 343$. This gives us two conditions on A and a. Solving them for A and a, we find that the only legitimate values for A and a are $A = 49$ and $a = 15$. Because t has value 0 at the CM-point of discriminant -4 on $X_0^{21}(2)/W_{21}$, the CM-point of -4 on $X_0^{21}(1)/W_{21}$ has coordinate 49. This proves the lemma. □

Case D = 26. We first recall a lemma of González and Rotger.

Lemma 17 [González and Rotger 2004, Proposition 2.1]. *Let C be a hyperelliptic curve of genus 2 defined over a field k of characteristic not equal to 2 or 3 and let w be its hyperelliptic involution. Assume that the group of automorphisms of C over k contains a subgroup $\langle u_1, u_2 = u_1 \cdot w\rangle$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and denote by C_i the elliptic quotient $C/\langle u_i\rangle$. If the two elliptic curves*

$$E_1 : y^2 = x^3 + A_1x + B_1, \quad E_2 : y^2 = x^3 + A_2x + B_2$$

are isomorphic to C_1 and C_2 over k, respectively, then C admits a hyperelliptic equation of the form $y^2 = ax^6 + bx^4 + cx^2 + d$, where $a \in k^*$, $b \in k$ are solutions of

$$\begin{aligned} 27a^3B_2 &= 2A_1^3 + 27B_1^2 + 9A_1B_1b + 2A_1^2b^2 - B_1b^3, \\ 9a^2A_2 &= -3A_1^2 + 9B_1b + A_1b^2, \end{aligned}$$

$c = (3A_1 + b^2)/(3a)$, $d = (27B_1 + 9A_1b + b^3)/(27a^2)$, and the involution u_1 on C is given by $(x, y) \mapsto (-x, y)$.

Lemma 18. *The Shimura curves* $X_1 : X_0^{26}(3)/\langle w_2, w_3\rangle$, $X_2 : X_0^{26}(3)/\langle w_2, w_{39}\rangle$, *and* $X_3 : X_0^{26}(3)/\langle w_6, w_{13}\rangle$ *are elliptic curves over* $\mathbb{Q}$ *with defining equations*

$$X_1 : y^2 = x^3 - 3403x - 83834,$$
$$X_2 : y^2 = x^3 - 43x + 166,$$
$$X_3 : y^2 = x^3 + 621x + 9774.$$

Moreover, on the equation for X_1, *the point at* ∞ *is the CM-point of discriminant* -312, *and the involution* $(x, y) \mapsto (x, -y)$ *is the Atkin–Lehner involution* $w_{13} = w_{26} = w_{39} = w_{78}$. *On the equation for* X_2, *the point at* ∞ *is the CM-point of discriminant* -24 *and the involution* $(x, y) \mapsto (x, -y)$ *is the Atkin–Lehner involution* $w_3 = w_6 = w_{13} = w_{26}$. *On the equation for* X_3, *the point at* ∞ *is the CM-point of discriminant* -8 *and the involution* $(x, y) \mapsto (x, -y)$ *is the Atkin–Lehner involution* $w_2 = w_3 = w_{26} = w_{39}$. *In all three cases, the* 2*-torsion points are the CM-points of discriminant* -104 *on their respective curves.*

Proof. The fact that the three curves in the lemma have genus one can be verified either by using the genus formula, together with Lemmas 6, 7, and 8, or by counting the dimensions of subspaces of $S_2(\Gamma_0(78))^{26\text{-new}}$ with appropriate eigenvalues for the Atkin–Lehner involutions. We omit the details.

On X_1, there is a unique CM-point of discriminant -312, which must be a $\mathbb{Q}$-rational point. Thus, X_1 is an elliptic curve over $\mathbb{Q}$. Likewise, X_2 and X_3 have unique CM-points of discriminants -24 and -8, respectively. They are also elliptic curves over $\mathbb{Q}$.

Observe that all cusp forms in $S_2(\Gamma_0(78))^{26\text{-new}}$ having -1 eigenvalue for w_2 are from the cusp form of level 26 corresponding to the isogeny class 26B of elliptic curves in Cremona's notation. Thus, X_1 and X_2 are isomorphic to either 26B1 or 26B2. Similarly, we find that the one-dimensional subspace of $S_2(\Gamma_0(78))^{26\text{-new}}$ that has eigenvalue $+1$ for both w_6 and w_{13} comes from the cusp form associated to 26A. Using the Cerednik–Drinfeld theory to compute the types of singular fibers at primes 2 and 13, we see that X_1 is isomorphic to the elliptic curve 26B2, X_2 is isomorphic to 26B1, and X_3 is isomorphic to 26A3. If we put the CM-point of discriminant -312 on X_1, that of discriminant -24 on X_2, and that of discriminant -8 on X_3 at ∞, respectively, and require that the Atkin–Lehner involutions w_{13}, w_3, and w_2 act by $(x, y) \to (x, -y)$ on the three curves, respectively, we get the equations for the three curves. $\square$

Lemma 19. (1) *An equation for the curve* $X_0^{26}(3)/\langle w_2\rangle$ *is*

$$y^2 = -\tfrac{2197}{3}x^6 - 362x^4 - 55x^2 - \tfrac{8}{3}$$

with the actions of the Atkin–Lehner involutions given by

$$w_3 : (x, y) \mapsto (-x, y), \quad w_{13} : (x, y) \mapsto (x, -y).$$

On this model, the two CM-points of discriminant -312 *are the two points at infinity, and the two CM-points of discriminant* -24 *are* $(0, \pm 2\sqrt{-6}/3)$.

(2) *An equation for the curve* $X_0^{26}(3)/\langle w_6 \rangle$ *is*

$$y^2 = \tfrac{2197}{72}x^6 - \tfrac{699}{8}x^4 - \tfrac{225}{8}x^2 - \tfrac{81}{8}$$

with the actions of the Atkin–Lehner involutions given by

$$w_2 : (x, y) \mapsto (-x, y), \quad w_{26} : (x, y) \mapsto (x, -y).$$

On this model, the two CM-points of discriminant -312 *are the two points at infinity, and the two CM-points of discriminant* -8 *are* $(0, \pm 9\sqrt{-2}/4)$.

(3) *An equation for* $X_0^{26}(3)/\langle w_{39} \rangle$ *is*

$$y^2 = \tfrac{8}{9}x^6 + 9x^4 - 18x^2 + 81$$

with the actions of the Atkin–Lehner involutions given by

$$w_2 : (x, y) \mapsto (-x, y), \quad w_6 : (x, y) \mapsto (x, -y).$$

On this model, the two CM-points of discriminant -24 *are the two points at infinity, and the two CM-points of discriminant* -8 *are* $(0, \pm 9)$.

Moreover, on each of these three curves, there are six CM-points of discriminant -104. *Their coordinates are* $(\alpha_j, 0)$, $j = 1, \ldots, 6$, *where* α_j *are the zeros of their respective polynomials of degree* 6.

Proof. We apply Proposition 2.1 of [González and Rotger 2004], cited as Lemma 17 above, with $C = X_0^{26}(3)/\langle w_2 \rangle$, w_{13}, $u_1 = w_3$, $u_2 = w_{39}$, $A_1 = -3403$, $B_1 = -83834$, $A_2 = -43$, and $B_2 = 166$. We find an equation for $X_0^{26}(3)/\langle w_2 \rangle$ is

$$y^2 = -\tfrac{2197}{3}x^6 - 362x^4 - 55x^2 - \tfrac{8}{3}$$

with the Atkin–Lehner involutions given by

$$w_3 : (x, y) \mapsto (-x, y), \quad w_{13} : (x, y) \mapsto (x, -y).$$

Since the CM-points of discriminant -24 are fixed by the involution $w_6 = w_3 : (x, y) \mapsto (-x, y)$, we see that their coordinates are $(0, \pm 2\sqrt{-6}/3)$. Likewise, the CM-points of discriminant -312 are the fixed points of $w_{78} = w_{39} : (x, y) \mapsto (-x, -y)$, so they are the two points at infinity. Also, the CM-points of discriminant -104 are the fixed points of $w_{26} = w_{13} : (x, y) \mapsto (x, -y)$. Their coordinates are $(\alpha_j, 0)$, $j = 1, \ldots, 6$, where α_j are the zeros of $-2197x^6/3 - 362x^4 - 55x^2 - 8/3$.

The equations of the other two curves are obtained in the same way. □

Lemma 20. *Let* $y^2 = -2197x^6/3 - 362x^4 - 55x^2 - 8/3$ *be the equation for* $X_0^{26}(3)/\langle w_2\rangle$ *given in the previous lemma. Then the coordinates of the four CM-points of discriminant* -8 *are* $(\pm 1/2\sqrt{-2}, \pm 3/16\sqrt{-2})$.

Proof. By Lemma 19, an equation for $X_0^{26}(3)/\langle w_2\rangle$ is $y^2 = -2197x^6/3 - 362x^4 - 55x^2 - 8/3$ with $w_3 : (x, y) \mapsto (-x, y)$ and $w_{13} : (x, y) \mapsto (x, -y)$. Thus, if we let $t_1 = x^2$, then t_1 is a Hauptmodul for $X_0^{26}(3)/W_{26,3}$. Likewise, if we let t_2 be the function x^2 in the equation $y^2 = 2197x^6/72 - 699x^4/8 - 225x^2/8 - 81/8$ for $X_0^{26}(3)/\langle w_6\rangle$, then t_2 is also a Hauptmodul for $X_0^{26}(3)/W_{26,3}$. It follows that $t_1 = (at_2 + b)/(ct_2 + d)$ for some a, b, c, d.

Now observe that the values of t_1 and t_2 at the CM-point of discriminant -312 are both ∞. Thus, $t_1 = at_2 + b$ for some a and b. The values of t_1 and t_2 at the CM-points of discriminant -104 are the zeros of $f_1(z) = -2197z^3/3 - 362z^2 - 55z - 8/3$ and $f_2(z) = 2197z^3/72 - 699z^2/8 - 225z/8 - 81/8$, respectively. Therefore, the constants a and b must satisfy $f_1(az+b) = Af_2(z)$ for some constant A. Comparing the coefficients, we find $A = 1/576$, $a = -1/24$ and $b = -1/8$. Since the value of t_2 at the CM-point of discriminant -8 is 0, the value of t_1 at the same point is $-1/8$, which implies that the four CM-points of discriminant -8 on $X_0^{26}(3)/\langle w_2\rangle$ have coordinates $(\pm 1/(2\sqrt{-2}), \pm 3/(16\sqrt{-2}))$ on the equation $y^2 = -2197x^6/3 - 362x^4 - 55x^2 - 8/3$ for $X_0^{26}(3)/\langle w_2\rangle$. □

Lemma 21. *There is a Hauptmodul* t_1 *for* $X_0^{26}(1)/W_{26}$ *that takes values* ∞, 0, *and the three zeros of* $-2x^3 + 19x^2 - 24x - 169$ *at the CM-point of discriminant* -8, *the CM-point of discriminant* -52, *and three CM-points of discriminant* -104, *respectively. Also, there is a Hauptmodul* $t = t_3$ *for* $X_0^{26}(3)/W_{26}$ *that takes values* $\pm 1/(2\sqrt{-2})$ *and the six zeros of* $-2197x^6/3 - 362x^4 - 55x^2 - 8/3$ *at the two CM-points of discriminant* -8 *and the six CM-points of discriminant* -104, *respectively. Moreover, the relation between* t_1 *and* t *and the action of* w_3 *on* t *are given by*

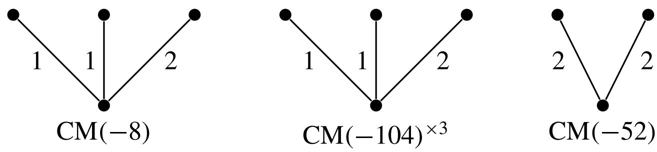
$$t_1 = -\frac{3(1 + t + 10t^2)^2}{(1 + 8t^2)(1 - t)^2}, \quad w_3 : t \mapsto -t.$$

Proof. According to Theorem 3.1 of [González and Rotger 2004], an equation for $X_0^{26}(1)$ is $y^2 = -2x^6 + 19x^4 - 24x^2 - 169$. In fact, the method used in that paper to deduce this equation also shows that the Atkin–Lehner involutions act by $w_{13} : (x, y) \mapsto (-x, y)$ and $w_{26} : (x, y) \mapsto (x, -y)$. Then the two points $(0, \pm 13\sqrt{-1})$ are the CM-points of discriminant -52, the two points at infinity are the fixed points of $w_2 : (x, y) \mapsto (-x, -y)$, that is, the two CM-points of discriminant -8, and the six points $(\alpha_j, 0)$, $j = 1, \ldots, 6$, are the six CM-points of discriminant -104, where α_j are the zeros of $-2x^6 + 19x^4 - 24x^2 - 169$. Thus, $t_1 = x^2$ is a Hauptmodul of $X_0^{26}(1)/W_{26}$ with values ∞, 0, the zeros of

$-2x^3+19x^2-24x-169$ at the CM-point of discriminant -8, the CM-point of discriminant -52, and the three CM-points of discriminant -104 on $X_0^{26}(1)/W_{26}$.

On the other hand, Lemmas 19 and 20 show that if we let t be the x in the equation $y^2=-2197x^6/3-362x^4-55x^2-8/3$ for $X_0^{26}(3)/\langle w_2\rangle$, then t is a Hauptmodul for $X_0^{26}(3)/W_{26}$ that takes values $\pm 1/(2\sqrt{-2})$ at the two CM-points of discriminant -8 and β_j, $j=1,\dots,6$, at the six CM-points of discriminant -104, where β_j are the six zeros of $-2197x^6/3-362x^4-55x^2-8/3$. It is clear that w_3 acts on t by $w_3: t\mapsto -t$.

The relation between t_1 and t is simple to determine. From the table at the end of Section 3, we know that the covering $X_0^{26}(3)/W_{26}\to X_0^{26}(1)/W_{26}$ is ramified precisely at the CM-points of discriminants -8, -52, and -104 of $X_0^{26}(1)/W_{26}$ with ramification types given by



It follows that

$$t_1=\frac{A(1+a_1t+a_2t^2)^2}{(1+8t^2)(1+bt)^2}$$

for some constants A, a_1, a_2, and b such that

$$\begin{aligned}-2f^3+19f^2g-24fg^2-169g^3\\ &=B(-2197t^6/3-362t^4-55t^2-8/3)(1+c_1t+c_2t^2+c_3t^3)^2\end{aligned}$$

for some constants B, c_1, c_2, and c_3, where $f=A(1+8t^2)(1+at)^2$ and $g=(1+b_1t+b_2t^2)^2$. Comparing the coefficients, we find

$$t_1=-\frac{3(1+t+10t^2)^2}{(1+8t^2)(1-t)^2}\quad\text{or}\quad t_1=-\frac{3(1-t+10t^2)^2}{(1+8t^2)(1+t)^2}.$$

Both are valid, since the action of w_3 sends one to the other. This gives us the lemma. □

Case $D=35$.

Lemma 22. *An equation for $X_0^{35}(1)/\langle w_5\rangle$ is*

$$y^2=-(x+12)(7x+4)(x^3+4x^2+144x+80)$$

with the action $w_7=w_{35}$ given by $w_7:(x,y)\mapsto(x,-y)$. The coordinates of the CM-points of discriminants -7, -28, -35, and -140 are $(-12,0)$, $(-4/7,0)$, ∞, and $(\alpha_j,0)$, respectively, where α_j are the three roots of $x^3+4x^2+144x+80$.

An equation for $X_0^{35}(2)/\langle w_7\rangle$ *is*

$$-2y^2 = (x^3+3x^2+11x+25)(x^3-3x^2+11x-25)$$

with the actions of $w_2 = w_{14}$ *and* $w_5 = w_{35}$ *given by* $w_2 : (x, y) \mapsto (-x, -y)$ *and* $w_5 : (x, y) \mapsto (x, -y)$. *The coordinates of the CM-points of discriminants* -7, -8, -140, *and* -280 *are* $(\pm\sqrt{-7}, \pm 8)$, *two points at* ∞, $(\beta_j, 0)$, $j = 1, \dots, 6$, *and* $(0, \pm 25/\sqrt{-2})$, *respectively, where* β_j *are the six roots of the polynomial* $(x^3+3x^2+11x+25)(x^3-3x^2+11x-25)$.

Proof. In Section 10.4 of [2012], Molina showed that an equation for $X_0^{35}(1)/\langle w_5\rangle$ is

$$y^2 = -x(9x+4)(4x+1)(172x^3+176x^2+60x+7),$$

where $w_7 : (x, y) \mapsto (x, -y)$ and the points $(0, 0)$, $(-4/9, 0)$, $(-1/4, 0)$, and $(\gamma_j, 0)$, $j = 1, \dots, 3$, are the CM-points of discriminant -7, -28, -35, and -140, respectively. Here γ_j are the zeros of $172x^3+176x^2+60x+7$. Setting

$$(x, y) = \left(-\frac{x'+12}{4x'+28}, \frac{5y'}{16(x'+7)^3}\right),$$

we get the equation in our lemma. The reason for this change of variable is the Shimura curve $X_0^{35}(1)/\langle w_7\rangle$ has genus 1 and the unique CM-point of discriminant -35 is a $\mathbb{Q}$-rational point. Thus, it is an elliptic curve over $\mathbb{Q}$. Computing the singular fibers at primes of bad reduction, we find that it is isomorphic to the elliptic curve 35A1, which, after a change of variables, has an equation $y^2 = x^3+4x^2+144x+80$. If we choose a Weierstrass equation for $X_0^{35}(1)/\langle w_7\rangle$ by requiring that the CM-point of discriminant -35 is the point at infinity and that w_5 acts by $(x, y) \to (x, -y)$, then up to a transformation of the form $x \to ax+b$, this Weierstrass equation must be $y^2 = x^3+4x^2+144x+80$ and the three 2-torsion points $(\alpha_j, 0)$ must be the three CM-points of discriminant -140. In view of this equation for $X_0^{35}(1)/\langle w_7\rangle$, we make the above change of variables for $X_0^{35}(1)/\langle w_5\rangle$.

We now consider the Shimura curve $X_0^{35}(2)/\langle w_7\rangle$. It is bielliptic with elliptic quotients $C_1 : X_0^{35}(2)/\langle w_7, w_{10}\rangle$ and $C_2 : X_0^{35}(2)/\langle w_2, w_7\rangle$. Here C_1 is an elliptic curve over $\mathbb{Q}$ because it has a unique CM-point of discriminant -8 and another two $\mathbb{Q}$-rational point coming from CM(-7). Likewise, C_2 is an elliptic curve over $\mathbb{Q}$ because C_2 has a unique CM-point of discriminant -280. By considering the eigenvalues of the Atkin–Lehner involutions associated to the eigenforms in $S_2(\Gamma_0(70))^{35\text{-new}}$, we find that both C_1 and C_2 fall in the isogeny class 35A, in Cremona's notation. Furthermore, by considering its singular fibers at primes of bad reduction using the Cerednik–Drinfeld theory, we find that C_1 is isomorphic to the elliptic curve 35A3 and C_2 is isomorphic to 35A2. We take $y^2 = x^3 - 1728x + 30672$

and $y^2 = x^3 - 170208x - 28273968$ to be (nonminimal) equations for 35A3 and 35A2, respectively.

Now if we choose a Weierstrass equation for C_1 by requiring that the CM-point of discriminant -8 is the infinity point and that the Atkin–Lehner involution w_2 acts by $(x, y) \mapsto (x, -y)$, then by a suitable transformation $x \mapsto ax + b$, the equation must be $y^2 = x^3 - 1728x + 30672$. Similarly, if we put the CM-point of discriminant -280 at infinity and require that w_5 acts by $(x, y) \mapsto (x, -y)$, then an equation for C_2 is $y^2 = x^3 - 170208x - 28273968$. Applying Lemma 17, we find an equation for $X_0^{35}(2)/\langle w_7\rangle$ is

$$y^2 = -\tfrac{9}{2}(x^6 + 13x^4 - 29x^2 - 625) = -\tfrac{9}{2}(x^3 + 3x^2 + 11x + 25)(x^3 - 3x^2 + 11x - 25).$$

Replacing y by $3y$, we get the equation

$$-2y^2 = (x^3 + 3x^2 + 11x + 25)(x^3 - 3x^2 + 11x - 25) \tag{2}$$

as claimed in the lemma. According to Lemma 17, the Atkin–Lehner involutions act by

$$w_{10} : (x, y) \mapsto (-x, y), \quad w_5 : (x, y) \mapsto (x, -y), \quad w_2 : (x, y) \mapsto (-x, -y).$$

Since the CM-points of discriminant -8, -140, and -280 on $X_0^{35}(2)/\langle w_7\rangle$ are fixed points of w_2, w_5, and w_{10}, respectively, we find that their coordinates are the two points at infinity, $(\beta_j, 0)$, $j = 1, \ldots, 6$, and $(0, \pm 25/\sqrt{-2})$, respectively, where β_j are the zeros of the polynomial on the right-hand side of (2).

To determine the coordinates of the four CM-points of discriminant -7, we observe that the curve $C_1 : X_0^{35}(2)/\langle w_7, w_{10}\rangle$ has exactly three $\mathbb{Q}$-rational points since it is isomorphic to the elliptic curve 35A3, which has precisely three $\mathbb{Q}$-rational points. Since we already know that C_1 has three $\mathbb{Q}$-rational points consisting of CM(-8) and CM(-7), any $\mathbb{Q}$-rational point of C_1 that is the CM-point of discriminant -8 will be a CM-point of discriminant -7. From the model $-2y^2 = x^6 + 13x^4 - 29x^2 - 625$ for $X_0^{35}(2)/\langle w_7\rangle$, we see that $-2y^2 = x^3 + 13x^2 - 29x - 625$ is also an equation for $X_0^{35}/\langle w_7, w_{10}\rangle$. On this model, the point at infinity is the CM-point of discriminant -8. Thus, the 3-torsion points $(-7, \pm 8)$ are the CM-points of discriminant -7 on $X_0^{35}(2)/\langle w_7, w_{10}\rangle$. This in turn implies that the four CM-points of discriminant -7 on $X_0^{35}(2)/\langle w_7\rangle$ have coordinates $(\pm\sqrt{-7}, \pm 8)$. This completes the proof of the lemma. □

Lemma 23. *There is a Hauptmodul t_1 for $X_0^{35}(1)/W_{35}$ that takes values -12, $-4/7$, ∞, and the three zeros of $x^3 + 4x^2 + 144x + 80$ at the CM-points of discriminants -7, -28, -35, and -140, respectively. Also, there is also a Hauptmodul t for $X_0^{35}(2)/W_{35}$ that takes values $\pm\sqrt{-7}$, ± 5, the six zeros of*

$$(x^3 + 3x^2 + 11x + 25)(x^3 - 3x^2 + 11x - 25),$$

and 0 *at the CM-points of discriminants* -7, -8, -140, *and* -280, *respectively. Moreover*, *the relation between* t_1 *and* t *is*

$$t_1 = -\frac{2(t-1)(t^2-6t+25)}{t^3+3t^2+11t+25}$$

and the Atkin–Lehner involution w_2 *on* t *is given by* $w_2 : t \mapsto -t$.

Proof. The existence of Hauptmoduln with the described values at CM-points follows immediately from Lemma 22. The fact that w_2 acts on t by $w_2 : t \mapsto -t$ also follows from the same lemma. We now determine the relation between Hauptmoduln.

The CM-point of discriminant -35 on $X_0^{35}(1)/W_{35}$ splits completely in the covering $X_0^{35}(2)/W_{35} \to X_0^{35}(1)/W_{35}$ and the three points lying above it are CM-points of discriminant -140 on $X_0^{35}(2)/W_{35}$. Replacing t by $-t$ if necessary, we may assume that the coordinates of these three points are the three zeros of $x^3+3x^2+11x+25$. Considering CM-points of discriminant -7, we have

$$t_1 + 12 = \frac{A(t^2+7)(t-a)}{t^3+3t^2+11t+25} \tag{3}$$

for some constants A and a. The point $t = a$ is a CM-point of discriminant -28. Thus, the point $t = -a$ is the other CM-point of discriminant -28 and this point lies above the CM-point of discriminant -28 on $X_0^{35}(1)/W_{35}$. Therefore, we have

$$t_1 + \tfrac{4}{7} = \frac{B(t+a)(t-b)^2}{t^3+3t^2+11t+25} \tag{4}$$

for some constants B and b. Comparing (3) and (4), we find $A = 10$, $B = -10/7$, $a = -5$, and $b = 3$. It follows that

$$t_1 = -\frac{2(t-1)(t^2-6t+25)}{t^3+3t^2+11t+25}.$$

To check the correctness, we observe that the point t with $t^3-3t^2+11t-25$ lies above CM-points of discriminant -140 on $X_0^{35}(1)/W_{35}$. Thus, if we write $t_1^3+4t_1^2+144t_1+80$ as a rational function of t, then $t^3-3t^2+11t-25$ should divide its numerator. Indeed, we find

$$t_1^3+4t_1^2+144t_1+80 = -\frac{200(t^3-t^2+11t-25)(t^3-t^2-5t-35)^2}{(t^3+3t^2+11t+25)^3}$$

as expected. This proves the lemma. □

***Case D* = 39.**

Lemma 24. *An equation for* $X_0^{39}(1)/\langle w_{13}\rangle$ *is*

$$y^2 = -(7x^2+23x+19)(x^2+x+1)$$

with $w_3 = w_{39} : (x, y) \mapsto (x, -y)$. *Moreover, the coordinates of the CM-points of discriminants* -52, -39, *and* -156 *are* $(\pm 2i, \pm\sqrt{13}(3+2i))$, $((-1\pm\sqrt{-3})/2, 0)$, *and* $((-23\pm\sqrt{-3})/14, 0)$, *respectively.*

Proof. By [Molina 2012], an equation for $X_0^{39}(1)$ is

$$y^2 = -(7x^4 + 79x^3 + 311x^2 + 497x + 277)(x^4 + 9x^3 + 29x^2 + 39x + 19)$$

with $w_{39} : (x, y) \mapsto (x, -y)$. Moreover, the coordinates of the CM-points of discriminants -39 and -156 are $(\alpha_j, 0)$ and $(\beta_j, 0)$, $j = 1, \dots, 4$, respectively, where α_j are the zeros of $x^4 + 9x^3 + 29x^2 + 39x + 19$ and β_j are the zeros of $7x^4 + 79x^3 + 311x^2 + 497x + 277$. Substituting x by $x - 2$, we obtain an equation

$$y^2 = -(7x^4 + 23x^3 + 5x^2 - 23x + 7)(x^4 + x^3 - x^2 - x + 1) \tag{5}$$

with smaller coefficients. This hyperelliptic curve has an obvious automorphism $(x, y) \mapsto (-1/x, y/x^4)$. We will show that this is the Atkin–Lehner involution w_{13}.

The Atkin–Lehner w_{13} permutes the CM-points of discriminant -39. It also permutes the CM-points of discriminant -156. Therefore, if w_{13} maps (x, y) to $((ax+b)/(cx+d), Cy/(cx+d)^4)$, then the constants a, b, c, and d must satisfy

$$(cx+d)^4 f_j\left(\frac{ax+b}{cx+d}\right) = C_j f_j(x)$$

for $f_1(x) = 7x^4 + 23x^3 + 5x^2 - 23x + 7$ and $f_2(x) = x^4 + x^3 - x^2 - x - 1$. We find w_{13} maps (x, y) to either $(-1/x, y/x^4)$ or $(-1/x, -y/x^4)$. The latter has no fixed points, so we conclude that w_{13} maps (x, y) to $(-1/x, y/x^4)$.

Now it is easy to show that $Y = y/x^2$ and $X = x - 1/x$ generate the function field of $X_0^{39}(1)/\langle w_{13}\rangle$. The relation between X and Y is also easy to find. It is

$$Y^2 = -(7X^2 + 23X + 19)(X^2 + X + 1), \tag{6}$$

which gives us an equation for $X_0^{39}(1)/\langle w_{13}\rangle$. The coordinates of the CM-points of discriminants -39 and -156 on $X_0^{39}(1)/\langle w_{13}\rangle$ are $((-1\pm\sqrt{-3})/2, 0)$ and $((-23\pm\sqrt{-3})/14, 0)$, respectively.

To find the coordinates of the CM-points of discriminant -52 on $X_0^{39}(1)/\langle w_{13}\rangle$, we first consider the CM-points of the same discriminant on $X_0^{39}(1)$. Since these points on $X_0^{39}(1)$ are the fixed points of w_{13} and on (5), the Atkin–Lehner involution w_{13} acts by $(x, y) \mapsto (-1/x, y/x^4)$, we find that the coordinates of the CM-points of discriminant -52 on (5) are $(\pm i, \pm\sqrt{13}(3+2i))$. This implies that the CM-points of discriminant -52 on $X_0^{39}(1)/\langle 13\rangle$ are $(\pm 2i, \pm\sqrt{13}(3+2i))$. The proof of the lemma is complete. □

Lemma 25. *There is a Hauptmodul t_1 on $X_0^{39}(1)/W_{39}$ that takes values*

$$\pm 2i, \quad \frac{-1\pm\sqrt{-3}}{2}, \quad \frac{-23\pm\sqrt{-3}}{14}$$

at the CM-points of discriminants -52, -39, *and* -156, *respectively. Also, there is a Hauptmodul t on $X_0^{39}(2)/W_{39}$ that takes values*

$$\pm 3i, \quad \frac{\pm 2\sqrt{-3}\pm\sqrt{-39}}{3}, \quad \pm 1\pm 2\sqrt{-3}$$

at the CM-points of discriminants -52, -39, *and* -156, *respectively. Moreover, the relation between t_1 and t is*

$$t_1 = -\frac{2(t^3+t^2+11t+3)}{(t^2+7)(t+3)}$$

and the Atkin–Lehner involution w_2 on t is $w_2 : t \mapsto -t$.

Proof. The existence of t_1 with the described properties follows from the previous lemma. Now let $s_1 = (t_1-2i)/(t_1+2i)$ so that s_1 takes values 0 and ∞ at the two CM-points of discriminant -52. Then the values of s_1 at the two CM-points of discriminant -156 are the zeros of

$$(9+46i)x^2 + 94x + (9-46i). \tag{7}$$

The covering $X_0^{39}(2)/W_{39} \to X_0^{39}(1)/W_{39}$ is ramified at $\mathrm{CM}(-52)\cup\mathrm{CM}(-156)$ of $X_0^{39}(1)/W_{39}$. There is a Hauptmodul s of $X_0^{39}(2)/W_{39}$ such that

$$s_1 = \frac{As(1-s)^2}{(1-as)^2}$$

for some complex numbers A and a. That is, s is determined by the property that it takes values 0 and 1 at the two points lying above the point $s_1 = 0$ with the point $s = 1$ having a ramification index 2 and value ∞ at the point lying above $s_1 = \infty$ with ramification index 1.

Now the condition that the CM-points of discriminant -156 are ramified implies that the discriminant of

$$As(1-s)^2 - x(1-as)^2$$

as a polynomial in s must be divisible by the polynomial in (7). This gives two relations between A and a. Solving them for A and a, we find that the only legitimate choice is $A = 9-46i$ and $a = 13$. Then we have

$$t_1 = \frac{2i(s_1+1)}{-s_1+1} = \frac{4394is^3 + (-15548-5746i)s^2 + (2392+3926i)s - 92 + 18i}{(13s-3+2i)(-169s^2+(416+624i)s+5-12i)}.$$

Let t be the Hauptmodul of $X_0^{39}(2)/W_{39}$ with

$$s=-\frac{3+2i}{13}\frac{(5+i)t+3-15i}{(5-i)t+3+15i}.$$

Then we have

$$t_1=-\frac{2(t^3+t^2+11t+3)}{(t+3)(t^2+7)}.$$

The values of t at CM(-52), CM(-39), and CM(-156) can be read off from

$$t_1^2+4=\frac{8(t^2+9)(t^2+2t+5)^2}{(t+3)^2(t^2+7)^2},$$

$$t_1^2+t_1+1=\frac{(t^2+2t+13)(3t^4+34t^2+27)}{(t+3)^2(t^2+7)^2},$$

and

$$7t_1^2+23t_1+19=\frac{(t^2-2t+13)(t^2-6t+21)^2}{(t+3)^2(t^2+7)^2},$$

respectively. To determine the action of w_2 on t, we recall that w_2 switches the two points in CM(-52). It also exchanges the two zeros of $x^2+2x+13$, corresponding to the two points in CM(-156) that lie above the CM-points of discriminant -39 on $X_0^{39}(1)/W_{39}$, with the two zeros of $x^2-2x+13$, corresponding to the other two points in CM(-156) that lie above the CM-points of discriminant -156 on $X_0^{39}(1)/W_{39}$. From this information, we can deduce that $w_2: t\mapsto -t$. □

5. Main results

5.1. *Schwarzian differential equations.*

Theorem. *Let Hauptmoduln for $X_0^D(N)/W_D$ be as in the lemmas. Then the automorphic derivatives associated to them are as follows. For $(D,N)=(6,1)$,*

$$Q(t)=\frac{108-113t+140t^2}{576t^2(1-t)^2}.$$

For $(D,N)=(6,5)$,

$$Q(t)=-\frac{15(23-456t^2+1608t^4)}{2(2+3t^2)^2(1+64t^2)^2}.$$

For $(D,N)=(6,7)$,

$$Q(t)=-\frac{3(267+6480t^2+64352t^4)}{4(1+27t^2)^2(3+32t^2)^2}.$$

For $(D,N)=(6,13)$,

$$Q(t) = -\frac{3(12492+43272t^2+37541t^4)}{(9+16t^2)^2(16+27t^2)^2}.$$

For $(D, N) = (10, 1)$,

$$Q(t) = \frac{3t^4-119t^3+3157t^2-7296t+10368}{16t^2(t-2)^2(t-27)^2}.$$

For $(D, N) = (10, 3)$,

$$Q(t) = \frac{8-303t^2-1200t^4-95840t^6}{36t^2(1+32t^2)^2(1+5t^2)^2}.$$

For $(D, N) = (10, 7)$,

$$Q(t) = -\frac{655+62410t^2+2237231t^4+35817920t^6+216522240t^8}{(1+27t^2)^2(1+20t^2)^2(5+128t^2)^2}.$$

For $(D, N) = (14, 1)$,

$$Q(t) = \frac{192+440t+43t^2+1036t^3+960t^4}{16t^2(8+13t+16t^2)^2}.$$

For $(D, N) = (14, 3)$,

$$Q(t) = -\frac{3(497-1988t^2+31494t^4+141436t^6+139601t^8)}{2(1+2t^2)^2(7+226t^2+343t^4)^2}.$$

For $(D, N) = (14, 5)$,

$$Q(t) = -\frac{623+16772t^2+55178t^4-853468t^6+97503t^8}{(1+16t^2)^2(7+114t^2+7t^4)^2}.$$

For $(D, N) = (15, 1)$,

$$Q(t) = \frac{177147-244944t+244242t^2-3680t^3+35t^4}{144t^2(1-t)^2(81-t)^2}.$$

For $(D, N) = (15, 2)$,

$$Q(t) = \frac{3(385+5500t^2-2042t^4+35196t^6-2175t^8)}{4(1-t)^2(1+t)^2(1-5t)^2(1+5t)^2(5+3t^2)^2}.$$

For $(D, N) = (15, 4)$,

$$Q(t) = -\frac{9(14+271t^2+2024t^4+7746t^6+19895t^{10}+16674t^8+10720t^{12})}{4(4t^2-t+1)^2(4t^2+t+1)^2(3t^2+1)^2(5t^2+3)^2}.$$

For $(D, N) = (21, 1)$,

$$Q(t) = \frac{21(40353607 - 17647350t + 3561369t^2 - 477652t^3 + 31833t^4 - 630t^5 + 7t^6)}{16t^2(49-t)^2(343-94t+7t^2)^2}.$$

For $(D, N) = (21, 2)$,

$$Q(t) = \frac{3(1 - 69t^2 - 4086t^4 + 23670t^6 + 6043653t^8 + 6781887t^{10})}{16t^2(1-t)^2(1+t)^2(1+27t^2)^2(1+63t^2)^2}.$$

For $(D, N) = (26, 1)$,

$$Q(t) = \frac{85683 + 15210t + 16694t^2 - 9480t^3 + 1363t^4 - 170t^5 + 12t^6}{16t^2(169+24t-19t^2+2t^3)^2}.$$

For $(D, N) = (26, 3)$,

$$Q(t) = -\frac{6(85+3528t^2+60543t^4+552448t^6+2850579t^8+7990200t^{10}+9677785t^{12})}{(1+8t^2)^2(8+165t^2+1086t^4+2197t^6)^2}.$$

For $(D, N) = (35, 1)$,

$$Q(t) = Q_1(t)/16(t+12)^2(7t+4)^2(t^3+4t^2+144t+80)^2,$$

where

$$Q_1(t) = 666427392t + 1132800t^4 + 181420032 - 753984t^5 + 24576t^6 + 147t^8 + 659096576t^2 + 85540864t^3 + 3808t^7.$$

For $(D, N) = (35, 2)$,

$$Q(t) = Q_1(t)/4(t^2+7)^2(t^2-25)^2(t^6+13t^4-29t^2-625)^2,$$

where

$$Q_1(t) = 2842805000t^2 + 91524600t^6 - 2082286t^8 - 217416t^{10} + 54644t^{12} + 3784t^{14} + 19t^{16} - 992578125 + 1017474100t^4.$$

For $(D, N) = (39, 1)$,

$$Q(t) = \frac{-3Q_1(t)}{4(4+t^2)^2(1+t+t^2)^2(19+23t+7t^2)^2},$$

where

$$Q_1(t) = 2596 + 7104t + 9692t^2 + 12348t^3 + 13149t^4 + 9522t^5 + 4367t^6 + 1086t^7 + 97t^8.$$

For $(D, N) = (39, 2)$,

$$Q(t) = \frac{-9Q_1(t)}{4(9+t^2)^2(13+2t+t^2)^2(13-2t+t^2)^2(27+34t^2+3t^4)^2},$$

where

$$Q_1(t) = 419253003 + 119984328t^2 + 89200020t^4 + 43676088t^6 + 10194786t^8 + 1272824t^{10} + 87380t^{12} + 3080t^{14} + 43t^{16}.$$

For these results, we take the Schwarzian differential equations on $X_0^{14}(1)/W_{14}$, $X_0^{14}(3)/W_{14}$, and $X_0^{14}(5)/W_{14}$ as examples for the proofs.

Proof. In Lemma 12, we see that there is a Hauptmodul t_1 on $X_0^{14}(1)/W_{14}$ with value ∞ at the elliptic point of order 4 and values 0 and $(-13 \pm 7\sqrt{-7})/32$ at the elliptic points of order 2. According to Proposition 3, the automorphic derivative $Q(t_1)$ associate to t_1 is

$$Q(t_1) = \frac{3}{16} - \frac{21+16B}{52t} + \frac{3(512t^2+416t-87)}{(16t^2+13t+8)^2} + \frac{4(21t+B(16t+13))}{13(16t^2+13t+8)},$$

for some constant B. We now use the covering $X_0^{14}(3)/W_{14} \to X_0^{14}(1)/W_{14}$ to determine the constant B. More precisely, according to Proposition 4, we have the relation between $Q(t_1)$ and the automorphic derivative $Q(t)$ associative to a Hauptmodul t of $X_0^{14}(3)/W_{14}$,

$$Q(t) = D(t_1, t) + Q(t_1)/(dt_1/dt)^2.$$

Note that there is a Hauptmodul t for $X_0^{14}(3)/W_{14}$ that takes values $\pm 1/\sqrt{-2}$, $(\pm 9\sqrt{-7} \pm 4\sqrt{-14})/49$ at the 6 elliptic points of order 6. Thus, the automorphic derivative $Q(t)$ is

$$Q(t) = \frac{3(2t^2-1)}{4(2t^2+1)^2} + \frac{3(18335t^2+38759t^4+117649t^6-791)}{4(7+226t^2+343t^4)^2} + \frac{343(686C_4t^3+109C_3t^2+109C_4t+109C_5)}{436(7+226t^2+343t^4)} - \frac{1372C_4t+981+218C_3}{436(2t^2+1)}$$

for some constants C_3, C_4, and C_5. Also, the action of the Atkin–Lehner involution w_3 is $w_3 : t \mapsto -t$. Thus, by Proposition 5, we can get the value $C_4 = 0$.

From the relations

$$t_1 = \frac{4(1+2t^2)(1-5t)^2}{9(1+t)^4} \quad \text{and} \quad Q(t) = D(t_1, t) + \frac{Q(t_1)}{(dt_1/dt)^2}$$

we find that

$$B = -\tfrac{373}{512}, \; C_3 = -\tfrac{91}{9}, \; \text{and} \; C_5 = -\tfrac{1301}{3087}.$$

For the case of $X_0^{14}(5)/X_{14}$, the chosen Hauptmodul t takes values $\pm i/4$ at the elliptic points of order 4, $(\pm 5\sqrt{-7} \pm 4\sqrt{-14})/7$ at the elliptic points of order 2, and the action of Atkin–Lehner involution w_5 is $t \mapsto -t$. Therefore, the automorphic derivative associative to t is

$$Q(t) = \frac{15(16t^2-1)}{2(16t^2+1)^2} + \frac{3(49t^6+399t^4+6351t^2-399)}{4(7t^4+114t^2+7)^2} - \frac{39+8B_1}{2(16t^2+1)} + \frac{7(B_1t^2+B_2)}{4(7t^4+114t^2+7)},$$

for some constants B_1 and B_2. From the relation

$$t_1 = -\frac{5(1-t+17t^2-13t^3)^2}{(1+16t^2)(1+3t)^4}$$

and Proposition 4, we can conclude that

$$Q(t) = -\frac{97503t^8 - 853468t^6 + 55178t^4 + 16772t^2 + 623}{(16t^2+1)^2(7t^4+114t^2+7)^2}. \qquad \square$$

5.2. ***Ramanujan-type formulae.*** Recall that if E is an elliptic curve defined over $\overline{\mathbb{Q}}$, which has CM by an imaginary quadratic field K of discriminant d, then up to an algebraic factor, the period of E can be expressed by

$$\Omega_d = \sqrt{\pi} \prod_{0<a<|d|} \Gamma\left(\frac{a}{|d|}\right)^{w_d \chi_d(a)/4h_d},$$

where w_d is the number of roots of unity in K, χ_d is the Kronecker character $\left(\frac{d}{\cdot}\right)$ associated to K, and h_d is the class number of K. Yang [2013a] contributes many Ramanujan-type series. For example,

$$\sum_{n=0}^{\infty} \left(74480n + \tfrac{6860}{3}\right) \frac{(1/12)_n(1/4)_n(5/12)_n}{(1/2)_n(3/4)_n n!} \left(\frac{-7^4}{3375}\right)^n = 7^3\sqrt{5}\sqrt[4]{3375}\frac{4\pi}{\sqrt[4]{12}\Omega_{-4}^2},$$

which is related to the period of an elliptic curve with CM by $\mathbb{Q}(\sqrt{-1})$. The power series

$$\sum_{n=0}^{\infty} \frac{(1/12)_n(1/4)_n(5/12)_n}{(1/2)_n(3/4)_n n!} t^n$$

mentioned above is the hypergeometric function

$${}_3F_2\left(\tfrac{1}{12}, \tfrac{1}{4}, \tfrac{5}{12}; \tfrac{1}{2}, \tfrac{3}{4}, t\right) = {}_2F_1\left(\tfrac{1}{24}, \tfrac{5}{24}; \tfrac{3}{4}; t\right)^2.$$

Note that the function ${}_2F_1\left(\tfrac{1}{24}, \tfrac{5}{24}; \tfrac{3}{4}; t\right)$ is related to the Schwarzian differential equation associated to the Hauptmodul t of $X_0^6(1)/W_6$ that takes values 0, 1, and ∞ at the CM-points of discriminants -4, -24, and -3, respectively. Yang also gave other similar identities related to Ω_{-4}, and also the Ramanujan-type series related to Ω_{-3} for the curve $X_0^6(1)/W_6$.

Yang [2013a] guesses that, in general, we can use the t-series expansion of a meromorphic form to obtain Ramanujan-type identities, which are related to certain periods of elliptic curves with CM. That is, we may have

$$\sum_{n=0}^{\infty}(R_1 n+R_2)A_n t_0^n = R_3\frac{\pi}{\Omega_d^2},$$

where R_1, R_2, $R_3 \in \overline{\mathbb{Q}}$, $\sum_0^\infty A_n t^n$ is the expansion of a meromorphic automorphic form of weight 2 with respect to a Hauptmodul t of a Shimura curve of genus zero such that t takes value 0 at a CM-point of discriminant d, and t_0 is the value of t at some CM-point of discriminant $d' \neq d$. To be more precise, let g_1 and g_2 be two linearly independent solutions of a given Schwarzian differential equation associated to a Shimura curve of genus 0. Write $g_1^2 = \sum_0^\infty A_n t^n$ and $g_2^2 = \sum_0^\infty B_n t^n$; then we expect that

$$\sum_{n=0}^{\infty}(R_1 n+R_2)\,A_n t_0^n = R_3\frac{\pi}{\Omega_d^2},$$
$$\sum_{n=0}^{\infty}(R_1 n+R_2+R_1/a)\,B_n t_0^n = R_3\frac{\Omega_d^2}{\pi},$$

for certain positive integer a. We remark that the series also converges p-adically for primes $p\,|\,M$ while $t_0 = M/N$. The p-adic numbers to which they converge should be related to the p-adic periods of certain elliptic curves with CM. Yang also gave some numerical examples of the p-adic analogues for the Ramanujan-type series obtained from $X_0^6(1)/W_6$. Here, let us see some numerical examples coming from $X_0^{14}(1)/W_{14}$.

From the Lemma 12, we know that there is a Hauptmodul t for $X_0^{14}(1)/W_{14}$ that takes values ∞, 0, and $(-13\pm 7\sqrt{-7})/32$ at CM-points of discriminants -4, -8, and -56, respectively. The t-series expansions of two linearly independent solutions of the Schwarzian differential equation associated to t (see Theorem),

$$\frac{d^2}{dt^2}f+Q(t)f=0,\qquad Q(t)=\frac{192+440t+43t^2+1036t^3+960t^4}{16t^2(8+13t+16t^2)^2},$$

are

$$g_1 = t^{1/4}\left(1+\tfrac{23}{64}t+\tfrac{1867}{8192}t^2-\tfrac{955937}{2621440}t^3+\tfrac{157030847}{671088640}t^4+\tfrac{3694251053}{42949672960}t^5+\dots\right)\text{ and}$$
$$g_2 = t^{3/4}\left(1+\tfrac{23}{192}t+\tfrac{3149}{24576}t^2-\tfrac{434593}{1572864}t^3+\tfrac{264972083}{1207959552}t^4+\tfrac{39014127761}{850403524608}t^5+\dots\right).$$

The Hauptmodul t takes value $t_0=-13/81$ at the CM-points of discriminants -91 (this is given in [Elkies 1998]). We now let

$$\sum_{n=0}^{\infty}A_n = t^{-1/2}g_1^2,\qquad \sum_{n=0}^{\infty}B_n = t^{-3/2}g_2^2,$$

and

$$C = \frac{81}{2548}\frac{\Gamma(5/8)\Gamma(7/8)}{\Gamma(1/8)\Gamma(3/8)} = \frac{81}{2548}\Omega_{-8}^2/\pi.$$

Then

$$\left(\sum_{n=0}^{\infty} R_1 n + R_2\right) A_n t_0^n = \tfrac{847}{18} 13^{3/4} 3C, \tag{8}$$

$$\left(\sum_{n=0} \infty R_1 n + R_1 + R_2\right) B_n t_0^n = \tfrac{847}{18} 13^{1/4} 27C^{-1}. \tag{9}$$

If we choose a Hauptmodul t that takes values 0, ∞, and $(-39 \pm 21\sqrt{-7})/16$ at CM-points of discriminant -4, -8, and -56, respectively, the Schwarzian differential equation associated to t is given by

$$\frac{d^2}{dt^2} f + Q(t) f = 0, \quad Q(t) = \frac{3(64t^4 + 440t^3 + 129t^2 + 9324t + 25920)}{16t^2(8t^2 + 39t + 144)^2},$$

and its two linearly independent solutions are

$$g_1 = t^{3/8}\left(1 + \tfrac{131}{2304}t + \tfrac{21631}{3538944}t^2 - \tfrac{49745249}{29896998912}t^3 + \tfrac{16603576771}{91843580657664}t^4 + \dots\right),$$
$$g_2 = t^{5/8}\left(1 + \tfrac{131}{3840}t + \tfrac{8923}{1966080}t^2 - \tfrac{257758957}{176664084480}t^3 + \tfrac{646181570409}{9226105147883520}t^4 + \dots\right).$$

The Hauptmodul t takes value $t_0 = 27/200$ at the CM-points of discriminants -168. Let

$$\sum_{n=0}^{\infty} C_n = t^{-3/4} g_1^2, \quad \sum_{n=0}^{\infty} D_n = t^{-5/4} g_2^2.$$

We have

$$\sum_{n=0}^{\infty} (R_1 n + R_2)\, C_n t_0^n = \frac{810000}{11^8} 27^{3/4} 200^{1/4} C,$$

$$\sum_{n=0}^{\infty} (R_1 n + R_2 + R_1/2)\, D_n t_0^n = \frac{810000}{11^8} 27^{1/4} 200^{3/4} C^{-1}$$

with $R_1 = 2904$, $R_2 = 12$, where

$$C = \frac{\Gamma(3/4)^2}{\Gamma(1/4)^2}\left(\frac{196}{3}\right)^{1/4} = \left(\frac{196}{3}\right)^{1/4} \Omega_{-4}^2/\pi.$$

Let $\Gamma_p(\cdot)$ stand for the p-adic Gamma function. The numerical results checked for 70 p-adic digits yield that

$$\sum_{n=0}^{\infty}(R_1 n + R_2)\, C_n t_0^n = \frac{2^4\cdot 11^8}{9}\left(27^3 200\frac{98\Gamma_3(1/4)}{27\Gamma_3(3/4)}\right)^{1/4},$$

$$\sum_{n=0}^{\infty}(R_1 n + R_2 + R_1/2)\, D_n t_0^n = \frac{2^4\cdot 11^8}{9}\left(27\cdot 200^3\frac{27\Gamma_3(3/4)}{98\Gamma_3(1/4)}\right)^{1/4},$$

hold 3-adically with $R_1 = 29040$ and $R_2 = 120$.

For the numbers $\sum nA_n t_0^n$, $\sum A_n t_0^n$, $\sum nB_n t_0^n$, and $\sum B_n t_0^n$, after numerical computation, we find that the equalities

$$\left(\sum_{n=0}^{\infty}(11011n + 7290) A_n t_0^n\right)^2 = 3^3\cdot 7\cdot 137\cdot 1571\frac{\Gamma_{13}(5/8)\Gamma_{13}(7/8)}{2\Gamma_{13}(1/8)\Gamma_{13}(3/8)},$$

$$\left(\sum_{n=0}^{\infty}(11011n + 75897)\, B_n t_0^n\right)^2 = 3^{12}\cdot 7\cdot 11^4\frac{\Gamma_{13}(1/8)\Gamma_{13}(3/8)}{8\Gamma_{13}(5/8)\Gamma_{13}(7/8)},$$

hold 13-adically.

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