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ZHAO XU-AN AND JIN CHUNHUA

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POLYNOMIAL INVARIANTS OF WEYL GROUPS FOR KAC–MOODY GROUPS

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We prove that the ring of polynomial invariants of Weyl group for an indecomposable and indefinite Kac–Moody Lie algebra is generated by the invariant symmetric bilinear form or is trivial depending on whether A is symmetrizable or not. The result was conjectured by Moody and assumed by Kac. As an application, we discuss the rational homotopy types of Kac–Moody groups and their flag manifolds.

1. Introduction

Let $A = (a_{ij})$ be an $n \times n$ integer matrix satisfying:

(1) For each i, $a_{ii} = 2$.

(2) For $i \neq j$, $a_{ij} \leq 0$.

(3) If $a_{ij} = 0$, then $a_{ji} = 0$.

Then A is called a Cartan matrix.

Let h be the real vector space with basis $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\}$, and denote the dual basis of $\Pi^\vee$ in the dual space h^* by $\{\omega_1, \omega_2, \ldots, \omega_n\}$; that is, $\omega_i(\alpha_j^\vee) = \delta_{ij}$ for $1 \leq i, j \leq n$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset h^*$ be given by $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for all i, j; then $\alpha_i = \sum_{j=1}^n a_{ji}\omega_j$. Note that if the Cartan matrix A is singular, then $\{\alpha_i \mid 1 \leq i \leq n\}$ is not a basis of h^*. Π and $\Pi^\vee$ are called the simple root system and simple coroot system associated to the Cartan matrix A, and α_i, $\alpha_i^\vee$, ω_i, $1 \leq i \leq n$ are the simple roots, simple coroots and fundamental dominant weights respectively.

By [Kac 1968] and [Moody 1968], for each Cartan matrix A, there is a Lie algebra $g(A)$ associated to A, which is called the Kac–Moody Lie algebra.

The Kac–Moody Lie algebra $g(A)$ is generated by $\alpha_i^\vee$, e_i, f_i, $1 \leq i \leq n$ over $\mathbb{C}$, with the defining relations

(1) $[\alpha_i^\vee, \alpha_j^\vee] = 0$,

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(2) $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$,

(3) $[\alpha_i^\vee, e_j] = a_{ij}e_j, \quad [\alpha_i^\vee, f_j] = -a_{ij}f_j$,

(4) $\mathrm{ad}(e_i)^{-a_{ij}+1}(e_j) = 0,\ 1 \le i \ne j \le n$,

(5) $\mathrm{ad}(f_i)^{-a_{ij}+1}(f_j) = 0,\ 1 \le i \ne j \le n$.

Kac and Peterson [1983; 1985] (see also [Kac 1985a]) constructed the Kac–Moody group $G(A)$ with Lie algebra $g(A)$. In this paper, for convenience we consider the quotient Lie algebra of $g(A)$ modulo its center $c(g(A))$ and the associated simply connected group $G(A)$ modulo $C(G(A))$. We still use the same symbols $g(A)$ and $G(A)$ and call them the Kac–Moody Lie algebra and the Kac–Moody group.

Cartan matrices and their associated Kac–Moody Lie algebras and Kac–Moody groups are divided into three types:

(1) Finite type, if A is positive definite. In this case, $G(A)$ is just the simply connected complex semisimple Lie group with Cartan matrix A.

(2) Affine type, if A is positive semidefinite and has rank $n-1$.

(3) Indefinite type otherwise.

A Cartan matrix A is called hyperbolic if all the proper principal submatrices of A are of finite or affine type. A Cartan matrix A is called symmetrizable if there exist an invertible diagonal matrix D and a symmetric matrix B such that $A = DB$. Also, $g(A)$ is called a symmetrizable Kac–Moody Lie algebra if A is symmetrizable.

The Weyl group $W(A)$ associated to a Cartan matrix A is the group generated by the Weyl reflections $\sigma_i : h^* \to h^*$ with respect to $\alpha_i^\vee$, for $1 \le i \le n$, where $\sigma_i(\alpha) = \alpha - \alpha(\alpha_i^\vee)\alpha_i$. $W(A)$ has a Coxeter presentation

$$W(A) = \langle \sigma_1, \dots, \sigma_n \mid \sigma_i^2 = e, 1 \le i \le n;\ (\sigma_i\sigma_j)^{m_{ij}} = e, 1 \le i < j \le n \rangle,$$

where $m_{ij} = 2, 3, 4, 6$ or ∞ as $a_{ij}a_{ji} = 0, 1, 2, 3$ or ≥ 4, respectively. The action of σ_i on fundamental dominant weights is given by $\sigma_i(\omega_j) = \omega_j - \omega_j(\alpha_i^\vee)\alpha_i = \omega_j - \delta_{ji}\alpha_i$. For details see [Kac 1983; Humphreys 1990].

The action of the Weyl group $W(A)$ on h^* induces an action of $W(A)$ on the polynomial ring $\mathbb{Q}[h^*] \cong \mathbb{Q}[\omega_1, \dots, \omega_n]$. If for each $\sigma \in W(A)$, $\sigma(f) = f$, then $f \in \mathbb{Q}[h^*]$ is called a $W(A)$-invariant polynomial. Since $W(A)$ is generated by σ_i, $1 \le i \le n$, f is a $W(A)$-invariant polynomial if and only if $\sigma_i(f) = f$ for $1 \le i \le n$. The $W(A)$-invariant polynomials form a ring, called the ring of $W(A)$ polynomial invariants, denoted by $I(A)$.

The invariant theory of Weyl groups has been a significant topic since the 1950s. It has important applications in the homology of Lie groups and their classifying spaces. Motivated by that study, Chevalley showed that the ring of invariants of a finite Weyl group is a polynomial algebra. A comprehensive study of the polynomial invariants was undertaken by Bourbaki, Solomon, Springer and Steinberg, etc.

Moody [1978] proved the following:

Theorem. *Let A be an indecomposable and symmetrizable $n \times n$ Cartan matrix whose associated invariant bilinear form ψ is nondegenerate and of signature* $(n-1, 1)$. *Then the ring of $W(A)$ polynomial invariants is $\mathbb{Q}[\psi]$.*

In the same paper, Moody further said: "*We conjecture that it is in fact true for all Weyl groups arising from nonsingular Cartan matrices of nonfinite type*."

Kac [1985b] also assumed that for an indecomposable and indefinite Cartan matrix, the ring of $W(A)$ polynomial invariants is $\mathbb{Q}[\psi]$ or trivial depending on whether A is symmetrizable or not.

In this paper, we prove the following:

Theorem. *Let A be an indecomposable and indefinite Cartan matrix A. If A is symmetrizable*, *then $I(A) = \mathbb{Q}[\psi]$*; *if A is nonsymmetrizable*, *then $I(A) = \mathbb{Q}$.*

The content of this paper is as follows. In Section 2, we discuss the general results about the polynomial invariants of Weyl group $W(A)$. In Sections 3 and 4, we consider the rank 2 and hyperbolic cases, respectively. The main theorem is proved in Section 5. In Section 6, we consider the applications of the theorem in determining the rational homotopy type of Kac–Moody groups and their flag manifolds.

2. Rings of polynomial invariants of Weyl groups: general case

In this section, we discuss some general properties of the ring of invariants of Weyl groups.

Lemma 2.1. *If a Cartan matrix A is the direct sum of Cartan matrices A_1, A_2, then $I(A)$ is isomorphic to the tensor product $I(A_1) \otimes I(A_2)$.*

So we only consider indecomposable Cartan matrices.

Lemma 2.2. *Write $f \in I(A)$ as $f = \sum_{i=0}^{l} f_i \in I(A)$, where f_i is the degree i homogeneous component of f. Then $f_i \in I(A)$.*

So $I(A)$ is a graded ring: $I(A) = \bigoplus_{i=0}^{\infty} I^i(A)$, where $I^i(A)$ is the subspace of homogeneous polynomials of degree i in $I(A)$. To determine the ring $I(A)$, we only need to consider homogeneous invariant polynomials.

Lemma 2.3. *For an indecomposable Cartan matrix A of affine or indefinite type*, *the orbit $\{\sigma(\omega) \mid \sigma \in W(A)\}$ of a nonzero element ω in the Tits cone $\left\{\sum_{i=1}^{n} \lambda_i \omega_i \mid \lambda_i \geq 0\right\}$ is an infinite set.*

Proof. If $\omega \neq 0$ is in the Tits cone, then we can assume $\omega = \sum_{i=1}^{n} \lambda_i \omega_i$, $\lambda_i \geq 0$. Let $S = \{1, 2, \ldots, n\}$, $J = \{i \in S \mid \lambda_i = 0\}$, $W_J(A)$ be the subgroup of $W(A)$ generated by $\{\sigma_i \mid i \in J\}$ and $G_J(A)$ be the parabolic subgroup of $G(A)$ corresponding to the Weyl group $W_J(A)$. Since $\{\sigma(\omega) \mid \sigma \in W(A)\} \cong W(A)/W_J(A)$ indexes the Schubert varieties of the generalized flag manifold $F_J(A) = G(A)/G_J(A)$, the lemma follows from the fact that the number of Schubert varieties in $F_J(A)$ is infinite for affine and indefinite type. For reference, see [Kumar 2002]. □

Corollary 2.4. *Let $f \in I(A)$ be a homogeneous invariant polynomial and $\omega \neq 0$ be in the Tits cone. If $\omega \mid f$, then $f = 0$.*

Proof. If $\omega \mid f$, then for any $\sigma \in W(A)$, $\sigma(\omega) \mid \sigma(f) = f$. Since $\{\sigma(\omega) \mid \sigma \in W(A)\}$ is an infinite set, if $f \neq 0$, this contradicts the condition that the degree of f is finite. □

Lemma 2.5. *For a Cartan matrix A, $I^1(A) = \{0\}$.*

Proof. Suppose $f = \sum_{i=1}^{n} \lambda_i \omega_i \in I^1(A)$; then for each j, $\sigma_j(f) = f - \lambda_j \alpha_j = f$. Since $\alpha_j \neq 0$, $\lambda_j = 0$. Hence $f = 0$. □

Lemma 2.6. *For a Cartan matrix A, we may write any homogeneous degree 2 polynomial f in the form $\sum_{i,j=1}^{n} \lambda_{ij} \omega_i \omega_j$, where $\lambda_{ij} = \lambda_{ji}$. Then $f \in I^2(A)$ if and only if $\partial f / \partial \omega_j = \frac{1}{2}(\partial^2 f / \partial \omega_j^2) \alpha_j$ for $1 \leq j \leq n$; that is, $2\lambda_{ij} = a_{ij} \lambda_{jj}$ for $1 \leq i, j \leq n$.*

Proof. If f is an invariant polynomial, then for each j,

$$\sigma_j(f) = f(\omega_1, \ldots, \omega_j - \alpha_j, \ldots, \omega_n) = f(\omega) - \frac{\partial f}{\partial \omega_j} \alpha_j + \frac{1}{2} \frac{\partial^2 f}{\partial \omega_j^2} \alpha_j^2 = f.$$

This is equivalent to $\dfrac{\partial f}{\partial \omega_j} = \dfrac{1}{2} \dfrac{\partial^2 f}{\partial \omega_j^2} \alpha_j$. That is, $2 \sum_{i=1}^{n} \lambda_{ij} \omega_i = \lambda_{jj} \alpha_j = \lambda_{jj} \sum_{i=1}^{n} a_{ij} \omega_i$; i.e., $2\lambda_{ij} = a_{ij} \lambda_{jj}$. □

Lemma 2.6 can be generalized:

Lemma 2.7. *Let A be a Cartan matrix. Then f is a degree l invariant polynomial if and only if for $1 \leq j \leq n$,*

$$\frac{\partial f}{\partial \omega_j} - \frac{1}{2!} \frac{\partial^2 f}{\partial \omega_j^2} \alpha_j + \cdots + (-1)^l \frac{1}{l!} \frac{\partial^l f}{\partial \omega_j^l} \alpha_j^{l-1} = 0. \tag{1}$$

Lemma 2.8. *An $n \times n$ Cartan matrix $A = \{a_{ij}\}$ is symmetrizable if and only if there exist nonzero $d_1, d_2, \ldots, d_n$ such that $a_{ij} d_j = a_{ji} d_i$ for all i, j.*

Proof. Suppose A is symmetrizable. Then there exist an invertible diagonal matrix $D = \mathrm{diag}(d_1, \dots, d_n)$ and a symmetric matrix B such that $A = DB$; that is, $a_{ij} = d_i b_{ij}$ for all i, j. So $a_{ij}/d_i = b_{ij} = b_{ji} = a_{ji}/d_j$. Equivalently, $a_{ij}d_j = a_{ji}d_i$.

If there exist nonzero $d_1, d_2, \dots, d_n$ such that $a_{ij}d_j = a_{ji}d_i$ for all i, j, let $D = \mathrm{diag}(d_1, \dots, d_n)$ and $B = (b_{ij})_{n\times n} = (a_{ij}/d_i)_{n\times n}$; then, $A = DB$. Therefore A is symmetrizable. □

Corollary 2.9. *If A is an indecomposable Cartan matrix, then* $\dim I^2(A) = 1$ *or* 0 *depending on whether A is symmetrizable or not. And if A is symmetrizable, then $I^2(A)$ is spanned by the invariant bilinear form ψ, which is unique up to a constant.*

Proof. If $\dim I^2(A) > 0$, choose $f(\omega) = \sum_{i,j=1}^n \lambda_{ij}\omega_i\omega_j \in I^2(A)$, $f(\omega) \neq 0$. By permuting the simple roots, we can assume that there exists an integer $k > 0$ such that $\lambda_{11}, \dots, \lambda_{kk} \neq 0$ but $\lambda_{k+1,k+1}, \dots, \lambda_{nn} = 0$. If $i \leq k$ and $j > k$, by Lemma 2.6 $2\lambda_{ij} = a_{ij}\lambda_{jj}$, therefore $\lambda_{ij} = 0$. By $0 = 2\lambda_{ij} = 2\lambda_{ji} = a_{ji}\lambda_{ii}$, we get $a_{ji} = 0$ for all $i \leq k$, $j > k$. Since A is indecomposable, k must equal n. Let $d_i = \lambda_{ii}$ for $1 \leq i \leq n$; then $a_{ij}d_j = a_{ji}d_i$ for all i, j. This means that A is symmetrizable.

If $\dim I^2(A) = 0$, then A is nonsymmetrizable (if A is symmetrizable, then the Killing form gives an element in $I^2(A)$).

Since A is indecomposable, for all i, j, the ratios $\lambda_{ij} : \lambda_{jj}$ and $\lambda_{ii} : \lambda_{jj}$ are determined by A. Therefore, if $\dim I^2(A) > 0$, then $\dim I^2(A) = 1$. □

Below, for an indecomposable and symmetrizable Cartan matrix A, we always fix a nonzero $\psi \in I^2(A)$.

3. Rings of polynomial invariants of Weyl groups, $n = 2$ case

A 2×2 Cartan matrix is of the form

$$A_{a,b} := \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

We say that $A_{a,b}$ is of finite, affine or indefinite type if $ab < 4$, $ab = 4$ or $ab > 4$, respectively. The action of reflections $\sigma_1, \sigma_2 \in W(A)$ on h^* is given by

$$\sigma_1(\omega_1) = -\omega_1 + b\omega_2, \quad \sigma_1(\omega_2) = \omega_2, \quad \sigma_2(\omega_1) = \omega_1, \quad \sigma_2(\omega_2) = -\omega_2 + a\omega_1.$$

Lemma 3.1. *The Weyl group $W_{a,b}$ of a Cartan matrix $A_{a,b}$ is the dihedral group D_m, where $m = 2, 3, 4, 6$ or ∞ when $ab = 0, 1, 2, 3$ or ≥ 4, respectively. If $A_{a,b}$ is of affine or indefinite type, then the ring of polynomial invariants of the Weyl group $W_{a,b}$ is $I(A_{a,b}) = \mathbb{Q}[\psi]$.*

Proof. For a Cartan matrix $A_{a,b}$ of affine or indefinite type, $ab \neq 0$. Since A is indecomposable and symmetrizable, $\dim I^2(A) = 1$ and $I^2(A)$ is spanned by $\psi = a\omega_1^2 - ab\omega_1\omega_2 + b\omega_2^2$. Suppose $f(\omega) = \sum_{i=0}^l \lambda_i \omega_1^i \omega_2^{l-i}$ is a degree l homogeneous invariant polynomial; then

$$\begin{aligned}\sigma_2(f) &= \sum_{i=0}^{l} \lambda_i \omega_1^i (-\omega_2 + a\omega_1)^{l-i} \\ &= \sum_{i=0}^{l} \sum_{j=0}^{l-i} (-1)^j \lambda_i \binom{l-i}{j} \omega_1^i \omega_2^j (a\omega_1)^{l-i-j} \\ &= \sum_{j=0}^{l} \sum_{i=0}^{l-j} (-1)^j \lambda_i \binom{l-i}{j} a^{l-i-j} \omega_1^{l-j} \omega_2^j \\ &= \sum_{j=0}^{l} \left(\sum_{i=0}^{j} (-1)^{l-j} \lambda_i \binom{l-i}{l-j} a^{j-i} \right) \omega_1^j \omega_2^{l-j}.\end{aligned}$$

So $\sigma_2(f) = f$ is equivalent to

$$\lambda_j = \sum_{i=0}^{j} (-1)^{l-j} \lambda_i \binom{l-i}{l-j} a^{j-i}, \quad 0 \le j \le l. \tag{2}$$

Letting $j = 0$, we get $\lambda_0 = (-1)^l \lambda_0$. So $\lambda_0 = 0$ or l is even.

(1) If $\lambda_0 = 0$, then $\omega_1 \mid f$. By Corollary 2.4, $f = 0$.

(2) If l is even, suppose $l = 2m$. There exists a constant λ such that $f - \lambda\psi^m$ is an invariant polynomial and $\omega_1 \mid (f - \lambda\psi^m)$, hence $f = \lambda\psi^m$.

This proves the lemma. □

4. Some results about hyperbolic Cartan matrices

Moody [1978] proved that for each indecomposable and symmetrizable hyperbolic Cartan matrix A, the ring of polynomial invariants $I(A)$ is $\mathbb{Q}[\psi]$, where ψ is the invariant symmetric bilinear form. So in this section we only consider nonsymmetrizable Cartan matrices.

Indecomposable $n \times n$ hyperbolic Cartan matrices exist only for $n \le 10$, and their number is finite for $3 \le n \le 10$. There are lists of hyperbolic Cartan matrices in [Wan 1991] and [Carbone et al. 2010].

Lemma 4.1. *Let A be an indecomposable and nonsymmetrizable hyperbolic Cartan matrix with $n \ge 4$.*

(C1) *The Dynkin diagram of A forms a circle. That is, $a_{ij} \neq 0$ if and only if $|i - j| = 0, 1$ or $n - 1$.*

(C2) *If $|i - j| = 1$ or $n - 1$, then $a_{ij} = -1$ or $a_{ji} = -1$.*

The lemma is proved by direct checking of the lists.

Remark 4.2. The lemma is not true for the case $n = 3$.

Lemma 4.3. *Let A be an indecomposable and nonsymmetrizable hyperbolic Cartan matrix with $n = 3$. Then A contains a 2×2 principal submatrix of affine type, or all the 2×2 principal submatrices of A are of finite type. In the latter case, A satisfies conditions (C1) and (C2).*

Feingold and Nicolai [2004] proved the following theorem:

Theorem 4.4. *Let $g(A)$ be the Kac–Moody Lie algebra associated to a symmetrizable Cartan matrix $A = (a_{ij})_{n\times n}$ which is generated by $\alpha_i^\vee$, e_i, f_i, $1 \le i \le n$, and let $\beta_1, \dots, \beta_m$ be a set of positive real roots of $g(A)$ such that $\beta_i - \beta_j$ is not a root for $1 \le i \ne j \le m$. Let E_i, F_i be root vectors in the one-dimensional root spaces corresponding to the positive real roots β_i and negative real roots $-\beta_i$, respectively, and let $H_i = [E_i, F_i]$. Then the Lie subalgebra of g generated by $\{E_i, F_i, H_i \mid 1 \le i \le m\}$ is a regular Kac–Moody subalgebra with Cartan matrix $B = (b_{ij})_{n\times n} = (2(\beta_j, \beta_i)/(\beta_i, \beta_i))_{n\times n}$.*

In the formulas below, subscripts not in the range from 1 to n are to be taken modulo n in that range. By using the ideas in the theorem of Feingold and Nicolai, we can prove the following lemma:

Lemma 4.5. *Let A be an $n \times n$ Cartan matrix satisfying conditions (C1) and (C2) in Lemma 4.1, with simple root system $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, then $\beta_i = \alpha_{i+1} + \alpha_{i+2}$, $1 \le i \le n$ is a set of positive real roots of $g(A)$, and $\beta_i - \beta_j$ for $i \ne j$ are not roots. Let $\alpha_i^\vee$, e_i, f_i, $1 \le i \le n$ be the generators of $g(A)$, $E_i = [e_{i+1}, e_{i+2}]$, $F_i = -[f_{i+1}, f_{i+2}]$ and $H_i = [E_i, F_i]$. Then H_i, E_i, F_i, $1 \le i \le n$ generate a full rank regular Kac–Moody subalgebra with simple root system $\{\beta_1, \beta_2, \dots, \beta_n\}$ and Cartan matrix $B = (b_{ij}) = (\beta_j(H_i))_{n\times n}$.*

Proof. For $\beta_i = \alpha_{i+1} + \alpha_{i+2}$,

$$
\begin{aligned}
H_i &= [E_i, F_i]\\
&= -[[e_{i+1}, e_{i+2}], [f_{i+1}, f_{i+2}]]\\
&= -\big[[[e_{i+1}, e_{i+2}], f_{i+1}], f_{i+2}\big] - [f_{i+1}, [[e_{i+1}, e_{i+2}], f_{i+2}]]\\
&= \big[[[f_{i+1}, e_{i+1}], e_{i+2}], f_{i+2}\big] + [[e_{i+1}, [f_{i+1}, e_{i+2}]], f_{i+2}]\\
&\quad + [f_{i+1}, [[f_{i+2}, e_{i+1}], e_{i+2}]] + \big[f_{i+1}, [e_{i+1}, [f_{i+2}, e_{i+2}]]\big]\\
&= -[[\alpha_{i+1}^\vee, e_{i+2}], f_{i+2}] - [f_{i+1}, [e_{i+1}, \alpha_{i+2}^\vee]]\\
&= -(a_{i+1,i+2}\alpha_{i+2}^\vee + a_{i+2,i+1}\alpha_{i+1}^\vee).
\end{aligned}
$$

Then $[H_i, E_i] = -2(a_{i+1,i+2} + a_{i+2,i+1} + a_{i+1,i+2}a_{i+2,i+1})E_i$.

Since, for each $1 \le i \le n$,

$$-2(a_{i+1,i+2}+a_{i+2,i+1}+a_{i+1,i+2}a_{i+2,i+1}) = 2(1-(a_{i+1,i+2}+1)(a_{i+2,i+1}+1)) = 2,$$

a routine check shows that $\beta_j(H_i) \le 0$. Therefore the matrix B with $b_{ij} = \beta_j(H_i)$ is a Cartan matrix. The Serre relations for H_i, E_i, F_i, $1 \le i \le n$ from B are checked as in [Feingold and Nicolai 2004], so H_i, E_i, F_i, $1 \le i \le n$ generate a Kac–Moody Lie algebra $g(B)$ inside $g(A)$. □

We can't say Lemma 4.5 is a corollary of the previous theorem, since we don't know whether the theorem is true for nonsymmetrizable Cartan matrices A. So we must prove Lemma 4.5 by direct computation.

Corollary 4.6. *Let A be an $n \times n$ indecomposable and nonsymmetrizable hyperbolic Cartan matrix. Then in $g(A)$ there is a full rank indecomposable and nonhyperbolic regular indefinite Kac–Moody subalgebra $g(B)$.*

This corollary is proved by using Lemma 4.5 and checking the list of indecomposable, nonsymmetrizable hyperbolic Cartan matrices. We have written a computer program to do the checking. The computation results show that except for the hyperbolic Lie algebras labeled 131, 132, 133, 137, 139, 141 in the list in [Carbone et al. 2010], all the subalgebras we constructed are nonsymmetrizable.

Below is a simple example:

Example 4.7. For the hyperbolic Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix},$$

we obtain a regular subalgebra $g(B)$ of $g(A)$ with simple roots

$$\beta_1 = \alpha_2 + \alpha_3, \quad \beta_2 = \alpha_3 + \alpha_1, \quad \beta_3 = \alpha_1 + \alpha_2,$$

and its Cartan matrix is

$$B = \begin{pmatrix} 2 & -2 & -2 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

It is nonsymmetrizable and indefinite, but nonhyperbolic.

5. Proof of the main theorem

Some preparatory lemmas. Let A be an $n \times n$ Cartan matrix and $S = \{1, 2, \dots, n\}$. For $J \subset S$, let A_J be the principal submatrix $(a_{ij})_{i,j \in J}$ corresponding to J. Then $A' = A_{S-\{n\}}$ is the upper-left $(n-1) \times (n-1)$ principal submatrix of A. Let h' be the subspace of h spanned by $\alpha_1^\vee, \dots, \alpha_{n-1}^\vee$ and h'^* the subspace of h^* spanned by $\omega_1, \dots, \omega_{n-1}$; then $h = h' \oplus \mathbb{R}\alpha_n^\vee$ and $h^* = h'^* \oplus \mathbb{R}\omega_n$. Let $\alpha_i \in h^*$, $1 \le i \le n$ and $\alpha_i' \in h'^*$, $1 \le i \le n-1$ be the simple roots of Cartan matrices A and A'

respectively, and σ_i, $1 \le i \le n$ and σ_i', $1 \le i \le n-1$ be the Weyl reflections in h^* and h'^* respectively. For $1 \le i \ne j \le n-1$, $\alpha_i = \alpha_i' + a_{ni}\omega_n$, $\sigma_i(\omega_j) = \sigma_i'(\omega_j)$, and $\sigma_i(\omega_i) = \omega_i - \alpha_i = \sigma_i'(\omega_i) - a_{ni}\omega_n$.

Lemma 5.1. *Let $\omega = (\omega_1, \dots, \omega_{n-1}, \omega_n)$ and $\omega' = (\omega_1, \dots, \omega_{n-1})$. If $f(\omega)$ is a degree l invariant polynomial under the action of $\sigma_1, \dots, \sigma_{n-1}$ and $f(\omega) = \sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$, with each $f_i(\omega')$ a degree i homogeneous polynomial in $S(h'^*)$, then $f_l(\omega')$ is invariant under the action of $\sigma_1', \dots, \sigma_{n-1}'$.*

Proof. For $k \ne n$,

$$\begin{aligned} f(\omega) = \sigma_k(f(\omega)) &= \sum_{i=0}^{l} \sigma_k(f_i(\omega'))\omega_n^{l-i} \\ &= \sum_{i=0}^{l} f_i(\sigma_k(\omega_1), \sigma_k(\omega_2), \dots, \sigma_k(\omega_k), \dots, \sigma_k(\omega_{n-1}))\omega_n^{l-i} \\ &= \sum_{i=0}^{l} f_i(\sigma_k'(\omega_1), \sigma_k'(\omega_2), \dots, \sigma_k'(\omega_k) - a_{nk}\omega_n, \dots, \sigma_k'(\omega_{n-1}))\omega_n^{l-i}. \end{aligned}$$

Setting $\omega_n = 0$, we get $f_l(\omega') = f_l(\sigma_k'(\omega')) = \sigma_k'(f_l(\omega'))$. □

Corollary 5.2. *If $f(\omega) = \sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i} \in I(A)$, then $f_l(\omega') \in I(A')$.*

Lemma 5.3. *If the degree l polynomial $f(\omega)$ is invariant under the action of $\sigma_1, \dots, \sigma_{n-1}$ and $f(\omega) = \sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$, then for $k \ne n$ and $0 \le i \le l$,*

$$\sigma_k'(f_i(\omega')) = \sum_{j=0}^{l-i} \frac{(-a_{nk})^j}{j!} \frac{\partial^j f_{i+j}(\omega')}{(\partial\omega_k)^j}. \tag{3}$$

Proof. Continuing the calculation of $f(w)$ as in Lemma 5.1, for $k \ne n$, we get

$$\begin{aligned} &\sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i} \\ &= f(\omega) = \sum_{i=0}^{l} f_i(\sigma_k'(\omega_1), \sigma_k'(\omega_2), \dots, \sigma_k'(\omega_k) - a_{nk}\omega_n, \dots, \sigma_k'(\omega_{n-1}))\omega_n^{l-i} \\ &= \sum_{i=0}^{l}\sum_{j=0}^{i} \frac{1}{j!}\frac{\partial^j f_i}{(\partial\omega_k)^j}(\sigma_k'(\omega_1), \sigma_k'(\omega_2), \dots, \sigma_k'(\omega_k), \dots, \sigma_k'(\omega_{n-1}))(-a_{nk}\omega_n)^j\omega_n^{l-i} \\ &= \sum_{i=0}^{l}\sum_{j=0}^{i} \frac{(-a_{nk})^j}{j!}\frac{\partial^j f_i}{(\partial\omega_k)^j}(\sigma_k'(\omega_1), \sigma_k'(\omega_2), \dots, \sigma_k'(\omega_k), \dots, \sigma_k'(\omega_{n-1}))\omega_n^{l-i+j} \\ &= \sum_{i=0}^{l}\left(\sum_{j=0}^{l-i} \frac{(-a_{nk})^j}{j!}\frac{\partial^j f_{i+j}}{(\partial\omega_k)^j}(\sigma_k'(\omega'))\right)\omega_n^{l-i}. \end{aligned}$$

By comparing the coefficients of ω_n^{l-i} in the two sides, we get

$$f_i(\omega') = \sum_{j=0}^{l-i} \frac{(-a_{nk})^j}{j!} \frac{\partial^j f_{i+j}}{(\partial\omega_k)^j}(\sigma_k'(\omega')).$$

Acting by σ_k' on both sides, we prove the lemma. □

In the following, given an $n \times n$ Cartan matrix A, we denote its upper-left $(n-1)\times(n-1)$ principal submatrix by A'.

Lemma 5.4. *Let A be an indefinite $n \times n$ Cartan matrix. If both A and A' are indecomposable and symmetrizable, then the restriction of the invariant symmetric bilinear form ψ to h' gives an invariant symmetric bilinear form ψ'.*

The proof is obvious by checking $\psi|_{h'} \neq 0$ and $\psi' = \psi|_{h'}$ is invariant under the action of $\sigma_1', \dots, \sigma_{n-1}'$.

Lemma 5.5. *Let f be a $W(A)$-invariant polynomial and $f(\omega) = \sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$. Then for $1 \leq j \leq l$,*

$$f_j(\omega') = \sum_{i=0}^{j} (-1)^{l-i} f_i(\omega')\binom{l-i}{l} - j{\omega_n'}^{j-i}, \tag{4}$$

where $\omega_n' = \sum_{j\neq n} a_{jn}\omega_j$.

Proof. We have $\sigma_n(\omega_n) = \omega_n - \alpha_n = -\omega_n - \sum_{j\neq n} a_{jn}\omega_j = -\omega_n - \omega_n'$, and

$$\begin{aligned}
f = \sigma_n(f) &= \sum_{i=0}^{l} f_i(\omega')\sigma_n(\omega_n^{l-i}) \\
&= \sum_{i=0}^{l} f_i(\omega')(-\omega_n - \sum_{j\neq n} a_{jn}\omega_j)^{l-i} \\
&= \sum_{i=0}^{l} f_i(\omega') \sum_{j=0}^{l-i} (-1)^{l-i}\binom{l-i}{j}\omega_n^j {\omega_n'}^{l-i-j} \\
&= \sum_{j=0}^{l}\Bigg[\sum_{i=0}^{l-j} (-1)^{l-i} f_i(\omega')\binom{l-i}{j}{\omega_n'}^{l-i-j}\Bigg]\omega_n^j \\
&= \sum_{j=0}^{l}\Bigg[\sum_{i=0}^{j} (-1)^{l-i} f_i(\omega')\binom{l-i}{l} - j{\omega_n'}^{j-i}\Bigg]\omega_n^{l-j}.
\end{aligned}$$

By comparing the coefficients of ω_n^{l-j}, we prove the lemma. □

Remark 5.6. In fact, equations (3) and (4) are just corollaries of (1) applied to $f(\omega)=\sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$.

Lemma 5.7. *Let f be a $W(A)$-invariant polynomial of degree l and let $f(\omega)=\sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$. If $l=2m$, then for each $i\le m-1$, there exist constants $a^i_j(l)$, $0\le j\le i$, depending on l, such that $f_{2i+1}=\sum_{j=0}^{i} a^i_j(l) f_{2(i-j)}\omega_n'^{2j+1}$. If $l=2m+1$, then $f_0=0$, and for each $i\le m$, there exist constants $b^i_j(l)$, $1\le j\le i$, such that $f_{2i}=\sum_{j=1}^{i} b^i_j(l) f_{2(i-j)+1}\omega_n'^{2j-1}$. And the coefficients $a^i_j(l)$ and $b^i_j(l)$ can be computed.*

Proof. Letting $j=0$ in (4), we get $f_0=(-1)^l f_0$. So there are two cases:

Case 1: l is even. Let $j=1$ in (4). Then $f_1=-f_1+\binom{l}{1} f_0\omega_n'$. That is,

$$f_1=\frac{1}{2}\binom{l}{1} f_0\omega_n'. \tag{5}$$

For $j=2$, we get $f_2=f_2-\binom{l-1}{1} f_1\omega_n'+\binom{l}{2} f_0\omega_n'^2$; equivalently, $f_1=\frac{1}{2}\binom{l}{1} f_0\omega_n'$. For $j=3$, we get

$$f_3=-f_3+\binom{l-2}{1} f_2\omega_n'-\binom{l-1}{2} f_1\omega_n'^2+\binom{l}{3} f_0\omega_n'^3.$$

Substituting (5), we get

$$f_3=\frac{1}{2}\binom{l-2}{1} f_2\omega_n'-\frac{1}{4}\binom{l}{3} f_0\omega_n'^3. \tag{6}$$

Continuing this procedure, we've proved the lemma when l is even.

Case 2: l is odd. Then $f_0=0$, and the proof is similar to the previous case. □

Corollary 5.8. *Let f be a $W(A)$-invariant polynomial and $f(\omega)=\sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$. Then $\omega_n' \mid f_{l-1}(\omega')$.*

Computation motivates us to make the following conjecture:

Conjecture. If l is even, then equation (4) for $j=2k$ can be derived from the set of equations for $j=0,1,2,\dots,2k-1$. If l is odd, Equation (4) for $j=2k-1$ can be derived from the set of equations for $j=0,1,2,\dots,2k-2$.

The conjecture is verified for $k\le 3$.

Lemma 5.9. *Let A be an $n\times n$ Cartan matrix. If $f(\omega), g(\omega)\in S(h^*)$ satisfy $\sigma_k(f(\omega))-f(\omega)=\sigma_k(g(\omega))-g(\omega)$ for each $1\le k\le n$, then $f-g\in I(A)$.*

The proof is trivial.

Lemma 5.10. *Let A be an indefinite $n\times n$ Cartan matrix and A' its upper-left $(n-1)\times(n-1)$ principal submatrix. If the ring of $W(A')$ polynomial invariants $I(A')$ is equal to $\mathbb{Q}[\psi']$ and $l=2m$, then for each $W(A)$-invariant polynomial*

$f(\omega)=\sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$ of degree l, there exists a constant k such that $f_l(\omega')=k\psi'^m$ and $f_{l-1}(\omega')=km\psi'^{m-1}\omega_n^*$, where $\omega_n^*=\sum_{k\neq n}\lambda_{kk}a_{nk}\omega_k$.

Proof. By Corollary 5.2, $f_l(\omega')\in I(A')$. Since $I(A')=\mathbb{Q}[\psi']$, there exists k such that $f_l(\omega')=k\psi'^m$. In (3), letting $j=l-1$, we get for $1\le k\le n-1$ that

$$\sigma_k'(f_{l-1}(\omega'))-f_{l-1}(\omega')=-a_{nk}\frac{\partial f_l}{\partial\omega_k}.$$

Let $g(\omega')=km\psi'^{m-1}\omega_n^*$; then it is easy to check

$$-a_{nk}\frac{\partial f_l}{\partial\omega_k}=\sigma_k'(g(\omega'))-g(\omega'),$$

so $\sigma_k'(f_{l-1}(\omega'))-f_{l-1}(\omega')=\sigma_k'(g(\omega'))-g(\omega')$. Applying Lemma 5.9 to f_{l-1}, g for the Cartan matrix A', we get $f_{l-1}(\omega')-g(\omega')\in I^{l-1}(A')$. But $I^{l-1}(A')=I^{2m-1}(A')=\{0\}$, hence $f_{l-1}(\omega')=g(\omega')$. □

Proof of three propositions.

Proposition 5.11. *Let A be an $n\times n$ indecomposable and indefinite Cartan matrix. If $I(A')=\mathbb{Q}$, then $I(A)=\mathbb{Q}$.*

Proof. Let f be a $W(A)$-invariant polynomial and $f(\omega)=\sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$. Then by Corollary 5.2, $f_l(\omega')\in I(A')$, so $f_l(\omega')=0$.

For $i=l-1$, Equation (3) is

$$\sigma_k'(f_{l-1}(\omega'))=f_{l-1}(\omega')-a_{nk}\frac{\partial f_l}{\partial\omega_k}(\omega').$$

Substituting $f_l(\omega')=0$ in the above equation, we get $\sigma_k'(f_{l-1}(\omega'))=f_{l-1}(\omega')$, so $f_{l-1}(\omega')=0$. Continuing this procedure, we show that $f_i(\omega')=0$ for all $i>0$ and f_0 is a constant. Hence $f(\omega)=f_0\omega_n^l$. By Corollary 2.4, $f=0$. □

Proposition 5.12. *Let A be an $n\times n$ symmetrizable and indefinite Cartan matrix. If $I(A')=\mathbb{Q}[\psi']$, then $I(A)=\mathbb{Q}[\psi]$.*

Proof. Let f be a $W(A)$-invariant polynomial and $f(\omega)=\sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$. Then by Corollary 5.2, $f_l(\omega')\in I(A')$, so $f_l(\omega')=0$ or there exists $\lambda\neq 0$ such that $f_l=\lambda\psi'^m$. If $f_l=0$, then $\omega_n\mid f$, so $f=0$. If $f_l=\lambda\psi'^m$, then by Lemma 5.4 we can assume $\psi|_{h'}=\psi'$, so $f-\lambda\psi^m$ is a $W(A)$-invariant polynomial and $\omega_n\mid(f-\lambda\psi^m)$. Hence $f=\lambda\psi^m$. □

Proposition 5.13. *Let A be an $n\times n$ indecomposable and nonsymmetrizable Cartan matrix. If A' is symmetrizable and $I(A')=\mathbb{Q}[\psi']$, then $I(A)=\mathbb{Q}$.*

Proof. Let f be a $W(A)$-invariant polynomial and $f(\omega)=\sum_{i=0}^{l} f_i(\omega')\omega_n^{l-i}$. Suppose $\psi'=\sum_{i,j=1}^{n-1}\lambda_{ij}\omega_i\omega_j$ with $\lambda_{ij}=\lambda_{ji}$ for all $1\le i,j\le n-1$.

If l is even, suppose $l=2m$. We prove $f_l=0$ first. Suppose $f_l\neq 0$; then by Lemma 5.10, there exists $k \neq 0$ such that $f_l = k\psi'^m$ and $f_{l-1} = km\psi'^{m-1}\omega_n^*$. By Corollary 5.8, $\omega_n' \mid f_{l-1} = km\psi'^{m-1}\omega_n^*$. So $\omega_n' \mid \psi'$ or $\omega_n' \mid \omega_n^*$. Since ψ' is $W(A)$-invariant and $-\omega_n'$ is in the Tits cone, by Corollary 2.4 $\omega_n' \mid \psi'$ is impossible. Therefore $\omega_n' \mid \omega_n^*$. Because A is indecomposable, both ω_n' and ω_n^* are not 0. Therefore there exists a constant $d_n \neq 0$ such that $\omega_n^* = d_n\omega_n'$. But $\omega_n' = \sum_{j\neq n} a_{jn}\omega_j$ and $\omega_n^* = \sum_{j\neq n} \lambda_{jj}a_{nj}\omega_j$, so $a_{jn}d_n = a_{nj}\lambda_{jj}$. Let $d_i = \lambda_{ii}$, $1 \leq i \leq n-1$; since A' is symmetrizable, by Lemma 2.8 we know $a_{ij}d_j = a_{ji}d_i$ for all $i, j \leq n-1$. Combining with $a_{jn}d_n = a_{nj}\lambda_{jj}$, we get $a_{ij}d_j = a_{ji}d_i$ for all $i, j \leq n$. This shows A is symmetrizable, contradicting our assumption, so $f_l = 0$.

If l is odd, then $f_l \in I^l(A')$ also implies $f_l = 0$.

If $f_l = 0$, then the remaining procedure of the proof is similar to the proof of Proposition 5.12. □

Proof of the main theorem. To prove the main theorem we need the following lemma:

Lemma 5.14. *Let A be a nonhyperbolic, indecomposable and indefinite Cartan matrix. Then there exists an integer k, $1 \leq k \leq n$ such that $A_{S-\{k\}}$ is an indecomposable and indefinite Cartan matrix.*

Proof. Since the Cartan matrix A is nonhyperbolic, there exists an integer k, $1 \leq k \leq n$ such that $A_{S-\{k\}}$ is indefinite. If $A_{S-\{k\}}$ is indecomposable, the lemma is proved. If $A_{S-\{k\}}$ is decomposable, then the Dynkin diagram of $A_{S-\{k\}}$ is split into r connected subdiagrams $\Gamma_1, \dots, \Gamma_r$, with $r > 1$, and there is an s_0, $1 \leq s_0 \leq r$, such that the principal submatrix corresponding to Γ_{s_0} is indefinite. Since A is indecomposable, the simple root α_k is connected to all Γ_s, $1 \leq s \leq r$.

We find a connected subdiagram Γ_s, $s \neq s_0$. There must be a vertex $\alpha_{k'}$ of Γ_s such that the subdiagram $\Gamma_s - \{\alpha_{k'}\}$ is connected (note that we can choose a vertex α from a connected finite graph Γ and the resulted subgraph $\Gamma - \{\alpha\}$ is still connected). It is obvious that $A_{S-\{k'\}}$ is indecomposable and indefinite. □

Now we can prove the main theorem:

Theorem. *Let A be an $n \times n$ indecomposable and indefinite Cartan matrix A. If A is symmetrizable, then $I(A) = \mathbb{Q}[\psi]$; if A is nonsymmetrizable, then $I(A) = \mathbb{Q}$.*

Proof. We prove this theorem by induction on n. For $n = 2$, this is Lemma 3.1.

Suppose this theorem is true for all $(n-1) \times (n-1)$ indecomposable and indefinite Cartan matrices.

For an $n \times n$ indecomposable and indefinite Cartan matrix A, if A is not hyperbolic, then by Lemma 5.14 we can find an $(n-1) \times (n-1)$ principal submatrix A' which is both indecomposable and indefinite. Without loss of generality, we can assume A' is the upper-left $(n-1) \times (n-1)$ principal submatrix.

Then by considering the symmetrizability of A' and A, there are three cases:

(1) Both A' and A are nonsymmetrizable.

(2) Both A' and A are symmetrizable.

(3) A' is symmetrizable and A is nonsymmetrizable.

The proof for these three cases are dealt with by combining the induction assumption and Proposition 5.11, Proposition 5.12, and Proposition 5.13 respectively.

So the theorem has been proven when A is nonhyperbolic. For the hyperbolic case, if A is symmetrizable, the proof is given in [Moody 1978]; if A is nonsymmetrizable, it is proved in the following:

Proposition 5.15. *For an $n \times n$ indecomposable, nonsymmetrizable hyperbolic Cartan matrix A, $I(A) = \mathbb{Q}$.*

Proof. For a Cartan matrix A with $n \geq 4$, by Corollary 4.6 we can find an $n \times n$ indecomposable, nonhyperbolic and indefinite Cartan matrix B such that the root system associated to B is a sub-root system of the root system associated to A, and the Weyl group $W(B)$ is a subgroup of $W(A)$. Therefore $I(A) \subset I(B)$.

If B is nonsymmetrizable, then by combining Lemma 5.14, Proposition 5.11 or Proposition 5.13 and the same induction procedure, we can prove $I(B) = \mathbb{Q}$. Hence $I(A) = \mathbb{Q}$.

If B is symmetrizable, then by combining Lemma 5.14 and Proposition 5.12, we prove $I(B) = \mathbb{Q}[\psi_B]$. To prove $I(A) = \mathbb{Q}$, it is sufficient to show the ψ_B^m, $m \geq 1$ are not $W(A)$ invariants.

Suppose ψ_B^m is a $W(A)$-invariant polynomial. If m is odd, we get $\psi_B = (\psi_B^m)^{1/m}$ is $W(A)$-invariant. If m is even, similarly we get for each $\sigma \in W(A)$ that $\sigma(\psi_B) = \psi_B$ or $-\psi_B$. But $\sigma(\psi_B) = -\psi_B$ is impossible (a symmetric bilinear form $\psi = \sum_{i,j=1}^{n} \lambda_{ij}\omega_i\omega_j$ with all the λ_{ii}, $1 \leq i \leq n$ having the same sign can't be linearly transformed to $-\psi$). So we get $\sigma(\psi_B) = \psi_B$. Therefore ψ_B is a $W(A)$-invariant polynomial. Since A is nonsymmetrizable, this is impossible. Hence $I(A) = \mathbb{Q}$.

For the $n = 3$ case, there are two possibilities. If A contains a 2×2 principal submatrix A' of affine type, then by combining Lemma 3.1 and Proposition 5.13, we show $I(A) = \mathbb{Q}$. If all the 2×2 principal submatrices of A are of finite type, then A satisfies conditions (C1) and (C2). So we can find an indecomposable, nonhyperbolic and indefinite Cartan matrix B such that $g(B)$ is a regular subalgebra of $g(A)$. By a similar method as for $n \geq 4$, we can also prove $I(A) = \mathbb{Q}$. This proves the proposition and with it the theorem. □

6. Applications to rational homotopy types of Kac–Moody groups and their flag manifolds of indefinite type

For the Kac–Moody Lie algebra $g(A)$, there is the Cartan decomposition $g(A) = h \oplus \sum_{\alpha\in\Delta} g_\alpha$, where h is the Cartan subalgebra and Δ is the root system of $g(A)$.

Let $b = h \oplus \sum_{\alpha \in \Delta^+} g_\alpha$ be the Borel subalgebra; then b corresponds to a Borel subgroup $B(A)$ in the Kac–Moody group $G(A)$. The homogeneous space $F(A) = G(A)/B(A)$ is called the flag manifold of $G(A)$. By [Kumar 2002], $F(A)$ is an ind-variety.

The cohomologies of Kac–Moody groups and their flag manifolds of finite and affine types are extensively studied. For reference see [Pontryagin 1935; Hopf 1941; Borel 1953a; 1953b; 1954; Bott and Samelson 1955; Bott 1956; Milnor and Moore 1965; Chevalley 1994]. But for the indefinite type, little is known.

The rational cohomology rings of Kac–Moody groups and their flag manifolds are also considered in [Kac 1985b] and [Kumar 1985]. The essentially new part of our work is that we study the properties of $P_A(q)$ and derive the explicit formula for i_k. For details see [Chunhua 2010; Chunhua and Xu-an 2012; Xu-an et al. 2013].

For a Kac–Moody group $G(A)$, $H^*(G(A))$ is a locally finite free graded commutative algebra over $\mathbb{Q}$. Let the odd-dimensional free generators of $H^*(G(A))$ be $y_1, \dots, y_l$ and the even-dimensional free generators be $z_1, \dots, z_k, \dots$. By [Kac 1985b; Kitchloo 1998], $l < n$. Denote the number of degree k generators of $H^*(G(A))$ by i_k; then the Poincaré series of $G(A)$ is

$$P_G(q) = \prod_{k=1}^{\infty} \frac{(1-q^{2k-1})^{i_{2k-1}}}{(1-q^{2k})^{i_{2k}}}.$$

The Poincaré series $P_G(q)$ determines the isometry type of the cohomology ring $H^*(G(A))$ and the rational homotopy type of $G(A)$.

Let $BB(A)$ be the classifying space of the Borel subgroup $B(A)$ and $j : F(A) \to BB(A)$ the classifying map of the principal $B(A)$-bundle $\pi : G(A) \to F(A)$. Denote the cohomology generators of $H^*(BB(A))$ by $\omega_1, \dots, \omega_n$, $\deg \omega_i = 2$. A routine computation on the Leray–Serre spectral sequences of the fibration $G(A) \xrightarrow{\pi} F(A) \xrightarrow{j} BB(A)$ shows

$$H^*(F(A)) \cong E_3^{*,*} \cong \mathbb{Q}[\omega_1, \dots, \omega_n]/\langle f_j \mid 1 \le j \le l \rangle \otimes \mathbb{Q}[z_1, \dots, z_k, \dots],$$

where each f_j corresponds to the differential of y_j and the collection of such f_j generates the ring $I(A)$ of $W(A)$ polynomial invariants.

By [Xu-an et al. 2013], there is the following theorem:

Theorem 6.1. *Let $P_A(q)$ be the Poincaré series of a flag manifold $F(A)$. Then the sequence $i_2 - i_1, i_4 - i_3, \dots, i_{2k} - i_{2k-1}, \dots$ can be derived from $P_A(q)$. In fact we can recover $P_A(q)$ from the sequence $i_2 - i_1, i_4 - i_3, \dots, i_{2k} - i_{2k-1}, \dots$.*

But to determine the rational homotopy type of $G(A)$, we need to determine the sequence $i_1, i_2, \dots, i_k, \dots$. So in addition to the Poincaré series $P_A(q)$, we need more ingredients. Note the number of generators of $I(A)$ of degree k is just the integer i_{2k-1}. So if we can determine the degrees of all the generators in $I(A)$, then we can determine the sequence $i_1, i_3, \dots, i_{2k-1}, \dots$. And the main theorem of this paper fills the gap. Now we have:

Theorem 6.2. *For an indecomposable and indefinite Cartan matrix A, $i_{2k-1} = 0$ for all $k > 0$ except for $k = 2$. And for $k = 2$, if A is symmetrizable, $i_3 = 1$; if A is nonsymmetrizable, $i_3 = 0$.*

Setting $\epsilon(A) = 1$ or 0 depending on whether A is symmetrizable or not as in [Kac 1985b], we get:

Theorem 6.3. *The sequence $i_1, i_2, i_3, \dots, i_k, \dots$ is determined by the Poincaré series $P_A(q)$ and $\epsilon(A)$.*

Theorem 6.4. *For an indecomposable and indefinite Cartan matrix A, the rational homotopy types of $G(A)$ are determined by the Poincaré series $P_A(q)$ and $\epsilon(A)$.*

Kumar [1985] proved that for a Kac–Moody Lie algebra $g(A)$, the Lie algebra cohomology $H^*(g(A), \mathbb{C})$ is given by $H^*(G(A)) \otimes \mathbb{C}$. So we also computed the cohomology of a Kac–Moody Lie algebra $g(A)$ with trivial coefficient.

For a Kac–Moody group $G(A)$, $i_1 = i_2 = 0$. And we have:

Corollary 6.5. *For an indecomposable and nonsymmetrizable indefinite Cartan matrix A, $G(A)$ is a 3-connected space.*

Corollary 6.6. *The dimension of the odd rational homotopy group $\pi_{\mathrm{odd}}(G(A))$ of an indefinite Kac–Moody group $G(A)$ is 1 or 0 depending on whether A is symmetrizable or not.*

Theorem 6.7. *For an indecomposable and indefinite Cartan matrix A, if A is symmetrizable, then*

$$H^*(G(A)) \cong \Lambda_{\mathbb{Q}}(y_3) \otimes \mathbb{Q}[z_1, \dots, z_k, \dots]$$

and

$$H^*(F(A)) \cong \mathbb{Q}[\omega_1, \dots, \omega_n]/\langle\psi\rangle \otimes \mathbb{Q}[z_1, \dots, z_k, \dots].$$

If A is nonsymmetrizable, then

$$H^*(G(A)) \cong \mathbb{Q}[z_1, \dots, z_k, \dots]$$

and

$$H^*(F(A)) \cong \mathbb{Q}[\omega_1, \dots, \omega_n] \otimes \mathbb{Q}[z_1, \dots, z_k, \dots],$$

where $\deg z_k \geq 4$ is even for all k and can be determined from the Poincaré series $P_A(q)$ and $\epsilon(A)$.

Note that the Poincaré series $P_A(q)$ can be computed easily by an inductive procedure. See [Chunhua 2010; Chunhua and Xu-an 2012] for details. So in principle the computation of rational homotopy types is solved for all indecomposable and indefinite Kac–Moody groups, whether they are symmetrizable or not.

Since Kac–Moody groups and their flag manifolds are products of indecomposable Kac–Moody groups and indecomposable Kac–Moody flag manifolds, by combining the known results for finite and affine types, we have determined the rational homotopy types of all Kac–Moody groups and their flag manifolds. Since $G(A)$ and $F(A)$ are rational formal (see [Sullivan 1977; Kumar 2002]), the rational homotopy groups and rational minimal model of the corresponding Kac–Moody group $G(A)$ and its flag manifold $F(A)$ can be directly computed from Theorem 6.7.

Theorem 6.8. *For an $n \times n$ indecomposable and indefinite Cartan matrix A satisfying $a_{ij}a_{ji} \geq 4$ for all $1 \leq i, j \leq n$, the rational homotopy type of $G(A)$ is determined by $\epsilon(A)$.*

Since there are a large number of Cartan matrices satisfying the condition of Theorem 6.8, this assertion may seem very surprising. But the proof is very simple. It is derived from the equality

$$P_A(q) = \frac{1+q}{1-(n-1)q}.$$

See [Chunhua 2010; Chunhua and Xu-an 2012] for explicit computations of $P_A(q)$.

It deserves to be mentioned that for a 3×3 nonsymmetrizable Cartan matrix A with $a_{ij}a_{ji} \geq 4$ for all i, j, the Kac–Moody group $G(A)$ is a 5-connected space.

For an indecomposable and symmetrizable Cartan matrix A, let p, q, r be the dimensions of the positive, negative and zero vector subspaces of the invariant bilinear form ψ, and set $\tau(A) = (p, q, r)$.

Theorem 6.9. *For an indecomposable and indefinite Cartan matrix A, if A is symmetrizable, then the cohomology ring $H^*(F(A), \mathbb{C})$ is determined by $P_A(q)$ and $\tau(A)$. If $g(A)$ is nonsymmetrizable, then the cohomology ring $H^*(F(A), \mathbb{C})$ is determined by $P_A(q)$.*

This is obtained from the Theorem 6.7 and the classification of real quadratic forms.

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ZHAO XU-AN
DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
KEY LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS, MINISTRY OF EDUCATION
BEIJING, 100875
CHINA
zhaoxa@bnu.edu.cn

JIN CHUNHUA
ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCES
BEIJING, 100190
CHINA
jinch@amss.ac.cn

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