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HERMITIAN CATEGORIES, EXTENSION OF SCALARS AND SYSTEMS OF SESQUILINEAR FORMS

EVA BAYER-FLUCKIGER, URIYA A. FIRST AND DANIEL A. MOLDOVAN

We prove that the category of *systems of sesquilinear forms* over a given hermitian category is equivalent to the category of *unimodular* 1-*hermitian forms* over another hermitian category. The sesquilinear forms are not required to be unimodular or defined on a reflexive object (i.e., the standard map from the object to its double dual is not assumed to be bijective), and the forms in the system can be defined with respect to different hermitian structures on the given category. This extends an earlier result of the first and third authors.

We use the equivalence to define a Witt group of sesquilinear forms over a hermitian category and to generalize results such as Witt's cancellation theorem, Springer's theorem, the weak Hasse principle, and finiteness of genus to systems of sesquilinear forms over hermitian categories.

Introduction

Quadratic and hermitian forms were studied extensively by various authors, who have developed a rich array of tools to study them. It is well known that in many cases (e.g., over fields), the theory of sesquilinear forms can be reduced to the theory of hermitian forms (e.g., see [Riehm 1974; Riehm and Shrader-Frechette 1976] and works based on them). In [Bayer-Fluckiger and Moldovan 2014], an explanation of this reduction was provided in the form of an equivalence between the category of sesquilinear forms over a ring and the category of unimodular 1-hermitian forms over a special hermitian category.

In this paper, we extend the equivalence of [Bayer-Fluckiger and Moldovan 2014] to hermitian categories, and, moreover, improve it in such a way that it applies to systems of sesquilinear forms in hermitian categories that admit *nonreflexive* objects (see Section 2). That is, we prove that the category of systems of sesquilinear forms over a hermitian category $\mathscr{C}$ is equivalent to the category of unimodular 1-hermitian

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forms over anther hermitian category $\mathscr{C}'$. The sesquilinear forms are not required to be unimodular or defined on a reflexive object, and the forms in the system can be defined with respect to different hermitian structures on the category $\mathscr{C}$.

Using the equivalence, we present a notion of a Witt group of sesquilinear forms, which is analogous to the standard Witt group of hermitian forms over rings with involution (e.g., see [Knus 1991; Scharlau 1985]). We also extend various results (Witt's cancellation theorem, Springer's theorem, finiteness of genus, the Hasse principle, etc.) to systems of sesquilinear forms over hermitian categories (and in particular to systems of sesquilinear forms over rings with a family of involutions).

Sections 1 and 2 recall the basics of sesquilinear forms over rings and hermitian categories, respectively. In Section 3, we prove the equivalence of the category of sesquilinear forms over a given hermitian category to a category of unimodular 1-hermitian forms over another hermitian category, and in Section 4 we extend this result to systems of sesquilinear forms. Section 5 presents applications of the equivalence.

1. Sesquilinear and hermitian forms

Let A be a ring. An *involution* on A is an additive map $\sigma : A \to A$ such that $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in A$ and $\sigma^2 = \mathrm{id}_A$. Let V be a right A-module. A *sesquilinear form* over (A, σ) is a biadditive map $s : V \times V \to A$ satisfying $s(xa, yb) = \sigma(a)s(x, y)b$ for all $x, y \in V$ and $a, b \in A$. The pair (V, s) is also called a sesquilinear form in this case.[1] The *orthogonal sum* of two sesquilinear forms (V, s) and (V', s') is defined to be $(V \oplus V', s \oplus s')$ where $s \oplus s'$ is given by

$$(s \oplus s')(x \oplus x', y \oplus y') = s(x, y) + s'(x', y')$$

for all $x, y \in V$ and $x', y' \in V'$. Two sesquilinear forms (V, s) and (V', s') are called *isometric* if there exists an isomorphism of A-modules $f : V \xrightarrow{\sim} V'$ such that $s'(f(x), f(y)) = s(x, y)$ for all $x, y \in V$.

Let $V^* = \mathrm{Hom}_A(V, A)$. Then V^* has a right A-module structure given by $(f \cdot a)(x) = \sigma(a)f(x)$ for all $f \in V^*$, $a \in A$. We say that V is *reflexive* if the homomorphism of right A-modules $\omega_V : V \to V^{**}$ defined by $\omega_V(x)(f) = \sigma(f(x))$ for all $x \in V$, $f \in V^*$ is bijective.

A sesquilinear space (V, s) over (A, σ) induces two homomorphisms of right A-modules $s_\ell, s_r : V \to V^*$, called the *left* and *right adjoint* of s, respectively. They are given by $s_\ell(x)(y) = s(x, y)$ and $s_r(x)(y) = \sigma(s(y, x))$ for all $x, y \in V$. Observe that $s_r = s_\ell^*\omega_V$ and $s_\ell = s_r^*\omega_V$. The form s is called *unimodular* if s_r and s_ℓ are isomorphisms. In this case, V must be reflexive.

[1] Some texts use the term *sesquilinear space*.

Let $\epsilon = \pm 1$. A sesquilinear form (V, s) over (A, σ) is called *ϵ-hermitian* if $\sigma(s(x, y)) = \epsilon s(y, x)$ for all $x, y \in V$, that is, if $s_r = \epsilon s_\ell$. A 1-hermitian form is also called a hermitian form.

There exists a classical notion of Witt group for unimodular ϵ-hermitian forms over (A, σ) (e.g., see [Knus 1991]): denote by $\mathrm{WG}^\epsilon(A, \sigma)$ the Grothendieck group of isometry classes of unimodular ϵ-hermitian forms (V, s) over (A, σ), with V finitely generated projective, addition being the orthogonal sum. A unimodular ϵ-hermitian form over (A, σ) is called *hyperbolic* if it is isometric to $(V \oplus V^*, \mathbb{H}_V^\epsilon)$ for some finitely generated projective right A-module V, where $\mathbb{H}_V^\epsilon$ is defined by

$$\mathbb{H}_V^\epsilon(x \oplus f, y \oplus g) = f(y) + \epsilon\sigma(g(x)) \quad \text{for all } x, y \in V, \; f, g \in V^*.$$

We let $\mathbb{H}_V = \mathbb{H}_V^1$. The quotient of $\mathrm{WG}^\epsilon(A, \sigma)$ by the subgroup generated by the unimodular ϵ-hermitian hyperbolic forms is called the *Witt group of unimodular ϵ-hermitian forms* over (A, σ) and is denoted by $\mathrm{W}^\epsilon(A, \sigma)$.

We denote by $\mathrm{Sesq}(A, \sigma)$ and $\mathrm{UH}^\epsilon(A, \sigma)$ the categories of sesquilinear and unimodular ϵ-hermitian forms over (A, σ), respectively. The morphisms of these categories are (bijective) isometries. For simplicity, we let $\mathrm{UH}(A, \sigma) := \mathrm{UH}^1(A, \sigma)$.

2. Hermitian categories

This section recalls some basic notions about hermitian categories, as presented in [Scharlau 1985] (see also [Knus 1991; Quebbemann et al. 1979]).

2A. ***Preliminaries.*** Recall that a *hermitian category* consists of a triple $(\mathscr{C}, *, \omega)$, where $\mathscr{C}$ is an additive category, $* : \mathscr{C} \to \mathscr{C}$ is a contravariant functor and $\omega = (\omega_C)_{C \in \mathscr{C}} : \mathrm{id} \to **$ is a natural transformation satisfying $\omega_C^* \omega_{C^*} = \mathrm{id}_{C^*}$ for all $C \in \mathscr{C}$. In this case, the pair $(*, \omega)$ is called a *hermitian structure* on $\mathscr{C}$. It is customary to assume that ω is a natural *isomorphism* rather than a natural *transformation*. Such hermitian categories will be called *reflexive*. In general, an object $C \in \mathscr{C}$ for which ω_C is an isomorphism is called *reflexive*, so the category $\mathscr{C}$ is reflexive precisely when all its objects are reflexive. We will often drop $*$ and ω from the notation and use these symbols to denote the functor and natural transformation associated with any hermitian category under discussion.

A *sesquilinear form* over the category $\mathscr{C}$ is a pair (C, s) with $C \in \mathscr{C}$ and $s : C \to C^*$. A sesquilinear form (C, s) is called *unimodular* if s and $s^* \omega_C$ are isomorphisms. (If C is reflexive, then s is bijective if and only if $s^* \omega_C$ is bijective.) Let $\epsilon = \pm 1$. A sesquilinear form (C, s) is called *ϵ-hermitian* if $s = \epsilon s^* \omega_C$. For brevity, 1-hermitian forms are often called hermitian forms. Orthogonal sums of forms are defined in the obvious way. Let (C, s) and (C', s') be two sesquilinear forms over $\mathscr{C}$. An *isometry* from (C, s) to (C', s') is an isomorphism $f : C \xrightarrow{\sim} C'$ satisfying $s = f^* s' f$. In this case, (C, s) and (C', s') are said to be *isometric*. We let $\mathrm{Sesq}(\mathscr{C})$ stand for the category of sesquilinear forms over $\mathscr{C}$ with isometries as morphisms.

Denote by $\mathrm{UH}^\epsilon(\mathscr{C})$ the category of unimodular ϵ-hermitian forms over $\mathscr{C}$. The morphisms are isometries. For brevity, let $\mathrm{UH}(\mathscr{C}) := \mathrm{UH}^1(\mathscr{C})$. The hyperbolic unimodular ϵ-hermitian forms over $\mathscr{C}$ are the forms isometric to $(Q \oplus Q^*, \mathbb{H}^\epsilon_Q)$, where Q is any reflexive object in $\mathscr{C}$ and $\mathbb{H}^\epsilon_Q$ is given by

$$\mathbb{H}^\epsilon_Q = \begin{bmatrix} 0 & \mathrm{id}_{Q^*} \\ \epsilon\omega_Q & 0 \end{bmatrix} : Q \oplus Q^* \to (Q \oplus Q^*)^* = Q^* \oplus Q^{**}.$$

Again, let $\mathbb{H}_Q = \mathbb{H}^1_Q$. The quotient of $\mathrm{WG}^\epsilon(\mathscr{C})$, the Grothendieck group of isometry classes of unimodular ϵ-hermitian forms over $\mathscr{C}$ (with respect to the orthogonal sum), by the subgroup generated by the hyperbolic forms is called the Witt group of unimodular ϵ-hermitian forms over $\mathscr{C}$ and is denoted by $\mathrm{W}^\epsilon(\mathscr{C})$. For brevity, set $\mathrm{W}(\mathscr{C}) = \mathrm{W}^1(\mathscr{C})$.

Example 2.1. Let (A, σ) be a ring with involution. If we take $\mathscr{C}$ to be Mod-A, the category of right A-modules, and define $*$ and ω as in Section 1, then $\mathscr{C}$ becomes a hermitian category. Furthermore, the sesquilinear forms (M, s) over (A, σ) correspond to the sesquilinear forms over $\mathscr{C}$ via $(M, s) \mapsto (M, s_r)$. This correspondence gives rise to isomorphisms of categories $\mathrm{Sesq}(A, \sigma) \cong \mathrm{Sesq}(\mathscr{C})$ and $\mathrm{UH}^\epsilon(A, \sigma) \cong \mathrm{UH}^\epsilon(\mathscr{C})$. Now let $\mathscr{C}$ be a subcategory of Mod-A such that $M \in \mathscr{C}$ implies $M^* \in \mathscr{C}$. Then $\mathscr{C}$ is still a hermitian category, and is reflexive if and only if $\mathscr{C}$ consists of reflexive A-modules (as defined in Section 1). For example, this happens if $\mathscr{C} = \mathscr{P}(A)$, the category of projective A-modules of finite type. In this case, the Witt group $\mathrm{W}^\epsilon(\mathscr{C}) = \mathrm{W}^\epsilon(\mathscr{P}(A))$ is isomorphic to $\mathrm{W}^\epsilon(A, \sigma)$.

2B. *Duality-preserving functors.* Let $\mathscr{C}$ and $\mathscr{C}'$ be two hermitian categories. A *duality-preserving functor* from $\mathscr{C}$ to $\mathscr{C}'$ is an additive functor $F : \mathscr{C} \to \mathscr{C}'$ together with a natural isomorphism $i = (i_M)_{M \in \mathscr{C}} : F* \to *F$. This means that for any $M \in \mathscr{C}$, there exists an isomorphism $i_M : F(M^*) \xrightarrow{\sim} (FM)^*$ such that for all $N \in \mathscr{C}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(M, N)$, the following diagram commutes:

$$\begin{tikzcd} F(N^*) \arrow[r, "F(f^*)"] \arrow[d, "i_N"] & F(M^*) \arrow[d, "i_M"] \\ (FN)^* \arrow[r, "(Ff)^*"] & (FM)^* \end{tikzcd}$$

Any duality-preserving functor induces a functor $\mathrm{Sesq}(\mathscr{C}) \to \mathrm{Sesq}(\mathscr{C}')$, which we also denote by F. It is given by

$$F(M, s) = (FM, i_M F(s))$$

for every $(M, s) \in \mathrm{Sesq}(\mathscr{C})$. If the functor $F : \mathscr{C} \to \mathscr{C}'$ is faithful, faithful and full, or induces an equivalence, then the functor $F : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{Sesq}(\mathscr{C}')$ shares the same property.

Let $\lambda = \pm 1$. A duality-preserving functor F is called *λ-hermitian* if

$$i_{M^*} F(\omega_M) = \lambda i_M^* \omega_{FM}$$

for all $M \in \mathscr{C}$. Let $\epsilon = \pm 1$. We recall from [Knus 1991, pp. 80–81] that in this case the functor $F : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{Sesq}(\mathscr{C}')$ maps $\mathrm{UH}^{\epsilon}(\mathscr{C})$ to $\mathrm{UH}^{\epsilon\lambda}(\mathscr{C}')$ and sends ϵ-hermitian hyperbolic forms to $\epsilon\lambda$-hermitian hyperbolic forms. Therefore, F induces a homomorphism between the corresponding Witt groups:

$$\mathrm{W}^{\epsilon}(F) : \mathrm{W}^{\epsilon}(\mathscr{C}) \to \mathrm{W}^{\epsilon\lambda}(\mathscr{C}').$$

If F is an equivalence of categories, then $F : \mathrm{UH}^{\epsilon}(\mathscr{C}) \to \mathrm{UH}^{\epsilon\lambda}(\mathscr{C}')$ is also an equivalence of categories and the induced group homomorphism $\mathrm{W}^{\epsilon}(F)$ is an isomorphism of groups.

2C. ***Transfer into the endomorphism ring.*** The aim of this subsection is to introduce the method of *transfer into the endomorphism ring*, which allows us to pass from the abstract setting of hermitian categories to that of a ring with involution, which is more concrete. This method will be applied repeatedly in Section 5. Note that it applies well only to reflexive hermitian categories.

Let $\mathscr{C}$ be a *reflexive* hermitian category, and let M be an object of $\mathscr{C}$, on which we suppose that there exists a unimodular ϵ_0-hermitian form h_0 for a certain $\epsilon_0 = \pm 1$. Put $E = \mathrm{End}_{\mathscr{C}}(M)$. According to [Quebbemann et al. 1979, Lemma 1.2], the form (M, h_0) induces on E an involution σ, defined by $\sigma(f) = h_0^{-1} f^* h_0$ for all $f \in E$. Let $\mathscr{P}(E)$ denote the category of projective right E-modules of finite type. Then, using σ, we can consider $\mathscr{P}(E)$ as a reflexive hermitian category (see Example 2.1).

Recall that an *idempotent* $e \in \mathrm{End}_{\mathscr{C}}(M)$ *splits* if there exist an object $M' \in \mathscr{C}$ and morphisms $i : M' \to M$, $j : M \to M'$ such that $ji = \mathrm{id}_{M'}$ and $ij = e$.

Denote by $\mathscr{C}|_M$ the full subcategory of $\mathscr{C}$ each object of which is isomorphic to a direct summand of a finite direct sum of copies of M. We consider the functor

$$\mathrm{T} = \mathrm{T}_{(M,h_0)} := \mathrm{Hom}(M, \_\,) : \mathscr{C}|_M \to \mathscr{P}(E)$$

given by

$$N \mapsto \mathrm{Hom}(M, N) \quad \text{for all } N \in \mathscr{C}|_M,$$
$$f \mapsto \mathrm{T}(f) \quad \text{for all } f \in \mathrm{Hom}(N, N'),\ N, N' \in \mathscr{C}|_M,$$

where for all $g \in \mathrm{Hom}(M, N)$, $\mathrm{T}(f)(g) = fg$. In [Quebbemann et al. 1979, Proposition 2.4], it was proved that the functor T is fully faithful and duality-preserving with respect to the natural isomorphism $i = (i_N)_{N \in \mathscr{C}|_M} : \mathrm{T}* \to *\mathrm{T}$ given by $i_N(f) = \mathrm{T}(h_0^{-1} f^*\omega_N)$ for every $N \in \mathscr{C}|_M$ and $f \in \mathrm{Hom}(M, N^*)$. In addition, if all the idempotents of $\mathscr{C}|_M$ split, then T is an equivalence of categories. By computation, we easily see that T is ϵ_0-hermitian.

Note that for any finite list of (reflexive) objects $M_1, \dots, M_t \in \mathscr{C}$ and any $\epsilon_0 = \pm 1$, there exists a unimodular ϵ_0-hermitian form (M, h_0) such that $M_1, \dots, M_t \in \mathscr{C}|_M$. Indeed, let $N = \bigoplus_{i=1}^t M_i$ and take $(M, h_0) = (N \oplus N^*, \mathbb{H}_N^{\epsilon_0})$. This means that as long as we treat finitely many hermitian forms, we may pass to the context of hermitian forms over rings with involution.

2D. ***Linear hermitian categories and ring extension.*** In this subsection we introduce the notion of extension of rings in hermitian categories.

Let K be a commutative ring. Recall that a *K-category* is an additive category $\mathscr{C}$ such that for every $A, B \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is endowed with a K-module structure such that composition is K-bilinear. For example, any additive category is in fact a $\mathbb{Z}$-category. An additive covariant functor $F : \mathscr{C} \to \mathscr{C}'$ between two K-categories is *K-linear* if the map $F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}'}(FA, FB)$ is K-linear for all $A, B \in \mathscr{C}$. K-linear contravariant functors are defined in the same manner. A *K-linear hermitian category* is a hermitian category $(\mathscr{C}, *, \omega)$ such that $\mathscr{C}$ is a K-category and $*$ is K-linear.

Fix a commutative ring K. Let $\mathscr{C}$ be an additive K-category and let R be a K-algebra (with unity, not necessarily commutative). We define the *extension of the category $\mathscr{C}$ to the ring R*, denoted $\mathscr{C} \otimes_K R$, to be the category whose objects are formal symbols $C \otimes_K R$, with $C \in \mathscr{C}$, and whose Hom-sets are defined by

$$\mathrm{Hom}_{\mathscr{C}\otimes_K R}(A \otimes_K R, B \otimes_K R) = \mathrm{Hom}_{\mathscr{C}}(A, B) \otimes_K R.$$

The composition in $\mathscr{C} \otimes_K R$ is defined in the obvious way. It is straightforward to check that $\mathscr{C} \otimes_K R$ is also a K-category. Moreover, when R is commutative, $\mathscr{C} \otimes_K R$ is an R-category. We define the *scalar extension functor*, $\mathscr{R}_{R/K} : \mathscr{C} \to \mathscr{C} \otimes_K R$ by

$$\begin{aligned} \mathscr{R}_{R/K} M &= M \otimes_K R \quad \text{for all } M \in \mathscr{C}, \\ \mathscr{R}_{R/K} f &= f \otimes_K 1 \quad \text{for all } f \in \mathrm{Hom}(M, N). \end{aligned}$$

The functor $\mathscr{R}_{R/K}$ is additive and K-linear.

In case K is obvious from the context, we write $\mathscr{C}_R$, M_R, f_R instead of $\mathscr{C} \otimes_K R$, $M \otimes_K R$, $f \otimes_K 1$, respectively. (Here, $M \in \mathscr{C}$ and f is a morphism in $\mathscr{C}$.)

Remark 2.2. The scalar extension we have just defined agrees with scalar extension of modules under mild assumptions, but not in general: Let S and R be two K-algebras, and write $S_R = S \otimes_K R$. There is an additive functor $G : (\text{Mod-}S)_R \to \text{Mod-}(S_R)$ given by

$$G(M_R) = M \otimes_S S_R \quad \text{and} \quad G(f \otimes a)(m \otimes b) = fm \otimes ab$$

for all $M, N \in \text{Mod-}S$, $f \in \text{Hom}_S(M, N)$, and $a, b \in R$, and the following diagram commutes:

$$\begin{tikzcd} \text{Mod-}S \arrow[r, "\mathcal{R}_{R/K}"] \arrow[d, equal] & (\text{Mod-}S)_R \arrow[d, "G"] \\ \text{Mod-}S \arrow[r, "\_\otimes_S S_R"] & \text{Mod-}(S_R) \end{tikzcd}$$

In general, G is neither full nor faithful. However, using standard tensor-Hom relations, it is easy to verify that the map

$$G : \text{Hom}_{(\text{Mod-}S)_R}(M_R, M'_R) \to \text{Hom}_{\text{Mod-}(S_R)}(GM_R, GM'_R) \tag{1}$$

is bijective if either (a) M is finitely generated projective, or (b) R is a flat K-module and M is finitely presented. In particular, if $\mathcal{C}$ is an additive subcategory of Mod-S consisting of finitely presented modules and R is flat as a K-module, then $\mathcal{C}_R$ can be understood as a full subcategory of Mod-(S_R) in the obvious way. An example in which the map G of (1) is neither injective nor surjective can be obtained by taking $S = K = \mathbb{Z}$, $R = \mathbb{Q}$ and $M = M' = \mathbb{Z}[1/p]/\mathbb{Z}$.

If $(\mathcal{C}, *, \omega)$ is a K-linear hermitian category and R/K is a *commutative* ring extension, then $\mathcal{C}_R$ also has a hermitian structure given by $(M_R)^* = (M^*)_R$, $(f \otimes a)^* = f^* \otimes a$ and $\omega_{M_R} = (\omega_M)_R = \omega_M \otimes 1$ for all $M, N \in \mathcal{C}$, $f \in \text{Hom}_{\mathcal{C}}(M, N)$ and $a \in R$. In this case, the functor $\mathcal{R}_{R/K}$ is a 1-hermitian duality-preserving functor (the natural transformation $i : \mathcal{R}_{R/K}* \to *\mathcal{R}_{R/K}$ is just the identity). In particular, we get a functor $\mathcal{R}_{R/K} : \text{Sesq}(\mathcal{C}) \to \text{Sesq}(\mathcal{C}_R)$ given by $\mathcal{R}_{R/K}(M, s) := (M_R, s_R)$, and $\mathcal{R}_{R/K}$ sends ϵ-hermitian (hyperbolic) forms to ϵ-hermitian (hyperbolic) forms.

2E. ***Scalar extension commutes with transfer.*** Let R/K be a commutative ring extension, let $\mathcal{C}$ be a reflexive K-linear hermitian category and let M be an object of $\mathcal{C}$ admitting a unimodular ϵ-hermitian form h. Then (M_R, h_R) is a unimodular ϵ-hermitian form over $\mathcal{C}_R$. Let $E = \text{End}_{\mathcal{C}}(M)$ and $E_R = \text{End}_{\mathcal{C}_R}(M_R) = E \otimes_K R$. It is easy to verify that the following diagram (of functors) commutes:

$$\begin{tikzcd} \mathcal{C}|_M \arrow[r, "\mathrm{T}_{(M,h)}"] \arrow[d, "\mathcal{R}_{R/K}"] & \mathcal{P}(E) \arrow[d, "\_\otimes_E E_R"] \\ \mathcal{C}_R|_{M_R} \arrow[r, "\mathrm{T}_{(M_R,h_R)}"] & \mathcal{P}(E_R) \end{tikzcd}$$

(Note that by Remark 2.2, $\mathcal{P}(E_R)$ and $\_ \otimes_E E_R$ can be understood as $\mathcal{P}(E)_R$ and $\mathcal{R}_{R/K}$, respectively.) Since all the functors are ϵ- or 1-hermitian, we get the following commutative diagram, in which the horizontal arrows are full and faithful:

$$\begin{tikzcd} \mathrm{UH}^{\lambda}(\mathscr{C}|_M) \arrow[r, "\mathrm{T}_{(M,h)}"] \arrow[d, "\mathscr{R}_{R/K}"] & \mathrm{UH}^{\lambda\epsilon}(\mathscr{P}(E)) \arrow[d, "\_\otimes_E E_R"] \\ \mathrm{UH}^{\lambda}(\mathscr{C}_R|_{M_R}) \arrow[r, "\mathrm{T}_{(M_R,h_R)}"] & \mathrm{UH}^{\lambda\epsilon}(\mathscr{P}(E_R)) \end{tikzcd}$$

This diagram means that in order to study the behavior of $\mathscr{R}_{R/K}$ on arbitrary K-linear hermitian categories, it is enough to study its behavior on hermitian categories obtained from K-algebras with K-involution (as in Example 2.1).

3. An equivalence of categories

Let $\mathscr{C}$ be a (not necessarily reflexive) hermitian category. In this section we prove that there exists a *reflexive* hermitian category $\mathscr{C}'$ such that the category $\mathrm{Sesq}(\mathscr{C})$ is equivalent to $\mathrm{UH}^1(\mathscr{C}')$. (We explain how to extend this result to *systems of sesquilinear forms* in the next section.)

The category $\mathscr{C}'$ resembles the category of double arrows presented in [Bayer-Fluckiger and Moldovan 2014, §3], but is not identical to it. This difference makes our construction work for nonreflexive hermitian categories and, as we shall explain in the next section, for systems of sesquilinear forms, where the forms can be defined with respect to different hermitian structures on $\mathscr{C}$.

3A. ***The category of twisted double arrows.*** Let $(\mathscr{C}, *, \omega)$ be a hermitian category. We construct the *category of twisted double arrows in* $\mathscr{C}$, denoted $\tilde{\mathrm{Ar}}_2(\mathscr{C})$, as follows: The objects of $\tilde{\mathrm{Ar}}_2(\mathscr{C})$ are quadruples (M, N, f, g) such that $f, g \in \mathrm{Hom}_{\mathscr{C}}(M, N^*)$. A morphism from (M, N, f, g) to (M', N', f', g') is a pair $(\phi, \psi^{\mathrm{op}})$ such that $\phi \in \mathrm{Hom}(M, M')$, $\psi \in \mathrm{Hom}(N', N)$, $f'\phi = \psi^* f$ and $g'\phi = \psi^* g$. The composition of two morphisms is given by $(\phi, \psi^{\mathrm{op}})(\phi', \psi'^{\mathrm{op}}) = (\phi\phi', (\psi'\psi)^{\mathrm{op}})$.

The category $\tilde{\mathrm{Ar}}_2(\mathscr{C})$ is easily seen to be an additive category. Moreover, it has a hermitian structure: For every $(M, N, f, g) \in \tilde{\mathrm{Ar}}_2(\mathscr{C})$, define $(M, N, f, g)^* = (N, M, g^*\omega_N, f^*\omega_N)$ and $\omega_{(M,N,f,g)} = \mathrm{id}_{(M,N,f,g)} = (\mathrm{id}_M, \mathrm{id}_N^{\mathrm{op}})$. In addition, for every morphism $(\phi, \psi^{\mathrm{op}}) : (M, N, f, g) \to (M', N', f', g')$, let $(\phi, \psi^{\mathrm{op}})^* = (\psi, \phi^{\mathrm{op}})$. It is now routine to check that $(\tilde{\mathrm{Ar}}_2(\mathscr{C}), *, \omega)$ is a *reflexive* hermitian category. Also observe that $**$ is just the identity functor on $\tilde{\mathrm{Ar}}_2(\mathscr{C})$. The following proposition describes the hermitian forms over $\tilde{\mathrm{Ar}}_2(\mathscr{C})$:

Proposition 3.1. *Let* $Z := (M, N, f, g) \in \tilde{\mathrm{Ar}}_2(\mathscr{C})$ *and let* $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(M, N)$. *Then* $(Z, (\alpha, \beta^{\mathrm{op}}))$ *is a hermitian form over* $\tilde{\mathrm{Ar}}_2(\mathscr{C})$ *if and only if* $\alpha = \beta$ *and* $\alpha^* f = g^*\omega_N\alpha$; *equivalently, if and only if* $\alpha = \beta$ *and* $\alpha^* g = f^*\omega_N\alpha$.

Proof. By definition, $Z^* = (N, M, g^*\omega_N, f^*\omega_N)$, so $(\alpha, \beta^{\mathrm{op}})$ is a morphism from Z to Z^* if and only if $\beta^* f = g^*\omega_N\alpha$ and $\beta^* g = f^*\omega_N\alpha$. In addition, by computation, we see $(\alpha, \beta^{\mathrm{op}}) = (\alpha, \beta^{\mathrm{op}})^* \circ \omega_Z$ precisely when $\alpha = \beta$. Therefore, $(Z, (\alpha, \beta^{\mathrm{op}}))$ is a hermitian form if and only if $\alpha=\beta$, $\alpha^* f = g^*\omega_N\alpha$ and $\alpha^* g = f^*\omega_N\alpha$. It is therefore enough to show $\alpha^* f = g^*\omega_N\alpha$ if and only if $\alpha^* g = f^*\omega_N\alpha$. Indeed, if $\alpha^* f = g^*\omega_N\alpha$, then $\alpha^*\omega_N^* g^{**} = f^*\alpha^{**}$. Therefore, $\alpha^* g = \alpha^*\omega_N^*\omega_{N^*} g = \alpha^*\omega_N^* g^{**}\omega_M = f^*\alpha^{**}\omega_M = f^*\omega_N\alpha$, as required (we used the naturality of ω and the identity $\omega_N^*\omega_{N^*} = \mathrm{id}_{N^*}$ in the computation). The other direction follows by symmetry. □

Theorem 3.2. *Let $\mathscr{C}$ be a hermitian category. Define a functor $F : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$ by*

$$F(M, s) = ((M, M, s^*\omega_M, s), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})) \quad \textit{and} \quad F(\psi) = (\psi, (\psi^{-1})^{\mathrm{op}})$$

for all $(M, s) \in \mathrm{Sesq}(\mathscr{C})$ and any morphism ψ in $\mathrm{Sesq}(\mathscr{C})$. Then F induces an equivalence of categories between $\mathrm{Sesq}(\mathscr{C})$ and $\mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$.

Proof. Let $(M, s) \in \mathrm{Sesq}(\mathscr{C})$. That $F(M, s)$ lies in $\mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$ follows from Proposition 3.1. Let $\psi : (M, s) \to (M', s')$ be an isometry. Then

$$\begin{aligned} F(\psi)^*(\mathrm{id}_{M'}, \mathrm{id}_{M'}^{\mathrm{op}})F(\psi) &= (\psi, (\psi^{-1})^{\mathrm{op}})^*(\mathrm{id}_{M'}, \mathrm{id}_{M'}^{\mathrm{op}})(\psi, (\psi^{-1})^{\mathrm{op}}) \\ &= (\psi^{-1}, \psi^{\mathrm{op}})(\mathrm{id}_{M'}, \mathrm{id}_{M'}^{\mathrm{op}})(\psi, (\psi^{-1})^{\mathrm{op}}) \\ &= (\psi^{-1}\,\mathrm{id}_{M'}\,\psi, (\psi^{-1}\,\mathrm{id}_{M'}\,\psi)^{\mathrm{op}}) \\ &= (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}}). \end{aligned}$$

Thus, $F(\psi)$ is an isometry from $F(M, s)$ to $F(M', s')$. It is clear that F respects composition, so we conclude that F is a functor.

To see that F induces an equivalence, we construct a functor G such that F and G are mutual inverses. Let $G : \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C})) \to \mathrm{Sesq}(\mathscr{C})$ be defined by

$$G((M, N, f, g), (\alpha, \alpha^{\mathrm{op}})) = (M, \alpha^* g) \quad \text{and} \quad G(\phi, \psi^{\mathrm{op}}) = \phi$$

for all $((M, N, f, g), (\alpha, \alpha^{\mathrm{op}})) \in \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$ and any morphism $(\phi, \psi^{\mathrm{op}})$ in $\mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$.

Let $(Z, (\alpha, \alpha^{\mathrm{op}})), (Z', (\alpha', \alpha'^{\mathrm{op}})) \in \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$ and let $(\phi, \psi^{\mathrm{op}})$ be a morphism $(Z, (\alpha, \alpha^{\mathrm{op}})) \to (Z', (\alpha', \alpha'^{\mathrm{op}}))$. It is easy to see that $G(Z, (\alpha, \alpha^{\mathrm{op}}))$ lies in $\mathrm{Sesq}(\mathscr{C})$, so we now check that $G(\phi, \psi^{\mathrm{op}})$ is an isometry from $G(Z, (\alpha, \alpha^{\mathrm{op}}))$ to $G(Z', (\alpha', \alpha'^{\mathrm{op}}))$. Writing $Z = (M, N, f, g)$ and $Z' = (M', N', f', g')$, this amounts to showing $\alpha^* g = \phi^*\alpha'^* g'\phi$. Indeed, since $(\phi, \psi^{\mathrm{op}})$ is a morphism from Z to Z', we have $g'\phi = \psi^* g$, and since $(\phi, \psi^{\mathrm{op}})$ is an isometry, we also have $(\phi, \psi^{\mathrm{op}})^*(\alpha', \alpha'^{\mathrm{op}})(\phi, \psi^{\mathrm{op}}) = (\alpha, \alpha^{\mathrm{op}})$, which in turn implies $\psi\alpha'\phi = \alpha$. We now have $\phi^*\alpha'^* g'\phi = \phi^*\alpha'^*\psi^* g = (\psi\alpha'\phi)^* g = \alpha^* g$, as required. That G preserves composition is straightforward.

It is easy to see that GF is the identity functor on $\mathrm{Sesq}(\mathscr{C})$, so it is left to show that there is a natural isomorphism from FG to $\mathrm{id}_{\mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))}$. Keeping the notation of the previous paragraph, we have

$$FG((M, N, f, g), (\alpha, \alpha^{\mathrm{op}})) = \big((M, M, (\alpha^* g)^* \omega_M, \alpha^* g), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})\big).$$

By Proposition 3.1 we have $\alpha^* f = g^* \omega_N \alpha$, hence $(\alpha^* g)^* \omega_M = g^* \alpha^{**} \omega_M = g^* \omega_N \alpha = \alpha^* f$. Thus,

$$FG((M, N, f, g), (\alpha, \alpha^{\mathrm{op}})) = \big((M, M, \alpha^* f, \alpha^* g), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})\big). \tag{2}$$

Define a natural isomorphism $t : \mathrm{id}_{\mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))} \to FG$ by $t_{(Z,(\alpha,\alpha^{\mathrm{op}}))} = (\mathrm{id}_M, \alpha^{\mathrm{op}})$. Using (2), it is easy to see that $t_{(Z,(\alpha,\alpha^{\mathrm{op}}))}$ is indeed an isometry from $(Z, (\alpha, \alpha^{\mathrm{op}}))$ to $FG(Z, (\alpha, \alpha^{\mathrm{op}}))$. The map t is natural, since for Z', $(\phi, \psi^{\mathrm{op}})$ as above, we have $FG(\phi, \psi^{\mathrm{op}}) t_{(Z,(\alpha,\alpha^{\mathrm{op}}))} = (\phi, (\phi^{-1})^{\mathrm{op}})(\mathrm{id}_M, \alpha^{\mathrm{op}}) = (\phi, (\alpha\phi^{-1})^{\mathrm{op}}) = (\phi, (\psi\alpha')^{\mathrm{op}}) = (\mathrm{id}_{M'}, \alpha'^{\mathrm{op}})(\phi, \psi^{\mathrm{op}}) = t_{(Z',(\alpha',\alpha'^{\mathrm{op}}))}(\phi, \psi^{\mathrm{op}})$ (we used the identity $\psi\alpha'\phi = \alpha$ verified above). □

Remark 3.3. Following [Bayer-Fluckiger and Moldovan 2014, §3], one can also construct the *category of (nontwisted) double arrows in* $\mathscr{C}$, denoted $\mathrm{Ar}_2(\mathscr{C})$. Its objects are quadruples (M, N, f, g) with $M, N \in \mathscr{C}$ and $f, g \in \mathrm{Hom}(M, N)$. A morphism from (M, N, f, g) to (M', N', f', g') is a pair (ϕ, ψ) where $\phi \in \mathrm{Hom}(M, M')$ and $\psi \in \mathrm{Hom}(N, N')$ satisfy $\psi f = f'\phi$ and $\psi g = g'\phi$. The category $\mathrm{Ar}_2(\mathscr{C})$ is obviously additive, and, moreover, it admits a hermitian structure given by $(M, N, f, g)^* = (N^*, M^*, g^*, f^*)$, $(\phi, \psi)^* = (\psi^*, \phi^*)$ and $\omega_{(M,N,f,g)} = (\omega_M, \omega_N)$.

There is a functor $T : \tilde{\mathrm{Ar}}_2(\mathscr{C}) \to \mathrm{Ar}_2(\mathscr{C})$ given by $T(M, N, f, g) = (M, N^*, f, g)$ and $T(\phi, \psi^{\mathrm{op}}) = (\phi, \psi^*)$. This functor induces an equivalence if $\mathscr{C}$ is reflexive, but otherwise it need neither be faithful nor full. In addition, provided $\mathscr{C}$ is reflexive, one can define a functor $F' : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{UH}(\mathrm{Ar}_2(\mathscr{C}))$ by $F'(M, s) = ((M, M^*, s^*\omega_M, s), (\omega_M, \mathrm{id}_{M^*}))$ and $F'(\psi) = (\psi, (\psi^{-1})^*)$. This functor induces an equivalence of categories; the proof is analogous to [Bayer-Fluckiger and Moldovan 2014, Theorem 4.1].

3B. *Hyperbolic sesquilinear forms.* Let $\mathscr{C}$ be a hermitian category. The equivalence $\mathrm{Sesq}(\mathscr{C}) \sim \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$ of Theorem 3.2 allows us to pull back notions defined for unimodular hermitian forms over $\tilde{\mathrm{Ar}}_2(\mathscr{C})$ to sesquilinear form over $\mathscr{C}$. In this subsection, we will do this for hyperbolicity, and thus obtain a notion of a Witt group of sesquilinear forms.

Throughout, F denotes the functor $\mathrm{Sesq}(\mathscr{C}) \to \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$ from Theorem 3.2.

Definition 3.4. A sesquilinear form (M, s) over $\mathscr{C}$ is called *hyperbolic* if $F(M, s)$ is hyperbolic as a unimodular hermitian form over $\tilde{\mathrm{Ar}}_2(\mathscr{C})$.

The following proposition gives a more concrete meaning to hyperbolicity of sesquilinear forms over $\mathscr{C}$.

Proposition 3.5. *Up to isometry, the hyperbolic sesquilinear forms over $\mathscr{C}$ are given by*

$$\left(M \oplus N, \begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix}\right),$$

where $M, N \in \mathscr{C}$, $f \in \operatorname{Hom}_{\mathscr{C}}(N, M^*)$, $g \in \operatorname{Hom}_{\mathscr{C}}(M, N^*)$ and $\left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right]$ is an element of $\operatorname{Hom}_{\mathscr{C}}(M \oplus N, M^* \oplus N^*)$ given in matrix form. Furthermore, a unimodular ϵ-hermitian form is hyperbolic as a sesquilinear form (i.e., in the sense of Definition 3.4) if and only if it is hyperbolic as a unimodular ϵ-hermitian form (see Section 2).

Proof. Let G be the functor $\mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C})) \to \mathrm{Sesq}(\mathscr{C})$ defined in the proof of Theorem 3.2. Since F and G are mutual inverses, the hyperbolic sesquilinear forms over $\mathscr{C}$ are the forms isometric to $G(Z \oplus Z^*, \mathbb{H}_Z)$ for $Z \in \tilde{\mathrm{Ar}}_2(\mathscr{C})$. Write $Z = (M, N, h, g)$. Then

$$(Z \oplus Z^*, \mathbb{H}_Z) = \left(\left(M \oplus N, N \oplus M, \begin{bmatrix} h & 0 \\ 0 & g^*\omega_N \end{bmatrix}, \begin{bmatrix} g & 0 \\ 0 & h^*\omega_N \end{bmatrix}\right), \begin{bmatrix} 0 & \mathrm{id}_{Z^*} \\ \omega_Z & 0 \end{bmatrix}\right).$$

Observe that

$$\begin{bmatrix} 0 & \mathrm{id}_{Z^*} \\ \omega_Z & 0 \end{bmatrix} = \begin{bmatrix} 0 & (\mathrm{id}_N, \mathrm{id}_M^{\mathrm{op}}) \\ (\mathrm{id}_M, \mathrm{id}_N^{\mathrm{op}}) & 0 \end{bmatrix} = \left(\begin{bmatrix} 0 & \mathrm{id}_N \\ \mathrm{id}_M & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathrm{id}_N \\ \mathrm{id}_M & 0 \end{bmatrix}^{\mathrm{op}}\right).$$

Thus,

$$G(Z \oplus Z^*, \mathbb{H}_Z) = \left(M \oplus N, \begin{bmatrix} 0 & \mathrm{id}_N \\ \mathrm{id}_M & 0 \end{bmatrix}^* \begin{bmatrix} g & 0 \\ 0 & h^*\omega_N \end{bmatrix}\right),$$

and since

$$\begin{bmatrix} 0 & \mathrm{id}_N \\ \mathrm{id}_M & 0 \end{bmatrix}^* \begin{bmatrix} g & 0 \\ 0 & h^*\omega_N \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{id}_{M^*} \\ \mathrm{id}_{N^*} & 0 \end{bmatrix} \begin{bmatrix} g & 0 \\ 0 & h^*\omega_N \end{bmatrix} = \begin{bmatrix} 0 & h^*\omega_N \\ g & 0 \end{bmatrix},$$

we see that $G(Z \oplus Z^*, \mathbb{H}_Z)$ matches the description in the proposition. Furthermore, by putting $h = f^*\omega_M$ for $f \in \operatorname{Hom}_{\mathscr{C}}(N, M^*)$, we get $h^*\omega_N = \omega_M^* f^{**}\omega_N = \omega_M^*\omega_{M^*} f = f$. Thus, $\left(M \oplus N, \left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right]\right)$ is hyperbolic for all M, N, f, g, as required.

To finish, note that we have clearly shown that $(Q \oplus Q^*, \mathbb{H}_Q^\epsilon)$ is hyperbolic as a sesquilinear form for every $Q \in \mathscr{C}$. To see the converse, assume $\left(M \oplus N, \left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right]\right)$ is ϵ-hermitian and unimodular. Then

$$\begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix} = \epsilon \begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix}^* \omega_{M \oplus N} = \epsilon \begin{bmatrix} 0 & g^* \\ f^* & 0 \end{bmatrix} \begin{bmatrix} \omega_M & 0 \\ 0 & \omega_N \end{bmatrix} = \begin{bmatrix} 0 & \epsilon g^*\omega_N \\ \epsilon f^*\omega_M & 0 \end{bmatrix},$$

hence $g = \epsilon f^*\omega_N$ and $f = \epsilon g^*\omega_M$. Since $\left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right]$ is unimodular, f and g are bijective and hence so are ω_N and ω_M. In particular, M is reflexive. It is now routine to verify that the map $\mathrm{id}_M \oplus f : M \oplus N \to M \oplus M^*$ is an isometry from $\left(M \oplus N, \left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right]\right)$ to $(M \oplus M^*, \mathbb{H}_M^\epsilon)$, so the former is hyperbolic in the sense of Section 2. $\square$

Let (A, σ) be a ring with involution. When $\mathscr{C}$ is the category of right A-modules, considered as a hermitian category as in Example 2.1, we obtain a notion of hyperbolic sesquilinear forms over (A, σ). These hyperbolic forms can be characterized as follows:

Proposition 3.6. *A sesquilinear form (M, s) over (A, σ) is hyperbolic if and only if there are submodules $M_1, M_2 \leq M$ such that $s(M_1, M_1) = s(M_2, M_2) = 0$ and $M = M_1 \oplus M_2$. Furthermore, if (M, s) is unimodular and ϵ-hermitian, then (M, s) is hyperbolic as a sesquilinear space if and only if it is hyperbolic as an ϵ-hermitian unimodular space.*

Proof. Recall that for any two right A-modules M_1, M_2, we identify $(M_1 \oplus M_2)^*$ with $M_1^* \oplus M_2^*$ via $f \leftrightarrow (f|_{M_1}, f|_{M_2})$. Let (M, s) be a sesquilinear space, and assume $M = M_1 \oplus M_2$. By straightforward computation, we see that s_r is of the form $\left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right] \in \mathrm{Hom}_A(M, M^*) = \mathrm{Hom}_A(M_1 \oplus M_2, M_1^* \oplus M_2^*)$ if and only if $s(M_1, M_1) = s(M_2, M_2) = 0$. The proposition therefore follows from Proposition 3.5. $\square$

3C. *Witt groups of sesquilinear forms.* Let $\mathscr{C}$ be a hermitian category. Denote by $\mathrm{WG}_S(\mathscr{C})$ the Grothendieck group of isometry classes of *sesquilinear* forms over $\mathscr{C}$, with respect to orthogonal sum. It is easy to see that the hyperbolic isometry classes span a subgroup of $\mathrm{WG}_S(\mathscr{C})$, which we denote by $\mathbb{H}(\mathscr{C})$. The *Witt group of sesquilinear forms* over $\mathscr{C}$ is defined to the quotient

$$\mathrm{W}_S(\mathscr{C}) = \mathrm{WG}_S(\mathscr{C})/\mathbb{H}(\mathscr{C}).$$

By definition, we have $\mathrm{W}_S(\mathscr{C}) \cong \mathrm{W}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$. Taking $\mathscr{C}$ to be the category of projective right A-modules (or, with a different result, reflexive right A-modules, or again arbitrary ones) of finite type and their duals, we obtain a notion of a Witt group for sesquilinear forms over (A, σ). Also observe that there is a homomorphism of groups $\mathrm{W}^{\epsilon}(\mathscr{C}) \to \mathrm{W}_S(\mathscr{C})$ given by sending the class of a unimodular ϵ-hermitian form to its corresponding class in $\mathrm{W}_S(\mathscr{C})$. Corollary 5.14 below presents sufficient conditions for the injectivity of this homomorphism.

3D. *Extension of scalars.* Let R/K be a commutative ring extension and let $\mathscr{C}$ be a K-linear hermitian category. Then the category $\tilde{\mathrm{Ar}}_2(\mathscr{C})$ is also K-linear. For later use, we now check that the scalar extension functor $\mathscr{R}_{R/K}$ of Section 2D "commutes" with the functor F of Theorem 3.2.

Proposition 3.7. *There is a 1-hermitian duality-preserving functor $J : \tilde{\mathrm{Ar}}_2(\mathscr{C})_R \to \tilde{\mathrm{Ar}}_2(\mathscr{C}_R)$ making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Sesq}(\mathscr{C}) \arrow[rr, "F"] \arrow[d, "\mathscr{R}_{R/K}"] & & \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C})) \arrow[d, "\mathscr{R}_{R/K}"] \\
\mathrm{Sesq}(\mathscr{C}_R) \arrow[r, "F"] & \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C}_R)) & \mathrm{UH}(\tilde{\mathrm{Ar}}_2(\mathscr{C})_R) \arrow[l, "J"']
\end{tikzcd}$$

It is given by

$$J((M,N,f,g)_R) = (M_R, N_R, f_R, g_R),$$
$$J((\phi,\psi^{\mathrm{op}})\otimes a) = (\phi\otimes a, (\psi\otimes a)^{\mathrm{op}})$$

for all $(M,N,f,g)\in \tilde{\mathrm{Ar}}_2(\mathscr{C})$ *and any morphism* $(\phi,\psi^{\mathrm{op}})$ *in* $\tilde{\mathrm{Ar}}_2(\mathscr{C})$. (*The associated natural isomorphism* $i : J* \to *J$ *is the identity map.*) *Furthermore, when* R *is flat as a* K*-module,* J *is faithful and full.*

Proof. We only check that J is faithful and full when R is flat as a K-module. All other assertions follow by computation. Let $Z=(M,N,f,g)$, $Z'=(M',N',f',g')$ be objects in $\tilde{\mathrm{Ar}}_2(\mathscr{C})$. Set

$$U = \{(f, g^{\mathrm{op}}) \mid (f,g)\in \mathrm{Hom}_{\mathscr{C}}(M,M')\times \mathrm{Hom}_{\mathscr{C}}(N',N)\},$$
$$V = \mathrm{Hom}_{\mathscr{C}}(M,N'^*)\times \mathrm{Hom}_{\mathscr{C}}(M,N'^*),$$

and define $\lambda : U\to V$ by

$$\lambda(\phi,\psi^{\mathrm{op}}) = (\psi^* f - f'\phi,\ \psi^* g - g'\phi).$$

Unfolding the definitions, we see that $\mathrm{Hom}_{\tilde{\mathrm{Ar}}_2(\mathscr{C})_R}(Z_R, Z'_R) = (\ker\lambda)\otimes_K R$ and $\mathrm{Hom}_{\tilde{\mathrm{Ar}}_2(\mathscr{C}_R)}(JZ_R, JZ'_R) = \ker(\lambda\otimes_K \mathrm{id}_R)$. Furthermore, the standard map from $(\ker\lambda)\otimes_K R$ to $\ker(\lambda\otimes_K \mathrm{id}_R)$ is just application of the functor J. When R is flat as a K-module, this map is an isomorphism; hence we are done. $\square$

Corollary 3.8. *Let* (M,s), (M',s') *be two sesquilinear forms over* $\mathscr{C}$, *and assume* R *is flat as a* K*-module. Then* $\mathscr{R}_{R/K}(M,s)$ *is isometric to* $\mathscr{R}_{R/K}(M',s')$ *if and only if* $\mathscr{R}_{R/K}F(M,s)$ *is isometric to* $\mathscr{R}_{R/K}F(M',s')$.

4. Systems of sesquilinear forms

In this section, we explain how to generalize the results of Section 3 to systems of sesquilinear forms.

Let A be a ring, and let $\{\sigma_i\}_{i\in I}$ be a nonempty family of (not necessarily distinct) involutions of A. A system of sesquilinear forms over $(A,\{\sigma_i\}_{i\in I})$ is a pair $(M,\{s_i\}_{i\in I})$ such that (M,s_i) is a sesquilinear space over (A,σ_i) for all i. An isometry between two systems of sesquilinear forms $(M,\{s_i\}_{i\in I})$, $(M',\{s'_i\}_{i\in I})$ is an isomorphism $f : M\to M'$ such that $s'_i(fx,fy) = s_i(x,y)$ for all $x,y\in M$, $i\in I$.

Observe that each of the involutions σ_i gives rise to a hermitian structure $(*_i,\omega_i)$ on Mod-A, the category of right A-modules. In particular, a system of sesquilinear forms $(M,\{s_i\})$ gives rise to homomorphisms $(s_i)_r, (s_i)_\ell : M\to M^{*_i}$ given by $(s_i)_r(x)(y) = \sigma_i(s_i(y,x))$ and $(s_i)_\ell(x)(y) = s_i(x,y)$, where $M^{*_i} = \mathrm{Hom}_A(M,A)$, considered as a right A-module via the action $(f\cdot a)m = \sigma_i(a)f(m)$. This leads to the notion of systems of sesquilinear forms over hermitian categories.

Let $\mathscr{C}$ be an additive category and let $\{*_i, \omega_i\}_{i\in I}$ be a nonempty family of hermitian structures on $\mathscr{C}$. A system of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\}_{i\in I})$ is a pair $(M, \{s_i\}_{i\in I})$ such that $M \in \mathscr{C}$ and (M, s_i) is a sesquilinear form over $(\mathscr{C}, *_i, \omega_i)$. An isometry between two systems of sesquilinear forms $(M, \{s_i\}_{i\in I})$ and $(M', \{s'_i\}_{i\in I})$ is an isomorphism $f : M \xrightarrow{\sim} M'$ such that $f^{*_i} s'_i f = s_i$ for all $i \in I$. We let $\mathrm{Sesq}_I(\mathscr{C})$ (or $\mathrm{Sesq}_I(\mathscr{C}, \{*_i, \omega_i\})$) denote the category of systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\}_{i\in I})$ with isometries as morphisms.

Keeping the notation of the previous paragraph, the results of Section 3 can be extended to systems of sesquilinear forms as follows: Define the category of *twisted double I-arrows* over $(\mathscr{C}, \{*_i, \omega_i\}_{i\in I})$, denoted $\tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$, to be the category whose objects are quadruples $(M, N, \{f_i\}_{i\in I}, \{g_i\}_{i\in I})$ with $M, N \in \mathscr{C}$ and $f_i, g_i \in \mathrm{Hom}_{\mathscr{C}}(M, N^{*_i})$. A morphism $(M, N, \{f_i\}, \{g_i\}) \to (M', N', \{f'_i\}, \{g'_i\})$ is a formal pair $(\phi, \psi^{\mathrm{op}})$ such that $\phi \in \mathrm{Hom}(M, M')$, $\psi \in \mathrm{Hom}(N', N)$ and $\psi^{*_i} f_i = f'_i \phi$, $\psi^{*_i} g_i = g'_i \phi$ for all $i \in I$. The composition is defined by the formula $(\phi, \psi^{\mathrm{op}})(\phi', \psi'^{\mathrm{op}}) = (\phi\phi', (\psi'\psi)^{\mathrm{op}})$.

The category $\tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$ can be made into a reflexive hermitian category by letting $(M, N, \{f_i\}, \{g_i\})^* = (N, M, \{g_i^{*_i}\omega_{i,N}\}, \{f_i^{*_i}\omega_{i,M}\})$, $(\phi, \psi^{\mathrm{op}})^* = (\psi, \phi^{\mathrm{op}})$ and $\omega_{(M,N,\{f_i\},\{g_i\})} = (\mathrm{id}_M, \mathrm{id}_N^{\mathrm{op}})$. It is now possible to prove the following theorem, whose proof is completely analogous to the proof of Theorem 3.2:

Theorem 4.1. *Define a functor* $F : \mathrm{Sesq}_I(\mathscr{C}) \to \mathrm{UH}(\tilde{\mathrm{Ar}}_{2I}(\mathscr{C}))$ *by*

$$F(M, \{s_i\}) = ((M, M, \{s_i^{*_i}\omega_{i,M}\}, \{s_i\}), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})) \quad \textit{and} \quad F(\psi) = (\psi, (\psi^{-1})^{\mathrm{op}}).$$

Then F induces an equivalence of categories.

Sketch of proof. It is easy to see that any hermitian form over $\mathrm{UH}(\tilde{\mathrm{Ar}}_{2I}(\mathscr{C}))$ has the form $((M, N, \{f_i\}, \{g_i\}), (\alpha, \alpha^{\mathrm{op}}))$. Define a functor $G : \mathrm{UH}(\tilde{\mathrm{Ar}}_{2I}(\mathscr{C})) \to \mathrm{Sesq}_I(\mathscr{C})$ by

$$G((M, N, \{f_i\}, \{g_i\}), (\alpha, \alpha^{\mathrm{op}})) = (M, \{\alpha^{*_i} g_i\}) \quad \text{and} \quad G(\phi, \psi^{\mathrm{op}}) = \phi.$$

Arguing as in the proof of Theorem 3.2, we see that F and G are mutual inverses. □

As we did in Section 3, we can use Theorem 4.1 to define hyperbolic systems of sesquilinear forms. Namely, a system of forms $(M, \{s_i\})$ over $\mathscr{C}$ will be called *hyperbolic* if $F(M, \{s_i\})$ is hyperbolic over $\tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$. The following two propositions are proved in the same manner as Propositions 3.5 and 3.6, respectively:

Proposition 4.2. *A system of sesquilinear forms $(M, \{s_i\})$ over $\mathscr{C}$ is hyperbolic if and only if there are $M_1, M_2 \in \mathscr{C}$, $f_i \in \mathrm{Hom}(M_2, M_1^{*_i})$, $g_i \in \mathrm{Hom}(M_1, M_2^{*_i})$ such that $M = M_1 \oplus M_2$ and, for all $i \in I$,*

$$s_i = \begin{bmatrix} 0 & f_i \\ g_i & 0 \end{bmatrix} \in \mathrm{Hom}(M, M^{*_i}) = \mathrm{Hom}(M_1 \oplus M_2, M_1^{*_i} \oplus M_2^{*_i}).$$

In this case, each of the sesquilinear forms (M, s_i) *(over* $(\mathscr{C}, *_i, \omega_i)$*) is hyperbolic.*

Proposition 4.3. *Let A be a ring and let* $\{\sigma_i\}_{i\in I}$ *be a nonempty family of involutions of A. A system of sesquilinear forms* $(M, \{s_i\})$ *over* $(A, \{\sigma_i\})$ *is hyperbolic if and only if there are submodules* $M_1, M_2 \leq M$ *such that* $M = M_1 \oplus M_2$ *and* $s_i(M_1, M_1) = s_i(M_2, M_2) = 0$ *for all* $i \in I$*. In this case, each of the sesquilinear forms* (M, s_i) *(over* (A, σ_i)*) is hyperbolic.*

The notion of hyperbolic systems of sesquilinear forms can be used to define Witt groups. We leave the details to the reader.

Let R/K be a commutative ring extension. If $\mathscr{C}$ and all the hermitian structures $\{*_i, \omega_i\}_{i\in I}$ are K-linear, then the scalar extension functor $\mathscr{R}_{R/K} : \mathscr{C} \to \mathscr{C}_R$ is 1-hermitian and duality-preserving with respect to $(*_i, \omega_i)$ for all $i \in I$. Therefore, we have a functor $\mathscr{R}_{R/K} : \mathrm{Sesq}_I(\mathscr{C}) \to \mathrm{Sesq}_I(\mathscr{C}_R)$ given by $\mathscr{R}_{R/K}(M, \{s_i\}_{i\in I}) = (M_R, \{(s_i)_R\}_{i\in I})$. We thus have a notion of scalar extension for systems of bilinear forms (and it agrees with the obvious scalar extension for systems of bilinear forms over a ring with a family of involutions, provided the assumptions of Remark 2.2 hold). Using the ideas of Section 3D, one can show:

Corollary 4.4. *Let* $(M, \{s_i\})$, $(M', \{s'_i\})$ *be two systems of sesquilinear forms over* $(\mathscr{C}, \{*_i, \omega_i\})$*, and assume R is flat as a K-module. Then* $\mathscr{R}_{R/K}(M, \{s_i\})$ *is isometric to* $\mathscr{R}_{R/K}(M', \{s'_i\})$ *if and only if* $\mathscr{R}_{R/K}F(M, \{s_i\})$ *is isometric to* $\mathscr{R}_{R/K}F(M', \{s'_i\})$*.*

5. Applications

This section uses the previous results to generalize various known results about hermitian forms (over rings or reflexive hermitian categories) to systems of sesquilinear forms over (not necessarily reflexive) hermitian categories. Some of the consequences to follow were obtained in [Bayer-Fluckiger and Moldovan 2014] for hermitian forms over rings. Here we rephrase them for hermitian categories, extend them to systems of sesquilinear forms and drop the assumption that the base module (or object) is reflexive.

5A. ***Witt's cancellation theorem.*** Quebbemann, Scharlau and Schulte [Quebbemann et al. 1979, §3.4] proved Witt's cancellation theorem for unimodular hermitian forms over hermitian categories $\mathscr{C}$ satisfying the following conditions:

(a) All idempotents in $\mathscr{C}$ split (see Section 2C).

(b) For all $C \in \mathscr{C}$, $E := \mathrm{End}_{\mathscr{C}}(C)$ is a *complete semilocal* ring in which 2 is invertible.

Recall that complete semilocal means that $E/\mathrm{Jac}(E)$ is semisimple (i.e., E is semilocal) and that the standard map $E \to \varprojlim\{E/\mathrm{Jac}(R)^n\}_{n\in\mathbb{N}}$ is an isomorphism (i.e., E is complete in the $\mathrm{Jac}(E)$-*adic topology*). In fact, condition (a) can be dropped since idempotents can be split artificially (see Section 5E below), or, alternatively, since by applying transfer (see Section 2C) one can move to a module category in which idempotents split.

We shall now use the Quebbemann–Scharlau–Schulte cancellation theorem together with Theorem 4.1 to give several conditions guaranteeing cancellation for systems of sesquilinear forms.

Our first criterion is based on the following well-known lemma:

Lemma 5.1. *Let K be a commutative noetherian complete semilocal ring* (*e.g.*, *a complete discrete valuation ring*). *Then any K-algebra A which is finitely generated as a K-module is complete semilocal.*

Proof. For brevity, write $I = \mathrm{Jac}(K)$ and $J = \mathrm{Jac}(A)$. By [Hinohara 1960, Theorem 2] and the proof of [First 2013, Proposition 8.8(i)] (for instance), $A = \varprojlim\{A/A(I^n)\}_{n\in\mathbb{N}}$. That $A = \varprojlim\{A/J^n\}_{n\in\mathbb{N}}$ follows if we verify that $J^m \subseteq AI \subseteq J$ for some $m \in \mathbb{N}$. The right inclusion holds since $1 + AI$ consists of right-invertible elements. Indeed, for all $a \in AI$, we have $aA + AI = A$, so by Nakayama's lemma (applied to the K-module A), $aA = A$. The existence of m, as well as the fact that A is semilocal, follows by arguing as in [Rowen 1988, Example 2.7.19′(ii)] (for instance). □

Theorem 5.2. *Let K be a commutative noetherian complete semilocal ring with* $2 \in K^\times$, *let $\mathscr{C}$ be a K-category equipped with K-linear hermitian structures $\{*_i, \omega_i\}_{i\in I}$, and let $(M, \{s_i\})$, $(M', \{s'_i\})$, $(M'', \{s''_i\})$ be systems of sesquilinear forms over* $(\mathscr{C}, \{*_i, \omega_i\})$. *Assume that* $\mathrm{Hom}_{\mathscr{C}}(M, N)$ *is finitely generated as a K-module for all $M, N \in \mathscr{C}$. Then*

$$(M, \{s_i\}) \oplus (M', \{s'_i\}) \simeq (M, \{s_i\}) \oplus (M'', \{s''_i\}) \iff (M', \{s'_i\}) \simeq (M'', \{s''_i\}).$$

Proof. In light of Theorem 4.1, it is enough to prove cancellation of unimodular 1-hermitian forms over the category $\tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$ (note that the equivalence of Theorem 4.1 respects orthogonal sums). This would follow from the cancellation theorem of [Quebbemann et al. 1979, §3.4] if we show that the endomorphism rings of objects in $\tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$ are complete semilocal rings in which 2 is invertible. Indeed, let $Z := (M, N, \{f_i\}, \{g_i\}) \in \tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$. Then $E := \mathrm{End}(Z)$ is a subring of $\mathrm{End}_{\mathscr{C}}(M) \times \mathrm{End}_{\mathscr{C}}(N)^{\mathrm{op}}$, which is a K-algebra by assumption. Since the hermitian structures $\{*_i, \omega_i\}$ are K-linear, E is in fact a K-subalgebra, which must be finitely generated as a K-module (because this is true for $\mathrm{End}_{\mathscr{C}}(M) \times \mathrm{End}_{\mathscr{C}}(N)^{\mathrm{op}}$ and K is noetherian). Thus, we are done by Lemma 5.1 and the fact that $2 \in K^\times$. □

As corollary, we get the following result, which resembles [Bayer-Fluckiger and Moldovan 2014, Theorem 8.1]:

Corollary 5.3. *Let K be a commutative noetherian complete semilocal ring with* $2 \in K^\times$, *let A be a K-algebra which is finitely generated as a K-module*, *and let*

$\{\sigma_i\}_{i\in I}$ be a family of K-involutions on A. Then cancellation holds for systems of sesquilinear forms over $(A, \{\sigma_i\})$ which are defined on finitely generated right A-modules.

For the next theorem, recall that a ring R is said to be *semiprimary* if R is semilocal and $\mathrm{Jac}(R)$ is nilpotent. For example, all artinian rings are semiprimary. Note that all semiprimary rings are complete semilocal. It is well-known that for a ring R and an idempotent $e \in R$, R is semiprimary if and only if eRe and $(1-e)R(1-e)$ are semiprimary. As a result, if M, N are two objects in an additive category, then $\mathrm{End}(M \oplus N)$ is semiprimary if and only if $\mathrm{End}(M)$ and $\mathrm{End}(N)$ are semiprimary.

Theorem 5.4. *Let $\mathscr{C}$ be an additive category with hermitian structures $\{*_i, \omega_i\}$, and let $(M, \{s_i\})$, $(M', \{s'_i\})$, $(M'', \{s''_i\})$ be systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$. Assume that $\mathrm{End}_{\mathscr{C}}(M)$, $\mathrm{End}_{\mathscr{C}}(M')$, $\mathrm{End}_{\mathscr{C}}(M'')$ are semiprimary rings in which 2 is invertible. Then*

$$(M, \{s_i\}) \oplus (M', \{s'_i\}) \simeq (M, \{s_i\}) \oplus (M'', \{s''_i\}) \iff (M', \{s'_i\}) \simeq (M'', \{s''_i\}).$$

Proof. As in the proof of Theorem 5.2, it is enough to show that the objects in $\tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$ have a complete semilocal endomorphism ring. In fact, we may restrict to those objects $Z := (M, N, \{f_i\}, \{g_i\})$ for which $\mathrm{End}_{\mathscr{C}}(M)$ and $\mathrm{End}_{\mathscr{C}}(N)$ are semiprimary. (These do form a hermitian subcategory of $\tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$ by the comments above.) Fix such a Z and let $H = \bigoplus_{i\in I} \mathrm{Hom}_{\mathscr{C}}(M, N^{*_i})$. We view the morphism $\{f_i\}$ and $\{g_i\}$ as elements of H in the obvious way. Let $A = \mathrm{End}(M)$ and $B = \mathrm{End}(N)$. We endow H with a (B^{op}, A)-bimodule structure by setting $b^{\mathrm{op}}(\bigoplus_{i\in I} h_i)a = \bigoplus_{i\in I}(b^{*_i} \circ h_i \circ a)$ for all $a \in A$, $b \in B$, $\bigoplus_i h_i \in H$. This allows us to construct the ring $S := \left[\begin{smallmatrix} A & \\ H & B^{\mathrm{op}} \end{smallmatrix}\right]$. It is now straightforward to check that $\mathrm{End}(Z)$ consists of those elements in $A \times B^{\mathrm{op}} = \left[\begin{smallmatrix} A & \\ & B^{\mathrm{op}} \end{smallmatrix}\right]$ that commute with $\left[\begin{smallmatrix} 0 & \\ f_i & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & \\ g_i & 0 \end{smallmatrix}\right]$ for all $i \in I$. Thus, $\mathrm{End}(Z)$ is a *semicentralizer* subring of $A \times B^{\mathrm{op}}$ in the sense of [First 2013, §1]. By [First 2013, Theorem 4.6], a semicentralizer subring of a semiprimary ring is semiprimary, so $\mathrm{End}(Z)$ is semiprimary, and in particular complete semilocal. □

Corollary 5.5. *Let A be a semiprimary ring with $2 \in A^\times$, and let $\{\sigma_i\}_{i\in I}$ be a family of involutions on A. Then cancellation holds for systems of sesquilinear forms over $(A, \{\sigma_i\})$ which are defined on finitely presented right A-modules.*

Proof. By [Björk 1971, Theorem 4.1] (or [First 2013, Theorem 7.3]), the endomorphism ring of a finitely presented A-module is semiprimary. Now apply Theorem 5.4. □

Corollary 5.6. *Let $\mathscr{C}$ be an* abelian *category equipped with hermitian structures $\{*_i, \omega_i\}$. Assume that $\mathscr{C}$ consists of objects of finite length. Then cancellation holds for systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$.*

Proof. By the Harada–Sai lemma [Rowen 1988, Proposition 2.9.29], the endomorphism ring of an object of finite length in an abelian category is semiprimary, so we are done by Theorem 5.4. Alternatively, one can check directly that the category $\mathrm{A}\tilde{\mathrm{r}}_{2I}(\mathscr{C})$ is abelian and consists of objects of finite length, apply the Harada–Sai lemma to $\mathrm{A}\tilde{\mathrm{r}}_{2I}(\mathscr{C})$, and then use the cancellation theorem of [Quebbemann et al. 1979, §3.4]. □

Remark 5.7. It is not hard to deduce from a theorem of Camps and Dicks [1993, Corollary 2] that if the endomorphism rings of $\mathscr{C}$ are semilocal, then so are the endomorphism rings of $\mathrm{A}\tilde{\mathrm{r}}_{2I}(\mathscr{C})$. (Simply check that $\mathrm{End}(M, N, \{f_i\}, \{g_i\})$ is a *rationally closed* subring of $\mathrm{End}_{\mathscr{C}}(M) \times \mathrm{End}_{\mathscr{C}}(N)^{\mathrm{op}}$ in the sense of [Camps and Dicks 1993, p. 204].) By applying transfer (see Section 2C) to $\mathrm{A}\tilde{\mathrm{r}}_{2I}(\mathscr{C})$, one can then move to the context of unimodular 1-hermitian forms over semilocal rings. Cancellation theorems for such forms were obtained by various authors, including Knebusch [1969], Reiter [1975] and Keller [1988]. However, none of these apply to the general case, as in fact cancellation is no longer true; see [Keller 1988, §2]. Nevertheless, the cancellation results of [ibid.] can still be used to get some partial results about systems of sesquilinear forms over $\mathscr{C}$; we leave the details to the reader.

5B. ***Finiteness results.*** In this subsection and the next, we generalize the finiteness results of [Bayer-Fluckiger and Moldovan 2014, §10] to systems of sesquilinear forms.

For a ring A, we denote by $T(A)$ the $\mathbb{Z}$-torsion subgroup of A. Recall that if R is a commutative ring, A is said to be *R-finite* if $A_R = A \otimes_{\mathbb{Z}} R$ is a finitely generated R-module and $T(A)$ is finite. Note that being R-finite passes to subrings.

The proofs of the results to follow are completely analogous to the proofs of the corresponding statements in [Bayer-Fluckiger and Moldovan 2014, §10]; they are based on applying the equivalence of Theorem 4.1 and then using the finiteness results of [Bayer-Fluckiger et al. 1989], possibly after applying transfer.

Throughout, $\mathscr{C}$ is an additive category and $\{*_i, \omega_i\}_{i\in I}$ is a nonempty family of hermitian structures on $\mathscr{C}$. Fix a system of sesquilinear forms $(V, \{s_i\}_{i\in I})$ over $(\mathscr{C}, \{{}_i, \omega_i\})$ and let $Z(V, \{s_i\}) = (V, V, \{s_i^{*_i}\omega_{i,V}\}, \{s_{i\,r}\}) \in \mathrm{A}\tilde{\mathrm{r}}_{2I}(\mathscr{C})$. (Note that $F(V, \{s_i\}) = (Z, (\mathrm{id}_V, \mathrm{id}_V^{\mathrm{op}}))$ with F as in Theorem 4.1.)

Theorem 5.8. *If there exists a nonzero integer m such that $\mathrm{End}_{\mathscr{C}}(V)$ is $\mathbb{Z}[1/m]$-finite, then there are finitely many isometry classes of summands of $(V, \{s_i\})$.*

Theorem 5.9. *Assume that there exists a nonzero integer m such that the ring $\mathrm{End}_{\mathrm{A}\tilde{\mathrm{r}}_{2I}(\mathscr{C})}(Z(V, \{s_i\}))$ is $\mathbb{Z}[1/m]$-finite (e.g., if $\mathrm{End}_{\mathscr{C}}(V)$ is $\mathbb{Z}[1/m]$-finite). Then there exist only finitely many isometry classes of systems of sesquilinear forms $(V', \{s'_i\}_{i\in I})$ over $\mathscr{C}$ such that $Z(V', \{s'_i\}) \simeq Z(V, \{s_i\})$ (as objects in $\mathrm{A}\tilde{\mathrm{r}}_{2I}(\mathscr{C})$).*

5C. *Finiteness of the genus.* Let $\mathscr{C}$ be a hermitian category admitting a nonempty family of hermitian structures $\{*_i, \omega_i\}_{i\in I}$. We say that two systems of sesquilinear forms $(M, \{s_i\})$, $(M', \{s'_i\})$ are *of the same genus* if they become isometric after applying $\mathscr{R}_{\mathbb{Z}_p/\mathbb{Z}}$ for every prime number p (where $\mathbb{Z}_p$ are the p-adic integer). (See Remark 2.2 for conditions under which this definition of genus agrees with the naive definition of genus for module categories.) As in [Bayer-Fluckiger and Moldovan 2014, Theorem 10.3], we have:

Theorem 5.10. *Let $(M, \{s_i\})$ be a system of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$, and assume that* $\mathrm{End}(M)$ *is $\mathbb{Q}$-finite. Then the genus of $(M, \{s_i\})$ contains only a finite number of isometry classes of systems of sesquilinear forms.*

5D. *Forms that are trivial in the Witt group.* Let $\mathscr{C}$ be a hermitian category. By definition, a unimodular ϵ-hermitian (resp. sesquilinear) form (M, s) is trivial in $\mathrm{W}^\epsilon(\mathscr{C})$ (resp. $\mathrm{W}_S(\mathscr{C})$) if and only if there are unimodular ϵ-hermitian (resp. sesquilinear) hyperbolic forms (H_1, h_1), (H_2, h_2) such that $(M, s) \oplus (H_1, h_1) \simeq (H_2, h_2)$. In this section, we will show that under mild assumptions, this implies that (M, s) is hyperbolic.

Lemma 5.11. *Let $M \in \mathscr{C}$, and assume that M is a (finite) direct sum of objects with local endomorphism ring. Then, up to isometry, there is at most one ϵ-hermitian hyperbolic form on M.*

Proof. For $X \in \mathscr{C}$, let $[X]$ denote the isomorphism class of X. The Krull–Schmidt theorem (e.g., see [Rowen 1988, p. 237 ff.]) implies that if $M \cong \bigoplus_{i=1}^t M_i$ with each M_i indecomposable, then the unordered list $[M_1], \dots, [M_t]$ is determined by M.

Let (M, s) be an ϵ-hermitian hyperbolic form on M, say $(M, s) \simeq (N \oplus N^*, \mathbb{H}^\epsilon_N)$. Write $N \cong \bigoplus_{i=1}^r N_i$ with each N_i indecomposable. Then $s \simeq \bigoplus_{i=1}^r \mathbb{H}^\epsilon_{N_i}$. It is easy to check that the isometry class of $\mathbb{H}^\epsilon_{N_i}$ depends only on the set $\{[N_i], [N_i^*]\}$. Furthermore, using the Krull–Schmidt theorem, one easily verifies that the unordered list $\{[N_1], [N_1^*]\}, \dots, \{[N_r], [N_r^*]\}$ is uniquely determined by M. It follows that (M, s) is isometric to a sesquilinear form which is determined by M up to isometry. $\square$

Proposition 5.12. *Let $\mathscr{C}$ be a hermitian category satisfying conditions* (a), (b) *on page 15. Then a unimodular ϵ-hermitian form (M, s) is trivial in* $\mathrm{W}^\epsilon(\mathscr{C})$ *if and only if it is hyperbolic.*

Proof. Note first that conditions (a) and (b) imply that every object of $\mathscr{C}$ is a sum of objects with local endomorphism rings, hence we may apply the Krull–Schmidt theorem to $\mathscr{C}$. (For example, this follows from [Rowen 1988, Theorem 2.8.40] since the endomorphism rings of $\mathscr{C}$ are *semiperfect*.) Let (M, s) be a unimodular ϵ-hermitian form such that $(M, s) \equiv 0$ in $\mathrm{W}^\epsilon(\mathscr{C})$. There are unimodular ϵ-hermitian hyperbolic forms (H_1, h_1), (H_2, h_2) such that $(M, s) \oplus (H_1, h_1) \simeq (H_2, h_2)$. Using the Krull–Schmidt theorem, it is easy to see that there is $N \in \mathscr{C}$ such that $M \cong N \oplus N^*$. Thus, we may consider $\mathbb{H}_N^\epsilon$ as a hermitian form on M. By Lemma 5.11, we have $\mathbb{H}_N^\epsilon \oplus h_1 \simeq h_2$, implying $\mathbb{H}_N^\epsilon \oplus h_2 \simeq s \oplus h_2$. Therefore, by the cancellation theorem of [Quebbemann et al. 1979, §3.4], $s \simeq \mathbb{H}_N^\epsilon$, as required. □

Proposition 5.13. *Let $\mathscr{C}$ be a hermitian category in which all idempotents split and such that either*

(1) *$\mathscr{C}$ is K-linear, where K is a noetherian complete semilocal ring with $2 \in K^\times$, and all* Hom*-sets in $\mathscr{C}$ are finitely generated as K-modules, or*

(2) *for all $M \in \mathscr{C}$, $\mathrm{End}_{\mathscr{C}}(M)$ is semiprimary and $2 \in \mathrm{End}_{\mathscr{C}}(M)^\times$.*

Then a sesquilinear form (M, s) is trivial in $\mathrm{W}_S(\mathscr{C})$ if and only if it is hyperbolic.

Proof. It is enough to verify that $F(M, s)$ is hyperbolic in $\tilde{\mathrm{Ar}}_2(\mathscr{C})$ (Theorem 3.2). The proofs of Theorems 5.2 and 5.4 imply that $\tilde{\mathrm{Ar}}_2(\mathscr{C})$ satisfies condition (b) of Section 5A, and condition (a) is routine (see also Lemma 5.17(ii) below). Therefore, $F(M, s)$ is hyperbolic by Proposition 5.12. □

Corollary 5.14. *Under the assumptions of Proposition 5.13, the map $\mathrm{W}(\mathscr{C}) \to \mathrm{W}_S(\mathscr{C})$ is injective.*

Proof. This follows from Propositions 5.13 and 3.5. □

5E. *Odd degree extensions.* Throughout this subsection, L/K is an odd degree field extension and char $K \neq 2$. A well-known theorem of Springer asserts that two unimodular hermitian forms over K become isometric over L if and only if they are already isometric over K. Moreover, the restriction map (the scalar extension map) $r_{L/K} : \mathrm{W}(K) \to \mathrm{W}(L)$ is injective. Both statements were extended to hermitian forms over finite-dimensional K-algebras with K-linear involution in [Bayer-Fluckiger and Lenstra 1990, Proposition 1.2 and Theorem 2.1] (see also [Fainsilber 1994] for a version in which L/K is replaced with an extension of complete discrete valuation rings). In this section, we extend these results to sesquilinear forms over hermitian categories.

Theorem 5.15. *Let $\mathscr{C}$ be an additive K-category such that $\dim_K \mathrm{Hom}(M, M')$ is finite for all $M, M' \in \mathscr{C}$. Let $\{*_i, \omega_i\}_{i \in I}$ be a nonempty family of K-linear hermitian structures on $\mathscr{C}$ and let $(M, \{s_i\})$, $(M', \{s_i'\})$ be two systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$. Then $\mathscr{R}_{L/K}(M, \{s_i\}) \simeq \mathscr{R}_{L/K}(M', \{s_i'\})$ if and only if $(M, \{s_i\}) \simeq (M', \{s_i'\})$.*

Proof. By Corollary 4.4, it is enough to prove $\mathscr{R}_{L/K} F(M, \{s_i\}) \simeq \mathscr{R}_{L/K} F(M', \{s_i'\})$ if and only if $F(M, \{s_i\}) \simeq F(M', \{s_i'\})$ (with F as in Theorem 4.1). Write $(Z, (\alpha, \alpha^{\mathrm{op}})) = F(M, \{s_i\}) \oplus F(M', \{s_i'\})$ and let $E = \mathrm{End}(Z)$. Then E is a K-subalgebra of $\mathrm{End}(M \oplus M') \times \mathrm{End}(M \oplus M')^{\mathrm{op}}$, which is finite-dimensional. By applying $\mathrm{T}_{(Z,(\alpha,\alpha^{\mathrm{op}}))}$ (see Section 2C), we reduce to showing that two 1-hermitian forms over E are isometric over $E \otimes_K L$ if and only if they are isometric over E, which is just [Bayer-Fluckiger and Lenstra 1990, Theorem 2.1]. (Note that we used the fact that transfer commutes with $\mathcal{R}_{L/K}$ in the sense of Section 2E.) $\square$

Corollary 5.16. *Let A be a finite-dimensional K-algebra and let $\{\sigma_i\}_{i\in I}$ be a nonempty family of K-involutions on A. Let $(M, \{s_i\})$, $(M', \{s'_i\})$ be two systems of sesquilinear forms over $(A, \{\sigma_i\})$. If M and M' are of finite type, then $\mathcal{R}_{L/K}(M, \{s_i\}) \simeq \mathcal{R}_{L/K}(M', \{s'_i\})$ if and only if $(M, \{s_i\}) \simeq (M', \{s'_i\})$.*

To state the analogue of the injectivity of $r_{L/K} : \mathrm{W}(K) \to \mathrm{W}(L)$ for hermitian categories, we need to introduce additional notation.

An additive category $\mathscr{C}$ is called *pseudoabelian* if all idempotents in $\mathscr{C}$ split. Any additive category $\mathscr{C}$ admits a *pseudoabelian closure* (e.g., see [Karoubi 1978, Theorem 6.10]), namely, a pseudoabelian additive category $\mathscr{C}^\circ$ equipped with an additive functor $A \mapsto A^\circ : \mathscr{C} \to \mathscr{C}^\circ$, such that the pair $(\mathscr{C}^\circ, A \mapsto A^\circ)$ is *universal*. The category $\mathscr{C}^\circ$ is unique up to equivalence and the functor $A \mapsto A^\circ$ turns out to be faithful and full. The category $\mathscr{C}^\circ$ can be realized as the category of pairs (M, e) with $M \in \mathscr{C}$ and $e \in \mathrm{End}_{\mathscr{C}}(M)$ an idempotent. The Hom-sets in $\mathscr{C}^\circ$ are given by $\mathrm{Hom}_{\mathscr{C}^\circ}((M, e), (M', e')) = e' \,\mathrm{Hom}_{\mathscr{C}}(M, M')e$ and the composition is the same as in $\mathscr{C}$. Finally, set $M^\circ = (M, \mathrm{id}_M)$ and $f^\circ = f$ for any object $M \in \mathscr{C}$ and any morphism f in $\mathscr{C}$. For simplicity, we will use only this particular realization of $\mathscr{C}^\circ$. Nevertheless, the universality implies that the statements to follow hold for any pseudoabelian closure.

Assume $\mathscr{C}$ admits a K-linear hermitian structure $(*, \omega)$. Then $\mathscr{C}^\circ$ is clearly a K-category, and, moreover, has a K-linear hermitian structure given by $(M, e)^* = (M^*, e^*)$ and $\omega_{(M,e)} = e^{**}\omega_M e \in \mathrm{Hom}_{\mathscr{C}^\circ}((M, e), (M^{**}, e^{**}))$. Also, the functor $M \mapsto M^\circ$ is 1-hermitian and duality-preserving (the isomorphism $(M^*)^\circ \to (M^\circ)^*$ being id_M), so we have a faithful and full functor $(M, s) \mapsto (M, s)^\circ = (M^\circ, s)$ from $\mathrm{Sesq}(\mathscr{C})$ to $\mathrm{Sesq}(\mathscr{C}^\circ)$. Henceforth, consider $\mathscr{C}$ and $\mathrm{Sesq}(\mathscr{C})$ as full subcategories of $\mathscr{C}^\circ$ and $\mathrm{Sesq}(\mathscr{C}^\circ)$, respectively; i.e., identify M° with M and $(M, s)^\circ$ with (M, s).

Lemma 5.17. *Let $\mathscr{C}$, $\mathscr{C}'$ be two hermitian categories and let $F : \mathscr{C} \to \mathscr{C}'$ be an ϵ-hermitian duality-preserving functor. Then*:

(i) *F extends to an ϵ-hermitian duality-preserving functor $F^\circ : \mathscr{C}^\circ \to \mathscr{C}'^\circ$. If F is faithful and full, then so is F°.*

(ii) *There is a 1-hermitian duality-preserving functor $G : \tilde{\mathrm{Ar}}_2(\mathscr{C})^\circ \to \tilde{\mathrm{Ar}}_2(\mathscr{C}^\circ)$. The functor G fixes $\tilde{\mathrm{Ar}}_2(\mathscr{C})$ and induces an equivalence of categories.*

Proof. (i) Define $F^\circ(M, e) = (FM, Fe) \in \mathscr{C}'^\circ$. The rest is routine.

(ii) Let G send $((M, M', f, g), (e, e'^{\mathrm{op}})) \in \tilde{\mathrm{Ar}}_2(\mathscr{C})^\circ$ to $((M, e), (M, e'), e'^* f e, e'^* g e)$ and any morphism to itself. The details are left to the reader. $\square$

Observe that the category $\mathscr{C}_L$ may not be pseudoabelian even when $\mathscr{C}$ is. We thus set $\mathscr{C}_L^\circ := (\mathscr{C}_L)^\circ$.

Theorem 5.18. *Let $(\mathscr{C}, *, \omega)$ be a pseudoabelian K-linear hermitian category such that $\dim_K \operatorname{Hom}(M, M')$ is finite for all $M, M' \in \mathscr{C}$. Then the maps*

$$\mathrm{W}^\epsilon(\mathscr{R}_{L/K}) : \mathrm{W}^\epsilon(\mathscr{C}) \to \mathrm{W}^\epsilon(\mathscr{C}_L^\circ) \quad \textit{and} \quad \mathrm{W}(\mathscr{R}_{L/K}) : \mathrm{W}_S(\mathscr{C}) \to \mathrm{W}_S(\mathscr{C}_L^\circ)$$

are injective.

Proof. We begin by showing that $\mathrm{W}^\epsilon(\mathscr{R}_{L/K}) : \mathrm{W}^\epsilon(\mathscr{C}) \to \mathrm{W}^\epsilon(\mathscr{C}_L^\circ)$ is injective. Let $(M, s) \in \mathrm{UH}^\epsilon(\mathscr{C})$ be such that $(M_L, s_L) \equiv 0$ in $\mathrm{W}^\epsilon(\mathscr{C}_L^\circ)$. Then there are objects $N, N' \in \mathscr{C}_L^\circ$ such that $s_L \oplus \mathbb{H}_N^\epsilon \simeq \mathbb{H}_{N'}^\epsilon$. Let

$$(U, h) = (M, s) \oplus (N' \oplus N'^*, \mathbb{H}_{N'}^\epsilon) \quad \text{and} \quad E = \operatorname{End}_{\mathscr{C}^\circ}(U),$$

and let σ be the involution induced by h on E. Set $E_L = E \otimes_K L = \operatorname{End}_{\mathscr{C}_L^\circ}(U_L)$ and $\sigma_L = \sigma \otimes_K \mathrm{id}_L$. Section 2E implies that $\mathscr{R}_{L/K}(\mathrm{T}_{(U,h)}(M, s)) = \mathrm{T}_{(U_L,h_L)}(M_L, s_L) \equiv 0$ in $\mathrm{W}^\epsilon(E_L, \sigma_L)$, and by [Bayer-Fluckiger and Lenstra 1990, Proposition 1.2], this means $\mathrm{T}_{(U,h)}(M, s) \equiv 0$ in $\mathrm{W}^\epsilon(E, \sigma)$ (here we need $\dim_K E < \infty$). Since $\mathscr{C}$ is pseudoabelian, the map $\mathrm{T}_{(U,h)} : \mathscr{C}|_U \to \mathscr{P}(E)$ is an equivalence of categories, hence the induced map $\mathrm{W}^\epsilon(\mathrm{T}_{(U,h)}) : \mathrm{W}^\epsilon(\mathscr{C}|_U) \to \mathrm{W}(\mathscr{P}(E)) = \mathrm{W}^\epsilon(E, \sigma)$ is an isomorphism of groups. Therefore, $(M, s) \equiv 0$ in $\mathrm{W}^\epsilon(\mathscr{C}|_U)$. In particular, the same identity holds in $\mathrm{W}^\epsilon(\mathscr{C})$.

Now let $(M, s) \in \operatorname{Sesq}(\mathscr{C})$ be such that $(M_L, s_L) \equiv 0$ in $\mathrm{W}_S(\mathscr{C}_L^\circ)$. Then by Proposition 5.13, (M_L, s_L) is hyperbolic in $\mathscr{C}_L^\circ$ (but not, a priori, in $\mathscr{C}_L$). Let F be the functor defined in Theorem 3.2 and let J be the functor $\tilde{\mathrm{Ar}}_2(\mathscr{C})_L \to \tilde{\mathrm{Ar}}_2(\mathscr{C}_L)$ of Proposition 3.7. By the lemma, there is a fully faithful 1-hermitian duality-preserving functor $J' := GJ^\circ : \tilde{\mathrm{Ar}}_2(\mathscr{C})_L^\circ \to \tilde{\mathrm{Ar}}_2(\mathscr{C}_L^\circ)$. Since (M_L, s_L) is hyperbolic in $\mathscr{C}_L^\circ$, there is $Q \in \tilde{\mathrm{Ar}}_2(\mathscr{C}_L^\circ)$ such that $F(M_L, s_L) \simeq (Q \oplus Q^*, \mathbb{H}_Q)$. Let $Z(M, s) := (M, M, s^*\omega_M, s)$ and $Z(M_L, s_L) = (M_L, M_L, s_L^*\omega_{M_L}, s_L)$. Recall that $F(M_L, s_L) = F\mathscr{R}_{L/K}(M, s) = J\mathscr{R}_{L/K}F(M, s)$ (Proposition 3.7) and hence $Q \oplus Q^* \simeq Z(M_L, s_L) = J(Z(M, s)_L) = J'(Z(M, s)_L)$. As J' is fully faithful and its image is pseudoabelian, we may assume $Q = J'H$ for some $H \in \tilde{\mathrm{Ar}}_2(\mathscr{C})_L^\circ$. We now have $J'(H \oplus H^*, \mathbb{H}_H) = (Q \oplus Q^*, \mathbb{H}_Q) \simeq F(M_L, s_L) = J'\mathscr{R}_{L/K}F(M, s)$, hence $(H \oplus H^*, \mathbb{H}_H) \simeq \mathscr{R}_{L/K}F(M, s)$ in $\tilde{\mathrm{Ar}}_2(\mathscr{C})_L^\circ$. In particular, $\mathscr{R}_{L/K}F(M, s) \equiv 0$ in $\mathrm{W}(\tilde{\mathrm{Ar}}_2(\mathscr{C})_L^\circ)$. By the previous paragraph, this means $F(M, s) \equiv 0$ in $\mathrm{W}(\tilde{\mathrm{Ar}}_2(\mathscr{C}))$ and hence; $(M, s) \equiv 0$ in $\mathrm{W}_S(\mathscr{C})$. ☐

We also have the following weaker version of Springer's theorem that works without assuming $\mathscr{C}$ is pseudoabelian:

Theorem 5.19. *Suppose that $(\mathscr{C}, *, \omega)$ is a K-linear hermitian category such that $\dim_K \operatorname{Hom}(M, M')$ is finite for all $M, M' \in \mathscr{C}$. Then $\mathrm{W}^\epsilon(\mathscr{R}_{L/K}) : \mathrm{W}^\epsilon(\mathscr{C}) \to \mathrm{W}^\epsilon(\mathscr{C}_L)$ is injective.*

Proof. Let $(M, s) \in \mathrm{UH}^\epsilon(\mathscr{C})$ be such that $(M_L, s_L) \equiv 0$ in $\mathrm{W}^\epsilon(\mathscr{C}_L)$. Then there are objects N_L, N'_L such that $s_L \oplus \mathbb{H}^\epsilon_{N_L} \simeq \mathbb{H}^\epsilon_{N'_L}$. Since $\mathbb{H}^\epsilon_{N_L} = (\mathbb{H}^\epsilon_N)_L$ and $\mathbb{H}^\epsilon_{N'_L} = (\mathbb{H}^\epsilon_{N'})_L$, we have $(s \oplus \mathbb{H}^\epsilon_N)_L \simeq (\mathbb{H}^\epsilon_{N'})_L$. By Theorem 5.15, this means that $s \oplus \mathbb{H}^\epsilon_N \simeq \mathbb{H}^\epsilon_{N'}$, hence $(M, s) \equiv 0$ in $\mathrm{W}^\epsilon(\mathscr{C})$. □

5F. *Weak Hasse principle.* In this final subsection, we prove a version of the *weak Hasse principle* for systems of sesquilinear forms over hermitian categories. Recall that the weak Hasse principle asserts that two quadratic forms over a global field k are isometric if and only if they are isometric over all completions of k. This actually fails for systems of quadratic forms, and we refer the reader to [Bayer-Fluckiger 1985; 1987] for necessary and sufficient conditions for the weak Hasse principle to hold in this case. A weak Hasse principle for *sesquilinear* forms defined over a skew field with a unitary involution was obtained in [Bayer-Fluckiger and Moldovan 2014].

Let K be a commutative ring admitting an involution σ, and let k be the fixed ring of σ. Let $\mathscr{C}$ be an additive K-category. A hermitian structure $(*, \omega)$ on $\mathscr{C}$ is called (K, σ)-*linear* if $(fa)^* = f^*\sigma(a)$ for all $a \in K$ and any morphism f in $\mathscr{C}$. (This means that the functor $*$ is k-linear.) In this case, $\mathrm{End}(M)$ is a K-algebra for all $M \in \mathscr{C}$, and for any unimodular ϵ-hermitian form (M, s) over $\mathscr{C}$, the restriction of the involution $f \mapsto s^{-1} f^* s$ to $K \cdot \mathrm{id}_M$ is σ.

Suppose now that K is a global field of characteristic not 2 admitting an involution σ of the second kind with fixed field k, and that $\mathscr{C}$ admits a nonempty family of (K, σ)-linear hermitian structures $\{*_i, \omega_i\}_{i \in I}$. For every prime spot p of k, let k_p be the completion of k at p, and set $K_p = K \otimes_k k_p$, $\sigma_p = \sigma \otimes_k \mathrm{id}_{k_p}$ and $\mathscr{C}_p = \mathscr{C} \otimes_k k_p$. Then each of the hermitian structures $(*_i, \omega_i)$ gives rise to a (K_p, σ_p)-linear hermitian structure on $\mathscr{C}_p$, which we also denote by $(*_i, \omega_i)$.

Theorem 5.20. *Let K be a global field of characteristic not* 2 *admitting an involution σ* of the second kind *with fixed field k. Let $\mathscr{C}$ be a K-category such that* $\dim_K \mathrm{Hom}(M, N)$ *is finite for all $M, N \in \mathscr{C}$, and assume there is a nonempty family* $\{*_i, \omega_i\}_{i \in I}$ *of (K, σ)-linear hermitian structures on $\mathscr{C}$. Then the weak Hasse principle* (*with respect to k*) *holds for systems of sesquilinear forms over* $(\mathscr{C}, \{*_i, \omega_i\})$. *That is, two systems of sesquilinear forms over* $(\mathscr{C}, \{*_i, \omega_i\})$ *are isometric if and only if they are isometric after applying $\mathscr{R}_{k_p/k}$ for all p.*

We will need the following lemma. (The lemma seems to be known, but we could not find an explicit reference, and hence included here an ad hoc proof.)

Lemma 5.21. *Let L/K be any field extension, and let $\mathscr{C}$ be an additive K-category such that* $\dim_K \mathrm{Hom}_{\mathscr{C}}(M, N)$ *is finite for all $M, N \in \mathscr{C}$. Then for all $N, M \in \mathscr{C}$, we have $N \cong M$ if and only if $N_L \cong M_L$.*

Sketch of proof. By applying $\mathrm{Hom}_{\mathscr{C}}(M \oplus N, \_)$, we may assume M and N are finitely generated projective right modules over $R := \mathrm{End}(M \oplus N)$, which is a finite-dimensional K-algebra by assumption. Let J be the Jacobson radical of R. By tensoring with R/J, we may assume R is semisimple. Let $\{V_i\}_i$ be a complete list of the simple right R-modules and write

$$(V_i)_L = \bigoplus_j W_{ij}^{n_{ij}},$$

the $\{W_{ij}\}_j$ being pairwise nonisomorphic indecomposable R_L-modules. The R_L-modules $\{W_{ij}\}_{i,j}$ are pairwise nonisomorphic because W_{ij} and $W_{i'j'}$ are nonisomorphic as R-modules when $i \neq i'$ (W_{ij} is isomorphic as an R-module to a direct sum of copies of V_i). Assume $M_L \cong N_L$ and write $M \cong \bigoplus_i V_i^{m_i}$, $N \cong \bigoplus_i V_i^{m'_i}$. Then $\bigoplus_{i,j} W_{ij}^{m_i n_{ij}} \cong M_L \cong N_L \cong \bigoplus_{i,j} W_{ij}^{m'_i n_{ij}}$. By the Krull–Schmidt theorem, we have $m_i n_{ij} = m'_i n_{ij}$ for all i, j, hence $m_i = m'_i$ and $M \cong N$. □

Proof of Theorem 5.20. By Corollary 4.4, it is enough to verify the Hasse principle (with respect to k) for 1-hermitian forms in the category $\mathscr{G} := \tilde{\mathrm{Ar}}_{2I}(\mathscr{C})$. Our assumptions imply that $\mathscr{G}$ is a (K, σ)-linear category such that $\dim_K \mathrm{Hom}(Z, Z')$ is finite for all $Z, Z' \in \mathscr{G}$. We now use the ideas developed in [Bayer-Fluckiger and Moldovan 2014, §9].

Let (Z, h), (Z', h') be two unimodular 1-hermitian forms over $\mathscr{G}$ such that $\mathscr{R}_{k_p/k}(Z, h) \simeq \mathscr{R}_{k_p/k}(Z', h')$ for all p. By Lemma 5.21, this implies that $Z \cong Z'$, so we may assume $Z = Z'$.

Fix a 1-hermitian form h_0 on Z and let τ be the involution induced by h_0 on $E := \mathrm{End}(Z)$ (i.e., $\tau(x) = h_0^{-1} x^* h_0$). There is an equivalence relation on the elements of E defined by $x \sim y$ if and only if there exists an invertible $z \in E$ such that $x = zy\tau(z)$. Let $H(\tau, E^\times)$ be the set of equivalence classes of invertible elements $x \in E^\times$ for which $x = \tau(x)$. In the same manner as in [Bayer-Fluckiger and Moldovan 2014, Theorem 5.1], we see that there is a one-to-one correspondence between isometry classes of unimodular 1-hermitian forms on Z and elements $H(\tau, E^\times)$. It is given by $(Z, t) \mapsto h_0^{-1} t$.

Applying the same argument to $Z_p = \mathscr{R}_{k_p/k} Z \in \mathscr{G}_p$, we see that the weak Hasse principle is equivalent to the injectivity of the standard map

$$\Phi : H(\tau, E^\times) \to \prod_p H(\tau_p, E_p^\times),$$

where $E_p = \mathrm{End}(Z_p) = E \otimes_k k_p$ and $\tau_p = \tau \otimes_k \mathrm{id}_{k_p}$. Observe that since $\mathscr{G}$ is (K, σ)-linear, τ is a unitary involution (and in fact, $\tau|_K = \sigma$). By [Bayer-Fluckiger and Moldovan 2014, §9], this means that Φ is injective, hence the weak Hasse holds. □

Corollary 5.22. *Let K be a global field of characteristic not* 2 *admitting an involution σ* of the second kind *with fixed field k. Let A be a finite-dimensional K-algebra admitting a nonempty family of involutions $\{\sigma_i\}_{i\in I}$ such that $\sigma_i|_K = \sigma$. Then the weak Hasse principle* (*with respect to k*) *holds for systems of sesquilinear forms over $(A, \{\sigma_i\})$.*

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REALIZATIONS OF THE THREE-POINT LIE ALGEBRA $\mathfrak{sl}(2, \mathcal{R}) \oplus (\Omega_{\mathcal{R}}/d\mathcal{R})$

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This paper is dedicated to Robert Wilson.

We describe the universal central extension of the three-point current algebra $\mathfrak{sl}(2, \mathcal{R})$, where $\mathcal{R} = \mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 + 4t]$, and construct realizations of it in terms of sums of partial differential operators.

1. Introduction

It is well known from the work of Kassel and Loday (see [Kassel and Loday 1982; 1984]) that if R is a commutative algebra and $\mathfrak{g}$ is a simple Lie algebra, both defined over the complex numbers, then the universal central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g} \otimes R$ is the vector space $(\mathfrak{g} \otimes R) \oplus \Omega^1_R/dR$, where Ω^1_R/dR is the space of Kähler differentials modulo exact forms (see [Kassel 1984]). The vector space $\hat{\mathfrak{g}}$ is made into a Lie algebra by defining

$$[x \otimes f, y \otimes g] := [xy] \otimes fg + (x, y)\overline{f\,dg}, \quad [x \otimes f, \omega] = 0$$

for $x, y \in \mathfrak{g}$, $f, g \in R$, $\omega \in \Omega^1_R/dR$, where $(-,-)$ denotes the Killing form on $\mathfrak{g}$. Here $\bar{a}$ denotes the image of $a \in \Omega^1_R$ in the quotient Ω^1_R/dR. A somewhat vague but natural question is whether there exist free field or Wakimoto-type realizations of these algebras. It is well known from the work of Wakimoto and of Feigin and Frenkel what the answer is when R is the ring of Laurent polynomials in one variable (see [Wakimoto 1986] and [Feigin and Frenkel 1990]). We find such a realization in the setting where $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 + 4t]$, and $\hat{\mathfrak{g}}$ is the three-point algebra.

In Kazhdan and Lusztig's explicit study [1991; 1993] of the tensor structure of modules for affine Lie algebras the ring of functions regular everywhere except at a finite number of points appears naturally. This algebra Bremner gave the name *n-point algebra*. In particular, in [Frenkel and Ben-Zvi 2001, Chapter 12], algebras of the form $\bigoplus_{i=1}^n \mathfrak{g}((t - x_i)) \oplus \mathbb{C}c$ appear in the description of the conformal blocks.

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These contain the n-point algebras $\mathfrak{g} \otimes \mathbb{C}[(t-x_1)^{-1}, \dots, (t-x_N)^{-1}] \oplus \mathbb{C}c$ modulo part of the center Ω_R/dR. Bremner [1994a] explicitly described the universal central extension of such an algebra.

Consider now the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with coordinate function s, and fix three distinct points a_1, a_2, a_3 on this Riemann sphere. Let R denote the ring of rational functions with poles only in the set $\{a_1, a_2, a_3\}$. It is known that the automorphism group $\mathrm{PGL}_2(\mathbb{C})$ of $\mathbb{C}(s)$ is simply 3-transitive, and R is a subring of $\mathbb{C}(s)$ that is isomorphic to the ring of rational functions with poles at $\{\infty, 0, 1, a\}$. Motivated by this isomorphism, one sets $a = a_4$ and here the *four-point ring* is $R = R_a = \mathbb{C}[s, s^{-1}, (s-1)^{-1}, (s-a)^{-1}]$, where $a \in \mathbb{C}\backslash\{0, 1\}$. Let $S := S_b = \mathbb{C}[t, t^{-1}, u]$, where $u^2 = t^2 - 2bt + 1$ with b a complex number not equal to ± 1. Then Bremner has shown us that $R_a \cong S_b$. As the latter, being $\mathbb{Z}_2$-graded, is a cousin to super Lie algebras, it is thus more immediately amendable to the theatrics of conformal field theory. Moreover, Bremner has given an explicit description of the universal central extension of $\mathfrak{g} \otimes R$ in terms of ultraspherical (Gegenbauer) polynomials where R is the four-point algebra (see [Bremner 1995]). In [Cox 2008] a realization was given for the four-point algebra where the center acts nontrivially.

In his study of the elliptic affine Lie algebras $\mathfrak{sl}(2, R) \oplus (\Omega_R/dR)$ where $R = \mathbb{C}[x, x^{-1}, y \mid y^2 = 4x^3 - g_2 x - g_3]$, Bremner [1994b] has also explicitly described the universal central extension of this algebra in terms of Pollaczek polynomials. Essentially the same algebras appear in [Fialowski and Schlichenmaier 2007; 2005]. Together with Bueno and Futorny, the first author described free-field-type realizations of the elliptic Lie algebra where $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = t^3 - 2bt^2 - t]$, $b \neq \pm 1$ (see [Bueno et al. 2009]).

Below, we study the three-point algebra case where R denotes the ring of rational functions with poles only in the set $\{a_1, a_2, a_3\}$. This algebra is isomorphic to $\mathbb{C}[s, s^{-1}, (s-1)^{-1}]$. Schlichenmaier [2003a] has a slightly different description of the three-point algebra as $\mathbb{C}[(z^2-a^2)^k, z(z^2-a^2)^k \mid k \in \mathbb{Z}]$, where $a \neq 0$. We show that $R \cong \mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 + 4t]$, thus resembling S_b above. Our main result, Theorem 5.1, provides a natural free field realization in terms of a β-γ-system and the oscillator algebra of the three-point affine Lie algebra when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Just as in the case of intermediate Wakimoto modules defined in [Cox and Futorny 2006], there are two different realizations depending on two different normal orderings. Besides Bremner's article mentioned above, other work on the universal central extension of three-point algebras can be found in [Benkart and Terwilliger 2007]. Previous related work on highest-weight modules of $\mathfrak{sl}(2, R)$ can be found in [Jakobsen and Kac 1985].

The three-point algebra is perhaps the simplest nontrivial example of a Krichever–Novikov algebra beyond an affine Kac–Moody algebra (see [Krichever and Novikov 1987a; 1987b; 1989]). A fair amount of interesting and fundamental work has been done by Krichever, Novikov, Schlichenmaier, and Sheinman on the representation theory of Krichever–Novikov algebras. In particular, Wess–Zumino–Witten–Novikov theory and analogues of the Knizhnik–Zamolodchikov (KZ) equations are developed for these algebras (see the survey article [Sheinman 2005], and, for example, [Schlichenmaier and Sheinman 1996; 1999; Sheinman 2003; Schlichenmaier 2003a; 2003b]).

The initial motivation for the use of Wakimoto's realization was to prove a conjecture of Kac and Kazhdan on the characters of certain irreducible representations of affine Kac–Moody algebras at the critical level (see [Wakimoto 1986] and [Frenkel 2005]). Another motivation for constructing free field realizations is that they are used to provide integral solutions to the KZ equations (see for example [Schechtman and Varchenko 1990] and [Etingof et al. 1998] and their references). A third is that they are used in determining the center of a certain completion of the enveloping algebra of an affine Lie algebra at the critical level, which is an important ingredient in the geometric Langland's correspondence [Frenkel 2007]. Yet a fourth is that free field realizations of an affine Lie algebra appear naturally in the context of the generalized AKNS hierarchies [Feigin and Frenkel 1999].

2. The three-point ring

The three-point algebra has at least four incarnations.

Three-point algebras. Fix a nonzero $a \in \mathbb{C}$. Let

$$\begin{aligned}\mathcal{S} &:= \mathbb{C}[s, s^{-1}, (s-1)^{-1}],\\ \mathcal{R} &:= \mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 + 4t],\\ \mathcal{A} &:= \mathcal{A}_a = \mathbb{C}[(z^2-a^2)^k, z(z^2-a^2)^j \mid k, j \in \mathbb{Z}].\end{aligned}$$

Note that Bremner introduced the ring $\mathcal{S}$ and Schlichenmaier [2003a] introduced $\mathcal{A}$. Variants of $\mathcal{R}$ were introduced by Bremner for elliptic and three-point algebras.

Proposition 2.1. (1) *The rings $\mathcal{R}$ and $\mathcal{S}$ are isomorphic by $t \mapsto s^{-1}(s-1)^2$ and $u \mapsto s - s^{-1}$.*

(2) *The rings $\mathcal{R}$ and $\mathcal{A}$ are isomorphic.*

Proof. (1) Let $\bar{f} : \mathbb{C}[t, u] \to \mathcal{S}$ be the ring homomorphism defined by $\bar{f}(t) = s^{-1}(s-1)^2 = s - 2 + s^{-1}$, $\bar{f}(u) = s - s^{-1}$.

We first check that

$$\bar{f}(u^2 - (t^2 + 4t)) = (s - s^{-1})^2 - (s - 2 + s^{-1})^2 - 4(s - 2 + s^{-1}) = 0$$

and $\bar{f}(t) = s^{-1}(s-1)^2$ is invertible in $\mathcal{S}$. Hence the map $\bar{f}$ descends to a well-defined ring homomorphism $f : \mathcal{R} \to \mathcal{S}$. To show that it is onto, we essentially solve for s and s^{-1} in terms of t and u. The inverse ring homomorphism of f is $\phi : \mathcal{S} \to \mathcal{R}$, given by

$$\phi(s) = \frac{t+2+u}{2}, \quad \phi(s^{-1}) = \frac{t+2-u}{2}.$$

In particular, $\phi((s-1)^{-1}) = (t^{-1}u - 1)/2$.

For part (2), observe $\mathcal{A} = \mathbb{C}[z, (z-a)^{-1}, (z+a)^{-1}]$, so, mapping z to $2as - a$, we get $\mathcal{A} \cong \mathbb{C}[s, s^{-1}, (s-1)^{-1}]$. Thus an isomorphism between $\mathcal{A}$ and $\mathcal{R}$ is implemented by the assignment

$$z \mapsto a(t+u), \quad (z+a)^{-1} \mapsto \frac{t+2-u}{4a}, \quad (z-a)^{-1} \mapsto \frac{t^{-1}u - 1}{4a}. \qquad \square$$

The universal central extension of the current algebra $\mathfrak{g} \otimes \mathcal{A}$. Let R be a commutative algebra defined over $\mathbb{C}$. Consider the left R-module $F = R \otimes R$ with left action given by $f(g \otimes h) = fg \otimes h$ for $f, g, h \in R$, and let K be the submodule generated by the elements $1 \otimes fg - f \otimes g - g \otimes f$. Then $\Omega_R^1 = F/K$ is the module of *Kähler differentials*. The element $f \otimes g + K$ is traditionally denoted by $f\,dg$. The canonical map $d : R \to \Omega_R^1$ is given by $df = 1 \otimes f + K$. The *exact differentials* are the elements of the subspace dR. The coset of $f\,dg$ modulo dR is denoted by $\overline{f\,dg}$. As Kassel has shown, the universal central extension of the current algebra $\mathfrak{g} \otimes R$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra defined over $\mathbb{C}$, is the vector space $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes R) \oplus \Omega_R^1/dR$, with Lie bracket given by

$$[x \otimes f, Y \otimes g] = [xy] \otimes fg + (x, y)\overline{f\,dg}, \quad [x \otimes f, \omega] = 0, \quad [\omega, \omega'] = 0,$$

where $x, y \in \mathfrak{g}$, $\omega, \omega' \in \Omega_R^1/dR$, and (x, y) denotes the Killing form on $\mathfrak{g}$.

There are at least four incarnations of the three-point algebras, three of which are defined as $\mathfrak{g} \otimes R \oplus \Omega_R/dR$ where $R = \mathcal{S}, \mathcal{R}, \mathcal{A}$ given above. The fourth incarnation appears in [Benkart and Terwilliger 2007] and is given in terms of the tetrahedron algebra. We will only work with $R = \mathcal{R}$.

Proposition 2.2 ([Bremner 1994a]; see also [Bremner 1995]). *Let $\mathcal{R}$ be as above. The set*

$$\{\omega_0 := \overline{t^{-1}\,dt},\ \omega_1 := \overline{t^{-1}u\,dt}\}$$

is a basis of $\Omega_{\mathcal{R}}^1/d\mathcal{R}$.

Proof. The proof follows almost exactly along the lines of [Bremner 1995] and [Bremner 1994a]. We know by the Riemann–Roch theorem that the space $\Omega_{\mathcal{R}}/d\mathcal{R}$ of Kähler differentials modulo exact forms on the sphere with three punctures has dimension 2 (see [Bremner 1994a]). We have the following formulae:

$$d(t^k)=kt^{k-1}\,dt,$$

$$d(t^k u)=t^k\,du+kt^{k-1}u\,dt, \tag{2-1}$$

$$t^k u\,dt\equiv -\frac{k+3}{4k+6}\,t^{k+1}u\,dt \qquad \text{mod } d\mathcal{R}, \tag{2-2}$$

$$t^{k-1}\,dt\equiv \frac{1}{k}d(t^k)\equiv 0 \qquad \text{mod } d\mathcal{R} \text{ for } k\neq 0. \tag{2-3}$$

By Equations (2-1), (2-2), and (2-3), we conclude that $\Omega_{\mathcal{R}}/d\mathcal{R}$ is spanned by $\{\overline{t^{-1}\,dt},\overline{t^{-1}u\,dt}\}$. □

Corollary 2.3. *In Ω^1_R/dR, one has*

$$\overline{t^k\,dt^l}=-k\delta_{l,-k}\,\omega_0, \tag{2-4}$$

$$\overline{t^k u\,d(t^l u)}=((l+1)\delta_{k+l,-2}+(4l+2)\delta_{k+l,-1})\omega_0, \tag{2-5}$$

$$\overline{t^k\,d(t^l u)}=-k\delta_{k,-l}\,\omega_1. \tag{2-6}$$

Proof. Using (2-1) above, we obtain

$$\begin{aligned}
t^k\,d(t^l u)&\equiv t^k(lt^{l-1}u\,dt+t^l\,du)\\
&\equiv lt^{l+k-1}u\,dt+t^{l+k}\,du\\
&\equiv lt^{l+k-1}u\,dt-(l+k)t^{l+k-1}u\,dt\\
&\equiv -kt^{l+k-1}u\,dt\\
&\equiv -k\delta_{l+k,0}t^{-1}u\,dt \quad \text{mod } d\mathcal{R}
\end{aligned}$$

in $\Omega_{\mathcal{R}}/\mathcal{R}$.

Next we observe $u\,du=\frac{1}{2}d(u^2)=\frac{1}{2}d(t^2+4t)=(t+2)\,dt$, so in $\Omega_{\mathcal{R}}$,

$$t^k u\,du=(t^{k+1}+2t^k)\,dt. \tag{2-7}$$

By (2-7) and (2-3),

$$\begin{aligned}
t^k u\,d(t^l u)&=t^k u(lt^{l-1}u\,dt+t^l\,du) \quad \text{in } \Omega_{\mathcal{R}}\\
&=(lt^{l+k-1}u^2\,dt+t^{l+k}u\,du)\\
&=(lt^{l+k-1}(t^2+4t)\,dt+(t^{l+k+1}+2t^{l+k})\,dt)\\
&=l(t^{k+l+1}+4t^{k+l})\,dt+(t^{l+k+1}+2t^{l+k})\,dt)\\
&=(l+1)t^{k+l+1}\,dt+(4l+2)t^{k+l}\,dt\\
&\equiv((l+1)\delta_{k+l,-2}+(4l+2)\delta_{k+l,-1})t^{-1}\,dt \quad \text{mod } \mathcal{R}.
\end{aligned}$$

This completes the proof of the corollary. □

Theorem 2.4. *The universal central extension of the algebra $\mathfrak{sl}(2, \mathbb{C}) \otimes \mathcal{R}$ is isomorphic to the Lie algebra with generators $e_n, e_n^1, f_n, f_n^1, h_n, h_n^1$, $n \in \mathbb{Z}$, ω_0, ω_1, and relations given by*

$$[x_m, x_n] := [x_m, x_n^1] = [x_m^1, x_n^1] = 0 \quad \text{for } x = e, f, \tag{2-8}$$

$$[h_m, h_n] := -2m\delta_{m,-n}\omega_0 = (n-m)\delta_{m,-n}\omega_0, \tag{2-9}$$

$$\begin{aligned}[h_m^1, h_n^1] &:= 2((n+1)\delta_{m+n,-2} + (4n+2)\delta_{m+n,-1})\omega_0 \\ &= (n-m)(\delta_{m+n,-2} + 4\delta_{m+n,-1})\omega_0,\end{aligned} \tag{2-10}$$

$$[h_m, h_n^1] := -2m\delta_{m,-n}\omega_1, \tag{2-11}$$

$$[\omega_i, x_m] := [\omega_i, \omega_j] = 0 \quad \text{for } x = e, f, h \text{ and } i, j \in \{0, 1\} \tag{2-12}$$

$$[e_m, f_n] := h_{m+n} - m\delta_{m,-n}\omega_0, \tag{2-13}$$

$$[e_m, f_n^1] := h_{m+n}^1 - m\delta_{m,-n}\omega_1 =: [e_m^1, f_n], \tag{2-14}$$

$$\begin{aligned}[e_m^1, f_n^1] &:= h_{m+n+2} + 4h_{m+n+1} + ((n+1)\delta_{m+n,-2} + (4n+2)\delta_{m+n,-1})\omega_0 \\ &= h_{m+n+2} + 4h_{m+n+1} + \tfrac{1}{2}(n-m)(\delta_{m+n,-2} + 4\delta_{m+n,-1})\omega_0,\end{aligned} \tag{2-15}$$

$$[h_m, e_n] := 2e_{m+n}, \tag{2-16}$$

$$[h_m, e_n^1] := 2e_{m+n}^1 =: [h_m^1, e_m], \tag{2-17}$$

$$[h_m^1, e_n^1] := 2e_{m+n+2} + 8e_{m+n+1}, \tag{2-18}$$

$$[h_m, f_n] := -2f_{m+n}, \tag{2-19}$$

$$[h_m, f_n^1] := -2f_{m+n}^1 =: [h_m^1, f_m], \tag{2-20}$$

$$[h_m^1, f_n^1] := -2f_{m+n+2} - 8f_{m+n+1}, \tag{2-21}$$

for all $m, n \in \mathbb{Z}$.

Proof. Let $\mathfrak{f}$ denote the free Lie algebra with generators $e_n, e_n^1, f_n, f_n^1, h_n, h_n^1$, $n \in \mathbb{Z}$, ω_0, ω_1, and relations given above in (2-8) through (2-21). The map

$$\phi : \mathfrak{f} \to (\mathfrak{sl}(2, \mathbb{C}) \otimes \mathcal{R}) \oplus (\Omega_{\mathcal{R}}/d\mathcal{R})$$

given by

$$\begin{aligned} \phi(e_n) &:= e \otimes t^n, & \phi(e_n^1) &= e \otimes ut^n, \\ \phi(f_n) &:= f \otimes t^n, & \phi(f_n^1) &= f \otimes ut^n, \\ \phi(h_n) &:= h \otimes t^n, & \phi(h_n^1) &= h \otimes ut^n, \\ \phi(\omega_0) &:= \overline{t^{-1}\,dt}, & \phi(\omega_1) &= \overline{t^{-1}u\,dt}, \end{aligned}$$

for $n \in \mathbb{Z}$, is a surjective Lie algebra homomorphism.

Consider the subalgebras $S_+ = \langle e_n, e_n^1 \mid n \in \mathbb{Z}\rangle$, $S_0 = \langle h_n, h_n^1, \omega_0, \omega_1 \mid n \in \mathbb{Z}\rangle$, and $S_- = \langle f_n, f_n^1 \mid n \in \mathbb{Z}\rangle$, and set $S = S_- + S_0 + S_+$. By (2-8) through (2-12), we have

$$S_+=\sum_{n\in\mathbb{Z}}\mathbb{C}e_n+\sum_{n\in\mathbb{Z}}\mathbb{C}e_n^1,\quad S_-=\sum_{n\in\mathbb{Z}}\mathbb{C}f_n+\sum_{n\in\mathbb{Z}}\mathbb{C}f_n^1,$$
$$S_0=\sum_{n\in\mathbb{Z}}\mathbb{C}h_n+\sum_{n\in\mathbb{Z}}\mathbb{C}h_n^1+\mathbb{C}\omega_0+\mathbb{C}\omega_1.$$

By (2-13) through (2-18), we see that

$$[e_n,S_+]=[e_n^1,S_+]=0,\quad [h_n,S_+]\subseteq S_+,\quad [h_n^1,S_+]\subseteq S_+,$$
$$[f_n,S_+]\subseteq S_0,\quad [f_n^1,S_+]\subseteq S_0,$$

and similarly $[x_n,S_-]=[x_n^1,S_-]\subseteq S$, $[x_n,S_0]=[x_n^1,S_0]\subseteq S$ for $x=e,f,h$. To sum up, we observe that $[x_n,S]\subseteq S$ and $[x_n^1,S]\subseteq S$ for $n\in\mathbb{Z}$, $x=h,e,f$. Thus $[S,S]\subset S$. Now, S contains the generators of $\mathfrak{f}$ and is a subalgebra. Hence $S=\mathfrak{f}$. Now it is clear that ϕ is a Lie algebra isomorphism. □

3. A triangular decomposition of the three-point loop algebras $\mathfrak{g}\otimes R$

From now on we identify R_a with $\mathcal{S}$ and set $R=\mathcal{S}$, which has a basis t^i, t^iu for $i\in\mathbb{Z}$. Let $p:R\to R$ be the automorphism given by $p(t)=t$ and $p(u)=-u$. Then one can decompose $R=R^0\oplus R^1$, where $R^0=\mathbb{C}[t^{\pm1}]=\{r\in R\mid p(r)=r\}$ and $R^1=\mathbb{C}[t^{\pm1}]u=\{r\in R\mid p(r)=-r\}$ are the eigenspaces of p. From now on, $\mathfrak{g}$ will denote a simple Lie algebra over $\mathbb{C}$ with triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, and then the *three-point loop algebra* $L(\mathfrak{g}):=\mathfrak{g}\otimes R$ has a corresponding $\mathbb{Z}/2\mathbb{Z}$-grading: $L(\mathfrak{g})^i:=\mathfrak{g}\otimes R^i$ for $i=0,1$. However, the degree of t does not render $L(\mathfrak{g})$ a $\mathbb{Z}$-graded Lie algebra. This leads us to the following notion:

Suppose I is an additive subgroup of the rational numbers $\mathbb{Q}$ and $\mathcal{A}$ is a $\mathbb{C}$-algebra such that $\mathcal{A}=\bigoplus_{i\in I}\mathcal{A}_i$, and that there exists a fixed $l\in\mathbb{N}$ with

$$\mathcal{A}_i\mathcal{A}_j\subset\bigoplus_{|k-(i+j)|\le l}\mathcal{A}_k$$

for all $i,j\in\mathbb{Z}$. Then $\mathcal{A}$ is said to be an *l-quasigraded algebra*. For nonzero $x\in\mathcal{A}_i$, one says that x is *homogeneous of degree* i and one writes $\deg x=i$.

For example, R has the structure of a 1-quasigraded algebra, where $I=\frac{1}{2}\mathbb{Z}$ and $\deg t^i=i$, $\deg t^iu=i+\frac{1}{2}$.

A *weak triangular decomposition* of a Lie algebra $\mathfrak{l}$ is a triple $(\mathfrak{H},\mathfrak{l}_+,\sigma)$ satisfying

(1) $\mathfrak{h}$ and $\mathfrak{l}_+$ are subalgebras of $\mathfrak{l}$,

(2) $\mathfrak{h}$ is abelian and $[\mathfrak{h},\mathfrak{l}_+]\subset\mathfrak{l}_+$,

(3) σ is an antiautomorphism of $\mathfrak{l}$ of order 2 which is the identity on $\mathfrak{h}$, and

(4) $\mathfrak{l}=\mathfrak{l}_+\oplus\mathfrak{h}\oplus\sigma(\mathfrak{l}_+)$.

We will let $\sigma(\mathfrak{l}_+)$ be denoted by $\mathfrak{l}_-$.

Theorem 3.1 [Bremner 1995, Theorem 2.1]. *The three-point loop algebra $L(\mathfrak{g})$ is a 1-quasigraded Lie algebra where* $\deg(x \otimes f) = \deg f$ *for f homogeneous. Set $R_+ = \mathbb{C}(1+u) \oplus \mathbb{C}[t,u]t$ and $R_- = p(R_+)$. Then $L(\mathfrak{g})$ has a weak triangular decomposition given by*

$$L(\mathfrak{g})_\pm = \mathfrak{g} \otimes R_\pm, \quad \mathcal{H} := \mathfrak{h} \otimes \mathbb{C}.$$

Formal distributions. We need some more notation that will simplify some of the arguments later. This notation follows roughly [Kac 1998] and [Matsuo and Nagatomo 1999]: The *formal delta function* $\delta(z/w)$ is the formal distribution

$$\delta(z/w) = z^{-1} \sum_{n \in \mathbb{Z}} z^{-n} w^n = w^{-1} \sum_{n \in \mathbb{Z}} z^n w^{-n}.$$

For any sequence of elements $\{a_m\}_{m \in \mathbb{Z}}$ in the ring $\mathrm{End}(V)$, V a vector space, the formal distribution

$$a(z) := \sum_{m \in \mathbb{Z}} a_m z^{-m-1}$$

is called a *field* if for any $v \in V$, $a_m v = 0$ for $m \gg 0$. If $a(z)$ is a field, then we set

$$a(z)_- := \sum_{m \geq 0} a_m z^{-m-1} \quad \text{and} \quad a(z)_+ := \sum_{m < 0} a_m z^{-m-1}. \tag{3-1}$$

The *normal ordered product* of two distributions $a(z)$ and $b(w)$ (and their coefficients) is defined by

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} {:}a_m b_n{:} z^{-m-1} w^{-n-1} = {:}a(z)b(w){:} = a(z)_+ b(w) + b(w) a(z)_-. \tag{3-2}$$

Now we should point out that while ${:}a^1(z_1) \cdots a^m(z_m){:}$ is always defined as a formal series, we will only define ${:}a(z)b(z){:} := \lim_{w \to z} {:}a(z)b(w){:}$ for certain pairs $(a(z), b(w))$.

Then one defines recursively

$${:}a^1(z_1) \cdots a^k(z_k){:} = {:}a^1(z_1)\big({:}a^2(z_2)({:} \cdots {:}a^{k-1}(z_{k-1})a^k(z_k){:}) \cdots {:}\big){:},$$

while the normal ordered product

$${:}a^1(z) \cdots a^k(z){:} = \lim_{z_1, z_2, \dots, z_k \to z} {:}a^1(z_1)\big({:}a^2(z_2)({:} \cdots {:}a^{k-1}(z_{k-1})a^k(z_k){:}) \cdots\big){:}$$

will only be defined for certain k-tuples $(a^1, \dots, a^k)$.

Let

$$\lfloor ab \rfloor = a(z)b(w) - {:}a(z)b(w){:} = [a(z)_-, b(w)], \tag{3-3}$$

(half of $[a(z), b(w)]$) denote the *contraction* of any two formal distributions $a(z)$ and $b(w)$. Note that the variables z, w are usually suppressed in this notation when no confusion will arise.

For $m=i-\frac{1}{2}$, $i\in\mathbb{Z}+\frac{1}{2}$ and $x\in\mathfrak{g}$, define $x_{m+\frac{1}{2}}=x\otimes t^{i-\frac{1}{2}}u=x^1_m$ and $x_m:=x\otimes t^m$. Motivated by conformal field theory, we set

$$x^1(z):=\sum_{m\in\mathbb{Z}}x_{m+\frac{1}{2}}z^{-m-1},\qquad x(z):=\sum_{m\in\mathbb{Z}}x_m z^{-m-1}.$$

Then the relations in Theorem 2.4 can be rewritten as

$$[x(z),y(w)]=[x,y](w)\delta(z/w)-(x,y)\omega_0\partial_w\delta(z/w),\tag{3-4}$$

$$\begin{aligned}[x^1(z),y^1(w)]=&\,P\big([x,y](w)\delta(z/w)-(x,y)\omega_0\partial_w\delta(z/w)\big)\\&-\tfrac{1}{2}(x,y)(\partial P)\omega_0\delta(z/w),\end{aligned}\tag{3-5}$$

$$[x(z),y^1(w)]=[x,y]^1(w)\delta(z/w)-(x,y)\omega_1\partial_w\delta(z/w)=[x^1(z),y(w)],\tag{3-6}$$

where $x,y\in\{e,f,h\}$.

4. Oscillator algebras

The β-γ system. In the physics literature, the following construction is often called the β-γ system, which corresponds to our a and a^* below. Let $\hat{\mathfrak{a}}$ be the infinite-dimensional oscillator algebra with generators $a_n, a_n^*, a_n^1, a_n^{1*}$, $n\in\mathbb{Z}$ together with $\mathbf{1}$, satisfying the relations

$$\begin{gathered}[a_n,a_m]=[a_m,a_n^1]=[a_m,a_n^{1*}]=[a_n^*,a_m^*]=[a_n^*,a_m^1]=[a_n^*,a_m^{1*}]=0,\\ [a_n^1,a_m^1]=[a_n^{1*},a_m^{1*}]=0=[\mathfrak{a},\mathbf{1}],\\ [a_n,a_m^*]=\delta_{m+n,0}\mathbf{1}=[a_n^1,a_m^{1*}].\end{gathered}$$

For $c=a,a^1$, and respectively $X=x,x^1$, with $r=0$ or $r=1$, we define $\mathbb{C}[\boldsymbol{x}]:=\mathbb{C}[x_n,x_n^1\mid n\in\mathbb{Z}]$ and $\rho_r:\hat{\mathfrak{a}}\to\mathfrak{gl}(\mathbb{C}[\boldsymbol{x}])$ by

$$\rho_r(c_m):=\begin{cases}\partial/\partial X_m & \text{if } m\ge 0 \text{ and } r=0,\\ X_m & \text{otherwise,}\end{cases}\tag{4-1}$$

$$\rho_r(c_m^*):=\begin{cases}X_{-m} & \text{if } m\le 0,\ \text{and } r=0,\\ -\partial/\partial X_{-m} & \text{otherwise,}\end{cases}\tag{4-2}$$

and $\rho_r(\mathbf{1})=1$. These two representations can be constructed using induction: For $r=0$, the representation ρ_0 is the $\hat{\mathfrak{a}}$-module generated by $1=:|0\rangle$, where

$$a_m|0\rangle=a_m^1|0\rangle=0 \text{ for } m\ge 0\quad\text{and}\quad a_m^*|0\rangle=a_m^{1*}|0\rangle=0\text{ for } m>0.$$

For $r=1$, the representation ρ_1 is the $\hat{\mathfrak{a}}$-module generated by $1=:|0\rangle$, where

$$a_m^*|0\rangle=a_m^{1*}|0\rangle=0\text{ for } m\in\mathbb{Z}.$$

If we define

$$\alpha(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad \alpha^*(z) := \sum_{n\in\mathbb{Z}} a_n^* z^{-n} \tag{4-3}$$

and

$$\alpha^1(z) := \sum_{n\in\mathbb{Z}} a_n^1 z^{-n-1}, \quad \alpha^{1*}(z) := \sum_{n\in\mathbb{Z}} a_n^{1*} z^{-n}, \tag{4-4}$$

then

$$\begin{aligned}
&[\alpha(z), \alpha(w)] = [\alpha^*(z), \alpha^*(w)] = [\alpha^1(z), \alpha^1(w)] = [\alpha^{1*}(z), \alpha^{1*}(w)] = 0,\\
&[\alpha(z), \alpha^*(w)] = [\alpha^1(z), \alpha^{1*}(w)] = \mathbf{1}\delta(z/w).
\end{aligned}$$

Note that $\rho_1(\alpha(z))$ and $\rho_1(\alpha^1(z))$ are not fields, whereas $\rho_r(\alpha^*(z))$ and $\rho_r(\alpha^{1*}(z))$ are always fields. Corresponding to these two representations there are two possible normal orderings: For $r = 0$ we use the usual normal ordering given by (3-1) and for $r = 1$ we define the *natural normal ordering* to be

$$\begin{aligned}
\alpha(z)_+ &= \alpha(z), & \alpha(z)_- &= 0,\\
\alpha^1(z)_+ &= \alpha^1(z), & \alpha^1(z)_- &= 0,\\
\alpha^*(z)_+ &= 0, & \alpha^*(z)_- &= \alpha^*(z),\\
\alpha^{1*}(z)_+ &= 0, & \alpha^{1*}(z)_- &= \alpha^{1*}(z).
\end{aligned}$$

This means in particular that for $r = 0$ we get

$$\begin{aligned}
\lfloor \alpha\alpha^* \rfloor = \lfloor \alpha(z), \alpha^*(w) \rfloor &= \sum_{m\geq 0} \delta_{m+n,0} z^{-m-1} w^{-n} \\
&= \delta_-(z/w) = \iota_{z,w}\Big(\frac{1}{z-w}\Big),
\end{aligned} \tag{4-5}$$

$$\lfloor \alpha^*\alpha \rfloor = -\sum_{m\geq 1} \delta_{m+n,0} z^{-m} w^{-n-1} = -\delta_+(w/z) = \iota_{z,w}\Big(\frac{1}{w-z}\Big), \tag{4-6}$$

(where $\iota_{z,w}$ denotes Taylor series expansion in the "region" $|z| > |w|$), and, for $r = 1$,

$$\lfloor \alpha\alpha^* \rfloor = [\alpha(z)_-, \alpha^*(w)] = 0, \tag{4-7}$$

$$\lfloor \alpha^*\alpha \rfloor = [\alpha^*(z)_-, \alpha(w)] = -\sum_{m,n\in\mathbb{Z}} \delta_{m+n,0} z^{-m} w^{-n-1} = -\delta(w/z), \tag{4-8}$$

while similar results hold for $\alpha^1(z)$. Notice that in both cases we have

$$[\alpha(z), \alpha^*(w)] = \lfloor \alpha(z)\alpha^*(w) \rfloor - \lfloor \alpha^*(w)\alpha(z) \rfloor = \delta(z/w).$$

Recall that the singular part of the *operator product expansion*

$$\lfloor ab\rfloor = \sum_{j=0}^{N-1} \iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right)c^j(w)$$

completely determines the bracket of mutually local formal distributions $a(z)$ and $b(w)$. (See Theorem A.3 of the appendix). One writes

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}.$$

The three-point Heisenberg algebra. The Cartan subalgebra $\mathfrak{h}$ tensored with $\mathcal{R}$ generates a subalgebra of $\hat{\mathfrak{g}}$ which is an extension of an oscillator algebra. This extension motivates the following definition: The Lie algebra with generators b_m, b^1_m, $m\in\mathbb{Z}$, $\mathbf{1}_0$, $\mathbf{1}_1$, and relations

$$[b_m, b_n] = (n-m)\,\delta_{m+n,0}\mathbf{1}_0 = -2m\,\delta_{m+n,0}\mathbf{1}_0, \tag{4-9}$$

$$\begin{aligned}[b^1_m, b^1_n] &= (n-m)(\delta_{m+n,-2}+4\delta_{m+n,-1})\mathbf{1}_0\\ &= 2((n+1)\delta_{m+n,-2}+(4n+2)\delta_{m+n,-1})\mathbf{1}_0,\end{aligned} \tag{4-10}$$

$$[b^1_m, b_n] = (n-m)\delta_{m,-n}\mathbf{1}_1 = -2m\delta_{m,-n}\mathbf{1}_1, \tag{4-11}$$

$$[b_m,\mathbf{1}_0] = [b^1_m,\mathbf{1}_0] = [b_m,\mathbf{1}_1] = [b^1_m,\mathbf{1}_1] = 0. \tag{4-12}$$

We will give it the appellation the *three-point (affine) Heisenberg algebra*, and denote it by $\hat{\mathfrak{h}}_3$.

If we introduce the formal distributions

$$\beta(z) := \sum_{n\in\mathbb{Z}} b_n z^{-n-1}, \quad \beta^1(z) := \sum_{n\in\mathbb{Z}} b^1_n z^{-n-1} = \sum_{n\in\mathbb{Z}} b_{n+\frac12} z^{-n-1}, \tag{4-13}$$

(where $b_{n+\frac12} := b^1_n$), then, using calculations done earlier for the three-point Lie algebra, we can see that the relations above can be rewritten in the form

$$\begin{aligned}[\beta(z),\beta(w)] &= 2\mathbf{1}_0\,\partial_z\delta(z/w) = -2\partial_w\delta(z/w)\mathbf{1}_0,\\ [\beta^1(z),\beta^1(w)] &= -2\big((w^2+4w)\partial_w\delta(z/w)+(2+w)\delta(z/w)\big)\mathbf{1}_0,\\ [\beta^1(z),\beta(w)] &= 2\partial_z\delta(z/w)\mathbf{1}_1 = -2\partial_w\delta(z/w)\mathbf{1}_1.\end{aligned}$$

Set

$$\hat{\mathfrak{h}}_3^{\pm} := \sum_{n\gtrless 0}(\mathbb{C}b_n+\mathbb{C}b^1_n), \quad \hat{\mathfrak{h}}_3^{0} := \mathbb{C}\mathbf{1}_0\oplus\mathbb{C}\mathbf{1}_1\oplus\mathbb{C}b_0\oplus\mathbb{C}b^1_0.$$

We introduce a Borel-type subalgebra

$$\hat{\mathfrak{b}}_3 = \hat{\mathfrak{h}}_3^{+}\oplus\hat{\mathfrak{h}}_3^{0}.$$

From the defining relations above, one can see that $\hat{\mathfrak{b}}_3$ is a subalgebra.

Lemma 4.1. *Let $\mathcal{V} = \mathbb{C}\boldsymbol{v}_0 \oplus \mathbb{C}\boldsymbol{v}_1$ be a two-dimensional representation of $\hat{\mathfrak{h}}_3^+$ with $\hat{\mathfrak{h}}_3^+\boldsymbol{v}_i = 0$ for $i = 0, 1$. Suppose $\lambda, \mu, \nu, \varkappa, \chi_1, \kappa_0 \in \mathbb{C}$ are such that*

$$\begin{aligned} b_0\boldsymbol{v}_0 &= \lambda\boldsymbol{v}_0, & b_0\boldsymbol{v}_1 &= \lambda\boldsymbol{v}_1, \\ b_0^1\boldsymbol{v}_0 &= \mu\boldsymbol{v}_0 + \nu\boldsymbol{v}_1, & b_0^1\boldsymbol{v}_1 &= \varkappa\boldsymbol{v}_0 + \mu\boldsymbol{v}_1, \\ \mathbf{1}_1\boldsymbol{v}_i &= \chi_1\boldsymbol{v}_i, & \mathbf{1}_0\boldsymbol{v}_i &= \kappa_0\boldsymbol{v}_i \quad \textit{for } i = 0, 1. \end{aligned}$$

Then the above equations define a representation of $\hat{\mathfrak{b}}_3$ on $\mathcal{V}$.

Proof. Since b_m acts by scalar multiplication for $m, n \geq 0$, the first defining relation (4-9) is satisfied for $m, n \geq 0$. The second relation (4-10) is also satisfied as the right-hand side is zero if $m, n \geq 0$. If $n = 0$, then since b_0 acts by a scalar, the relation (4-11) leads to no condition on $\lambda, \mu, \nu, \varkappa, \chi_1, \kappa_0 \in \mathfrak{h}_4^0$. If $m \geq 0$ and $n > 0$, the third relation doesn't give us a condition on χ_1 as

$$0 = b_m^1 b_n\boldsymbol{v}_i - b_n b_m^1\boldsymbol{v}_i = [b_m^1, b_n]\boldsymbol{v}_i = -2\delta_{m,-n} m\chi_1\boldsymbol{v}_i = 0.$$

If $m = n = 0$, the third relation however becomes

$$0 = \lambda b_0^1\boldsymbol{v}_i - b_0^1\lambda\boldsymbol{v}_i = b_0^1 b_0\boldsymbol{v}_i - b_0 b_0^1\boldsymbol{v}_i = [b_0^1, b_0]\boldsymbol{v}_i = -2\cdot 0\chi_1\boldsymbol{v}_i = 0,$$

so there is no condition on χ_1. □

Let B_0^1 denote the linear transformation on $\mathcal{V}$ that agrees with the action of b_0^1. If we define the notion of a $\hat{\mathfrak{b}}_3$-submodule as is done in [Sheinman 1995, Definition 1.2], then $\mathcal{V}$ above is an irreducible $\hat{\mathfrak{b}}_3$-module when $\varkappa\nu \neq 0$, that is, when $\det B_0^1 \neq \mu^2$. If one induces from $\mathcal{V}$, the resulting representation for the three-point affine algebra cannot be irreducible if $\mathcal{V}$ is not irreducible as a quasigraded module itself.

Let $\mathbb{C}[\boldsymbol{y}] := \mathbb{C}[y_{-n}, y_{-m}^1 \mid m, n \in \mathbb{N}^*]$. The following is a straightforward computation:

Lemma 4.2. *The linear map $\rho : \hat{\mathfrak{b}}_3 \to \mathrm{End}(\mathbb{C}[\boldsymbol{y}] \otimes \mathcal{V})$ defined by*

$$\rho(b_n) = y_n \qquad \textit{for } n < 0, \tag{4-14}$$

$$\rho(b_n^1) = y_n^1 + \delta_{n,-1}\partial_{y_{-3}^1}\chi_0 - \delta_{n,-3}\partial_{y_{-1}^1}\chi_0 \qquad \textit{for } n < 0, \tag{4-15}$$

$$\rho(b_n) = -n(2\partial_{y_{-n}}\chi_0 + 2\partial_{y_{-n}^1}\chi_1) \qquad \textit{for } n > 0, \tag{4-16}$$

$$\rho(b_n^1) = -2n\partial_{y_{-n}}\chi_1 + 2(n+2)\partial_{y_{-n-4}^1}\chi_0 - 4c(n+1)\partial_{y_{-n-2}^1}\chi_0 + 2n\partial_{y_{-n}^1}\chi_0 \qquad \textit{for } n > 0, \tag{4-17}$$

$$\rho(b_0) = \lambda, \tag{4-18}$$

$$\rho(b_0^1) = 4\partial_{y_{-4}^1}\chi_0 - 2c\partial_{y_{-2}^1}\chi_0 + B_0^1, \tag{4-19}$$

is a representation of $\hat{\mathfrak{b}}_3$.

5. Two realizations of the affine three-point algebra $\hat{\mathfrak{g}}$

Assume that $\chi_0\in\mathbb{C}$, and define $\mathcal{V}$ as in Lemma 4.1. The $\alpha(z)$, $\alpha^1(z)$, $\alpha^*(z)$, and $\alpha^{1*}(z)$ are generating series of oscillator algebra elements as in (4-3) and (4-4). Our main result is the following:

Theorem 5.1. *Fix $r\in\{0,1\}$, which then fixes the corresponding normal ordering convention defined in the previous section. Set $\hat{\mathfrak{g}}=(\mathfrak{sl}(2,\mathbb{C})\otimes\mathcal{R})\oplus\mathbb{C}\omega_0\oplus\mathbb{C}\omega_1$. Then, using* (4-1), (4-2) *and Lemma 4.2, the following defines a representation of the three-point algebra $\hat{\mathfrak{g}}$ on $\mathbb{C}[\boldsymbol{x}]\otimes\mathbb{C}[\boldsymbol{y}]\otimes\mathcal{V}$:*

$$\tau(\omega_1)=0,\qquad \tau(\omega_0)=\chi_0=\kappa_0+4\delta_{r,0},$$
$$\tau(f(z))=-\alpha(z),\qquad \tau(f^1(z))=-\alpha^1(z),$$
$$\tau(h(z))=2(:\alpha(z)\alpha^*(z):+:\alpha^1(z)\alpha^{1*}(z):)+\beta,$$
$$\tau(h^1(z))=2(:\alpha^1(z)\alpha^*(z):+(z^2+4z):\alpha(z)\alpha^{1*}(z):)+\beta^1(z),$$
$$\begin{aligned}\tau(e(z))=\,&:\alpha(z)(\alpha^*(z))^2:+(z^2+4z):\alpha(z)(\alpha^{1*}(z))^2:+2:\alpha^1(z)\alpha^*(z)\alpha^{1*}(z):\\&+\beta(z)\alpha^*(z)+\beta^1(z)\alpha^{1*}(z)+\chi_0\partial\alpha^*(z),\end{aligned}$$
$$\begin{aligned}\tau(e^1(z))=\,&\alpha^1(z)\alpha^*(z)\alpha^*(z)+(z^2+4z)(\alpha^1(z)(\alpha^{1*}(z))^2+2:\alpha(z)\alpha^*(z)\alpha^{1*}(z):)\\&+\beta^1(z)\alpha^*+(z^2+4z)\beta(z)\alpha^{1*}(z)+\chi_0((z^2+4z)\partial_z\alpha^{1*}(z)+(z+2)\alpha^{1*}(z)).\end{aligned}$$

Before we go through the proof, it will be fruitful to review Kac's λ-notation (see [Kac 1998, Section 2.2] and [Wakimoto 2001] for some of its properties), used in operator product expansions. If $a(z)$ and $b(w)$ are formal distributions, then

$$[a(z),b(w)]=\sum_{j=0}^{\infty}\frac{(a_{(j)}b)(w)}{(z-w)^{j+1}}$$

is transformed under the *formal Fourier transform*

$$F^{\lambda}_{z,w}a(z,w)=\operatorname{Res}_z e^{\lambda(z-w)}a(z,w)$$

into the sum

$$[a_\lambda b]=\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}a_{(j)}b.$$

Set

$$P(w)=w^2+4w.$$

So for example we have the following:

Lemma 5.2. *Given the definitions in the previous section, we have*

(1) $[\beta^1{}_\lambda \beta^1] = -(2(w^2+4w)\lambda + (2w+4))\kappa_0 = -(2P\lambda + \partial P)\kappa_0$,

(2) $[:\alpha\alpha^*:_\lambda :\alpha\alpha^*:] = -\delta_{r,0}\lambda$,

(3) $[:\alpha(\alpha^*)^2:_\lambda :\alpha(\alpha^*)^2:] = -4\delta_{r,0}:\alpha^*\partial\alpha^*: - 4\delta_{r,0}:(\alpha^*)^2:\lambda$.

Note that similar expressions hold for $\alpha^1(z)$ and $\alpha^{1*}(z)$ (the λ-notation suppresses the variables z and w, which are understood).

Proof. We'll prove (2) and (3). By Wick's theorem,

$$
\begin{aligned}
&:\alpha(z)\alpha^*(z)::\alpha(w)\alpha^*(w): \\
&\qquad = :\alpha(z)\alpha^*(z)\alpha(w)\alpha^*(w): + \lfloor \alpha(z), \alpha^*(w)\rfloor :\alpha(w)\alpha^*(z): \\
&\qquad\quad + \lfloor \alpha^*(z), \alpha(w)\rfloor :\alpha(z)\alpha^*(w): + \lfloor \alpha(z), \alpha^*(w)\rfloor \lfloor \alpha^*(z), \alpha(w)\rfloor \\
&\qquad = :\alpha(z)\alpha(w)\alpha^*(z)\alpha^*(w): + :\alpha(w)\alpha^*(z): \iota_{z,w}\Big(\frac{1}{z-w}\Big) \\
&\qquad\quad + :\alpha(z)\alpha^*(w): \iota_{z,w}\Big(\frac{1}{w-z}\Big) - \delta_{r,0}\iota_{z,w}\Big(\frac{1}{z-w}\Big)^2
\end{aligned}
$$

and

$$
\begin{aligned}
&[:\alpha(z)\alpha^*(z)^2:, :\alpha(w)\alpha^*(w)^2:] \\
&\quad = 2:\alpha(z)\alpha^*(z)\alpha^*(w)^2:\delta(z/w) - 2:\alpha(w)\alpha^*(z)^2\alpha^*(w):\delta(z/w) \\
&\qquad - 4\delta_{r,0}:\alpha^*(z)\alpha^*(w):\partial_w\delta(z/w) \\
&\quad = -4\delta_{r,0}:\alpha^*(z)\partial_w(\alpha^*(w):\delta(z/w)) + 4\delta_{r,0}:\alpha^*(z)\partial_w\alpha^*(w):\delta(z/w) \\
&\quad = -4\delta_{r,0}:\partial_w(\alpha^*(w)\alpha^*(w):\delta(z/w)) + 4\delta_{r,0}:\alpha^*(w)\partial_w\alpha^*(w):\delta(z/w) \\
&\quad = -4\delta_{r,0}:\partial_w\alpha^*(w)\alpha^*(w):\delta(z/w) - 4\delta_{r,0}:\alpha^*(w)\alpha^*(w):\partial_w\delta(z/w). \qquad \square
\end{aligned}
$$

Proof of Theorem 5.1. We need to check the following table is preserved under τ:

$[\cdot_\lambda\cdot]$	$f(w)$	$f^1(w)$	$h(w)$	$h^1(w)$	$e(w)$	$e^1(w)$
$f(z)$	0	0	$*$	$*$	$*$	$*$
$f^1(z)$		0	$*$	$*$	$*$	$*$
$h(z)$			$*$	$*$	$*$	$*$
$h^1(z)$				$*$	$*$	$*$
$e(z)$					0	0
$e^1(z)$						0

Here, $*$ indicates nonzero formal distributions that are obtained from the defining relations (3-4), (3-5), and (3-6). The proof is carried out using Wick's theorem, Taylor's theorem, and Lemma 5.2, as one can see below:

$$
\begin{aligned}
[\tau(f)_\lambda\tau(f)]&=0,\quad [\tau(f)_\lambda\tau(f^1)]=0,\quad [\tau(f^1)_\lambda\tau(f^1)]=0,\\
[\tau(f)_\lambda\tau(h)]&=-[\alpha_\lambda(2(\alpha\alpha^*+\alpha^1\alpha^{1*})+\beta)]=-2\alpha=2\tau(f),\\
[\tau(f)_\lambda\tau(h^1)]&=-[\alpha_\lambda(2(\alpha^1\alpha^*+P\alpha\alpha^{1*})+\beta^1)]=-2\alpha^1=2\tau(f^1),\\
[\tau(f)_\lambda\tau(e)]&=-\big[\alpha_\lambda(:\alpha(\alpha^*)^2:+P:\alpha(\alpha^{1*})^2:\\
&\qquad+2:\alpha^1\alpha^*\alpha^{1*}:+\beta\alpha^*+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*)\big]\\
&=-2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)-\beta-\chi_0\lambda=-\tau(h)-\chi_0\lambda,\\
[\tau(f)_\lambda\tau(e^1)]&=-\big[\alpha_\lambda(\alpha^1(\alpha^*)^2+P(\alpha^1(\alpha^{1*})^2+2:\alpha\alpha^*\alpha^{1*}:)\\
&\qquad+\beta^1\alpha^*+P\beta\alpha^{1*}+\chi_0(P\partial\alpha^{1*}+\tfrac{1}{2}\partial P\alpha^{1*}))\big]\\
&=-2(:\alpha^1\alpha^*:+P:\alpha\alpha^{1*}:)-\beta^1=-\tau(h^1),\\
[\tau(f^1)_\lambda\tau(h)]&=-[\alpha^1_\lambda(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)]=-2\alpha^1=2\tau(f^1),\\
[\tau(f^1)_\lambda\tau(h^1)]&=-[\alpha^1_\lambda(2(:\alpha^1\alpha^*:+P:\alpha\alpha^{1*}:)+\beta^1)]=-2P\alpha^1=2P\tau(f^1),\\
[\tau(f^1)_\lambda\tau(e)]&=-[\alpha^1_\lambda(:\alpha(\alpha^*)^2:+P:\alpha(\alpha^{1*})^2:\\
&\qquad+2:\alpha^1\alpha^*\alpha^{1*}:+\beta\alpha^*+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*)]\\
&=-(2P:\alpha\alpha^{1*}:+2:\alpha^1\alpha^*:+\beta^1)=-\tau(h^1),\\
[\tau(f^1)_\lambda\tau(e^1)]&=-\big[\alpha^1_\lambda(\alpha^1(\alpha^*)^2+P(\alpha^1(\alpha^{1*})^2+2:\alpha\alpha^*\alpha^{1*}:)\\
&\qquad+\beta^1\alpha^*+P\beta\alpha^{1*}+\chi_0(P\partial\alpha^{1*}+\tfrac{1}{2}(\partial P)\alpha^{1*}))\big]\\
&=-(P(2(:\alpha^1\alpha^{1*}:+:\alpha\alpha^*:)+\beta+\chi_0\lambda)+\tfrac{1}{2}\chi_0\partial P)\\
&=-(P\tau(h)+P\chi_0\lambda+\chi_0\tfrac{1}{2}\partial P).
\end{aligned}
$$

Note that $:a(z)b(z):$ and $:b(z)a(z):$ are usually not equal, but $:\alpha^1(w)\alpha^{*1}(w):=$ $:\alpha^{1*}(w)\alpha^1(w):$ and $:\alpha(w)\alpha^*(w):=:\alpha^*(w)\alpha(w):$. Thus, we calculate

$$
\begin{aligned}
[\tau(h)_\lambda\tau(h)]&=[(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)_\lambda(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)]\\
&=4(-:\alpha\alpha^*:+:\alpha^*\alpha:-:\alpha^1\alpha^{1*}:+:\alpha^{1*}\alpha^1:)-8\delta_{r,0}\lambda+[\beta_\lambda\beta]\\
&=-2(4\delta_{r,0}+\kappa_0)\lambda,
\end{aligned}
$$

which can be put into the form of (3-4):

$$
\begin{aligned}
[\tau(h(z)),\tau(h(w))]&=-2(4\delta_{r,0}+\kappa_0)\partial_w\delta(z/w)\\
&=-2\chi_0\partial_w\delta(z/w)=\tau(-2\omega_0\partial_w\delta(z/w)).
\end{aligned}
$$

Next, we calculate

$$
[\tau(h)_\lambda\tau(h^1)]=4\big((:\alpha^*\alpha^1:-:\alpha^1\alpha^*:)+P(-:\alpha\alpha^{1*}:+:\alpha^{1*}\alpha:)\big)+[\beta_\lambda\beta^1].
$$

Since $[a_n, a_m^{1*}] = [a_n^1, a_m^*] = 0$, we have $[\tau(h(z)), \tau(h^1(w))] = [\beta(z), \beta^1(w)] = 0$. As $\tau(\omega_1) = 0$, relation (3-6) is satisfied.

We continue with

$$\begin{aligned}[\tau(h^1)_\lambda \tau(h^1)] &= [(2(:\alpha^1\alpha^*: + P:\alpha\alpha^{1*}:) + \beta^1)_\lambda(2(:\alpha^1\alpha^*: + P:\alpha\alpha^{1*}:) + \beta^1)] \\ &= -8\delta_{r,0}P\lambda - 4\delta_{r,0}\partial P - 2\kappa_0(P\lambda + \tfrac{1}{2}\partial P),\end{aligned}$$

yielding the relation

$$\begin{aligned}[\tau(h^1(z)), \tau(h^1(w))] &= -2(4\delta_{r,0} + \kappa_0)((w^2+4w)\partial_w\delta(z/w) + (w+2))\delta(z/w)) \\ &= \tau(-(h,h)\omega_0 P\partial_w\delta(z/w) - \tfrac{1}{2}(h,h)\partial P\omega_0\delta(z/w)).\end{aligned}$$

Next we calculate the h paired with the e:

$$\begin{aligned}[\tau(h)_\lambda\tau(e)] &= \big[(2(:\alpha\alpha^*: + :\alpha^1\alpha^{1*}:) + \beta)_\lambda \\ &\qquad (:\alpha(\alpha^*)^2: + P:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta\alpha^* + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*)\big] \\ &= 4:\alpha(\alpha^*)^2: - 2:\alpha(\alpha^*)^2: - 4\delta_{r,0}\alpha^*\lambda \\ &\qquad - 2P:\alpha(\alpha^{1*})^2: + 4:\alpha^*\alpha^1\alpha^{1*}: + 2\alpha^*\beta + 2\chi_0\alpha^*\lambda + 2\chi_0\partial\alpha^* \\ &\qquad\quad + 4P:\alpha(\alpha^{1*})^2: - 4\delta_{r,0}\alpha^*\lambda + 2\beta^1\alpha^{1*} - 2\lambda\alpha^*\kappa_0 \\ &= 2\tau(e)\end{aligned}$$

and

$$\begin{aligned}[\tau(h^1)_\lambda\tau(e)] &= 2:\alpha^1(\alpha^*)^2: + 2P:\alpha^1(\alpha^{1*})^2: + 4P:\alpha\alpha^*\alpha^{1*}: \\ &\qquad + 2\delta(z/w)\alpha^*\beta^1 + 2P\beta\alpha^{1*} + 2P\chi_0\partial\alpha^{1*} + \partial P\alpha^{1*}\chi_0 \\ &= 2\tau(e^1)\end{aligned}$$

Similarly, $[\tau(h)_\lambda\tau(e^1)] = 2\tau(e^1)$ and $[\tau(h^1)_\lambda\tau(e^1)] = 2\tau(e^1)$.

We prove the Serre relation for just one of the relations, $[\tau(e)_\lambda\tau(e^1)]$; the proof of the others, $[\tau(e)_\lambda\tau(e)]$ and $[\tau(e^1)_\lambda\tau(e^1)]$, is similar, as the reader can verify.

After expanding the definitions and collecting terms, we have

$$\begin{aligned}&[\tau(e)_\lambda\tau(e^1)] \\ &\quad = [:\alpha(\alpha^*)^2:_\lambda(:\alpha^1(\alpha^*)^2: + 2P:\alpha\alpha^*\alpha^{1*}: + \beta^1\alpha^*)] \\ &\qquad + [P:\alpha(\alpha^{1*})^2:_\lambda(:\alpha^1(\alpha^*)^2: + P(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}:) + \beta^1\alpha^*)] \\ &\qquad + \big[2:\alpha^1\alpha^*\alpha^{1*}:_\lambda(\alpha^1(\alpha^*)^2 + P(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}:) + P\beta\alpha^{1*} \\ &\qquad\quad + \chi_0((w^2+4w)\partial_w\alpha^{1*} + (w+2)\alpha^{1*}))\big] \\ &\qquad + [\beta\alpha^*{}_\lambda(2P:\alpha\alpha^*\alpha^{1*}: + \beta^1\alpha^* + P\beta\alpha^{1*})] \\ &\qquad + [\beta^1\alpha^{1*}{}_\lambda(:\alpha^1(\alpha^*)^2: + P\alpha^1(\alpha^{1*})^2 + \beta^1\alpha^* + P\beta\alpha^{1*})] \\ &\qquad + [\chi_0\partial\alpha^*{}_\lambda(2P:\alpha\alpha^*\alpha^{1*}:)]\end{aligned}$$

$$
\begin{aligned}
&= 2{:}\alpha^1\alpha^*(\alpha^*)^2{:} + 2P{:}\alpha(\alpha^*)^2\alpha^{1*}{:} - 4P{:}\alpha(\alpha^*)^2\alpha^{1*}{:} - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda \\
&\quad - 4\delta_{r,0}P{:}\partial(\alpha^*)\alpha^{1*}{:} + \beta^1(\alpha^*)^2 - 2P{:}\alpha(\alpha^*)^2\alpha^{1*}{:} + 2P\alpha^1\alpha^*(\alpha^{1*})^2 \\
&\quad - 4\delta_{r,0}P{:}\alpha^{1*}\alpha^*{:}\lambda - 4\delta_{r,0}\partial P{:}\alpha^{1*}\alpha^*{:} - 4\delta_{r,0}P{:}\partial\alpha^{1*}\alpha^*{:} \\
&\quad - 2P^2{:}\alpha(\alpha^{1*})^3{:} + 2P^2{:}\alpha(\alpha^{1*})^3{:} + P\beta^1(\alpha^{1*})^2 \\
&\quad - 2{:}\alpha^1\alpha^*(\alpha^*)^2{:} + 4P{:}\alpha^1\alpha^*(\alpha^{1*})^2{:} - 2P{:}\alpha^1\alpha^*(\alpha^{1*})^2{:} \\
&\quad - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\partial(\alpha^*)\alpha^{1*}{:} \\
&\quad + 4P{:}\alpha(\alpha^*)^2\alpha^{1*}{:} - 4P{:}\alpha^1\alpha^*(\alpha^{1*})^2{:} \\
&\quad - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\alpha^*\partial\alpha^{1*}{:} + 2P\beta{:}\alpha^*\alpha^{1*}{:} \\
&\quad + 2\chi_0(P{:}\partial\alpha^*\alpha^{1*}{:} + P{:}\alpha^*\partial\alpha^{1*}{:} + P{:}\alpha^*\alpha^{1*}{:}\lambda + \tfrac{1}{2}(\partial P){:}\alpha^*\alpha^{1*}{:}) \\
&\quad - 2P\beta\alpha^*\alpha^{1*} - 2\kappa_0P\alpha^*\alpha^{1*}\lambda - 2\kappa_0P\partial\alpha^*\alpha^{1*} \\
&\quad - \beta^1(\alpha^*)^2 - P\beta^1(\alpha^{1*})^2 - \kappa_0(2P\alpha^*\alpha^{1*}\lambda + 2P\alpha^*\partial\alpha^{1*} + \partial P\alpha^*\alpha^{1*}) \\
&\quad + 2\chi_0P\alpha^*\alpha^{1*}\lambda \\
&= -4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\partial(\alpha^*)\alpha^{1*}{:} + \chi_1(2{:}\alpha^*\partial\alpha^*{:} + {:}(\alpha^*)^2{:}\lambda) \\
&\quad - 4\delta_{r,0}P{:}\alpha^{1*}\alpha^*{:}\lambda - 4\delta_{r,0}\partial P{:}\alpha^{1*}\alpha^*{:} - 4\delta_{r,0}P{:}\partial\alpha^{1*}\alpha^*{:} \\
&\quad - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\partial(\alpha^*)\alpha^{1*}{:} - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\alpha^*\partial\alpha^{1*}{:} \\
&\quad + 2\chi_0(P{:}\partial\alpha^*\alpha^{1*}{:} + P{:}\alpha^*\partial\alpha^{1*}{:} + P{:}\alpha^*\alpha^{1*}{:}\lambda + \tfrac{1}{2}(\partial P){:}\alpha^*\alpha^{1*}{:}) \\
&\quad - \kappa_0P\alpha^*\alpha^{1*}\lambda - \kappa_0P\partial\alpha^*\alpha^{1*} \\
&\quad - \kappa_0(2P\alpha^*\alpha^{1*}\lambda + 2P\alpha^*\partial\alpha^{1*} + \partial P\alpha^*\alpha^{1*}) \\
&\quad - 2\chi_0P\alpha^*\alpha^{1*}\lambda \\
&= 0. \qquad \square
\end{aligned}
$$

6. Further comments

We plan to use the above construction to help elucidate the structure of these representations of a three-point algebra, describe the space of their intertwining operators, and eventually describe the center of a certain completion of the universal enveloping algebra for the three-point algebra.

Appendix

For the convenience of the reader we include the following results, which are useful for performing the computations necessary for proving our results:

Theorem A.1 (Wick's theorem [Bogoliubov and Shirkov 1980; Huang 1998; Kac 1998]). *Let $a^i(z)$ and $b^j(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\boldsymbol{x}]\otimes\mathbb{C}[\boldsymbol{y}])$, *satisfying:*

(1) $[\lfloor a^i(z)b^j(w)\rfloor, c^k(x)_\pm] = [\lfloor a^i b^j\rfloor, c^k(x)_\pm] = 0$ *for all* i, j, k *and* $c^k(x) = a^k(z)$ *or* $c^k(x) = b^k(w)$.

(2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ *for all* i *and* j.

(3) *The products*

$$\lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{i_s} \rfloor : a^1(z) \cdots a^M(z) b^1(w) \cdots b^N(w) :_{(i_1,\dots,i_s;\, j_1,\dots,j_s)}$$

have coefficients in $\mathrm{End}(\mathbb{C}[\boldsymbol{x}] \otimes \mathbb{C}[\boldsymbol{y}])$ *for all subsets* $\{i_1, \dots, i_s\} \subset \{1, \dots, M\}$, $\{j_1, \dots, j_s\} \subset \{1, \cdots N\}$. *Here, the subscript* $(i_1, \dots, i_s;\, j_1, \dots, j_s)$ *means that those factors* $a^i(z)$, $b^j(w)$ *with indices* $i \in \{i_1, \dots, i_s\}$, $j \in \{j_1, \dots, j_s\}$ *are to be omitted from the product* $: a^1 \cdots a^M b^1 \cdots b^N :$, *and when* $s = 0$ *we do not omit any factors.*

Then

$$: a^1(z) \cdots a^M(z) :: b^1(w) \cdots b^N(w) :$$
$$= \sum_{s=0}^{\min(M,N)} \sum_{\substack{i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{j_s} \rfloor : a^1(z) \cdots a^M(z) b^1(w) \cdots b^N(w) :_{(i_1,\dots,i_s;\, j_1,\dots,j_s)} .$$

Theorem A.2 (Taylor's theorem [Kac 1998, Theorem 2.4.3]). *Let* $a(z)$ *be a formal distribution. Then, in the region* $|z - w| < |w|$,

$$a(z) = \sum_{j=0}^{\infty} \partial_w^{(j)} a(w) (z - w)^j. \tag{A-1}$$

Theorem A.3 [Kac 1998, Theorem 2.3.2]. *Set* $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_n, x_n^1 \mid n \in \mathbb{Z}]$ *and* $\mathbb{C}[\boldsymbol{y}] = C[y_m, y_m^1 \mid m \in \mathbb{N}^*]$. *Let* $a(z)$ *and* $b(z)$ *be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\boldsymbol{x}] \otimes \mathbb{C}[\boldsymbol{y}])$, *where we are using the usual normal ordering. The following are equivalent*:

(i)
$$[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z - w) c^j(w),$$

where $c^j(w) \in \mathrm{End}(\mathbb{C}[\boldsymbol{x}] \otimes \mathbb{C}[\boldsymbol{y}])[\![w, w^{-1}]\!]$.

(ii)
$$\lfloor ab \rfloor = \sum_{j=0}^{N-1} \iota_{z,w} \left(\frac{1}{(z - w)^{j+1}} \right) c^j(w).$$

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MULTI-BUMP BOUND STATE SOLUTIONS FOR THE QUASILINEAR SCHRÖDINGER EQUATION WITH CRITICAL FREQUENCY

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We study the existence of single- and multi-bump solutions of quasilinear Schrödinger equations

$$-\Delta u+\lambda V(x)u-\tfrac{1}{2}(\Delta|u|^2)u=|u|^{p-2}u,$$

the function V being a critical frequency in the sense that $\inf_{x\in\mathbb{R}^N} V(x)=0$. We show that if the zero set of V has several isolated connected components $\Omega_1,\dots,\Omega_k$ such that the interior of Ω_i is not empty and $\partial\Omega_i$ is smooth, then for $\lambda>0$ large, there exists, for any nonempty subset $J\subset\{1,2,\dots,k\}$, a standing wave solution trapped in a neighborhood of $\bigcup_{j\in J}\Omega_j$.

1. Introduction and main results

Consider the following quasilinear Schrödinger equation:

$$-\Delta u+\lambda V(x)u-\tfrac{1}{2}(\Delta|u|^2)u=|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1-1}$$

where $N\geq 3$, $\lambda>0$ is a parameter, $4<p<2\cdot 2^*$, and 2^* is the critical Sobolev exponent.

We are interested in the ground state solutions for (1-1), i.e., the positive solutions with least energy. Solutions of this type are related to the existence of standing wave solutions for the following quasilinear Schrödinger equation:

$$i\partial_t w=-\hbar^2\Delta w+V(x)w-f(|w|^2)w-k\Delta h(|w|^2)h'(|w|^2)w \quad \text{in } \mathbb{R}^N, \tag{1-2}$$

where V is a given potential, $\hbar$ is the Planck constant, k is a real constant, and f, h are real functions. Such quasilinear equations appear naturally in mathematical physics, and have been derived as models of several physical phenomena corresponding to various types of h (see, for example, [Brizhik et al. 2003; Brihaye and Hartmann 2006; Brüll and Lange 1986; Hartmann and Zakrzewski 2003; Kurihura 1981], and the references therein).

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Due to its significant application in mathematical physics, the equation (1-2) with $k = 0$ (the semilinear case) has attracted much attention in recent years. Many authors have obtained existence results for one-bump or multi-bump bound state solutions under different assumptions on the potential function V. We refer the readers to [Ambrosetti et al. 1997; Ambrosetti et al. 2001; Bartsch and Wang 2000; Cingolani and Lazzo 2000; Cingolani and Nolasco 1998; del Pino and Felmer 1998; del Pino and Felmer 1997], and the references therein.

In the quasilinear case (that is, the equation (1-2) with $k \neq 0$) we observe that, due to the presence of the quasilinear term, there is a different critical exponent than in the semilinear case, as observed in [Liu et al. 2003]; the number $q = 2\cdot 2^* = 4N/(N-2)$ behaves as a critical exponent for the quasilinear equation. There has been much recent work concerned with the quasilinear Schrödinger equations (1-1) and (1-2). For instance, in [Colin 2003], a change of variables was used to prove the existence of soliton wave solutions; see also the paper by Liu, Wang and Wang [2003], where a change of variables was also used. In [Colin and Jeanjean 2004], various existence results for standing wave solutions to (1-1) for special f and h are obtained. For the stability and instability results for a special case of (1-2), we also refer the reader to [Colin et al. 2010].

For more recent related work on the quasilinear Schrödinger equation with critical exponents, we refer the reader to, for instance, [Liu et al. 2013; 2012; do Ó et al. 2010a; 2010b, Lins and Silva 2009], and to the references therein.

The current paper is concerned with the existence of one-bump or multi-bump bound states for the following quasilinear equation with frequency V:

$$-\Delta u + \lambda V(x)u - \tfrac{1}{2}(\Delta |u|^2)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Our hypotheses on V are:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $V(x) \geq 0$ and $\liminf_{|x|\to\infty} V(x) > 0$;

(V_2) $\Omega := \operatorname{int} V^{-1}(0)$ is nonempty, bounded, has smooth boundary, and $\overline{\Omega} = V^{-1}(0)$;

(V_3) Ω consists of k components:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k,$$

and $\overline{\Omega}_i \cap \overline{\Omega}_j = \varnothing$ for all $i \neq j$.

For the proof of the main theorem, we follow the idea of Y. Ding and K. Tanaka [2003] to modify the nonlinearity and use the decay flow. We point out that, although this idea has been used before to deal with other problems, it is not at all trivial to adapt the procedure for our problem. The appearance of the quasilinear term $\Delta(|u|^2)u$ forces us to consider our problem in an Orlicz space, and more delicate estimates are also needed.

To state the main results, we first introduce some necessary notation. We denote $\lambda V(x)$ by $V_\lambda(x)$. Formally, we define the functional J_λ by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1+u^2)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda(x)u^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,dx, \tag{1-3}$$

where $u \in X := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V_\lambda(x)u^2 < \infty\}$. Note that, under our assumptions, the functional J_λ is not well defined on X. We make the following change of variables, which was first used by Liu, Wang, and Wang [2003].

Let $v = h(u) = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2})$, so $dv = \sqrt{1+u^2}du$. Moreover, $h(u)$ satisfies

$$h(u) \sim \begin{cases} u & \text{if } |u| \ll 1, \\ \frac{1}{2}u|u| & \text{if } |u| \gg 1. \end{cases} \tag{1-4}$$

Since $h'(u) > 0$, $h(u)$ is strictly monotone and hence has an inverse function denoted by $u = f(v)$. Obviously,

$$f(v) \sim \begin{cases} v & \text{if } |u| \ll 1, \\ \sqrt{2/|v|}\,v & \text{if } |v| \gg 1, \end{cases} \qquad f'(v) = \frac{1}{\sqrt{1+f^2(v)}}. \tag{1-5}$$

Let $G(v) = f^2(v)$. Then

$$G(v) = f^2(v) \sim \begin{cases} v^2 & \text{if } |v| \ll 1, \\ 2|v| & \text{if } |v| \gg 1, \end{cases} \tag{1-6}$$

and $G(v)$ is convex, so there exists $C_0 > 0$ such that $G(2v) \le C_0 G(v)$,

$$G'(v) = \frac{2f(v)}{\sqrt{1+f^2(v)}}, \qquad G''(v) = \frac{2}{(1+f^2(v))^2} > 0. \tag{1-7}$$

Now we introduce the Orlicz space (see [Rao and Ren 1991])

$$E_G^\lambda = \left\{ v \,\Big|\, \int_{\mathbb{R}^N} V_\lambda G(v) < +\infty \right\}$$

equipped with the norm

$$|v|_G^\lambda := \inf_{\xi>0} \xi\left(1 + \int_{\mathbb{R}^N} V_\lambda G(\xi^{-1}v)\,dx\right).$$

Then E_G^λ is a Banach space (see [Liu et al. 2003]).

Let

$$H_G^\lambda := \left\{ v \in E_G^\lambda \,\Big|\, \int_{\mathbb{R}^N} |\nabla v|^2\,dx < \infty \right\},$$

equipped with the norm

$$\|v\|_\lambda = \|\nabla v\|_{L^2} + |v|_G^\lambda.$$

Using the change of variable, we define the functional Φ_λ on H_G^λ by

$$\Phi_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V_\lambda f^2(v))\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |f(v)|^p\,dx. \tag{1-8}$$

Then Φ_λ is Gâteaux differentiable, and the Gâteaux derivative $\Phi'_\lambda(v)$ has the form

$$\langle \Phi'_\lambda(v), w\rangle = \int_{\mathbb{R}^N} \nabla v \nabla w\,dx + \int_{\mathbb{R}^N} V_\lambda(x) f(v) f'(v) w\,dx - \int_{\mathbb{R}^N} |f(v)|^{p-2} f(v) f'(v) w\,dx. \tag{1-9}$$

Obviously, $v \in H_G^\lambda$ is a critical point of Φ_λ if and only if v is a solution of the following equation:

$$-\Delta v + V_\lambda f(v) f'(v) = |f(v)|^{p-2} f(v) f'(v), \quad x \in \mathbb{R}^N. \tag{1-10}$$

Moreover, one can easily check that v is solution of (1-10) if and only if $u = f(v)$ is a solution of (1-1).

We define the Nehari manifold N_λ by $N_\lambda = \{v \in H_G^\lambda \setminus \{0\} \mid \langle \Phi'_\lambda(v), v\rangle = 0\}$, and let

$$c_\lambda = \inf_{v \in N_\lambda} \Phi_\lambda(v).$$

We say that $u = f(v)$ is a least energy solution of (1-1) if $v \in N_\lambda$ is such that c_λ is achieved.

Note that under our assumptions, for λ large enough, the following Dirichlet problem is a kind of *limit* problem:

$$\begin{cases} -\Delta u - \frac{1}{2}(\Delta |u|^2) u = |u|^{p-2} u, \ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \partial\Omega, \end{cases} \tag{1-11}$$

where $\Omega = \operatorname{int}\{V^{-1}(0)\}$.

In fact, by a minor change of the arguments in Guo and Tang [2012], one can easily see that under the conditions (V_1), (V_2), and $4 < p < 2 \cdot 2^*$, for λ large, c_λ is achieved by a critical point v_λ of Φ_λ such that $u_\lambda = f(v_\lambda)$ is a solution of (1-1). Furthermore, for any sequence $\lambda_n \to +\infty$, $\{v_{\lambda_n}\}$ has a subsequence converging to v such that $u = f(v)$ is a least energy solution of (1-11). Thus by assumption (V_3), there is Ω_{i_0} $(1 \le i_0 \le k)$ such that $u = f(v)$ is indeed a least energy solution defined on Ω_{i_0} and $u = f(v) = 0$ elsewhere. Thus it is natural to ask whether, for a given $j \in \{1, 2, \ldots, k\}$, (1-1) has a family of solutions $\{u_\lambda\}$ which converges to a least energy solution in Ω_j and to 0 elsewhere. In this paper, we answer this question in the affirmative. Moreover, we can also construct multi-bump type solutions.

Our main results are:

Theorem 1.1. *Suppose* (V_1)–(V_3) *hold. Then for any* $\varepsilon > 0$ *and any nonempty subset* J *of* $\{1, 2, \dots, k\}$, *there exists* $\Lambda = \Lambda(\varepsilon) > 0$ *such that, for* $\lambda \geq \Lambda$, (1-1) *has a solution* u_λ *such that* $v_\lambda = h(u_\lambda)$ *satisfies*

$$\Big|\Phi_\lambda(v_\lambda) - \sum_{j\in J} c(\Omega_j)\Big| \leq \varepsilon, \tag{1-12}$$

$$\int_{\mathbb{R}^N\setminus\Omega_J} (|\nabla v_\lambda|^2 + V_\lambda f^2(v_\lambda))\,dx \leq \varepsilon, \tag{1-13}$$

where $\Omega_J = \bigcup_{j\in J}\Omega_j$. *Moreover, for any sequence* $\lambda_n \to \infty$, *we can extract a subsequence* $\{\lambda_{n_i}\}$ *such that* $v_{\lambda_{n_i}}$ *converges strongly in* H^1_G *to a function* v *that satisfies* $v(x) = 0$ *for* $x \notin \Omega_J$, *and* $u = f(v)|_{\Omega_j}$ *is a least energy solution of*

$$\begin{cases} -\Delta u - \frac{1}{2}(\Delta|u|^2)u = |u|^{p-2}u, \ u > 0 & \text{in } \Omega_j, \\ \qquad u = 0 & \text{in } \partial\Omega_j, \end{cases} \tag{1-14}$$

for $j \in J$. *Here* $c(\Omega_j)$ *in* (1-12) *is the least energy of* (1-14).

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, there exists* $\Lambda > 0$ *such that for* $\lambda > \Lambda$, (1-1) *has at least* $2^k - 1$ *bound states.*

The paper is organized as follows. In Section 2, we give some estimates in Orlicz space. In Section 3, we modify the functional by penalizing the nonlinearity. In Section 4, we consider compactness for the modified functional. In Section 5, we give some asymptotic properties for some sequences and prove that, for λ large, the critical points of the modified functional are indeed critical points of the original one. Section 6 is devoted to the properties of the limit problem. In Section 7, we give a minimax argument. In Section 8, we prove the existence of critical points by a flow argument; the proofs of the main results are also delivered in this section.

In the following, without specific notification, all the integral variables are x, and for simplicity we omit dx in every integral.

2. Some estimates in the Orlicz space

We begin with a precise estimate between the Orlicz norm and some integrals in Orlicz space H^λ_G, namely:

Lemma 2.1 [Guo and Tang 2012]. *There exist constants* $C_1, C_2 > 0$ *such that, for any* $v \in H^\lambda_G$,

$$C_1 \min\{\|v\|_\lambda, \|v\|^2_\lambda\} \leq \int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V_\lambda f^2(v) \leq C_2 \max\{\|v\|_\lambda, \|v\|^2_\lambda\}. \tag{2-1}$$

Let Ω'_j $(1 \leq j \leq k)$ be bounded open subsets with smooth boundary such that

$\overline{\Omega'_i}$ and $\overline{\Omega'_j}$ are disjoint if $i \neq j$ and that $\overline{\Omega}_j \subset \Omega'_j$ for all j. Let K be one of the following sets:

$$\mathbb{R}^N, \quad \Omega'_j \ (j = 1, 2, \ldots, k), \quad \text{or} \quad \mathbb{R}^N \setminus \bigcup_{j \in J} \Omega'_j \ (J \subset \{1, 2, \ldots, k\}). \tag{2-2}$$

Lemma 2.2. *There exist* $\delta_0 > 0$, $\nu_0 > 0$ *such that, for* $\lambda \geq 1$,

$$\delta_0 \int_K \big(|\nabla v|^2 + V_\lambda f^2(v)\big) \leq \int_K \big(|\nabla v|^2 + V_\lambda f^2(v)\big) - \nu_0 \int_K f^2(v). \tag{2-3}$$

Proof. We follow similar arguments as in the proof of Proposition 3.1 in [Tang 2008], but with necessary modifications. We omit it. □

3. Penalization of the functional

To proceed, we introduce the cut-off function $l(t) : \mathbb{R} \to \mathbb{R}$ defined by

$$l(t) = \begin{cases} \min\{t^{(p-2)/2}, \nu_0\} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}$$

where ν_0 is as in Lemma 2.2. For a fixed nonempty subset $J \subset \{1, 2, \ldots, k\}$, set

$$\Omega_J = \bigcup_{j \in J} \Omega_j, \quad \Omega'_J = \bigcup_{j \in J} \Omega'_j, \quad \chi_{\Omega'_J}(x) = \begin{cases} 1 & \text{for } x \in \Omega'_J, \\ 0 & \text{for } x \notin \Omega'_J, \end{cases}$$

and

$$w(x, \xi^2) = \chi_{\Omega'_J}(x)\xi^{p-2} + (1 - \chi_{\Omega'_J}(x))l(\xi^2),$$
$$W(x, \xi^2) = \int_0^{\xi^2} w(x, t)\, dt.$$

We define $\Psi_\lambda : H^\lambda_G \to \mathbb{R}$ by

$$\Psi_\lambda(v) = \tfrac{1}{2} \int_{\mathbb{R}^N} \big(|\nabla v|^2 + V_\lambda f^2(v)\big) - \tfrac{1}{2} \int_{\mathbb{R}^N} W(x, f^2(v)).$$

Then one can check that $\Psi_\lambda \in C^2(H^\lambda_G, \mathbb{R})$ and that its critical points are solutions of

$$-\Delta v + V_\lambda f(v) f'(v) = w(x, f^2(v)) f(v) f'(v) \quad \text{in } \mathbb{R}^N.$$

Note that $l(t) = t^{(p-2)/2}$ for $t \in \big[0, \nu_0^{2/(p-2)}\big]$, hence a critical point v of Ψ_λ is a solution of (1-10) if and only if $|f(v)|^2 \leq \nu_0^{2/(p-2)}$ in $\mathbb{R}^N \setminus \Omega'_J$.

4. Compactness of the modified functional

Proposition 4.1. *For $\lambda \geq 1$, Ψ_λ satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$. That is, any sequence $\{v_n\} \subset H_G^\lambda$ satisfying*

$$\Psi_\lambda(v_n) \to c, \tag{4-1}$$

$$\Psi'_\lambda(v_n) \to 0 \text{ strongly in } (H_G^\lambda)^*, \tag{4-2}$$

has a strongly convergent subsequence in H_G^λ, where $(H_G^\lambda)^*$ is the dual space of H_G^λ.

To prove Proposition 4.1, we require the following lemma:

Lemma 4.2. *Suppose that $\{v_n\} \subset H_G^\lambda$ is a $(\mathrm{PS})_c$ sequence. Then there exist two positive constants, $m(c)$ and $M(c)$, which are independent of $\lambda \geq 1$, such that*

$$m(c) \leq \liminf_{n\to\infty} \|v_n\|_\lambda^2 \leq \limsup_{n\to\infty} \|v_n\|_\lambda^2 \leq M(c).$$

Proof. Let $w_n = f(v_n)/f'(v_n)$. It follows from (4-1) and (4-2) that

$$\Psi_\lambda(v_n) - \frac{1}{p}\Psi'_\lambda(v_n)w_n = c + o(1) + \varepsilon_n\|w_n\|_\lambda,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Thus

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\frac{1}{2}-\frac{1}{p}\Big(1+\frac{f^2(v_n)}{1+f^2(v_n)}\Big)\Big)|\nabla v_n|^2 + \Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\mathbb{R}^N} V_\lambda f^2(v_n)\\ -\frac{1}{2}\int_{\mathbb{R}^N} W(x, f^2(v_n)) + \frac{1}{p}\int_{\mathbb{R}^N} w(x, f^2(v_n))f^2(v_n)\\ = c + o(1) + \varepsilon_n\|u_n\|_\lambda.\end{aligned}$$

Let $L(t) = \int_0^t l(t)\,dt$; we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N} W(x, f^2(v_n)) - \frac{1}{p}\int_{\mathbb{R}^N} w(x, f^2(v_n))f^2(v_n)\\ = \int_{\mathbb{R}^N\setminus\Omega'_J}\Big(\frac{1}{2}L(f^2(v_n)) - \frac{1}{p}l(f^2(v_n))f^2(v_n)\Big).\end{aligned}$$

Note that for $t \in \big[\nu_0^{2/(p-2)}, \infty\big)$,

$$\begin{aligned}\frac{1}{2}L(t^2) - \frac{1}{p}l(t^2)t^2 &= \frac{1}{2}\Big(\nu_0 t^2 - \frac{p-2}{p}\nu_0^{p/(p-2)}\Big) - \frac{1}{p}t^2\\ &= \Big(\frac{1}{2}-\frac{1}{p}\Big)\big(\nu_0 t^2 - \nu_0^{p/(p-2)}\big) \leq \Big(\frac{1}{2}-\frac{1}{p}\Big)\nu_0 t^2,\end{aligned}$$

and for $t \leq \nu_0^{2/(p-2)}$,

$$\frac{1}{2}L(t^2) - \frac{1}{p}l(t^2)t^2 = 0.$$

We obtain that

$$\Big(\frac{1}{2}-\frac{2}{p}\Big)\int_{\mathbb{R}^N}|\nabla v_n|^2+\Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\mathbb{R}^N}V_\lambda f^2(v_n)-\Big(\frac{1}{2}-\frac{1}{p}\Big)\nu_0\int_{\mathbb{R}^N}f^2(v_n) \\ \leq c+o(1)+\varepsilon_n\|v_n\|_\lambda.$$

Since $4<p<4N/(N-2)$, we have

$$\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V_\lambda f^2(v_n)-\nu_0\int_{\mathbb{R}^N}f^2(v_n)\leq\Big(\frac{p-4}{2p}\Big)^{-1}c+o(1)+o(\|v_n\|_\lambda).$$

By Lemma 2.2, we get

$$\delta_0\int_{\mathbb{R}^N}\big(|\nabla v_n|^2\,dx+V_\lambda f^2(v_n)\big)\leq\Big(\frac{p-4}{2p}\Big)^{-1}c+o(1)+\varepsilon_n\|v_n\|_\lambda.$$

It follows from Lemma 2.1 that

$$C_1\min\{\|v_n\|_\lambda,\,\|v_n\|_\lambda^2\}\leq\delta_0^{-1}\Big(\frac{p-4}{2p}\Big)^{-1}c+o(1)+o(\|v_n\|_\lambda).$$

Thus $\|v_n\|_\lambda$ is bounded as $n\to\infty$, and

$$\limsup_{n\to\infty}\|v_n\|_\lambda\leq M(c):=\max\bigg\{\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}\delta_0^{-1}c,\sqrt{\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}\delta_0^{-1}c}\bigg\}.$$

On the other hand, since

$$\frac{1}{2}L(t^2)-\frac{1}{p}l(t^2)t^2\geq 0\quad\text{for all } t\in\mathbb{R},$$

we have

$$c+o(1)+\varepsilon_n\|w_n\|_\lambda\leq\Big(\frac{1}{2}-\frac{1}{p}\Big)C_2\max\{\|v_n\|_\lambda,\,\|v_n\|_\lambda^2\}.$$

Therefore

$$\liminf_{n\to\infty}\|v_n\|_\lambda^2\geq m(c):=\min\bigg\{\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}C_2^{-1}c,\sqrt{\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}C_2^{-1}c}\bigg\}.$$

This completes the proof of Lemma 4.2. □

Proof of Proposition 4.1. By Lemma 4.2, we know that $\{v_n\}$ is bounded in H_G^λ and thus is bounded in $D^{1,2}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, so there exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that:

$$\nabla v_n\rightharpoonup\nabla v\quad\text{weakly in } L^2(\mathbb{R}^N),$$
$$v_n\to v\quad\text{a.e. in }\mathbb{R}^N,$$
$$f(v_n)\rightharpoonup f(v)\quad\text{weakly in } L^q(\mathbb{R}^N)\text{ for }2\leq q\leq 2\cdot 2^*,$$
$$f(v_n)\to f(v)\quad\text{strongly in } L^p_{\text{loc}}(\mathbb{R}^N).$$

Moreover, by Proposition 2.8 of [Guo and Tang 2012], v is a critical point of Ψ_λ, that is, for any $\psi \in H_G^\lambda$,

$$\int_{\mathbb{R}^N} \big(\nabla v \nabla \psi + V_\lambda f(v) f'(v)\psi\big) = \int_{\mathbb{R}^N} w(x, f^2(v)) f(v) f'(v)\psi.$$

Next we show that $v_n \to v$ strongly in H_G^λ. Indeed, it follows from (4-1) and (4-2) that

$$\begin{aligned}
o(1) &= (\Psi'_\lambda(v_n) - \Psi'_\lambda(v))\left(\frac{f(v_n)}{f'(v_n)} - \frac{f(v)}{f'(v)}\right)\\
&= \Psi'_\lambda(v_n)\frac{f(v_n)}{f'(v_n)} - \Psi'_\lambda(v_n)\frac{f(v)}{f'(v)} - \Psi'_\lambda(v)\frac{f(v_n)}{f'(v_n)} - \Psi'_\lambda(v)\frac{f(v)}{f'(v)}\\
&= \int_{\mathbb{R}^N}\left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n|^2 + \int_{\mathbb{R}^N} V_\lambda f^2(v_n) - \int_{\mathbb{R}^N} w(x, f^2(v_n)) f^2(v_n)\\
&\quad - \int_{\mathbb{R}^N}\left(1 + \frac{f^2(v)}{1+f^2(v)}\right)\nabla v_n \nabla v - \int_{\mathbb{R}^N} V_\lambda \frac{f(v_n)}{\sqrt{1+f^2(v_n)}} f(v)\sqrt{1+f^2(v)}\\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v_n)) f(v_n) f'(v_n)\frac{f(v)}{f'(v)} - \int_{\mathbb{R}^N}\left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)\nabla v_n \nabla v\\
&\quad - \int_{\mathbb{R}^N} V_\lambda \frac{f(v)}{\sqrt{1+f^2(v)}} f(v_n)\sqrt{1+f^2(v_n)}\\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v)) f(v) f'(v)\frac{f(v_n)}{f'(v_n)}\\
&\quad + \int_{\mathbb{R}^N}\left(1 + \frac{f^2(v)}{1+f^2(v)}\right)|\nabla v|^2 + \int_{\mathbb{R}^N} V_\lambda f^2(v) - \int_{\mathbb{R}^N} w(x, f^2(v)) f^2(v)\\
&= \int_{\mathbb{R}^N}\left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)(\nabla v_n - \nabla v)^2\\
&\quad + \int_{\mathbb{R}^N}\left(\frac{f^2(v_n)}{1+f^2(v_n)} - \frac{f^2(v)}{1+f^2(v)}\right)\nabla v(\nabla v_n - \nabla v) && \text{(I)}\\
&\quad + \int_{\mathbb{R}^N} V_\lambda\left(f^2(v_n) - \frac{f(v)}{\sqrt{1+f^2(v)}} f(v_n)\sqrt{1+f^2(v_n)}\right) && \text{(II)}\\
&\quad + \int_{\mathbb{R}^N} V_\lambda\left(f^2(v) - \frac{f(v_n)}{\sqrt{1+f^2(v_n)}} f(v)\sqrt{1+f^2(v)}\right) && \text{(III)}\\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v_n))\left(f(v_n) f'(v_n)\frac{f(v)}{f'(v)} - f^2(v_n)\right) && \text{(IV)}\\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v))\left(f(v) f'(v)\frac{f(v_n)}{f'(v_n)} - f^2(v)\right) && \text{(V)}\\
&=: \int_{\mathbb{R}^N}\left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right)(\nabla v_n - \nabla v)^2 + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.
\end{aligned}$$

In the following we shall estimate the above terms one by one. First of all, note that since $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^2(\mathbb{R}^N)$ and

$$\frac{f^2(v_n)}{1+f^2(v_n)} - \frac{f^2(v)}{1+f^2(v)}$$

is bounded, we have $\mathrm{I} = o(1)$ as $n \to \infty$. Moreover,

$$\begin{aligned}
\mathrm{II}+\mathrm{III} &= \int_{\mathbb{R}^N} V_\lambda \Big(f^2(v_n) - \frac{f(v)}{\sqrt{1+f^2(v)}} f(v_n)\sqrt{1+f^2(v_n)} \Big) \\
&\quad + \int_{\mathbb{R}^N} V_\lambda \Big(f^2(v) - \frac{f(v_n)}{\sqrt{1+f^2(v_n)}} f(v)\sqrt{1+f^2(v)} \Big) \\
&= \int_{\mathbb{R}^N} V_\lambda f(v_n)(f(v_n)-f(v)) + V_\lambda f(v_n) f(v) \Big(1 - \frac{\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v)}} \Big) \\
&\quad + \int_{\mathbb{R}^N} V_\lambda f(v)(f(v)-f(v_n)) + V_\lambda f(v) f(v_n) \Big(1 - \frac{\sqrt{1+f^2(v)}}{\sqrt{1+f^2(v_n)}} \Big) \\
&= \int_{\mathbb{R}^N} V_\lambda (f(v_n)-f(v))^2 \\
&\quad + \int_{\mathbb{R}^N} V_\lambda \frac{f(v_n)f(v)}{\sqrt{1+f^2(v)}\big(\sqrt{1+f^2(v_n)}+\sqrt{1+f^2(v)}\big)} (f^2(v)-f^2(v_n)) \\
&\quad + \int_{\mathbb{R}^N} V_\lambda \frac{f(v)f(v_n)}{\sqrt{1+f^2(v_n)}\big(\sqrt{1+f^2(v_n)}+\sqrt{1+f^2(v)}\big)} (f^2(v_n)-f^2(v)) \\
&= \int_{\mathbb{R}^N} V_\lambda (f(v_n)-f(v))^2 + o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

In the last equality, we use the facts that $f^2(v_n) \rightharpoonup f^2(v)$ weakly and that the two terms

$$\frac{f(v)f(v_n)}{\sqrt{1+f^2(v)}\big(\sqrt{1+f^2(v_n)}+\sqrt{1+f^2(v)}\big)}$$

and

$$\frac{f(v)f(v_n)}{\sqrt{1+f^2(v_n)}\big(\sqrt{1+f^2(v_n)}+\sqrt{1+f^2(v)}\big)}$$

are bounded. For the last two terms, we have

$$\begin{aligned}
\mathrm{IV}+\mathrm{V} &= \int_{\mathbb{R}^N} w(x, f^2(v_n)) f(v_n) \Big(\frac{f(v_n)}{f'(v)} f(v) - f(v_n) \Big) \\
&\qquad + \int_{\mathbb{R}^N} w(x, f^2(v)) f(v) \Big(\frac{f(v)}{f'(v_n)} f(v_n) - f(v) \Big) \\
&= \int_{\Omega'_J} |f(v_n)|^{p-2} f(v_n) \Big(\frac{f'(v_n)}{f'(v)} f(v) - f(v_n) \Big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v_n))f(v_n)\left(\frac{f(v_n')}{f'(v)}f(v)-f(v_n)\right)\\
&\quad + \int_{\Omega'_J} |f(v)|^{p-2}f(v)\left(\frac{f'(v)}{f'(v_n)}f(v_n)-f(v)\right)\\
&\quad + \int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v))f(v)\left(\frac{f'(v)}{f'(v_n)}f(v_n)-f(v)\right)\\
&= \int_{\Omega'_J} (f(v_n)-f(v))(f^{p-1}(v)-f^{p-1}(v_n))\\
&\quad + \int_{\Omega'_J} |f(v_n)|^{p-2}f(v_n)\left(\frac{f'(v_n)}{f'(v)}-1\right)f(v) && (\mathrm{I}_1)\\
&\quad + \int_{\Omega'_J} |f(v)|^{p-2}f(v)\left(\frac{f'(v)}{f'(v_n)}-1\right)f(v_n) && (\mathrm{I}_2)\\
&\quad + \int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v_n))f(v_n)\left(\frac{f(v_n')}{f'(v)}f(v)-f(v_n)\right) && (\mathrm{I}_3)\\
&\quad + \int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v))f(v)\left(\frac{f'(v)}{f'(v_n)}f(v_n)-f(v)\right) && (\mathrm{I}_4)\\
&= \int_{\Omega'_J} (f(v_n)-f(v))(f^{p-1}(v)-f^{p-1}(v_n))+\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{I}_1 &= \int_{\Omega'_J} |f(v_n)|^{p-2}f(v_n)\left(\frac{f'(v_n)}{f'(v)}-1\right)f(v)\\
&= \int_{\Omega'_J} |f(v_n)|^{p-2}f(v_n)\frac{\sqrt{1+f^2(v)}-\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v_n)}}f(v)\\
&= \int_{\Omega'_J} |f(v_n)|^{p-2}f(v)(f(v)-f(v_n))\frac{f(v_n)(f(v)+f(v_n))}{\sqrt{1+f^2(v_n)}\big(\sqrt{1+f^2(v)}+\sqrt{1+f^2(v_n)}\big)}\\
&\le C\left(\int_{\Omega'_J} f^p(v_n)\right)^{(p-2/p)}\left(\int_{\Omega'_J} f^p(v)\right)^{1/p}\left(\int_{\Omega'_J} (f(v)-f(v_n))^p\right)^{1/p}\\
&= o(1) \text{ as } n\to\infty \quad (\text{since } f(v_n)\to f(v) \text{ strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^N)).
\end{aligned}
$$

Similarly, we have $\mathrm{I}_2 = o(1)$ as $n\to\infty$.

As for $\mathrm{I}_3+\mathrm{I}_4$, we have

$$
\mathrm{I}_3+\mathrm{I}_4 = \int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v_n))f(v_n)\left(\frac{f(v_n')}{f'(v)}f(v)-f(v_n)\right)
$$

$$
\begin{aligned}
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v)) f(v) \left(\frac{f'(v)}{f'(v_n)} f(v_n) - f(v) \right) \\
&= \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n)) f(v_n)(f(v) - f(v_n)) \\
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v)) f(v)(f(v_n) - f(v)) \\
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n)) f(v_n) \left(\frac{f'(v_n)}{f'(v)} - 1 \right) f(v) \\
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v)) f(v) \left(\frac{f'(v)}{f'(v_n)} - 1 \right) f(v_n).
\end{aligned}
$$

For the same reasons that we used in the above estimates for I_1, we can see that the last two terms in the above equalities go to zero as n goes to infinity.

Thus

$$
\begin{aligned}
\mathrm{I}_3 + \mathrm{I}_4 &= \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n)) f(v_n)(f(v) - f(v_n)) \\
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v)) f(v)(f(v_n) - f(v)) + o(1) \\
&= \int_{\mathbb{R}^N \setminus \Omega'_J} (l(f^2(v_n)) - l(f^2(v))) f(v)(f(v) - f(v_n)) \\
&\quad - \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n))(f(v) - f(v_n))^2 + o(1).
\end{aligned}
$$

On the other hand, since $f(v_n) \to f(v)$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $f(v_n) \rightharpoonup f(v)$ weakly in $L^q(\mathbb{R}^N)$ for $2 \le q \le 2 \cdot 2^*$, and $l(t) \le \nu_0$ for all $t \ge 0$, we have

$$
\int_{\Omega'_J} (f(v_n) - f(v))(f^{p-1}(v) - f^{p-1}(v_n)) = o(1)
$$

and

$$
\int_{\mathbb{R}^N \setminus \Omega'_J} (l(f^2(v_n)) - l(f^2(v))) f(v)(f(v) - f(v_n)) = o(1).
$$

At last, we obtain the following estimate:

$$
\begin{aligned}
o(1) &= \int_{\mathbb{R}^N} \left(1 + \frac{f^2(v_n)}{1 + f^2(v_n)} \right) |\nabla v_n - \nabla v|^2 \\
&\quad + \int_{\mathbb{R}^N} V_\lambda (f(v) - f(v_n))^2 - \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n))(f(v) - f(v_n))^2.
\end{aligned}
$$

On the other hand, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(f(v)-f(v_n))|^2 \\
&\quad= \int_{\mathbb{R}^N} \left|\frac{\nabla v}{\sqrt{1+f^2(v)}} - \frac{\nabla v_n}{\sqrt{1+f^2(v_n)}}\right|^2 \\
&\quad= \int_{\mathbb{R}^N} \frac{1}{\sqrt{1+f^2(v_n)}} \left|\nabla v_n - \nabla v + \left(1-\frac{\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v)}}\right)\nabla v\right|^2 \\
&\quad= \int_{\mathbb{R}^N} \frac{|\nabla v-\nabla v_n|^2}{\sqrt{1+f^2(v_n)}} + 2\int_{\mathbb{R}^N} \frac{1}{\sqrt{1+f^2(v_n)}}\left(1-\frac{\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v)}}\right)\nabla v(\nabla v-\nabla v_n) \\
&\qquad\quad + \int_{\mathbb{R}^N} \frac{1}{\sqrt{1+f^2(v_n)}}\left(1-\frac{\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v)}}\right)^2 |\nabla v|^2.
\end{aligned}$$

We claim that both of the last two terms in the above last equality are $o(1)$ as $n \to \infty$. In fact, the first term goes to zero because $\nabla v_n \rightharpoonup \nabla v$, while the second term goes to zero by the dominated convergence theorem.

Thus we have

$$\int_{\mathbb{R}^N} |\nabla(f(v)-f(v_n))|^2 \le \int_{\mathbb{R}^N} |\nabla v - \nabla v_n|^2$$

by Lemma 2.2 and the definition of $l(t)$, so we get

$$\begin{aligned}
&\delta_0 \int_{\mathbb{R}^N} \left(|\nabla(f(v)-f(v_n))|^2 + V_\lambda (f(v)-f(v_n))^2\right) \\
&\qquad< \int_{\mathbb{R}^N}\left(1+\frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n - \nabla v|^2 + \int_{\mathbb{R}^N} V_\lambda (f(v)-f(v_n))^2 \\
&\qquad\qquad\qquad\qquad - \nu_0 \int_{\mathbb{R}^N\setminus\Omega'_J} (f(v)-f(v_n))^2 = o(1).
\end{aligned}$$

Obviously, $\int_{\mathbb{R}^N} V_\lambda\big(f(v_n)-f(v)\big)^2 \to 0$ as $n \to \infty$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^N} V_\lambda (f^2(v_n) - f^2(v)) &= \int_{\mathbb{R}^N} (f(v_n)-f(v))(f(v_n)+f(v)) \\
&\le C\left(\int_{\mathbb{R}^N} V_\lambda (f(v_n)-f(v))^2\right)^{1/2}
\end{aligned}$$

for some constant C. By Proposition 2.1(3) of [Liu et al. 2003], we have $v_n \to v$ strongly in H_G^λ. This completes the proof of Proposition 4.1. □

5. Some asymptotic behavior

We denote by $H_G^{0,1}(\Omega_j)$ the closure of $C_0^\infty(\Omega)$ under the norm of $H_G^1(\Omega)$.

Proposition 5.1. *Assume that the sequences* $\{v_n\} \subset H_G^1$ *and* $\{\lambda_n\} \subset [0, \infty)$ *satisfy*

$$\lambda_n \to \infty, \tag{5-1}$$

$$\Psi_{\lambda_n}(v_n) \to c, \tag{5-2}$$

$$\|\Psi'_{\lambda_n}(v_n)\|^*_{\lambda_n} \to 0. \tag{5-3}$$

Then there exists a subsequence of $\{v_n\}$ (*still denoted by* $\{v_n\}$) *such that*

$$v_n \rightharpoonup v \quad \text{weakly in } H_G^1$$

for some $v \in H_G^1$. *Moreover, we have*:

(i) $v \equiv 0$ *in* $\mathbb{R}^N \setminus \Omega_J$, *and* v *is a solution of*

$$\begin{cases} -\Delta v = |f(v)|^{p-2} f(v) f'(v), & \text{in } \Omega_j, \\ v \in H_G^{0,1}(\Omega_j) & \text{for } j \in J. \end{cases} \tag{5-4}$$

(ii) v_n *converges to* v *in a stronger sense*, *namely*

$$v_n \to v \quad \text{strongly in } H_G^1 \text{ as } \lambda_n \to \infty.$$

(iii) *The functions* $\{v_n\}$ *satisfy*:

$$\int_{\mathbb{R}^N} V_{\lambda_n} f^2(v_n) \to 0,$$

$$\Psi_\lambda(v_n) \to \sum_{j \in J} I_{\Omega_j}(v),$$

$$\|v_n\|_{\lambda_n, \mathbb{R}^N \setminus \Omega'_J} \to 0,$$

$$\|v_n\|_{\lambda_n, \Omega'_j} \to \int_{\Omega_j} |\nabla v|^2 \quad \text{for } j \in J, \text{ as } n \to \infty.$$

Proof. By arguments similar to those used in the proof of Lemma 4.2, we have

$$m(c) \le \liminf_{n \to \infty} \|v_n\|^2_{\lambda_n} \le \limsup_{n \to \infty} \|v_n\|^2_{\lambda_n} \le M(c).$$

Thus $\{v_n\}$ is bounded in H_G^1. Hence there is a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that:

$$\begin{gathered} \nabla v_n \rightharpoonup \nabla v \text{ weakly in } L^2(\mathbb{R}^N), \\ v_n \rightharpoonup v \text{ weakly in } L^q(\mathbb{R}^N) \text{ for } 2 \le q \le 2 \cdot 2^*, \\ v_n \to v \text{ a.e. in } \mathbb{R}^N, \\ f(v_n) \to f(v) \text{ strongly in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for } 2 \le q < 2 \cdot 2^*, \\ f(v_n) \rightharpoonup f(v) \text{ weakly in } L^q(\mathbb{R}^N) \text{ for } 2 \le q \le 2 \cdot 2^*. \end{gathered}$$

(i) Let $C_m := \{x \in \mathbb{R}^N \mid V(x) \geq 1/m\}$. Then for n large, we have

$$\begin{aligned}\int_{C_m} f^2(v_n) \leq \frac{m}{\lambda_n}\int_{\mathbb{R}^N} \lambda_n V f^2(v_n) &\leq \frac{m}{\lambda_n}\int_{\mathbb{R}^N} (V_{\lambda_n} f^2(v_n) + |\nabla v_n|^2) \\ &\leq \frac{m}{\lambda_n} C \max\{\|v_n\|_{\lambda_n}, \|v_n\|_{\lambda_n}^2\} \to 0 \quad \text{as } \lambda_n \to \infty.\end{aligned}$$

Thus

$$0 \leq \int_{C_m} f^2(v) \leq \lim_{n\to\infty} \int_{C_m} f^2(v_n) = 0.$$

Hence $f(v) = 0$ on $\bigcup_{m=1}^{\infty} C_m = \mathbb{R}^N \setminus \bar{\Omega}$. Note that $\Psi'_{\lambda_n}(v_n) \to 0$ as $\lambda_n \to \infty$, so we have

$$\begin{aligned}o(1) &= \Psi'_{\lambda_n}(v_n) \cdot \frac{f(v)}{f'(v)} \\ &= \int_{\Omega_j} \Big(1 + \frac{f^2(v)}{1+f^2(v)}\Big)\nabla v_n \nabla v - \int_{\Omega_j} w(x, f^2(v_n)) f(v_n) f'(v_n) \frac{f(v)}{f'(v)} \\ &= \int_{\Omega_j} \Big(1 + \frac{f^2(v)}{1+f^2(v)}\Big)|\nabla v|^2 - \int_{\Omega_j} w(x, f^2(v)) f^2(v) + o(1);\end{aligned}$$

here we use the fact that $f(v_n) \to f(v)$ strongly in $L^q_{\text{loc}}(\mathbb{R}^N)$.

On the other hand, by Lemma 2.2, we have

$$\begin{aligned}\delta_0 \|v\|_{\lambda_n} &\leq \int_{\Omega_j} (|\nabla v|^2 - \nu_0 f^2(v)) \\ &\leq \int_{\Omega_j} \Big(1 + \frac{f^2(v)}{1+f^2(v)}\Big)|\nabla v|^2 - \int_{\Omega_j} w(x, f^2(v)) f^2(v) = 0\end{aligned}$$

Note that $\|v\|_{\lambda_n}$ indeed does not dependent on λ_n. We have that $v \equiv 0$ in Ω_j for $j \in \{1, 2, \dots, k\} \setminus J$, and this completes the proof of part (i).

(ii) Indeed, by a similar argument as in the proof of Proposition 4.1, for n large, we have

$$\begin{aligned}o(1) &= \int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^2 + \int_{\mathbb{R}^N} V_{\lambda_n}(f(v_n) - f(v))^2 - \nu_0 \int_{\mathbb{R}^N \setminus \Omega_J} (f(v) - f(v_n))^2 \\ &\geq \delta_0 \Big(\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^2 + \int_{\mathbb{R}^N} V_{\lambda_n}(f(v_n) - f(v))^2\Big) \\ &\geq \delta_0\, C \min\big\{\|v_n - v\|_{H_G^{\lambda_n}}, \|v_n - v\|^2_{H_G^{\lambda_n}}\big\}.\end{aligned}$$

Hence $\|v_n - v\|_{H^1_G} \to 0$ as $n \to \infty$. This completes the proof of part (ii).

(iii) This is a direct consequence of parts (i) and (ii). In fact, from (ii) and (i), one can see that

$$\tfrac{1}{2}\int_{\mathbb{R}^N} V_{\lambda_n} f^2(v_n) = \tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_j} V_{\lambda_n} f^2(v_n)$$
$$= \int_{\mathbb{R}^N\setminus\Omega_j} V_{\lambda_n} f^2(v_n)(f(v_n)-f(v))^2 \to 0 \quad \text{as } n\to\infty.$$

Thus we have

$$\lim_{n\to\infty}\tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_j} V_{\lambda_n} f^2(v_n) = \tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_j} V_{\lambda_n} f^2(v) = 0.$$

Obviously, we get

$$\Psi_{\lambda_n}(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 + V_{\lambda_n} f^2(v_n) - \frac{1}{p}\int_{\mathbb{R}^N} W(x, f^2(v_n)) \to \sum_{j\in J} I_{\Omega_j}(v),$$

where $I_{\Omega_j}(v) = \frac{1}{2}\int_{\Omega_j}|\nabla v|^2 - (1/p)\int_{\Omega_j}|f(v)|^p$. Furthermore,

$$\lim_{n\to\infty}\|f^2(v_n)\|_{H_G^{\lambda_n}(\mathbb{R}^N\setminus\Omega'_J)} = 0,$$
$$\lim_{n\to\infty}\|\nabla v_n\| = \int_{\Omega_j}|\nabla v|^2 \ \text{ for } j\in J.$$

This completes the proof of Proposition 5.1. □

Proposition 5.2. *There exist constants $M>0$, $\Lambda_0>0$ such that if v_λ is a critical point of Ψ_λ for $\lambda\ge\Lambda_0$, then $|f(v_\lambda)|^2\le \nu_0^{2/(p-2)}$ and $\Psi_\lambda(v)\le M$. In particular, v_λ solves the problem* (1-10).

Proof. Let $B_r(x)=\{y\in\mathbb{R}^N \mid |x-y|<r\}$. Since v_λ is a critical point of Ψ_λ, we have

$$-\Delta v_\lambda + V_\lambda f(v_\lambda) f'(v_\lambda)$$
$$= \chi(\Omega_j)|f(v_\lambda)|^{p-1} f'(v_\lambda) + (1-\chi(\Omega_j)) l(x, f^2(v_\lambda)) f(v_\lambda) f'(v_\lambda).$$

That is,

$$-\Delta v_\lambda + \big(V_\lambda - \chi(\Omega_j)|f(v_\lambda)|^{p-12} - (1-\chi(\Omega_j)) l(x, f^2(v_\lambda))\big)\frac{f(v_\lambda) f'(v_\lambda)}{v_\lambda} v_\lambda = 0.$$

Let

$$V_0 = \big(V_\lambda - \chi(\Omega_j)|f(v_\lambda)|^{p-12} - (1-\chi(\Omega_j)) l(x, f^2(v_\lambda))\big)\frac{f(v_\lambda) f'(v_\lambda)}{v_\lambda}.$$

Then our assumptions on V imply that V_0 belongs to K_N^{loc}, the local Kato class, and thus $|v_\lambda(x)|_{L^\infty}$ is bounded (see Theorem C1.2 of [Simon 1982]). It follows from Theorem 8.17 of [Gilbarg and Trudinger 1983] that

$$|v_\lambda(x)| \le C \int_{B(x,r)} |v_\lambda(y)|^p \, dy.$$

By Proposition 5.1, we see that for any sequence $\lambda_n \to \infty$, we can extract a subsequence of $\{\lambda_n\}$ (still denoted by $\{\lambda_n\}$) such that $v_{\lambda_n} \to v \in H_0^1(\Omega_j)$ strongly in $L^2(\mathbb{R}^N \setminus \Omega_j)$. Since the sequence $\{\lambda_n\}$ can be chosen arbitrarily, we conclude that

$$v_\lambda \to v \in H_0^1(\Omega_j) \quad \text{strongly as } \lambda \to \infty.$$

Now choose $r \in (0, \operatorname{dist}(\Omega_J, \mathbb{R}^N \setminus \Omega'_J))$; we have, uniformly in $x \in \mathbb{R}^N \setminus \Omega'_J$, that

$$\begin{aligned}
|v_\lambda(x)| &\le C(r) \int_{B_r(x)} |v_\lambda(x)|^p \\
&\le C(r)(\text{meas } B_r(x))^{1-q/2^*} \biggl(\int_{B(x,r)} |v_\lambda(x)|^{2^*} \biggr)^{p/2^*} \\
&\le C(r)(\text{meas } B_r(x))^{1-q/2^*} \biggl(\int_{B(x,r)} |\nabla v_\lambda(x)|^2 \biggr)^{p/2} \\
&\le C(r) \biggl(\biggl(\int_{\mathbb{R}^N \setminus \Omega'_J} |\nabla v_\lambda|^2 \biggr)^{1/2} + V_\lambda f^2(v_\lambda) \biggr) \\
&\le C \max\{\|v_\lambda\|, \|v_\lambda\|^{1/2}\} \\
&\to 0 \quad \text{uniformly in } x \in \mathbb{R}^N \setminus \Omega'_J,
\end{aligned}$$

which implies that $f(|v_\lambda|) \to 0$ uniformly in $x \in \mathbb{R}^N \setminus \Omega'_J$. This completes the proof of Proposition 5.2. □

Remark 5.3. The critical points of Ψ_λ are not necessarily positive. In fact, if we replace the function v by its positive part v^+ in the nonlinearity term $W(x, f^2(v))$ of Ψ_λ, and the new functional is denoted by Ψ_λ^+, then by arguments similar to those above, one can see that the new functional Ψ_λ^+ still satisfies properties analogous to all those proved for Ψ_λ in previous sections. As a consequence, the critical points of Ψ^+ are positive. In the following, for convenience we only consider Ψ_λ instead of Ψ_λ^+.

Remark 5.4. Proposition 4.1 shows that Ψ_λ satisfies the Palais–Smale condition. We can easily check that Ψ_λ has mountain pass geometry. Hence, a mountain pass argument shows that, for each $\lambda > 0$, Ψ_λ admits a nontrivial critical point u_λ. In fact, $\Psi_\lambda(u_\lambda) \le \max_{t>0} I_{\Omega_j}(t\omega_j)$ (see Section 6 for the definition of I_{Ω_j} and ω_j) and thus $\Psi_\lambda(u_\lambda) \le M$, where M is independent of λ. As a result, by Proposition 5.2, we deduce the existence of a positive solution to (1-10) and thus a positive solution to the original problem (1-1) for $\lambda > \Lambda$. However, it is not clear whether such solutions concentrate on the set Ω_J. The aim of the following parts of the paper is to focus on the solutions with such properties.

6. Limit problem

For $j \in J$ we define the following two functionals:

$$I_{\Omega_j}(v) = \frac{1}{2}\int_{\Omega_j} |\nabla v|^2 - \frac{1}{p}\int_{\Omega_j} |f(v)|^p \quad \text{for } v \in H_G^{1,0}(\Omega_j),$$

and

$$\Psi_{\lambda,\Omega'_j}(u) = \frac{1}{2}\int_{\Omega'_j} (|\nabla v|^2 + V_\lambda f^2(v)) - \frac{1}{p}\int_{\Omega'_j} |f(v)|^p \quad \text{for } v \in H_G^1(\Omega'_j). \tag{6-1}$$

By Lemma 2.2 of [Guo and Tang 2012] and the following inequality

$$\|f(v)\|_{L^p} \le \|f(v)\|_2^{\theta} \|f(v)\|_{L^{2\cdot 2^*}}^{1-\theta} \quad \text{for } 0 < \theta < 1,$$

following a standard argument (see [Tang 2008]), one can see that both I_{Ω_j} and $\Psi_{\Omega'_j}$ satisfy the mountain pass geometry conditions. That is:

(i) $I_{\Omega_j}(0) = \Psi_{\lambda,\Omega'_j}(0) = 0$.

(ii) There exist $\rho_0 > 0$ and $\rho_1 > 0$, independent of $\lambda \ge 0$, such that

$$\begin{aligned} \|v\|_{H_G^{1,0}(\Omega_j)} \le \rho_0 &\Longrightarrow I_{\Omega_j}(v) \ge 0, \\ \|v\|_{H_G^{1,0}(\Omega_j)} = \rho_0 &\Longrightarrow I_{\Omega_j}(v) \ge \rho_1, \end{aligned} \tag{6-2}$$

and

$$\begin{aligned} \|v\|_{H_G^{1}(\Omega'_j)} \le \rho_0 &\Longrightarrow \Psi_{\lambda,\Omega'_j}(v) \ge 0, \\ \|v\|_{H_G^{1}(\Omega'_j)} = \rho_0 &\Longrightarrow \Psi_{\lambda,\Omega'_j}(v) \ge \rho_1. \end{aligned} \tag{6-3}$$

Here we use the notation

$$\|v\|_{H_G^{1,0}(\Omega_j)} = \int_{\Omega_j} |\nabla v|^2 \quad \text{for } v \in H_G^{0,1}(\Omega_j).$$

(iii) There exists $\psi_j \in C_0^\infty(\Omega_j)$ such that

$$\begin{gathered} \|\psi_j(x)\|_{H_G^{\lambda}(\Omega_j)} = \|\psi_j(x)\|_{H_G^{\lambda,0}(\Omega_j)} \ge \rho_1, \\ \Psi_{\lambda,\Omega'_j}(\psi_j) = I_{\Omega_j}(\psi_j) < 0. \end{gathered}$$

We define

$$\begin{aligned} c_j &= \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} I_{\Omega_j}(\gamma(t)), \\ c_{\lambda,j} &= \inf_{\gamma\in\Gamma_{\lambda,j}} \max_{t\in[0,1]} \Psi_{\lambda,\Omega'_j}(\gamma(t)), \end{aligned} \tag{6-4}$$

where

$$\Gamma_j = \{\gamma \in C([0,1], H_G^{0,1}(\Omega_j)) \mid \gamma(0) = 0,\ I_{\Omega_j}(\gamma(1)) < 0\},$$
$$\Gamma_{\lambda,j} = \{\gamma \in C([0,1], H_G^1(\Omega'_j)) \mid \gamma(0) = 0,\ \Psi_{\lambda,\Omega'_j}(\gamma(1)) < 0\}.$$

By Proposition 2.3 and Lemma 2.2 of [Guo and Tang 2012], it is standard to verify that Φ_{λ,Ω'_j} and I_{Ω_j} satisfy the Palais–Smale condition and that c_j, $c_{\lambda,j}$ are achieved by critical points. We denote the corresponding critical points by ω_j and $\omega_{\lambda,j}$ respectively.

Lemma 6.1. (i) *$0 < \rho_1 \le c_{\lambda,j} \le c_j$ for all $\lambda \ge 0$.*

(ii) *c_j and $c_{\lambda,j}$ are least energy levels for I_{Ω_j} and Φ_{λ,Ω'_j}, respectively, i.e.,*

$$c_j = \inf\{I_{\Omega_j}(v) \mid v \in H_G^{0,1}(\Omega_j) \setminus \{0\} \text{ is a critical point of } I_{\Omega_j}\},$$
$$c_{\lambda,j} = \inf\{\Psi_{\lambda,\Omega'_j}(v) \mid v \in H_G^1(\Omega'_j) \setminus \{0\} \text{ is a critical point of } \Psi_{\lambda,\Omega'_j}\}.$$

(iii) *$c_j = \max_{r>0} I_{\Omega_j}(r\omega_j)$, $c_{\lambda,j} = \max_{r>0} \Phi_{\lambda,\Omega'_j}(r\omega_{\lambda,j})$.*

(iv) *$c_{\lambda,j} \to c_j$ as $\lambda \to \infty$.*

Proof. By (6-3), it is easy to see that $c_{\lambda,j} \ge \rho_1$. On the other hand, for any $v \in H_G^{0,1}(\Omega_j)$, we may extend v to $\tilde{v} \in H_G^1(\Omega'_j)$ by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega_j, \\ 0 & \text{if } x \in \Omega'_j \setminus \bar{\Omega}_j, \end{cases}$$

so we may regard $H_G^{0,1}(\Omega_j) \subset H_G^1(\Omega'_j)$. Thus we have $\Gamma_j \subset \Gamma_{\lambda,j}$ and

$$\begin{aligned} c_{\lambda,j} &= \inf_{\gamma\in\Gamma_{\lambda,j}} \max_{t\in[0,1]} \Psi_{\lambda,\Omega'_j}(\gamma(t)) \\ &\le \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} \Psi_{\lambda,\Omega'_j}(\gamma(t)) \\ &= \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} I_{\Omega_j}(\gamma(t)) = c_j. \end{aligned} \tag{6-5}$$

This proves (i).

Note that, since $f(v)$ is monotone with respect to v, and so is $|f(v)|^p$ with respect to $|f(v)|$, the proofs of (ii) and (iii) are standard; see [Tang 2008].

Now we prove (iv). Using Proposition 5.1, we may extract a subsequence $\lambda_n \to \infty$ such that

$$\omega_{\lambda_n,j} \to v_0 \quad \text{strongly in } H_G^1(\Omega'_j),$$

where $v_0 \in H_G^{0,1}(\Omega_j)$ is a solution of (5-4) and

$$\Psi_{\lambda_n,\Omega'_j}(\omega_{\lambda_n,j}) \to I_{\Omega_j}(v_0).$$

By the definition of c_j, we have

$$\limsup_{\lambda\to\infty} c_{\lambda,j} = \limsup_{\lambda\to\infty} \Psi_{\lambda,\Omega'_j}(\omega_{\lambda,j}) \geq I_{\Omega_j}(u_0) \geq c_j.$$

Comparing with (6-5), we get (iv). This completes the proof of Lemma 6.1. □

7. Minimax arguments

Now we give a minimax argument for Φ_λ (see (1-8)).

We choose $R \geq 2$ such that

$$I_{\Omega_j}(R\omega_j) < 0. \tag{7-1}$$

Without loss of generality, we assume that $J = \{1, 2, \dots, l\}$ ($l \leq k$). Set

$$\gamma_0(s_1, s_2, \dots, s_l) = \sum_{j=1}^{l} s_j R\omega_j \quad \text{for } (s_1, s_2, \dots, s_l) \in [0, 1]^l, \tag{7-2}$$

$$\Gamma_J = \left\{ \gamma \in C([0,1]^l, H^1_G) \;\middle|\; \begin{array}{l} \gamma(s_1, s_2, \dots, s_l) = \gamma_0(s_1, s_2, \dots, s_l) \\ \text{for } (s_1, s_2, \dots, s_l) \in \partial([0, l]^l) \end{array} \right\}.$$

We define

$$b_{\lambda,J} = \inf_{\gamma\in\Gamma_J} \max_{(s_1,s_2,\dots,s_l)\in\partial([0,1]^l)} \Phi_\lambda(\gamma(s_1, s_2, \dots, s_l)).$$

Note that the projection $t \mapsto tR\omega_j$ belongs to Γ_j and satisfies

$$\max_{t\in[0,1]} I_{\Omega_j}(tR\omega_j) = c_j$$

for any $j \in J$. Hence $\gamma_0 \in \Gamma_J$, $\Gamma_J \neq \varnothing$, and $b_{\lambda,J}$ is well defined. We denote $c_J = \sum_{j=1}^{l} c_j$. Then we have:

Lemma 7.1. (i) $\sum_{j=1}^{l} c_{\lambda,j} \leq b_{\lambda,J} \leq c_J$ *for all* $\lambda \geq 0$.

(ii) $\Psi_\lambda(\gamma(s_1, s_2, \dots, s_l)) \leq c_J - \rho_1$ *for all* $\lambda \geq 0$, $\gamma \in \Gamma_J$ *and* $(s_1, s_2, \dots, s_l) \in \partial([0, 1]^l)$, *where* ρ_1 *is given in* (6-2), (6-3).

Proof. For any given $\gamma \in \Gamma_J$, let

$$T_j(s_1, \dots, s_l) = \frac{\int_{\Omega'_j} |f(\gamma(s_1, \dots, s_l))|^{p-1} f'(\gamma(s_1, \dots, s_l))\gamma(s_1, \dots, s_l)}{\int_{\Omega'_j} |\nabla\gamma(s_1, \dots, s_l)|^2 + V_\lambda f(\gamma(s_1, \dots, s_l)) f'(\gamma(s_1, \dots, s_l))\gamma(s_1, \dots, s_l)}$$

for $j = 1, 2, \dots, l$.

We define a map $\mathcal{T} : [0, 1]^l \to \mathbb{R}^l$ by

$$\mathcal{T}(\cdot) = (T_1(\cdot), \dots, T_l(\cdot)).$$

Thus for $(s_1, s_2, \dots, s_l) \in \partial([0,1]^l)$, we have

$$\mathcal{T}(s_1,\dots,s_l) = \left(\frac{\int_{\Omega'_1} |f(s_1R\omega_1)|^{p-1} f'(s_1R\omega_1)s_1R\omega_1}{\int_{\Omega'_1} |\nabla(s_1R\omega_1)|^2 + V_\lambda f(s_1R\omega_1)s_1R\omega_1}, \dots, \frac{\int_{\Omega'_l} |f(s_lR\omega_l)|^{p-1} f'(s_lR\omega_l)s_lR\omega_l}{\int_{\Omega'_l} |\nabla(s_lR\omega_l)|^2 + V_\lambda f(s_lR\omega_l)s_lR\omega_l}\right).$$

To proceed, we consider the function ρ defined by

$$\rho(\alpha) = \frac{\int_{\Omega_j} |f(\alpha v)|^{p-1} f'(\alpha v)\alpha v}{\alpha^2 \int_{\Omega_j} |\nabla v|^2 + \int_{\Omega_j} V_\lambda f(\alpha v) f'(\alpha v)\alpha v} = \frac{\rho_1(\alpha)}{\int_{\Omega_j} |\nabla v|^2 + \rho_2(\alpha)},$$

where

$$\rho_1(\alpha) = \int_{\Omega_j} \frac{f(\alpha v)|f(\alpha v)|^{p-1} v}{\alpha\sqrt{1+f^2(\alpha v)}}, \qquad \rho_2(\alpha) = \int_{\Omega_j} V_\lambda \frac{f(\alpha v)v}{\alpha\sqrt{1+f^2(\alpha v)}}.$$

By the proof of Lemma 3.2 of [Guo and Tang 2012], we see that ρ_1 is monotone increasing and ρ_2 is monotone decreasing; as a result, we see that ρ is monotone with respect to α. On the other hand, we note that $I_{\Omega_j}(R\omega_j) < 0$, $j = 1, 2, \dots, l$, for the same reason as in the proof of Lemma 4.2 of [Tang 2008], so we obtain

$$\deg(\mathcal{T}, [0,1]^l, (1, 1, \dots, 1)) = 1.$$

Hence there exists $(s_1, s_2, \dots, s_l) \in [0,1]^l$ such that

$$T_j(s_1, s_2, \dots, s_l) = 1 \quad \text{for } j = 1, 2, \dots, l. \tag{7-3}$$

Now we prove (i).

Since $\gamma_0 \in \Gamma_J$, we have

$$\begin{aligned} b_{\lambda,J} &\le \max_{(s_1,s_2,\dots,s_l)\in[0,1]^l} \Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)) \\ &= \max_{(s_1,s_2,\dots,s_l)\in[0,1]^l} \sum_{j=1}^{l} I_{\Omega_j}(s_jR\omega_j) = \sum_{j=1}^{l} c_j = c_J. \end{aligned}$$

On the other hand, by (7-3), for any $\gamma \in \Gamma_J$, there exists $s_\gamma \in [0,1]^l$ such that

$$\frac{\int_{\Omega'_j} |f(\gamma(s_\gamma))|^{p-1} f'(\gamma(s_\gamma))\gamma(s_\gamma)}{\int_{\Omega'_j} |\nabla\gamma(s_\gamma)|^2 + V_\lambda f(\gamma s_\gamma)\gamma(s_\gamma)} = 1 \quad \text{for} \quad j = 1, 2, \dots, l.$$

This implies that $\Psi'_{\lambda,\Omega'_j}(\gamma(s_\gamma)) \cdot \gamma(s_\gamma) = 0$ for $j = 1, 2, \dots, l$. Thus, if we define $u(x) = \gamma(s_\gamma)(x)$, we have

$$\Psi_\lambda(u) = \Psi_{\lambda,\mathbb{R}^N\setminus\Omega'_J}(u) + \sum_{j=1}^{l} \Psi_{\lambda,\Omega'_j}(u),$$

where

$$\Psi_{\lambda,\mathbb{R}^N\setminus\Omega'_J}(u) = \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'_J}(|\nabla v|^2 + V_\lambda f^2(v)) - \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'_J} W(f^2(v)).$$

Since $W(f^2(v)) \le \nu_0 f^2(v)$, we have

$$\begin{aligned}\Psi_{\lambda,\mathbb{R}^N\setminus\Omega'_J}(u) &= \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'_J}(|\nabla v|^2 + V_\lambda f^2(v)) - \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'_J} W(f^2(v)) \\ &\ge \frac{1}{2}\|u\|^2_{H^\lambda_G(\mathbb{R}^N\setminus\Omega'_j)} - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^N\setminus\Omega'_j)} \\ &\ge \frac{\delta_0}{2}\|u\|^2_{H^\lambda_G(\mathbb{R}^N\setminus\Omega'_j)} \ge 0.\end{aligned}$$

Thus

$$\begin{aligned}\Psi_\lambda(u) = \Psi_{\lambda,\mathbb{R}^N\setminus\Omega'_J}(u) + \sum_{j=1}^{l}\Psi_{\lambda,\Omega'_j}(u) &\ge \sum_{j=1}^{l}\Psi_{\lambda,\Omega'_j}(u) \\ &\ge \sum_{j=1}^{l}\inf\bigl\{\Psi_{\lambda,\Omega'_j}(v) \mid v \in H^1_G(\Omega'_j),\ \Psi'_{\lambda,\Omega'_j}(v)\cdot v = 0\bigr\} = \sum_{j=1}^{l} c_{\lambda,j}.\end{aligned}$$

Since $\gamma \in \Gamma_J$ is arbitrary, we have $b_{\lambda,J} \ge c_{\lambda,J}$.

For (ii), by the definition of γ_0, for $(s_1, s_2, \dots, s_l) \in \partial([0,1]^l)$ we have

$$\Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)) = \sum_{j=1}^{l} I_{\Omega_j}(s_j R\omega_j),$$

and $I_{\Omega_j}(s_j R\omega_j) \le c_j$ for $j = 1, 2, \dots, l$. On the other hand, for some $j_0 \in J$, either $s_{j_0} = 1$ or $s_{j_0} = 0$, and thus $I_{\Omega_{j_0}}(s_{j_0} R\omega_{j_0}) \le 0$. Therefore

$$\Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)) \le \sum_{j\ne j_0} I_{\Omega_j}(s_j R\omega_j) \le c_J - \rho_1.$$

This completes the proof of Lemma 7.1. □

Corollary 7.2. *We have $b_{\lambda,J} \to c_J$ as $\lambda \to \infty$. Moreover, $b_{\lambda,J}$ is a critical value of Ψ_λ for large λ.*

Proof. From Lemma 6.1, we know that $c_{\lambda,j} \to c_j$ as $\lambda \to \infty$. It follows from Lemma 7.1 that $b_{\lambda,J} \to c_J$ as $\lambda \to \infty$. Thus, we may choose Λ large enough such that for all $\lambda \ge \Lambda$, we have $b_{\lambda,J} > c_J - \rho_1$. Since Ψ_λ satisfies the Palais–Smale condition, by the standard deformation argument we can see that $b_{\lambda,J}$ is a critical value of Ψ_λ for $\lambda \ge \Lambda$. □

8. Flow arguments and the proofs of the main results

Let

$$\Psi_\lambda^{c_J} = \{v \in H_G^\lambda \mid \Psi_\lambda(v) \le c_J\}.$$

We choose

$$0 < \mu < \tfrac{1}{3} \min_{j \in J} c_j, \tag{8-1}$$

and define

$$D_\lambda^\mu = \{v \in H_G^\lambda \mid \|v\|_{H_G^\lambda(\mathbb{R}^N \setminus \Omega'_J)} \le \mu,\ |\Psi_{\lambda,\Omega'_j}(v) - c_j| \le \mu \text{ for all } j \in J\}.$$

Note that ω_j is the least energy solution of (5-4), and

$$\Psi_{\lambda,\Omega'_j}(\omega_j) = \frac{1}{2}\int_{\Omega_j} |\nabla \omega_j|^2 - \int_{\Omega_j} |f(\omega_j)|^p = c_j.$$

Thus $D_\lambda^\mu \cap \Psi_\lambda^{c_J}$ contains all the functions of the following form:

$$\omega(x) = \begin{cases} \omega_j(x) & \text{if } x \in \Omega_j, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega_J. \end{cases}$$

Lemma 8.1. *There exists $\sigma_0 > 0$ and $\Lambda_0 \ge 0$, independent of λ, such that*

$$\|\Psi'_\lambda(u)\|_\lambda^* \ge \sigma_0 \quad \textit{for all } \lambda \ge \Lambda_0 \textit{ and for all } u \in (D_\lambda^{2\mu} \setminus D_\lambda^\mu) \cap \Psi_\lambda^{c_J}. \tag{8-2}$$

Proof. We prove it by contradiction. Suppose that there exist $\lambda_n \to \infty$ and $v_n \in (D_{\lambda_n}^{2\mu} \setminus D_{\lambda_n}^\mu) \cap \Psi_{\lambda_n}^{c_J}$ such that $\|\Psi'_{\lambda_n}(u)\|_{\lambda_n}^* \to 0$. Since $v_n \in D_{\lambda_n}^{2\mu}$, thus v_n is bounded in H_G^1, and it turns out that $\Psi_{\lambda_n}(v_n)$ stays bounded as $n \to \infty$. We may assume that (up to a subsequence)

$$\Psi_{\lambda_n}(v_n) \to c \le c_J.$$

Applying Proposition 5.1, we can extract a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that $v_n \to v$ in H_G^1 and such that the following hold:

$$\lim_{n\to\infty} \Psi_{\lambda_n}(v_n) = \sum_{j=1}^{l} I_{\Omega_j}(v) \le c_J, \tag{8-3}$$

$$\lim_{n\to\infty} \|v_n\|^2_{H_G^{\lambda_n}(\Omega'_j)} = \int_{\Omega_j} |\nabla v|^2 \quad \text{for all } j \in J, \tag{8-4}$$

$$\lim_{n\to\infty} \int_{\Omega'_j} |f(v_n)|^p = \int_{\Omega_j} |f(v)|^p, \tag{8-5}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega'_J} (|\nabla v_n|^2 + V_{\lambda_n} f^2(v_n)) = 0. \tag{8-6}$$

Since $c_J = \sum_{j=1}^{l} c_j$ and c_j is the least energy level for $I_{\Omega_j}(u)$, we have two possibilities:

(1) $I_{\Omega_j}(v|_{\Omega_j}) = c_j$ for all $j \in J$.

(2) $I_{\Omega_{j_0}}(v|_{\Omega_{j_0}}) = 0$, that is, $u|_{\Omega_{j_0}} = 0$ for some $j_0 \in J$.

In case (1), we have

$$\frac{1}{2}\int_{\Omega_j} |\nabla v|^2 - \frac{1}{2}\int_{\Omega_j} |f(v)|^p = c_j \text{ for all } j \in J$$

and it follows from (8-3), (8-4), and (8-6) that $v_n \in D^{\mu}_{\lambda_n}$ for large n, which contradicts the fact that to $v_n \in D^{2\mu}_{\lambda_n} \setminus D^{\mu}_{\lambda_n}$.

In case (2), it follows from (8-3) and (8-4) that

$$|\Psi_{H^{\lambda_n}_G(\Omega'_j)}(v_n) - c_{j_0}| \to c_{j_0} \geq 3\mu.$$

This also contradicts the fact that $v_n \in D^{2\mu}_{\lambda_n} \setminus D^{\mu}_{\lambda_n}$. This completes the proof. □

Proposition 8.2. *Let μ satisfy* (7-3) *and let Λ_0 be the constant given in Lemma 8.1. Then for $\lambda \geq \Lambda_0$, there exists a solution v_λ of* (1-1) *such that $v_\lambda \in D^{\mu}_{\lambda} \cap \Psi^{c_J}_{\lambda}$.*

Proof. Assume, to the contrary, that Ψ_λ has no critical points in $D^{\mu}_{\lambda} \cap \Psi^{c_J}_{\lambda}$. Since Ψ_λ satisfies the Palais–Smale condition, there exists a constant $d_\lambda > 0$ such that

$$\|\Psi'_\lambda(v)\|^*_\lambda \geq d_\lambda \quad \text{for all } v \in D^{\mu}_{\lambda} \cap \Psi^{c_J}_{\lambda},$$

where $\|\cdot\|^*_\lambda$ is the norm of the dual space of H^λ_G. By Lemma 8.1 we have

$$\|\Psi'_\lambda(v)\|^*_\lambda \geq \sigma_0 \quad \text{for all } v \in (D^{2\mu}_{\lambda} \setminus D^{\mu}_{\lambda}) \cap \Psi^{c_J}_{\lambda}.$$

Let $\varphi : H^\lambda_G \to \mathbb{R}$ be a Lipschitz continuous function such that

$$\varphi(v) = \begin{cases} 1 & \text{for } v \in D^{3\mu/2}_{\lambda}, \\ 0 & \text{for } v \notin D^{2\mu}_{\lambda}, \end{cases}$$

and $0 \leq \varphi(v) \leq 1$ for any $v \in H^\lambda_G$.

Since $\Psi_\lambda \in C^1(H^\lambda_G, \mathbb{R})$, we denote by $\mathscr{G} : H^{\lambda *}_G \to H^\lambda_G$ the pseudogradient field of Ψ, which satisfies

$$\|\mathscr{G}(u)\|_{H^\lambda_G} \leq 2\|\Psi'(u)\|^*_\lambda, \quad \langle \Psi'(u), \mathscr{G}(u)\rangle \geq (\|\Psi'(u)\|^*_\lambda)^2. \tag{8-7}$$

Now for $v \in \Psi^{c_J}_{\lambda}$, we define $\widetilde{W}(v) : \Psi^{c_J}_{\lambda} \to H^\lambda_G$ by

$$\widetilde{W}(v) = -\varphi(v)\frac{\mathscr{G}(u)}{\|\Psi'_\lambda(v)\|^*_\lambda}.$$

We consider the deformation $\eta : [0, \infty) \times \Psi^{c_J}_{\lambda} \to \Psi^{c_J}_{\lambda}$ defined by

$$\frac{d\eta}{dt} = \widetilde{W}(\eta(t, v)), \quad \eta(0, v) = v \in \Psi_\lambda^{c_J}.$$

Then $\eta(t, v)$ satisfies

$$\frac{d}{dt}\Psi_\lambda(\eta(t, v)) = -\varphi(\eta(t, v))\frac{\langle \Psi_\lambda'(\eta(t, v)), \mathcal{G}(\eta(t, v))\rangle}{\|\Psi_\lambda'(u)(\eta(t, v))\|_\lambda^*} \le 0, \tag{8-8}$$

$$\left\|\frac{d\eta}{dt}\right\|_\lambda \le 2 \quad \text{for all } t, v, \tag{8-9}$$

$$\eta(t, v) = v \quad \text{for all } t \ge 0 \text{ and } v \in \Psi_\lambda^{c_J} \setminus D_\lambda^{2\mu}. \tag{8-10}$$

Let $\gamma_0(s_1, s_2, \dots, s_l) \in \Gamma_J$ be the path defined in (7-2). We consider

$$\eta(t, \gamma_0(s_1, s_2, \dots, s_l))$$

for large t. Since for all $(s_1, s_2, \dots, s_l) \in \partial([0, 1]^l)$, $\gamma_0(s_1, s_2, \dots, s_l) \notin D_\lambda^{2\mu}$, we have by (8-10) that

$$\eta(t, \gamma_0(s_1, s_2, \dots, s_l)) = \gamma_0(s_1, s_2, \dots, s_l) \quad \text{for all } (s_1, s_2, \dots, s_l) \in \partial([0, 1]^l),$$

and $\eta(t, \gamma_0(s_1, s_2, \dots, s_l)) \in \Gamma_J$ for all $t \ge 0$.

Since $\operatorname{supp}\gamma_0(s_1, s_2, \dots, s_l)(x) \subset \overline{\Omega}_J$ for all $(s_1, s_2, \dots, s_l) \in \partial([0, 1]^l)$, it follows that $\Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)(x))$ and $\|\gamma_0(s_1, s_2, \dots, s_l)(x)\|_{H_G^\lambda(\Omega_j')}$ do not depend on $\lambda \ge 0$. On the other hand,

$$\Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)(x)) \le c_J \quad \text{for all } (s_1, s_2, \dots, s_l) \in [0, 1]^l,$$

and $\Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)(x)) = c_J$ if and only if $s_j = 1/R$; that is,

$$\gamma_0(s_1, s_2, \dots, s_l)(x)|_{\Omega_j} = \omega_j$$

for all $j \in J$. Thus we have that

$$m_0 := \max\{\Psi_\lambda(v) \mid v \in \gamma_0([0, 1]^l) \setminus D_\lambda^\mu\} \tag{8-11}$$

is independent of λ, and $m_0 < c_J$.

By (8-9), one can see that for any $t > 0$,

$$\|\eta(0, \gamma_0(s_1, \dots, s_l)) - \eta(t, \gamma_0(s_1, \dots, s_l))\|_{H_G^\lambda} \le 2t.$$

Since $\Psi_{\lambda,\Omega_j'} \in C^2(H_G^\lambda)$ for all $j = 1, \dots, l$, by the same arguments as in Proposition 4.5 of [Tang 2008], we have that for a large number T, there exists a positive number μ_0, which is independent of λ, such that for all $j = 1, 2, \dots, l$ and $t \in [0, T]$,

$$\|\Psi_{\lambda,\Omega_j'}'(\eta(t, \gamma_0(s_1, \dots, s_l)))\|_{H_G^\lambda}^* \le \mu_0.$$

We claim that for large T,

$$\max_{(s_1,s_2,\ldots,s_l)\in[0,1]^l} \Psi_\lambda(\eta(T,\gamma_0(s_1,s_2,\ldots,s_l)(x))) \le \max\bigl\{m_0, c_J - \tfrac{1}{2}\tau_0\mu\bigr\}, \tag{8-12}$$

where $\tau_0 = \max\{\sigma_0, \sigma_0/\mu_0\}$, and m_0 is given in (8-11) .

In fact, if $\gamma_0(s_1,s_2,\ldots,s_l)(x) \notin D_\lambda^\mu$, then by (8-11) we have

$$\Psi_\lambda(\eta(T,\gamma_0(s_1,s_2,\ldots,s_l)(x))) \le m_0,$$

and thus (8-12) holds.

Now we consider the case when $\gamma_0(s_1,s_2,\ldots,s_l)(x) \in D_\lambda^\mu$. Set

$$\tilde{d}_\lambda := \min\{d_\lambda, \sigma_0\}, \qquad T = \frac{\sigma_0\mu}{4\tilde{d}_\lambda}, \qquad \text{and} \qquad \tilde{\eta}(t) := \eta(t,\gamma_0(s_1,s_2,\ldots,s_l)).$$

We have two cases:

(1) $\tilde{\eta}(t) \in D_\lambda^{3\mu/2}$ for all $t \in [0,T]$.

(2) $\tilde{\eta}(t_0) \in \partial D_\lambda^{3\mu/2}$ for some $t_0 \in [0,T]$.

If (1) holds, then $\varphi(\tilde{\eta}(t)) = 1$ and $\|\Psi_\lambda'(\tilde{\eta}(t))\|_\lambda^* \ge \tilde{d}_\lambda$ for all $t \in [0,T]$. It follows from (8-8) that

$$\begin{aligned}\Psi_\lambda(\tilde{\eta}(T)) &= \Psi_\lambda(\gamma_0(s_1,s_2,\ldots,s_l)) + \int_0^T \frac{d}{ds}\Psi_\lambda(\tilde{\eta}(t)) \\ &\le c_J - 2\int_0^T \tilde{d}_\lambda\,ds = c_J - 2\tilde{d}_\lambda T \le c_J - \tfrac{1}{2}\tau_0\mu.\end{aligned}$$

If (2) holds, there exists $0 \le t_1 < t_2 \le T$ such that

$$\tilde{\eta}(t_1) \in \partial D_\lambda^\mu, \tag{8-13}$$

$$\tilde{\eta}(t_2) \in \partial D_\lambda^{3\mu/2}, \tag{8-14}$$

$$\tilde{\eta}(t) \in D_\lambda^{3\mu/2} \setminus D_\lambda^\mu \quad \text{for all } t \in [t_1,t_2]. \tag{8-15}$$

By (8-14), either

$$\|\tilde{\eta}(t_2)\|_{H_G^\lambda(\mathbb{R}^N\setminus\Omega_j')} = \frac{3\mu}{2}$$

or

$$|\Psi_{\lambda,\Omega_{j_0}'}(\tilde{\eta}(t_2)) - c_{j_0}| = \frac{3\mu}{2} \quad \text{for some } j_0 \in J.$$

We only address the latter case; the former can be proved in a similar way. By (8-14), we have

$$|\Psi_{\lambda,\Omega_{j_0}'}(\tilde{\eta}(t_1)) - c_{j_0}| \le \mu,$$

and hence

$$|\Psi_{\lambda,\Omega_{j_0}'}(\tilde{\eta}(t_2)) - \Psi_{\lambda,\Omega_{j_0}'}(\tilde{\eta}(t_1))| \ge |\Psi_{\lambda,\Omega_{j_0}'}(\tilde{\eta}(t_2)) - c_{j_0}| - |\Psi_{\lambda,\Omega_{j_0}'}(\tilde{\eta}(t_1)) - c_{j_0}| \ge \tfrac{1}{2}\mu.$$

On the other hand, by the mean value theorem, there exists $t' \in (t_1, t_2)$ such that

$$|\Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - \Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1))| = \left|\Psi'_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t')) \cdot \frac{d\tilde{\eta}}{dt}\right|(t_2 - t_1).$$

Thus we have

$$\begin{aligned}\Psi_\lambda(\tilde{\eta}(T)) &= \Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)(x)) - \int_0^T \varphi(\tilde{\eta}(s)))\frac{\langle \Psi'(\tilde{\eta}(s))), \mathcal{G}(\tilde{\eta}(s)))\rangle}{\|\Psi'_\lambda(\tilde{\eta}(s)))\|^*_\lambda} v \\ &\le c_J - \int_{t_1}^{t_2} \varphi(\tilde{\eta}(s)))\|\Psi'_\lambda(\tilde{\eta}(s))\|^*_\lambda \, ds \\ &= c_J - \sigma_0(t_2 - t_1) \le c_J - \tfrac{1}{2}\tau_0\mu.\end{aligned}$$

Thus (8-12) is proved. Recall that $\tilde{\eta}(T) = \eta(T, \gamma_0(s_1, s_2, \dots, s_l)) \in \Gamma_J$. Hence

$$b_{\lambda,J} \le \Psi_\lambda(\tilde{\eta}(T)) \le \max\{m_0, c_J - \tfrac{1}{2}\tau_0\mu\}. \tag{8-16}$$

However, by Corollary 7.2, we have $b_{\lambda,J} \to c_J$ as $\lambda \to \infty$. This contradicts (8-16), and hence Ψ_λ has a critical point $v_\lambda \in D^\mu_\lambda$ for large λ, so by Proposition 5.2, v_λ is a solution of the problem (1-10). □

Proof of Theorem 1.1. Let v_λ be a solution to the problem (1-1) obtained in Proposition 8.2. For any given sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$, we can extract a subsequence (still denoted by $\{\lambda_n\}$). Arguing as in the proof of Proposition 5.1, we can extract a subsequence of $\{v_{\lambda_n}\}$ (still denoted $\{v_{\lambda_n}\}$) such that $v_{\lambda_n} \to v$ in H^1_G and

$$\lim_{n\to\infty} \Psi_{\lambda_n}(v_n) = c_j \quad \text{for all } j \in J, \tag{8-17}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega'_J} (|\nabla v_{\lambda_n}|^2 + V_{\lambda_n}|f(v_{\lambda_n})|^2) = 0. \tag{8-18}$$

Since the limits in (8-17) and (8-18) do not depend on the choice of sequence $\{\lambda_n\}$ ($\lambda_n \to \infty$), then both (1-12) and (1-13) hold, and the limit function $v(x)$ satisfies:

(1) $v(x) = 0$ for $x \in \mathbb{R}^N \setminus \Omega_J$.

(2) $v|_{\Omega_j}$ is a least energy solution of

$$\begin{cases} -\Delta v(x) = |f(v)|^{p-1} f(v), & x \in \Omega_j, \\ v(x) \in H^{0,1}_G(\Omega_j) \end{cases}$$

for $j \in J$.

This completes the proof of Theorem 1.1. □

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ON STABLE SOLUTIONS OF THE BIHARMONIC PROBLEM WITH POLYNOMIAL GROWTH

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We prove the nonexistence of smooth stable solutions to the biharmonic problem $\Delta^2 u = u^p$, $u > 0$ in $\mathbb{R}^N$ for $1 < p < \infty$ and $N < 2(1 + x_0)$, where x_0 is the largest root of the equation

$$x^4 - \frac{32p(p+1)}{(p-1)^2}x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3}x - \frac{64p(p+1)^2}{(p-1)^4} = 0.$$

In particular, as $x_0 > 5$ when $p > 1$, we obtain the nonexistence of smooth stable solutions for any $N \le 12$ and $p > 1$. Moreover, we consider also the corresponding problem in the half-space $\mathbb{R}^N_+$, and the elliptic problem $\Delta^2 u = \lambda(u+1)^p$ on a bounded smooth domain Ω with the Navier boundary conditions. We prove the regularity of the extremal solution in lower dimensions.

1. Introduction

Consider the biharmonic equation

$$\Delta^2 u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N \tag{1-1}$$

where $N \ge 5$ and $p > 1$. Let

$$\Lambda(\phi) := \int_{\mathbb{R}^N} |\Delta\phi|^2\,dx - p\int_{\mathbb{R}^N} u^{p-1}\phi^2\,dx \quad \text{for all } \phi \in H^2(\mathbb{R}^N). \tag{1-2}$$

A solution u is said to be stable if $\Lambda(\phi) \ge 0$ for any test function $\phi \in H^2(\mathbb{R}^N)$.

In this note, we prove the following classification result.

Theorem 1.1. *Let $N \ge 5$ and $p > 1$. Equation* (1-1) *has no classical stable solution if $N < 2 + 2x_0$, where x_0 is the largest root of the polynomial*

$$H(x) = x^4 - \frac{32p(p+1)}{(p-1)^2}x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3}x - \frac{64p(p+1)^2}{(p-1)^4}. \tag{1-3}$$

Moreover, we have $x_0 > 5$ for any $p > 1$. Consequently, if $N \le 12$, (1-1) *has no classical stable solution for all $p > 1$.*

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For the corresponding second-order problem,

$$\Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad p > 1, \tag{1-4}$$

Farina has obtained the optimal Liouville type result for all finite Morse index solutions. He proved in [Farina 2007] that a smooth finite Morse index solution to (1-4) exists if and only if $p \geq p_{JL}$ and $N \geq 11$, or $p = \frac{N+2}{N-2}$ and $N \geq 3$. Here p_{JL} is the so-called Joseph–Lundgren exponent; see (1.11) in [Gui et al. 1992].

The nonexistence of positive solutions to (1-1) is shown if $p < \frac{N+4}{N-4}$, and all entire solutions are classified if $p = \frac{N+4}{N-4}$; see [Lin 1998; Wei and Xu 1999]. On the other hand, the radially symmetric solutions to (1-1) are studied in [Ferrero et al. 2009; Gazzola and Grunau 2006; Guo and Wei 2010; Karageorgis 2009]. In particular, Karageorgis [2009] proved that the radial entire solution to (1-1) is stable if and only if $p \geq p_{JL_4}$ and $N \geq 13$. Here p_{JL_4} stands for the corresponding Joseph–Lundgren exponent to Δ^2.

The general fourth-order case (1-1) is more delicate, since the integration by parts argument used by Farina cannot be adapted easily. The first nonexistence result for general stable solutions was proved by Wei and Ye [2013], who proposed we consider (1-1) as a system

$$-\Delta u = v, \quad -\Delta v = u^p \quad \text{in } \mathbb{R}^N, \tag{1-5}$$

and introduced the idea to use different test functions with u but also v. Using estimates in [Souplet 2009] they showed that for $N \leq 8$, (1-1) has no smooth stable solutions. For $N \geq 9$, using a blow-up argument, they proved that the classification holds still for $p < N/(N-8) + \epsilon_N$ with $\epsilon_N > 0$, but without any explicit value of ϵ_N. This result was improved by Wei, Xu and Yang in [Wei et al. 2013] for $N \geq 20$ with a more explicit bound.

Using the stability of system (1-5) and an interesting iteration argument, Cowan [2013, Theorem 2] proved that there is no smooth stable solution to (1-1) if $N < 2 + \frac{4(p+1)}{p-1} t_0$, where

$$t_0 = \sqrt{\frac{2p}{p+1}} + \sqrt{\frac{2p}{p+1} - \sqrt{\frac{2p}{p+1}}} \quad \text{for all } p > 1. \tag{1-6}$$

In particular, if $N \leq 10$, (1-1) has no stable solution for any $p > 1$.

However, the study for radial solutions in [Karageorgis 2009] suggests the following conjecture.

Conjecture. A smooth stable solution to (1-1) exists if and only if $p \geq p_{JL_4}$ and $N \geq 13$.

Consequently, the Liouville type result for stable solutions of (1-1) should hold true for $N \leq 12$ with any $p > 1$; that's what we prove here. More precisely, by

[Karageorgis 2009, Theorem 1], the radial entire solutions to (1-1) are unstable if and only if

$$\frac{N^2(N-4)^2}{16} < pQ_4\Big(-\frac{4}{p-1}\Big), \tag{1-7}$$

where $Q_4(m) = m(m-2)(m+N-2)(m+N-4)$. The left-hand side comes from the best constant of the Hardy–Rellich inequality (see [Rellich 1969]): Let $N \geq 5$,

$$\int_{\mathbb{R}^N} |\Delta\varphi|^2\, dx \geq \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^4}\, dx \quad \text{for all } \varphi \in H^2(\mathbb{R}^N).$$

The right-hand side of (1-7) comes from the weak radial solution $w(x) = |x|^{-4/(p-1)}$. When $p > \frac{N+4}{N-4}$, we can check that $w \in H^2_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\Delta^2 w = Q_4\Big(-\frac{4}{p-1}\Big)w^p \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

Since $w^{p-1}(x) = |x|^{-4}$, and in view of the Hardy–Rellich inequality, the condition (1-7) means just that w is not a stable solution in $\mathbb{R}^N$, that is, there exists $\varphi \in H^2(\mathbb{R}^N)$ such that

$$\Lambda_w(\varphi) := \int_{\mathbb{R}^N} |\Delta\varphi|^2\, dx - p\int_{\mathbb{R}^N} Q_4\Big(-\frac{4}{p-1}\Big)w^{p-1}\varphi^2\, dx < 0.$$

If we set $N = 2+2x$, a direct calculation shows that (1-7) is equivalent to $H_{JL_4}(x) < 0$, where

$$H_{JL_4}(x) := (x^2-1)^2 - \frac{32p(p+1)}{(p-1)^2}x^2 + \frac{32p(p+1)(p+3)}{(p-1)^3}x - \frac{64p(p+1)^2}{(p-1)^4}.$$

By [Gazzola and Grunau 2006], (1-7) is equivalent to $N < 2+2x_1$ if x_1 denotes the largest root of H_{JL_4}. Note that closeness between the fourth-order polynomials H_{JL_4} and H (in Theorem 1.1); they differ only by $H(x) - H_{JL_4}(x) = 2x^2 - 1$.

Furthermore, Theorem 1.1 improves the bound given in [Cowan 2013] for all $p > 1$. Indeed, Lemmas 2.2 and 2.4 below imply that $x_0 > \frac{2(p+1)}{p-1}t_0$.

Recall that to handle the equation (1-1), we prove in general that $v = -\Delta u > 0$ in $\mathbb{R}^N$ by studying function averages on the sphere; see [Wei and Xu 1999]. Applying the blow-up argument as in [Souplet 2009; Wei and Ye 2013], we can assume that u and v are uniformly bounded in $\mathbb{R}^N$. Therefore the following Souplet's estimate [2009] holds true in $\mathbb{R}^N$, which was established for any *bounded* solution u of (1-1):

$$v \geq \sqrt{\frac{2}{p+1}}\,u^{(p+1)/2}. \tag{1-8}$$

Here we propose a new approach. Without assuming the boundedness of u or showing immediately the positivity of v, we prove first some integral estimates for stable solutions of (1-1), which will enable us the estimate (1-8). This idea permits us to handle more general biharmonic equations: let $N \geq 5$ and $p > 1$, and consider

$$\Delta^2 u = u^p, \quad u > 0 \text{ in } \Sigma \subset \mathbb{R}^N, \quad u = \Delta u = 0 \text{ on } \partial\Sigma. \tag{1-9}$$

Let $E = H^2(\Sigma) \cap H_0^1(\Sigma)$ and

$$\Lambda_0(\phi) := \int_\Sigma |\Delta\phi|^2\,dx - p\int_\Sigma u^{p-1}\phi^2\,dx \quad \text{for all } \phi \in E. \tag{1-10}$$

A solution u of (1-9) is said to be *stable* if $\Lambda_0(\phi) \geq 0$ for any $\phi \in E$.

Proposition 1.2. *Let u be a classical stable solution of* (1-9) *where Σ is one of $\mathbb{R}^N$, the half-space $\Sigma = \mathbb{R}^N_+$, or the exterior domain $\Sigma = \mathbb{R}^N \setminus \overline{\Omega}$ or $\mathbb{R}^N_+ \setminus \overline{\Omega}$, where Ω is a bounded smooth domain of $\mathbb{R}^N$. Then the inequality* (1-8) *holds in Σ, and consequently $v > 0$ in Σ.*

Using this, we obtain a Liouville type result for (1-9) in the half-space situation, which improves the result in [Wei and Ye 2013] for a wider range of N, and without assuming the boundedness of u or $v = -\Delta u$.

Theorem 1.3. *Let x_0 be defined as in Theorem 1.1. If $N < 2 + 2x_0$, there exists no classical stable solution of* (1-9) *if $\Sigma = \mathbb{R}^N_+$.*

Our proof combines also many ideas from [Wei and Ye 2013; Cowan and Ghoussoub 2014; Cowan 2013]. Briefly, for (1-1), we apply different test functions to both equations of the system (1-5) and make use of the following inequality in [Cowan and Ghoussoub 2014] (see also [Cowan 2013; Dupaigne et al. 2013a]): if u is a stable solution of (1-1), then

$$\int_{\mathbb{R}^N} \sqrt{p}\,u^{(p-1)/2}\varphi^2\,dx \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx \quad \text{for all } \varphi \in C_0^1(\mathbb{R}^N). \tag{1-11}$$

This will enable us to make two estimates. From these estimates, we prove that for any stable solution u of (1-1), $\phi \in C_0^2(\mathbb{R}^N)$ and $s \geq 1$,

$$L(s) < 0 \Rightarrow \int_{\mathbb{R}^N} u^p v^{s-1}\phi^2\,dx \leq C\int_{\mathbb{R}^N} v^s\big(|\Delta(\phi^2)| + |\nabla\phi|^2\big)\,dx. \tag{1-12}$$

Here L is a polynomial of degree 4, see (2-9) below, and the constant C depends only on p and s. Applying then the iteration argument of Cowan [2013], we show that $u \equiv 0$ if $N < 2 + 2x_0$, which is a contradiction, since u is positive.

Using similar ideas, we consider the elliptic equation on bounded domains:

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u = \lambda(u+1)^p & \text{in a bounded smooth domain } \Omega \subset \mathbb{R}^N,\ N \geq 1 \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known (see [Berchio and Gazzola 2005; Gazzola et al. 2010]) that there exists a critical value $\lambda^* > 0$ depending on $p > 1$ and Ω such that:

- If $\lambda \in (0, \lambda^*)$, (P_λ) has a minimal and classical solution u_λ which is stable.
- If $\lambda = \lambda^*$, then $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ is a weak solution to (P_{λ^*}); u^* is called the *extremal solution*.
- No solution of (P_λ) exists whenever $\lambda > \lambda^*$.

In [Cowan et al. 2010; Wei and Ye 2013], it was proved that if $1 < p < ((N-8)/N)_+^{-1}$, or equivalently $N < 8p/(p-1)$, the extremal solution u^* is smooth. Recently, Cowan and Ghoussoub improved the above result by showing that u^* is smooth if $N < 2 + 4(p+1)/(p-1)t_0$ with t_0 in (1-6), so u^* is smooth for any $p > 1$ when $N \le 10$. Our result is this:

Theorem 1.4. *The extremal solution u^* is smooth if $N < 2 + 2x_0$ with x_0 given by Theorem 1.1. In particular, u^* is smooth for any $p > 1$ if $N \le 12$.*

We remark that our proof does not use the *a priori* estimate of $v = -\Delta u$ as in [Cowan et al. 2010; Cowan and Ghoussoub 2014].

The paper is organized as follows. We prove some preliminary results and Proposition 1.2 in Section 2. The proofs of Theorems 1.1, 1.3 and 1.4 are given in Sections 3 and 4.

2. Preliminaries

We show first how to obtain the estimate (1-8) for stable solutions of (1-9). Our idea is to use the stability condition (1-10) to get some decay estimates for stable solutions of (1-9). In the following, we denote by B_r the ball of center 0 and radius $r > 0$.

Lemma 2.1. *Let u be a stable solution to* (1-9) *and set $v = -\Delta u$. Then*

$$\int_{\Sigma \cap B_R} (v^2 + u^{p+1})\, dx \le C R^{N-4-8/(p-1)} \quad \textit{for all } R > 0. \tag{2-1}$$

Proof. We proceed similarly as in Step 1 of the proof for [Wei and Ye 2013, Theorem 1.1], but we do not assume here that $v > 0$ or u is bounded in Σ. For any $\xi \in C^4(\Sigma)$ satisfying $\xi = \Delta\xi = 0$ on $\partial\Sigma$ and $\eta \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_\Sigma (\Delta^2 \xi)\xi\eta^2\, dx = \int_\Sigma [\Delta(\xi\eta)]^2\, dx + \int_\Sigma \bigl[-4(\nabla\xi \cdot \nabla\eta)^2 + 2\xi\Delta\xi|\nabla\eta|^2\bigr]\, dx \\ + \int_\Sigma \xi^2\bigl[2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\bigr]\, dx.\end{aligned} \tag{2-2}$$

The proof is direct as in [Wei and Ye 2013, Lemma 2.3], noticing just that in the integrations by parts, all boundary integration terms on $\partial\Sigma$ vanish under the Navier conditions for ξ.

Let u be a solution of (1-9). Taking $\xi = u$ in (2-2), we have

$$\int_\Sigma [\Delta(u\eta)]^2\,dx - \int_\Sigma u^{p+1}\eta^2\,dx$$
$$= 4\int_\Sigma (\nabla u\nabla\eta)^2\,dx + 2\int_\Sigma uv|\nabla\eta|^2\,dx - \int_\Sigma u^2\big[2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\big]\,dx,$$

where $v = -\Delta u$. Using $\phi = u\eta$ in (1-10), we obtain easily

$$\text{(2-3)}\quad \int_\Sigma \big[(\Delta(u\eta))^2 + u^{p+1}\eta^2\big]\,dx$$
$$\le C_1\int_\Sigma \big[|\nabla u|^2|\nabla\eta|^2 + u^2|\nabla(\Delta\eta)\cdot\nabla\eta| + u^2(\Delta\eta)^2\big]\,dx + C_2\int_\Sigma uv|\nabla\eta|^2\,dx.$$

Here and below, C and C_i denote generic positive constants independent of u, which can change from one line to another. Since $\Delta(u\eta) = 2\nabla u\cdot\nabla\eta + u\Delta\eta - v\eta$ we get from (2-3)

$$\text{(2-4)}\quad \int_\Sigma \big[v^2\eta^2 + u^{p+1}\eta^2\big]\,dx$$
$$\le C_1\int_\Sigma \big[|\nabla u|^2|\nabla\eta|^2 + u^2|\nabla(\Delta\eta)\cdot\nabla\eta| + u^2(\Delta\eta)^2\big]\,dx + C_2\int_\Sigma uv|\nabla\eta|^2\,dx.$$

On the other hand, since $u = 0$ on $\partial\Sigma$,

$$2\int_\Sigma |\nabla u|^2|\nabla\eta|^2\,dx = \int_\Sigma \Delta(u^2)|\nabla\eta|^2\,dx + 2\int_\Sigma uv|\nabla\eta|^2\,dx$$
$$= \int_\Sigma u^2\Delta(|\nabla\eta|^2)\,dx + 2\int_\Sigma uv|\nabla\eta|^2\,dx.$$

By inputting this into (2-4), we arrive at

$$\text{(2-5)}\quad \int_\Sigma \big[v^2\eta^2 + u^{p+1}\eta^2\big]\,dx$$
$$\le C_1\int_\Sigma u^2\big[|\nabla(\Delta\eta)\cdot\nabla\eta| + (\Delta\eta)^2 + |\Delta(|\nabla\eta|^2)|\big]\,dx + C_2\int_\Sigma uv|\nabla\eta|^2\,dx.$$

If we let $\eta = \varphi^m$ with $m > 2$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \ge 0$, it follows that

$$\int_\Sigma uv|\nabla\eta|^2\,dx = m^2\int_\Sigma uv\varphi^{2(m-1)}|\nabla\varphi|^2\,dx$$
$$\le \frac{1}{2C}\int_\Sigma (v\varphi^m)^2\,dx + C\int_\Sigma u^2\varphi^{2(m-2)}|\nabla\varphi|^4\,dx.$$

Now choose a cutoff function φ_0 in $C_0^\infty(B_2)$ satisfying $0 \le \varphi_0 \le 1$ and $\varphi_0 = 1$ for $|x| < 1$. Inputting the above inequality into (2-5) with $\varphi = \varphi_0(R^{-1}x)$ for $R > 0$ and

$\eta = \varphi^m$ with $m = (2p+2)/(p-1) > 2$, we arrive at

$$\begin{aligned}(2\text{-}6)\qquad \int_\Sigma (v^2+u^{p+1})\varphi^{2m}\,dx &\le \frac{C}{R^4}\int_\Sigma u^2\varphi^{2m-4}\,dx\\ &\le \frac{C}{R^4}\left(\int_\Sigma u^{p+1}\varphi^{(p+1)(m-2)}\,dx\right)^{2/(p+1)} R^{N(p-1)/(p+1)}\\ &= \frac{C}{R^4}\left(\int_\Sigma u^{p+1}\varphi^{2m}\,dx\right)^{2/(p+1)} R^{N(p-1)/(p+1)}.\end{aligned}$$

Hence

$$\int_\Sigma u^{p+1}\varphi^{2m}\,dx \le C R^{N-4(p+1)/(p-1)}.$$

Combining with (2-6) we get (2-1), since $\varphi^{2m} = 1$ for $x \in B_R := \{x \in \mathbb{R}^N : |x| \le R\}$. □

Proof of Proposition 1.2. Let

$$\zeta = \beta u^{(p+1)/2} - v, \quad \text{where } \beta = \sqrt{\frac{2}{p+1}}.$$

Then a direct computation shows that $\Delta\zeta \ge \beta^{-1}u^{(p-1)/2}\zeta$ in Σ. Consider $\zeta_+ := \max(\zeta, 0)$. For any $R > 0$, we have

$$\begin{aligned}(2\text{-}7)\qquad \int_{\Sigma\cap B_R} |\nabla\zeta_+|^2\,dx &= -\int_{\Sigma\cap B_R} \zeta_+\Delta\zeta\,dx + \int_{\partial(\Sigma\cap B_R)} \zeta_+\frac{\partial\zeta}{\partial\nu}\,d\sigma\\ &\le \int_{\Sigma\cap\partial B_R} \zeta_+\frac{\partial\zeta}{\partial\nu}\,d\sigma.\end{aligned}$$

Here we used $\zeta_+\Delta\zeta \ge 0$ in Σ and $\zeta = 0$ on $\partial\Sigma$. Now let S^{N-1} denote the unit sphere in $\mathbb{R}^N$ and

$$e(r) = \int_{S^{N-1}\cap(r^{-1}\Sigma)} \zeta_+^2(r\sigma)\,d\sigma \quad \text{for } r > 0.$$

We remark that there exists an $R_0 > 0$ satisfying

$$(2\text{-}8)\qquad \int_{\Sigma\cap\partial B_r} \zeta_+\frac{\partial\zeta}{\partial\nu}\,d\sigma = \frac{r^{N-1}}{2}e'(r) \quad \text{for all } r \ge R_0.$$

Moreover, for $R \ge R_0$, we deduce from (2-1) that

$$\begin{aligned}\int_{R_0}^R r^{N-1}e(r)\,dr \le \int_{B_R\cap\Sigma} \zeta_+^2\,dx &\le C\int_{B_R\cap\Sigma}(v^2+u^{p+1})\,dx\\ &\le CR^{N-4-8/(p-1)} = o(R^N).\end{aligned}$$

This means that the function e cannot be nondecreasing at infinity, so there exists a sequence $R_j \to \infty$ satisfying $e'(R_j) \le 0$. Combining (2-7) and (2-8) with

$R = R_j \to \infty$, we obtain

$$\int_\Sigma |\nabla \zeta_+|^2 \, dx = 0.$$

Using $\zeta = 0$ on $\partial\Sigma$, we have $\zeta_+ \equiv 0$ in Σ, or equivalently (1-8) holds true in Σ. Clearly $v > 0$ in Σ by (1-8). □

In the following, we show some properties of the polynomials L and H, useful for our proofs. Let

$$L(s) = s^4 - 32\frac{p}{p+1}s^2 + 32\frac{p(p+3)}{(p+1)^2}s - 64\frac{p}{(p+1)^2}, \quad s \in \mathbb{R}. \tag{2-9}$$

Lemma 2.2. *$L(2t_0) < 0$ and L has a unique root s_0 in the interval $(2t_0, \infty)$.*

Proof. Obviously

$$L(2t_0) = 16t_0^4 - 128\frac{p}{p+1}t_0^2 + 64\frac{p(p+3)}{(p+1)^2}t_0 - 64\frac{p}{(p+1)^2}.$$

Since $t_0^2/(2t_0-1) = \sqrt{2p/(p+1)}$ (see [Cowan 2013]), we have $t_0^4 = \dfrac{2p}{p+1}(2t_0-1)^2$. A direct computation yields

$$\begin{aligned}\frac{(p+1)^2 L(2t_0)}{32p} &= (p+1)(2t_0-1)^2 - 4(p+1)t_0^2 + 2(p+3)t_0 - 2\\ &= (p-1)(1-2t_0).\end{aligned}$$

Since $t_0 > 1$ for any $p > 1$, we have $L(2t_0) < 0$. Furthermore, for all $p > 1$, $s \ge 2t_0$, we have

$$\begin{aligned}(p+1)L''(s) = 12(p+1)s^2 - 64p &\ge 48(p+1)t_0^2 - 64p\\ &\ge 48(p+1)\frac{2p}{p+1} - 64p = 32p > 0\end{aligned}$$

in $[2t_0, \infty)$, where we used $t_0^2 \ge 2p/(p+1)$, which holds by (1-6). Therefore L is convex in $[2t_0, \infty)$. Since $\lim_{s\to\infty} L(s) = \infty$ and $L(2t_0) < 0$, it's clear that L admits a unique root in $(2t_0, \infty)$. □

Remark 2.3. After the change of variable $x = \dfrac{p+1}{p-1}s$, a direct calculation gives

$$H(x) = \left(\frac{p+1}{p-1}\right)^4 L(s),$$

hence $H(x) < 0$ if and only if $L(s) < 0$. Using the lemma above, we see that $x_0 = \frac{p+1}{p-1}s_0$ is the largest root of H, and x_0 is the only root of H for $x \ge \frac{2(p+1)}{p-1}t_0$.

Lemma 2.4. *If $x_0 = \dfrac{p+1}{p-1}s_0$ is the largest root of H, then $x_0 > 5$ for any $p > 1$.*

Proof. Since x_0 is the largest root of H, to have $x_0 > 5$ it suffices to show $H(5) < 0$. Let $J(p) = (p-1)^4 H(5)$; then $J(p) = -15p^4 - 1284p^3 + 4262p^2 - 3844p + 625$. Therefore,

$$J'(p) = -60p^3 - 3852p^2 + 8524p - 3844, \quad J''(p) = -180p^2 - 7704p + 8524.$$

We see that $J'' < 0$ in $[2, \infty)$. Consequently $J'(p) < 0$ and $J(p) < 0$ for $p \ge 2$. Hence $x_0 > 5$ if $p \ge 2$. For $p \in (1, 2)$, we have $x_0 > \frac{2(p+1)}{p-1} t_0 \ge 6t_0$, which exceeds 5 since $t_0 > 1$. □

3. Proof of Theorems 1.1 and 1.3

We will prove only Theorem 1.1, since the proof of Theorem 1.3 is completely similar, just changing B_r to $B_r \cap \mathbb{R}^N_+$.

The following result generalizes [Cowan 2013, Lemma 4], which is a crucial argument for our proof. As above, the constant C always denotes a positive number which may change term by term, but does not depend on the solution u. For $k \in \mathbb{N}$, let $R_k := 2^k R$ with $R > 0$.

Lemma 3.1. *Assume that u is a classical stable solution of* (1-1). *Then for all $2 \le s < s_0$, there is $C < \infty$ such that*

$$\int_{B_{R_k}} u^p v^{s-1}\, dx \le \frac{C}{R^2} \int_{B_{R_{k+1}}} v^s\, dx \quad \text{for all } R > 0. \tag{3-1}$$

Proof. Let u be a classical stable solution of (1-1). Let $\phi \in C_0^2(\mathbb{R}^N)$ and $\varphi = u^{(q+1)/2}\phi$ with $q \ge 1$. With this φ, the stability inequality (1-11) gives

$$\begin{aligned}\sqrt{p}\int_{\mathbb{R}^N} & u^{(p-1)/2}u^{q+1}\phi^2 \\ &\le \int_{\mathbb{R}^N} u^{q+1}|\nabla\phi|^2 + \int_{\mathbb{R}^N} \left|\nabla u^{(q+1)/2}\right|^2\phi^2 + (q+1)\int_{\mathbb{R}^N} u^q\phi\nabla u\nabla\phi.\end{aligned} \tag{3-2}$$

Integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \left|\nabla u^{\frac{q+1}{2}}\right|^2\phi^2\, dx &= \frac{(q+1)^2}{4}\int_{\mathbb{R}^N} u^{q-1}|\nabla u|^2\phi^2\, dx \\ &= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} \phi^2\nabla(u^q)\nabla u\, dx \\ &= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} u^q v\phi^2\, dx - \frac{q+1}{4q}\int_{\mathbb{R}^N} \nabla(u^{q+1})\nabla(\phi^2)\, dx \\ &= \frac{(q+1)^2}{4q}\int_{\mathbb{R}^N} u^q v\phi^2\, dx + \frac{q+1}{4q}\int_{\mathbb{R}^N} u^{q+1}\Delta(\phi^2)\, dx\end{aligned} \tag{3-3}$$

and

$$(q+1)\int_{\mathbb{R}^N} u^q\phi\nabla u\nabla\phi\,dx = \frac{1}{2}\int_{\mathbb{R}^N}\nabla(u^{q+1})\nabla(\phi^2)\,dx = -\frac{1}{2}\int_{\mathbb{R}^N} u^{q+1}\Delta(\phi^2)\,dx. \tag{3-4}$$

Combining (3-2)–(3-4), we conclude that

$$a_1\int_{\mathbb{R}^N} u^{(p-1)/2}u^{q+1}\phi^2\,dx \le \int_{\mathbb{R}^N} u^q v\phi^2\,dx + C\int_{\mathbb{R}^N} u^{q+1}\big(|\Delta(\phi^2)|+|\nabla\phi|^2\big)\,dx \tag{3-5}$$

where $a_1 = (4q\sqrt{p}\,)/(q+1)^2$. Now choose $\phi(x) = h(R_k^{-1}x)$, where $h \in C_0^\infty(B_2)$ is such that $h \equiv 1$ in B_1. Then

$$\int_{\mathbb{R}^N} u^{(p-1)/2}u^{q+1}\phi^2\,dx \le \frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2\,dx + \frac{C}{R^2}\int_{B_{R_{k+1}}} u^{q+1}\,dx. \tag{3-6}$$

Now, apply the stability inequality (1-11) with $\varphi = v^{(r+1)/2}\phi$, $r \ge 1$, to obtain

$$\sqrt{p}\int_{\mathbb{R}^N} u^{(p-1)/2}v^{r+1}\phi^2 \le \int_{\mathbb{R}^N} v^{r+1}|\nabla\phi|^2 + \int_{\mathbb{R}^N}\big|\nabla v^{(r+1)/2}\big|^2\phi^2 + (r+1)\int_{\mathbb{R}^N} v^r\phi\nabla v\nabla\phi.$$

By a very similar computation (recalling that $-\Delta v = u^p$), we have

$$\int_{\mathbb{R}^N} u^{(p-1)/2}v^{r+1}\phi^2\,dx \le \frac{1}{a_2}\int_{\mathbb{R}^N} u^p v^r\phi^2\,dx + \frac{C}{R^2}\int_{B_{R_{k+1}}} v^{r+1}\,dx \tag{3-7}$$

where $a_2 = (4r\sqrt{p}\,)/(r+1)^2$.

Using (3-6) and (3-7), we get

$$\begin{aligned} I_1 + a_2^{r+1}I_2 := \int_{\mathbb{R}^N} u^{(p-1)/2}u^{q+1}\phi^2\,dx + a_2^{r+1}\int_{\mathbb{R}^N} u^{(p-1)/2}v^{r+1}\phi^2\,dx \\ \le \frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2\,dx + a_2^r\int_{\mathbb{R}^N} u^p v^r\phi^2\,dx \\ + \frac{C}{R^2}\int_{B_{R_{k+1}}}(u^{q+1}+v^{r+1})\,dx. \end{aligned} \tag{3-8}$$

Now fix

$$2q = (p+1)r + p - 1, \quad \text{or equivalently} \quad q+1 = \tfrac{1}{2}(p+1)(r+1). \tag{3-9}$$

By Young's inequality, we get

$$
\begin{aligned}
\frac{1}{a_1}\int_{\mathbb{R}^N} u^q v\phi^2\,dx
&= \frac{1}{a_1}\int_{\mathbb{R}^N} u^{(p-1)/2}u^{(p+1)/2r}v\phi^2\,dx\\
&= \frac{1}{a_1}\int_{\mathbb{R}^N} u^{(p-1)/2}u^{(q+1)r/(r+1)}v\phi^2\,dx\\
&\le \frac{r}{r+1}\int_{\mathbb{R}^N} u^{(p-1)/2}u^{q+1}\phi^2\,dx+\frac{1}{a_1^{r+1}(r+1)}\int_{\mathbb{R}^N} u^{(p-1)/2}v^{r+1}\phi^2\,dx\\
&= \frac{r}{r+1}I_1+\frac{1}{a_1^{r+1}(r+1)}I_2,
\end{aligned}
$$

and similarly

$$a_2^r\int_{\mathbb{R}^N} u^p v^r\phi^2\,dx \le \frac{1}{r+1}I_1+\frac{a_2^{r+1}r}{r+1}I_2.$$

Combining the above two inequalities and (3-8), we deduce that

$$a_2^{r+1}I_2 \le \left(\frac{a_2^{r+1}r}{r+1}+\frac{1}{a_1^{r+1}(r+1)}\right)I_2+\frac{C}{R^2}\int_{B_{R_{k+1}}}(u^{q+1}+v^{r+1})\,dx;$$

hence

$$\frac{(a_1a_2)^{r+1}-1}{r+1}I_2 \le \frac{Ca_1^{r+1}}{R^2}\int_{B_{R_{k+1}}}(u^{q+1}+v^{r+1})\,dx.$$

Thus, if $a_1a_2>1$, by the choice of ϕ,

$$\int_{B_{R_k}} u^{(p-1)/2}v^{r+1}\,dx \le I_2 \le \frac{C}{R^2}\int_{B_{R_{k+1}}}(u^{q+1}+v^{r+1})\,dx.$$

From (1-8) and (3-9), we get $u^{q+1}\le Cv^{r+1}$. Setting $s=r+1$, we can conclude that if $a_1a_2>1$,

$$
\begin{aligned}
\int_{B_{R_k}} u^p v^{s-1}\,dx \le C_1\int_{B_{R_k}} u^{(p-1)/2}v^s\,dx &\le \frac{C_2}{R^2}\int_{B_{R_{k+1}}}(u^{q+1}+v^{r+1})\,dx\\
&\le \frac{C_3}{R^2}\int_{B_{R_{k+1}}} v^s\,dx.
\end{aligned}
\tag{3-10}
$$

On the other hand, a simple verification shows that $a_1a_2>1$ is equivalent to $L(s)<0$. By Lemma 2.2, for $s\in[2t_0,s_0)$, this last inequality holds. So the inequality (3-10), which is (3-1), holds for any $2t_0\le s<s_0$. On the other hand, the estimate (3-1) is valid for $2\le s<2t_0$ [Cowan 2013, Lemma 4], hence for $2\le s<s_0$. □

We can then follow the iteration process in [Cowan 2013] (see Proposition 1 or Corollary 2 there) to obtain this consequence:

Corollary 3.2. *Suppose u is a classical stable solution of* (1-1). *For all* $2 \le \beta < \frac{N}{N-2} s_0$, *there are* $\ell \in \mathbb{N}$ *and* $C < \infty$ *such that*

$$\left(\int_{B_R} v^\beta \, dx\right)^{1/\beta} \le C R^{\frac{1}{2} N(2/\beta - 1)} \left(\int_{B_{R_{3\ell}}} v^2 \, dx\right)^{1/2} \quad \textit{for all } R > 0.$$

Now we are in position to complete the proof of Theorem 1.1. Let u be a smooth stable solution to (1-1). Corollary 3.2 and (2-1) imply that for any $2 \le \beta < \frac{N}{N-2} s_0$, there exists $C > 0$ such that

$$\left(\int_{B_R} v^\beta \, dx\right)^{1/\beta} \le C R^{\frac{1}{2} N(2/\beta - 1) + \frac{1}{2} N - 2 - 4/(p-1)} \quad \text{for all } R > 0.$$

Note that

$$\frac{1}{2} N(2/\beta - 1) + \frac{1}{2} N - 2 - \frac{4}{p-1} < 0 \iff N < \frac{2(p+1)}{p-1} \beta.$$

Considering the allowable range of β given in Corollary 3.2, if $N < 2 + \frac{2(p+1)}{p-1} s_0$, after sending $R \to \infty$ we get $\|v\|_{L^\beta(\mathbb{R}^N)} = 0$, which is impossible since v is positive. To conclude, the equation (1-1) has no classical stable solution if $N < 2 + 2x_0$ where $x_0 = \frac{p+1}{p-1} s_0$.

Moreover, by Lemma 2.4, $x_0 > 5$ for any $p > 1$, which means that if $N \le 12$, (1-1) has no classical stable solution for all $p > 1$. $\square$

4. Proof of Theorem 1.4

In this section, we consider the elliptic problem (P_λ). Let u_λ be the minimal solution of (P_λ). It is well known that u_λ is stable. To simplify the presentation, we erase the index λ. By [Cowan and Ghoussoub 2014; Dupaigne et al. 2013a],

$$\sqrt{\lambda p} \int_\Omega (u+1)^{(p-1)/2} \varphi^2 \, dx \le \int_\Omega |\nabla \varphi|^2 \, dx \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{4-1}$$

Using $\varphi = u^{(q+1)/2}$ as a test function in (3-2), by similar computation as for (3-5) in Section 3, we obtain

$$a_1 \sqrt{\lambda} \int_\Omega (u+1)^{(p-1)/2} u^{q+1} \, dx \le \int_\Omega u^q v \, dx, \quad \text{where } a_1 = \frac{4q\sqrt{p}}{(q+1)^2}. \tag{4-2}$$

Here we do not need a cutoff function ϕ, because all boundary terms appearing in the integrations by parts vanish under the Navier boundary conditions, hence the calculations are even easier. We can use Young's inequality as for Theorem 1.1, but we show here a proof inspired by [Dupaigne et al. 2013b].

Similarly as for (3-7), using $\varphi = v^{(r+1)/2}$ in (4-1), we have

$$a_2\sqrt{\lambda}\int_\Omega (u+1)^{(p-1)/2}v^{r+1}\,dx \le \int_\Omega \lambda(u+1)^p v^r\,dx, \quad \text{where } a_2 = \frac{4r\sqrt{p}}{(r+1)^2}. \tag{4-3}$$

Fix $2q = (p+1)r + p - 1$. Applying Hölder's inequality,

$$\begin{aligned}\int_\Omega u^q v\,dx &\le \left(\int_\Omega u^{(p-1)/2}v^{r+1}\,dx\right)^{1/(r+1)}\left(\int_\Omega u^{(p-1)/2+q+1}\,dx\right)^{r/(r+1)}\\ &\le \left(\int_\Omega (u+1)^{(p-1)/2}v^{r+1}\,dx\right)^{1/(r+1)}\left(\int_\Omega u^{(p-1)/2+q+1}\,dx\right)^{r/(r+1)}\end{aligned} \tag{4-4}$$

and

$$\int_\Omega (u+1)^p v^r\,dx \le \left(\int_\Omega (u+1)^{(p-1)/2}v^{r+1}\,dx\right)^{r/(r+1)}\left(\int_\Omega (u+1)^{(p-1)/2+q+1}\,dx\right)^{1/(r+1)}. \tag{4-5}$$

Multiplying (4-2) with (4-3), using (4-4) and (4-5), we get immediately

$$\left(\int_\Omega (u+1)^{(p-1)/2}u^{q+1}\,dx\right)^{1/(r+1)} \le \frac{1}{a_1a_2}\left(\int_\Omega (u+1)^{(p-1)/2+q+1}\,dx\right)^{1/(r+1)}. \tag{4-6}$$

On the other hand, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$(u+1)^{(p-1)/2+q+1} \le (1+\varepsilon)(u+1)^{(p-1)/2}u^{q+1} + C_\varepsilon \quad \text{in } \mathbb{R}_+.$$

If $a_1a_2 > 1$, there exists $\varepsilon_0 > 0$ satisfying $1+\varepsilon_0 < (a_1a_2)^{r+1}$. We deduce from (4-6) that

$$\left(1 - \frac{1+\varepsilon_0}{(a_1a_2)^{r+1}}\right)\int_\Omega (u+1)^{(p-1)/2}u^{q+1}\,dx \le C.$$

Therefore, when $L(s) < 0$, or equivalently when $a_1a_2 > 1$, there is $C > 0$ such that

$$\int_\Omega u^{(p-1)/2+q+1}\,dx \le \int_\Omega (u+1)^{(p-1)/2}u^{q+1}\,dx \le C.$$

Since $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$, we conclude, using Lemma 2.2,

$$u^* \in L^{(p-1)/2+q+1}(\Omega) \quad \text{for all } q \text{ satisfying } \frac{2(q+1)}{p+1} = r+1 = s < s_0. \tag{4-7}$$

Furthermore, by [Gazzola et al. 2010], we know that $u^* \in H^2(\Omega)$. Since $u^* \ge 0$ satisfies $\Delta^2 u^* = \lambda^*(u^*+1)^p \le C(u^*)^{p-1}u^* + C$ with $u^* = \Delta u^* = 0$ on $\partial\Omega$, by standard elliptic estimate, we know that u^* is smooth if

$$\frac{N}{4} < \Big(\frac{p-1}{2} + q + 1\Big)\frac{1}{p-1} = \frac{1}{2}\Big(1 + \frac{p+1}{p-1}s\Big).$$

Therefore, u^* is smooth if $N < 2 + 2x_0$. By Lemma 2.4, u^* is smooth for any $p > 1$ if $N \leq 12$. □

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VALUATIVE MULTIPLIER IDEALS

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The main goal of this paper is to construct an algebraic analogue of quasi-plurisubharmonic function (qpsh for short) from complex analysis and geometry. We define a notion of qpsh function on a valuation space associated to a quite general scheme. We then define the multiplier ideals of these functions and prove some basic results about them, such as subadditivity property, the approximation theorem. We also treat some applications in complex algebraic geometry.

1. Introduction

Given a line bundle L on a smooth projective complex variety, a classical theorem of Kodaira asserts that if L carries on a smooth metric with positive curvature, then L is ample; equivalently, the global sections of a multiple of L give an embedding to a projective space and hence induce such a metric on L. More generally, global sections of a multiple of L induce a semipositive singular metric. Conversely, given a semipositive singular metric h, the local weight function φ, which is plurisubharmonic (psh for short), should be related to sections of multiples of L, or perhaps of a small perturbation of L. See [Lehmann 2011] for more details.

On the other hand, if we work locally near the origin of $\mathbb{C}^n$, then Section 5 of [Boucksom et al. 2008] shows that we can transform a psh germ φ to a formal psh function $\widehat{\varphi}$ on quasimonomial valuations centred at the origin. This valuative transform usually loses much information on the original psh function, however, it preserves the information on the singularity of φ. In particular, they give the same multiplier ideals which essentially means that they characterize the same singularity because of the Demailly's approximation. The idea of studying psh functions using valuations was systematically developed in the work just cited and its predecessors [Favre and Jonsson 2004; 2005a; 2005b]. The main purpose of this paper is to define a similar notion of qpsh functions on a separated, regular, connected and excellent schemes over $\mathbb{Q}$, and we then study these functions.

Although we don't discuss Berkovich spaces in this paper, our work should be related to the qpsh functions (or metrics on line bundles) on the Berkovich

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space associated to a smooth projective variety over a trivially valued field. See [Boucksom et al. 2012b; 2012c].

Let us briefly introduce some terminology. Roughly speaking, we consider a function φ on divisorial valuations on a scheme X such that $\varphi(t\,\mathrm{ord}_E) = t\varphi(\mathrm{ord}_E)$ and $\sup_E |\varphi(\mathrm{ord}_E)|/A(\mathrm{ord}_E) < +\infty$, where E runs over all prime divisors over X. We prove that such functions form a Banach space $\mathrm{BH}(X)$ if we equip it with the norm $\|\varphi\| = \sup_E |\varphi(\mathrm{ord}_E)|/A(\mathrm{ord}_E)$ (see Proposition 3.2). By convention we set $\log|\mathfrak{a}|(\mathrm{ord}_E) = -\mathrm{ord}_E(\mathfrak{a})$ for a nonzero coherent ideal sheaf $\mathfrak{a}$, and one can easily check that $\log|\mathfrak{a}|$ is a valuative function in $\mathrm{BH}(X)$. We define the set of qpsh functions $\mathrm{QPSH}(X)$ to be the closed convex cone generated by functions of the form $\log|\mathfrak{a}|$. We then define the multiplier ideal $\mathcal{J}(\varphi)$ of a qpsh function φ to be the largest ideal $\mathfrak{a}$ such that $\sup_E(-\mathrm{ord}_E(\mathfrak{a}) - \varphi(\mathrm{ord}_E))/A(\mathrm{ord}_E) < 1$. This definition is reasonable because of Proposition 4.3 and Corollary 4.14.

Our first main result is that a qpsh function is a decreasing limit of a sequence of qpsh functions of the form $c_k \log|\mathfrak{b}_k|$. In complex analysis and geometry, such a regularization is crucial. See [Demailly 1992; 1993]. Moreover, we prove that we can actually choose $\mathfrak{b}_k = \mathcal{J}(k\varphi)$ satisfying the subadditivity property. See Proposition 4.22(1). Readers can compare this result with [Demailly et al. 2000].

Theorem 1.1 (cf. Theorem 4.24). *Let φ be a bounded homogeneous function. Then φ is qpsh if and only if φ is the limit function, in norm, of a decreasing sequence of qpsh functions of the form $c_k \log|\mathfrak{b}_k|$. Furthermore, we can choose $c_k = 1/k$ and $\mathfrak{b}_k = \mathcal{J}(k\varphi)$ which form a subadditive sequence of ideals.*

Given an ideal $\mathfrak{a}$ on a scheme X, the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ is a fundamental invariant both in singularity theory and birational geometry (see [Lazarsfeld 2004; Kollár and Mori 1998], etc.). The log canonical threshold admits the following description in terms of valuations:

$$\mathrm{lct}(\mathfrak{a}) = \inf_E \frac{A(\mathrm{ord}_E)}{\mathrm{ord}_E(\mathfrak{a})},$$

where E runs over all prime divisors over X and $A(\mathrm{ord}_E) = \mathrm{ord}_E(K_{Y/X}) + 1$. In fact in the above formulae one can take the infimum over all real valuations centred on X. It is well-known that if Y is a log resolution of $\mathfrak{a}$, then there exists some prime divisor E on Y such that ord_E computes the log canonical threshold, that is, $\mathrm{lct}(\mathfrak{a}) = A(\mathrm{ord}_E)/\mathrm{ord}_E(\mathfrak{a})$. Given a qpsh function φ, we can define the log canonical threshold $\mathrm{lct}(\varphi)$ as the limit of $1/c_k\,\mathrm{lct}(\mathfrak{a}_k)$, where $c_k \log|\mathfrak{a}_k|$ converges to φ strongly in norm. We show that

$$\mathrm{lct}(\varphi) = \inf_E \frac{A(\mathrm{ord}_E)}{-\varphi(\mathrm{ord}_E)}.$$

Unfortunately, there might be no divisorial valuation that computes the log canonical threshold in general. However, we can prove that there exists a real valuation that computes the log canonical threshold. This has been heavily studied in [Jonsson and Mustață 2012; 2014] and other references. Conjecture B of [Jonsson and Mustață 2012] suggests that a valuation that computes the norm is quasimonomial (see Conjecture 5.9). Equivalently we consider the reciprocal of the log canonical threshold, which is exactly the norm of φ by definition. More generally, for a nonzero ideal $\mathfrak{q}$ we consider $\|\varphi\|_{\mathfrak{q}} := \sup_E(-\varphi(\mathrm{ord}_E)/(A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q}))$, and we prove that there exists a real valuation that computes this norm. The proof in this paper mainly follows the strategy of [Jonsson and Mustață 2012]. A similar result appears in [Jonsson and Mustață 2014].

Theorem 1.2 (Theorem 5.2). *Let $\varphi \in \mathrm{QPSH}(X)$ be a qpsh function and let $\mathfrak{q}$ be a nonzero ideal on X. Then there exists a nontrivial tempered valuation v that computes $\|\varphi\|_{\mathfrak{q}}$.*

If X is a complex projective variety, then we can provide $\mathrm{QPSH}(X)$ with more structures. Namely, given a $\mathbb{Q}$-line bundle L on X, we say that the function $\lambda \log|\mathfrak{a}|$ is L-psh if λ is a nonnegative rational number and $L \otimes \mathfrak{a}^\lambda$ is semi-ample. We can then define $\mathrm{PSH}(L) \subseteq \mathrm{QPSH}(X)$ as the closure of the set of such functions. We also define the set of pseudo L-psh functions as $\mathrm{PSH}_\sigma(L) := \bigcap_{\epsilon>0} \mathrm{PSH}(L + \epsilon A)$, where A is an ample line bundle. See the section on D-psh functions (page 118) for more details.

In this setting, we show that there exists the maximal L-psh function φ that can be written explicitly as $\varphi(v) = -v(\|L\|)$, and that there exists the maximal pseudo L-psh function ϕ that can be written explicitly as $\phi(v) = -\sigma_v(\|L\|)$ (see Propositions 6.10 and 6.11). As an immediate corollary we generalize Theorem 6.14 of [Lehmann 2011] as follows (see that paper for the definitions of the perturbed ideal and the diminished ideal).

Theorem 1.3 (Theorem 6.16). *Let D be a pseudo-effective divisor. Assume that $\phi_{\max}$ is the maximal pseudo D-psh function. Then the perturbed ideal and the diminished ideal are $\mathcal{J}_{\sigma,-}(D) = \mathcal{J}_-(\phi_{\max})$ and $\mathcal{J}_\sigma(D) = \mathcal{J}(\phi_{\max})$, respectively. In particular, we can write $\mathcal{J}_\sigma(D)$ explicitly as*

$$\Gamma(U, \mathcal{J}_\sigma(L)) = \{f \in \Gamma(U, \mathcal{O}_X) \mid v(f) + A(v) - \sigma_v(\|L\|) > 0 \text{ for all } v \in \mathrm{V}_U^*\}.$$

Further, a nonzero ideal $\mathfrak{q} \subseteq \mathcal{J}_\sigma(\|L\|)$ if and only if $v(\mathfrak{q}) + A(v) - \sigma_v(\|L\|) > 0$ for all $v \in \mathrm{V}_X^*$.

In the last subsection of this paper, we prove the finite generation of a divisorial module as another application. The proposition below can also be obtained using minimal model theory (see Remark 6.21). Note that our proof here avoids using "the length of extremal rays" (see [Birkar and Hu 2012]).

Proposition 1.4 (Proposition 6.18). *Let (X, B) be a log canonical pair. Assume that $K_X + B$ is $\mathbb{Q}$-Cartier and abundant, and that $R(K_X + B)$ is finitely generated. Then, if $\mathcal{F}$ is any reflexive sheaf, $M^p_{\mathcal{F}}(K_X + B)$ is a finitely generated $R(K_X + B)$-module.*

This proposition can be slightly generalized (see Proposition 6.24).

2. Valuation spaces

Throughout this paper, all schemes are assumed to be separated, regular, connected and excellent schemes over $\mathbb{Q}$. All rings are assumed to be integral, regular and excellent rings containing $\mathbb{Q}$. An ideal on a scheme means a coherent ideal sheaf on a scheme. A birational model of a scheme is a model birational to and proper over this scheme, and a divisor over a scheme is a divisor on a birational model of the scheme. For definitions and properties of valuations, multiplier ideals, singularities in birational geometry, etc., see [Kollár and Mori 1998; Lazarsfeld 2004; Jonsson and Mustaţă 2012]. From now on we abbreviate this last reference as [JM12].

Real valuations. Let X be a scheme, and let $K(X)$ be its function field. A *real valuation* v is a function $v : K(X)^* \to \mathbb{R}$ such that $v(fg) = v(f) + v(g)$ and $v(f + g) \geq \min\{v(f), v(g)\}$. By convention we set $v(0) := +\infty$. Let

$$\mathcal{O}_v := \{f \mid v(f) \geq 0\}$$

be its valuation ring. If there exists a point $\xi \in X$ such that the morphism $\mathcal{O}_{X,\xi} \hookrightarrow \mathcal{O}_v$ is a local homomorphism, then ξ is called the *centre* of v on X and denoted by $c_X(v)$. Note that ξ is unique since X is separated, and also note that the centre always exists provided that X is complete. A real valuation with centred on X is called a real valuation on X or simply a valuation on X, and we denote by Val_X the set of valuations on X. The set of valuations Val_X is independent of the choice of a birational model of X. More precisely, if $Y \to X$ is a proper birational morphism of schemes, then $\mathrm{Val}_X = \mathrm{Val}_Y$. A valuation v on X is said to be the *trivial* valuation if its centre $c_X(v)$ is the generic point of X. We denote by $\mathrm{Val}^*_X \subseteq \mathrm{Val}_X$ the set of nontrivial valuations on X.

The set Val_X can be equipped with an induced topology defined by the maps $v \to v(f)$ for all rational functions $f \in K(X)^*$. For every nonzero ideal $\mathfrak{a}$, we have that $v(\mathfrak{a})$ is well defined and $v(\mathfrak{a}) = v(\bar{\mathfrak{a}})$, where $\bar{\mathfrak{a}}$ denotes the integral closure of $\mathfrak{a}$. Note that the topology on Val_X defined by pointwise convergence on ideals on X is equivalent to that on functions in $K(X)$. Readers can consult [JM12, Section 1] for more details.

In this topology, the map $c_X : \mathrm{Val}_X \to X$ is anti-continuous. That is, the inverse image of an open subset is closed. More precisely, if $U \subseteq X$ is an open subset and $\mathfrak{m}$ is the defining ideal of $X \setminus U$, then $\mathrm{Val}_U = \{v \in \mathrm{Val}_X \mid v(\mathfrak{m}) = 0\}$ and Val_U is closed in Val_X.

For two valuations v, w on X, we say that $v \leq w$ if $v(\mathfrak{a}) \leq w(\mathfrak{a})$ for every nonzero ideal $\mathfrak{a}$. This is equivalent to that the centre $\eta := c_X(w) \in \overline{c_X(v)}$ and that $v(f) \leq w(f)$ for every nonzero local function $f \in \mathcal{O}_{X,\eta}$.

Quasimonomial valuations. Let X be a scheme, let $\xi \in X$ be a point, and let $\underline{x} = (x_1, \dots, x_r)$ be a regular system of parameters at ξ. If $f \in \mathcal{O}_{X,\xi}$ is a local regular function, then f can be expressed as $f = \sum_\beta c_\beta x^\beta$ in $\widehat{\mathcal{O}_{X,\xi}}$ with each coefficient c_β either zero or a unit. For each $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r_{\geq 0}$, we define a real valuation by $\mathrm{val}_{\xi,\alpha}(f) = \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}$, where $\langle \alpha, \beta \rangle := \sum_i \alpha_i \beta^i$, which is called a *monomial* valuation on X.

A pair (Y, D) is called log smooth if Y is a scheme and D is a reduced divisor whose components are regular subschemes intersecting each other transversally. A pair (Y, D) is called a log resolution of X if there is a birational projective morphism $\pi : Y \to X$ and $(Y, D + K_{Y/X})$ is log smooth. Let (Y', D') be another log resolution of X, we say $(Y', D') \succeq (Y, D)$ if Y' is projective over Y and the support of D' contains the support of the pullback of D. Note that log resolutions of X form an inverse system.

Let (Y, D) be a log resolution of X, and let η be the generic point of an irreducible component of the intersection of some prime components of D. We denote by $\mathrm{QM}_\eta(Y, D)$ the set of real valuations which can be defined as a monomial valuation at η. Note that $\eta \in \overline{c_X(v)}$ and $\mathrm{QM}_\eta(Y, D) \cong \mathbb{R}^r_{\geq 0}$ as topological spaces. We also define

$$\mathrm{QM}(Y, D) = \textstyle\bigcup_\eta \mathrm{QM}_\eta(Y, D),$$

where η runs over every generic point of some component of the intersection of some prime components of D. A real valuation v is said to be *quasimonomial* if there exists a log resolution (Y, D) such that $v \in \mathrm{QM}(Y, D)$.

Remark 2.1. Let $\Gamma_v = v(K(X)^*) \subseteq \mathbb{R}$ be the value group of v. Denote by $\mathrm{ratrk}(v) = \dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q})$ the rational rank of v, and let k_v, $k(\xi)$ be the residue fields of $\mathcal{O}_v$, $\mathcal{O}_{X,\xi}$ respectively, where $\xi = c_X(v)$. If we let $\mathrm{trdeg}_X(v) = \mathrm{trdeg}(k_v/k(\xi))$ be the transcendental degree of v over X, we have Abhyankar's inequality $\mathrm{ratrk}(v) + \mathrm{trdeg}_X(v) \leq \dim(\mathcal{O}_{X,\xi})$. Quasimonomial valuations are exactly the ones that give equality in the Abhyankar's inequality; see [JM12, Proposition 3.7].

Let $v \in \mathrm{Val}_X$ be a quasimonomial valuation. A log smooth pair (Y, D) is said to be adapted to v if $v \in \mathrm{QM}(Y, D)$. We say (Y, D) is a good pair adapted to v if $\{v(D_i) \mid v(D_i) > 0\}$ are rationally independent.

Lemma 2.2 [JM12, Lemma 3.6]. *Let $v \in \mathrm{Val}_X$ be a quasimonomial valuation. There exists a good pair (Y, D) adapted to v. If $(Y', D') \succeq (Y, D)$ and (Y, D) is a good pair adapted to v, then (Y', D') is also a good pair adapted to v.*

An important class of valuations are divisorial valuations. A valuation is called *divisorial* if it is positively proportional to ord_E for some prime divisor E over X, where ord_E is the vanishing order along E. One easily verifies that the trivial valuation is quasimonomial of rational rank zero, and a divisorial valuation is quasimonomial of rational rank one. Let (Y, D) be a log smooth pair adapted to v. It can be verified that v is divisorial if and only if $\mathbb{R}_{\geq 0}[v] \subseteq \mathrm{QM}_\eta(Y, D) \cong \mathbb{R}^r_{\geq 0}$ is a rational ray, that is, $\mathbb{R}_{\geq 0}[v]$ contains some rational point in $\mathbb{R}^r_{\geq 0}$.

For every log resolution (Y, D) we can define the retraction map

$$r_{Y,D} : \mathrm{Val}_X \to \mathrm{QM}(Y, D)$$

by taking v to a quasimonomial valuation in $\mathrm{QM}(Y, D)$ with $r_{Y,D}(v)(D_i) = v(D_i)$. Note that $r_{Y,D}$ is continuous and that $v \geq r_{Y,D}(v)$ with equality if and only if $v \in \mathrm{QM}(Y, D)$. Furthermore, if $(Y', D') \succeq (Y, D)$ is another resolution, then the retraction map $r_{Y,D} : \mathrm{QM}(Y', D') \to \mathrm{QM}(Y, D)$ (by abuse of notation if without confusion) is a surjective mapping that is integral linear on every $\mathrm{QM}_{\eta'}(Y', D')$ and we have that $r_{Y,D} \circ r_{Y',D'} = r_{Y,D}$. Therefore we can naturally regard $\mathrm{QM}(Y, D)$ as a subset of $\mathrm{QM}(Y', D')$, and hence of the set of quasimonomial valuations on X. Also note that $v(\mathfrak{a}) \geq r_{Y,D}(v)(\mathfrak{a})$ for an ideal $\mathfrak{a}$ on X, with equality if (Y, D) is a log resolution of $\mathfrak{a}$; see [Lazarsfeld 2004] and [JM12, Corollary 4.8].

Tempered valuations. We first introduce the log discrepancy on an arbitrary scheme. Let $\pi : Y \to X$ be a birational proper morphism. The 0-th Fitting ideal $\mathrm{Fitt}_0(\Omega_{Y/X})$ is a locally principal ideal with its corresponding effective divisor denoted by $K_{Y/X}$; see [JM12, Section 1.3]. For a quasimonomial valuation $v \in \mathrm{QM}(Y, D)$, we define the log discrepancy

$$A_X(v) = \sum v(D_i) \cdot A_X(\mathrm{ord}_{D_i}) = \sum v(D_i) \cdot (1 + \mathrm{ord}_{D_i}(K_{Y/X})).$$

We simply denote this by A when the scheme X is obvious. Note that A is strictly positive linear on every $\mathrm{QM}_\eta(Y, D)$, and in particular continuous on every $\mathrm{QM}_\eta(Y, D)$ (or is *weakly continuous* according to Definition 3.4). Also note that if $(Y', D') \succeq (Y, D)$ and $v \in \mathrm{QM}(Y', D')$, then $A(v) \geq A(r_{Y,D}(v))$ and equality holds only when $v \in \mathrm{QM}(Y, D)$. For an arbitrary valuation $v \in \mathrm{Val}_X$, we define

$$A(v) = \sup_{(Y,D)} A(r_{Y,D}(v)) \in [0, +\infty].$$

Note that A is lower-semicontinuous (lsc) as a valuative function.

Definition 2.3. A valuation v is said to be *tempered* if $A(v) < \infty$. The valuation space V_X of X is defined to be the space of tempered valuations as a subspace of Val_X.

We similarly denote by V_X^* the subset of nontrivial tempered valuations. If $f : X' \to X$ is a proper birational morphism, then $A_X(v) = A_{X'}(v) + v(K_{X'/X})$ (see [JM12, Proposition 5.1(3)]) and hence $\mathrm{V}_{X'} = \mathrm{V}_X$. Since V_X is a topological subspace of Val_X, it is naturally a subspace of the Berkovich space X^{an}. See [JM12, Section 6.3] for a comparison.

With the aid of the log discrepancy, we can normalize V_X^* by letting $A(v) = 1$, that is, we define $\Lambda_X := \{v \in \mathrm{V}_X^* \mid A(v) = 1\}$. In particular, we normalize every cone complex $\mathrm{QM}(Y, D)$ by setting $\Delta(Y, D) := \{v \in \mathrm{QM}(Y, D) \mid A(v) = 1\}$. It is clear that $\Delta(Y, D)$ naturally possesses the structure of a simplicial complex, and by convention we say that $\Delta(Y, D)$ is a *dual complex*. Readers can compare the constructions here with [Boucksom et al. 2008; 2012b; 2012c].

The following lemma allows us to compare v and ord_ξ, where $\xi = c_X(v)$, which is quite useful (see [Lazarsfeld 2004] or [JM12, Section 5.3] for the definition of ord_ξ). See [JM12, Proposition 5.10] for a proof. Recently S. Boucksom, C. Favre and M. Jonsson [Boucksom et al. 2012a] gave a refinement of the following lemma.

Lemma 2.4 (Izumi-type inequality). *Let $\xi = c_X(v)$ and $\mathfrak{m}_\xi$ be the defining ideal of $\overline{\{\xi\}}$. Then we have $v(\mathfrak{m}_\xi)\,\mathrm{ord}_\xi \le v \le A(v)\,\mathrm{ord}_\xi$.*

Passing to the completion. A morphism $f : X' \to X$ is *regular* if it is flat and its fibres are geometrically regular (see [JM12, Section 1.1]). The following lemma on log discrepancy is essential for finding a valuation that computes the log canonical threshold or norms in Section 5.

Lemma 2.5 [JM12, Proposition 5.13]. *Let $f : X' \to X$ be a regular morphism, and let $f_* : \mathrm{Val}_{X'} \to \mathrm{Val}_X$ be the induced map. If $v' \in \mathrm{Val}_{X'}$ is a valuation on X', then $A(v') \ge A(f_*(v'))$. If we assume further that $X' = \mathrm{Spec}\,\widehat{\mathcal{O}_{X,\xi}}$ and v' is centred at the closed point of X', then $A(v') = A(f_*(v'))$.*

Definition 2.6. If $\xi \in X$ is a point, then we define $\mathrm{V}_{X,\xi} := c_X^{-1}(\xi)$ as a subspace of V_X. We can normalize $\mathrm{V}_{X,\xi}$ by letting $v(\mathfrak{m}) = 1$, where $\mathfrak{m}$ is the defining ideal of $\overline{\{\xi\}}$. More precisely we define $\mathbb{V}_{X,\xi} := \{v \in \mathrm{V}_{X,\xi} \mid v(\mathfrak{m}) = 1\}$. Let $M > 0$ be a positive real number. We also define $\mathbb{V}_{X,\xi,M} := \{v \in \mathbb{V}_{X,\xi} \mid A(v) \le M\}$. According to [JM12, Proposition 5.9] the space $\mathbb{V}_{X,\xi,M}$ is compact. If $X = \mathrm{Spec}\,A$ and $\mathfrak{m}$ is the defining ideal of $\overline{\{\xi\}}$, we often use the notation $\mathbb{V}_{A,\mathfrak{m}}$ instead of $\mathbb{V}_{X,\xi}$.

Let $(R, \mathfrak{m})$ be a local ring. Given a tempered valuation $v \in \mathrm{V}_{R,\mathfrak{m}}$, we define $v'(f) = \lim_{k\to\infty} v(\mathfrak{a}_k)$ for every $f \in \widehat{R}$, where $\mathfrak{a}_k \cdot \widehat{R} = f + \widehat{\mathfrak{m}}^k$. This is well-defined since $v(\mathfrak{a}_k) \le A(v)\,\mathrm{ord}_\xi(\mathfrak{a}_k) \le A(v)\,\mathrm{ord}_{\xi'}(f) < \infty$ by Lemma 2.4. The above definition leads to a correspondence between the valuation spaces of $\mathrm{Spec}\,R$ and $\mathrm{Spec}\,\widehat{R}$ as follows. Throughout this paper we will use the notations v and v' to indicate that $v = f_* v'$ for simplicity if without confusion.

Proposition 2.7. *Let $(R, \mathfrak{m})$ be a local ring, and let $(\widehat{R}, \widehat{\mathfrak{m}})$ be its $\mathfrak{m}$-adic completion. If we denote by $f : \operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$ the canonical morphism, then the induced map $f_* : \mathbb{V}_{\widehat{R},\widehat{\mathfrak{m}}} \to \mathbb{V}_{R,\mathfrak{m}}$ is bijective. If K' is a compact subspace of $\mathbb{V}_{\widehat{R},\widehat{\mathfrak{m}}}$, then f_* is a homeomorphism from K' to its image. In particular, $\mathbb{V}_{\widehat{R},\widehat{\mathfrak{m}},M} \cong \mathbb{V}_{R,\mathfrak{m},M}$ for any positive number M.*

Proof. The bijectivity of f_* follows from [JM12, Corollary 5.11], and we will prove the latter statement. Let $K = f_*(K')$. It suffices to show that K' is homeomorphic to K. Let $h \in \widehat{R}$ be a nonzero function. We have that $\max_{v' \in K'} v'(h) = \alpha < \infty$ since K' is compact. If $g \in R$ is a nonzero function such that $g - h \in \widehat{\mathfrak{m}}^n$ in $\widehat{R}$ for some $n > \alpha$. Then $v'(g - h) \geq n v'(\widehat{\mathfrak{m}}) > v'(h)$ for all $v' \in K'$. It follows that $v(g) = v'(g) = v'(h)$ for all $v' \in K'$ and hence they induce the same topology. $\square$

3. Functions on valuation spaces

In this section we will discuss various classes of functions on valuation space with an emphasis on the quasi-plurisubharmonic (qpsh for short) functions.

Bounded homogeneous functions. Let X be a scheme and V_X be its valuation space. A valuative function φ is *homogeneous* if $\varphi(tv) = t\varphi(v)$ for all $v \in \mathrm{V}_X$ and $t \in \mathbb{R}_+$. A valuative function φ is *bounded* if $\sup_{v \in \mathrm{V}_X^*} |\varphi(v)|/A(v) < \infty$. The set of bounded homogeneous functions (denoted by $\mathrm{BH}(X)$) forms an $\mathbb{R}$-linear space, which can be equipped with the norm $\|\varphi\| = \sup_{v \in \mathrm{V}_X^*} |\varphi(v)|/A(v)$. If $\mathfrak{q}$ is a nonzero ideal on X, then we define the $\mathfrak{q}$-norm as $\|\varphi\|_{\mathfrak{q}} = \sup_{v \in \mathrm{V}_X^*} |\varphi(v)|/(A(v) + v(\mathfrak{q}))$.

We also define

$$\|\varphi\|_{\mathfrak{q}}^+ := \sup_{v \in \mathrm{V}_X^*} \frac{\varphi(v)}{A(v) + v(\mathfrak{q})}$$

and

$$\|\varphi\|_{\mathfrak{q}}^- := \sup_{v \in \mathrm{V}_X^*} \frac{-\varphi(v)}{A(v) + v(\mathfrak{q})}.$$

Clearly, $\|\varphi\|_{\mathfrak{q}}^+ = \|-\varphi\|_{\mathfrak{q}}^-$ and $\|\cdot\|_{\mathfrak{q}} = \max\{\|\cdot\|_{\mathfrak{q}}^+, \|\cdot\|_{\mathfrak{q}}^-\}$.

Lemma 3.1. *Given two nonzero ideals $\mathfrak{p}$, $\mathfrak{q}$ on X, the $\mathfrak{p}$-norm and the $\mathfrak{q}$-norm are equivalent.*

Proof. We first assume that $\mathfrak{p} = \mathcal{O}_X$. Then we have the inequalities

$$\|\cdot\|_{\mathfrak{q}} \leq \|\cdot\| \leq \left(1 + \sup_{v \in \mathrm{V}_X^*} \frac{v(\mathfrak{q})}{A(v)}\right) \|\cdot\|_{\mathfrak{q}}.$$

Note that $\sup_{v \in \mathrm{V}_X^*} v(\mathfrak{q})/A(v) = \max_{D_i}(\operatorname{ord}_{D_i}(\mathfrak{q})/A(\operatorname{ord}_{D_i})) < \infty$, where D_i runs over all irreducible components of D on a birational model Y such that (Y, D) is a log resolution of $\mathfrak{q}$. This implies that $1 + \sup_{v \in \mathrm{V}_X^*} v(\mathfrak{q})/A(v) < \infty$ and leads to the conclusion. $\square$

Proposition 3.2. *Given a scheme X, $\mathrm{BH}(X)$ is a Banach space. If $f : X' \to X$ is a regular morphism and $f_* : \mathrm{V}_{X'} \to \mathrm{V}_X$ is the induced map, then the induced map $f^* : \mathrm{BH}(X) \to \mathrm{BH}(X')$ by taking φ to $\varphi \circ f_*$ is a bounded linear operator of Banach spaces. More precisely, $\| f^*(\varphi)\|_{\mathfrak{q}\cdot\mathcal{O}_{X'}} \le \|\varphi\|_{\mathfrak{q}}$ for any nonzero ideal $\mathfrak{q}$ on X.*

Proof. Note that a bounded homogeneous function φ is also a function on Λ_X, defined by $\{v \in \mathrm{V}_X^* \mid A(v) = 1\}$, with the norm $\sup_{v\in\Lambda_X} |\varphi(v)| < \infty$. If $\{\varphi_m\}$ is a Cauchy sequence in $\mathrm{BH}(X)$, then φ_m converges pointwise to a homogeneous function φ. Since $\sup_{v\in\Lambda_X}|\varphi(v)| \le \sup_{v\in\Lambda_X}|\varphi(v)-\varphi_m(v)|+\sup_{v\in\Lambda_X}|\varphi_m(v)| < \infty$, φ is a bounded homogeneous function. This proves that $\mathrm{BH}(X)$ is a Banach space. For the second statement, simply note that

$$\| f^*(\varphi)\|_{\mathfrak{q}\cdot\mathcal{O}_{X'}} = \sup_{v'\in \mathrm{V}_{X'}^*} \frac{|\varphi(v)|}{A(v') + v'(\mathfrak{q}\cdot\mathcal{O}_{X'})} \le \sup_{v\in \mathrm{V}_X^*} \frac{|\varphi(v)|}{A(v) + v(\mathfrak{q})} = \|\varphi\|_{\mathfrak{q}}.$$

by Lemma 2.5. □

Remark 3.3. Let φ be a bounded homogeneous function such that $\varphi(v) = -v(\mathfrak{a})$ for some nonzero ideal $\mathfrak{a}$ on X. It is easy to see that the norm $\|\varphi\|_{\mathfrak{q}}$ is exactly the Arnold multiplicity $\mathrm{Arn}^{\mathfrak{q}}\mathfrak{a}$, and its reciprocal is the log canonical threshold $\mathrm{lct}^{\mathfrak{q}}\mathfrak{a}$. We will discuss this type of functions in detail later.

Definition 3.4. A bounded homogeneous function φ is said to be *weakly continuous* if φ is continuous on every dual complex $\Delta(Y, D)$.

Example 3.5. (1) As we already mentioned, the log discrepancy A is a weakly bounded homogeneous function.

(2) If $\{\varphi_k\}$ is a sequence of continuous, bounded homogeneous functions that converges to a function φ strongly in norm, then φ is weakly continuous.

Ideal functions and qpsh functions. Given a nonzero ideal $\mathfrak{a}$, we define $|\mathfrak{a}|(v) = -e^{v(\mathfrak{a})}$ by convention. It is obvious that $\log|\mathfrak{a}|$ is a continuous, bounded homogeneous function.

Definition 3.6. A bounded homogeneous function φ is said to be an *ideal function* if there exists a finite number of nonzero ideals $\mathfrak{a}_j$ and positive real numbers c_j such that $\varphi = \sum_{j=1}^{l} c_j \log|\mathfrak{a}_j|$.

Lemma 3.7. *Let $\varphi = \sum_{j=1}^{l} c_j \log|\mathfrak{a}_j|$ be an ideal function on X and $\mathfrak{q}$ be a nonzero ideal. Then,*

$$\|\varphi\|_{\mathfrak{q}} = \max_E \left\{ \frac{\sum_{j=1}^{l} c_j \operatorname{ord}_E \mathfrak{a}_j}{A(\operatorname{ord}_E) + \operatorname{ord}_E \mathfrak{q}} \right\}$$

for some prime divisor E over X.

Proof. Let (Y, D) be a log resolution of $\mathfrak{q} \cdot \prod_{j=1}^{l} \mathfrak{a}_j$, and let D_i's be the irreducible components of D. By an easy computation, we see that

$$\|\varphi\|_{\mathfrak{q}} = \max_{D_i} \left\{ \frac{\sum_{j=1}^{l} c_j \operatorname{ord}_{D_i} \mathfrak{a}_j}{A(\operatorname{ord}_{D_i}) + \operatorname{ord}_{D_i} \mathfrak{q}} \right\},$$

where D_i runs over all irreducible components of D. □

Lemma 3.8. *Let φ be a bounded homogeneous function which is determined on some dual complex $\Delta(Y, D)$ in the sense of $\varphi = \varphi \circ r_{Y,D}$. Assume that φ is affine on each face of the dual complex $\Delta(Y, D)$ and that $(Y', D') \succeq (Y, D)$. Then $\varphi = \varphi \circ r_{Y',D'}$ which is also affine on each face of the dual complex $\Delta(Y', D')$.*

Proof. The assumption that φ is affine on each face of the dual complex $\Delta(Y, D)$ is equivalent to that φ is linear on each simplicial cone of $\mathrm{QM}(Y, D)$. The conclusion follows from the fact that $r_{Y,D}$ is linear on each simplicial cone of $\mathrm{QM}(Y', D')$. □

Definition 3.9. A bounded homogeneous function φ is a *quasi-plurisubharmonic* (qpsh for short) function if there exists a sequence of ideal functions that converges to φ strongly in norm. The set of qpsh functions, which is a closed convex cone in $\mathrm{BH}(X)$, is denoted by $\mathrm{QPSH}(X)$.

Definition 3.10. The *support* of a qpsh function is the set of elements of the form $c_X(v)$, for some nontrivial tempered valuation v such that $\varphi(v) < 0$.

If $\varphi = \sum_{j=1}^{l} c_i \log |\mathfrak{a}_i|$ is an ideal function, then the support of φ is the union of the vanishing loci $V(\mathfrak{a}_j)$ and hence proper closed. We will see that the support of a qpsh function is a countable union of proper closed subsets. See Corollary 4.26.

Proposition 3.11. *Let $\varphi \in \mathrm{QPSH}(X)$ be a qpsh function. Then, φ is convex on each face of every dual complex $\Delta(Y, D)$. Moreover, $\varphi \circ r_{Y,D}$ form a decreasing net of continuous functions that converges to φ strongly in norm. In particular, φ is weakly continuous and upper-semicontinuous* (*usc for short*).

Proof. To show that φ is convex on each face of every dual complex $\Delta(Y, D)$, it suffices to prove this when φ is an ideal function. We can assume that $\varphi = c \log |\mathfrak{a}|$. Let η be a generic point of the intersection of $D_1, \dots, D_l$. We will prove that φ is convex on $\mathrm{QM}_\eta(Y, D)$, which essentially implies the convexity on $\Delta(Y, D)$. To this end, assume that $v = \sum_{j=1}^{k} \lambda_j v_j$ such that $v, v_j \in \mathrm{QM}_\eta(Y, D)$, $\lambda_j > 0$ for every j and $\sum_{j=1}^{k} \lambda_j = 1$. Assume further that $\mathfrak{a} \cdot \mathcal{O}_Y$ is principal near η generated by f. If we consider the local coordinates $\underline{y} = \{y_1, \dots, y_l\}$ with the origin η, then v and v_j can be represented by $\alpha = (\alpha^1, \dots, \alpha^l)$ and $\alpha_j = (\alpha_j^1, \dots, \alpha_j^l)$ with the values $v(f) = \min\{\langle \alpha, \beta \rangle \mid f = \sum c_\beta y^\beta\}$ and $v_j(f) = \min\{\langle \alpha_j, \beta \rangle \mid f = \sum c_\beta y^\beta\}$. Obviously, $v(f) \geq \sum_{j=1}^{k} \lambda_j v_j(f)$ and we obtain the required convexity. If $\mathfrak{a} \cdot \mathcal{O}_Y$ is not principal, then φ is the maximum of a finite number of convex functions and hence convex.

Given an arbitrary qpsh function φ, the functions $\varphi \circ r_{Y,D}$ form a decreasing net because $v \le r_{Y,D}(v)$, and φ is continuous on $\Delta(Y, D)$ because it is the uniform limit function of continuous functions. It suffices to show that $\varphi \circ r_{Y,D}$ converges to φ strongly in norm. To this end, consider a sequence of ideal functions $\varphi_j = c_j \log|\mathfrak{a}_j|$ that converges to φ strongly in norm. We then obtain that

$$\|\varphi - \varphi \circ r_{Y,D}\| \le \|\varphi - \varphi_j\| + \|\varphi_j - \varphi \circ r_{Y,D}\| \le 2\|\varphi - \varphi_j\|$$

if (Y, D) is a log resolution of $\mathfrak{a}_j$, which completes the proof. □

Remark 3.12. The proposition above implies that a qpsh function is uniquely determined by its values on divisorial valuations. In fact, if φ and ϕ have the same values on divisorial valuations, then $\varphi = \phi$ on every dual complex $\Delta(Y, D)$ by continuity and hence $\varphi = \phi$ on V_X.

The following example shows that the pointwise limit of a decreasing sequence of ideal functions is not qpsh in general.

Example 3.13. Let $X = \operatorname{Spec} k[x]$ be an affine line, and let $\phi_k = \sum_{j=1}^{k} \log|f_j|$, where $f_j = x - j$. We see that ϕ_k is a decreasing sequence of ideal functions and the pointwise limit function φ exists. But φ is not qpsh because $\|\varphi - \phi\| \ge 1$ for any ideal function ϕ.

If $f : X' \to X$ is a regular morphism and φ is a qpsh function on X, then $f^*\varphi$ is a qpsh function on $\mathrm{V}_{X'}$ by Proposition 3.2. In particular if $f : U \to X$ (resp. $f : \operatorname{Spec} \mathbb{O}_{X,\xi} \to X$) is an open inclusion, we say that $f^*\varphi$ is the restriction (resp. germ) of φ, denoted by $\varphi|_U$ (resp. φ_ξ). Also, restrictions to the neighbourhoods of a point ξ induce a map $\mathrm{QPSH}(X) \to \varinjlim_{U \ni \xi} \mathrm{QPSH}(U)$, and the image of φ is also said to be the germ of φ, denoted by $\varphi|_\xi$.

If ξ is not contained in the support of a qpsh function φ, then $\varphi_\xi = 0$ by Proposition 3.11. However, the following example shows that it could happen that the germ of φ is nonzero in the set $\varinjlim_{U \ni \xi} \mathrm{QPSH}(U)$.

Example 3.14. Let $X = \operatorname{Spec} k[x]$ be an affine line, and let $\phi_k = \sum_{m=1}^{k} 2^{-m} \log|f_m|$, where $f_m = x - 1/m$. It is easy to see that ϕ_k converges to a function ϕ strongly in norm. Note that the origin is not contained in the support of ϕ, but the germ of ϕ in $\varinjlim_{U \ni 0} \mathrm{QPSH}(U)$ is nonzero.

From this example we see that if we define $\|\varphi|_\xi\| := \inf_{U \ni \xi} \|\varphi|_U\|$, then $\|\cdot\|$ is only a seminorm.

Proposition 3.15. *There is a surjective map of convex cones*

$$r : \varinjlim_{U \ni \xi} \mathrm{QPSH}(U) \to \mathrm{QPSH}(\operatorname{Spec} \mathbb{O}_{X,\xi})$$

which preserves the seminorm, and also preserves $\|\cdot\|^+$ *and* $\|\cdot\|^-$.

Proof. If $\varphi = c\log|\mathfrak{a}|$ and $\varphi' = c'\log|\mathfrak{a}|$, then we claim that $\|\varphi|_\xi - \varphi'|_\xi\| = \|\varphi_\xi - \varphi'_\xi\|$. To this end, let $\mu : (Y, D) \to X$ be a log resolution of $\mathfrak{a}\cdot\mathfrak{a}'$, and let $\mathfrak{a}\cdot\mathcal{O}_Y = \mathcal{O}_Y(-F)$ and $\mathfrak{a}'\cdot\mathcal{O}_Y = \mathcal{O}_Y(-F')$. One can easily check that

$$\|\varphi|_\xi - \varphi'|_\xi\| = \max_{D_i\in\mathcal{S}} \frac{|\operatorname{ord}_{D_i} F - \operatorname{ord}_{D_i} F'|}{A(\operatorname{ord}_{D_i})},$$

where $\mathcal{S}$ consists of irreducible components D_i of D such that $\mu(D_i)$ contains ξ in its support. This implies the claim.

Given a qpsh function $\varphi_\xi \in \mathrm{QPSH}(\operatorname{Spec}\mathcal{O}_{X,\xi})$, there exists a sequence of ideal functions $\varphi_{\xi,i} = c_i \log|\mathfrak{a}_{\xi,i}|$ that converges to φ_ξ strongly in norm. Let $\mathfrak{a}_i$ be ideals on X such that $\mathfrak{a}_i\cdot\mathcal{O}_{X,\xi} = \mathfrak{a}_{\xi,i}$. We have that $\varphi_i = c_i\log|\mathfrak{a}_i|$ converges to a qpsh function in $\varinjlim_{U\ni\xi}\mathrm{QPSH}(U)$ strongly in norm due to the previous claim. Therefore we obtain the surjectivity of r.

Finally, for two qpsh functions φ and φ' on an open neighbourhood of ξ, the equality $\|\varphi|_\xi - \varphi'|_\xi\| = \|\varphi_\xi - \varphi'_\xi\|$ follows from the claim in the first paragraph. Apply a similar argument to $\|\cdot\|^+$ and $\|\cdot\|^-$, we obtain the conclusion. □

From the discussion above, we see that $\varphi|_\xi$ provides more information while it is not a valuative function. We sometimes identify $\varphi|_\xi$ and φ_ξ as the germ of φ at ξ.

4. Multiplier ideals

We will now discuss the multiplier ideals of qpsh functions. Recall that a graded sequence of ideals $\mathfrak{a}_\bullet$ is a sequence of ideals that satisfies $\mathfrak{a}_m\cdot\mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$. By convention we put $\mathfrak{a}_0 = \mathcal{O}_X$, and we say $\mathfrak{a}_\bullet$ is nontrivial if $\mathfrak{a}_m \neq 0$ for some positive integer m. Note that in this case there are infinitely many m such that $\mathfrak{a}_m \neq 0$. A subadditive sequence of ideals $\mathfrak{b}_\bullet$ is a one-parameter family $\mathfrak{b}_t$ satisfying $\mathfrak{b}_s\cdot\mathfrak{b}_t \supseteq \mathfrak{b}_{s+t}$ for every $s, t \in \mathbb{R}_+$. Similarly, we put $\mathfrak{b}_0 = \mathcal{O}_X$ and we say that $\mathfrak{b}_\bullet$ is nontrivial if $\mathfrak{b}_t \neq 0$ for all $t\in\mathbb{R}_+$. Throughout this paper, every sequence of ideals is assumed to be nontrivial. We define $v(\mathfrak{a}_\bullet) = \inf_{m\geq 1} v(\mathfrak{a}_m)/m$ and $v(\mathfrak{b}_\bullet) = \sup_{t>0} v(\mathfrak{b}_t)/t$ as in [Ein et al. 2006]. We similarly define $|\mathfrak{a}_\bullet|(v) = e^{-v(\mathfrak{a}_\bullet)}$ and $|\mathfrak{b}_\bullet|(v) = e^{-v(\mathfrak{b}_\bullet)}$ for a graded sequence and a subadditive sequence of ideals respectively.

Multiplier ideals.

Definition 4.1. For a bounded homogeneous function $\varphi \in \mathrm{BH}(X)$, the multiplier ideal $\mathcal{J}(\varphi)$ of φ is the largest ideal in the set of nonzero ideals $\{\mathfrak{a} \mid \|\log|\mathfrak{a}| - \varphi\|^+ < 1\}$. If this set is empty, then we define $\mathcal{J}(\varphi) = (0)$.

Remark 4.2. We will see that the set above is always nonempty when φ is qpsh and $\mathcal{J}(\varphi)$ is therefore nonzero (see Remark 4.21). Moreover, we have the inequality $\varphi \leq \log|\mathcal{J}(\varphi)|$ (see Remark 4.21), and hence $\|\log|\mathcal{J}(\varphi)| - \varphi\| < 1$ holds.

The following proposition shows that our definition of multiplier ideals coincides with the classical definition of multiplier ideals.

Proposition 4.3. *If φ is an ideal function and we write $\varphi = \sum_{i=1}^{l} c_i \log|\mathfrak{a}_i|$, then $\mathcal{J}(\varphi) = \mathcal{J}\big(\prod_{i=1}^{l} \mathfrak{a}_i{}^{c_i}\big)$.*

Proof. Let $\pi : (Y, D) \to X$ be a log resolution of $\prod_{i=1}^{l} \mathfrak{a}_i$, and $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$ with F_i being supported in D. Since $\mathcal{J}\big(\prod_{i=1}^{l} \mathfrak{a}_i{}^{c_i}\big) = \pi_* \mathcal{O}_Y\big(K_{Y/X} - \llcorner \sum_{i=1}^{l} c_i F_i \lrcorner\big)$, it is easy to check that $\big\|\log\big|\mathcal{J}\big(\prod_{i=1}^{l} \mathfrak{a}_i{}^{c_i}\big)\big| - \varphi\big\|^+ < 1$, which immediately implies that $\mathcal{J}(\varphi) \supseteq \mathcal{J}\big(\prod_{i=1}^{l} \mathfrak{a}_i{}^{c_i}\big)$.

Conversely, if $f \in \Gamma(U, \mathcal{J}(\varphi))$ is a regular function on an affine open subset U, then $\|\log|f| - \varphi|_U\|^+ < 1$. It follows that $v(f) + A(v) + \varphi(f) > 0$ for all nontrivial tempered valuations v on U. In particular, $\mathrm{ord}_E\, f + \mathrm{ord}_E\, K_{Y/X} + 1 > -\varphi(\mathrm{ord}_E)$ for any prime divisor E on $\pi^{-1}U$. Thus $f \in \Gamma\big(U, \mathcal{J}\big(\prod_{i=1}^{l} \mathfrak{a}_i{}^{c_i}\big)\big)$ and it follows that $\mathcal{J}(\varphi) \subseteq \mathcal{J}\big(\prod_{i=1}^{l} \mathfrak{a}_i{}^{c_i}\big)$. $\square$

The lemmas below will be frequently used in this paper.

Lemma 4.4. *Given a nonzero ideal $\mathfrak{q}$ and a qpsh function $\varphi \in \mathrm{QPSH}(X)$, $\mathfrak{q} \subseteq \mathcal{J}(\lambda\varphi)$ if and only if $\lambda^{-1} > \|\varphi\|_{\mathfrak{q}}$. Thus $\|\varphi\|_{\mathfrak{q}}^{-1} = \min\{t \mid \mathfrak{q} \nsubseteq \mathcal{J}(t\varphi)\}$.*

Proof. If $\mathfrak{q} \subseteq \mathcal{J}(\lambda\varphi)$, then $\|\log|\mathfrak{q}| - \lambda\varphi\|^+ < 1$ by definition. That is,

$$\sup_{v \in \mathrm{V}_X^*} \frac{-v(\mathfrak{q}) - \lambda\varphi(v)}{A(v)} < 1.$$

This implies that $-v(\mathfrak{q}) - \lambda\varphi(v) \le (1-\varepsilon)A(v)$ for every $v \in \mathrm{V}_X^*$. Thus

$$\frac{-\lambda\varphi(v)}{A(v) + v(\mathfrak{q})} \le \frac{(1-\varepsilon)A(v) + v(\mathfrak{q})}{A(v) + v(\mathfrak{q})} \le (1-\varepsilon) + \varepsilon\|\log|\mathfrak{q}|\|_{\mathfrak{q}} < 1$$

by Lemma 3.7. We obtain $\lambda^{-1} > \|\varphi\|_{\mathfrak{q}}$ by definition.

Conversely we assume that $\|\varphi\|_{\mathfrak{q}} = \sup_{v \in \mathrm{V}_X^*} (-\lambda\varphi(v))/(A(v) + v(\mathfrak{q})) < 1$. Then $-\lambda\varphi(v) \le (1-\varepsilon)(A(v) + v(\mathfrak{q}))$. Therefore

$$\frac{-v(\mathfrak{q}) - \lambda\varphi(v)}{A(v)} \le 1 - \varepsilon - \varepsilon\frac{v(\mathfrak{q})}{A(v)} \le 1 - \varepsilon$$

for a sufficiently small ε which leads to the conclusion $\mathfrak{q} \subseteq \mathcal{J}(\lambda\varphi)$. $\square$

Lemma 4.5. *Let ξ be a point of a scheme X, and φ be a qpsh function. Assume that the multiplier ideal $\mathcal{J}(\varphi)$ is nonzero.* (In fact, this assumption automatically holds by Lemma 4.20 and Remark 4.21.) *Then*:

(1) $\mathcal{J}(\varphi|_U) = \mathcal{J}(\varphi) \cdot \mathcal{O}_U$.

(2) $\mathcal{J}(\varphi_\xi) = \mathcal{J}(\varphi) \cdot \mathcal{O}_{X,\xi}$.

(3) *Set* $\lambda^{-1} := \|\varphi\|_{\mathfrak{q}}$. *If* $\xi \in \mathrm{V}(\mathcal{J}(\lambda\varphi) : \mathfrak{q})$, *then* $\|\varphi\|_{\mathfrak{q}} = \|\varphi_\xi\|_{\mathfrak{q} \cdot \mathcal{O}_{X,\xi}}$.

Proof. (1) Since $\|\log|\mathcal{J}(\varphi)\cdot\mathcal{O}_U| - \varphi|_U\|^+ \leq \|\log|\mathcal{J}(\varphi)| - \varphi\|^+ < 1$, we have $\mathcal{J}(\varphi)\cdot\mathcal{O}_U \subseteq \mathcal{J}(\varphi|_U)$. On the other hand, if we denote by $\mathfrak{m}$ the defining ideal of $X \setminus U$, then there exists a sufficiently large integer k such that $v(\mathcal{J}(\varphi)) \leq v(\mathfrak{m}^k)$ for all valuations v centred in $X \setminus U$. Now we extend $\mathcal{J}(\varphi|_U)$ to X and still denote it by $\mathcal{J}(\varphi|_U)$. Therefore $\|\log|\mathcal{J}(\varphi|_U)\cdot\mathfrak{m}^k| - \varphi\|^+ < 1$, which implies $\mathcal{J}(\varphi|_U) \subseteq \mathcal{J}(\varphi)\cdot\mathcal{O}_U$.

(2) First note that $\|\log|\mathcal{J}(\varphi)\cdot\mathcal{O}_{X,\xi}| - \varphi_\xi\|^+ \leq \|\log|\mathcal{J}(\varphi)| - \varphi\|^+ < 1$, and it follows that $\mathcal{J}(\varphi)\cdot\mathcal{O}_{X,\xi} \subseteq \mathcal{J}(\varphi_\xi)$. For the inverse inclusion, we see that if $f \in \mathcal{J}(\varphi_\xi)$, then there exists an open neighbourhood U of ξ such that $\|\log|f| - \varphi|_U\|^+ < 1$ by Proposition 3.15. Thus $f \in \mathcal{J}(\varphi|_U)\cdot\mathcal{O}_{X,\xi} = \mathcal{J}(\varphi)\cdot\mathcal{O}_{X,\xi}$.

(3) It is obvious that $\|\varphi\|_{\mathfrak{q}} \geq \|\varphi_\xi\|_{\mathfrak{q}\cdot\mathcal{O}_{X,\xi}}$ by Proposition 3.2. If $\xi \in \mathrm{V}(\mathcal{J}(\lambda\varphi) : \mathfrak{q})$, then $(\mathcal{J}(\lambda\varphi_\xi) : \mathfrak{q}\cdot\mathcal{O}_{X,\xi}) = (\mathcal{J}(\lambda\varphi) : \mathfrak{q})\cdot\mathcal{O}_{X,\xi} \neq \mathcal{O}_{X,\xi}$. Therefore $\mathfrak{q}\cdot\mathcal{O}_{X,\xi} \not\subseteq \mathcal{J}(\lambda\varphi|_\xi)$ and $\lambda^{-1} \leq \|\varphi_\xi\|_{\mathfrak{q}\cdot\mathcal{O}_{X,\xi}}$ by Lemma 4.4. □

Algebraic qpsh functions.

Definition 4.6. A qpsh function $\varphi \in \mathrm{QPSH}(X)$ is *algebraic* if it is the limit function of an increasing sequence of ideal functions $\varphi = \lim_{m\to\infty}\varphi_m$ (in norm). Note that φ being algebraic implies that $t\varphi$ is algebraic for any $t \in \mathbb{R}_{>0}$, and that $\varphi + \psi$ is algebraic provided that ψ is another algebraic qpsh function. Thus the set of algebraic qpsh functions is a convex subcone of $\mathrm{QPSH}(X)$, denoted by $\mathrm{QPSH}^a(X)$.

An algebraic function is lower-semicontinuous (lsc for short) on V_X by its definition, and usc by Proposition 3.11, so it is continuous. We will see that in Definition 4.6 the phrase "in norm" is not necessary; that is, the pointwise limit of an increasing sequence of ideal functions is algebraic qpsh (see Lemma 4.15). One can compare this fact with Remark 4.25. The following example shows that a qpsh function is not necessarily algebraic.

Example 4.7. Let $X = \operatorname{Spec} k[x_1, x_2]$ be the affine plane. If we set

$$\phi_k = \sum_{l=1}^{k} \frac{1}{2^l}\log|f_l|, \quad \text{where } f_l = x_1 + x_2^{2^l},$$

then ϕ_k converges to a qpsh function ϕ strongly in norm. However, the qpsh function ϕ is not algebraic since there is no ideal function $\varphi \leq \phi$.

The following lemma shows that a graded system of ideals naturally induces an algebraic qpsh function.

Lemma 4.8. *Let* $\mathfrak{a}_\bullet$ *be a graded sequence of ideals. If we let* $\log|\mathfrak{a}_\bullet|(v) = v(\mathfrak{a}_\bullet)$, *then* $\log|\mathfrak{a}_\bullet|$ *is an algebraic qpsh function.*

Proof. It suffices to find a subsequence of $\{\mathfrak{a}_m\}$ such that $\{\varphi_k := (1/m_k)\log|\mathfrak{a}_{m_k}|\}$ is an increasing sequence of ideal functions that converges to a qpsh function strongly in norm. Let $\mathfrak{b}_\bullet$ be a sequence of ideals such that $\mathfrak{b}_t = \mathcal{J}(\mathfrak{a}_\bullet^t)$ (see [Lazarsfeld

2004]). Note that $\mathfrak{b}_\bullet$ is subadditive of controlled growth (see [JM12, Section 2, Section 6, Appendix]). Now we fix an integer m such that $\mathfrak{a}_m \neq 0$. Since $\mathfrak{b}_m \supseteq \mathcal{J}(\mathfrak{a}_m) \supseteq \mathfrak{a}_m$, we have $v(\mathfrak{b}_m) \leq v(\mathfrak{a}_m)$. Since $v(\mathfrak{b}_m) + A(v) - (1/k)v(\mathfrak{a}_{mk}) > 0$ for all nontrivial tempered valuations v, where k is a sufficiently divisible integer, we have $v(\mathfrak{a}_{mk})/(mk) < v(\mathfrak{b}_m)/m + A(v)/m$. From the inequalities

$$\frac{v(\mathfrak{b}_m)}{m} \leq \frac{v(\mathfrak{a}_{mk})}{mk} < \frac{v(\mathfrak{b}_m)}{m} + \frac{A(v)}{m}$$

we have $\left\| \frac{1}{mk} \log|\mathfrak{a}_{mk}| - \frac{1}{mkl} \log|\mathfrak{a}_{mkl}| \right\| < \frac{1}{m}$ for every positive integer l. As we multiply m, we obtain the desired sequence of ideal functions. □

Definition 4.9. Let $\varphi \in \mathrm{BH}(X)$ be a bounded homogeneous function. Its envelope ideal $\mathfrak{a}(\varphi)$ is the largest ideal in the set $\{\mathfrak{a} | \log|\mathfrak{a}| \leq \varphi\}$ if this set is nonempty. If it is empty, we set $\mathfrak{a}(\varphi) = 0$.

Proposition 4.10. *If φ is qpsh and $\mathfrak{a}(\varphi)$ is nonzero, then the envelope ideal can be written explicitly as $\Gamma(U, \mathfrak{a}(\varphi)) := \{f \in \mathcal{O}_X(U) \mid v(f) + \varphi(v) \geq 0$ for every $v \in \mathrm{V}_U^*\}$ on every open subset U.*

Proof. Since the question is local, we can assume that $X = \operatorname{Spec} A$ is affine. It suffices to prove that the ideal $\mathfrak{a}$, defined by

$$\mathfrak{a}(U) := \{f \in \mathcal{O}_X(U) \mid v(f) + \varphi(v) \geq 0 \text{ for every } v \in \mathrm{V}_U^*\}$$

on every open subset U, is coherent. To this end, we write $I := \mathfrak{a}(X)$, and we will prove that $\mathfrak{a}(U_g) = I_g$ for any nonzero regular function $g \in A$, where U_g denotes the affine open subset defined by g. Since $\mathfrak{a}(U_g) \supseteq I_g$ by definition, we only need to prove the converse inclusion. Note that there exists a large integer k such that $kv(g) \geq v(\mathfrak{a}(\varphi))$ for every nontrivial tempered valuation v centred in the locus $V(g)$, and hence $k \log|g|(v) \leq \varphi(v)$. If f is a regular function on U_g such that $v(f) + \varphi(v) \geq 0$ for every $v \in \mathrm{V}_{U_g}^*$, then $v(fg^k) + \varphi(v) \geq 0$ for every $v \in \mathrm{V}_X^*$. This implies that $f \in I_g$. □

If we set $\mathfrak{a}(\varphi)_m = \mathfrak{a}(m\varphi)$, then $\{\mathfrak{a}(\varphi)_\bullet\}$ is a (possibly trivial) graded sequence of ideals. The following lemma shows that every algebraic qpsh function is of the form $\log|\mathfrak{a}_\bullet|$.

Lemma 4.11. *If $\varphi \in \mathrm{QPSH}^a(X)$ is an algebraic qpsh function, then $\varphi = \log|\mathfrak{a}(\varphi)_\bullet|$.*

Proof. Given an arbitrarily small positive number ε, there exist an ideal $\mathfrak{a}$ on X and an integer m such that $\frac{1}{m} \log|\mathfrak{a}| \leq \varphi$ and $\left\| \frac{1}{m} \log|\mathfrak{a}| - \varphi \right\| < \varepsilon$. We have $\frac{1}{m} \log|\mathfrak{a}(\varphi)_m| \geq \frac{1}{m} \log|\mathfrak{a}|$ by definition and the conclusion follows. □

By combining Proposition 4.3, Lemmas 4.8 and 4.11, we see that a bounded homogeneous function is algebraic qpsh if and only if it is derived from a graded sequence of ideals. Readers can compare the following theorem with Theorem 4.24.

Theorem 4.12. *If φ is a bounded homogeneous function, the following statements are equivalent.*

(1) $\varphi \in \mathrm{QPSH}^a(X)$ *is algebraic qpsh.*

(2) *There exists a graded sequence of ideals* $\mathfrak{a}_\bullet$ *such that* $\varphi = \log|\mathfrak{a}_\bullet|$.

(3) *The graded system of ideals* $\mathfrak{a}(\varphi)_\bullet$ *is nontrivial and* $\varphi = \log|\mathfrak{a}(\varphi)_\bullet|$.

Proof. If we assume (1), then (3) holds by Lemma 4.11. Note that (3) implies (2) if we simply put $\mathfrak{a}_\bullet = \mathfrak{a}(\varphi)_\bullet$. Finally, (1) follows from (2) by Lemma 4.8. □

We will use the following easy lemma. For the convenience of readers we present a proof here.

Lemma 4.13. *Let* $\varphi \in \mathrm{QPSH}^a(X)$ *be an algebraic qpsh function.*

(1) *Assume that* $\{\varphi_m\}$ *is an increasing sequence of qpsh functions that converges to* φ *strongly in norm. Then* $\mathcal{J}(\varphi) = \mathcal{J}(\varphi_m)$ *for sufficiently large* m.

(2) *Assume that* $f : X' \to X$ *is a regular morphism of schemes. Then* $f^*\varphi$ *is algebraic qpsh.*

Proof. (1) We see that $\|\log|\mathcal{J}(\varphi)| - \varphi\|^+ = 1 - \varepsilon$ for some positive number ε. If $\|\varphi - \varphi_m\| < \varepsilon$, then $\|\log|\mathcal{J}(\varphi)| - \varphi_m\|^+ < 1$ and $\mathcal{J}(\varphi) \subseteq \mathcal{J}(\varphi_m)$. The inverse inclusion $\mathcal{J}(\varphi) \supseteq \mathcal{J}(\varphi_m)$ is obvious because $\varphi \geq \varphi_m$.

(2) Assume φ_m is an increasing sequence of ideal functions that converges to φ strongly in norm. Then $f^*\varphi_m$ is also an increasing sequence of ideal functions that converges to $f^*\varphi$ strongly in norm by Proposition 3.2. This implies that $f^*\varphi$ is algebraic qpsh. □

By combining Lemmas 4.8 and 4.13(1), we see that the definition of valuative multiplier ideals of algebraic functions coincides with the classical definition of asymptotic multiplier ideals.

Corollary 4.14. *Let* $\mathfrak{a}_\bullet$ *be a graded sequence of ideals. If we write* $\varphi = \log|\mathfrak{a}_\bullet|$, *then* $\mathcal{J}(\varphi) = \mathcal{J}(\mathfrak{a}_\bullet)$.

General qpsh functions.

Lemma 4.15. *If* $\{\varphi_\lambda\}$ *is a family of* (*algebraic*) *qpsh functions, then* $\sup_\lambda \varphi_\lambda$ *is an* (*algebraic*) *qpsh function. Therefore, the convex cone* $\mathrm{QPSH}(X)$ (*resp.* $\mathrm{QPSH}^a(X)$) *is closed under taking the supremum.*

Proof. We firstly assume that $\{\varphi_\lambda\}$ is a family of algebraic qpsh functions, and we write $\psi = \sup_\lambda \varphi_\lambda$. Since $\psi \geq \varphi_\lambda$ for every λ, $\mathfrak{a}(\psi)_m \supseteq \mathfrak{a}(\varphi_\lambda)_m$. It follows that $\log|\mathfrak{a}(\psi)_\bullet| \geq \log|\mathfrak{a}(\varphi_\lambda)_\bullet| = \varphi_\lambda$. Therefore $\psi = \log|\mathfrak{a}(\psi)_\bullet|$ is algebraic qpsh.

Now we treat the case when $\{\varphi_\lambda\}$ is a family of general qpsh functions. For each λ, we assume that $\{\varphi_{\lambda,m}\}$ is a sequence of ideal functions that converges to φ_λ

strongly in norm such that $\|\varphi_\lambda - \varphi_{\lambda,m}\| < \frac{1}{m}$. If we set $\psi_m = \sup_\lambda \varphi_{\lambda,m}$, which is an algebraic qpsh by the previous argument, then $\|\psi - \psi_m\| \leq \frac{1}{m}$ and it follows that $\{\psi_m\}$ is a sequence that converges to ψ strongly in norm. $\square$

Since the convex cones $\mathrm{QPSH}(X)$ and $\mathrm{QPSH}^a(X)$ are closed under taking the supremum by Lemma 4.15, we can introduce the following definition.

Definition 4.16. Let φ be a bounded homogeneous function. Assume that the set $\{\psi \in \mathrm{QPSH}(X) \mid \psi \leq \varphi\}$ is nonempty. Then we say the maximal function in this set the qpsh envelope function. We similarly define the algebraic qpsh envelope function of φ if it exists.

In general, we cannot ensure the sets defined as above are nonempty. For instance, the function in Example 3.13 is bounded homogeneous but its qpsh envelope function does not exist. Also note that the function ϕ in Example 4.7 is qpsh itself but its algebraic qpsh envelope function does not exist.

Lemma 4.17. *Let φ be a bounded homogeneous function that is determined on some dual complex $\Delta(Y, D)$ in the sense of $\varphi = \varphi \circ r_{Y,D}$. Then, its qpsh envelope function ψ exists. Further, ψ is algebraic qpsh.*

Proof. Let $Z \subseteq X$ be the image of the reduced divisor D on X, and $\mathfrak{m}$ be the defining ideal of Z. Since $\log|\mathfrak{m}|$ is strictly negative on $\Delta(Y, D)$ and φ is bounded on $\Delta(Y, D)$, there exists an integer k such that $k \log|\mathfrak{m}| \leq \varphi$ on $\Delta(Y, D)$. Because φ is determined on the dual complex $\Delta(Y, D)$ in the sense of $\varphi = \varphi \circ r_{Y,D}$, we have that $k \log|\mathfrak{m}| \leq \varphi$ on V_X. It follows that its algebraic qpsh envelope function ϕ exists. In particular, its qpsh envelope function ψ exists.

Now we will show that $\psi = \phi$. Set

$$\mu_1 = \max_{v \in \Delta(Y,D)} |v(\mathfrak{m})| \quad \text{and} \quad \mu_2 = \min_{v \in \Delta(Y,D)} |v(\mathfrak{m})|.$$

For any small number $\varepsilon > 0$, we choose $\delta \ll 1$ such that $(1 + \mu_1/\mu_2)\delta < \varepsilon$ and an ideal function ϕ' such that $\|\phi' - \psi\| < \delta$. Note that for every valuation $v \in \Delta(Y, D)$ we have

$$\psi(v) > \phi'(v) - \frac{\delta}{\mu_2} v(\mathfrak{m}) \geq \phi'(v) - \frac{\delta\mu_1}{\mu_2} > \psi(v) - \left(1 + \frac{\mu_1}{\mu_2}\right)\delta.$$

We can assume that $\phi' \leq \psi$ and $|\psi(v) - \phi'(v)| < \varepsilon$ on $\Delta(Y, D)$ after replacing ϕ' by $\phi' + (\delta/\mu_2)\log|\mathfrak{m}|$. Because $\varphi = \varphi \circ r_{Y,D}$, we obtain that $\phi' \leq \varphi$. It follows that $\phi' \leq \phi \leq \psi$ by the definition of the qpsh envelope function. Since ε can be chosen arbitrarily small, this forces $\phi = \psi$ on $\Delta(Y, D)$. If we pick any higher log resolution (Y', D'), we can show that $\phi = \psi$ on $\Delta(Y', D')$ by the same argument. The conclusion therefore follows from Proposition 3.11. $\square$

The above lemma leads to the following definition.

Definition 4.18. Let $\varphi \in \mathrm{QPSH}(X)$ be a qpsh function. We denote by $\varphi_{Y,D}$ the qpsh envelop function of $\varphi \circ r_{Y,D}$.

Lemma 4.19. *Let φ be a qpsh function. Then there exists a decreasing sequence of algebraic functions that converges to φ strongly in norm.*

Proof. Let $\{\varphi_m\}$ be a sequence of ideal functions that converges to φ strongly in norm. We can assume that $\varphi_m = c_m \log|\mathfrak{a}_m|$ and $\|\varphi - \varphi_m\| < \frac{1}{m}$. Let (Y, D) be a log resolution $\mathfrak{a}_1$. It is easy to see that $\|\varphi \circ r_{Y,D} - \varphi_1\| < 1$, and therefore $\|\varphi_{Y,D} - \varphi_1\| < 1$. We deduce that $\|\varphi_{Y,D} - \varphi\| < 2$. Now we replace φ_1 by $\varphi_{Y,D}$ and continue this process. Note that if $(Y', D') \succeq (Y, D)$, then $\varphi_{Y',D'} \leq \varphi_{Y,D}$ by Proposition 3.11. We easily obtain the required decreasing sequence of algebraic functions. □

Lemma 4.20. *Let $\{\varphi_m\}$ be a sequence of qpsh functions that converges to a qpsh function φ strongly in norm. Then $\mathcal{J}(\varphi) = \mathcal{J}((1+\varepsilon)\varphi_m)$ for a sufficiently small positive real number $\varepsilon > 0$ and a sufficiently large integer m.*

Proof. First we prove that $\mathcal{J}(\varphi) \subseteq \mathcal{J}((1+\varepsilon)\varphi_m)$ for a sufficiently small number $\varepsilon > 0$ and a sufficiently large integer m. To this end, we pick a sufficiently small number $\varepsilon > 0$ such that $\mathcal{J}(\varphi) = \mathcal{J}((1+\varepsilon)\varphi)$. Since $\mathcal{J}((1+\varepsilon)\varphi) \subseteq \mathcal{J}((1+\varepsilon)\varphi_m)$ provided that m is sufficiently large, we have $\mathcal{J}(\varphi) \subseteq \mathcal{J}((1+\varepsilon)\varphi_m)$. Conversely, we pick a sufficiently large integer m such that $\|\varphi - \varphi_m\| < 1 - \frac{1}{1+\varepsilon}$. Applying Lemma 4.4 again, we see that if $f \in \mathcal{J}((1+\varepsilon)\varphi_m)$ then $\|\varphi_m\|_f < \frac{1}{1+\varepsilon}$ and hence $\|\varphi\|_f \leq \|\varphi_m\|_f + \|\varphi - \varphi_m\|_f < 1$, which implies that $f \in \mathcal{J}(\varphi)$. □

Remark 4.21. Note that we always have $\phi \leq \log|\mathcal{J}(\phi)|$ for an algebraic qpsh function ϕ by [JM12, Propositions 6.2 and 6.5]. It follows by Lemmas 4.19 and 4.20 that $\mathcal{J}(\varphi)$ is nonzero and $(1+\varepsilon)\varphi \leq (1+\varepsilon)\varphi_m \leq \log|\mathcal{J}(\varphi)|$, where $\{\varphi_m\}$ is a decreasing sequence of algebraic functions that converges to a qpsh function φ strongly in norm. Since ε can be chosen arbitrarily small, we immediately obtain that $\varphi \leq \log|\mathcal{J}(\varphi)|$.

Now we discuss the multiplier ideals of general qpsh functions.

Proposition 4.22. *Let $\varphi \in \mathrm{QPSH}(X)$ be a qpsh function on X.*

(1) *Assume that ψ is another qpsh function on X. Then,*

$$\mathcal{J}(\varphi + \psi) \subseteq \mathcal{J}(\varphi) \cdot \mathcal{J}(\psi).$$

(2) *Assume that $f : X' \to X$ is a regular morphism of schemes. Then,*

$$\mathcal{J}(\varphi) \cdot \mathcal{O}_{X'} = \mathcal{J}(f^*\varphi).$$

Proof. (1) By Lemma 4.19 we can assume that there are decreasing sequences of algebraic functions $\{\varphi_m\}$ and $\{\psi_m\}$ converging to φ and ψ strongly in norm respectively. Then for some sufficiently large integer m, by Lemma 4.20 we have $\mathcal{J}(\varphi+\psi)=\mathcal{J}((1+\varepsilon)(\varphi_m+\psi_m))\subseteq\mathcal{J}((1+\varepsilon)\varphi_m)\cdot\mathcal{J}((1+\varepsilon)\psi_m)=\mathcal{J}(\varphi)\cdot\mathcal{J}(\psi)$ since $\varphi_m+\psi_m$ converges to $\varphi+\psi$ strongly in norm. The inclusion appearing in this expression follows from [JM12, Theorem A.2].

(2) Since f is regular, for any ideal function $\phi=\sum_i c_i \log \mathfrak{a}_i$, we have

$$\mathcal{J}(\phi)\cdot\mathcal{O}_{X'}=\mathcal{J}\left(\prod_i \mathfrak{a}_i^{c_i}\right)\cdot\mathcal{O}_{X'}=\mathcal{J}\left(\prod_i (\mathfrak{a}_i\cdot\mathcal{O}_{X'})^{c_i}\right)=\mathcal{J}(f^*\phi)$$

by the argument of [JM12, Proposition 1.9]. If $\{\varphi_m\}$ is a sequence of ideal functions that converges to φ strongly in norm, then $f^*\varphi_m$ is a decreasing sequence of ideal functions that converges to $f^*\varphi$ strongly in norm by Proposition 3.2. Therefore we have $\mathcal{J}(\varphi)\cdot\mathcal{O}_{X'}=\mathcal{J}((1+\varepsilon)\varphi_m)\cdot\mathcal{O}_{X'}=\mathcal{J}((1+\varepsilon)f^*\varphi_m)=\mathcal{J}(f^*\varphi)$. □

Recall from [JM12] that if $\mathfrak{b}_\bullet$ is subadditive, then the limit

$$v(\mathfrak{b}_\bullet):=\lim_{m\to\infty}\frac{1}{m}v(\mathfrak{b}_m)\in[0,+\infty]$$

is well-defined. For the purpose of constructing a "good" valuative function, we introduce the notion of a subadditive sequence of ideals of controlled growth as follows.

Definition 4.23 [JM12, Definition 2.9]. A subadditive sequence of ideals $\mathfrak{b}_\bullet$ is *of controlled growth* if

$$\frac{v(\mathfrak{b}_t)}{t}>v(\mathfrak{b}_\bullet)-\frac{A(v)}{t}$$

for every nontrivial tempered valuation v and every $t>0$.

We see that $v(\mathfrak{b}_\bullet):=\lim_{m\to\infty}\frac{1}{m}v(\mathfrak{b}_m)<+\infty$ for every nontrivial tempered valuation v. Furthermore, if we define $\log|\mathfrak{b}_\bullet|(v)=-v(\mathfrak{b}_\bullet)$, then $\log|\mathfrak{b}_\bullet|$ is approximated by $\frac{1}{m}\log|\mathfrak{b}_m|$ strongly in norm and hence qpsh. Given a qpsh function, if we define $\mathcal{J}(\varphi)_t:=\mathcal{J}(t\varphi)$, then $\mathcal{J}(\varphi)_\bullet$ is a subadditive sequence of controlled growth by Proposition 4.22, Definition 4.1 and Remark 4.2. This allows us to give a characterization of qpsh functions as follows. Readers could compare the following theorem with Theorem 4.12.

Theorem 4.24. *If φ is a bounded homogeneous function, the following statements are equivalent.*

(1) *φ is qpsh.*

(2) *There is a subadditive sequence of ideals $\mathfrak{b}_\bullet$ of controlled growth such that $\varphi=\log|\mathfrak{b}_\bullet|$.*

(3) *The ideal $\mathcal{J}(t\varphi)$ is nonzero for every $t>0$ and $\varphi=\log|\mathcal{J}(\varphi)_\bullet|$.*

Proof. If we assume (1), then (3) follows from the previous argument together with Definition 4.1 and Remark 4.2. Note that (3) implies (2) if we simply put $\mathfrak{b}_\bullet = \mathcal{J}(\varphi)_\bullet$. Finally, (1) follows from (2) by the previous argument. □

Remark 4.25. From the theorem we see that every qpsh function φ can be approximated by a decreasing sequence of ideal functions φ_k in norm. Indeed, we can take $\varphi_k = \frac{1}{2^k} \log |\mathcal{J}(2^k \varphi)|$. However, if φ is only the pointwise limit of a decreasing sequence of ideal functions on V_X, then φ is not necessarily qpsh (see Example 3.13).

An immediate application of the preceding discussion is the following result on the support of a qpsh function.

Corollary 4.26. *Let φ be a qpsh function. Then its support* $\operatorname{supp} \varphi$ *is a countable union of proper Zariski closed subsets of X.*

Remark 4.27. Readers can compare the constructions here with [Boucksom et al. 2008]. If we work on $X = \operatorname{Spec} \widehat{R}$, where R is the localization of $\mathbb{C}[x_1, \ldots, x_n]$ at the origin, then our definition of qpsh functions coincides the notion of *formal psh functions*. A brief argument is as follows. Given a formal psh function g, we have a subadditive sequence of ideals $\{\mathcal{L}^2(tg)\}_{t>0}$ in $\widehat{R}$, that satisfies $v(\mathcal{L}^2(tg)) + A(v) + (1+\epsilon)g(v) \geq 0$ for every quasimonomial valuation v centred at the origin and an arbitrarily small ϵ; see [Boucksom et al. 2008, Theorems 3.10 and 3.9]. It follows that $\{\mathcal{L}^2(tg)\}_{t>0}$ form a subadditive sequence of ideals of controlled growth that induces to a qpsh function φ on X. Therefore $\varphi(v) = g(v)$ for every divisorial valuation v centred at the origin. Conversely, a qpsh function can be naturally viewed as a formal psh function by definition. Therefore we construct an one-to-one correspondence. The details are left to the readers.

Remark 4.28. Recall from complex geometry that a function $\varphi : X \to [-\infty, +\infty)$ from a complex manifold is *qpsh* if it is locally equal to the sum of a smooth function and a psh function. If X is a smooth complex variety, we should be able to define the valuative transform of φ, which is expected to be a qpsh function on the valuation space V_X as defined in this paper. This was done locally in [Boucksom et al. 2008] and its predecessors [Favre and Jonsson 2004; 2005a; 2005b]. However, the global situation is not fully understood by us at this point.

5. Computing norms

Generalities.

Definition 5.1. Let φ be a bounded homogeneous function and $\mathfrak{q}$ be a nonzero ideal on X. We say a nontrivial tempered valuation $v \in \mathrm{V}_X^*$ *computes* $\|\varphi\|_{\mathfrak{q}}$ if the equality $\|\varphi\|_{\mathfrak{q}} = |\varphi(v)|/(A(v) + v(\mathfrak{q}))$ holds.

The main result of this section is the following theorem.

Theorem 5.2. *Let $\varphi \in \mathrm{QPSH}(X)$ be a qpsh function and let $\mathfrak{q}$ be a nonzero ideal on X. Then there exists a nontrivial tempered valuation v that computes $\|\varphi\|_{\mathfrak{q}}$.*

Before we prove this theorem, we need some preparations.

Proposition 5.3. *Let φ be a bounded homogeneous function that is determined on some dual complex $\Delta(Y, D)$ in the sense of $\varphi = \varphi \circ r_{Y,D}$. Assume that φ is weakly continuous (see Definition 3.4). Then there exists a quasimonomial valuation v that computes $\|\varphi\|_{\mathfrak{q}}$. If we assume further that φ is affine on each face of $\Delta(Y, D)$, then there exists a divisorial valuation v that computes $\|\varphi\|_{\mathfrak{q}}$.*

Proof. For every nontrivial tempered valuation $v \in V_X^*$, we have

$$\frac{|\varphi(v)|}{A(v) + v(\mathfrak{q})} \geq \frac{|\varphi \circ r_{Y,D}(v)|}{A(r_{Y,D}(v)) + r_{Y,D}(v)(\mathfrak{q})}$$

with equality if and only if $v \in \mathrm{QM}(Y, D)$. Thus

$$\|\varphi\|_{\mathfrak{q}} = \sup_{v \in \mathrm{QM}(Y,D)} \frac{|\varphi(v)|}{A(v) + v(\mathfrak{q})} = \sup_{v \in \Delta(Y,D)} \frac{|\varphi(v)|}{1 + v(\mathfrak{q})}.$$

Since φ is weakly continuous, the function $v \to |\varphi(v)|/(A(v) + v(\mathfrak{q}))$ is continuous on $\mathrm{QM}(Y, D)$. Therefore the function $v \to |\varphi(v)|/(1 + v(\mathfrak{q}))$ is continuous on the dual complex $\Delta(Y, D)$ and hence achieves its maximum in $\Delta(Y, D)$.

Assume that φ is affine on $\Delta(Y, D)$, and denote by $\{D_i\}$ the irreducible components of D. After replacing (Y, D) by some higher log resolution, we can assume that (Y, D) is a log resolution of $\mathfrak{q}$ by Lemma 3.8. Then we have $\|\varphi\|_{\mathfrak{q}} = \max_{D_i}(|\varphi(\mathrm{ord}_{D_i})|/(A(\mathrm{ord}_{D_i}) + \mathrm{ord}_{D_i}(\mathfrak{q})))$, where D_i runs over all irreducible components of D since the functions φ, A and $\log|\mathfrak{q}|$ are all affine on $\Delta(Y, D)$. $\square$

Computing norms of qpsh functions. This subsection is devoted to the proof of Theorem 5.2. The proof here follows the strategy of [JM12]. We first consider the local case.

Lemma 5.4. *Let $(R, \mathfrak{m})$ be a local ring, let $\varphi \in \mathrm{QPSH}(\mathrm{Spec}\, R)$ be a qpsh function, and let $\mathfrak{q}$ be a nonzero ideal on $\mathrm{Spec}\, R$. We set $\lambda^{-1} = \|\varphi\|_{\mathfrak{q}}$ and assume that $\sqrt{(\mathcal{J}(\lambda\varphi) : \mathfrak{q})} = \mathfrak{m}$. If we define another qpsh function $\psi = \max\{\varphi, p \log|\mathfrak{m}|\}$ for a sufficiently large integer p, then $\|\varphi\|_{\mathfrak{q}} = \|\psi\|_{\mathfrak{q}}$. Moreover, if a nontrivial tempered valuation v computes $\|\psi\|_{\mathfrak{q}}$, then v also computes $\|\varphi\|_{\mathfrak{q}}$.*

Proof. Since $\sqrt{(\mathcal{J}(\lambda\varphi) : \mathfrak{q})} = \mathfrak{m}$, we have $\mathfrak{m}^n \cdot \mathfrak{q} \subseteq \mathcal{J}(\lambda\varphi)$ for some integer n. Set $\lambda'^{-1} = \|\varphi\|_{\mathfrak{m}^n \cdot \mathfrak{q}}$, it follows that $\lambda' > \lambda$ by Lemma 4.4. Pick an integer $p > n/(\lambda' - \lambda)$, and fix a sufficiently small number $\varepsilon \ll 1$ such that $p > n/((1-\varepsilon)\lambda' - \lambda)$. Observe that

$$\|\psi\|_{\mathfrak{q}} = \sup_{v \in V_R^*} \frac{\min\{-\varphi(v), pv(\mathfrak{m})\}}{A(v) + v(\mathfrak{q})} \geq \sup_{v \in V_\varepsilon^*} \frac{\min\{-\varphi(v), pv(\mathfrak{m})\}}{A(v) + v(\mathfrak{q})},$$

where V_ε^* is the set of $v \in V_R^*$ satisfying $-\varphi(v)/(A(v)+v(\mathfrak{q})) \geq (1-\varepsilon)/\lambda$.

By the definition of λ' we have $nv(\mathfrak{m})/(-\varphi(v)) \geq \lambda' - (A(v)+v(\mathfrak{q}))/(-\varphi(v))$ for every nontrivial tempered valuation v. This implies that

$$\begin{aligned}\|\psi\|_{\mathfrak{q}} &\geq \sup_{v\in V_\varepsilon} \frac{-\varphi(v)}{A(v)+v(\mathfrak{q})} \min\left\{1, \frac{p}{n}\left(\lambda' - \frac{A(v)+v(\mathfrak{q})}{-\varphi(v)}\right)\right\} \\ &\geq \sup_{v\in V_\varepsilon} \frac{-\varphi(v)}{A(v)+v(\mathfrak{q})} \min\left\{1, \frac{p}{n}\left(\lambda' - \frac{\lambda}{1-\varepsilon}\right)\right\} = \sup_{v\in V_\varepsilon} \frac{-\varphi(v)}{A(v)+v(\mathfrak{q})} = \|\varphi\|_{\mathfrak{q}}.\end{aligned}$$

Moreover, if a nontrivial tempered valuation v computes $\|\psi\|_{\mathfrak{q}}$, we see from these inequalities that v also computes $\|\varphi\|_{\mathfrak{q}}$. □

Lemma 5.5. *Let* $(R, \mathfrak{m})$ *be a local ring, let* φ *be an ideal function on* Spec R *such that* $\varphi \geq p \log|\mathfrak{m}|$ *for some integer* p, *and let* $\mathfrak{q}$ *be a nonzero ideal on* Spec R. *Then there exists a nontrivial tempered valuation* $v \in \mathbb{V}_{R,\mathfrak{m},M}$ (*see Definition 2.6*) *that computes* $\|\psi\|_{\mathfrak{q}}$ *provided that* $M > p \cdot \|\varphi\|_{\mathfrak{q}}^{-1}$.

Proof. If we write $c = p/M$, then $0 < c < \|\psi\|_{\mathfrak{q}}$. For every $v \in \mathbb{V}_{R,\mathfrak{m}}$ such that $-\varphi(v)/(A(v)+v(\mathfrak{q})) > c$, we have $A(v) \leq A(v)+v(\mathfrak{q}) < p/c = M$. Thus $\|\varphi\|_{\mathfrak{q}} = \sup v \in \mathbb{V}_{R,\mathfrak{m},M} - \varphi(v)/(A(v)+v(\mathfrak{q}))$. Note that $\mathbb{V}_{R,\mathfrak{m},M}$ is compact. Since the function $v \to -\varphi(v)/(A(v)+v(\mathfrak{q}))$ is usc as the valuative function A is lsc, the maximum can be achieved in $\mathbb{V}_{R,\mathfrak{m},M}$. □

Lemma 5.6. *Let* $\varphi \in$ QPSH(X) *be a qpsh function on* X *and* $\{\varphi_m\}$ *be a decreasing sequence of algebraic functions converging to* φ *strongly in norm. Set* $\lambda^{-1} = \|\varphi\|_{\mathfrak{q}}$ *and* $\lambda_m^{-1} = \|\varphi_m\|_{\mathfrak{q}}$. *Then,* $\mathcal{J}(\lambda\varphi) \subseteq \mathcal{J}(\lambda_m\varphi_m)$ *for every sufficiently large integer* m.

Proof. If $f \in \mathcal{J}(\lambda\varphi)$, then $\|\varphi\|_f < (1-\varepsilon)/\lambda$ for a sufficiently small number $\varepsilon > 0$. We have $\|\varphi_m\|_f \leq \|\varphi\|_f < (1-\varepsilon)/\lambda < \lambda_m^{-1}$ since $\lambda_m < \lambda/(1-\varepsilon)$ for sufficiently large m. It follows that $f \in \mathcal{J}(\lambda_m\varphi_m)$ by Lemma 4.4. □

Lemma 5.7. *Let* $(R, \mathfrak{m})$ *be a local ring, let* φ *be a qpsh function on* Spec R *such that* $\varphi \geq p \log|\mathfrak{m}|$, *and let* $\mathfrak{q}$ *be a nonzero ideal on* Spec R. *Then there exists a nontrivial tempered valuation* $v \in \mathbb{V}_{R,\mathfrak{m},M}$ *which computes* $\|\varphi\|_{\mathfrak{q}}$ *provided that* $M > p \cdot \|\varphi\|_{\mathfrak{q}}^{-1}$.

Proof. Assume that $\{\varphi_m\}$ is a decreasing sequence of ideal functions which converges to ψ strongly in norm. Then $\mathfrak{m}^n \cdot \mathfrak{q} \subseteq \mathcal{J}(\lambda\varphi) \subseteq \mathcal{J}(\lambda_m\varphi_m)$ for every sufficiently large integer m by Lemma 5.6. We set $\lambda'^{-1} = \|\varphi\|_{\mathfrak{m}^n\cdot\mathfrak{q}}$ and $\lambda_m'^{-1} = \|\varphi_m\|_{\mathfrak{m}^n\cdot\mathfrak{q}}$. Note that $M > p \cdot \lambda_m$ for every sufficiently large integer m. Therefore for every sufficiently large integer m, there exists $v_m \in \mathbb{V}_{R,\mathfrak{m},M}$ that computes $\|\varphi_m\|_{\mathfrak{q}}$ by Lemma 5.5. By passing $\{\varphi_m\}$ to a subsequence, we can assume $\{v_m\}$ is a sequence of points that converges to some point $v \in \mathbb{V}_{R,\mathfrak{m},M}$. Note that

$$\begin{aligned}\frac{-\lambda\varphi(v)}{A(v)+v(\mathfrak{q})} \geq \frac{-\lambda\varphi_m(v)}{A(v)+v(\mathfrak{q})} &\geq \frac{-\lambda\varphi_m(v_n)}{A(v_n)+v_n(\mathfrak{q})} - \delta \\ &\geq 1 - \|\lambda\varphi_m - \lambda_n\varphi_n\|_{\mathfrak{q}} - \delta \\ &\geq 1 - \lambda\|\varphi_m - \varphi_n\|_{\mathfrak{q}} - \delta - (\lambda_n - \lambda)\|\varphi\|_{\mathfrak{q}},\end{aligned}$$

where the second inequality holds because the function $v \to -\lambda\varphi_m(v)/(A(v)+v(\mathfrak{q}))$ is usc. Since $\|\psi_m - \psi_n\|_{\mathfrak{q}}$, δ and $\lambda_n - \lambda$ can be chosen arbitrarily small, we have $-\lambda\psi(v)/(A(v)+v(\mathfrak{q})) \geq 1$ and the conclusion follows. □

Now we turn to treat the global case.

Proof of Theorem 5.2. Pick a generic point ξ of $V(\mathcal{J}(\lambda\varphi) : \mathfrak{q})$. Note that by Lemma 4.5(3) $\|\varphi\|_{\mathfrak{q}} = \|\varphi_\xi\|_{\mathfrak{q}\cdot\mathcal{O}_{X,\xi}}$. After replacing X and φ by $\operatorname{Spec}\mathcal{O}_{X,\xi}$ and φ_ξ, respectively, we reduce the global case to the local case. After replacing φ by $\max\{\varphi, p\log|\mathfrak{m}|\}$ for a sufficiently large integer p by Lemma 5.4, we can assume that $\varphi \geq p\log|\mathfrak{m}|$. Finally by Lemma 5.7, there exists a valuation $v \in \mathbb{V}_{X,\xi,M}$ that computes $\|\varphi\|_{\mathfrak{q}}$. □

An immediate consequence of Theorem 5.2 is the following corollary.

Corollary 5.8. *Let φ be a qpsh function on X. Then, on every open subset U, we can explicitly write*

$$\Gamma(U, \mathcal{J}(\varphi)) = \{f \in \Gamma(U, \mathcal{O}_X) \mid v(f) + A(v) + \varphi(v) > 0 \textit{ for every } v \in \mathrm{V}_U^*\}.$$

Let $\mathfrak{q}$ be a nonzero ideal on X. Then, $\mathfrak{q} \subseteq \mathcal{J}(\varphi)$ if and only if $v(\mathfrak{q}) + A(v) + \varphi(v) > 0$ for every $v \in \mathrm{V}_X^*$.

The following conjecture was raised as [JM12, Conjecture B] (cf. [JM12, Theorem 7.8]). It is already known for several special cases (see [JM12, Sections 8 and 9]).

Conjecture 5.9. *Let φ be a qpsh function on X and $\mathfrak{q}$ be a nonzero ideal on X. Then there exists a nontrivial quasimonomial valuation v which computes $\|\varphi\|_{\mathfrak{q}}$. Conversely, if a nontrivial tempered valuation v computes the norm of some qpsh function, then v is quasimonomial.*

6. Applications

If X is a smooth complex projective variety, we are interested in associating a qpsh function to a line bundle that plays the role of a semipositive singular metric. The starting point is the following easy observation. Given a pseudo-effective line bundle L, an ideal $\mathfrak{a}$ together with a nonnegative rational number λ such that $L \otimes \mathfrak{a}^\lambda$ is semi-ample corresponds to a semipositive singular metric h in the sense that they give the same multiplier ideals $\mathcal{J}(\mathfrak{a}^{\lambda m}) = \mathcal{J}(h^{\otimes m})$ for every integer $m > 0$. However in general, this correspondence becomes quite mysterious since many analogue notions cannot be constructed. This has been studied in many relevant references [Boucksom 2004; Ein et al. 2006; 2009; Ein and Popa 2008; Lehmann 2011; Nakayama 2004]. We will discuss the qpsh function associated to a line bundle in detail within this section. Besides, it might be possible to generalize the results in this section to varieties with mild singularities such as klt singularities (see [Boucksom et al. 2012d; Boucksom et al. 2013b]).

Throughout this section X will be a projective smooth variety over $\mathbb{C}$ for simplicity. The term "divisor" will always refer to a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Given a section $s \in H^0(X, L)$ of a line bundle, the notation $\log|s|$ denotes the qpsh function defined locally by a regular function corresponding to s.

D-psh functions.

Definition 6.1. Let D be a divisor. We define the set

$$\mathcal{L}_D := \left\{ \frac{\log|\mathfrak{a}|}{k} \,\middle|\, kmD \otimes \mathfrak{a}^m \text{ is globally generated for every sufficiently divisible } m \right\}.$$

We then define set of D-psh functions to be the closure $\mathrm{PSH}(D) = \overline{\mathcal{L}_D}$ in norm.

Lemma 6.2. (1) $\mathrm{PSH}(D)$ *is compact and convex in* $\mathrm{QPSH}(X)$.

(2) $\mathrm{PSH}(tD) = t\mathrm{PSH}(D)$ *for any* $t \in \mathbb{Q}_{>0}$.

(3) $\mathrm{PSH}(D) + \mathrm{PSH}(D') \subseteq \mathrm{PSH}(D + D')$.

(4) *If A is a semiample divisor*, *then* $\mathrm{PSH}(D) \subseteq \mathrm{PSH}(D + A)$.

Proof. We firstly prove (1). To prove that $\mathrm{PSH}(D)$ is convex, it suffices to show that $\mathcal{L}_D$ is convex. Given qpsh functions $\varphi, \varphi' \in \mathcal{L}_D$ and a rational number $0 < \lambda < 1$, we will show that $\lambda\varphi + (1-\lambda)\varphi' \in \mathcal{L}_D$. If we write $\varphi = \frac{1}{k}\log|\mathfrak{a}|$, $\varphi' = \frac{1}{k'}\log|\mathfrak{a}'|$ and $\lambda = q/p$, then

$$\begin{aligned}\lambda\varphi + (1-\lambda)\varphi' &= \frac{1}{kp}\log|\mathfrak{a}^q| + \frac{1}{k'p}\log|\mathfrak{a}'^{p-q}| \\ &= \frac{1}{kk'p}\log|\mathfrak{a}^{qk'} \cdot \mathfrak{a}'^{k(p-q)}|.\end{aligned}$$

It is easy to check that $kk'pmL \otimes \mathfrak{a}^{mqk'} \cdot \mathfrak{a}'^{mk(p-q)}$ is globally generated for every sufficiently divisible integer m and the conclusion follows. Note that (2), (3) and (4) can be proved in a similar way. $\square$

Question 6.3. Let φ be a general qpsh function. Does there exist a divisor D such that $\varphi \in \mathrm{PSH}(D)$?

Definition 6.4. For an ample divisor A, the set of pseudo D-psh functions is defined to be $\mathrm{PSH}_\sigma(D) := \bigcap_{\varepsilon>0} \mathrm{PSH}(D+\varepsilon A)$.

Note that this definition is independent of the choice of the ample divisor A, and that the set $\mathrm{PSH}_\sigma(D)$ also satisfies the properties listed in Lemma 6.2.

Theorem 6.5 (Nadel vanishing). *Let L be a line bundle on a smooth projective variety X and $L \equiv A + D$, where A is a nef and big $\mathbb{Q}$-divisor. Assume that $\varphi \in \mathrm{PSH}_\sigma(D)$. Then*

$$H^i(X, (K_X + L) \otimes \mathcal{J}(\varphi)) = 0$$

for all $i > 0$.

Proof. By Kodaira's lemma, $A - \delta E$ is ample for some effective divisor E and every sufficiently small number $\delta > 0$. If we write $\varphi_E = \log|\mathbb{O}_X(-E)|$, then by semicontinuity of multiplier ideals we have $\mathcal{J}(\varphi) = \mathcal{J}(\varphi+\delta\varphi_E)$ for every sufficiently small number $\delta > 0$. After replacing A and φ with $A-\delta E$ and $\varphi+\delta\varphi_E$, respectively, we can assume that A is ample.

By definition we can assume that there exists a sequence of ideal functions $\{\varphi_k\}$ that converges to φ strongly in norm, such that $\varphi_k \in \mathcal{L}_{D+\epsilon_k A}$ and $\epsilon_k \to 0^+$. Choose $\varepsilon \ll 1$ such that $A - \varepsilon D$ is ample. We see that $\mathcal{J}(\varphi) = \mathcal{J}((1+\varepsilon)\varphi_k)$ for every sufficiently large integer k by Lemma 4.20. Note that $(1+\varepsilon)\varphi_k \in \mathcal{L}_{(1+\varepsilon)(D+\epsilon_k A)}$. For a sufficiently large integer k, $A - \varepsilon D - (1+\varepsilon)\epsilon_k A$ is ample. After replacing A and φ by $A - \varepsilon D - (1+\varepsilon)\epsilon_k A$ and $(1+\varepsilon)\varphi_k$, respectively, we reduce to the classical Nadel vanishing (see [Lazarsfeld 2004]). ☐

As an application of this theorem, one can easily deduce the following theorem by letting $G = K_X + (n+1)H$, where H is a hypersurface of X and $n = \dim X$, with the aid of the Castelnuovo–Mumford regularity.

Theorem 6.6 (global generation). *Let D be a divisor on X. A qpsh function φ lies in $\mathrm{PSH}_\sigma(D)$ if and only if there exists a line bundle G such that $(mD+G)\otimes\mathcal{J}(m\varphi)$ is globally generated for all $m \in \mathbb{Z}_+$ with mD integral.*

Given a qpsh function φ, a positive real number λ is said to be the (higher) jumping number of φ if $\mathcal{J}((\lambda-\epsilon)\varphi) \supsetneq \mathcal{J}(\lambda\varphi)$ for every positive real number ϵ.

Definition 6.7. Let φ be a qpsh function. We define the ideal $\mathcal{J}_-(\varphi)$ to be the largest ideal in the set $\{\mathfrak{a} \mid \|\log|\mathfrak{a}| - \varphi\| \le 1\}$. One can see that $\mathcal{J}_-(\varphi)$ can be written explicitly as

$$\Gamma(U, \mathcal{J}_-(\varphi)) = \{f \in \mathbb{O}_X(U) \mid v(f) + A(v) + \varphi(v) \ge 0 \, \textit{for every } v \in V_U^*\}$$

for every open subset U.

Lemma 6.8. *If φ is D-psh for some divisor D, then the descending chain of ideals $\mathcal{J}((1-\epsilon)\varphi)$ stabilizes as $\epsilon \to 0^+$. Further, $\mathcal{J}((1-\epsilon)\varphi) = \mathcal{J}_-(\varphi)$ for $\epsilon \ll 1$. It follows that the set of its (higher) jumping numbers is discrete.*

Proof. By adding an ample divisor to D, we can assume that D is Cartier. By Theorem 6.5 and the Castelnuovo–Mumford regularity there exists an ample line bundle G such that $\mathcal{O}_X(D+G) \otimes \mathcal{J}((1-\epsilon)\varphi)$ is globally generated for $\epsilon \ll 1$. Since the descending chain of vector spaces $H^0(X, \mathcal{O}_X(D+G) \otimes \mathcal{J}((1-\epsilon)\varphi))$ will stabilize as $\epsilon \to 0^+$, the descending chain of ideals $\mathcal{J}((1-\epsilon)\varphi)$ will stabilize. The reader can find more details in [Lehmann 2011, Theorem 4.2].

Fix a sufficiently small number ϵ'. Since $\|\log|\mathcal{J}((1-\epsilon')\varphi)| - (1-\epsilon)\varphi\| < 1$ for every sufficiently small number ϵ, we see that $\|\log|\mathcal{J}((1-\epsilon')\varphi)| - \varphi\| \leq 1$. It follows that $\mathcal{J}((1-\epsilon)\varphi) \subseteq \mathcal{J}_-(\varphi)$. To prove the converse inclusion, simply notice that

$$\Gamma(U, \mathcal{J}_-(\varphi)) = \{f \in \mathcal{O}_X(U) | v(f) + A(v) + \varphi(v) \geq 0 \text{ for every } v \in V_U^*\}$$

and hence $\mathcal{J}((1-\epsilon)\varphi) \supseteq \mathcal{J}_-(\varphi)$ for $\epsilon \ll 1$ by Corollary 5.8. □

To investigate the structure of the sets $\mathrm{PSH}(D)$ and $\mathrm{PSH}_\sigma(D)$, we need the following construction. Given an integer k, a divisor D and a qpsh function φ, we define the linear system $V_m(D, \varphi, t) := \{L \in |\llcorner mD \lrcorner| \mid \frac{1}{m}\log|s_L| \leq \frac{1}{t}\log|\mathcal{J}_-(t\varphi)|\}$, where s_L is the section associated to the divisor L and $\epsilon \ll 1$. If we choose $\mathfrak{a}(D, \varphi, t)_m := \mathfrak{b}(V_m(D, \varphi, t))$, the base ideal of the linear system $V_m(D, \varphi, t)$, then $\mathfrak{a}(D, \varphi, t)_\bullet$ is a graded sequence of ideals. Moreover, for every positive rational number t, we define $\varphi_t^D := \log|\mathfrak{a}(D, \varphi, t)_\bullet|$.

Lemma 6.9. *Let D be a divisor on X and φ be a qpsh function. Then, $\varphi \in \mathrm{PSH}(D)$ if and only if $\varphi = \lim_{t\to\infty} \varphi_t^D$ pointwise.*

Proof. First assume that $\varphi \in \mathrm{PSH}(D)$. Let $\{\varphi_m\}$ be a sequence of ideal functions that converges to φ such that each $\varphi_m \in \mathcal{L}_D$. If t is not a (higher) jumping number of φ, then, by Lemma 4.20 we have

$$\mathcal{J}_-(t\varphi) = \mathcal{J}((t-\epsilon)\varphi) = \mathcal{J}((t-\epsilon+\epsilon')\varphi_m) \supseteq \mathcal{J}_-(t\varphi_m)$$

and

$$\mathcal{J}_-(t\varphi) = \mathcal{J}(t\varphi) = \mathcal{J}((t+\epsilon)\varphi_m) \subseteq \mathcal{J}_-(t\varphi_m)$$

for every sufficiently large integer m. It follows that $\mathcal{J}_-(t\varphi) = \mathcal{J}_-(t\varphi_m)$ and $\varphi_t^D = \varphi_{m,t}^D$. Note that $\varphi_{m.t}^D \geq \varphi_m$, and hence $\frac{1}{t}\log|\mathcal{J}_-(t\varphi)| \geq \varphi_t^D \geq \varphi$. If t is a (higher) jumping number, then $\varphi_t^D \geq \varphi_{t-\epsilon}^D \geq \varphi$. Therefore, we have $\|\varphi_t^D - \varphi\| \leq \frac{1}{t}$ and hence $\varphi = \lim_{t\to\infty} \varphi_t^D$.
Conversely, we assume that $\varphi = \lim_{t\to\infty} \varphi_t^D$. Since φ_t^D is algebraic from $\mathfrak{a}(D, \varphi, t)_\bullet$ for each t, φ_t^D is D-psh for every $t > 0$. Since $\frac{1}{t}\log|\mathcal{J}_-(t\varphi)| \geq \varphi_t^D$ and φ_t^D has a

decreasing subsequence, φ_t^D converges to φ strongly in norm, which implies the conclusion immediately. $\square$

For every nontrivial tempered valuation v, we define $v(\|D\|) = v(\mathfrak{a}_\bullet)$, where $\mathfrak{a}_m = \mathfrak{b}(|\llcorner mD \lrcorner|)$.

Proposition 6.10. *The set* $\mathrm{PSH}(D)$ *is closed under taking the supremum. The maximal D-psh function $\varphi_{\max}$ can be written explicitly as $\varphi_{\max}(v) = -v(\|D\|)$ for all $v \in \mathrm{V}_X^*$.*

Proof. Let φ_λ be a family of D-psh functions. By Lemma 6.9 $\varphi_\lambda = \lim_{t\to\infty} \varphi_{\lambda,t}^D$. Note that $\varphi_{\lambda,t}^D = \log|\mathfrak{a}(D, \varphi_\lambda, t)_\bullet|$, where $\mathfrak{a}(D, \varphi_\lambda, t)_m = \mathfrak{b}(V_m(D, \varphi_\lambda, t))$.
If we write $\varphi = \sup_\lambda \varphi_\lambda$, then $\mathcal{J}_-(t\varphi_\lambda) \subseteq \mathcal{J}_-(t\varphi)$ for every λ and every t. It follows that $\mathfrak{b}(V_m(D, \varphi_\lambda, t)) \subseteq \mathfrak{b}(V_m(D, \varphi, t))$ for every m, λ and t. We deduce that $\sup_\lambda \varphi_{\lambda,t}^D \leq \varphi_t^D$ and hence

$$\varphi(v) = \sup_\lambda \lim_{t\to\infty} \varphi_{\lambda,t}^D(v) \leq \lim_{t\to\infty} \sup_\lambda \varphi_{\lambda,t}^D(v) \leq \lim_{t\to\infty} \varphi_t^D(v)$$

for every $v \in \mathrm{V}_X^*$. Note that the pointwise limits appearing in these inequalities exist because we can take decreasing subsequences which are bounded from below. Since $\frac{1}{t}\log|\mathcal{J}_-(t\varphi)| \geq \varphi_t^D$, we obtain the equality $\varphi = \lim_{t\to\infty} \varphi_t^D$ and φ is D-psh by Lemma 6.9.
Now we prove that $\varphi_{\max}(v) = -v(\|D\|)$ for all $v \in V_X^*$. Let φ be a qpsh function such that $\varphi(v) = -v(\|D\|)$. Because φ is algebraic from $\mathfrak{a}_\bullet$, where $\mathfrak{a}_m = \mathfrak{b}(|\llcorner mD \lrcorner|)$, φ is D-psh. It suffices to show that $\varphi_{\max} \leq \varphi$. For each t, $\varphi_{\max,t}^D = \log|\mathfrak{a}(D, \varphi_{\max}, t)_\bullet|$, where $\mathfrak{a}(D, \varphi_{\max}, t)_m = \mathfrak{b}(V_m(D, \varphi_{\max}, t))$. It follows that $\mathfrak{a}(D, \varphi_{\max}, t)_m \subseteq \mathfrak{a}_m$ and $\varphi_{\max,t}^D \leq \varphi$. Therefore, $\varphi_{\max} = \lim_{t\to\infty} \varphi_{\max,t}^D \leq \varphi$, which forces $\varphi_{\max} = \varphi$. $\square$

For every nontrivial tempered valuation v, we define

$$\sigma_v(\|D\|) := \lim_{\varepsilon\to 0^+} v(\|D + \varepsilon A\|)$$

for some ample divisor A. Note that [Nakayama 2004] verifies that this definition is independent of the choice of the ample divisor A.

Proposition 6.11. *The set* $\mathrm{PSH}_\sigma(D)$ *is closed under taking the supremum. The maximal pseudo D-psh function $\phi_{\max}$ can be expressed as $\phi_{\max}(v) = -\sigma_v(\|D\|)$ explicitly for all $v \in V_X^*$.*

Proof. Let φ_λ be a family of pseudo D-psh functions, and let $\varphi = \sup_\lambda \varphi_\lambda$. By Theorem 6.6 there exists an ample divisor G such that $\varphi_{\lambda,k} \in \mathrm{PSH}(D + \frac{1}{k}G)$, where $\varphi_{\lambda,k} = \frac{1}{k}\log|\mathcal{J}(k\varphi_\lambda)|$. We have $\sup_\lambda \varphi_{\lambda,k} \in \mathrm{PSH}(D + \frac{1}{k}G)$ for every k by Proposition 6.10. Since $\sum_\lambda \mathcal{J}(k\varphi_\lambda) \subseteq \mathcal{J}(k\varphi)$, we have $\varphi_k \geq \sup_\lambda \varphi_{\lambda,k} \geq \varphi$. Hence

$$\varphi = \lim_{k\to\infty}(\sup\nolimits_\lambda \varphi_{\lambda,k}) \in \mathrm{PSH}_\sigma(D).$$

Now we prove that $\phi_{\max}(v) = -\sigma_v(\|D\|)$ for all $v \in V_X^*$. Let $\phi(v) = -\sigma_v(\|D\|)$, and let $\varphi_{\max}^{\epsilon}$ be the maximal $(D+\epsilon A)$-psh function for every $\epsilon \ll 1$. We see that $\phi = \lim_{\epsilon\to 0^+} \varphi_{\max}^{\epsilon}$ pointwise. Because $\varphi_{\max}^{\epsilon}$ is decreasing as $\epsilon \to 0^+$, $\mathcal{J}(m\varphi_{\max}^{\epsilon})$ form a descending chain of ideals as $\epsilon \to 0^+$ for every integer $m > 0$. If we fix an integer m and a sequence $\epsilon_1 > \epsilon_2 > \cdots$ such that $\lim_{k\to\infty} \epsilon_k = 0$, then the descending chain stabilizes when $k \gg 0$ because there exists an ample divisor G such that $mD + G$ is Cartier and $\mathcal{O}_X(mD+G) \otimes \mathcal{J}(m\varphi_{\max}^{\epsilon_k})$ is globally generated for every $k \gg 0$. It follows that $\|\varphi_{\max}^{\epsilon_k} - \varphi_{\max}^{\epsilon_{k'}}\| < \frac{1}{m}$ for all sufficiently large k and k'. Equivalently, $\varphi_{\max}^{\epsilon_k}$ form a Cauchy sequence with respect to the norm. Therefore $\varphi_{\max}^{\epsilon_k}$ converges to ϕ strongly in norm, and hence $\phi \in \mathrm{PSH}_\sigma(D)$. Note that $\phi_{\max} \le \varphi_{\max}^{\epsilon}$, and hence $\phi_{\max} \le \phi$, which implies the conclusion. □

Question 6.12 [Lehmann 2011, Question 6.15]. Is the maximal pseudo D-psh function algebraic?

Abundant divisors, introduced in [Nakayama 2004; Boucksom et al. 2013a], form a class of pseudo-effective divisors with nice asymptotic behaviour. We denote by $\kappa_\sigma(D)$ the numerical Kodaira dimension. A pseudo-effective divisor D is said to be *abundant* if $\kappa(D) = \kappa_\sigma(D)$. We present the following easy corollary for the reader's convenience.

Corollary 6.13. (1) *The set* $\mathrm{PSH}(D)$ *is nonempty if and only if D is $\mathbb{Q}$-effective.*

(2) $0 \in \mathrm{PSH}(D)$ *if and only if D is nef and abundant.*

(3) *The set* $\mathrm{PSH}_\sigma(D)$ *is nonempty if and only if D is pseudo-effective.*

(4) $0 \in \mathrm{PSH}_\sigma(D)$ *if and only if D is nef.*

(5) *Let $\varphi_{\max}$ be the maximal D-psh function, and $\phi_{\max}$ be the maximal pseudo D-psh function. Then, D is abundant if and only if $\varphi_{\max}=\phi_{\max}$.*

Proof. The first statement is trivial. The second is a consequence of the main result of [Russo 2009], and (4) follows from (2) immediately. If D is not pseudo-effective, then $\mathrm{PSH}_\sigma(D)$ is empty from (1). We prove (3) as follows. If D pseudo-effective, then $\mathrm{PSH}_\sigma(D)$ is nonempty by Proposition 6.11. To prove (5), simply notice that D is abundant if and only if $v(\|D\|) = \sigma_v(\|D\|)$ for every divisorial valuation v by [Lehmann 2011, Proposition 6.18] and the last statement follows by Propositions 6.10 and 6.11. □

Question 6.14. Assume that the divisor D is abundant. Is the set $\mathrm{PSH}(D)$ equal to the set $\mathrm{PSH}_\sigma(D)$?

We introduce the following definition of the perturbed ideal and the diminished ideal as [Lehmann 2011, Definitions 4.3 and 6.2]. We use the notation $\mathcal{J}_{\sigma,-}(D)$ instead of $\mathcal{J}_-(D)$ to avoid that readers may confuse it with the notation $\mathcal{J}_-(\varphi)$.

Definition 6.15. Let D be a pseudo-effective divisor. In the finite descending chain $\{\mathcal{J}(\|L + \frac{1}{m}A\|)\}_{m=1}^{\infty}$, we define the perturbed ideal $\mathcal{J}_{\sigma,-}(D)$ to be the smallest ideal, and we define the diminished ideal $\mathcal{J}_\sigma(D)$ to be the largest ideal in the set $\{\mathcal{J}_{\sigma,-}((1+\epsilon)D)\}_{\epsilon>0}$.

Finally, we obtain a generalization of [Lehmann 2011, Theorem 6.14].

Theorem 6.16. *Let D be a pseudo-effective divisor. Assume that $\phi_{\max}$ is the maximal pseudo D-psh function. Then, the perturbed ideal $\mathcal{J}_{\sigma,-}(D) = \mathcal{J}_{-}(\phi_{\max})$, and the diminished ideal $\mathcal{J}_\sigma(D) = \mathcal{J}(\phi_{\max})$. In particular, we can write $\mathcal{J}_\sigma(D)$ explicitly as $\Gamma(U, \mathcal{J}_\sigma(L)) = \{f \in \Gamma(U, \mathcal{O}_X) \mid v(f) + A(v) - \sigma_v(\|L\|) > 0$ for all $v \in \mathrm{V}_U^*\}$. Further, a nonzero ideal $\mathfrak{q} \subseteq \mathcal{J}_\sigma(\|L\|)$ if and only if $v(\mathfrak{q}) + A(v) - \sigma_v(\|L\|) > 0$ for all $v \in \mathrm{V}_X^*$.*

Proof. That $\mathcal{J}_{\sigma,-}(D) = \mathcal{J}_{-}(\phi_{\max})$ follows from [Lehmann 2011, Proposition 4.7]. To prove the second equality, note that by definition $\mathcal{J}_\sigma(D) = \mathcal{J}((1+\epsilon)\varphi_{\max}^\delta)$, where $\varphi_{\max}^\delta$ denotes the maximal $(D+\delta A)$-psh function for an ample divisor A, sufficiently small ϵ and sufficiently small $\delta = \delta(\epsilon)$. From the proof of Proposition 6.11, $\varphi_{\max}^\delta$ converges to $\phi_{\max}$ strongly in norm. Therefore, Lemma 4.20 asserts that $\mathcal{J}(\phi_{\max}) = \mathcal{J}((1+\epsilon)\varphi_{\max}^\delta) = \mathcal{J}_\sigma(D)$ as $\delta \to 0^+$. The last statement is obvious by Corollary 5.8. ☐

Remark 6.17. It should not be too difficult to generalize most results in this subsection from $\mathbb{Q}$-divisors to $\mathbb{R}$-divisors, that is, one can define D-psh functions for an $\mathbb{R}$-divisor D and obtain similar results.

Finite generation. The goal of this subsection is to prove the finite generation proposition below as an application of qpsh functions. For definitions and properties of different types of Zariski decompositions, divisorial algebras and modules, we refer to [Nakayama 2004].

Proposition 6.18. *Let (X, B) be a log canonical pair. Assume that $K_X + B$ is $\mathbb{Q}$-Cartier and abundant, and that $R(K_X + B)$ is finitely generated. Then, for any reflexive sheaf $\mathcal{F}$, $M_{\mathcal{F}}^p(K_X + B)$ is a finitely generated $R(K_X + B)$-module.*

Before we prove the proposition, we need a lemma.

Lemma 6.19 (global division). *Let X be a smooth projective variety of dimension n. Consider line bundles L and D, a linear system $V \subseteq |L|$ spanned by the sections $\{s_1, \dots, s_l\}$, and a D-psh function φ. If we denote by ϕ_V the L-psh function* $\max_{1\le j\le l} \log|s_j|$, *then for every integer $m \geq n+2$, any section σ in*

$$H^0(X, \mathcal{O}_X(K_X + mL + D) \otimes \mathcal{J}(m\phi_V + \varphi))$$

can be written as a linear combination $\sum_j s_j g_j$ of sections g_j in

$$H^0(X, \mathcal{O}_X(K_X + (m-1)L + D)).$$

Proof. Let $\{\varphi_k \in \mathscr{L}_D\}$ be a sequence of ideal functions that converges to φ strongly in norm. Since $\mathscr{J}(m\phi_V+\varphi_k) \supseteq \mathscr{J}(m\phi_V+\varphi)$, the section σ vanishes along the ideal $\mathscr{J}(m\phi_V+\varphi_k)$. If we denote by $\mathfrak{a}$ the base ideal $\mathfrak{b}(V)$, then $\phi_V = \log|\mathfrak{a}|$. Apply [Ein and Popa 2008, Theorem 4.1], and we deduce the conclusion. □

Remark 6.20. In the statement of the theorem just cited, one can verify that the assumption that $D \otimes \mathfrak{b}^\lambda$ is nef and abundant implies that $\lambda \log|\mathfrak{b}|$ is D-psh. Note that Lemma 6.19 is not a generalization of the theorem because we did not obtain that every g_j is in

$$H^0(X, \mathbb{O}_X(K_X+(m-1)L+D)\otimes \mathscr{J}((m-1)\phi_V+\varphi)).$$

Nonetheless, it should be possible to generalize in the sense that

$$g_j \in H^0(X, \mathbb{O}_X(K_X+(m-1)L+D)\otimes \mathscr{J}((m-1)\phi_V+\varphi)),$$

if one can develop a theory on the restriction of qpsh functions to subvarieties (see the proof of [Ein and Popa 2008, Theorem 3.2]).

Proof of Proposition 6.18. We can assume that (X, B) is log smooth of dimension n; K_X+B is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor; and $\mathscr{F} = \mathbb{O}_X(A)$ is a very ample line bundle by [Birkar 2010, Theorem 1.1]. Since $R = R(K_X+B)$ is finitely generated, after a possible truncation we can assume that R is generated by $R_1 = H^0(m_0(K_X+B))$ for some integer m_0 such that $m_0(K_X+B)$ is Cartier (see [ibid., Remarks 2.2 and 2.3]). If we set $\mathfrak{a} = \mathfrak{b}(|m_0(K_X+B)|)$ and $L := m_0(K_X+B)$, then $\phi := \log|\mathfrak{a}|$ is the maximal L-psh function. The rest of the proof is an analogue of [Demailly et al. 2013, Section 3]. Let m be a sufficiently large integer (to be specified later), and let σ be a global section of $m(K_X+B)+A$. We have

$$m(K_X+B)+A = K_X+(n+2)L+D,$$

where

$$D := B+(m-(n+2)m_0-1)\Big(K_X+B+\frac{1}{m}A\Big)+\frac{m_0(n+2)+1}{m}A.$$

Set

$$\varphi = \psi_m + (m-(n+2)m_0-1)\varphi_m,$$

where ψ_m is $(B+\frac{1}{m}(m_0(n+2)+1)A)$-psh such that $\|\psi_m\| < 1$, and φ_m is the maximal $(K_X+B+\frac{1}{m}A)$-psh function. Notice that

$$\|\log|\sigma|-(n+2)\phi-\varphi\|^+ \le \|(m_0(n+2)+1)\varphi_m-(n+2)\phi-\psi_m\|^+.$$

We will show that $(m_0(n+2)+1)\varphi_m \le (n+2)\phi$ for sufficiently large m, which implies that $\|\log|\sigma|-(n+2)\phi-\varphi\|^+ < 1$ and that by definition σ vanishes along $\mathscr{J}((n+2)\phi+\varphi)$. Since ϕ is determined on some dual complex $\Delta(Y, D)$, it suffices to prove that $(m_0(n+2)+1)\varphi_m \leq (n+2)\phi$ on $\Delta(Y, D)$. Further, we can assume that ϕ is affine on $\Delta(Y, D)$. It suffices to check the inequality at vertices because φ_m is convex on the dual complex. From the argument of Proposition 6.11, we see that $m_0\varphi_m$ converges to ϕ strongly in norm since $K_X + B$ is abundant. Therefore for sufficiently large m the inequality $(m_0(n+2)+1)/(n+2)\varphi_m \leq \phi$ holds at vertices of $\Delta(Y, D)$, and hence for every nontrivial tempered valuation. Finally, σ can be written as a linear combination $\sum_j s_j g_j$, where g_j are sections in $H^0(X, \mathbb{O}_X((m-m_0)(K_X+B)+A)$ by Lemma 6.19, which completes the proof. □

Remark 6.21. The above finite generation proposition can be proved in another way as follows. Since the conclusion that $M^p_{\mathcal{F}}(K_X + B)$ is a finitely generated $R(K_X + B)$-module is equivalent to that (X, B) has a good minimal model by [Birkar 2010, Theorem 1.3], it suffices to prove that (X, B) has a good minimal model. By [Birkar and Hu 2012, Theorem 5.3] we conclude that (X, B) has a log minimal model (X', B'). Since the positive part of the CKM-Zariski decomposition is semi-ample, the log minimal model (X, B) is good. We here give a different proof without using the minimal model theory, in particular the length of extremal rays.

Proposition 6.18 can be slightly generalized as follows.

Definition 6.22. [Birkar et al. 2010, Definitions 3.6.4 and 3.6.6] Let D be a divisor on X. A normal projective variety Z is said to be the *ample model* of D if there is a rational map $g : X \dashrightarrow Z$ and an ample $\mathbb{R}$-divisor H on Z such that if $p : W \to X$ and $q : W \to Z$ resolve g then q is a contraction and we can write $p^*D = q^*H + N$, where $N \geq 0$ is an $\mathbb{R}$-divisor and for every $B \sim_{\mathbb{Q}} p^*D$ then $B \geq N$. Let (X, B) be a pair. A normal variety Z is said to be the *log canonical model* of (X, B) if it is the ample model of $K_X + B$.

Lemma 6.23. *Let D be an abundant divisor on a normal projective variety X. Assume that D has the ample model. Then, $R(D)$ is finitely generated.*

Proof. After replacing X by a log resolution, we can assume that $g : X \dashrightarrow Z$ is a morphism and $D = P + N = g^*H + N$, where H is an ample $\mathbb{R}$-divisor on the ample model Z and $N \geq 0$ is an $\mathbb{R}$-divisor such that for every $B \sim_{\mathbb{Q}} D$ we have $B \geq N$. Note that $D = P + N$ is a CKM-Zariski decomposition. Since D is abundant, we have that $\mathrm{Fix}\|D\| = N_\sigma(D) \leq N \leq \mathrm{Fix}\|D\|$ by [Lehmann 2011, Proposition 6.18] and hence $P = P_\sigma(D)$. Furthermore, we can assume that there exist a smooth projective variety T and a big $\mathbb{Q}$-divisor G on T such that $\mu : X \to T$ is a contraction and $P_\sigma(D) = P_\sigma(\mu^*G)$ by [Lehmann 2014, Theorems 5.7 and 6.1]. It follows that Z is also the ample model of G. Notice that the rational map $h : T \dashrightarrow Z$ is birational. Therefore $H = p_*G$ is an $\mathbb{R}$-Cartier $\mathbb{Q}$-divisor and hence $\mathbb{Q}$-Cartier, which completes the proof. □

Finally, we obtain the proposition below by combining Proposition 6.18 and Lemma 6.23.

Proposition 6.24. *Let (X, B) be a log canonical pair. Assume that $K_X + B$ is $\mathbb{Q}$-Cartier and abundant, and that (X, B) has the log canonical model. Then, $R(K_X + B)$ is finitely generated. Further, for any reflexive sheaf $\mathscr{F}$, $M^p_{\mathscr{F}}(K_X + B)$ is a finitely generated $R(K_X + B)$-module.*

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QUASICONFORMAL CONJUGACY CLASSES OF PARABOLIC ISOMETRIES OF COMPLEX HYPERBOLIC SPACE

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We investigate the quasiconformal conjugacy classes of parabolic isometries acting on complex hyperbolic space. Our main result is that a screw parabolic isometry cannot be quasiconformally conjugate to a translation. This implies that a cyclic group generated by a screw parabolic isometry is not quasiconformally stable in its deformation space.

We are interested in the quasiconformal deformation theory of a complex hyperbolic quasi-Fuchsian group. We mainly focus on the case that the group is a cyclic group generated by a parabolic isometry.

We recall the definition of quasiconformal stability from Kleinian group theory (see, for instance, [Bers 1970; Kapovich 2008; Marden 1974; Maskit 1988]). Let Γ be a finitely generated discrete subgroup of the orientation-preserving isometry group $\mathrm{Isom}(\mathbb{H}^{n+1})$ acting on real hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$. Such a group Γ is called a Kleinian group. A representation $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^{n+1})$ is said to be a *deformation* if it is a discrete, faithful and type-preserving representation. The Kleinian group Γ is said to be *quasiconformally stable* if any deformation $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^{n+1})$ sufficiently near the identity deformation is obtained by a quasiconformal conjugation. That is, there is a quasiconformal mapping of the boundary at infinity, $\phi : \partial\mathbb{H}^{n+1} \to \partial\mathbb{H}^{n+1}$, such that $\rho(g) = \phi \circ g \circ \phi^{-1}$ for any $g \in \Gamma$.

In $\mathbb{H}^2$ and $\mathbb{H}^3$, a geometrically finite Kleinian group is quasiconformally stable [Bers 1970; Marden 1974]. This is one of the fundamental results in the deformation theory of Kleinian groups. However, there is a nonelementary geometrically finite Kleinian group of hyperbolic 4-space which is not quasiconformally stable [Kim 2011]. This is mainly due to the presence of screw parabolic isometries in hyperbolic 4-space.

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Hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ has the natural boundary at infinity $\hat{\mathbb{R}}^n$. Every isometry of $\mathbb{H}^{n+1}$ extends continuously to a Möbius transformation of $\hat{\mathbb{R}}^n$ which is a finite composition of reflections in codimension-1 spheres or hyperplanes, and vice versa. On the boundary at infinity $\hat{\mathbb{R}}^n$, a parabolic isometry is Möbius conjugate to $x \mapsto Ax + e_1$ with $A \in \mathrm{SO}(n)$, $A(e_1) = e_1$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. If $A = I$, then it is called a *strictly* parabolic isometry or a *translation*; otherwise it is a *screw* parabolic isometry. There are no screw parabolic isometries if $n < 3$. This means that there is only one conformal, and hence quasiconformal, conjugacy class of parabolic isometries in lower dimensions. In $\mathbb{H}^4$, screw parabolic isometries are not quasiconformally conjugate to translations. Furthermore, there are infinitely many distinct quasiconformal conjugacy classes of screw parabolic isometries. Let Γ be a cyclic group generated by a translation. Then we can deform Γ into a cyclic group Γ' generated by a screw parabolic isometry such that Γ is arbitrary close to Γ'. Hence, the cyclic group Γ is not quasiconformally stable in its deformation space. We can generalize this to a nonelementary Kleinian group of $\mathbb{H}^4$ (see [Kim 2011] for details). On the other hand, it is known that a convex cocompact (i.e., geometrically finite without parabolic isometries) Kleinian group is quasiconformally stable in any dimension [Izeki 2000].

Now, we consider the case of complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$. A complex hyperbolic quasi-Fuchsian group is a discrete, faithful, type-preserving and geometrically finite representation of the fundamental group of a surface in the group $\mathrm{PU}(2, 1)$ of holomorphic isometries acting on complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ [Goldman 1999; Parker and Platis 2010; Schwartz 2007]. It is the complex counterpart of a Kleinian group of real hyperbolic space. The deformation space is the set of all such groups factored by the conjugation action of the holomorphic isometry group $\mathrm{PU}(2, 1)$. Naturally, we can ask if a complex hyperbolic quasi-Fuchsian group is quasiconformally stable in its deformation space (see [Parker and Platis 2010] for more related questions). To that end, we consider a cyclic group generated by a parabolic isometry of $\mathbb{H}^2_{\mathbb{C}}$.

The boundary at infinity of complex hyperbolic space can be identified with the one-point compactification of the Heisenberg group $\mathscr{H}$: $\partial\mathbb{H}^2_{\mathbb{C}} = \mathscr{H} \cup \{\infty\}$. A holomorphic isometry of $\mathbb{H}^2_{\mathbb{C}}$ extends continuously to an extended Heisenberg group automorphism of $\partial\mathbb{H}^2_{\mathbb{C}}$, and vice versa. On $\partial\mathbb{H}^2_{\mathbb{C}}$, a parabolic isometry of $\mathbb{H}^2_{\mathbb{C}}$ is conjugate to either a Heisenberg translation or the composition of a vertical translation and a rotation by an element of $\mathrm{PU}(2, 1)$. We call the latter a screw parabolic isometry.

A Heisenberg translation can be conjugate to either a horizontal translation or a vertical translation by an element of $\mathrm{PU}(2, 1)$. We can conjugate a horizontal translation (or a vertical translation) further by an element of $\mathrm{PU}(2, 1)$ so that the translation length is 1 with respect to the Cygan norm of the Heisenberg group.

Therefore, we have the following classification of conformal classes of parabolic isometries up to the conjugation action of PU(2, 1):

(1)
- a horizontal translation $T_{(1,0)}$,
- a vertical translation $T_{(0,1)}$,
- a 1-parameter family of screw parabolic isometries $\{A_\theta : \theta \in (0, 2\pi)\}$,

where

$$T_{(\zeta,\nu)} = \begin{pmatrix} 1 & -\sqrt{2}\bar{\zeta} & -|\zeta|^2 + i\nu \\ 0 & 1 & \sqrt{2}\zeta \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(2, 1) \tag{2}$$

for $\zeta \in \mathbb{C}$, $\nu \in \mathbb{R}$, and

$$A_\theta = \begin{pmatrix} 1 & 0 & i \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(2, 1) \tag{3}$$

for $\theta \in (0, 2\pi)$ (see Section 1B for details).

Miner [1994] showed that a horizontal translation and a vertical translation are not quasiconformally conjugate. That is, no quasiconformal mapping of the Heisenberg group conjugates a horizontal translation to a vertical one. We prove here that a screw parabolic isometry is not quasiconformally conjugate to a translation, as follows:

Theorem 3.3. *Let $T_{(0,1)}(z, t) = (z, t + 1)$ be a vertical translation and $A(z, t) = (e^{i\theta}z, t + 1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism of the Heisenberg group $\mathcal{H}$. Then A is not quasiconformally conjugate to $T_{(0,1)}$.*

Theorem 3.7. *Let $T_{(1,0)}(z, t) = (z + 1, t + 2\,\mathrm{Im}\,\bar{z})$ be a horizontal translation and $A_\theta(z, t) = (e^{i\theta}z, t + 1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism of the Heisenberg group $\mathcal{H}$. Then A_θ is not quasiconformally conjugate to $T_{(1,0)}$.*

A screw parabolic isometry is called *rational* if some iteration of it becomes a translation; otherwise, it is called *irrational*. For a rational screw parabolic isometry A, the *order* of A is the smallest positive integer n such that A^n becomes a translation. For the 1-parameter family of screw parabolic isometries from (1), we prove that a rational screw parabolic isometry cannot be quasiconformally conjugate to an irrational screw parabolic isometry in Corollary 3.4, that two distinct rational screw parabolic isometries are quasiconformally conjugate only if they have the same order in Corollary 3.5, and that two distinct irrational screw parabolic isometries are not quasiconformally conjugate to each other in Proposition 3.6. In summary, together with the result of [ibid.], we have the following distinct quasiconformal conjugacy classes of parabolic isometries of $\mathbb{H}^2_{\mathbb{C}}$ (compare with the list (1)):

- a horizontal translation $T_{(1,0)}$;
- a vertical translation $T_{(0,1)}$;
- a subfamily of irrational screw parabolic isometries $\{A_\vartheta : \vartheta \in (0, 2\pi) \text{ irrational}\}$;
- a subfamily of rational screw parabolic isometries $\{A_{2\pi i/n} : n = 2, 3, \dots\}$.

Let $\Gamma < \mathrm{PU}(2, 1)$ be a cyclic group generated by a vertical translation. Then we can deform Γ into a cyclic group Γ' generated by a screw parabolic isometry such that Γ is arbitrary close to Γ' with respect to the l^2 norm of $\mathrm{PU}(2, 1)$. Applying Theorem 3.3, this shows that Γ is not quasiconformally stable in its deformation space. Thus, we have:

Theorem. *Let $\Gamma < \mathrm{PU}(2, 1)$ be a cyclic group generated by a vertical translation. Then it is not quasiconformally stable in its deformation space.*

This paper is organized as follows. In Section 1, we recall some basic facts related to complex hyperbolic geometry, the Heisenberg group and the theory of quasiconformal mappings. In Section 2, we construct a family of horizontal curves in a cylindrical region and compute the modulus of the curve family. This curve family will be used to prove Theorem 3.3 in Section 3. We will also prove Theorem 3.7 in Section 3.

1. Preliminaries

Throughout this section, we use [Goldman 1999] as references for the basic definitions of complex hyperbolic geometry and [Korányi and Reimann 1985; 1995] for the theory of quasiconformal mappings.

1A. *Complex hyperbolic space.* Let $\mathbb{C}^{2,1}$ be the complex vector space $\mathbb{C}^3$ with the Hermitian form of signature (2, 1) given by

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle = \boldsymbol{w}^* J \boldsymbol{z} = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1, \tag{4}$$

where the Hermitian matrix is

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consider the following subspaces of $\mathbb{C}^{2,1}$:

$$\begin{aligned} V_- &= \{\boldsymbol{z} \in \mathbb{C}^{2,1} : \langle \boldsymbol{z}, \boldsymbol{z} \rangle < 0\}, \\ V_0 &= \{\boldsymbol{z} \in \mathbb{C}^{2,1} - \{0\} : \langle \boldsymbol{z}, \boldsymbol{z} \rangle = 0\}. \end{aligned} \tag{5}$$

Let $\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \to \mathbb{CP}^2$ be the canonical projection onto complex projective space. Then complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ is defined to be $\mathbb{P}V_-$ and the boundary at infinity $\partial\mathbb{H}^2_{\mathbb{C}}$ to be $\mathbb{P}V_0$. We define the Siegel domain model of complex hyperbolic space by considering the section defined by $z_3 = 1$. For any $z = (z_1, z_2) \in \mathbb{C}^2$, we lift the point z to $\boldsymbol{z} = (z_1, z_2, 1) \in \mathbb{C}^{2,1}$, called the *standard lift of* z. Then $\langle \boldsymbol{z}, \boldsymbol{z}\rangle = z_1 + z_2\bar{z}_2 + \bar{z}_1$. Hence the Siegel domain model of complex hyperbolic space is defined by

$$\mathbb{H}^2_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\operatorname{Re}(z_1) + |z_2|^2 < 0\}. \tag{6}$$

The boundary is the one-point compactification of the paraboloid defined by $\{(z_1, z_2) \in \mathbb{C}^2 : 2\operatorname{Re}(z_1) + |z_2|^2 = 0\}$. The standard lift of ∞ is $(1, 0, 0) \in \mathbb{C}^{2,1}$.

The *Bergman metric* ρ on $\mathbb{H}^2_{\mathbb{C}}$ is defined by

$$\cosh^2\left(\frac{\rho(z, w)}{2}\right) = \frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle\langle \boldsymbol{w}, \boldsymbol{z}\rangle}{\langle \boldsymbol{z}, \boldsymbol{z}\rangle\langle \boldsymbol{w}, \boldsymbol{w}\rangle}, \tag{7}$$

where $\boldsymbol{z}$ and $\boldsymbol{w}$ are the standard lifts of z and $w \in \mathbb{H}^2_{\mathbb{C}}$. Let $\mathrm{SU}(2, 1)$ be the group of unitary matrices which preserve the given Hermitian form with determinant 1. Then the group of holomorphic isometries of $\mathbb{H}^2_{\mathbb{C}}$ is $\mathrm{PU}(2, 1) = \mathrm{SU}(2, 1)/\{I, \omega I, \omega^2 I\}$, where $\omega = (-1 + i\sqrt{3})/2$ is a cube root of unity.

Let $z = (z_1, z_2) \in \partial\mathbb{H}^2_{\mathbb{C}}$ be a finite point with standard lift $\boldsymbol{z} = (z_1, z_2, 1)$ satisfying

$$2\operatorname{Re}(z_1) + |z_2|^2 = 0. \tag{8}$$

We write $\zeta = z_2/\sqrt{2} \in \mathbb{C}$. Then (8) implies that $2\operatorname{Re}(z_1) = -2|\zeta|^2$. We can also write $z_1 = -|\zeta|^2 + i\nu$ for $\nu \in \mathbb{R}$. Thus,

$$\boldsymbol{z} = \begin{pmatrix} -|\zeta|^2 + i\nu \\ \sqrt{2}\zeta \\ 1 \end{pmatrix} \tag{9}$$

for $\zeta \in \mathbb{C}$ and $\nu \in \mathbb{R}$. Thus, we identify the boundary $\partial\mathbb{H}^2_{\mathbb{C}}$ with the one-point compactification of $\mathbb{C} \times \mathbb{R}$. Furthermore, an element $T_{(\zeta,\nu)} \in \mathrm{SU}(2, 1)$ of (2) is the unique unipotent upper triangular matrix which sends $(0, 0) \in \mathbb{C} \times \mathbb{R}$ to the finite point $(\zeta, \nu) \in \mathbb{C} \times \mathbb{R}$. The group structure of the unipotent upper triangular matrices induces a group multiplication on $\mathbb{C} \times \mathbb{R}$, which is the Heisenberg group structure.

1B. *Heisenberg group.* The Heisenberg group $\mathscr{H}$ can be described as the set of pairs $(z, t) \in \mathbb{C} \times \mathbb{R}$ with the group multiplication

$$(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\operatorname{Im} z_1\bar{z}_2). \tag{10}$$

The *Cygan* norm on $\mathscr{H}$ is defined by $|(z, t)| = (|z|^4 + t^2)^{1/4}$, and the Cygan metric d is given by

$$d((z_1, t_1), (z_2, t_2)) = |(z_1, t_1)^{-1} \cdot (z_2, t_2)|. \tag{11}$$

The Heisenberg group $\mathcal{H}$ acts on itself by left translation: $T_{(z_0,t_0)}(z,t) = (z_0,t_0)\cdot(z,t)$ for $(z_0,t_0) \in \mathcal{H}$. A Heisenberg translation of the form $T_{(0,t)}$ for $t \in \mathbb{R}$ is called a *vertical translation*. The unitary group $U(1)$ acts by *rotations*: $(z,t) \mapsto (\lambda z, t)$ for a unit $\lambda \in \mathbb{C} - \{1\}$. *Real dilation* is defined by $(z,t) \mapsto (rz, r^2 t)$ for $r \in \mathbb{R}_+ - \{1\}$. A parabolic Heisenberg group automorphism is either a Heisenberg translation or the composition of a vertical translation and a rotation. We call the latter type *screw parabolic*. A screw parabolic automorphism $A(z,t) = (e^{i\theta}z, t+s)$, for $\theta \in (0, 2\pi)$, $s \in \mathbb{R}$, is said to be *rational* if some iteration of it becomes a Heisenberg translation. Otherwise, it is said to be *irrational*. The *Heisenberg similarity group* is generated by Heisenberg translations, rotations, and real dilations.

It is known to many people that there are two conformal conjugacy classes of Heisenberg translations. More precisely, we can conjugate a Heisenberg translation by a holomorphic isometry of $\mathbb{H}^2_{\mathbb{C}}$ to obtain a horizontal translation or a vertical translation in the following way. Let T be a nonvertical translation. We may conjugate T by a Heisenberg automorphism $m(z,t) = (\lambda e^{i\theta} z, \lambda^2 t)$ for $\lambda \in \mathbb{R}_+$, $\theta \in [0, 2\pi)$, such that

$$m \circ T \circ m^{-1} = T_{(r,s)}, \tag{12}$$

where $T_{(r,s)}(z,t) = (z+r, t+s+2r \operatorname{Im} \bar{z})$ for some real numbers r and s with $r \neq 0$. For a computation, we note that for $w \in \mathbb{C}$, $(w,t)(r,s)(-w,-t) = (r, s+4r \operatorname{Im} w)$ and hence $s + 4r \operatorname{Im} w = 0$ if $\operatorname{Im} w = -s/4r$. We conjugate both sides of (12) by a Heisenberg translation $T_{(w,t)}$ with $\operatorname{Im} w = -s/4r$ as follows:

$$T_{(w,t)} m T m^{-1} T^{-1}_{(w,t)} \;=\; T_{(w,t)} T_{(r,s)} T^{-1}_{(w,t)} \;=\; T_{(r,0)}. \tag{13}$$

Conjugating both sides of (13) by a dilation $L(z,t) = (Lz, L^2 t)$ for some $L \in \mathbb{R}_+$, we have

$$L T_{(w,t)} m T m^{-1} T^{-1}_{(w,t)} L^{-1} \;=\; L T_{(r,0)} L^{-1} \;=\; T_{(1,0)}, \tag{14}$$

where $T_{(1,0)}(z,t) = (z+1, t+2\operatorname{Im}\bar{z})$. Thus, any nonvertical translation T is conjugate to $T_{(1,0)}$ by a Heisenberg automorphism.

A screw parabolic isometry can be conjugated to $A_\theta(z,t) = (e^{i\theta}z, t+1)$, with $\theta \in (0, 2\pi)$, by an element of $\mathrm{SU}(2,1)$. In addition, two distinct normalized screw parabolic isometries are not $\mathrm{SU}(2,1)$-conjugate to each other. Therefore, we have the following classification of conformal classes of parabolic isometries up to the conjugation action of the holomorphic isometries of $\mathbb{H}^2_{\mathbb{C}}$:

- a horizontal translation $T_{(1,0)}(z,t) = (z+1, t+2\operatorname{Im}\bar{z})$;
- a vertical translation $T_{(0,1)}(z,t) = (z, t+1)$;

- a 1-parameter family of screw parabolic isometries

$$\{A_\theta(z,t) = (e^{i\theta}z, t+1) : \theta \in (0, 2\pi)\}.$$

1C. ***Quasiconformal mappings.*** Let $\phi : \mathcal{H} \to \mathcal{H}$ be a homeomorphism. We define

$$M(p,r) = \sup_{\{q:d(p,q)=r\}} d(\phi p, \phi q) \quad \text{and} \quad m(p,r) = \inf_{\{q:d(p,q)=r\}} d(\phi p, \phi q) \tag{15}$$

for $p \in \mathcal{H}$ and $r > 0$.

Definition 1.1. A homeomorphism $\phi : \mathcal{H} \to \mathcal{H}$ is called K-quasiconformal if the function

$$H(p) = \limsup_{r\to 0} \frac{M(p,r)}{m(p,r)} \tag{16}$$

for $p \in \mathcal{H}$ is uniformly bounded by K.

We also need to use the Carnot–Carathéodory metric d_{cc} for our proof with quasiconformal mappings. A smooth curve $\gamma : [0, 1] \to \mathcal{H}$ is *horizontal* if, for all $t \in [0, 1]$, its tangent vector $\dot\gamma(t)$ lies in the subspace of the tangent space spanned by the vector fields $X = \partial/\partial x + 2y\, \partial/\partial t$ and $Y = \partial/\partial y - 2x\, \partial/\partial t$ for $(x, y, t) \in \mathbb{C} \times \mathbb{R}$. We define a quadratic form g on the planes generated by vector fields X and Y such that X and Y are orthonormal. Then the Carnot–Carathéodory length of γ is given by

$$l(\gamma) = \int_0^1 g(\dot\gamma(t), \dot\gamma(t))^{1/2}\, dt \tag{17}$$

and the Carnot–Carathéodory distance d_{cc} between two points $p, q \in \mathcal{H}$ is the infimum of the Carnot–Carathéodory lengths of all horizontal curves connecting p to q.

Let Γ be a family of piecewise-C^1 horizontal curves. Denote by Σ_Γ the collection of nonnegative Borel measurable functions $\sigma : \mathcal{H} \to \mathbb{R}$ such that $\int_\gamma \sigma \geq 1$ for all $\gamma \in \Gamma$. These are the so-called *admissible functions*. Then we define the *modulus* of Γ by

$$\mathrm{M}(\Gamma) = \inf_{\sigma\in\Sigma_\Gamma} \int_{\mathcal{H}} \sigma^4\, d\mathrm{vol}. \tag{18}$$

We now relate the modulus of a curve family to a quasiconformal mapping.

Theorem 1.2 [Korányi and Reimann 1995]. *If a homeomorphism $\phi : \mathcal{H} \to \mathcal{H}$ is K-quasiconformal, then*

$$\frac{1}{K^2}\mathrm{M}(\Gamma) \leq \mathrm{M}(\phi\Gamma) \leq K^2\mathrm{M}(\Gamma) \tag{19}$$

for any curve family Γ.

The Cygan metric d and the Carnot–Carathéodory metric d_{cc} give us the same classes of quasiconformal mappings since they are bi-Lipschitz related:

Theorem 1.3 [Basmajian and Miner 1998]. *For any $p, q \in \mathcal{H}$,*

$$d(p,q) \le d_{cc}(p,q) \le \sqrt{\pi}\, d(p,q).$$

Finally, we need the following property of quasiconformal mappings.

Proposition 1.4 [Korányi and Reimann 1995]. *There exists a constant C such that for any K-quasiconformal mapping $\phi : \mathcal{H} \to \mathcal{H}$,*

$$\frac{M(p,r)}{m(p,r)} \le e^{KC}$$

for any $p \in \mathcal{H}$ and $r > 0$.

2. The modulus of a cylinder

We construct here a family of piecewise smooth horizontal curves in a cylindrical region and compute its modulus. Let $\alpha_0 : [0,1] \to \mathcal{H}$ be a piecewise smooth horizontal curve defined by $\alpha_0(t) = \alpha^1(t) * \alpha^2(t) * \alpha^3(t) * \alpha^4(t)$ (see Figure 1), where

$$\begin{aligned}
&\alpha^1(t) = (2ti, 0), && 0 \le t \le \tfrac14,\\
&\alpha^2(t) = \left(2t - \tfrac12 + \tfrac12 i,\, 2t - \tfrac12\right), && \tfrac14 \le t \le \tfrac12,\\
&\alpha^3(t) = \left(\tfrac12 + \left(\tfrac32 - 2t\right)i,\, 2t - \tfrac12\right), && \tfrac12 \le t \le \tfrac34,\\
&\alpha^4(t) = (2-2t, 1), && \tfrac34 \le t \le 1.
\end{aligned} \tag{20}$$

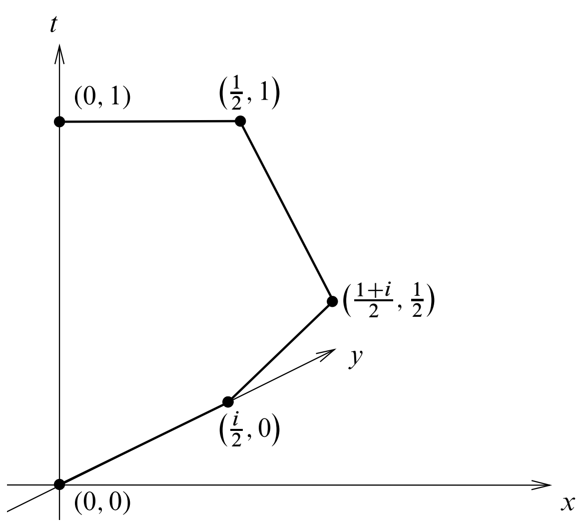

Figure 1. A piecewise smooth horizontal curve $\alpha_0(t)$.

Note that $\alpha_0(0) = (0, 0)$, $\alpha_0(1) = (0, 1)$, and $|\dot{\alpha}^i(t)| = 2$. The Carnot–Carathéodory length of α_0 is

$$l(\alpha_0) = \sum_{i=1}^{4} \int_0^{1/4} |\dot{\alpha}^i(t)|\, dt = 2.$$

Translating α_0 by $T_{(z,0)}$, for $(z, 0) \in \mathcal{H}$, produces a piecewise smooth horizontal curve α_z, given by

$$\alpha_z(t) = T_{(z,0)}\alpha_0(t), \quad 0 \le t \le 1, \tag{21}$$

from $\alpha_z(0) = (z, 0)$ to $\alpha_z(1) = (z, 1)$. Let $\alpha_z^i(t) = T_{(z,0)}\alpha^i(t)$. Then $\alpha_z(t) = \alpha_z^1(t) * \alpha_z^2(t) * \alpha_z^3(t) * \alpha_z^4(t)$, where

$$\alpha_z^1(t) = (x, y + 2t, -4xt), \qquad 0 \le t \le \tfrac{1}{4}, \tag{22}$$

$$\alpha_z^2(t) = \big(x + 2t - \tfrac{1}{2}, y + \tfrac{1}{2}, 2t - \tfrac{1}{2} - x + 4ty - y\big), \qquad \tfrac{1}{4} \le t \le \tfrac{1}{2}, \tag{23}$$

$$\alpha_z^3(t) = \big(x + \tfrac{1}{2}, y + \tfrac{3}{2} - 2t, 2t - \tfrac{1}{2} - 3x + 4tx + y\big), \qquad \tfrac{1}{2} \le t \le \tfrac{3}{4}, \tag{24}$$

$$\alpha_z^4(t) = (x + 2 - 2t, y, 1 + 2(2 - 2t)y), \qquad \tfrac{3}{4} \le t \le 1. \tag{25}$$

Since Heisenberg translations are isometries with respect to the Carnot–Carathéodory metric, all curves α_z have Carnot–Carathéodory length 2. Define a family of curves $\Gamma_{r,R}$ for $0 < r < R$ by

$$\Gamma_{r,R} = \{\alpha_z : r \le |z| \le R\}. \tag{26}$$

This family of curves defines a mapping α from the cylindrical region

$$D = \{(x, y) \in \mathbb{C} : r^2 < x^2 + y^2 < R^2\} \times [0, 1]$$

to $\mathcal{H}$, given by

$$\alpha(x, y, t) = \alpha_{x+yi}(t). \tag{27}$$

Let $D_i = \{(x, y) : r^2 < x^2 + y^2 < R^2\} \times [(i - 1)/4, i/4]$, $i = 1, 2, 3, 4$, so that $D = \bigcup_{i=1}^4 D_i$. Then the Jacobian determinant of α is given by

$$|\mathrm{J}\alpha(x, y, t)| = \begin{cases} 4|x| & \text{on } D_1, \\ 4|1 + y| & \text{on } D_2, \\ 4|1 + x| & \text{on } D_3, \\ 4|y| & \text{on } D_4. \end{cases} \tag{28}$$

Lemma 2.1. *For $1 < r < R$, we have the following lower bound for the modulus of the curve family*:

$$\mathrm{M}(\Gamma_{r,R}) \ge \tfrac{1}{256}(R^2 - r^2)\left(\frac{\pi}{2} - 2\arctan\frac{1}{\sqrt{r^2 - 1}}\right).$$

Proof. Let σ be an arbitrary admissible function in $\Sigma_{\Gamma_{\sqrt{r},\sqrt{R}}}$. By Hölder's inequality,

$$1 \leq \int_{\alpha_z} \sigma \leq \left(\int_0^1 \sigma^2(\alpha_z(t))\,dt\right)^{1/2}\left(\int_0^1 |\dot{\alpha}_z(t)|^2\,dt\right)^{1/2}. \tag{29}$$

Since $\int_0^1 |\dot{\alpha}_z(t)|^2\,dt = 2$,

$$\frac{1}{2} \leq \int_0^1 \sigma^2(\alpha_z(t))\,dt = \sum_{i=1}^{4}\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^2(\alpha_z^i(t))\,dt. \tag{30}$$

Applying Hölder's inequality to each term of the right-hand side, we have

$$\begin{aligned}\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^2(\alpha_z^i(t))|\mathrm{J}\alpha|^{1/2}\frac{1}{|\mathrm{J}\alpha|^{1/2}}\,dt \\ \leq \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))|\mathrm{J}\alpha|\,dt\right)^{1/2}\left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \frac{1}{|\mathrm{J}\alpha|}\,dt\right)^{1/2}.\end{aligned} \tag{31}$$

From (30), using the Jacobian determinant (28) and (31), we have

$$\begin{aligned}\frac{1}{2} &\leq \sum_{i=1}^{4}\left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{\frac{1}{2}}\left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \frac{1}{|\mathrm{J}\alpha|}\,dt\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^{4}\left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{\frac{1}{2}}\right)\left(\sum_{i=1}^{4}\left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \frac{1}{|\mathrm{J}\alpha|}\,dt\right)^{\frac{1}{2}}\right) \\ &\leq \frac{1}{4}\left(\sum_{i=1}^{4}\left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{\frac{1}{2}}\right)\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x+1|}}+\frac{1}{\sqrt{|y|}}+\frac{1}{\sqrt{|y+1|}}\right).\end{aligned} \tag{32}$$

Thus,

$$\begin{aligned}2\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x+1|}}+\frac{1}{\sqrt{|y|}}+\frac{1}{\sqrt{|y+1|}}\right)^{-1} \\ \leq \sum_{i=1}^{4}\left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{1/2} \leq 4\left(\sum_{i=1}^{4}\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{1/2}.\end{aligned} \tag{33}$$

Equations (31), (32) and (33) only hold if $|\mathrm{J}\alpha| \neq 0$. However, when we estimate a lower bound of the modulus in (35), we will restrict the domain of the integration so that we may assume $|\mathrm{J}\alpha| \neq 0$.

Using the trivial inequality

$$4\int_{\alpha(D)} \sigma^4\,d\mathrm{vol} \geq \sum_{i=1}^{4}\int_{\alpha(D_i)} \sigma^4\,d\mathrm{vol} \tag{34}$$

and defining $U = \{x, y : r^2 \le x^2 + y^2 \le R^2,\ x \ge 1,\ y \ge 1\}$, we have

$$
\begin{aligned}
(35)\qquad \int_{\mathcal{H}} \sigma^4\, d\mathrm{vol} &\ge \int_{\alpha(D)} \sigma^4\, d\mathrm{vol} = \frac{1}{4}\sum_{i=1}^{4}\int_{\alpha(D_i)} \sigma^4\, d\mathrm{vol} \\
&\ge \sum_{i=1}^{4} \iint_U \int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\, |\mathrm{J}\alpha|\, dt\, dx\, dy \\
&\ge \iint_U \frac{1}{4}\Big(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x+1|}} + \frac{1}{\sqrt{|y|}} + \frac{1}{\sqrt{|y+1|}}\Big)^{-2} dx\, dy \\
&\ge \tfrac{1}{256}\,\mathrm{Area}(U);
\end{aligned}
$$

the third inequality follows from (33) and the fact that $d\mathrm{vol} = 4\, dx\, dy\, dt$; for the last inequality we argue as follows:

$$
\begin{aligned}
&\Big(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x+1|}} + \frac{1}{\sqrt{|y|}} + \frac{1}{\sqrt{|y+1|}}\Big)^{-2} \\
&\ge \bigg(\frac{\sqrt{|x(x+1)y(y+1)}}{\sqrt{|(x+1)y(y+1)|}+\sqrt{|xy(y+1)|}+\sqrt{|x(x+1)(y+1)|}+\sqrt{|x(x+1)y|}}\bigg)^2 \\
&\ge \bigg(\frac{\sqrt{|x(x+1)y(y+1)|}}{4\sqrt{|(x+1)y(y+1)| + |xy(y+1)| + |x(x+1)(y+1)| + |x(x+1)y|}}\bigg)^2 \\
&\ge \frac{1}{16}\cdot\frac{|x(x+1)y(y+1)|}{|(x+1)y(y+1)| + |xy(y+1)| + |x(x+1)(y+1)| + |x(x+1)y|} \\
&\ge \frac{1}{16}\cdot\frac{x(x+1)y(y+1)}{4x(x+1)y(y+1)} = \frac{1}{64} \quad \text{if } x \ge 1,\, y \ge 1.
\end{aligned}
$$

Since σ was arbitrary, we obtain (see Figure 2)

$$
\mathrm{M}(\Gamma_{r,R}) \ge \tfrac{1}{256}(R^2 - r^2)\Big(\frac{\pi}{2} - 2\arctan\frac{1}{\sqrt{r^2-1}}\Big) \qquad \square
$$

3. Parabolic quasiconformal conjugacy classes

Throughout this section, let $A(z,t) = A_\theta(z,t) = (e^{i\theta}z, t+1)$ be a screw parabolic automorphism of the Heisenberg group $\mathcal{H}$ for $\theta \in (0, 2\pi)$, and

$$
T_{(z_0,t_0)}(z,t) = (z + z_0, t + t_0 + 2\,\mathrm{Im}\, z_0\bar{z})
$$

be a Heisenberg translation for $(z_0, t_0) \in \mathcal{H}$. An injective map $\phi : \mathcal{H} \to \mathcal{H}$ is called *quasisymmetric* if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$
(36)\qquad d(x,y) \le t\, d(x,z) \implies d(\phi x, \phi y) \le \eta(t)\, d(\phi x, \phi z)
$$

for $x, y, z \in \mathcal{H}$, $t \in [0,\infty)$.

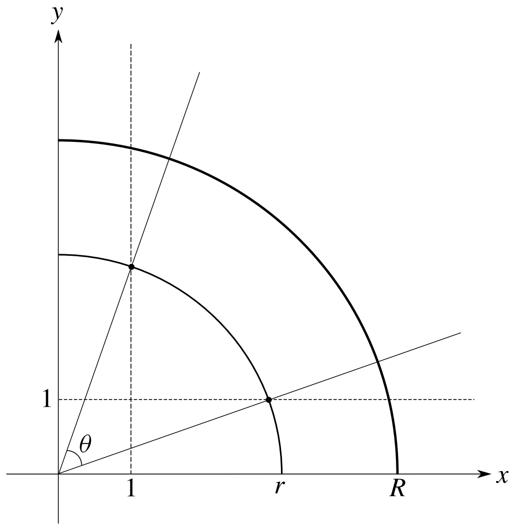

Figure 2. $\theta = \dfrac{\pi}{2} - 2\arctan\dfrac{1}{\sqrt{r^2-1}}$ (see previous page).

Theorem 3.1 [Heinonen and Holopainen 1997, Theorem 6.21]. *If $\phi : \mathcal{H} \to \mathcal{H}$ is quasiconformal, then it is quasisymmetric.*

Lemma 3.2. *Let $\phi : \mathcal{H} \to \mathcal{H}$ be a quasiconformal map that fixes all integer points $(0, n)$ on the vertical axis. Then there exist a nondecreasing function $c : [0, \infty) \to [0, \infty)$ and a constant $r_0 > 0$ satisfying:*

- $\lim_{r\to\infty} c(r) = \infty$,
- *for any $re^{i\theta} \in \mathbb{C}$ with $r > r_0$,*

$$|p(\phi(re^{i\theta}, 0))| \geq c(r), \tag{37}$$

where $p : \mathcal{H} \to \mathbb{C}$ is the vertical projection.

Proof. Throughout the proof, $[x]$ denotes the greatest integer less than or equal to x for any $x \in \mathbb{R}$ and $B(p, r)$ the ball of radius $r \geq 0$ centered at $p \in \mathcal{H}$.

We use the property that the quasiconformal map ϕ is quasisymmetric for a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ (Theorem 3.1). For any $re^{i\theta} \in \mathbb{C}$, $r > 0$,

$$\frac{d((0,0), (0, [r]^2))}{d((0,0), (re^{i\theta}, 0))} = \frac{[r]}{r} \leq 1$$

implies that

$$\frac{d(\phi(0,0), \phi(0, [r]^2))}{d(\phi(0,0), \phi(re^{i\theta}, 0))} = \frac{[r]}{d((0,0), \phi(re^{i\theta}, 0))} \leq \eta(1).$$

Thus we have

$$\frac{[r]}{\eta(1)} \leq d((0,0), \phi(re^{i\theta}, 0)), \tag{38}$$

and hence $\phi(re^{i\theta}, 0)$ lies in the complement of the ball $B((0,0), [r]/\eta(1))$.

Similarly, for any $re^{i\theta} \in \mathbb{C}$ and any integer n,

$$\frac{d((0,n),(0,0))}{d((0,n),(re^{i\theta},0))} = \frac{\sqrt{|n|}}{(r^4+n^2)^{1/4}} \leq 1 \tag{39}$$

implies

$$\frac{d(\phi(0,n),\phi(0,0))}{d(\phi(0,n),\phi(re^{i\theta},0))} = \frac{\sqrt{|n|}}{d((0,n),\phi(re^{i\theta},0))} \leq \eta(1). \tag{40}$$

Thus,

$$\frac{\sqrt{|n|}}{\eta(1)} \leq d((0,n), \phi(re^{i\theta}, 0)), \tag{41}$$

and hence $\phi(re^{i\theta}, 0)$ lies in the complement of the ball $B((0,n), \sqrt{|n|}/\eta(1))$. Since the integer n was arbitrary, the image $\phi(re^{i\theta}, 0)$ also lies in the complement of the set

$$\bigcup_{n\in\mathbb{Z}} B\left((0,n), \frac{\sqrt{|n|}}{\eta(1)}\right).$$

Therefore, together with (38), the image $\phi(re^{i\theta}, 0)$ should lie in the complement of

$$D_r = B\left((0,0), \frac{[r]}{\eta(1)}\right) \cup \bigcup_{n\in\mathbb{Z}} B\left((0,n), \frac{\sqrt{|n|}}{\eta(1)}\right).$$

Note that the t-intersects of the sphere of radius $[r]/\eta(1)$ centered at $(0,0)$ are $\pm(0, [r]^2/\eta^2(1))$. We put

$$n_r = \left[\frac{[r]^2}{\eta^2(1)}\right] \in \mathbb{N}.$$

Take a positive real number r_0 large enough that $n_{r_0} > \eta(1)$.

To finish the proof, we will show that for $r > r_0$, D_r contains an infinite cylinder

$$C_r = \{(z,t) \in \mathcal{H} : |z| \leq c(r), t \in \mathbb{R}\},$$

where $c(r)$ is a positive function such that $\lim_{r\to\infty} c(r) = \infty$. Since D_r is symmetric with respect to the z-plane of $\mathcal{H}$, it suffices to show that the upper half of D_r, denoted by $\frac{1}{2}D_r$, contains a half cylinder $\frac{1}{2}C_r = \{(z,t) \in \mathcal{H} : |z| \leq c(r), t \geq 0\}$.

Since

$$B\left((0,n), \frac{\sqrt{n_r}}{\eta(1)}\right) \subseteq B\left((0,n), \frac{\sqrt{n}}{\eta(1)}\right)$$

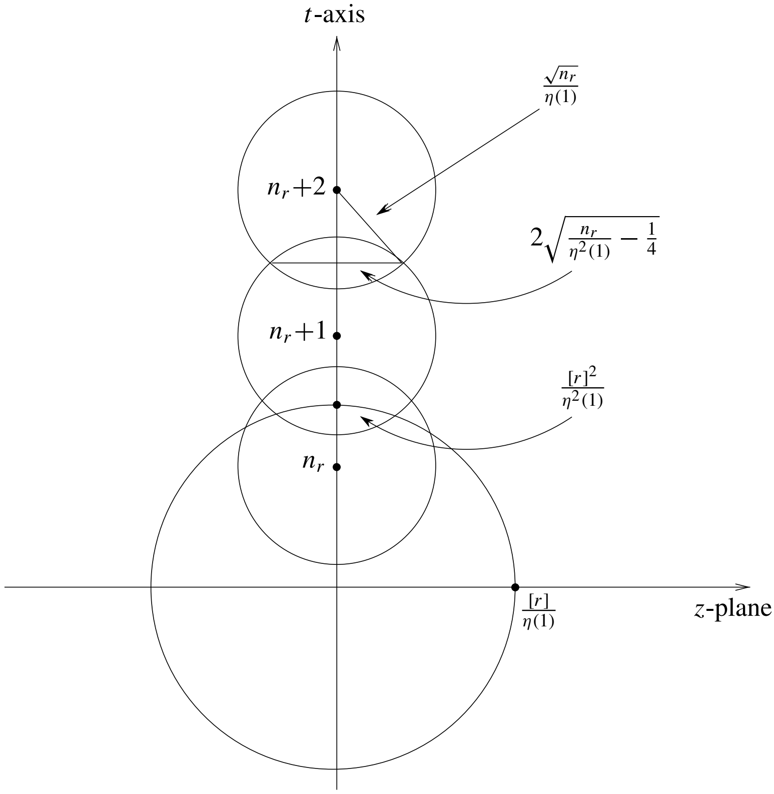

Figure 3. A set $\frac{1}{2}D'_r$.

for $n > n_r > 0$, the set $\frac{1}{2}D_r$ contains a proper subset $\frac{1}{2}D'_r$ (see Figure 3):

$$\tfrac{1}{2}D'_r = B\left((0,0), \frac{[r]}{\eta(1)}\right) \cup \bigcup_{n \geq n_r} B\left((0,n), \frac{\sqrt{n_r}}{\eta(1)}\right).$$

Take

$$c(r) = \min\left\{\sqrt{\frac{n_r}{\eta^2(1)} - \frac{1}{4}}, \frac{[r]}{\eta(1)}\right\}.$$

Then we see that $\frac{1}{2}D'_r$ contains the half cylinder $\frac{1}{2}C_r = \{(z,t) \in \mathscr{H} : |z| \leq c(r), t \geq 0\}$. Therefore, we have the lemma. □

Theorem 3.3. *Let $T_{(0,1)}(z,t) = (z, t+1)$ be a vertical translation and $A(z,t) = (e^{i\theta}z, t+1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism of the Heisenberg group $\mathscr{H}$. Then A is not quasiconformally conjugate to $T_{(0,1)}$.*

Proof. Suppose, to the contrary, that a K-quasiconformal map $\phi : \mathscr{H} \to \mathscr{H}$ exists such that

$$\phi \circ A \circ \phi^{-1} = T_{(0,1)}. \tag{42}$$

Let Γ_1 and Γ_2 be the cyclic groups generated by A and $T_{(0,1)}$, respectively. Then ϕ projects to a K-quasiconformal mapping, called ϕ again, between the quotients. That is,

$$\phi : \mathcal{H}/\Gamma_1 \to \mathcal{H}/\Gamma_2. \tag{43}$$

If ϕ does not fix $(0, 0)$, we compose (42) with a Heisenberg translation m that sends $\phi(0, 0)$ to $(0, 0)$, so that we have

$$m \circ \phi \circ A = m \circ T_{(0,1)} \circ \phi. \tag{44}$$

Because the vertical translation $T_{(0,1)}$ commutes with all Heisenberg translations,

$$m \circ \phi \circ A = T_{(0,1)} \circ m \circ \phi, \tag{45}$$

and since m is conformal, $m \circ \phi$ is also K-quasiconformal and fixes $(0, 0)$. Hence, without loss of generality we may assume that the quasiconformal mapping ϕ of (42) fixes $(0, 0)$.

Evaluating (42) at $(0, 0)$ shows that $\phi(0, 1) = (0, 1)$. By induction, ϕ fixes all integer points $\{(0, n)\}$ on the vertical axes. The global estimate of Proposition 1.4 implies that there exists a constant c_0 such that for any given integer r, there is some r' for which

$$B_{r'} \subseteq \phi(B_{\sqrt{r}}) \subseteq B_{c_0 r'}, \tag{46}$$

where B_t is a ball of radius t centered at the origin. Since the integer point $(0, r)$ is fixed by ϕ, the point $(0, r)$ also lies in $\phi(S_{\sqrt{r}})$, where S_t is the sphere of radius t centered at the origin. Hence, (46) implies that

$$r' \le \sqrt{r} \le c_0 r'. \tag{47}$$

We consider the curve family $\Gamma_{\sqrt{r},\sqrt{R}}$ from (26), where r and R are square integers satisfying $r_0 < \sqrt{r} < \sqrt{R}$ and r_0 is the constant from Lemma 3.2. We put $\Gamma = \Gamma_{\sqrt{r},\sqrt{R}}$ during this proof. All curves α_z in Γ have length $\sqrt{\pi}$ and are homotopic to the generator of $\pi_1(\mathcal{H}/\Gamma_1)$.

We now compute the modulus of the family $\phi\Gamma$ consisting of the images of curves in Γ under ϕ. For any $r > 0$, let l_r denote the Carnot–Carathéodory distance from $(r, 0)$ to $A(r, 0) = (e^{i\theta} r, 1)$. Since the Carnot–Carathéodory distance is larger than or equal to the Cygan distance (Theorem 1.3), we have

$$l_r \ge d\big((r, 0), (re^{i\theta}, 1)\big) = \Big(2^4 r^4 \sin^4 \frac{\theta}{2} + 1\Big)^{1/4}. \tag{48}$$

Since the Carnot–Carathéodory distance and the Cygan distance are invariant under Heisenberg translations, the length of any horizontal curve from (z, t) to $A(z, t) = (e^{i\theta} z, t+1)$ is at least $l_{|z|}$. Note that $\phi\Gamma$ is the family of curves connecting

$\phi(p)$ to $A\phi(p)$, where p belongs to the annulus $\{(z,0) \in \mathcal{H} : \sqrt{r} \le |z| \le \sqrt{R}\}$. Using Lemma 3.2, we see that every curve $\gamma \in \phi\Gamma$ has length at least $l_{c(\sqrt{r})}$, where $c : [0,\infty) \to [0,\infty)$ is the function from the lemma.

We denote by D the support of the curve family $\phi\Gamma$:

$$D = \phi \circ \alpha\bigl(\bigl\{z \in \mathbb{C} : \sqrt{r} < |z| < \sqrt{R}\bigr\} \times [0,1]\bigr),$$

where α is the mapping of (27). Since each curve of Γ is contained in a fundamental domain for the action of the cyclic group $\langle A\rangle$, and the quasiconformal homeomorphism ϕ conjugates A to T (see (42)), D is also contained in a fundamental domain for the action of the cyclic group $\langle T\rangle$. Note that T is the vertical translation by 1. Thus, the intersection of a vertical line with D might have several components, but the total length is bounded by 1.

Now, let $\sigma = 1/l_{c(\sqrt{r})}$ be a constant function whose support is D. Then for any $\gamma \in \phi\Gamma$,

$$\int_\gamma \sigma = \frac{1}{l_{c(\sqrt{r})}} l(\gamma) \ge 1, \tag{49}$$

and hence σ is an admissible function of $\phi\Gamma$. Therefore,

$$\mathrm{M}(\phi\Gamma) \le \int_{\mathcal{H}} \sigma^4 \, d\mathrm{vol} = \int_D \sigma^4 \, d\mathrm{vol} \le \sigma^4 \int_{p(D)} 1 \, dx \, dy, \tag{50}$$

where $p : \mathcal{H} \to \mathbb{C}$ is the vertical projection.

Since the curves in Γ belong to the ball $B_{\sqrt{R+1}}$, $D \subseteq \phi(B_{\sqrt{R+1}})$. Again, Proposition 1.4 implies that

$$B_{\widetilde{R}} \subseteq \phi(B_{\sqrt{R+1}}) \subseteq B_{c_0\widetilde{R}} \tag{51}$$

for some $\widetilde{R} > 0$. Because the integer point $(0, R+1)$ is fixed by ϕ, $(0, R+1)$ lies in the image of the sphere $\phi(S_{\sqrt{R+1}})$ and hence $\widetilde{R} \le \sqrt{R+1} \le c_0\widetilde{R}$. In particular, we have $c_0\widetilde{R} \le c_0\sqrt{R+1}$. Therefore, we have $p(D) \subseteq p(B_{c_0\sqrt{R+1}})$. From (50),

$$\begin{aligned}\sigma^4 \int_{p(D)} 1 \, dx \, dy &\le \frac{1}{l^4_{c(\sqrt{r})}} \int_{p(B_{c_0\sqrt{R+1}})} 1 \, dx \, dy \\ &= \frac{\pi c_0^2 (R+1)}{l^4_{c(\sqrt{r})}} \le \frac{\pi c_0^2 (R+1)}{2^4 c^4(\sqrt{r}) \sin^4 \frac{\theta}{2} + 1}.\end{aligned} \tag{52}$$

Now we finish the proof by deriving a contradiction. Since ϕ is K-quasiconformal,

$$M(\Gamma) \le K^2 M(\phi\Gamma). \tag{53}$$

Combining Lemma 2.1, (52), and (53), we have

$$\tfrac{1}{256}(R-r)\Big(\frac{\pi}{2}-2\arctan\frac{1}{\sqrt{r-1}}\Big)\le\frac{\pi c_0^2K^2(R+1)}{2^4c^4(\sqrt{r})\sin^4\frac{\theta}{2}+1}. \tag{54}$$

Because the square integers $r<R$ are arbitrary, we take $R=4r$. Lemma 3.2 implies that $c(\sqrt{r})\to\infty$ as $r\to\infty$, and hence we have a contradiction. □

For a positive real number $n\in\mathbb{R}_+-\{1\}$, we will denote simply by n the real dilation $(z,t)\mapsto(nz,n^2t)$ We will use the following normalization repeatedly:

$$\begin{aligned}(\sqrt{n})^{-1}A_\theta^n(\sqrt{n})(z,t)&=(\sqrt{n})^{-1}A_\theta^n(\sqrt{n}z,nt)\\&=(\sqrt{n})^{-1}(e^{ni\theta}\sqrt{n}z,nt+n)\\&=(e^{ni\theta}z,t+1)=A_{n\theta}(z,t),\end{aligned} \tag{55}$$

$$\begin{aligned}n^{-1}T_{(r,s)}n(z,t)&=n^{-1}T_{(r,s)}(nz,n^2t)\\&=n^{-1}(nz+r,n^2t+s+2rn\operatorname{Im}\bar{z})\\&=\Big(z+\frac{r}{n},t+\frac{s}{n^2}+\frac{2r}{n}\operatorname{Im}\bar{z}\Big)=T_{(r/n,s/n^2)}(z,t),\end{aligned} \tag{56}$$

where $n\in\mathbb{Z}$, $A_\theta(z,t)=(e^{i\theta}z,t+1)$ for $\theta\in[0,2\pi)$, and

$$T_{(r,s)}(z,t)=(z+r,t+s+2r\operatorname{Im}\bar{z})$$

for $r,s\in\mathbb{R}$.

Corollary 3.4. *A rational screw parabolic automorphism is not quasiconformally conjugate to an irrational screw parabolic automorphism.*

Proof. Let A_θ be a rational screw parabolic automorphism and A_ϑ be an irrational screw parabolic automorphism of $\mathcal{H}$. Suppose, to the contrary, that a K-quasiconformal map $\phi:\mathcal{H}\to\mathcal{H}$ exists such that $\phi\circ A_\vartheta\circ\phi^{-1}=A_\theta$. Then for any integer n,

$$\phi\circ A_\vartheta^n\circ\phi^{-1}=A_\theta^n. \tag{57}$$

Because A_θ is a rational screw parabolic automorphism, $A_\theta^{n_0}=T_{(0,n_0)}$ for some integer n_0. We conjugate both sides of (57) by a real dilation $\sqrt{n_0}$ and use (55) and (56) as follows:

$$\begin{aligned}(\sqrt{n_0})^{-1}\phi A_\vartheta^{n_0}\phi^{-1}\sqrt{n_0}&=(\sqrt{n_0})^{-1}T_{(0,n_0)}\sqrt{n_0},\\(\sqrt{n_0})^{-1}\phi(\sqrt{n_0}A_{n_0\vartheta}(\sqrt{n_0})^{-1})\phi^{-1}\sqrt{n_0}&=T_{(0,1)}.\end{aligned} \tag{58}$$

This implies that a screw parabolic $A_{n_0\vartheta}(z,t)=(e^{n_0\vartheta i}z,t+1)$ is conjugate to a vertical translation $T_{(0,1)}$ by a quasiconformal mapping $(\sqrt{n_0})^{-1}\phi\sqrt{n_0}$, which is a contradiction to Theorem 3.3. □

Applying the same idea as above, we also have:

Corollary 3.5. *If two rational screw parabolic automorphisms are quasiconformally conjugate, then they have the same order.*

Proposition 3.6. *Let A_θ and A_ϑ be two distinct irrational screw parabolic automorphisms for $\theta, \vartheta \in (0, 2\pi)$. Then A_θ and A_ϑ are not quasiconformally conjugate to each other.*

Proof. Using the normalization of (55), the proof follows the same idea of Proposition 4.15 of [Kim 2011]. □

We need the following theorem to prove that a screw parabolic automorphism is not quasiconformally conjugate to a horizontal translation.

Theorem F [Korányi and Reimann 1995]. *If $\{\varphi_n : G \to \hat{\mathcal{H}}\}$, for a proper subset $G \subset \mathcal{H}$, is a sequence of K-quasiconformal mappings such that every mapping φ_n omits two points a_n and b_n (depending on φ_n) with a distance at least l (l a fixed positive number independent of φ_n), then there exists a locally uniformly convergent subsequence converging to a K-quasiconformal mapping or to a constant.*

Theorem 3.7. *Let $T_{(1,0)}(z, t) = (z + 1, t + 2\operatorname{Im} \bar{z})$ be a horizontal translation and $A_\theta(z, t) = (e^{i\theta} z, t + 1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism of the Heisenberg group $\mathcal{H}$. Then A_θ is not quasiconformally conjugate to $T_{(1,0)}$.*

Proof. Suppose, to the contrary, that a K-quasiconformal map $\phi : \mathcal{H} \to \mathcal{H}$ exists such that

$$\phi \circ A_\theta \circ \phi^{-1} = T_{(1,0)}. \tag{59}$$

Then for any integer n, we also have

$$\phi \circ A_\theta^n \circ \phi^{-1} = T^n_{(1,0)} = T_{(n,0)}. \tag{60}$$

First consider the case that A_θ is a rational parabolic automorphism. Then there is a positive integer n_0 such that $A_\theta^{n_0} = T_{(0,n_0)}$. We conjugate both sides of (60) by a real dilation n as follows:

$$n^{-1}(\phi A_\theta^n \phi^{-1})n = n^{-1} T_{(n,0)} n = T_{(1,0)}. \tag{61}$$

In particular, when $n = n_0$,

$$n_0^{-1} \phi T_{(0,n_0)} \phi^{-1} n_0 = T_{(1,0)}. \tag{62}$$

Using that $T_{(0,n_0)} = \sqrt{n_0}\, T_{(0,1)} (\sqrt{n_0})^{-1}$, we rewrite the left-hand side of (62) as

$$(n_0^{-1} \phi \sqrt{n_0}) T_{(0,1)} ((\sqrt{n_0})^{-1} \phi^{-1} n_0) = T_{(1,0)}. \tag{63}$$

Because $(n_0)^{-1}\phi\sqrt{n_0}$ is also a K-quasiconformal mapping, (63) implies that the vertical translation $T_{(0,1)}$ is conjugate to the horizontal translation $T_{(1,0)}$ by the quasiconformal mapping $n_0^{-1}\phi\sqrt{n_0}$. This is a contradiction to Theorem 5.1 of [Miner 1994].

The second case is that A_θ is an irrational screw parabolic automorphism. Here we use the property that, under a mild condition, an infinite sequence of K-quasiconformal mappings is a normal family; see Theorem F.

It is possible that the quasiconformal mapping ϕ of (59) does not fix the origin $(0, 0)$. Hence, we conjugate both sides of (59) by a Heisenberg translation m which sends $\phi(0, 0)$ to $(0, 0)$ (m might be the identity map) so that we have

$$m\phi A_\theta \phi^{-1} m^{-1} = m \circ T_{(1,0)} \circ m^{-1}. \tag{64}$$

If $m \circ T_{(1,0)} \circ m^{-1}$ is a vertical translation, then we have proved the theorem. Otherwise, $mT_{(1,0)}m^{-1}$ is a nonvertical translation. Now we conjugate (64) by a rotation $\lambda : (z, t) \mapsto (\lambda z, t)$ for a unit $\lambda \in \mathbb{C}$ so that $\lambda m T_{(1,0)} m^{-1}\lambda = T_{(r,s)}$ for some real numbers $r \neq 0$ and s:

$$\lambda m\phi A_\theta \phi^{-1} m^{-1}\lambda^{-1} = \lambda m T_{(1,0)} m^{-1}\lambda^{-1} = T_{(r,s)}. \tag{65}$$

Let $\varphi = \lambda m\phi$. Then φ is a K-quasiconformal mapping, fixes the origin $(0, 0)$ and

$$\varphi \circ A_\theta \circ \varphi^{-1} = T_{(r,s)}. \tag{66}$$

(We note that if ϕ fixes $(0, 0)$, then m and λ are the identity map, $T_{(r,s)} = T_{(1,0)}$, and $\varphi = \phi$.)

Let n be any integer; then from (66), we have

$$\varphi A_\theta^n \varphi^{-1} = T_{(r,s)}^n = T_{(nr,ns)} \tag{67}$$

because r and s are real numbers. Evaluating (67) at $(0, 0)$ shows that

$$\varphi(0, n) = (nr, ns). \tag{68}$$

We conjugate both sides of (67) by a real dilation n and use equations (55) and (56) as follows:

$$\begin{aligned} n^{-1}\varphi A_\theta^n \varphi^{-1} n &= n^{-1} T_{(nr,ns)} n, \\ n^{-1}\varphi(\sqrt{n} A_{n\theta} (\sqrt{n})^{-1})\varphi^{-1} n &= T_{(r,s/n)}. \end{aligned} \tag{69}$$

Because A_θ is an irrational screw parabolic, there is a subsequence $\{A_{n_k\theta} : k \in \mathbb{N}\}$ which converges to the vertical translation $T_{(0,1)}$. For each $k \in \mathbb{N}$, let $\psi_k = n_k^{-1}\varphi\sqrt{n_k}$. Then each ψ_k is again K-quasiconformal, fixes $(0, 0)$, and

$$\psi_k A_{n_k\theta} \psi_k^{-1} = T_{(r,s/n_k)}. \tag{70}$$

To apply Theorem F, let $G = \mathcal{H} - \{(0, 0)\}$ and restrict each ψ_k on G. Thus, we have an infinite sequence of K-quasiconformal mappings, $\mathcal{F} = \{\psi_k : G \to \hat{\mathcal{H}} \mid k \in \mathbb{N}\}$. Note that each $\psi_k|_G$ omits $(0, 0)$ and ∞ in $\hat{\mathscr{H}}$. Hence, the sequence $\mathscr{F}$ has a convergent subsequence whose limit, say ψ, is a K-quasiconformal homeomorphism for the following reason: for any integer m,

$$\psi_k(0, m) = n_k^{-1} \varphi_{\sqrt{n_k}}(0, m) = \left(mr, \frac{ms}{n_k}\right)$$

converges to $(mr, 0)$ as $k \to \infty$. Thus, $\psi(0, m) = (mr, 0)$ for any integer m, and hence ψ is not a constant function. We now extend ψ to $\mathscr{H}$ by defining $\psi(0, 0) = (0, 0)$. From (70), we have $\psi \circ T_{(0,1)} \circ \psi^{-1} = T_{(r,0)}$ which is a contradiction by Theorem 5.1 of [Miner 1994]. □

Corollary 3.8. *Let $T_{(1,0)}(z, t) = (z + 1, t + 2 \operatorname{Im} \bar{z})$ be a horizontal translation and $A(z, t) = (e^{i\theta} z, t + 1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism in the Heisenberg group $\mathscr{H}$. Let Γ_1 and Γ_2 be the cyclic groups generated by $T_{(1,0)}$ and A, respectively. Then there exists no quasiconformal mapping between $\mathscr{H}/\Gamma_1$ and $\mathscr{H}/\Gamma_2$. In particular, Γ_1 is not quasiconformally conjugate to Γ_2.*

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ON THE DISTRIBUTIONAL HESSIAN OF THE DISTANCE FUNCTION

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We describe the precise structure of the distributional Hessian of the distance function from a point of a Riemannian manifold. At the same time we discuss some geometrical properties of the cut locus of a point, and compare some different weak notions of the Hessian and Laplacian.

1. Introduction

Let (M, g) be an n-dimensional, smooth, complete Riemannian manifold; for any point $p \in M$, we define $d_p : M \to \mathbb{R}$ to be the distance function from p.

Such distance functions and their relatives, the Busemann functions, come into several arguments in differential geometry. With few exceptions they are not smooth in $M \setminus \{p\}$ (and are obviously singular at p), but it is easy to see that they are 1-Lipschitz and so (by Rademacher's theorem) differentiable almost everywhere, with unit gradient.

In this note we are concerned with the precise description of the distributional Hessian of d_p, having in mind the following *Laplacian and Hessian comparison theorems* (see [Petersen 1998], for instance):

Theorem 1.1. *If* (M, g) *satisfies* $\mathrm{Ric} \geq (n-1)K$ *then, considering polar coordinates around the points* $p \in M$ *and* P *in the simply connected,* n*-dimensional space* S^K *of constant curvature* $K \in \mathbb{R}$, *we have*

$$\Delta d_p(r) \leq \Delta^K d_P^K(r).$$

If the sectional curvature of (M, g) *is greater than or equal to* K, *then*

$$\mathrm{Hess}\, d_p(r) \leq \mathrm{Hess}^K\, d_P^K(r).$$

Here $\Delta^K d_P^K(r)$ *and* $\mathrm{Hess}^K\, d_P^K(r)$ *denote respectively the Laplacian and the Hessian of the distance function* $d_P^K(\,\cdot\,) = d^K(P, \cdot\,)$ *in* S^K, *at distance r from* P.

It is often stated that these inequalities actually hold on the whole manifold (M, g) in some weak sense, say in the sense of distributions, or viscosity, or barriers. Such

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results can simplify, and sometimes are necessary for, global arguments involving this comparison theorem. More generally, one often would like to use the (weak or strong) maximum principle for the Laplacian in situations where the functions involved are not smooth, for instance in Eschenburg and Heintze's proof [1984] of the splitting theorem (first proved in [Cheeger and Gromoll 1971]), or proofs of the Toponogov theorem and the soul theorem [Cheeger and Gromoll 1972; Gromoll and Meyer 1969].

To be precise, we give definitions of these notions:

Definition 1.2. Let A be a smooth, symmetric (0, 2)-tensor field on a Riemannian manifold (M, g).

- We say that a function $f : M \to \mathbb{R}$ satisfies $\operatorname{Hess} f \leq A$ in the *distributional sense* if for every smooth vector field V with compact support we have $\int_M f\, \nabla^2_{ji}(V^i V^j)\, d\mathrm{Vol} \leq \int_M A_{ij} V^i V^j\, d\mathrm{Vol}$.
- For a continuous function $f : M \to \mathbb{R}$, we say that $\operatorname{Hess} f \leq A$ at the point $p \in M$ in the *barrier sense* if for every $\varepsilon > 0$ there exists a neighborhood U_ε of the point p and a C^2-function $h_\varepsilon : U_\varepsilon \to \mathbb{R}$ such that $h_\varepsilon(p) = f(p)$, $h_\varepsilon \geq f$ in U_ε and $\operatorname{Hess} h_\varepsilon(p) \leq A(p) + \varepsilon g(p)$ as (0, 2)-tensor fields. (Such a function h_ε is called an *upper barrier*.)
- For a continuous function $f : M \to \mathbb{R}$, we say that $\operatorname{Hess} f \leq A$ at the point $p \in M$ in the *viscosity sense* if for every C^2-function h from a neighborhood U of the point p such that $h(p) = f(p)$ and $h \leq f$ in U, we have $\operatorname{Hess} h(p) \leq A(p)$.

The weak notions of the inequality $\Delta f \leq \alpha$ for some smooth function $\alpha : M \to \mathbb{R}$ are defined analogously:

- We say that a function $f : M \to \mathbb{R}$ satisfies $\Delta f \leq \alpha$ in the *distributional sense* if for every smooth, nonnegative function $\varphi : M \to \mathbb{R}$ with compact support we have $\int_M f\, \Delta\varphi\, d\mathrm{Vol} \leq \int_M \alpha\varphi\, d\mathrm{Vol}$.
- For a continuous function $f : M \to \mathbb{R}$, we say that $\Delta f \leq \alpha$ at the point $p \in M$ in the *barrier sense* if for every $\varepsilon > 0$ there exists a neighborhood U_ε of the point p and a C^2-function $h_\varepsilon : U_\varepsilon \to \mathbb{R}$ such that $h_\varepsilon(p) = f(p)$, $h_\varepsilon \geq f$ in U_ε and $\Delta h_\varepsilon(p) \leq \alpha(p) + \varepsilon$.
- For a continuous function $f : M \to \mathbb{R}$, we say that $\Delta f \leq \alpha$ at the point $p \in M$ in the *viscosity sense* if for every C^2-function h from a neighborhood U of the point p such that $h(p) = f(p)$ and $h \leq f$ in U, we have $\Delta h(p) \leq \alpha(p)$.

In this definition and the rest of this paper we have used the Einstein summation convention on repeated indices. In particular, by $\nabla^2_{ij}(V^i V^j)$ we mean $\nabla^2_{ij}(V \otimes V)^{ij}$, the function obtained by contracting twice the second covariant derivative of the tensor product $V \otimes V$.

The notion of inequalities in the barrier sense was defined by Calabi [1958] for the Laplacian (he used the terminology "weak sense" rather than "barrier sense"). He also proved the relative global "weak" Laplacian comparison theorem (see [Petersen 1998, Section 9.3]).

The notion of a viscosity solution (which is connected to the definition of inequality "in the viscosity sense"; see the Appendix) was introduced by Crandall and Lions [1983, Definition 3.2] for partial differential *equations*; the above definition for the Hessian is a generalization to a very special *system* of PDEs.

The distributional notion is useful when integrations (by parts) are involved, the other two concepts when the arguments are based on the maximum principle.

From the definitions it is easy to see that the barrier sense implies the viscosity sense; moreover, by [Ishii 1995], if $f : M \to \mathbb{R}$ satisfies $\Delta f \leq \alpha$ in the viscosity sense it also satisfies $\Delta f \leq \alpha$ as distributions, and vice versa. In the Appendix we will discuss in detail the relations between these definitions.

In the next section we will describe the distributional structure of the Hessian (and hence of the Laplacian) of d_p, which will imply the mentioned validity of the above inequalities on the whole manifold.

It is a standard fact that the function d_p is smooth in the set $M \setminus (\{p\} \cup \mathrm{Cut}_p)$, where Cut_p is the *cut locus* of the point p, which we are now going to define and state some general properties of (we keep [Gallot et al. 1990; Sakai 1996] as general references). It is anyway well known that Cut_p is a closed set of zero (canonical) measure. Hence, in the open set $M \setminus (\{p\} \cup \mathrm{Cut}_p)$ the Hessian and Laplacian of d_p are the usual ones (even seen as distributions or using other weak definitions), and all the analysis is concerned with what happens on Cut_p (the situation at the point p is straightforward, as d_p is easily seen to behave as the function $\|x\|$ at the origin of $\mathbb{R}^n$).

We let $U_p = \{v \in T_pM \mid g_p(v, v) = 1\}$ be the set of unit tangent vectors to M at p. Given $v \in U_p$, we consider the geodesic $\gamma_v(t) = \exp_p(tv)$, and we let $\sigma_v \in \mathbb{R}^+$ (or possibly equal to $+\infty$) be the maximal time such that $\gamma_v([0, \sigma_v])$ is minimal between any pair of its points. This defines a map $\sigma : U_p \to \mathbb{R}^+ \cup \{+\infty\}$, and the point $\gamma_v(\sigma_v)$ (when $\sigma_v < +\infty$) is called the *cut point* of the geodesic γ_v.

Definition 1.3. The set of all cut points $\gamma_v(\sigma_v)$ for $v \in U_p$ with $\sigma_v < +\infty$ is called the *cut locus* of the point $p \in M$.

There are two reasons why a geodesic can cease to be minimal:

Proposition 1.4. *If for a geodesic $\gamma_v(t)$ from the point $p \in M$ we have $\sigma_v < +\infty$, at least one of the following two conditions is satisfied*:

(1) *Another minimal geodesic from p arrives at the cut point $q = \gamma_v(\sigma_v v)$.*

(2) *The differential $d\exp_p$ is not invertible at the point $\sigma_v v \in T_pM$.*

Conversely, if at least one of these conditions is satisfied, the geodesic $\gamma_v(t)$ *cannot be minimal on any interval larger that* $[0, \sigma_v]$.

It is well known that the subset of points $q \in \mathrm{Cut}_p$ where more than one minimal geodesic from p arrive coincides with Sing, the singular set of the distance function d_p in $M \setminus \{p\}$. We also define Conj, the set of points $q = \gamma_v(\sigma_v) \in \mathrm{Cut}_p$ with $d\exp_p$ not invertible at $\sigma_v v \in T_pM$; we call Conj the *locus of optimal conjugate points*. See [Gallot et al. 1990; Sakai 1996].

2. The structure of the distributional Hessian of the distance function

The following properties of the function d_p and the cut locus of $p \in M$ are proved in [Mantegazza and Mennucci 2003, Section 3] (see also the wonderful [Li and Nirenberg 2005] for other fine properties, notably the local Lipschitz continuity of the function $\sigma : U_p \to \mathbb{R}^+ \cup \{+\infty\}$ in Theorem 1.1 there).

Given an open set $\Omega \subset \mathbb{R}^n$, we say that a continuous function $u : \Omega \to \mathbb{R}$ is *locally semiconcave* if for any open convex set $K \subset \Omega$ with compact closure in Ω, the function $u|_K$ is the sum of a C^2 function and a concave function.

A continuous function $u : M \to \mathbb{R}$ is called *locally semiconcave* if for any local chart $\psi : \mathbb{R}^n \to U \subset M$, the function $u \circ \psi$ is locally semiconcave in $\mathbb{R}^n$ according to the above definition.

Proposition 2.1 [Mantegazza and Mennucci 2003, Proposition 3.4]. *The function* d_p *is locally semiconcave in* $M \setminus \{p\}$.

This fact, which follows from recognizing d_p as a viscosity solution of the *eikonal equation* $|\nabla u| = 1$ (see [Mantegazza and Mennucci 2003]), has some significant consequences; we need some definitions for the precise statements.

Given a continuous function $u : \Omega \to \mathbb{R}$ and a point $q \in M$, the *superdifferential* of u at q is the subset of T_q^*M defined by

$$\partial^+ u(q) = \{d\varphi(q) \mid \varphi \in C^1(M),\, \varphi(q) - u(q) = \min_M(\varphi - u)\}.$$

For any locally Lipschitz function u, the set $\partial^+ u(q)$ is a compact convex set, almost everywhere coinciding with the differential of the function u, by Rademacher's theorem.

Proposition 2.2 [Alberti et al. 1992, Proposition 2.1]. *Let the function* $u : M \to \mathbb{R}$ *be semiconcave. Then the superdifferential* $\partial^+ u$ *is not empty at each point*; *moreover*, $\partial^+ v$ *is upper semicontinuous*, *that is*,

$$q_k \to q, \quad v_k \to v, \quad v_k \in \partial^+ u(q_k) \quad \Longrightarrow \quad v \in \partial^+ u(q).$$

In particular, *if the differential* du *exists at* **every** *point of* M, *then* $u \in C^1(M)$.

Proposition 2.3 [Alberti et al. 1992, Remark 3.6]. *The set* $\mathrm{Ext}(\partial^+ d_p(q))$ *of extremal points of the* (*convex*) *superdifferential set of* d_p *at* q *is in one-to-one correspondence with the family* $\mathcal{G}(q)$ *of minimal geodesics from* p *to* q. *In symbols*,

$$\mathcal{G}(q) = \{t \mapsto \exp_q(-vt) \mid v \in \mathrm{Ext}(\partial^+ d_p(q))\},$$

where $t \in [0, 1]$.

We now deal with the structure of the cut locus of $p \in M$. Let $\mathcal{H}^{n-1}$ denote the $(n-1)$-dimensional Hausdorff measure on (M, g) (see [Federer 1969; Simon 1983]).

Definition 2.4. We say that a subset $S \subset M$ is C^r*-rectifiable*, for $r \geq 1$, if it can be covered by a countable family of embedded C^r-submanifolds of dimension $n-1$, with the exception of a set of $\mathcal{H}^{n-1}$-measure zero. (See the references just cited for a complete discussion of the notion of rectifiability.)

Proposition 2.5 [Mantegazza and Mennucci 2003, Theorem 4.10]. *The cut locus of* $p \in M$ *is* C^∞*-rectifiable. Hence*, *its Hausdorff dimension is at most* $n-1$. *Moreover*, *for any compact subset* K *of* M, *the measure* $\mathcal{H}^{n-1}(\mathrm{Cut}_p \cap K)$ *is finite* [Li and Nirenberg 2005, Corollary 1.3].

To explain the following consequence of such rectifiability, we need to briefly introduce the theory of functions with *bounded variation*; see [Ambrosio et al. 2000; Braides 1998; Federer 1969; Simon 1983] for details. We say that a function $u : \mathbb{R}^n \to \mathbb{R}^m$ is a function with *locally bounded variation*, denoted $u \in \mathrm{BV}_{\mathrm{loc}}$, if its distributional derivative Du is a Radon measure. This notion can be easily extended to maps between manifolds using smooth local charts.

A standard result says that the derivative of a locally semiconcave function stays in $\mathrm{BV}_{\mathrm{loc}}$; in view of Proposition 2.1, this implies that the vector field ∇d_p belongs to $\mathrm{BV}_{\mathrm{loc}}$ in the open set $M \setminus \{p\}$.

Then we define the subspace of $\mathrm{BV}_{\mathrm{loc}}$ of functions (or vector fields, as before) with locally *special bounded variation*, called $\mathrm{SBV}_{\mathrm{loc}}$ (see [Ambrosio 1989a; 1989b; 1990; Ambrosio et al. 2000; Braides 1998]).

The Radon measure representing the distributional derivative Du of a function $u : \mathbb{R}^n \to \mathbb{R}^m$ with locally bounded variation can be always uniquely separated into three mutually singular measures

$$Du = \widetilde{Du} + \mathrm{J}u + \mathrm{C}u,$$

where the first term is the part absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$, $\mathrm{J}u$ is a measure concentrated on an $(n-1)$-rectifiable set and $\mathrm{C}u$, called the *Cantor part*, vanishes on subsets of Hausdorff dimension $n-1$.

The space $\mathrm{SBV}_{\mathrm{loc}}$ is defined as the class of functions $u \in \mathrm{BV}_{\mathrm{loc}}$ such that $\mathrm{C}u = 0$; that is, the Cantor part of the distributional derivative of u is zero. Again, by means of local charts, this notion is easily generalized to Riemannian manifolds.

Proposition 2.6 [Mantegazza and Mennucci 2003, Corollary 4.13]. *The* ($\mathcal{H}^{n-1}$-*almost everywhere defined*) *measurable unit vector field* ∇d_p *belongs to the space* $\mathrm{SBV}_{\mathrm{loc}}(M \setminus \{p\})$ *of vector fields with locally special bounded variation.*

An immediate consequence of this proposition is that the (0, 2)-tensor field valued distribution $\operatorname{Hess} d_p$ is actually a Radon measure with an absolutely continuous part, with respect to the canonical volume measure Vol of (M, g), concentrated in $M \setminus (\{p\} \cup \mathrm{Cut}_p)$, where d_p is a smooth function. Hence in this set $\operatorname{Hess} d_p$ coincides with the standard Hessian $\widetilde{\operatorname{Hess}}\, d_p$ times the volume measure Vol. When the dimension of M is at least two, the singular part of the measure $\operatorname{Hess} d_p$ does not "see" the singular point p; hence, it is concentrated on Cut_p and absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ restricted to Cut_p.

By the properties of rectifiable sets, at $\mathcal{H}^{n-1}$-almost every point $q \in \mathrm{Cut}_p$, there exists an $(n-1)$-dimensional *approximate tangent space* $\mathrm{ap}T_q\mathrm{Cut}_p \subset T_qM$ (in the sense of geometric measure theory; see [Federer 1969; Simon 1983] for details). To give an example, we say that a hyperplane $T \subset \mathbb{R}^n$ is the approximate tangent space to an $(n-1)$-dimensional rectifiable set $K \in \mathbb{R}^n$ at the point x_0 if $\mathcal{H}^{n-1} \llcorner T$ is the limit as $\rho \to +\infty$, in the sense of Radon measures, of the blow-up measures $\mathcal{H}^{n-1} \llcorner \rho(K - x_0)$ around the point x_0. With some technicalities, this notion can be extended also to Riemannian manifolds.

Moreover (see [Ambrosio et al. 2000]), at $\mathcal{H}^{n-1}$-almost every point $q \in \mathrm{Cut}_p$, the field ∇d_p has two distinct *approximate* (in the sense of the Lebesgue differentiation theorem) limits "on the two sides" of $\mathrm{ap}T_q\mathrm{Cut}_p \subset T_qM$, given by ∇d_p^+ and ∇d_p^-.

We want to see now that exactly two distinct geodesics and no more arrive at $\mathcal{H}^{n-1}$-almost every point of Cut_p. We underline that a stronger form of this theorem was already obtained in [Ardoy and Guijarro 2011] and [Figalli et al. 2011], concluding that the set $\mathrm{Cut}_p \setminus U$ (where U is as in the following statement) has Hausdorff dimension not greater that $n-2$.

Theorem 2.7. *There is an open set* $U \subset M$ *such that* $\mathcal{H}^{n-1}(\mathrm{Cut}_p \setminus U) = 0$ *and satisfying these conditions*:

(i) *The subset* $\mathrm{Cut}_p \cap U$ *does not contain conjugate points*; *hence the set of optimal conjugate points has* $\mathcal{H}^{n-1}$-*measure zero.*

(ii) *Exactly two minimal geodesics from* $p \in M$ *arrive at every point of* $\mathrm{Cut}_p \cap U$.

(iii) *Locally around every point of* $\mathrm{Cut}_p \cap U$ *the set* Cut_p *is a smooth* $(n-1)$-*dimensional hypersurface*; *hence* $\mathrm{ap}T_q\mathrm{Cut}_p$ *is actually the classical tangent space to a hypersurface.*

Proof. First we show that the set of optimal conjugate points Conj is a closed subset of $\mathcal{H}^{n-1}$-measure zero, then we will see that the points of Sing $\setminus$ Conj where more than two geodesics arrive also form a closed subset of $\mathcal{H}^{n-1}$-measure zero. Claim (iii) then follows by the analysis in the proof of Proposition 4.7 in [Mantegazza and Mennucci 2003].

Recalling that $U_p = \{v \in T_pM \mid g_p(v, v) = 1\}$ is the set of unit tangent vectors to M at p, we define the function $c : U_p \to \mathbb{R}^+ \cup \{+\infty\}$ such that the point $\gamma_v(c_v)$ is the first conjugate point (if it exists) along the geodesic γ_v; that is, the differential $d\exp_p$ is not invertible at the point $c_v v \in T_pM$. By Lemma 4.11 and the proof of Proposition 4.9 in [Mantegazza and Mennucci 2003], in the open subset $V \subset U_p$ where the rank of the differential of the map $F : U_p \to M$ defined by $F(v) = \exp_p(c_v v)$ is $n-1$, the map $c : U_p \to \mathbb{R}^+ \cup \{+\infty\}$ is smooth; hence $F(V)$ is locally a smooth hypersurface. Since, by Sard's theorem, the image of $U_p \setminus V$ is a closed set of $\mathcal{H}^{n-1}$-measure zero, we only have to deal with the images $F(v)$ of unit vectors $v \in V$ such that $c_v = \sigma_v$ (see end of the introduction), that is, with $F(V) \cap \mathrm{Cut}_p$, which is a closed set.

We then consider the set

$$D \subset (F(V) \cap \mathrm{Cut}_p)$$

of points q where $\mathrm{ap}T_q\mathrm{Cut}_p$ exists and the *density* of the rectifiable set $F(V) \cap \mathrm{Cut}_p$ in the cut locus of the point p with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ is 1 (see [Federer 1969; Simon 1983]). It is well known that D and $F(V) \cap \mathrm{Cut}_p$ only differ by a set of $\mathcal{H}^{n-1}$-measure zero. If $F(v) = q \in D$, then $c_v = \sigma_v$ and, by the above density property, the hypersurface $F(V)$ is "tangent" to Cut_p at the point q; that is, $T_qF(V) = \mathrm{ap}T_q\mathrm{Cut}_p$.

We now claim that the minimal geodesic γ_v is tangent to the hypersurface $F(V)$, hence to the cut locus, at the point q. Indeed, since $d\exp_p$ is not invertible at $c_v v \in T_pM$, by the Gauss lemma there exists a vector $w \in T_vU_p$ such that $d\exp_p[c_v v](w) = 0$, hence

$$dF_v(w) = (dc[v](w))\dot{\gamma}_v(c_v) + d\exp_p[c_v v](c_v w) = (dc[v](w))\dot{\gamma}_v(c_v);$$

thus, $\dot{\gamma}_v(c_v)$ belongs to the tangent space $dF(T_vU_p)$ to the hypersurface $F(V)$ at the point q, which coincides with $\mathrm{ap}T_q\mathrm{Cut}_p$, as we claimed.

By the properties of SBV functions described before, at $\mathcal{H}^{n-1}$-almost every point $q \in D$, the blow-up of the function d_p is a "roof", meaning that exactly two minimal geodesics arrive at q, both intersecting the cut locus transversally (the vectors ∇d_p^+ and ∇d_p^- do not belong to $\mathrm{ap}T_qM$); hence the above minimal geodesic γ_v cannot coincide with any of these two.

We then conclude that $\mathcal{H}^{n-1}(D) = 0$, and the same for the set Conj.

Now suppose that $q \in \mathrm{Cut}_p \setminus \mathrm{Conj} \subset \mathrm{Sing}$; by the analysis in the proof of Proposition 4.7 in [Mantegazza and Mennucci 2003] (and Lemma 4.8), a *finite* number $m \geq 2$ of distinct minimal geodesics arrive at the point q, and when $m > 2$ the cut locus of p is given by the union of at least m smooth hypersurfaces with Lipschitz boundary going through the point q. In particular, the above blow-up at q cannot be a single hyperplane $\mathrm{ap}T_q\mathrm{Cut}_p$. By the preceding discussion, the set of such points with $m > 2$ is then of $\mathcal{H}^{n-1}$-measure zero; moreover, by Propositions 2.2 and 2.3, the set of points in $\mathrm{Cut}_p \setminus \mathrm{Conj}$ with only two minimal geodesics is open, and we are done. □

Remark 2.8. In the special two-dimensional and analytic case, more can be said: the number of optimal conjugate points is locally finite and the cut locus is a locally finite graph with smooth edges; see [Myers 1935; 1936]. We conjecture that in general the set of optimal conjugate points is an $(n-2)$-dimensional rectifiable set.

By Theorem 2.7(iii), in the open set U the two side limits ∇d_p^+ and ∇d_p^- of the gradient field ∇d_p are actually smooth and classical limits; moreover, there is a locally defined smoothly varying unit normal vector $\nu_q \in T_qM$ orthogonal to $T_q\mathrm{Cut}_p$, with the convention that $g_q(\nu_q, v)$ is positive for every vector $v \in T_qM$ belonging to the half-space corresponding to the side associated to ∇d_p^+. Hence, since $\mathcal{H}^{n-1}(\mathrm{Cut}_p \setminus U) = 0$, we have a precise description of the singular jump part as follows:

$$\mathrm{J}\nabla d_p = -((\nabla d_p^+ - \nabla d_p^-) \otimes \nu)\, \mathcal{H}^{n-1} \llcorner \mathrm{Cut}_p,$$

and, noticing that the jump in the gradient of d_p in U must be orthogonal to the tangent space $T_q\mathrm{Cut}_p$, and thus parallel to the unit normal vector $\nu_q \in T_qM$, we conclude

$$\mathrm{J}\nabla d_p = -(\nu \otimes \nu) \left|\nabla d_p^+ - \nabla d_p^-\right|_g \mathcal{H}^{n-1} \llcorner \mathrm{Cut}_p.$$

Notice that the singular part of the distributional Hessian of d_p is a rank-1 symmetric $(0, 2)$-tensor field.

Remark 2.9. This description of the jump part of the singular measure follows directly from the structure theorem for BV functions (see [Ambrosio et al. 2000]), even if we didn't know from Theorem 2.7 that the cut locus is $\mathcal{H}^{n-1}$-almost everywhere smooth.

Theorem 2.10. *If $n \geq 2$, the distributional Hessian of the distance from a point $p \in M$ is given by the Radon measure*

$$\mathrm{Hess}\, d_p = \widetilde{\mathrm{Hess}}\, d_p \,\mathrm{Vol} - (\nu \otimes \nu) \left|\nabla d_p^+ - \nabla d_p^-\right|_g \mathcal{H}^{n-1} \llcorner \mathrm{Cut}_p,$$

where $\widetilde{\mathrm{Hess}}\, d_p$ is the standard Hessian of d_p, where it exists ($\mathcal{H}^{n-1}$-almost everywhere on M), and ∇d_p^+, ∇d_p^-, ν are defined above.

Corollary 2.11. *If $n \geq 2$, the distributional Laplacian of d_p is the Radon measure*

$$\Delta d_p = \widetilde{\Delta} d_p \operatorname{Vol} - \left|\nabla d_p^+ - \nabla d_p^-\right|_g \mathcal{H}^{n-1} \llcorner \operatorname{Cut}_p,$$

where $\widetilde{\Delta} d_p$ is the standard Laplacian of d_p, where it exists.

Corollary 2.12. *We have*

$$\Delta d_p \leq \widetilde{\Delta} d_p \operatorname{Vol}$$

and

$$\operatorname{Hess} d_p \leq \widetilde{\operatorname{Hess}}\, d_p \operatorname{Vol},$$

as $(0, 2)$*-tensor fields. Hence the Hessian and Laplacian inequalities in Theorem 1.1 hold in the sense of distributions. Moreover*,

$$\Delta d_p \geq \widetilde{\Delta} d_p \operatorname{Vol} - 2\mathcal{H}^{n-1} \llcorner \operatorname{Cut}_p$$

and

$$\operatorname{Hess} d_p \geq \widetilde{\operatorname{Hess}}\, d_p \operatorname{Vol} - 2(\nu \otimes \nu)\, \mathcal{H}^{n-1} \llcorner \operatorname{Cut}_p \geq \widetilde{\operatorname{Hess}}\, d_p \operatorname{Vol} - 2g\, \mathcal{H}^{n-1} \llcorner \operatorname{Cut}_p,$$

as $(0, 2)$*-tensor fields.*

Remark 2.13. From their definition, it is easy to see that the same inequalities hold also for the Busemann functions; see for instance [Petersen 1998, Subsection 9.3.4] (in Section 9.3 of the same book, it is shown that the above Laplacian comparison holds on all of M in the barrier sense, while an analogous result for the Hessian can be found in Section 11.2). We stress here that Propositions 2.1, 2.2 and 2.3 about the semiconcavity and the structure of the superdifferential of the distance function d_p can also be used to show that the above inequalities hold in the barrier and viscosity senses.

Remark 2.14. Several of the conclusions of this paper also hold for the distance function from a closed *subset* of M with boundary of class at least C^3; see [Mantegazza and Mennucci 2003] for details.

Appendix: Weak definitions of sub/supersolutions of PDEs

Let (M, g) be a smooth, complete, Riemannian manifold and let A be a smooth $(0, 2)$-tensor field.

If $f : M \to \mathbb{R}$ satisfies $\operatorname{Hess} f \leq A$ at the point $p \in M$ in the barrier sense, for every $\varepsilon > 0$ there exists a neighborhood U_ε of the point p and a C^2-function $h_\varepsilon : U_\varepsilon \to \mathbb{R}$ such that $h_\varepsilon(p) = f(p)$, $h_\varepsilon \geq f$ in U_ε and $\operatorname{Hess} h_\varepsilon(p) \leq A(p) + \varepsilon g(p)$; hence, every C^2-function h from a neighborhood U of the point p such that $h(p) = f(p)$ and $h \leq f$ in U satisfies $h(p) = h_\varepsilon(p)$ and $h \leq h_\varepsilon$ in $U \cap U_\varepsilon$. It is then easy to see that

$\operatorname{Hess} h(p) \le \operatorname{Hess} h_\varepsilon(p) \le A(p) + \varepsilon g(p)$ for every $\varepsilon > 0$, hence $\operatorname{Hess} h(p) \le A(p)$. This shows that $\operatorname{Hess} f \le A$ at the point $p \in M$ also in the *viscosity sense*.

The converse is not true; indeed, it is straightforward to check that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$ satisfies $f''(0) \le 0$ in the viscosity sense but not in the barrier sense.

The same argument clearly also applies to the two definitions of $\Delta f \le \alpha$ for a smooth function $\alpha : M \to \mathbb{R}$.

Nonetheless, the notions of viscosity sense and distributional sense coincide:

Proposition A.1. *If $f : M \to \mathbb{R}$ satisfies* $\operatorname{Hess} f \le A$ *in the viscosity sense*, *it also satisfies* $\operatorname{Hess} f \le A$ *in the distributional sense*, *and vice versa. The same holds for $\Delta f \le \alpha$.*

In order to show the proposition, we recall the definitions of *viscosity* (sub/super) solutions to a second order PDE. Take a continuous map $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$, where Ω is an open subset of $\mathbb{R}^n$ and S^n denotes the space of real $n \times n$ symmetric matrices; also suppose that F satisfies the *monotonicity condition*

$$X \ge Y \quad \Longrightarrow \quad F(x, r, p, X) \le F(x, r, p, Y)$$

for every $(x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where $X \ge Y$ means that the difference matrix $X - Y$ is nonnegative definite. We consider then the second order PDE given by $F(x, f, \nabla f, \nabla^2 f) = 0$.

A continuous function $f : \Omega \to \mathbb{R}$ is said to be a *viscosity subsolution* of the above PDE if for every point $x \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $f(x) - \varphi(x) = \sup_\Omega(f - \varphi)$, we have $F(x, \varphi, \nabla\varphi, \nabla^2\varphi) \le 0$ (see [Crandall et al. 1992; Ishii 1995]). Analogously, $f \in C^0(\Omega)$ is a *viscosity supersolution* if for every point $x \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $f(x) - \varphi(x) = \inf_\Omega(f - \varphi)$, we have $F(x, \varphi, \nabla\varphi, \nabla^2\varphi) \ge 0$. If $f \in C^0(\Omega)$ is both a viscosity subsolution and supersolution, it is then a *viscosity solution* of $F(x, f, \nabla f, \nabla^2 f) = 0$ in Ω.

It is easy to see that the functions $f \in C^0(\Omega)$ such that $\Delta f \le \alpha$ in the *viscosity sense* at any point of Ω, as in Definition 1.2, coincide with the viscosity supersolutions of the equation $-\Delta f + \alpha = 0$ at the same point (here the function F is given by $F(x, r, p, X) = -\operatorname{trace} X + \alpha(x)$).

In the case of a Riemannian manifold (M, g), one works in local charts, and the operators we are interested in become

$$\operatorname{Hess}^M_{ij} f(x) = \frac{\partial^2 f(x)}{\partial x^i \partial x^j} - \Gamma^k_{ij}(x) \frac{\partial f}{\partial x^k}$$

and

$$\Delta^M f(x) = g^{ij}(x) \operatorname{Hess}^M_{ij} f(x),$$

where Γ^k_{ij} are the Christoffel symbols.

Analogously to the case of $\mathbb{R}^n$, taking

$$F(x, r, p, X) = -g^{ij}(x)X_{ij} + g^{ij}(x)\Gamma^k_{ij}(x)p_k + \alpha(x)$$

(which is a smooth function independent of the variable r), we see that, according to Definition 1.2, f satisfies $\Delta^M f \leq \alpha$ in the *viscosity sense* at any point of M if and only if it is a viscosity supersolution of the equation $F(x, f, \nabla f, \nabla^2 f) = 0$ at the same point.

Getting back to $\mathbb{R}^n$, given a linear, degenerate elliptic operator L with smooth coefficients, that is, defined by

$$Lf(x) = -a^{ij}(x)\nabla^2_{ij} f(x) + b^k(x)\nabla_k f(x) + c(x)f(x),$$

and a smooth function $\alpha : \Omega \to \mathbb{R}$, we say that $f \in C^0(\Omega)$ is a distributional supersolution of the equation $Lf + \alpha = 0$ if

$$\int_\Omega (f L^*\varphi + \alpha\varphi)\, dx \geq 0$$

for every nonnegative, smooth function $\varphi \in C^\infty_c(\Omega)$. Here L^* is the formal adjoint operator of L:

$$L^*\varphi(x) = -\nabla^2_{ji}(a^{ij}\varphi)(x) - \nabla_k(b^k\varphi)(x) + c(x)\varphi(x).$$

Under the hypothesis that the matrix of coefficients (a_{ij}) (which is nonnegative definite) has a "square root" matrix belonging to $C^1(\Omega, S^n)$, Ishii [1995] showed the equivalence of the class of continuous viscosity subsolutions and the class of continuous distributional subsolutions of the equation $Lf + \alpha = 0$. More precisely, he proved the following two theorems (see also [Lions 1983]):

Theorem A.2 [Ishii 1995, Theorem 1]. *If $f \in C^0(\Omega)$ is a viscosity subsolution of the equation $Lf + \alpha = 0$, then it is a distribution subsolution of the same equation.*

Theorem A.3 [Ishii 1995, Theorem 2]. *Assume that the "square root" of the matrix of coefficients (a_{ij}) belongs to $C^1(\Omega)$. If $f \in C^0(\Omega)$ is a distributional subsolution of the equation $Lf + \alpha = 0$, then it is a viscosity subsolution of the same equation.*

As the PDE is linear, a function $f \in C^0(\Omega)$ is a viscosity (distributional) supersolution of the equation $Lf + \alpha = 0$ if and only if the function $-f$ is a viscosity (distributional) subsolution of $L(-f) - \alpha = 0$; in the above theorems every occurrence of the term "subsolution" can replaced with "supersolution" (and also with "solution").

For simplicity, we will work in a single coordinate chart of M mapping onto $\Omega \subseteq \mathbb{R}^n$, while the general situation can be dealt with by standard partition of unity arguments.

Consider $f \in C^0(M)$ which is a viscosity supersolution of $-\Delta^M f + \alpha = 0$. It is a straightforward computation to check that this happens if and only if f is a viscosity supersolution of $-\sqrt{g}\Delta^M f + \alpha\sqrt{g} = 0$, where $\sqrt{g} = \sqrt{\det g_{ij}}$ is the density of Riemannian volume of (M, g), and vice versa. Moreover, notice that setting $L = -\sqrt{g}\Delta^M$, we have that $L^* = L$; that is, L is a self-adjoint operator. L also satisfies the hypotheses of Ishii's theorems, since the matrices g_{ij} and g^{ij} are smooth and positive definite in Ω. See [Horn and Johnson 1994, Chapter 6], in particular Example 6.2.14, for instance.

Then, in local coordinates, Ishii's theorems guarantee that f is a distributional supersolution of the same equation. That is, for each $\varphi \in C_c^\infty(\Omega)$, f satisfies

$$\int_\Omega f L^* \varphi \, dx \geq -\int_\Omega \alpha\sqrt{g}\varphi \, dx;$$

hence,

$$\int_M -f\Delta^M \varphi \, d\mathrm{Vol} = \int_\Omega -f\sqrt{g}\Delta^M \varphi \, dx \geq -\int_\Omega \alpha\sqrt{g}\varphi \, dx = -\int_M \alpha\varphi \, d\mathrm{Vol}\,.$$

This shows that then f satisfies $\Delta^M f \leq \alpha$ in the distributional sense, as in Definition 1.2.

Following these steps in reverse order, one gets the converse. Hence, the notions of $\Delta^M \leq \alpha$ in the viscosity and distributional senses coincide.

Now we turn our attention to the Hessian inequality; it is not covered directly by Ishii's theorems, which are peculiar to PDEs and do not deal with *systems* (like the general theory of viscosity solutions). For simplicity, we discuss the case of an open set $\Omega \subset \mathbb{R}^n$ (with its canonical flat metric), since all the arguments can be extended to any Riemannian manifold (M, g) by localization and introduction of the first-order correction given by Christoffel symbols, as above.

The idea is to transform the matrix inequality $\mathrm{Hess}\, f \leq A$ into a family of scalar inequalities; indeed, if everything is smooth, such an inequality is satisfied if and only if for every compactly supported, smooth vector field W we have $W^i W^j \,\mathrm{Hess}_{ij}\, f \leq A_{ij} W^i W^j$. The only price to pay is that we lose the constant coefficients of the Hessian, hence making the linear operator L^W, acting on $f \in C^2(\Omega)$ as $L^W f = -W^i W^j \,\mathrm{Hess}_{ij}\, f$, only *degenerate* elliptic. Notice that Ishii's condition in Theorem A.3 is satisfied for every smooth vector field W such that $\|W\| \in C_c^1(\Omega)$, but not by any arbitrary smooth vector field. This has the collateral effect of making the proof of the Hessian case in Proposition A.1 slightly asymmetric.

Lemma A.4. *Let $f \in C^0(\Omega)$. If for every compactly supported, smooth vector field W with $\|W\| \in C_c^1(\Omega)$, we have that f is a viscosity supersolution of the equation $-W^i W^j \,\mathrm{Hess}_{ij}\, f + A_{ij} W^i W^j = 0$, then the function f satisfies* $\mathrm{Hess}\, f \leq A$ *in the viscosity sense in all of Ω.*

Vice versa, if $f \in C^0(\Omega)$ *satisfies* $\operatorname{Hess} f \leq A$ *in the viscosity sense in* Ω, *then* f *is a viscosity supersolution of the equation* $-V^i V^j \operatorname{Hess}_{ij} f + A_{ij} V^i V^j = 0$ *for every compactly supported, smooth vector field* V.

Proof. Let us take a point $x \in \Omega$ and a C^2-function h in a neighborhood U of the point x such that $h(x) = f(x)$ and $h \leq f$. Choosing a unit vector W_x and a smooth, nonnegative function φ which is 1 at x and zero outside a small ball inside U, we consider the smooth vector field $W(y) = W_x \varphi^2(y)$ for every $y \in \Omega$, which clearly satisfies $\|W\| = \varphi \in C_c^1(\Omega)$. By the hypothesis of the first statement, the function f is then a viscosity supersolution of the equation $-W^i W^j \operatorname{Hess}_{ij} f + A_{ij} W^i W^j = 0$, which implies that $-W_x^i W_x^j \operatorname{Hess}_{ij} h(x) + A_{ij}(x) W_x^i W_x^j \geq 0$. Since this holds for every point $x \in \Omega$ and unit vector W_x, we conclude that $\operatorname{Hess} h(x) \leq A(x)$ as $(0, 2)$-tensor fields, and hence $\operatorname{Hess} f \leq A$ in the viscosity sense in Ω.

The argument to show the second statement is analogous: given a compactly supported, smooth vector field V, a point $x \in \Omega$ and a function h as above, the hypothesis implies that $-V_x^i V_x^j \operatorname{Hess}_{ij} h(x) + A_{ij}(x) V_x^i V_x^j \geq 0$, hence the thesis. □

Suppose now that $f \in C^0(\Omega)$ satisfies $\operatorname{Hess} f \leq A$ in the viscosity sense on the whole Ω; hence, by this lemma, for every compactly supported, smooth vector field V, the function f is a viscosity supersolution of the equation $-V^i V^j \operatorname{Hess}_{ij} f + A_{ij} V^i V^j = 0$. By Theorem A.2 and the subsequent discussion, it is then a distributional supersolution of the same equation; that is,

$$\int_\Omega \big[-f \nabla^2_{ji}(V^i V^j \varphi) + A_{ij} V^i V^j \varphi\big]\, dx \geq 0$$

for every nonnegative, smooth function $\varphi \in C_c^\infty(\Omega)$.

Considering a nonnegative, smooth function $\varphi \in C_c^\infty(\Omega)$ such that it is 1 on the support of the vector field V, we conclude

$$\int_\Omega f \nabla^2_{ji}(V^i V^j)\, dx \leq \int_\Omega A_{ij} V^i V^j\, dx,$$

which means that $\operatorname{Hess} f \leq A$ in the distributional sense.

Conversely, if $f \in C^0(\Omega)$ satisfies $\operatorname{Hess} f \leq A$ in the distributional sense, then for every compactly supported, smooth vector field W with $\|W\| \in C_c^1(\Omega)$ and every smooth, nonnegative function $\varphi \in C_c^\infty(\Omega)$, we define the smooth, nonnegative functions $\varphi_n = \varphi + \psi/n$, where ψ is a smooth, nonnegative and compactly supported function such that $\psi \equiv 1$ on the support of W. It follows that the vector field $V = W\sqrt{\varphi_n}$ is smooth; hence, applying the definition of $\operatorname{Hess} f \leq A$ in the distributional sense, we get

$$\int_\Omega \big[-f \nabla^2_{ji}(W^i W^j \varphi_n) + A_{ij} W^i W^j \varphi_n\big]\, dx \geq 0.$$

As $\varphi_n \to \varphi$ in $C_c^\infty(\Omega)$ and f is continuous, we can pass to the limit as $n \to \infty$ and conclude that

$$\int_\Omega \big[-f\nabla^2_{ji}(W^iW^j\varphi) + A_{ij}W^iW^j\varphi\big]\,dx \geq 0$$

for every nonnegative, smooth function $\varphi \in C_c^\infty(\Omega)$ and every compactly supported, smooth vector field W with $\|W\| \in C_c^1(\Omega)$. That is, for any vector field W as above, we have that f is a distributional supersolution of the equation $-W^iW^j\,\mathrm{Hess}_{ij}\, f + A_{ij}W^iW^j = 0$.

By Theorem A.3 and the subsequent discussion, it is then a viscosity supersolution of the same equation and, by Lemma A.4, we conclude that the function f satisfies $\mathrm{Hess}\, f \leq A$ in the viscosity sense.

Summarizing, we have the following sharp relations among the weak notions of the partial differential inequalities $\mathrm{Hess}\, f \leq A$ and $\Delta f \leq \alpha$:

barrier sense $\Longrightarrow$ viscosity sense $\Longleftrightarrow$ distributional sense.

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NOETHER'S PROBLEM FOR ABELIAN EXTENSIONS OF CYCLIC p-GROUPS

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In loving memory of my dear mother

Let K be a field and G a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K-automorphisms defined by $g \cdot x(h) = x(gh)$ for any g, $h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. Noether's problem then asks whether $K(G)$ is rational (i.e., purely transcendental) over K. The first main result of this article is that $K(G)$ is rational over K for a certain class of p-groups having an abelian subgroup of index p. The second main result is that $K(G)$ is rational over K for any group of order p^5 or p^6 (where p is an odd prime) having an abelian normal subgroup such that its quotient group is cyclic. (In both theorems we assume that if char $K \neq p$ then K contains a primitive p^e-th root of unity, where p^e is the exponent of G.)

1. Introduction

Let K be a field. A field extension L of K is called rational over K (or K-rational, for short) if $L \simeq K(x_1, \dots, x_n)$ for some integer n, with $x_1, \dots, x_n$ algebraically independent over K. Now let G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K-automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. *Noether's problem* then asks whether $K(G)$ is rational over K. This is related to the inverse Galois problem, to the existence of generic G-Galois extensions over K, and to the existence of versal G-torsors over K-rational field extensions [Swan 1983; Saltman 1982; Garibaldi et al. 2003, §33.1, p. 86]. Noether's problem for abelian groups was studied extensively by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. The reader is referred to [Swan 1983] for a survey of this problem. Fischer's theorem is a starting point of investigating Noether's problem for finite abelian groups in general.

Theorem 1.1 (Fischer [Swan 1983, Theorem 6.1]). *Let G be a finite abelian group of exponent e. Assume that* (i) *either* char $K = 0$ *or* char $K > 0$ *with* char $K \nmid e$, *and* (ii) *K contains a primitive e-th root of unity. Then $K(G)$ is rational over K.*

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On the other hand, just a handful of results about Noether's problem have been obtained when the groups are nonabelian. This is the case even when the group G is a p-group. The reader is referred to [Chu and Kang 2001; Hu and Kang 2010; Kang 2006; 2011; 2009] for previous results on Noether's problem for p-groups. The following theorem of Kang generalizes Fischer's theorem for the metacyclic p-groups.

Theorem 1.2 [Kang 2006, Theorem 1.5]. *Let G be a metacyclic p-group with exponent p^e, and let K be any field such that* (i) $\operatorname{char} K = p$, *or* (ii) $\operatorname{char} K \neq p$ *and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

The next job is to study Noether's problem for metabelian groups. Three results due to Haeuslein, Hajja and Kang, respectively, are known.

Theorem 1.3 [Haeuslein 1971]. *Let K be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H such that G/H is cyclic of prime order p*, (ii) *$\mathbb{Z}[\zeta_p]$ is a unique factorization domain*, *and* (iii) *$\zeta_{p^e} \in K$, where e is the exponent of G. If $G \to \mathrm{GL}(V)$ is any finite-dimensional linear representation of G over K, then $K(V)^G$ is rational over K.*

Theorem 1.4 [Hajja 1983]. *Let K be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H such that G/H is cyclic of order n*, (ii) *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain*, *and* (iii) *K is algebraically closed with* $\operatorname{char} K = 0$. *If $G \to \mathrm{GL}(V)$ is any finite-dimensional linear representation of G over K, then $K(V)^G$ is rational over K.*

Theorem 1.5 [Kang 2009, Theorem 1.4]. *Let K be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H such that G/H is cyclic of order n*, (ii) *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain*, *and* (iii) *$\zeta_e \in K$, where e is the exponent of G. If $G \to \mathrm{GL}(V)$ is any finite-dimensional linear representation of G over K, then $K(V)^G$ is rational over K.*

Note that those integers n for which $\mathbb{Z}[\zeta_n]$ is a unique factorization domain are determined by Masley and Montgomery.

Theorem 1.6 [Masley and Montgomery 1976]. *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain if and only if* $1 \leq n \leq 22$, *or* $n = 24, 25, 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 40, 42, 45, 48, 50, 54, 60, 66, 70, 84, 90$.

Therefore, Theorem 1.3 holds only for primes p such that $1 \leq p \leq 19$. One of the goals of our paper is to show that the this condition can be waived, under some additional assumptions regarding the structure of the abelian subgroup H.

Consider the following situation. Let G be a group of order p^n for $n \geq 2$ with an abelian subgroup H of order p^{n-1}. Bender [1927/28] determined some interesting properties of these groups. We study further the case when the p-th lower central subgroup $G_{(p)}$ is trivial. (Recall that $G_{(0)} = G$ and $G_{(i)} = [G, G_{(i-1)}]$ for $i \geq 1$ form the so-called lower central series.) For our purposes we need to classify with generators and relations these groups. We achieve this in the following lemma.

Lemma 1.7. *Let G be a group of order p^n for $n \geq 2$ with an abelian subgroup H of order p^{n-1}. Choose any $\alpha \in G$ such that α generates G/H, that is, $\alpha \notin H$, $\alpha^p \in H$. Define $H(p) = \{h \in H : h^p = 1, h \notin H^p\} \cup \{1\}$, and assume that $[H(p), \alpha] \subset H(p)$. Assume also that the p-th lower central subgroup $G_{(p)}$ is trivial. Then H is a direct product of normal subgroups of G belonging to four types*:

(1) *$(C_p)^s$ for some $s \geq 1$. There exist generators $\alpha_1, \dots, \alpha_s$ of $(C_p)^s$ such that $[\alpha_j, \alpha] = \alpha_{j+1}$ for $1 \leq j \leq s-1$ and $\alpha_s \in Z(G)$.*

(2) *C_{p^a} for some $a \geq 1$. There exists a generator β of C_{p^a} such that $[\beta, \alpha] = \beta^{bp^{a-1}}$ for some $b : 0 \leq b \leq p-1$.*

(3) *$C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}} \times (C_p)^s$ for some $k \geq 1$, $a_i \geq 2$, $s \geq 1$. There exist generators $\alpha_{11}, \alpha_{21}, \dots, \alpha_{k1}$ of $C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$ such that $[\alpha_{i,1}, \alpha] = \alpha_{i+1,1}^{p^{a_{i+1}-1}} \in Z(G)$ for $i = 1, \dots, k-1$. There also exist generators $\alpha_{k,2}, \dots, \alpha_{k,s+1}$ of $(C_p)^s$ such that $[\alpha_{k,j}, \alpha] = \alpha_{k,j+1}$ for $1 \leq j \leq s$ and $\alpha_{k,s+1} \in Z(G)$.*

(4) *$C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$ for some $k \geq 2$, $a_i \geq 2$. For any $i : 1 \leq i \leq k$ there exists a generator $\alpha_{i,1}$ of the factor $C_{p^{a_i}}$ such that $[\alpha_{i,1}, \alpha] = \alpha_{i+1,1}^{p^{a_i-1}} \in Z(G)$ and $[\alpha_{k,1}, \alpha] \in \langle \alpha_{1,1}^{p^{a_1-1}}, \dots, \alpha_{k,1}^{p^{a_k-1}} \rangle$.*

The first main result of this paper is a generalization of Theorem 1.3:

Theorem 1.8. *Let G be a group of order p^n for $n \geq 2$ with an abelian subgroup H of order p^{n-1}, and let G be of exponent p^e. Choose any $\alpha \in G$ such that α generates G/H, that is, $\alpha \notin H$, $\alpha^p \in H$. Define $H(p) = \{h \in H : h^p = 1, h \notin H^p\} \cup \{1\}$, and assume that $[H(p), \alpha] \subset H(p)$. Let the p-th lower central subgroup $G_{(p)}$ be trivial. Assume that* (i) $\operatorname{char} K = p > 0$, *or* (ii) $\operatorname{char} K \neq p$ *and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

The key idea to prove Theorem 1.8 is to find a faithful G-subspace W of the regular representation space $\bigoplus_{g \in G} K \cdot x(g)$ and to show that W^G is rational over K. The subspace W is obtained as an induced representation from H by applying Lemma 1.7.

The next goal of our article is to study Noether's problem for some groups of orders p^5 and p^6 for any odd prime p. We use the list of generators and relations for these groups, given by James [1980]. It is known that $K(G)$ is always rational if G is a p-group of order at most p^4 and $\zeta_e \in K$, where e is the exponent of G

(see [Chu and Kang 2001]). However, in [Hoshi and Kang 2011] it is shown that there exists a group G of order p^5 such that $\mathbb{C}(G)$ is not rational over $\mathbb{C}$.

The second main result of this article is the following rationality criterion for the groups of orders p^5 and p^6 having an abelian normal subgroup such that its quotient group is cyclic.

Theorem 1.9. *Let G be a group of order p^n for $n \leq 6$ with an abelian normal subgroup H such that G/H is cyclic. Let G be of exponent p^e. Assume that* (i) $\operatorname{char} K = p > 0$, *or* (ii) $\operatorname{char} K \neq p$ *and K contains a primitive p^e-th root of unity. Then $K(G)$ is rational over K.*

We do not know whether Theorem 1.9 holds for any $n \geq 7$. However, we should not "overgeneralize" Theorem 1.9 to the case of any metabelian group because of the following theorem of Saltman.

Theorem 1.10 [Saltman 1984]. *For any prime number p and for any field K with* $\operatorname{char} K \neq p$ (*in particular*, *K may be an algebraically closed field*), *there is a metabelian p-group G of order p^9 such that $K(G)$ is not rational over K.*

We organize this paper as follows. We recall some preliminaries in Section 2 that will be used in the proofs of Theorems 1.8 and 1.9. There we also prove Lemma 2.5, which is a generalization of Kang's argument [2011, Case 5, Step II]. In Section 3 we prove Lemma 1.7, which is of independent interest, since it provides a list of generators and relations for any p-group G having an abelian subgroup H of index p, provided that $[H(p), \alpha] \subset H(p)$ and $G_{(p)} = 1$. Our main results — Theorems 1.8 and 1.9 — are proved in Sections 4 and 5, respectively.

2. Preliminaries

We list several results which will be used in the sequel.

Theorem 2.1 [Hajja and Kang 1995, Theorem 1]. *Let G be a finite group acting on $L(x_1, \dots, x_m)$, the rational function field of m variables over a field L, such that*

(1) *for any $\sigma \in G$, $\sigma(L) \subset L$,*

(2) *the restriction of the action of G to L is faithful,*

(3) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma),$$

where $A(\sigma) \in \mathrm{GL}_m(L)$ and $B(\sigma)$ is an $m \times 1$ matrix over L. Then there exist $z_1, \dots, z_m \in L(x_1, \dots, x_m)$ such that $L(x_1, \dots, x_m)^G = L^G(z_1, \dots, z_m)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$ and $1 \leq i \leq m$.

Theorem 2.2 [Ahmad et al. 2000, Theorem 3.1]. *Let G be a finite group acting on $L(x)$, the rational function field of one variable over a field L. Assume that, for any $\sigma \in G$, $\sigma(L) \subset L$ and $\sigma(x) = a_\sigma x + b_\sigma$ for any $a_\sigma, b_\sigma \in L$ with $a_\sigma \neq 0$. Then $L(x)^G = L^G(z)$ for some $z \in L[x]$.*

Theorem 2.3 [Chu and Kang 2001, Theorem 1.7]. *If* $\operatorname{char} K = p > 0$ *and G is a finite p-group, then $K(G)$ is rational over K.*

The following lemma can be extracted from some proofs in [Kang 2011; Hu and Kang 2010].

Lemma 2.4. *Let $\langle\tau\rangle$ be a cyclic group of order $n > 1$, acting on $K(v_1, \dots, v_{n-1})$, the rational function field of $n-1$ variables over a field K, such that*

$$\tau : v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{n-1} \mapsto (v_1 \cdots v_{n-1})^{-1} \mapsto v_1.$$

Suppose that K contains a primitive n-th root of unity ξ. Then $K(v_1, \dots, v_{n-1}) = K(s_1, \dots, s_{n-1})$, where $\tau : s_i \mapsto \xi^i s_i$ for $1 \le i \le n-1$.

Proof. Define $w_0 = 1 + v_1 + v_1 v_2 + \cdots + v_1 v_2 \cdots v_{n-1}$, $w_1 = (1/w_0) - 1/n$, $w_{i+1} = (v_1 v_2 \cdots v_i / w_0) - 1/n$ for $1 \le i \le n-1$. Thus $K(v_1, \dots, v_{n-1}) = K(w_1, \dots, w_n)$ with $w_1 + w_2 + \cdots + w_n = 0$ and

$$\tau : w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{n-1} \mapsto w_n \mapsto w_1.$$

Define $s_i = \sum_{1 \le j \le n} \xi^{-ij} w_j$ for $1 \le i \le n-1$. Then $\tau : s_i \mapsto \xi^i s_i$ for $1 \le i \le n-1$ and $K(w_1, \dots, w_n) = K(s_1, \dots, s_{n-1})$. $\square$

Next, generalizing an argument used in [Kang 2011, Case 5, Step II], we obtain a result that will play an important role in our work.

Lemma 2.5. *Let $k > 1$, let p be any prime and let $\langle\alpha\rangle$ be a cyclic group of order p, acting on $K(y_{1i}, y_{2i}, \dots, y_{ki} : 1 \le i \le p-1)$, the rational function field of $k(p-1)$ variables over a field K, such that*

$$\alpha : y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto (y_{j1} y_{j2} \cdots y_{jp-1})^{-1} \quad \text{for } 1 \le j \le k.$$

Assume that $K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \le i \le p-1) = K(y_{1i}, y_{2i}, \dots, y_{ki} : 1 \le i \le p-1)$ where for any $j : 1 \le j \le k$ and for any $i : 1 \le i \le p-1$ the variable v_{ji} is a monomial in the variables $y_{1i}, y_{2i}, \dots, y_{ki}$. Assume also that the action of α on $K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \le i \le p-1)$ is given by

$$\alpha : v_{j1} \mapsto v_{j1} v_{j2}^p,\ v_{j2} \mapsto v_{j3} \mapsto \cdots \mapsto v_{jp-1} \mapsto A_j (v_{j1} v_{j2}^{p-1} v_{j3}^{p-2} \cdots v_{jp-1}^2)^{-1}$$

for $1 \le j \le k$, where A_j is some monomial in $v_{1i}, \dots, v_{j-1i}$ for $2 \le j \le k$ and $A_1 = 1$. If K contains a primitive p-th root of unity ζ, then

$$K(v_{1i}, v_{2i}, \dots, v_{ki} : 1 \le i \le p-1) = K(s_{1i}, s_{2i}, \dots, s_{ki} : 1 \le i \le p-1),$$

where $\alpha : s_{ji} \mapsto \zeta^i s_{ji}$ *for* $1 \le j \le k$, $1 \le i \le p-1$.

Proof. We write the additive version of the multiplication action of α; that is, consider the $\mathbb{Z}[\pi]$-module $M = \bigoplus_{1\le m\le k}\big(\bigoplus_{1\le i\le p-1}\mathbb{Z}\cdot v_{mi}\big)$, where $\pi = \langle\alpha\rangle$. Define submodules $M_j = \bigoplus_{1\le m\le j}\big(\bigoplus_{1\le i\le p-1}\mathbb{Z}\cdot v_{mi}\big)$ for $1 \le j \le k$. Thus α has the following additive action

$$\alpha : v_{j1} \mapsto v_{j1} + p v_{j2},$$
$$v_{j2} \mapsto v_{j3} \mapsto \cdots \mapsto v_{jp-1} \mapsto A_j - v_{j1} - (p-1)v_{j2} - (p-2)v_{j3} - \cdots - 2v_{jp-1},$$

where $A_j \in M_{j-1}$.

By Lemma 2.4, M_1 is isomorphic to the $\mathbb{Z}[\pi]$-module $N = \bigoplus_{1\le i\le p-1}\mathbb{Z}\cdot u_i$, where $u_1 = v_{12}$, $u_i = \alpha^{i-1}\cdot v_{12}$ for $2 \le i \le p-1$, and

$$\alpha : u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p-1} \mapsto -u_1 - u_2 - \cdots - u_{p-1} \mapsto u_1.$$

Let $\Phi_p(T) \in \mathbb{Z}[T]$ be the p-th cyclotomic polynomial. Since $\mathbb{Z}[\pi]$ is isomorphic to $\mathbb{Z}[T]/(T^p - 1)$, we find that $\mathbb{Z}[\pi]/\Phi_p(\alpha) \simeq \mathbb{Z}[T]/\Phi_p(T) \simeq \mathbb{Z}[\omega]$, the ring of p-th cyclotomic integers. As $\Phi_p(\alpha)\cdot x = 0$ for any $x \in N$, the $\mathbb{Z}[\pi]$-module N can be regarded as a $\mathbb{Z}[\omega]$-module through the morphism $\mathbb{Z}[\pi] \to \mathbb{Z}[\pi]/\Phi_p(\alpha)$. When N is regarded as a $\mathbb{Z}[\omega]$-module, we have $N \simeq \mathbb{Z}[\omega]$, the rank-one free $\mathbb{Z}[\omega]$-module.

We claim that M itself can be regarded as a $\mathbb{Z}[\omega]$-module, that is, $\Phi_p(\alpha)\cdot M = 0$.

We return to multiplicative notation. Note that all v_{ji} are monomials in the y_{ji}. The action of α on y_{ji} given in the statement satisfies $\prod_{0\le m\le p-1}\alpha^m(y_{ji}) = 1$ for any $1 \le j \le k$, $1 \le i \le p-1$. Using the additive notations, we get $\Phi_p(\alpha)\cdot y_{ji} = 0$. Hence $\Phi_p(\alpha)\cdot M = 0$.

Define $M' = M/M_{k-1}$. We have a short exact sequence of $\mathbb{Z}[\pi]$-modules

$$0 \to M_{k-1} \to M \to M' \to 0. \tag{2-1}$$

Since M is a $\mathbb{Z}[\omega]$-module, (2-1) is a short exact sequence of $\mathbb{Z}[\omega]$-modules. Proceeding by induction, we obtain that M is a direct sum of free $\mathbb{Z}[\omega]$-modules isomorphic to N. Hence, $M \simeq \bigoplus_{1\le j\le k} N_j$, where $N_j \simeq N$ is a free $\mathbb{Z}[\omega]$-module and so a $\mathbb{Z}[\pi]$-module also (for $1 \le j \le k$).

Finally, we interpret the additive version of $M \simeq \bigoplus_{1\le j\le k} N_j \simeq N^k$ in terms of the multiplicative version as follows: There exist w_{ji} that are monomials in v_{ji} for $1 \le j \le k$, $1 \le i \le p-1$ such that $K(w_{ji}) = K(v_{ji})$ and α acts as

$$\alpha : w_{j1} \mapsto w_{j2} \mapsto \cdots \mapsto w_{jp-1} \mapsto (w_{j1}w_{j2}\cdots w_{jp-1})^{-1} \quad \text{for } 1 \le j \le k.$$

According to Lemma 2.4, the above action can be linearized as pointed out in the statement. □

Now, let G be any metacyclic p-group generated by two elements σ and τ with relations $\sigma^{p^a}=1$, $\tau^{p^b}=\sigma^{p^c}$ and $\tau^{-1}\sigma\tau=\sigma^{\varepsilon+\delta p^r}$ where $\varepsilon=1$ if p is odd, $\varepsilon=\pm 1$ if $p=2$, $\delta=0,1$ and $a, b, c, r\geq 0$ are subject to some restrictions. For the description of these restrictions see, for example, [Kang 2006, p. 564].

Theorem 2.6 [Kang 2006, Theorem 4.1]. *Let p be a prime number, m, n and r positive integers, $k=1+p^r$ if $(p,r)\neq(2,1)$ or $k=-1+2^r$ if $p=2$ and $r\geq 2$. Let G be a split metacyclic p-group of order p^{m+n} and exponent p^e defined by $G=\langle\sigma,\tau:\sigma^{p^m}=\tau^{p^n}=1, \tau^{-1}\sigma\tau=\sigma^k\rangle$. Let K be any field such that* char $K\neq p$ *and K contains a primitive p^e-th root of unity, and let ζ be a primitive p^m-th root of unity. Then $K(x_0,x_1,\ldots,x_{p^n-1})^G$ is rational over K, where G acts on $x_0,\ldots,x_{p^n-1}$ by*

$$\sigma: x_i\mapsto\zeta^{k^i}x_i,\quad \tau: x_0\mapsto x_1\mapsto\cdots\mapsto x_{p^n-1}\mapsto x_0.$$

3. Proof of Lemma 1.7

It is well known that H is a normal subgroup of G. We divide the proof into steps.

Step I. Let β_1 be any element of H that is not central. Since $G_{(p)}=\{1\}$, there exist $\beta_2,\ldots,\beta_k\in H$ for some $k: 2\leq k\leq p$ such that $[\beta_j,\alpha]=\beta_{j+1}$, where $1\leq j\leq k-1$ and $\beta_k\neq 1$ is central. We are going to show now that the order of β_2 is not greater than p. In particular, from the multiplication rule $[a,\alpha][b,\alpha]=[ab,\alpha]$ (for any $a,b\in H$) it follows that all p-th powers are contained in the center of G.

From $[\beta_j,\alpha]=\beta_{j+1}$ there follows the well known formula

$$\alpha^{-p}\beta_1\alpha^p=\beta_1\beta_2^{\binom{p}{1}}\beta_3^{\binom{p}{2}}\cdots\beta_p^{\binom{p}{p-1}}\beta_{p+1}, \tag{3-1}$$

where we put $\beta_{k+1}=\cdots=\beta_{p+1}=1$. Since α^p is in H, we obtain the formula

$$\beta_2^{\binom{p}{1}}\beta_3^{\binom{p}{2}}\cdots\beta_k^{\binom{p}{k-1}}=1.$$

Hence $\bigl(\beta_2\cdot\prod_{j\neq 2}\beta_j^{a_j}\bigr)^p=1$ for some integers a_j. It is not hard to see that this identity is impossible if the order of β_2 exceeds p. Indeed, if $\ell=\max\{j:\beta_j^p\neq 1\}$, then β_ℓ^p is in the subgroup generated by $\beta_2^p,\ldots,\beta_{\ell-1}^p$. Thus $[\beta_\ell^p,\alpha]=[\beta_2^{b_2p}\cdots\beta_{\ell-1}^{b_{\ell-1}p},\alpha]=\beta_3^{b_2p}\cdots\beta_\ell^{b_{\ell-1}p}\neq 1$ for some $b_2,\ldots,b_{\ell-1}\in\mathbb{Z}_p$. On the other hand, $[\beta_\ell^p,\alpha]=\beta_{\ell+1}^p=1$, which is a contradiction.

Step II. Let us write the decomposition of H as a direct product of cyclic subgroups (not necessarily normal in G): $H\simeq(C_p)^t\times C_{p^{a_1}}\times C_{p^{a_2}}\times\cdots\times C_{p^{a_s}}$ for $0\leq t$, $2\leq a_1\leq a_2\leq\cdots\leq a_s$. Choose a generator $\alpha_{11}\in C_{p^{a_1}}$. Since $G_{(p)}=\{1\}$, there exist $\alpha_{12},\ldots,\alpha_{1k}\in H$ for some $k: 2\leq k\leq p$ such that $[\alpha_{1j},\alpha]=\alpha_{1j+1}$, where $1\leq j\leq k-1$ and $\alpha_{1k}\neq 1$ is central. From Step I it follows that the order of α_{12} is not greater than p. We are going to define a normal subgroup of G which depends on the nature of the element α_{12}. We will denote it by $\langle\langle\alpha_{11}\rangle\rangle$, and call it *the commutator chain of* α_{11}. Simultaneously, we will define a complement in H denoted by $\overline{\langle\langle\alpha_{11}\rangle\rangle}$.

Case II.1. Let $\alpha_{12} = \alpha_{11}^{p^{a_1-1}c_1}$ for some $c_1 : 0 \le c_1 \le p-1$. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}\rangle$ and $\overline{\langle\langle\alpha_{11}\rangle\rangle} = (C_p)^t \cdot \langle\alpha_{21}, \dots, \alpha_{s1}\rangle$. Clearly, $\langle\langle\alpha_{11}\rangle\rangle$ is a normal subgroup of type 2.

Case II.2. Let $\alpha_{12} \notin H^p$. According to the assumptions of our lemma, we have $[H(p), \alpha] \cap H^p = \{1\}$, so $\alpha_{1j} \notin H^p$ for all j. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \dots, \alpha_{1k}\rangle$. Then $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times (C_p)^{k-1}$ is a normal subgroup of type 3. Define $\overline{\langle\langle\alpha_{11}\rangle\rangle} = (C_p)^{t-k+1} \cdot \langle\alpha_{21}, \dots, \alpha_{s1}\rangle$, where $(C_p)^{t-k+1}$ is the complement of $(C_p)^{k-1}$ in $(C_p)^t$.

Case II.3. Let $\alpha_{12} \in H^p$. Then $\alpha_{12} = \prod_{i\in A} \alpha_{i1}^{p^{a_i-1}d_i}$, where $A \subset \{1, 2, \dots, s\}$, $1 \le d_i \le p-1$. Put $i_0 = \min\{i \in A\}$.

If $i_0 = 1$, then $\alpha_{12} = \bigl(\alpha_{11}^{d_1} \prod_{i\in A, i\ne 1} \alpha_{i1}^{p^{a_i-a_1}} d_i\bigr)^{p^{a_1-1}}$. We replace the generator α_{11} with $\alpha'_{11} = \alpha_{11}^{d_1} \prod_{i\in A, i\ne 1} \alpha_{i1}^{p^{a_i-a_1}d_i}$. Clearly, $\operatorname{ord}\alpha'_{11} = \operatorname{ord}\alpha_{11}$ and $[\alpha'_{11}, \alpha] \in \langle\alpha'_{11}\rangle$, so this case is reduced to Case I.

If $i_0 > 1$, then $\alpha_{12} = \bigl(\alpha_{i_01}^{d_{i_0}} \prod_{i\in A, i\ne i_0} \alpha_{i1}^{p^{a_i-a_{i_0}}d_i}\bigr)^{p^{a_{i_0}-1}}$. We replace the generator α_{i_01} with $\alpha'_{i_01} = \alpha_{i_01}^{d_{i_0}} \prod_{i\in A, i\ne i_0} \alpha_{i1}^{p^{a_i-a_{i_0}}d_i}$. Clearly, $\operatorname{ord}\alpha'_{i_01} = \operatorname{ord}\alpha_{i_01}$ and $\alpha'^{p^{a_{i_0}-1}}_{i_01} = \alpha_{12}$.

Abusing notation we will assume henceforth that $i_0 = 2$ and $\alpha_{21}^{p^{a_2-1}} = \alpha_{12}$. Consider $\alpha_{22} = [\alpha_{21}, \alpha]$. We have three possibilities now.

Subcase II.3.1. If $\alpha_{22} \in \langle\alpha_{11}^{p^{a_1-1}}, \alpha_{21}^{p^{a_1-1}}\rangle$, define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \alpha_{21}\rangle$. Then $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}}$ is a normal subgroup of type 4.

Subcase II.3.2. If $\alpha_{22} \notin H^p$, there exist $\alpha_{22}, \dots, \alpha_{2\ell} \in H$ for some $\ell : 2 \le \ell \le p$ such that $[\alpha_{2j}, \alpha] = \alpha_{2j+1}$, where $1 \le j \le \ell-1$ and $\alpha_{2\ell} \ne 1$ is central. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \alpha_{21}, \alpha_{22}, \dots, \alpha_{2\ell}\rangle$. Then $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times (C_p)^{\ell-1}$ is a normal subgroup of type 3.

Subcase II.3.3. $\alpha_{22} \in H^p$. According to the observations we have just made, this subcase leads to the following two final possibilities.

- $\alpha_{22} = \alpha_{31}^{p^{a_3-1}}, \dots, \alpha_{r-12} = \alpha_{r1}^{p^{a_r-1}}, \alpha_{r2} \in \langle\alpha_{11}^{p^{a_1-1}}, \dots, \alpha_{r1}^{p^{a_r-1}}\rangle$. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \alpha_{21}, \dots, \alpha_{r1}\rangle$. Then $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_r}}$ is a normal subgroup of type 4. Define $\overline{\langle\langle\alpha_{11}\rangle\rangle} = (C_p)^t \cdot \langle\alpha_{r+11}, \dots, \alpha_{s1}\rangle$.
- $\alpha_{22} = \alpha_{31}^{p^{a_3-1}}, \dots, \alpha_{r-12} = \alpha_{r1}^{p^{a_r-1}}, \alpha_{r2} \notin H^p$. Then there exist $\alpha_{r2}, \dots, \alpha_{r\ell} \in H$ for some $\ell : 2 \le \ell \le p$ such that $[\alpha_{rj}, \alpha] = \alpha_{rj+1}$, where $1 \le j \le \ell-1$ and $\alpha_{r\ell} \ne 1$ is central. Define $\langle\langle\alpha_{11}\rangle\rangle = \langle\alpha_{11}, \alpha_{21}, \dots, \alpha_{r1}, \alpha_{r2}, \dots, \alpha_{r\ell}\rangle$. In this case $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_1}} \times \cdots \times C_{p^{a_r}} \times (C_p)^{\ell-1}$ is a normal subgroup of type 3. Define $\overline{\langle\langle\alpha_{11}\rangle\rangle} = (C_p)^{t-\ell+1} \cdot \langle\alpha_{r+11}, \dots, \alpha_{s1}\rangle$, where $(C_p)^{t-\ell+1}$ is the complement of $(C_p)^{\ell-1}$ in $(C_p)^t$.

Step III. Put $H_1 = \langle\langle\alpha_{11}\rangle\rangle$ and $H_2 = \overline{\langle\langle\alpha_{11}\rangle\rangle}$. Note that $H_1 \cap H_2 = \{1\}$. However, H_2 may not be a normal subgroup of G. That is why we need to show that there exist a commutator chain $\mathcal{H}_1$ and a normal subgroup $\mathcal{H}_2$ of G such that $H = \mathcal{H}_1 \times \mathcal{H}_2$. In this step, we will describe a somewhat algorithmic approach which replaces the generators of H until the desired result is obtained.

Assume henceforth that H_2 is not normal in G. Then there exists a generator $\beta \in H_2$ such that $\alpha^{-1}\beta\alpha = hh_1$ for some $h \in H_2$, $h_1 \in H_1$, $h_1 \notin H_2$. Since $h = \beta h_2$ for some $h_2 \in H_2$, we get $[\beta, \alpha] = h_1 h_2$.

Let us assume first that $\operatorname{ord}\beta = p$. If $h_1 \in H^p$, then $h_2 \notin [H(p), \alpha]$; otherwise $[H(p), \alpha] \cap H^p \neq \{1\}$. In other words, h_2 does not appear in similar chains, so we can simply put $h_1 h_2$, instead of h_2, as a generator of H_2. In this way we obtain a group that is G-isomorphic to H_2. Thus we get that $[\beta, \alpha]$ is in this new copy of H_2. Similarly, if $h_1 \in H(p)$ and $h_2 \notin [H(p), \alpha]$, we can obtain a new copy of H_2 such that $[\beta, \alpha]$ is in H_2. If $h_2 \in [H(p), \alpha]$, we may assume that $[\beta, \alpha] \in H_1$. In this case $\langle\langle \alpha_{11} \rangle\rangle$ must be of type 3. Let $\langle\langle \alpha_{11} \rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}} \times (C_p)^s$ be generated by elements $\alpha_{11}, \ldots, \alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{ks+1}$ with relations given in the statement of the lemma. Assume that $\alpha_{k\ell} = [\beta, \alpha]$ for some $\ell : 2 \leq \ell \leq s+1$. If $\ell > 2$, replace β with $\beta' = \beta\alpha_{k\ell-1}^{-1}$. Hence $[\beta', \alpha] = 1$. If $\ell = 2$, we can put $\alpha'_{k1} = \alpha_{k1}\beta^{-1}$, instead of α_{k1}, as a generator of H_1. In this way we obtain a group of type 4, since $[\alpha'_{k1}, \alpha] = 1$. Clearly, $[\beta, \alpha]$ is not in this new commutator chain $\mathcal{H}_1$. It is not hard to see that with similar replacements we can treat the general case $[\beta, \alpha] = \prod_i \alpha_{i1}^{p^{a_i-1}c_i} \cdot \prod_j \alpha_{kj}$. Thus we obtain the decomposition $H = \mathcal{H}_1 \times \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are normal subgroups of G.

Next, we are going to assume that $\operatorname{ord}\beta > p$. According to the definition of the commutator chain of α_{11} we need to consider the three cases of Step II separately.

Case III.1. $\alpha_{12} = \alpha_{11}^{p^{a_1-1}c_1}$ for some $c_1 : 1 \leq c_1 \leq p-1$. Here we must have $h_1 = \alpha_{11}^{p^{a_1-1}d_1}$ for some $d_1 : 1 \leq d_1 \leq p-1$. We can replace β with $\beta' = \beta\alpha_{11}^{-d_1/c_1}$, so $[\beta', \alpha] = h_2$.

Case III.2. $\alpha_{12} \notin H^p$. If $h_1 = \prod_{j\geq 2} \alpha_{1j}^{d_j}$ for some $d_j : 0 \leq d_j \leq p-1$, we can replace β with $\beta' = \beta \prod_{j\geq 2} \alpha_{1j-1}^{-d_j}$. Hence $[\beta', \alpha] = h_2$. This reduces the analysis to the case $h_1 = \alpha_{11}^{p^{a_1-1}d_1}$ for some $d_1 : 0 \leq d_1 \leq p-1$. We now have three possibilities for h_2.

Subcase III.2.1. Let $h_2 \notin H^p$ and $h_2 \notin [H, \alpha]$. We can put $h_1 h_2$, instead of h_2, as a generator of H_2. In this way we obtain a group that is G-isomorphic to H_2. Thus we get that $[\beta, \alpha]$ is in this new copy of H_2.

Subcase III.2.2. Let $h_2 \notin H^p$ and $h_2 \in [H, \alpha]$, that is, there exists $\gamma \notin H^p$ such that $[\gamma, \alpha] = h_2$. Put $\beta' = \beta\gamma^{-1}$. Then $[\beta', \alpha] = h_1 = \alpha_{11}^{p^{a_1-1}d_1}$. Hence the commutator chain of α_{11} is contained in the commutator chain $\langle\langle \beta' \rangle\rangle$ which is a normal subgroup of G of type 3.

Subcase III.2.3. Let $h_2 \in H^p$; that is, $h_2 = \prod_{i\in B} \alpha_{i1}^{p^{a_i-1}d_i}$, where $B = \{i : \alpha_{i1} \in H_2\}$,

$0 \le d_i \le p-1$. We can replace α_{11} with $\alpha'_{11} = \alpha_{11}^{d_1} \prod_{i \in B} \alpha_{i1}^{p^{a_i - a_1} d_i}$. Now we have $[\beta, \alpha] = \alpha'^{p^{a_1-1}}_{11}$, so the commutator chain of α'_{11} is contained in the commutator chain $\langle\langle\beta\rangle\rangle$, which is a normal subgroup of G of type 3.

Case III.3. $\alpha_{12} \in H^p$. We have that either $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_r}}$ is a normal subgroup of type 4, or $\langle\langle\alpha_{11}\rangle\rangle \simeq C_{p^{a_1}} \times C_{p^{a_1}} \times \cdots \times C_{p^{a_r}} \times (C_p)^{\ell-1}$ is a normal subgroup of type 3.

Similarly to Case III.2, if h_1 is a product of elements of order p that are not in $\langle\alpha_{11}^{p^{a_1-1}}\rangle$, by a suitable change of the generator β we will obtain $[\beta,\alpha] = h_2$. Thus we again reduce the considerations to the case $h_1 = \alpha_{11}^{p^{a_1-1}d_1}$ for some $d_1 : 0 \le d_1 \le p-1$. We have three possibilities for h_2, which are identical to the three subcases in Case III.2. The only slight difference is that the new commutator chain here can be of type 3 or type 4.

In this way, we have investigated all possibilities for the proper construction of the normal factors of H. The construction is algorithmic in nature. When we define a new commutator chain $\langle\langle\beta'\rangle\rangle$ or $\langle\langle\beta\rangle\rangle$ (as in Subcases III.2.2 and III.2.3), we have to start the same process all over again until we can not get a new commutator chain that contains the previous one. Denote by $\mathcal{H}_1$ the last commutator chain obtained by the described algorithm from H_1. We have that $\mathcal{H}_1$ is a normal subgroup of G of one of the types 1–4. Denote by $\mathcal{H}_2$ the subgroup obtained from H_2 by the replacements described above. Then H is a direct product of $\mathcal{H}_1$ and $\mathcal{H}_2$, where $\mathcal{H}_2$ is normal in G. Proceeding by induction we will obtain the decomposition given in the statement.

4. Proof of Theorem 1.8

If $\operatorname{char} K = p > 0$, we can apply Theorem 2.3. Therefore, we will assume that $\operatorname{char} K \neq p$.

According to Lemma 1.7, $H \simeq \mathcal{H}_1 \times \cdots \times \mathcal{H}_t$, where $\mathcal{H}_1, \ldots, \mathcal{H}_t$ are normal subgroups of G that are isomorphic to any of the four types described in Lemma 1.7.

Let V be a K-vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in G} K \cdot x(g)$, where G acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Therefore $K(V)^G = K(x(g) : g \in G)^G = K(G)$.

Now, for any subgroup $\mathcal{H}_i$ $(1 \le i \le t)$ we can define a faithful representation subspace $V_i = \bigoplus_{1 \le j \le k_i} K \cdot Y_j$, where k_i is the number of the generators of $\mathcal{H}_i$ as an abelian group. (For details see Cases I–IV.) Therefore, $\bigoplus_{1 \le i \le t} V_i$ is a faithful representation space of the subgroup H.

Next, for any subgroup $\mathcal{H}_i$ $(1 \le i \le t)$ we define $x_{jk} = \alpha^k \cdot Y_j$ for $1 \le j \le k_i$, $0 \le k \le p-1$. Define $W_i = \bigoplus_{j,k} K \cdot x_{jk} \subset V^*$. Then $W = \bigoplus_{1 \le i \le t} W_i$ is a faithful G-subspace of V^*. Thus, by Theorem 2.1 it suffices to show that W^G is rational over K. Note that $W^G = (W^H)^{\langle\alpha\rangle} = ((\cdots(W^{\mathcal{H}_1})^{\mathcal{H}_2}\cdots)^{\mathcal{H}_t})^{\langle\alpha\rangle} = ((\cdots(W_1^{\mathcal{H}_1} \bigoplus_{2 \le j \le t} W_j)^{\mathcal{H}_2}\cdots)^{\mathcal{H}_t})^{\langle\alpha\rangle} = \cdots = \bigoplus_{1 \le j \le t} (W_j^{\mathcal{H}_j})^{\langle\alpha\rangle}$. Therefore, we need to calculate $W_j^{\mathcal{H}_j}$ when $\mathcal{H}_j$ is isomorphic to any of the four types described in Lemma 1.7. Finally, we will show that the action of α on W^H can be linearized.

Case I. Assume that $\mathcal{H}_1$ is of type 3; that is, for some $k \ge 1$, $a_i \ge 2$, $s \ge 1$, $\mathcal{H}_1 \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}} \times (C_p)^s$. Denote by $\alpha_1, \ldots, \alpha_k$ the generators of $C_{p^{a_1}} \times \cdots \times C_{p^{a_k}}$, and by $\alpha_{k+1}, \ldots, \alpha_{k+s}$ the generators of $(C_p)^s$. According to Lemma 1.7, we have the relations $[\alpha_i, \alpha] = \alpha_{i+1}^{p^{a_{i+1}-1}} \in Z(G)$ for $1 \le i \le k-1$; $[\alpha_{k+j}, \alpha] = \alpha_{k+j+1}$ for $0 \le j \le s-1$; and $\alpha_{k+s} \in Z(G)$. Because of the frequent use of $k+s$ in this case, we put $r = k+s$.

We divide the proof into several steps.

Step 1. Define $X_1, X_2, \ldots, X_r \in V^*$ by

$$X_j = \sum_{\ell_1,\ldots,\ell_r} x\Big(\prod_{i\ne j} \alpha_i^{\ell_i}\Big) \quad \text{for } 1 \le j \le r.$$

Note that $\alpha_i \cdot X_j = X_j$ for $j \ne i$. Let $\zeta_{p^{a_i}} \in K$ be a primitive p^{a_i}-th root of unity for $1 \le i \le k$, and let ζ be a primitive p-th root of unity. Define $Y_1, Y_2, \ldots, Y_r \in V^*$ by

$$Y_i = \sum_{m=0}^{p^{a_i}-1} \zeta_{p^{a_i}}^{-m} \alpha_i^m \cdot X_i, \quad Y_j = \sum_{m=0}^{p-1} \zeta^{-m} \alpha_j^m \cdot X_j,$$

for $1 \le i \le k$ and $k+1 \le j \le r$.

It follows that

$$\begin{aligned}
&\alpha_i : \ Y_i \mapsto \zeta_{p^{a_i}} Y_i,\ Y_j \mapsto Y_j && \text{for } j \ne i \text{ and } 1 \le i \le k,\\
&\alpha_j : \ Y_j \mapsto \zeta Y_j,\ Y_i \mapsto Y_i && \text{for } i \ne j \text{ and } k+1 \le j \le r.
\end{aligned}$$

Thus $V_1 = \bigoplus_{1\le j\le r} K \cdot Y_j$ is a faithful representation space of the subgroup $\mathcal{H}_1$.

Define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \le j \le r$ and $0 \le i \le p-1$. Recall that $[\alpha_i, \alpha] = \alpha_{i+1}^{p^{a_{i+1}-1}} \in Z(G)$ for $1 \le i \le k-1$; $[\alpha_{k+j}, \alpha] = \alpha_{k+j+1}$ for $0 \le j \le s-1$; and $\alpha_r \in Z(G)$. Hence

$$\alpha^{-i} \alpha_j \alpha^i = \alpha_j \alpha_{j+1}^{i p^{a_{i+1}-1}} \quad \text{for } 1 \le j \le k-1,\ 1 \le i \le p-1$$

and

$$\alpha^{-i} \alpha_j \alpha^i = \alpha_j \alpha_{j+1}^{\binom{i}{1}} \alpha_{j+2}^{\binom{i}{2}} \cdots \alpha_r^{\binom{i}{r-j}} \quad \text{for } k \le j \le r-1,\ 1 \le i \le p-1.$$

It follows that

$$\begin{aligned}
&\alpha_\ell : x_{\ell i} \mapsto \zeta_{p^{a_\ell}} x_{\ell i},\ x_{\ell+1 i} \mapsto \zeta^i x_{\ell+1 i},\ x_{ji} \mapsto x_{ji} && \text{for } 1 \le \ell \le k-1,\ j \ne \ell, \ell+1,\\
&\alpha_k : x_{ki} \mapsto \zeta_{p^{a_k}} x_{ki},\ x_{wi} \mapsto \zeta^{\binom{i}{w-k}} x_{wi},\ x_{vi} \mapsto x_{vi} && \text{for } 1 \le v \le k-1,\\
& && \phantom{\text{for }} k+1 \le w \le r,
\end{aligned}$$

$$\alpha_m : x_{ui} \mapsto \zeta^{\binom{i}{u-m}} x_{ui},\ x_{vi} \mapsto x_{vi} \qquad \text{for } k+1 \le m \le r,\ 1 \le v \le m-1, m \le u \le r,$$

$$\alpha : x_{j0} \mapsto x_{j1} \mapsto \cdots \mapsto x_{jp-1} \mapsto \zeta_{p^{c_j}}^{b_j} x_{j0} \qquad \text{for } 1 \le j \le r,$$

where $0 \le i \le p-1$, and c_j, b_j are some integers such that $0 \le b_j < p^{c_j} \le p^{a_j}$.

Let $W_1 = \bigoplus_{j,i} K \cdot x_{ji} \subset V^*$. As noted at the start of the proof, we must find $W_1^{\mathcal{H}_1}$.

Step 2. For $1 \le j \le r$ and for $1 \le i \le p-1$ define $y_{ji} = x_{ji}/x_{ji-1}$. Thus $W_1 = K(x_{j0}, y_{ji} : 1 \le j \le r, 1 \le i \le p-1)$ and for every $g \in G$,

$$g \cdot x_{j0} \in K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1) \cdot x_{j0} \quad \text{for } 1 \le j \le r,$$

while the subfield $K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1)$ is invariant by the action of G, that is,

$$\alpha_\ell : y_{\ell+1i} \mapsto \zeta y_{\ell+1i},\ y_{ji} \mapsto y_{ji} \qquad \text{for } 1 \le \ell \le k-1,\ j \ne \ell+1,$$

$$\alpha_m : y_{ui} \mapsto \zeta^{\binom{i-1}{u-m-1}} y_{ui},\ y_{vi} \mapsto y_{vi} \qquad \text{for } k \le m \le r-1,\ 1 \le v \le m,\ m+1 \le u \le r,$$

$$\alpha_r : y_{vi} \mapsto y_{vi} \qquad \text{for } 1 \le v \le r,$$

$$\alpha : y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto \zeta_{p^{c_j}}^{b_j} (y_{j1} \cdots y_{jp-1})^{-1} \qquad \text{for } 1 \le j \le r.$$

From Theorem 2.2 it follows that if $K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1)^G$ is rational over K, so is $K(x_{j0}, y_{ji} : 1 \le j \le r, 1 \le i \le p-1)^G$ over K.

Since K contains a primitive p^e-th root of unity ζ_{p^e}, where p^e is the exponent of G, K contains as well a primitive p^{c_j+1}-th root of unity, and we may replace the variables y_{ji} by $y_{ji}/\zeta_{p^{c_j+1}}^{b_j}$ so that we obtain a more convenient action of α without changing the actions of the α_j. Namely we may assume that

$$\alpha :\ y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto (y_{j1} y_{j2} \cdots y_{jp-1})^{-1} \quad \text{for } 1 \le j \le r.$$

Define $u_{r1} = y_{r1}^p$, $u_{ri} = y_{ri}/y_{ri-1}$ for $2 \le i \le p-1$. Then

$$K(y_{ji}, u_{ri} : 1 \le j \le r-1, 1 \le i \le p-1) = K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1)^{\langle \alpha_{r-1} \rangle}.$$

From Theorem 2.2 it follows that if $K(y_{ji}, u_{ri} : 1 \le j \le r-1, 2 \le i \le p-1)^G$ is rational over K, so is $K(y_{ji}, u_{ri} : 1 \le j \le r-1, 1 \le i \le p-1)^G$ over K. We have the actions

$$\alpha_\ell : u_{ri} \mapsto u_{ri} \qquad \text{for } 1 \le \ell \le k-1,$$

$$\alpha_m : u_{ri} \mapsto \zeta^{\binom{i-2}{r-m-2}} u_{ri} \qquad \text{for } 2 \le i \le p-1, k \le m \le r-2,$$

$$\alpha : u_{r2} \mapsto u_{r3} \mapsto \cdots \mapsto u_{rp-1} \mapsto (u_{r1}u_{r2}^{p-1}u_{r3}^{p-2}\cdots u_{rp-1}^{2})^{-1} \mapsto u_{r1}u_{r2}^{p-2}u_{r3}^{p-3}\cdots u_{rp-2}^{2}u_{rp-1}.$$

For $2 \le i \le p-1$ define

$$v_{ri} = u_{ri}y_{r-1i}^{-1}y_{r-2i}y_{r-3i}^{-1}\cdots y_{k+2i}^{(-1)^{r-k}}y_{k+1i}^{(-1)^{r-k+1}},$$

and put $v_{r1} = u_{r1}$.

With the aid of the well known property $\binom{n}{m} - \binom{n-1}{m} = \binom{n-1}{m-1}$, it is not hard to verify the identity

$$\binom{i-2}{r-m-2} - \binom{i-1}{r-m-2} + \binom{i-1}{r-m-3} - \binom{i-1}{r-m-4} + \cdots \\ \cdots + (-1)^{r-m-1}\binom{i-1}{2} + (-1)^{r-m}\binom{i-1}{1} + (-1)^{r-m+1}\binom{i-1}{0} = 0.$$

It follows that

$$\alpha_m :\ v_{ri} \mapsto v_{ri} \qquad \text{for } 1 \le i \le p-1 \text{ and } 1 \le m \le r-2,$$
$$\alpha :\ v_{r2} \mapsto v_{r3} \mapsto \cdots \mapsto v_{rp-1} \mapsto A_r \cdot (v_{r1}v_{r2}^{p-1}v_{r3}^{p-2}\cdots v_{rp-1}^{2})^{-1}.$$

where A_r is some monomial in y_{ji} for $2 \le j \le r-1$, $1 \le i \le p-1$.

Define $u_{r-11} = y_{r-11}^{p}$, $u_{r-1i} = y_{r-1i}/y_{r-1i-1}$ for $2 \le i \le p-1$. Then

$$K(y_{ji}, u_{r-1i} : 1 \le j \le r-2, 1 \le i \le p-1) = K(y_{ji} : 1 \le j \le r-1, 1 \le i \le p-1)^{\langle \alpha_{r-2} \rangle}.$$

From Theorem 2.2 it follows that if $K(y_{ji}, u_{r-1i} : 1 \le j \le r-2, 2 \le i \le p-1)^G$ is rational over K, so is $K(y_{ji}, u_{r-1i} : 1 \le j \le r-2, 1 \le i \le p-1)^G$ over K. Similarly to the definition of v_{ri}, we can define v_{r-1i} so that $\alpha_m(v_{r-1i}) = v_{r-1i}$ for $2 \le i \le p-1$ and $1 \le m \le r-3$. It is obvious that we can proceed in the same way, defining elements $v_{r-2i}, v_{r-3i}, \ldots, v_{k+1i}$ such that α_m acts trivially on all the v_{ji} for $k \le m \le r-3$.

Recall that the actions of α_ℓ on the y_{ji} for $1 \le \ell \le k-1$ are

$$\alpha_\ell :\ y_{\ell+1i} \mapsto \zeta y_{\ell+1i},\ \ y_{ji} \mapsto y_{ji}, \quad \text{for } 1 \le i \le p-1, 1 \le \ell \le k-1, j \ne \ell+1.$$

For any $1 \le \ell \le k-1$ define $v_{\ell+11} = y_{\ell+11}^{p}$, $v_{\ell+1i} = y_{\ell+1i}/y_{\ell+1i-1}$, where $2 \le i \le p-1$. Put also $v_{1i} = y_{1i}$ for $1 \le i \le p-1$. Then

$$K(v_{ji} : 1 \le j \le r, 1 \le i \le p-1) = K(y_{ji} : 1 \le j \le r, 1 \le i \le p-1)^{\mathcal{H}_1}.$$

The action of α is given by

$$\alpha : v_{11} \mapsto v_{12} \mapsto \cdots \mapsto v_{1p-1} \mapsto (v_{11}v_{12}\cdots v_{1p-1})^{-1},\ \ v_{m1} \mapsto v_{m1}v_{m2}^{p},$$
$$v_{m2} \mapsto v_{m3} \mapsto \cdots \mapsto v_{mp-1} \mapsto A_m \cdot (v_{m1}v_{m2}^{p-1}v_{m3}^{p-2}\cdots v_{mp-1}^{2})^{-1},$$

for $2 \le m \le r$, where A_m is some monomial in $v_{k+1i}, \dots, v_{m-1i}$ for $k+2 \le m \le r$ and $A_2 = A_3 = \cdots = A_{k+1} = 1$. From Lemmas 2.4 and 2.5 it follows that the action of α on $K(v_{ji} : 1 \le j \le r, 1 \le i \le p-1)$ can be linearized.

Case II. Assume that $\mathcal{H}_1$ is of type 1; that is, $\mathcal{H}_1 \simeq (C_p)^{s+1}$ for some $s \ge 0$. Denote by $\beta_1, \dots, \beta_{s+1}$ the generators of $(C_p)^{s+1}$. According to Lemma 1.7, we have the relations $[\beta_j, \alpha] = \beta_{j+1}$ for $1 \le j \le s$ and $\beta_{s+1} \in Z(G)$.

Define $X_1, X_2, \dots, X_{s+1} \in V^*$ by

$$X_j = \sum_{\ell_1, \dots, \ell_{s+1}} x\Big(\prod_{m \neq j} \beta_m^{\ell_m}\Big)$$

for $1 \le j \le s+1$. Note that $\beta_j \cdot X_i = X_i$ for $j \neq i$. Let ζ be a primitive p-th root of unity. Define $Y_1, Y_2, \dots, Y_{s+1} \in V^*$ by

$$Y_j = \sum_{r=0}^{p-1} \zeta^{-r} \beta_j^r \cdot X_j$$

for $1 \le j \le s+1$.

It follows that

$$\beta_j : Y_j \mapsto \zeta Y_j, \;\; Y_i \mapsto Y_i \qquad \text{for } i \neq j \text{ and } 1 \le j \le s+1.$$

Thus $V_1 = \bigoplus_{1 \le j \le s+1} K \cdot Y_j$ is a representation space of the subgroup $\mathcal{H}_1$.

Define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \le j \le s+1$, $0 \le i \le p-1$. Recall that $[\beta_j, \alpha] = \beta_{j-1}$. Hence

$$\alpha^{-i} \beta_j \alpha^i = \beta_j \beta_{j+1}^{\binom{i}{1}} \beta_{j+2}^{\binom{i}{2}} \cdots \beta_{s+1}^{\binom{i}{s+1-j}}.$$

It follows that

$$\begin{aligned}
&\beta_1 : x_{1i} \mapsto \zeta x_{1i}, \;\; x_{ji} \mapsto \zeta^{\binom{i}{j-1}} x_{ji} && \text{for } 2 \le j \le s+1, 0 \le i \le p-1,\\
&\beta_j : x_{\ell i} \mapsto x_{\ell i}, \;\; x_{mi} \mapsto \zeta^{\binom{i}{m-j}} x_{mi} && \text{for } \; 1 \le \ell \le j-1,\\
& && \quad j \le m \le s+1, 0 \le i \le p-1,\\
&\alpha : x_{j0} \mapsto x_{j1} \mapsto \cdots \mapsto x_{jp-1} \mapsto \zeta^{b_j} x_{j0} && \text{for } 1 \le j \le s+1, 0 \le b_j \le p-1.
\end{aligned}$$

Compare the actions of $\alpha, \beta_1, \dots, \beta_{s+1}$ with the actions of $\alpha, \alpha_k, \dots, \alpha_{k+s}$ from Case I, Step 1. They are almost the same. Apply the proof of Case I.

Case III. Assume that $\mathcal{H}_1$ is of type 2; that is, $\mathcal{H}_1 \simeq C_{p^a}$ for some $a \ge 1$. Denote by β the generator of C_{p^a}. Then $[\beta, \alpha] = \beta^{bp^{a-1}}$ for some $b : 0 \le b \le p-1$. Let $\zeta_{p^a} \in K$ be a primitive p^a-th root of unity, and let ζ be a primitive p-th root of unity. Define $X = \sum_i \zeta_{p^a}^{-i} x(\beta^i)$. Then $\beta(X) = \zeta_{p^a} X$, and define $x_i = \alpha^i \cdot X$ for

$0 \le i \le p-1$. It follows that

$$\begin{aligned} \beta &: x_i \mapsto \zeta_{p^a}\zeta^{ib}x_i && \text{for } 0 \le i \le p-1, \\ \alpha &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto \zeta_{p^a}^c x_0 && \text{for } 0 \le c \le p^a-1. \end{aligned}$$

Define $W_1 = \bigoplus_i K \cdot x_i \subset V^*$. For $1 \le i \le p-1$ define $y_i = x_i/x_{i-1}$. Thus $W_1 = K(x_0, y_i : 1 \le i \le p-1)$ and for every $g \in G$

$$g \cdot x_0 \in K(y_i : 1 \le i \le p-1) \cdot x_0,$$

while the subfield $K(y_i : 1 \le i \le p-1)$ is invariant by the action of G, that is,

$$\begin{aligned} \beta &: y_i \mapsto \zeta^b y_i && \text{for } 1 \le i \le p-1, \\ \alpha &: y_1 \mapsto y_2 \mapsto \cdots \mapsto \zeta_{p^a}^c (y_1 \cdots y_{p-1})^{-1} && \text{for } 0 \le c \le p^a-1. \end{aligned}$$

From Theorem 2.2 it follows that if $K(y_i : 1 \le i \le p-1)^G$ is rational over K, so is $K(x_0, y_i : 1 \le i \le p-1)^G$ over K.

Since K contains a primitive p^e-th root of unity ζ_{p^e}, where p^e is the exponent of G, K contains $\zeta_{p^{a+1}}^c$ as well. We may replace the variables y_i by $y_i/\zeta_{p^{a+1}}^c$ so that we obtain

$$\alpha : y_1 \mapsto y_2 \mapsto \cdots \mapsto y_{p-1} \mapsto (y_1 y_2 \cdots y_{p-1})^{-1}.$$

Define $u_1 = y_1^p$, $u_i = y_i/y_{i-1}$ for $2 \le i \le p-1$. Then $K(u_i : 1 \le i \le p-1) = K(y_i : 1 \le i \le p-1)^{\langle \beta \rangle}$. The action of α is given by

$$\alpha : u_1 \mapsto u_1 u_2^p,\ u_2 \mapsto u_3 \mapsto \cdots \mapsto u_{p-1} \mapsto (u_1 u_2^{p-1} u_3^{p-2} \cdots u_{p-1}^2)^{-1}.$$

From Lemma 2.4 (or 2.5) it follows that the action of α can be linearized.

Case IV. Assume that $\mathcal{H}_1$ is of type 4, that is, $\mathcal{H}_1 \simeq C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$ for some $k \ge 2$. Denote by $\alpha_1, \dots, \alpha_k$ the generators of $\mathcal{H}_1$. According to Lemma 1.7, we have the relations $[\alpha_i, \alpha] = \alpha_{i+1}^{p^{a_{i+1}-1}} \in Z(G)$ for $1 \le i \le k-1$ and $[\alpha_k, \alpha] = \prod_{j=1}^k \alpha_j^{p^{a_j-1}c_j} \in Z(G)$ for some $0 \le c_j \le p-1$.

Similarly to the previous cases, define $Y_1, Y_2, \dots, Y_k \in V^*$ so that

$$\alpha_i :\ Y_i \mapsto \zeta_{p^{a_i}} Y_i,\ Y_j \mapsto Y_j \quad \text{for } j \ne i \text{ and } 1 \le i \le k.$$

Thus $V_1 = \bigoplus_{1 \le j \le k} K \cdot Y_j$ is a faithful representation space of the subgroup $\mathcal{H}_1$.

Next, define $x_{ji} = \alpha^i \cdot Y_j$ for $1 \le j \le k$, $0 \le i \le p-1$. Note that

$$\alpha^{-i}\alpha_j\alpha^i = \alpha_j \alpha_{j+1}^{ip^{a_{j+1}-1}} \quad \text{for } 1 \le j \le k-1,\ 1 \le i \le p-1$$

and

$$\alpha^{-i}\alpha_k\alpha^i = \alpha_k \prod_{j=1}^k \alpha_j^{ip^{a_j-1}c_j} \quad \text{for } 1 \le i \le p-1.$$

It follows that

$$
\begin{array}{ll}
\alpha_\ell : x_{\ell i} \mapsto \zeta_{p^{a_\ell}} x_{\ell i},\ x_{\ell+1 i} \mapsto \zeta^i x_{\ell+1 i},\ x_{ji} \mapsto x_{ji} & \text{for } 1 \le \ell \le k-1,\ j \ne \ell, \ell+1, \\
\alpha_k : x_{ki} \mapsto \zeta_{p^{a_k}} \zeta^{ic_k} x_{ki},\ x_{ji} \mapsto \zeta^{ic_j} x_{ji} & \text{for } 1 \le j \le k-1, \\
\alpha : x_{j0} \mapsto x_{j1} \mapsto \cdots \mapsto x_{jp-1} \mapsto \zeta_{p^{a_j}}^{b_j} x_{j0} & \text{for } 1 \le j \le k,
\end{array}
$$

where $0 \le i \le p-1$, $0 \le c_j \le p-1$ and $0 \le b_j \le p^{a_j} - 1$.

Define $W_1 = \bigoplus_{j,i} K \cdot x_{ji} \subset V^*$, and for $1 \le i \le p-1$ define $y_i = x_i / x_{i-1}$. Thus $W_1 = K(x_{j0}, y_{ji} : 1 \le j \le k, 1 \le i \le p-1)$ and for every $g \in G$,

$$g \cdot x_{j0} \in K(y_{ji} : 1 \le j \le k, 1 \le i \le p-1) \cdot x_{j0} \quad \text{for } 1 \le j \le k,$$

while the subfield $K(y_{ji} : 1 \le j \le k, 1 \le i \le p-1)$ is invariant by the action of G, that is,

$$
\begin{array}{ll}
\alpha_\ell : y_{\ell+1 i} \mapsto \zeta y_{\ell+1 i},\ y_{ji} \mapsto y_{ji} & \text{for } 1 \le i \le p-1, \\
 & \qquad 1 \le \ell \le k-1,\ j \ne \ell+1, \\
\alpha_k : y_{ji} \mapsto \zeta^{c_j} y_{ji} & \text{for } 1 \le i \le p-1,\ 1 \le j \le k, \\
\alpha : y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto \zeta_{p^{a_j}}^{b_j} (y_{j1} \cdots y_{jp-1})^{-1}. &
\end{array}
$$

From Theorem 2.2 it follows that if $K(y_{ji} : 1 \le j \le k, 1 \le i \le p-1)^G$ is rational over K, so is $K(x_{j0}, y_{ji} : 1 \le j \le k, 1 \le i \le p-1)^G$ over K. As before, we can again assume that α acts in this way:

$$\alpha : \ y_{j1} \mapsto y_{j2} \mapsto \cdots \mapsto y_{jp-1} \mapsto (y_{j1} y_{j2} \cdots y_{jp-1})^{-1}.$$

Now, assume that $0 < c_1 \le p-1$. For $2 \le j \le k$ choose e_j such that $c_1 e_j + c_j \equiv 0 \pmod{p}$, and define $u_{1i} = y_{1i}$, $u_{ji} = y_{1i}^{e_j} y_{ji}$. It follows that

$$
\begin{array}{ll}
\alpha_\ell : u_{\ell+1 i} \mapsto \zeta u_{\ell+1 i},\ u_{ji} \mapsto u_{ji} & \text{for } 1 \le i \le p-1, \\
 & \qquad 1 \le \ell \le k-1,\ j \ne \ell+1, \\
\alpha_k : u_{1i} \mapsto \zeta^{c_1} u_{1i},\ u_{ji} \mapsto u_{ji} & \text{for } 1 \le i \le p-1,\ 2 \le j \le k.
\end{array}
$$

Define $v_{j1} = u_{j1}^p$, $v_{ji} = u_{ji} / u_{ji-1}$ for $2 \le i \le p-1$, $1 \le j \le k$. Then

$$K(v_{ji} : 1 \le j \le k, 1 \le i \le p-1) = K(u_{ji} : 1 \le j \le k, 1 \le i \le p-1)^{\mathcal{H}_1}.$$

The action of α is given by

$$\alpha : v_{j1} \mapsto v_{j1} v_{j2}^p,\ v_{j2} \mapsto v_{j3} \mapsto \cdots \mapsto v_{jp-1} \mapsto (v_{j1} v_{j2}^{p-1} v_{j3}^{p-2} \cdots v_{jp-1}^2)^{-1}$$

for $2 \le j \le k$. Lemma 2.5 implies the action of α on $K(v_{ji} : 1 \le j \le k, 1 \le i \le p-1)$ can be linearized.

Finally, let $c_1 = 0$. Define $v_{j1} = u_{j1}^p$, $v_{ji} = u_{ji}/u_{ji-1}$ for $2 \le i \le p-1$, $2 \le j \le k$. Then $K(u_{1i}, v_{ji} : 2 \le j \le k, 1 \le i \le p-1) = K(u_{ji} : 1 \le j \le k, 1 \le i \le p-1)^{\mathcal{H}_1}$. The action of α again can be linearized as before. We are done.

5. Proof of Theorem 1.9

By studying the classification of all groups of order p^5 made by James [1980], we see that the nonabelian groups with an abelian subgroup of index p and that are not direct products of smaller groups are precisely the groups from the isoclinic families with numbers 2, 3, 4 and 9. Notice that all these groups satisfy the conditions of Theorem 1.8. The isoclinic family 8 contains only the group $\Phi_8(32)$ which is metacyclic, so we can apply Theorem 1.2. It is not hard to see that there are no other groups of order p^5 containing a normal abelian subgroup H such that G/H is cyclic.

The groups of order p^6 with an abelian subgroup of index p and that are not direct products of smaller groups are precisely the groups from the isoclinic families with numbers 2, 3, 4 and 9. Again, all these groups satisfy the conditions of Theorem 1.8. The groups of order p^6, containing a normal abelian subgroup H such that G/H is cyclic of order $> p$ are precisely the groups from the isoclinic families with numbers 8 and 14. Note that the groups $\Phi_8(42)$, $\Phi_8(33)$, $\Phi_{14}(42)$ are metacyclic, and the group $\Phi_8(321)a$ is a direct product of the metacyclic group $\Phi_8(32)$ and the cyclic group C_p. Therefore, we need to consider the remaining groups, whose presentations we write down for convenience of the reader.

$$\Phi_8(321)b = \langle \alpha_1, \alpha_2, \beta, \gamma : [\alpha_1, \alpha_2] = \beta = \alpha_1^p,\ [\beta, \alpha_2] = \beta^p = \gamma^p,\ \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_8(321)c_r = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ [\beta, \alpha_2]^{r+1} = \beta^{p(r+1)} = \alpha_1^{p^2},\ \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_8(321)c_{p-1} = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ [\beta, \alpha_2] = \beta^p = \alpha_2^{p^2},\ \alpha_1^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_8(222) = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ [\beta, \alpha_2] = \beta^p,\ \alpha_1^{p^2} = \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_{14}(321) = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ \alpha_1^{p^2} = \beta^p,\ \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle,$$
$$\Phi_{14}(222) = \langle \alpha_1, \alpha_2, \beta : [\alpha_1, \alpha_2] = \beta,\ \alpha_1^{p^2} = \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle.$$

Case I. $G = \Phi_8(321)b$. Denote by H the abelian normal subgroup of G generated by α_1 and γ. Then $H = \langle \alpha_1, \gamma\beta^{-1} \rangle \simeq C_{p^3} \times C_p$ and $G/H = \langle \alpha_2 \rangle \simeq C_{p^2}$.

Let V be a K-vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in G} K \cdot x(g)$, where G acts on V^* by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Thus we have $K(V)^G = K(x(g) : g \in G)^G = K(G)$.

Define $X_1, X_2 \in V^*$ by

$$X_1 = \sum_{i=0}^{p-1} x((\gamma\beta^{-1})^i), \quad X_2 = \sum_{i=0}^{p^3-1} x(\alpha_1^i).$$

Note that $\gamma\beta^{-1} \cdot X_1 = X_1$ and $\alpha_1 \cdot X_2 = X_2$.

Let $\zeta_{p^3} \in K$ be a primitive p^3-th root of unity and put $\zeta = \zeta_{p^3}^{p^2}$, a primitive p-th root of unity. Define $Y_1, Y_2 \in V^*$ by

$$Y_1 = \sum_{i=0}^{p^3-1} \zeta_{p^3}^{-i} \alpha_1^i \cdot X_1, \quad Y_2 = \sum_{i=0}^{p-1} \zeta^{-i} (\gamma\beta^{-1})^i \cdot X_2.$$

It follows that

$$\begin{aligned} \alpha_1 &: Y_1 \mapsto \zeta_{p^3} Y_1, \ \ Y_2 \mapsto Y_2, \\ \gamma\beta^{-1} &: Y_1 \mapsto Y_1, \qquad Y_2 \mapsto \zeta Y_2, \\ \gamma &: Y_1 \mapsto \zeta_{p^2} Y_1, \ \ Y_2 \mapsto \zeta Y_2. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H.

Define $x_i = \alpha_2^i \cdot Y_1$, $y_i = \alpha_2^i \cdot Y_2$ for $0 \le i \le p^2 - 1$. From the relations $\alpha_1 \alpha_2^i = \alpha_2^i \alpha_1 \beta^i \beta^{\binom{i}{2} p}$ it follows that

$$\begin{aligned} \alpha_1 &: x_i \mapsto \zeta_{p^3} \zeta_{p^2}^i \zeta^{\binom{i}{2}} x_i, \ \ y_i \mapsto y_i, \\ \gamma &: x_i \mapsto \zeta_{p^2} x_i, \qquad y_i \mapsto \zeta y_i, \\ \alpha_2 &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^2-1} \mapsto x_0, \\ & \quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^2-1} \mapsto y_0, \end{aligned}$$

for $0 \le i \le p^2 - 1$.

We find that $Y = \left(\bigoplus_{0 \le i \le p^2-1} K \cdot x_i\right) \oplus \left(\bigoplus_{0 \le i \le p^2-1} K \cdot y_i\right)$ is a faithful G-subspace of V^*. Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0 \le i \le p^2-1)^G$ is rational over K.

For $1 \le i \le p^2 - 1$, define $u_i = x_i / x_{i-1}$ and $v_i = y_i / y_{i-1}$. Thus

$$K(x_i, y_i : 0 \le i \le p^2 - 1) = K(x_0, y_0, u_i, v_i : 1 \le i \le p^2 - 1)$$

and for every $g \in G$

$$g \cdot x_0 \in K(u_i, v_i : 1 \le i \le p^2 - 1) \cdot x_0, \ \ g \cdot y_0 \in K(u_i, v_i : 1 \le i \le p^2 - 1) \cdot y_0,$$

while the subfield $K(u_i, v_i : 1 \le i \le p^2 - 1)$ is invariant by the action of G. Thus $K(x_i, y_i : 0 \le i \le p^2 - 1)^G = K(u_i, v_i : 1 \le i \le p^2 - 1)^G(u, v)$ for some u, v such that $\alpha_1(v) = \gamma(v) = \alpha_2(v) = v$ and $\alpha_1(u) = \gamma(u) = \alpha_2(u) = u$. We now have

$$\begin{aligned} \alpha_1 &: u_i \mapsto \zeta_{p^2} \zeta^{i-1} u_i, \ \ v_i \mapsto v_i, \\ \gamma &: u_i \mapsto u_i, \qquad v_i \mapsto v_i, \\ \alpha_2 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1 u_2 \cdots u_{p^2-1})^{-1}, \\ & \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^2-1} \mapsto (v_1 v_2 \cdots v_{p^2-1})^{-1}, \end{aligned} \tag{5-1}$$

for $1 \le i \le p^2-1$. If $K(u_i, v_i : 1 \le i \le p^2-1)^G(u, v)$ is rational over K, it follows from Theorem 2.2 that $K(x_i, y_i : 0 \le i \le p^2-1)^G$ is rational over K.

Since γ acts trivially on $K(u_i, v_i : 1 \le i \le p^2-1)$, we find that

$$K(u_i, v_i : 1 \le i \le p^2-1)^G = K(u_i, v_i : 1 \le i \le p^2-1)^{\langle \alpha_1, \alpha_2 \rangle}.$$

Now, consider the metacyclic p-group

$$\widetilde{G} = \langle \sigma, \tau : \sigma^{p^3} = \tau^{p^2} = 1, \tau^{-1}\sigma\tau = \sigma^k, k = 1+p \rangle.$$

Define $X = \sum_{0 \le j \le p^3-1} \zeta_{p^3}^{-j} x(\sigma^j)$, $V_i = \tau^i X$ for $0 \le i \le p^2-1$. It follows that

$$\begin{aligned} \sigma &: V_i \mapsto \zeta_{p^3}^{k^i} V_i, \\ \tau &: V_0 \mapsto V_1 \mapsto \cdots \mapsto V_{p^2-1} \mapsto V_0. \end{aligned}$$

Note that $K(V_0, V_1, \ldots, V_{p^2-1})^{\widetilde{G}}$ is rational by Theorem 2.6.

Define $U_i = V_i/V_{i-1}$ for $1 \le i \le p^2-1$. Then $K(V_0, V_1, \ldots, V_{p^2-1})^{\widetilde{G}} = K(U_1, U_2, \ldots, U_{p^2-1})^{\widetilde{G}}(U)$, where

$$\begin{aligned} &\sigma : U \mapsto U, \ \ U_i \mapsto \zeta_{p^3}^{k^i - k^{i-1}} U_i, \\ &\tau : U \mapsto U, \ \ U_1 \mapsto U_2 \mapsto \cdots \mapsto U_{p^2-1} \mapsto (U_1 U_2 \cdots U_{p^2-1})^{-1}. \end{aligned}$$

Notice that $k^i - k^{i-1} = (1+p)^{i-1}p \equiv (1+(i-1)p)p \pmod{p^3}$, so $\zeta_{p^3}^{k^i-k^{i-1}} = \zeta_{p^2}^{1+(i-1)p}$. Compare the first and third entries of (5-1) (i.e., the actions of α_1, α_2 on $K(u_i : 1 \le i \le p^2-1)$) with the actions of $\widetilde{G}$ on $K(U_i : 1 \le i \le p^2-1)$. They are the same. Hence, according to Theorem 2.6, we get that $K(u_1, \ldots, u_{p^2-1})^G(u) \cong K(U_1, \ldots, U_{p^2-1})^{\widetilde{G}}(U) = K(V_0, V_1, \ldots, V_{p^2-1})^{\widetilde{G}}$ is rational over K. Since by Lemma 2.4 we can linearize the action of α_2 on $K(v_i : 1 \le i \le p^2-1)$, we finally obtain that $K(u_i, v_i : 1 \le i \le p^2-1)^{\langle \alpha_1, \alpha_2 \rangle}$ is rational over K.

Case II. $G = \Phi_8(321)c_r$. Denote by H the abelian normal subgroup of G generated by α_1 and β. Then $H = \langle \alpha_1, \alpha_1^{-p}\beta^{r+1} \rangle \simeq C_{p^3} \times C_p$ and $G/H = \langle \alpha_2 \rangle \simeq C_{p^2}$. Let $a = (r+1)^{-1} \in \mathbb{Z}_{p^2}$, hence $\beta = \alpha_1^{ap}(\alpha_1^{-p}\beta^{r+1})^a$. Similarly to Case I, we can define $Y_1, Y_2 \in V^*$ such that

$$\begin{aligned} \alpha_1 &: \ Y_1 \mapsto \zeta_{p^3} Y_1, \ \ Y_2 \mapsto Y_2, \\ \alpha_1^{-p}\beta^{r+1} &: \ Y_1 \mapsto Y_1, \qquad Y_2 \mapsto \zeta Y_2, \\ \beta &: \ Y_1 \mapsto \zeta_{p^2}^a Y_1, \ \ Y_2 \mapsto \zeta^a Y_2. \end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H.

Define $x_i = \alpha_2^i \cdot Y_1$, $y_i = \alpha_2^i \cdot Y_2$ for $0 \le i \le p^2-1$. From the relations $\alpha_1\alpha_2^i = \alpha_2^i\alpha_1\beta^i\beta^{\binom{i}{2}p}$ and $\beta\alpha_2^i = \alpha_2^i\beta^{1+ip}$ it follows that, for $0 \le i \le p^2-1$,

$$\begin{aligned}
\alpha_1 &: x_i \mapsto \zeta_{p^3}\zeta_{p^2}^{ai}\zeta^{a\binom{i}{2}}x_i, \quad y_i \mapsto \zeta^{ai}y_i,\\
\beta &: x_i \mapsto \zeta_{p^2}^{a}\zeta^{ai}x_i, \quad y_i \mapsto \zeta^{a}y_i,\\
\alpha_2 &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^2-1} \mapsto x_0,\\
&\quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^2-1} \mapsto y_0.
\end{aligned}$$

We find that $Y = \left(\bigoplus_{0\le i\le p^2-1} K\cdot x_i\right) \oplus \left(\bigoplus_{0\le i\le p^2-1} K\cdot y_i\right)$ is a faithful G-subspace of V^*. Thus, by Theorem 2.1, it suffices to show that $K(x_i, y_i : 0\le i\le p^2-1)^G$ is rational over K.

For $1\le i\le p^2-1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. We now have

$$\begin{aligned}
\alpha_1 &: u_i \mapsto \zeta_{p^2}^{a}\zeta^{a(i-1)}u_i, \quad v_i \mapsto \zeta^{a}v_i,\\
\beta &: u_i \mapsto \zeta^{a}u_i, \quad v_i \mapsto v_i,\\
\alpha_2 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1u_2\cdots u_{p^2-1})^{-1},\\
&\quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^2-1} \mapsto (v_1v_2\cdots v_{p^2-1})^{-1},
\end{aligned}$$

for $1\le i\le p^2-1$. Theorem 2.2 implies that if $K(u_i, v_i : 1\le i\le p^2-1)^G(u, v)$ is rational over K, so is $K(x_i, y_i : 0\le i\le p^2-1)^G$ over K.

Since β acts in the same way as α_1^p on $K(u_i, v_i : 1\le i\le p^2-1)$, we find that $K(u_i, v_i : 1\le i\le p^2-1)^G = K(u_i, v_i : 1\le i\le p^2-1)^{\langle\alpha_1,\alpha_2\rangle}$.

For $1\le i\le p^2-1$ define $V_i = v_i/u_i^p$. It follows that, for $1\le i\le p^2-1$,

$$\begin{aligned}
\alpha_1 &: u_i \mapsto \zeta_{p^2}^{a}\zeta^{a(i-1)}u_i, \quad V_i \mapsto V_i,\\
\alpha_2 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1u_2\cdots u_{p^2-1})^{-1},\\
&\quad V_1 \mapsto V_2 \mapsto \cdots \mapsto V_{p^2-1} \mapsto (V_1V_2\cdots V_{p^2-1})^{-1}.
\end{aligned} \tag{5-2}$$

Compare (5-2) with (5-1). They look almost the same. Apply the proof of Case I.

Case III. $G = \Phi_8(321)c_{p-1}$. Denote by H the abelian normal subgroup of G generated by α_1 and β. Then $H \simeq C_{p^2}\times C_{p^2}$ and $G/H \simeq C_{p^2}$. Similarly to Case I, we can define $Y_1, Y_2 \in V^*$ such that

$$\begin{aligned}
\alpha_1 &: Y_1 \mapsto \zeta_{p^2}Y_1, \quad Y_2 \mapsto Y_2,\\
\beta &: Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_{p^2}Y_2.
\end{aligned}$$

Thus $K\cdot Y_1 + K\cdot Y_2$ is a representation space of the subgroup H.

Define $x_i = \alpha_2^i\cdot Y_1$, $y_i = \alpha_2^i\cdot Y_2$ for $0\le i\le p^2-1$. From the relations $\alpha_1\alpha_2^i = \alpha_2^i\alpha_1\beta^i\beta^{\binom{i}{2}p}$ and $\beta\alpha_2^i = \alpha_2^i\beta^{1+ip}$ it follows that, for $0\le i\le p^2-1$,

$$\begin{aligned}
\alpha_1 &: x_i \mapsto \zeta_{p^2}x_i, \quad y_i \mapsto \zeta_{p^2}^{i}\zeta^{\binom{i}{2}}y_i,\\
\beta &: x_i \mapsto x_i, \qquad y_i \mapsto \zeta_{p^2}\zeta^{i}y_i,
\end{aligned}$$

$$\begin{aligned}\alpha_2 : x_0 &\mapsto x_1 \mapsto \cdots \mapsto x_{p^2-1} \mapsto x_0,\\ y_0 &\mapsto y_1 \mapsto \cdots \mapsto y_{p^2-1} \mapsto \zeta y_0.\end{aligned}$$

For $1 \le i \le p^2 - 1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. We now have

$$\begin{aligned}\alpha_1 &: u_i \mapsto u_i, \quad v_i \mapsto \zeta_{p^2}\zeta^{i-1}v_i,\\ \beta &: u_i \mapsto u_i, \quad v_i \mapsto \zeta v_i,\\ \alpha_2 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1u_2\cdots u_{p^2-1})^{-1},\\ &\quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^2-1} \mapsto \zeta(v_1v_2\cdots v_{p^2-1})^{-1},\end{aligned}$$

for $1 \le i \le p^2-1$. Since β acts in the same way as α_1^p on $K(u_i, v_i : 1 \le i \le p^2-1)$, we find that $K(u_i, v_i : 1 \le i \le p^2-1)^G = K(u_i, v_i : 1 \le i \le p^2-1)^{\langle\alpha_1,\alpha_2\rangle}$.

Let $\zeta_{p^3} \in K$ be a primitive p^3-th root of unity such that $\zeta_{p^3}^{p^2} = \zeta$. For $1 \le i \le p^2-1$ define $w_i = v_i/\zeta_{p^3}$. It follows that

$$\begin{aligned}\alpha_1 &: \ u_i \mapsto u_i, \quad w_i \mapsto \zeta_{p^2}\zeta^{i-1}w_i,\\ \alpha_2 &: \ u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1u_2\cdots u_{p^2-1})^{-1},\\ &\quad\ w_1 \mapsto w_2 \mapsto \cdots \mapsto w_{p^2-1} \mapsto (w_1w_2\cdots w_{p^2-1})^{-1},\end{aligned} \tag{5-3}$$

for $1 \le i \le p^2 - 1$. Compare (5-3) with (5-1) or (5-2). They look almost the same. Apply the proof of Case I.

Case IV. $G = \Phi_8(222)$. Denote by H the abelian normal subgroup of G generated by α_1 and β. Then $H \simeq C_{p^2} \times C_{p^2}$ and $G/H \simeq C_{p^2}$. The proof henceforth is almost the same as Case III.

Case V. $G = \Phi_{14}(321)$. Denote by H the abelian normal subgroup of G generated by α_2 and β. Then $H \simeq C_{p^2} \times C_{p^2}$ and $G/H \simeq C_{p^2}$.

As before, we can define $Y_1, Y_2 \in V^*$ such that

$$\begin{aligned}\alpha_2 &: \ Y_1 \mapsto \zeta_{p^2}Y_1, \quad Y_2 \mapsto Y_2,\\ \beta &: \ Y_1 \mapsto Y_1, \quad Y_2 \mapsto \zeta_{p^2}Y_2.\end{aligned}$$

Thus $K \cdot Y_1 + K \cdot Y_2$ is a representation space of the subgroup H.

Define $x_i = \alpha_1^i \cdot Y_1$, $y_i = \alpha_1^i \cdot Y_2$ for $0 \le i \le p^2 - 1$. From the relations $\alpha_2\alpha_1^i = \alpha_1^i\alpha_2\beta^{-i}$ it follows that, for $0 \le i \le p^2 - 1$,

$$\begin{aligned}\alpha_2 &: x_i \mapsto \zeta_{p^2}x_i, \quad y_i \mapsto \zeta_{p^2}^{-i}y_i,\\ \beta &: x_i \mapsto x_i, \qquad y_i \mapsto \zeta_{p^2}y_i,\\ \alpha_1 &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p^2-1} \mapsto x_0,\\ &\quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p^2-1} \mapsto \zeta y_0.\end{aligned}$$

For $1 \le i \le p^2-1$, define $u_i = x_i/x_{i-1}$ and $v_i = y_i/y_{i-1}$. We now have

$$\begin{aligned}
\alpha_2 &: u_i \mapsto u_i, \quad v_i \mapsto \zeta_{p^2}^{-1} v_i, \\
\beta &: u_i \mapsto u_i, \quad v_i \mapsto v_i, \\
\alpha_1 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p^2-1} \mapsto (u_1 u_2 \cdots u_{p^2-1})^{-1}, \\
&\quad\; v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p^2-1} \mapsto \zeta (v_1 v_2 \cdots v_{p^2-1})^{-1},
\end{aligned}$$

for $1 \le i \le p^2-1$. Since β acts trivially on $K(u_i, v_i : 1 \le i \le p^2-1)$, we find that $K(u_i, v_i : 1 \le i \le p^2-1)^G = K(u_i, v_i : 1 \le i \le p^2-1)^{\langle \alpha_1, \alpha_2 \rangle}$.

Define $w_1 = v_1^{p^2} \zeta^{-1}$, $w_i = v_i/v_{i-1}$ for $2 \le i \le p^2-1$. We now have

$$K(v_1, \ldots, v_{p^2-1})^{\langle \alpha_2 \rangle} = K(w_1, \ldots, w_{p^2-1})$$

and

$$\begin{aligned}
\alpha_1 &: w_1 \mapsto w_2^{p^2} w_1, \\
&\quad\; w_2 \mapsto w_3 \mapsto \cdots \mapsto w_{p^2-1} \mapsto 1/(w_1 w_2^{p^2-1} w_3^{p^2-2} \cdots w_{p^2-1}^2).
\end{aligned}$$

Define $z_1 = w_2$, $z_i = \alpha_1^{i-1} \cdot w_2$ for $2 \le i \le p^2-1$. Then $K(w_i : 1 \le i \le p^2-1) = K(z_i : 1 \le i \le p^2-1)$ and

$$\alpha_1 : \; z_1 \mapsto z_2 \mapsto \cdots \mapsto z_{p^t-1} \mapsto (z_1 z_2 \cdots z_{p^2-1})^{-1}.$$

The action of α_1 can be linearized by Lemma 2.4. Thus $K(u_i, z_i : 1 \le i \le p^2-1)^{\langle \alpha_1 \rangle}$ is rational over K by Theorem 2.1. We are done.

Case VI. $G = \Phi_{14}(222)$. Denote by H the abelian normal subgroup of G generated by α_2 and β. Then $H \simeq C_{p^2} \times C_{p^2}$ and $G/H \simeq C_{p^2}$. The proof henceforth is almost the same as Case V.

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LEGENDRIAN θ-GRAPHS

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We give necessary and sufficient conditions for two triples of integers to be realized as the Thurston–Bennequin number and the rotation number of a Legendrian θ-graph with all cycles unknotted. We show that these invariants are not enough to determine the Legendrian class of a topologically planar θ-graph. We define the transverse push-off of a Legendrian graph, and we determine its self linking number for Legendrian θ-graphs. In the case of topologically planar θ-graphs, we prove that the topological type of the transverse push-off is that of a pretzel link.

1. Introduction

In this paper, we continue the systematic study of Legendrian graphs in $(\mathbb{R}^3, \xi_{\text{std}})$ initiated in [O'Donnol and Pavelescu 2012]. Legendrian graphs have appeared naturally in several important contexts in the study of contact manifolds. They are used in Giroux's proof [2002] of existence of open book decompositions compatible with a given contact structure. Legendrian graphs also appeared in Eliashberg and Fraser's proof [2009] of the Legendrian simplicity of the unknot.

In this article we focus on Legendrian θ-graphs. We predominantly work with topologically planar embeddings and embeddings where all the cycles are unknots. In the first part, we investigate questions about realizability of the classical invariants and whether the Legendrian type can be determined by these invariants. In the second part, we introduce the transverse push-off a Legendrian graph and investigate its properties in the case of θ-graphs.

O'Donnol and Pavelescu [2012] extended the classical invariants Thurston–Bennequin number, tb, and rotation number, rot, from Legendrian knots to Legendrian graphs. Here we prove that all possible pairs of (tb, rot) for a θ-graph with unknotted cycles are realized. It is easily shown that all pairs of integers (tb, rot) of different parities and such that $\text{tb} + |\text{rot}| \leq -1$ can be realized as the Thurston–Bennequin number and the rotation number of a Legendrian unknot. We call a pair of integers *acceptable* if they satisfy the two restrictions above. For

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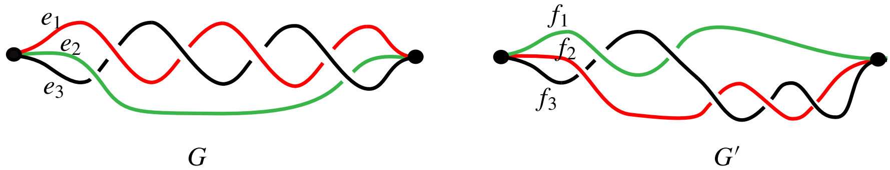


Figure 1. Non-Legendrian isotopic graphs with the same invariants.

θ-graphs, we show the following:

Theorem 1. *Any two triples of integers* $(\mathrm{tb}_1, \mathrm{tb}_2, \mathrm{tb}_3)$ *and* $(\mathrm{rot}_1, \mathrm{rot}_2, \mathrm{rot}_3)$ *for which* $(\mathrm{tb}_i, \mathrm{rot}_i)$ *are acceptable and* $R = \mathrm{rot}_1 - \mathrm{rot}_2 + \mathrm{rot}_3 \in \{0, -1\}$ *can be realized as the Thurston–Bennequin number and the rotation number of a Legendrian* θ*-graph with all cycles unknotted.*

It is known that certain Legendrian knots and links are determined by the invariants tb and rot: the unknot [Eliashberg and Fraser 2009], torus knots and the figure eight knot [Etnyre and Honda 2001], and links consisting of an unknot and a cable of that unknot [Ding and Geiges 2007]. To ask the same question in the context of Legendrian graphs, we restrict to topologically planar Legendrian θ-graphs. A *topologically planar graph* is one which is ambient isotopic to a planar embedding. The answer is no, the Thurston–Bennequin number and the rotation number do not determine the Legendrian type of a topologically planar θ-graph. The pair of graphs in Figure 1 provides a counterexample.

The second part of this article is concerned with Legendrian ribbons of Legendrian θ-graphs and their boundary. Roughly, a ribbon of a Legendrian graph g is a compact oriented surface R_g containing g in its interior, such that there is a natural contraction of R_g to g and ∂R_g is a transverse knot or link. We define the *transverse push-off of* g to be the boundary of R_g. This introduces two new invariants of Legendrian graphs, the transverse push-off and its self linking number. In the case of a Legendrian knot, this definition gives a two component link consisting of both the positive and the negative transverse push-offs. However, with graphs the transverse push-off can have various numbers of components, depending on the structure of the abstract graph and Legendrian type.

We show the push-off of a Legendrian θ-graph is either a transverse knot K with $\mathrm{sl} = 1$ or a three component transverse link whose three components are the positive transverse push-offs of the three Legendrian cycles given the correct orientation. For topologically planar graphs, the topological type of ∂R_g is determined solely by the Thurston–Bennequin number of g, thus:

Theorem 2. *Let* G *represent a topologically planar Legendrian* θ*-graph with* $\mathrm{tb} = (\mathrm{tb}_1, \mathrm{tb}_2, \mathrm{tb}_3)$. *Then the transverse push-off of* G *is an* (a_1, a_2, a_3)*-pretzel link, where* $a_1 = \mathrm{tb}_1 + \mathrm{tb}_2 - \mathrm{tb}_3$, $a_2 = \mathrm{tb}_1 + \mathrm{tb}_3 - \mathrm{tb}_2$, $a_3 = \mathrm{tb}_2 + \mathrm{tb}_3 - \mathrm{tb}_1$.

This elegant relation is specific to θ-graphs and does not generalize to θ_n-graphs for $n > 3$. We give examples to sustain this claim in the last part of the article. This phenomenon is due to the relationship between flat vertex graphs and pliable vertex graphs in the special case of all vertices of degree at most three.

2. Background

We give a short overview of contact structures, Legendrian and transverse knots and their invariants. We recall how the invariants of Legendrian knots can be extended to Legendrian graphs. Let M be an oriented 3-manifold and let ξ be a 2-plane field on M. If $\xi = \ker\alpha$ for some 1-form α on M satisfying $\alpha \wedge d\alpha > 0$, then ξ is a *contact structure* on M. On $\mathbb{R}^3$, the 1-form $\alpha = dz - y\,dx$ defines a contact structure called the standard contact structure, ξ_{std}. Throughout this article we work in $(\mathbb{R}^3, \xi_{\text{std}})$.

A knot $K \subset (M, \xi)$ is called *Legendrian* if, for all $p \in K$, the tangent T_pK is contained in the contact plane ξ_p at p. A spatial graph G is called *Legendrian* if all its edges are Legendrian curves that are nontangent to each other at the vertices. If all edges around a vertex are oriented outward, then no two tangent vectors at the vertex coincide in the contact plane. However, two tangent vectors may have the same direction but different orientations resulting in a smooth arc through the vertex. It is a result of this structure that the order of the edges around a vertex in a contact plane is not changed up to cyclic permutation under Legendrian isotopy. We study Legendrian knots and graphs via their front projection, the projection on the xz-plane. Two generic front projections of a Legendrian graph are related by Reidemeister moves I, II and III, together with three moves IV, V and VI, given by the mutual position of vertices and edges [Baader and Ishikawa 2009]; see Figure 2. Here forward we will refer to these moves as RI, RII, RIII, RIV, RV and RVI.

Apart from the topological knot class, there are two classical invariants of Legendrian knots: the Thurston–Bennequin number, tb, and the rotation number, rot. The Thurston–Bennequin number is independent of the orientation on K and measures the twisting of the contact framing on K with respect to the Seifert framing. To compute tb of a Legendrian knot K, consider a nonzero vector field v transverse to ξ, take the push-off K' of K in the direction of v, and define $\text{tb}(K) := \text{lk}(K, K')$. For a Legendrian knot K, $\text{tb}(K)$ can be computed in terms of the writhe and the number of cusps in its front projection $\tilde{K}$ as

$$\text{tb}(K) = \text{writhe}(\tilde{K}) - \tfrac{1}{2}\text{cusps}(\tilde{K}).$$

To define the rotation number, $\text{rot}(K)$, consider the positively oriented trivialization $\{d_1 = \partial/\partial y, d_2 = -y\,\partial/\partial z - \partial/\partial x\}$ for ξ_{std}. Let v be a nonzero vector field tangent to K pointing in the direction of the orientation on K. The winding number

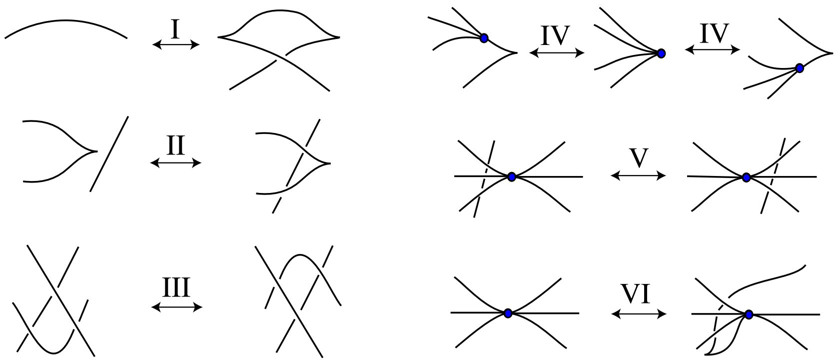


Figure 2. Legendrian isotopy moves for graphs: RI, RII and RIII, a vertex passing through a cusp (RIV), an edge passing under or over a vertex (RV), an edge adjacent to a vertex rotates to the other side of the vertex (RVI). Reflections of these moves that are Legendrian front projections are also allowed.

of v about the origin with respect to this trivialization is the rotation number of K, denoted $\operatorname{rot}(K)$. One can check that for $\tilde{K}$ the front projection for K,

$$\operatorname{rot}(K) = \tfrac{1}{2}(\downarrow\text{cusps}(\tilde{K}) - \uparrow\text{cusps}(\tilde{K})),$$

where $\downarrow$cusps ($\uparrow$cusps) denotes the number of down (up) cusps in the diagram.

Given a Legendrian knot K, Legendrian knots in the same topological class as K can be obtained by stabilizations. A *stabilization* means replacing a strand of K in the front projection of K by one of the zig-zags in Figure 3. The stabilization is said to be positive if down cusps are introduced and negative if up cusps are introduced. The Legendrian isotopy type of K changes through stabilization and so do the Thurston–Bennequin number and rotation number: $\operatorname{tb}(S_\pm(K)) = \operatorname{tb}(K) - 1$ and $\operatorname{rot}(S_\pm(K)) = \operatorname{rot}(K) \pm 1$.

Both the Thurston–Bennequin number and the rotation number can be extended to piecewise smooth Legendrian knots and to Legendrian graphs [O'Donnol and Pavelescu 2012]. For a Legendrian graph G, fix an order on the cycles of G and

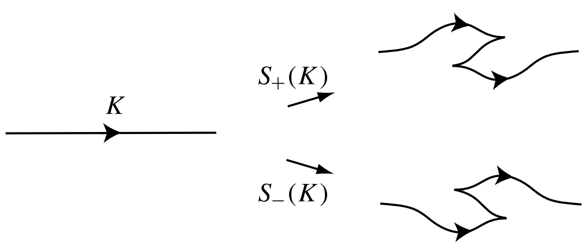


Figure 3. Positive and negative stabilizations in the front projection.

define $\mathrm{tb}(G)$ as the ordered list of Thurston–Bennequin numbers of the cycles of G. Once we fix an order on the cycles of G with orientation, we define $\mathrm{rot}(G)$ to be the ordered list of rotation numbers of the cycles of G. If G has no cycles, define both $\mathrm{tb}(G)$ and $\mathrm{rot}(G)$ to be the empty list.

An oriented knot $t \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ is called *transverse* if, for all $p \in t$, the tangent $T_p t$ is positively transverse to the contact plane ξ_p at p. If t is transverse, we let Σ be an oriented surface with $t = \partial\Sigma$. Then, $\xi|_\Sigma$ is trivial, so there is a nonzero vector field v over Σ in ξ. If t' is obtained by pushing t slightly in the direction of v, then the *self linking number* of t is $\mathrm{sl}(t) = \mathrm{lk}(t, t')$. It is easily seen that if $\tilde{t}$ is the front projection of t, then $\mathrm{sl}(t) = \mathrm{writhe}(\tilde{t})$.

For an embedded surface $\Sigma \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$, the intersection $l_x = T_x\Sigma \cap \xi_x$ is a line for most $x \in \Sigma$, except where the contact plane and the plane tangent to Σ coincide. We denote by $l := \bigcup l_x \subset T\Sigma$ this singular line field, where the union includes lines of intersection only. Then, there is a singular foliation $\mathcal{F}$, called *the characteristic foliation on* Σ, whose leaves are tangent to l. The characteristic foliation is used in the precise definition of Legendrian ribbon, given in Section 4.

3. Realization theorem

In this section we find which triples of integers can be realized as tb and rot of Legendrian θ-graphs with all cycles unknotted. Both the structure of the θ-graph and the required unknotted cycles impose restrictions on these integers. We also investigate whether tb and rot uniquely determine the Legendrian type.

Eliashberg and Fraser [2009] showed that a Legendrian unknot K is Legendrian isotopic to a unique unknot in standard form. The standard forms are shown in Figure 4. The number of cusps and the number of crossings of the unknot in standard form are uniquely determined by $\mathrm{tb}(K)$ and $\mathrm{rot}(K)$ as follows:

(1) If $\mathrm{rot}(K) \neq 0$ (Figure 4, left), then $\mathrm{tb}(K) = -(2t+1+s)$ and

$$\mathrm{rot}(K) = \begin{cases} s & \text{if the leftmost cusp is a down cusp,} \\ -s & \text{if the leftmost cusp is an up cusp.} \end{cases}$$

(2) If $\mathrm{rot}(K) = 0$ (Figure 4, right), then

$$\mathrm{tb}(K) = -(2t+1).$$

The following lemma identifies restrictions on the invariants of Legendrian unknots.

Lemma 3. *A pair of integers* $(\mathrm{tb}, \mathrm{rot})$ *can be realized as the Thurston–Bennequin number and the rotation number of a Legendrian unknot if and only if they are of different parities and*

$$\mathrm{tb} + |\mathrm{rot}| \leq -1.$$

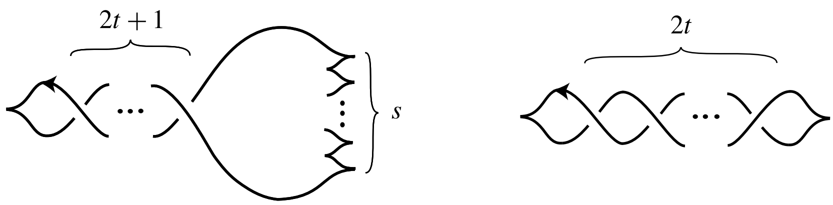


Figure 4. Legendrian unknot in standard form. Left: $\mathrm{rot}(K) > 0$ (the reverse orientation gives $\mathrm{rot}(K) < 0$). Right: $\mathrm{rot}(K) = 0$.

Proof. We know from [Eliashberg 1992] that for a Legendrian unknot K in $(\mathbb{R}^3, \xi_{\mathrm{std}})$, we have $\mathrm{tb}(K) + |\mathrm{rot}(K)| \le -1$. From [Eliashberg and Fraser 2009], explained above, we see that tb and rot have different parities.

For a pair (tb, rot):

- If $\mathrm{rot} > 0$, the pair (tb, rot) is realized via the Legendrian unknot with front projection as in Figure 4, left, for $(t, s) = \left(-\frac{1}{2}(\mathrm{tb} + \mathrm{rot} + 1), \mathrm{rot}\right)$.
- If $\mathrm{rot} < 0$, the pair (tb, rot) is realized via the Legendrian unknot with front projection as in Figure 4, left, for $(t, s) = \left(-\frac{1}{2}(\mathrm{tb} - \mathrm{rot} + 1), -\mathrm{rot}\right)$.
- If $\mathrm{rot} = 0$, the pair (tb, rot) is realized via the Legendrian unknot with front projection as in Figure 4, right, for $t = -\frac{1}{2}(\mathrm{tb} + 1)$. □

We have described the pairs (tb, rot) that can occur for the unknot.

Towards the proof of Theorem 1, we show in the next lemma that Legendrian θ-graphs can be standardized near their two vertices.

Lemma 4. *Any Legendrian θ-graph G can be Legendrian isotoped to a graph $\tilde{G}$ whose front projection looks as in Figure 5 in the neighborhood of its two vertices.*

Proof. Label the vertices of G by a and b. In the front projection of G, use RVI, if necessary, to move the three strands on the right of vertex a while near a and on the left of vertex b while near b. Then, small enough neighborhoods of the two vertices look as in Figure 5. □

For the remainder of this section, we assume that near its two vertices, a and b, the front projection of the graph looks as in Figure 5. We fix notation: e_1, e_2, e_3 are

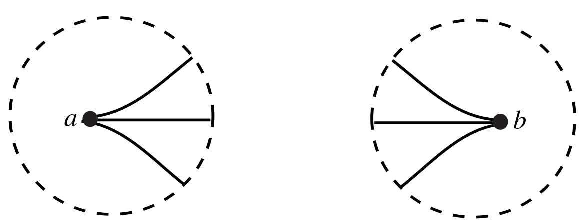


Figure 5. A Legendrian θ-graph near its two vertices.

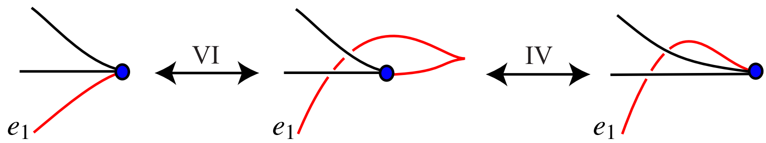


Figure 6. Moving edge e_1 at the right vertex.

respectively the top, middle, and lower strands at a in the front projection; $\mathscr{C}_1$ is the oriented cycle going out of vertex a along e_1 and into a along e_2; $\mathscr{C}_2$ exits a along e_1 and enters a along e_3; and $\mathscr{C}_3$ exits a along e_2 and enters a along e_3. We note that there is no consistent way of orienting the three edges which gives three oriented cycles. It should also be noted that the above edge labeling is given after the graph is embedded. If a labeled graph is embedded, relabeling of the graph and reorienting of the cycles may be necessary in order to have Lemma 6 apply.

Remark 5. Once the edges at the left vertex a are labeled e_1, e_2, e_3 from top to bottom, the edges can be moved around the right vertex (using a combination of RVI and RIV) so that edge e_1 is also in top position at vertex b in the front projection. An example of moving e_1 from bottom position to top position next to b is shown in Figure 6. There are two possibilities for the order of edges at the right vertex b. The first case, where the edges are e_1, e_2, e_3 from top to bottom, we will call *parallel vertices*. The second case, where the edges are e_1, e_3, e_2 from top to bottom, we will call *antiparallel vertices*.

In the next lemma we show what additional restrictions occur as a result of the structure of the θ-graph.

Lemma 6. *Let* $\mathrm{rot}_1, \mathrm{rot}_2$ *and* rot_3 *be integers representing rotation numbers for cycles* $\mathscr{C}_1, \mathscr{C}_2$ *and* $\mathscr{C}_3$, *in the above notation. Then* $\mathrm{rot}_1 - \mathrm{rot}_2 + \mathrm{rot}_3 \in \{0, -1\}$.

Proof. Consider an arbitrary Legendrian θ-graph in front projection that has been labeled and isotoped as described above. For $i = 1, 2, 3$, let k_i (k_i') represent the number of down (up) cusps along the edge e_i when oriented from vertex a to vertex b. Let $s_i := \frac{1}{2}(k_i - k_i')$ for $i = 1, 2, 3$. Then, since $\mathscr{C}_1$ has a down cusp at b, we know $\mathrm{rot}_1 = s_1 - s_2$; since $\mathscr{C}_2$ has a down cusp at b, we know $\mathrm{rot}_2 = s_1 - s_3$; and

$$\mathrm{rot}_3 = \begin{cases} s_2 - s_3 & \text{if } \mathscr{C}_3 \text{ has a down cusp at } b \text{ (parallel vertices)}, \\ s_2 - s_3 - 1 & \text{if } \mathscr{C}_3 \text{ has an up cusp at } b \text{ (antiparallel vertices)}. \end{cases}$$

This gives two possible values for $R = \mathrm{rot}_1 - \mathrm{rot}_2 + \mathrm{rot}_3 \in \{0, -1\}$. □

Remark 7. The proof of Lemma 6 implies for Legendrian θ-graphs that the cyclic order of the edges at one vertex is determined by the cyclic order of edges at the other vertex and the parity of the sum of the rotation numbers.

	rot_1	rot_2	rot_3	$R=0$	$R=-1$
(i)	+	+	+	$r_1-r_2+r_3=0$	$r_1+r_3+1=r_2$
(ii)	+	+	−	$r_1-r_2-r_3=0$	$r_1+1=r_2+r_3$
(iii)	+	−	+	$r_1+r_2+r_3=0$	$r_1+r_2+r_3+1=0$
(iv)	+	−	−	$r_1+r_2-r_3=0$	$r_1+r_2+1=r_3$
(v)	−	+	+	$-r_1-r_2+r_3=0$	$r_1+r_2=r_3+1$
(vi)	−	+	−	$-r_1-r_2-r_3=0$	$r_1+r_2+r_3=1$
(vii)	−	−	+	$-r_1+r_2+r_3=0$	$r_1=r_2+r_3+1$
(viii)	−	−	−	$-r_1+r_2-r_3=0$	$r_1+r_3=r_2+1$

Table 1. The $+$ stands for $\mathrm{rot}_i \geq 0$, and the $-$ stands for $\mathrm{rot}_i < 0$.

Theorem 8. *Any two triples of integers* $(\mathrm{tb}_1, \mathrm{tb}_2, \mathrm{tb}_3)$ *and* $(\mathrm{rot}_1, \mathrm{rot}_2, \mathrm{rot}_3)$ *for which* $\mathrm{tb}_i + |\mathrm{rot}_i| \leq -1$, tb_i *and* rot_i *are of different parities for* $i = 1, 2, 3$ *and* $R = \mathrm{rot}_1 - \mathrm{rot}_2 + \mathrm{rot}_3 \in \{0, -1\}$ *can be realized as the Thurston–Bennequin number and the rotation number of a Legendrian* θ*-graph with all cycles unknotted.*

Proof. Let $\mathrm{tb} = (\mathrm{tb}_1, \mathrm{tb}_2, \mathrm{tb}_3)$ and $\mathrm{rot} = (\mathrm{rot}_1, \mathrm{rot}_2, \mathrm{rot}_3)$ be triples of integers as in the hypothesis. We give front projections of Legendrian θ-graphs realizing these triples. In these projections the edges at vertex a are labeled e_1, e_2, e_3 from top to bottom and are in varying order at vertex b. Let $r_i := |\mathrm{rot}_i|$ for $i = 1, 2, 3$. We differentiate our examples according to the values of rot_1, rot_2 and rot_3 and the relationship between r_1, r_2 and r_3.

When $R = 0$, for the sign combinations (i)–(viii) shown in Table 1 there is a choice of indices i, j, k with $\{i, j, k\} = \{1, 2, 3\}$ such that $r_i \geq r_j + r_k$ (in fact, $r_i = r_j + r_k$).

When $R = -1$, for the sign combinations (i), (iv), (vi) and (vii) there is a choice of indices i, j, k with $\{i, j, k\} = \{1, 2, 3\}$ such that $r_i \geq r_j + r_k$; combination (iii) is not realized; and for each combination (ii), (v) and (viii), there is a choice of indices i, j, k with $\{i, j, k\} = \{1, 2, 3\}$ such that $r_i + 1 = r_j + r_k$.

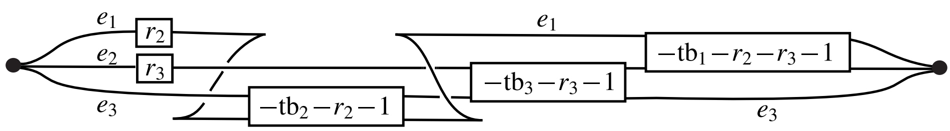
Thus any realizable $(\mathrm{rot}_1, \mathrm{rot}_2, \mathrm{rot}_3)$ falls into at least one of the following six cases: (1) $r_1 \geq r_2 + r_3$, (2) $r_2 \geq r_1 + r_3$, (3) $r_3 \geq r_1 + r_2$, (4) $r_1 + 1 = r_2 + r_3$, (5) $r_2 + 1 = r_1 + r_3$ and (6) $r_3 + 1 = r_1 + r_2$. We describe ways of realizing the invariants for these six cases.

The cycles $\mathscr{C}_1, \mathscr{C}_2$ and $\mathscr{C}_3$ are as described earlier. The choice of orientations for the three cycles implies that e_1 is oriented from a to b in both $\mathscr{C}_1$ and $\mathscr{C}_2$, while e_3 is oriented from b to a in both $\mathscr{C}_2$ and $\mathscr{C}_3$. A box along a single strand designates the number of stabilizations along the strand. We take

- r_i positive stabilizations if $\mathrm{rot}_i \geq 0$,
- r_i negative stabilizations if $\mathrm{rot}_i < 0$,

when edges e_1, e_2 and e_3 are oriented as in cycle $\mathscr{C}_i$. A box along a pair of strands designates the number of crossings between the two strands. All the crossings are as those in Figure 4.

Case 1 ($r_1 \geq r_2 + r_3$). The figure represents the front projection of a Legendrian θ-graph with the prescribed tb and rot.



Since $\mathrm{tb}_i + |\mathrm{rot}_i| \leq -1$, the integers $-\mathrm{tb}_2 - r_2 - 1$ and $-\mathrm{tb}_3 - r_3 - 1$ are nonnegative. Since $r_1 \geq r_2 + r_3$, we have $-\mathrm{tb}_1 - r_2 - r_3 - 1 \geq -\mathrm{tb}_1 - r_1 - 1 \geq 0$. So all of the indicated number of half twists are nonnegative integers as needed. The number $-\mathrm{tb}_1 - r_2 - r_3 - 1$ changes parity according to whether $\mathrm{rot}_1 - \mathrm{rot}_2 + \mathrm{rot}_3$ equals -1 or 0.

We check that the Thurston–Bennequin number and the rotation number for this embedding have the correct values. For a cycle $\mathscr{C}$ we use

$$\mathrm{tb}(\mathscr{C}) = w(\mathscr{C}) - \tfrac{1}{2}\mathrm{cusps}(\mathscr{C}), \qquad \mathrm{rot}(\mathscr{C}) = \tfrac{1}{2}(\downarrow\mathrm{cusps}(\mathscr{C}) - \uparrow\mathrm{cusps}(\mathscr{C})),$$

where $w =$ writhe, cusps $=$ total number of cusps, $\downarrow$cusps $=$ number of down cusps, $\uparrow$cusps $=$ number of up cusps.

- $\mathrm{tb}(\mathscr{C}_1) = w(\mathscr{C}_1) - \frac{1}{2}\mathrm{cusps}(\mathscr{C}_1) = (\mathrm{tb}_1 + r_2 + r_3 + 3) - (r_2 + r_3 + 3) = \mathrm{tb}_1$.
- $\mathrm{tb}(\mathscr{C}_2) = w(\mathscr{C}_2) - \frac{1}{2}\mathrm{cusps}(\mathscr{C}_2) = (\mathrm{tb}_2 + r_2 + 3) - (r_2 + 3) = \mathrm{tb}_2$.
- $\mathrm{tb}(\mathscr{C}_3) = w(\mathscr{C}_3) - \frac{1}{2}\mathrm{cusps}(\mathscr{C}_3) = (\mathrm{tb}_3 + r_3 + 1) - (r_3 + 1) = \mathrm{tb}_3$.

If $\mathrm{rot}_1 - \mathrm{rot}_2 + \mathrm{rot}_3 = 0$, then $-\mathrm{tb}_1 - r_2 - r_3 - 1$ has the same parity as $-\mathrm{tb}_1 - r_1 - 1$. They are both even, since tb_1 and rot_1 have different parities. This implies that at vertex b the upper strand is e_1 and the middle strand is e_2.

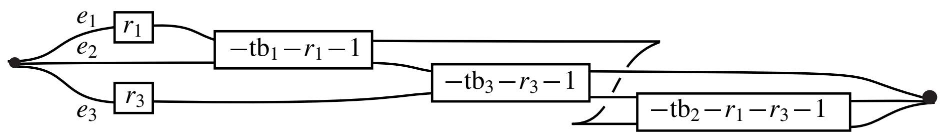
- $\mathrm{rot}(\mathscr{C}_1) = \frac{1}{2}(\downarrow\mathrm{cusps}(\mathscr{C}_1) - \uparrow\mathrm{cusps}(\mathscr{C}_1))$
  $= \frac{1}{2}(2 \cdot \mathrm{sgn}(\mathrm{rot}_2) \cdot r_2 + 3 - 2 \cdot \mathrm{sgn}(\mathrm{rot}_3) \cdot r_3 - 3) = \mathrm{rot}_2 - \mathrm{rot}_3 = \mathrm{rot}_1$.
- $\mathrm{rot}(\mathscr{C}_2) = \frac{1}{2}(\downarrow\mathrm{cusps}(\mathscr{C}_2) - \uparrow\mathrm{cusps}(\mathscr{C}_2)) = \frac{1}{2}(2 \cdot \mathrm{sgn}(\mathrm{rot}_2) \cdot r_2 + 3 - 3) = \mathrm{rot}_2$.
- $\mathrm{rot}(\mathscr{C}_3) = \frac{1}{2}(\downarrow\mathrm{cusps}(\mathscr{C}_3) - \uparrow\mathrm{cusps}(\mathscr{C}_3)) = \frac{1}{2}(2 \cdot \mathrm{sgn}(\mathrm{rot}_3) \cdot r_3 + 1 - 1) = \mathrm{rot}_3$.

If $\mathrm{rot}_1 - \mathrm{rot}_2 + \mathrm{rot}_3 = -1$, then $-\mathrm{tb}_1 - r_2 - r_3 - 1$ has different parity than $-\mathrm{tb}_1 - r_1 - 1$. Since tb_1 and rot_1 have different parities, $-\mathrm{tb}_1 - r_1 - 1$ is even and $-\mathrm{tb}_1 - r_2 - r_3 - 1$ is odd. This implies that at vertex b the upper strand is e_2 and the middle strand is e_1. Computations for $\mathrm{rot}(\mathscr{C}_2)$ and $\mathrm{rot}(\mathscr{C}_3)$ are the same as above.

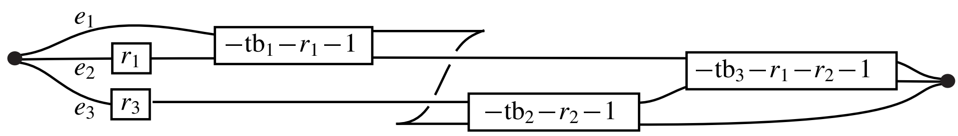
- $\mathrm{rot}(\mathscr{C}_1) = \frac{1}{2}(\downarrow\mathrm{cusps}(\mathscr{C}_1) - \uparrow\mathrm{cusps}(\mathscr{C}_1))$
  $= \frac{1}{2}(2 \cdot \mathrm{sgn}(\mathrm{rot}_2) \cdot r_2 + 2 - 2 \cdot \mathrm{sgn}(\mathrm{rot}_3) \cdot r_3 - 4) = \mathrm{rot}_2 - \mathrm{rot}_3 - 1 = \mathrm{rot}_1$.

In the remaining cases, a similar check may be done to verify that they have the correct tb and rot.

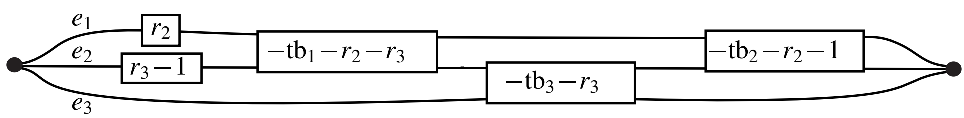
Case 2 ($r_2 \geq r_1 + r_3$). The figure below represents the front projection of a Legendrian θ-graph with the prescribed tb and rot. Since $r_2 \geq r_1 + r_3$, we have $-\mathrm{tb}_2 - r_1 - r_3 - 1 \geq -\mathrm{tb}_2 - r_2 - 1 \geq 0$.



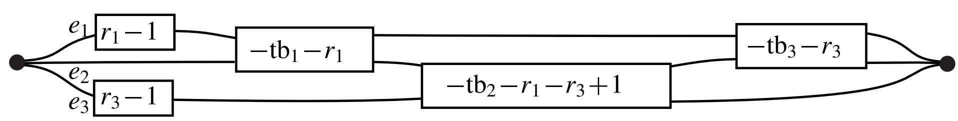
Case 3 ($r_3 \geq r_1 + r_2$). Again, the figure represents the front projection of a Legendrian θ-graph with the prescribed tb and rot. Since $r_3 \geq r_1 + r_2$, we have $-\mathrm{tb}_3 - r_1 - r_2 - 1 \geq -\mathrm{tb}_3 - r_3 - 1 \geq 0$.



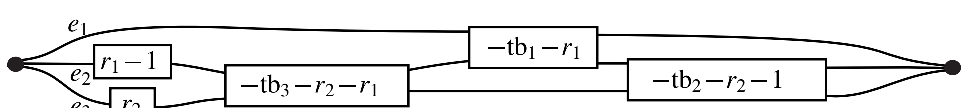
Case 4 ($r_1 + 1 = r_2 + r_3$). In this case the graph below realizes (tb, rot). Since $r_2 + r_3 = r_1 + 1$, we have $-\mathrm{tb}_1 - r_2 - r_3 = -\mathrm{tb}_1 - r_1 - 1 \geq 0$.



Case 5 ($r_2 + 1 = r_1 + r_3$). For this case the graph below realizes (tb, rot). Given $r_1 + r_3 = r_2 + 1$, we have $-\mathrm{tb}_2 - r_1 - r_3 + 1 = -\mathrm{tb}_2 - r_2 > 0$.



Case 6 ($r_3 + 1 = r_1 + r_2$). In this case the graph below realizes (tb, rot). Since $r_1 + r_2 = r_3 + 1$, we have $-\mathrm{tb}_3 - r_1 - r_2 = -\mathrm{tb}_3 - r_3 - 1 \geq 0$.



This completes the proof. □

3.1. ***Topologically planar θ-graphs are not Legendrian simple.*** We ask whether the invariants tb and rot determine the Legendrian type of a planar θ-graph. If we do not require that the cyclic order of the edges around the vertex a (or b) is the same in both embeddings, the answer is negative. We give a counterexample:

Example 9. The two graphs in Figure 1 have the same invariants but they are not Legendrian isotopic. Let $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ be the three cycles of G determined by the pairs of edges $\{e_1, e_2\}$, $\{e_1, e_3\}$ and $\{e_2, e_3\}$, respectively. Let $\mathscr{C}'_1$, $\mathscr{C}'_2$ and $\mathscr{C}'_3$ be the three cycles of G' determined by $\{f_2, f_1\}$, $\{f_2, f_3\}$ and $\{f_1, f_3\}$, respectively. The cycles have $\mathrm{tb}(\mathscr{C}_1) = \mathrm{tb}(\mathscr{C}'_1) = -1$, $\mathrm{tb}(\mathscr{C}_2) = \mathrm{tb}(\mathscr{C}'_2) = -5$, $\mathrm{tb}(\mathscr{C}_3) = \mathrm{tb}(\mathscr{C}'_3) = -3$ and $\mathrm{rot}(\mathscr{C}_i) = \mathrm{rot}(\mathscr{C}'_i) = 0$ for $i = 1, 2, 3$.

Assume the two graphs are Legendrian isotopic. Since the cycles with same invariants should correspond to each other via the Legendrian isotopy (which we denote by ι), the edges correspond as $e_1 \leftrightarrow \iota(e_1) = f_2$, $e_2 \leftrightarrow \iota(e_2) = f_1$ and $e_3 \leftrightarrow \iota(e_3) = f_3$. But at both vertices of G the (counterclockwise) order of edges in the contact plane is $e_1 - e_2 - e_3$ and at both vertices of G' the (counterclockwise) order of edges in the contact plane is $\iota(e_1) - \iota(e_3) - \iota(e_2)$. This leads to a contradiction, since a Legendrian isotopy preserves the cyclic order of edges at each vertex.

Corollary 10. *The invariants* tb *and* rot *are not enough to distinguish the Legendrian class of an θ_n-graph for $n \geq 3$.*

Proof. For $n \geq 4$, a pair of graphs with the same invariants but of different Legendrian type can be obtained from (G, G') in Example 9 by adding $n - 3$ unknotted edges at the top of the three existing ones. □

4. Legendrian ribbons and transverse push-offs

In this section we work with Legendrian ribbons of θ-graphs. We examine the relationship between the Legendrian graph and the boundary of its ribbon, the transverse push-off. The transverse push-off is another invariant of Legendrian graphs. We explore whether it contains more information than the classical invariants rotation number and Thurston–Bennequin number. We determine the number of components and the self linking number for the push-off of a Legendrian θ-graph. In the special case of topologically planar graphs, we prove that the topological type of the transverse push-off of a θ-graph is that of a pretzel-type curve whose coefficients are determined by the Thurston–Bennequin invariant of the graph.

Let g be a Legendrian graph. A *ribbon for g* is a compact oriented surface R_g such that:

(1) g in contained in the interior of R_g;

(2) there exists a choice of orientations for R_g such that ξ has no negative tangency with R_g;

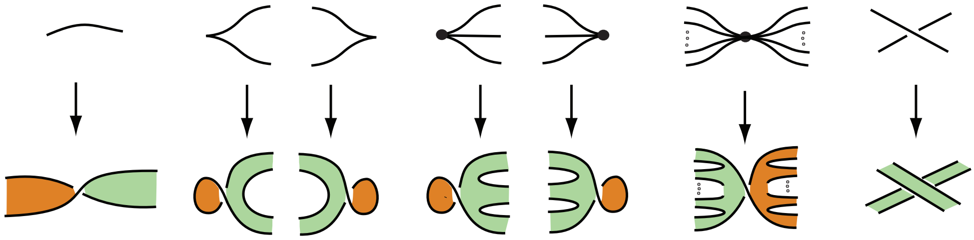


Figure 6. Attaching a ribbon surface to a Legendrian graph. The two sides of the surface are marked by different colors.

(3) there exists a vector field X on R_g tangent to the characteristic foliation whose time flow ϕ_t satisfies $\bigcap_{t\geq 0} \phi_t(R_g) = g$; and

(4) the boundary of R_g is transverse to the contact structure.

The following is a construction which takes a graph in the front projection and produces its ribbon viewed in the front projection. Portions of this construction were previously examined by Avdek [2013, Algorithm 2, Steps 4–6]. Starting with a front projection of the graph, we construct a ribbon surface containing the graph as described in Figure 6:

(a) To each arc between consecutive cusps of an edge we attach a band with a single negative half twist.

(b) To each left and right cusp along a strand we attach disks containing a positive half twist.

(c,d) To each vertex we attach twisted disks as in Figure 6(c,d).

(e) Crossings in the diagram of the graph are preserved.

Legendrian ribbons were first introduced by Giroux [2002] to have a well-defined way to contract a contact handlebody onto the Legendrian graph at the core of the handlebody. We are interested in some particular features of Legendrian ribbons. The boundary of a Legendrian ribbon is an oriented transverse link with the orientation inherited from the ribbon surface. The ribbon associated with a given Legendrian graph is unique up to isotopy and therefore gives a natural way to associate a transverse link to the graph.

Definition 11. The *transverse push-off* of a Legendrian graph is the boundary of its ribbon.

In the case of Legendrian knots the above definition gives a two component link of both the positive and negative transverse push-offs. However, with graphs the transverse push-off can have various numbers of components, depending on the

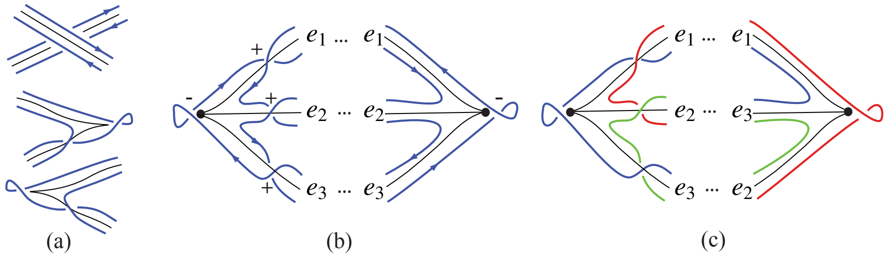


Figure 7. Transverse push-off: (a) at cusps and crossings (b) of a Legendrian θ-graph with one component (c) of a Legendrian θ-graph with three components.

abstract graph and Legendrian type. The transverse push-off is a new invariant of Legendrian graphs.

4.1. ***Self linking of transverse push-offs.*** Here we determine possible self linking numbers and the number of components of the transverse push-off of a Legendrian θ-graph.

Theorem 12. *The transverse push-off of a Legendrian θ-graph is either a transverse knot K with* $\mathrm{sl} = 1$ *or a three component transverse link whose three components are the transverse push-offs of the three Legendrian cycles given the correct orientation.*

Proof. Given an arbitrary Legendrian θ-graph, by Lemma 4, it can be isotoped to an embedding where near the vertices it has a projection like that shown in Figure 5, where the edges at the left vertex are labeled e_1, e_2, e_3 from top to bottom. Then using Remark 5, move the edges around the right vertex so that edge e_1 is also in the top position in the front projection. There are two possibilities for the order of edges at the right vertex: parallel vertices, shown in Figure 7(b), and antiparallel vertices, shown in Figure 7(c).

Now we will focus on the number of components of the transverse push-off. For simplicity of bookkeeping we will place the negative half twists that occur on each arc between consecutive cusps to the left on that portion of the edge. For the projections shown in Figure 7(b,c) the portion of the graph not pictured could have any number of crossings and cusps. Along each edge, the top and bottom positions of the strands are preserved through cusps and crossings. See Figure 7(a). So we see that the arc of the transverse push-off which lies above (resp. below) the Legendrian arc in the projection on one side of the diagram still lies above (resp. below) on the other side. Thus the number of components in the transverse push-off can be determined by a careful tracing of the diagrams in Figure 7(b,c). Graphs with parallel vertices have a transverse push-off with one component, and graphs with antiparallel vertices have a transverse push-off with three components.

If the boundary of the Legendrian ribbon is a knot T, then $\mathrm{sl}(T)$ equals the signed count of crossings in a front diagram for T. Crossings in the diagram of the graph and cusps along the three edges do not contribute to this count. A crossing in the diagram of the graph contributes two negative and two positive crossings. A cusp contributes a canceling pair of positive and negative crossings; see Figure 7(a). Apart from these, there is one positive crossing along each edge and one negative crossing for every disk at each vertex, giving $\mathrm{sl}(T) = 1$; see Figure 7(b).

If the boundary has three components T_1, T_2 and T_3, then they have the same self linking as the transverse push-offs of the cycles of the Legendrian graph with the correct orientation. Let $\overline{\mathscr{C}}_i$ be the cycle $\mathscr{C}_i$ with the opposite orientation. Then T_1, T_2 and T_3, are the positive transverse push-offs of $\overline{\mathscr{C}}_1$, $\mathscr{C}_2$ and $\overline{\mathscr{C}}_3$, respectively. □

4.2. ***Topologically planar Legendrian θ-graphs.*** To be able to better understand the topological type of a Legendrian ribbon and the transverse push-off (its boundary) we will model the ribbon with a flat vertex graph. A *flat vertex graph* (or *rigid vertex graph*) is an embedded graph where the vertices are rigid disks with the edges being flexible tubes or strings between the vertices. This is in contrast with pliable vertex graphs (or just spatial graphs) where the edges have freedom of motion at the vertices. Both flat vertex and pliable vertex graphs are studied up to ambient isotopy and have sets of five Reidemeister moves. For both of them, the first three Reidemeister moves are the same as those for knots and links and RIV consists of moving an edge over or under a vertex; see Figure 8. For flat vertex graphs, RV is the move where the flat vertex is flipped over. For pliable vertex graphs, RV is the move where two of the edges are moved near the vertex in such a way that their order around the vertex is changed in the projection.

For a *Legendrian ribbon, the associated flat vertex graph* is given by the following construction: a vertex is placed on each twisted disk where the original vertices were, and an edge replaces each band in the ribbon. The information that is lost with this model is the amount of twisting that occurs on each edge. The flat vertex graph model is particularly useful when working with the θ-graph because it is a trivalent graph. We see with the following lemma the relationship between trivalent flat vertex and trivalent pliable vertex graphs.

Lemma 13. *For graphs with all vertices of degree* 3 *or less, the set of equivalent diagrams is the same for both pliable and flat vertex spatial graphs.*

Proof. We follow notation in [Kauffman 1989, pages 699, 704]. The lemma can be reformulated to say, given the diagrams of two ambient isotopic pliable vertex graphs with maximal degree 3, these are also ambient isotopic as flat vertex graphs, and vice versa. The Reidemeister moves for pliable vertex graphs and flat vertex graphs differ only in RV; see Figure 8. For pliable vertex graphs, RV is the move where two of the edges are moved near the vertex in such a way that this changes

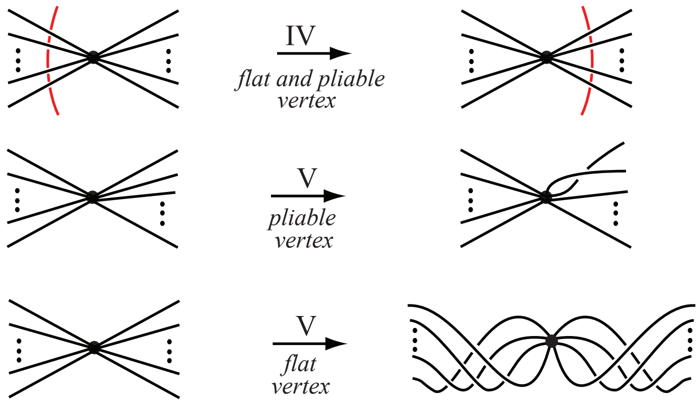


Figure 8. RIV and RV for pliable and flat vertex graphs.

their order around the vertex in the projection. For flat vertex graphs, RV is the move where the flat vertex is flipped over. For vertices of valence at most 3, these two moves give the same diagrammatic results. Thus the same sequence of Reidemeister moves can be used in the special case of graphs with maximal degree 3. □

Here we set up the notation that will be used in the following theorem. For a Legendrian θ-graph G, we consider a front projection in which the neighborhoods of the two vertices are as those in Figure 5 and we denote its three cycles by $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$, following the notation of Section 2. Let $\mathrm{cr}[e_i, e_j]$ be the signed intersection count of edges e_i and e_j in the cycle $\mathscr{C}_1$, $\mathscr{C}_2$ or $\mathscr{C}_3$ which they determine. Let $\mathrm{cr}[e_i]$ be the signed self-intersection count of e_i. Let tb_1, tb_2 and tb_3 be the Thurston–Bennequin numbers of $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$.

Theorem 14. *Let G represent a topologically planar Legendrian θ-graph with* $\mathrm{tb} = (\mathrm{tb}_1, \mathrm{tb}_2, \mathrm{tb}_3)$. *Then the transverse push-off of G is an* (a_1, a_2, a_3)*-pretzel link, where* $a_1 = \mathrm{tb}_1 + \mathrm{tb}_2 - \mathrm{tb}_3$, $a_2 = \mathrm{tb}_1 + \mathrm{tb}_3 - \mathrm{tb}_2$ *and* $a_3 = \mathrm{tb}_2 + \mathrm{tb}_3 - \mathrm{tb}_1$.

Proof. The proof will be done in two parts. First, the transverse push-off will be shown to be a pretzel knot or link. Second, it will be shown to be of a particular type of pretzel link, an (a_1, a_2, a_3)-pretzel knot or link.

We first look at the ribbon as a topological object. If the ribbon can be moved through ambient isotopy to a projection where the three bands do not cross over each other and come together along a flat disk, then the boundary of the ribbon would be a pretzel link with crossings only occurring as twists on each band. If we model the ribbon with a flat vertex graph this simplifies our question to whether the resulting flat vertex graph can be moved so that it is embedded in the plane. The resulting graph is topologically planar because it is coming from a topologically planar Legendrian graph. Thus by Lemma 13, it can be moved to a planar embedding.

In order to show the pretzel knot (or link) is an (a_1, a_2, a_3)-pretzel link, we will look at what happens to the ribbon as the associated flat vertex graph is moved to a planar embedding. We will work with the Legendrian θ-graph in the form shown in Figure 5 near its vertices. We need to count the number of twists in the bands of the Legendrian ribbon once it has been moved to the embedding where the associated flat vertex graph is planar. We will prove $a_1 = \mathrm{tb}_1 + \mathrm{tb}_2 - \mathrm{tb}_3$ by writing each of these numbers in terms of the number of cusps and the number of signed crossings between the edges of the Legendrian graph. The proofs for a_2 and a_3 are similar.

We will use the following observations to be able to write a_1, the number of half twists in the band associated with edge e_1, in terms of the number of cusps, $\mathrm{cr}[e_i]$ and $\mathrm{cr}[e_i, e_j]$.

(1) Based on the construction of the ribbon surface, c cusps on one of the edges contribute with $c+1$ negative half twists to the corresponding band.

(2) We look at each of the Reidemeister moves for flat vertex graphs and see how they change the number of twists on the associated band of the ribbon surface.
   (a) A positive (negative) RI adds a full positive (negative) twist to the band; see Figure 9(a,b).
   (b) RII, RIII and RIV do not change the number of twists in any of the bands.
   (c) RV adds a half twist on each of the three bands; see Figure 9(c,d). The sign of the half twists depends on the crossing and which bands are crossed. If two bands have a positive (resp. negative) crossing, then they each have the addition of a positive (resp. negative) half twist, and the third band has the addition of a negative (resp. positive) half twist.

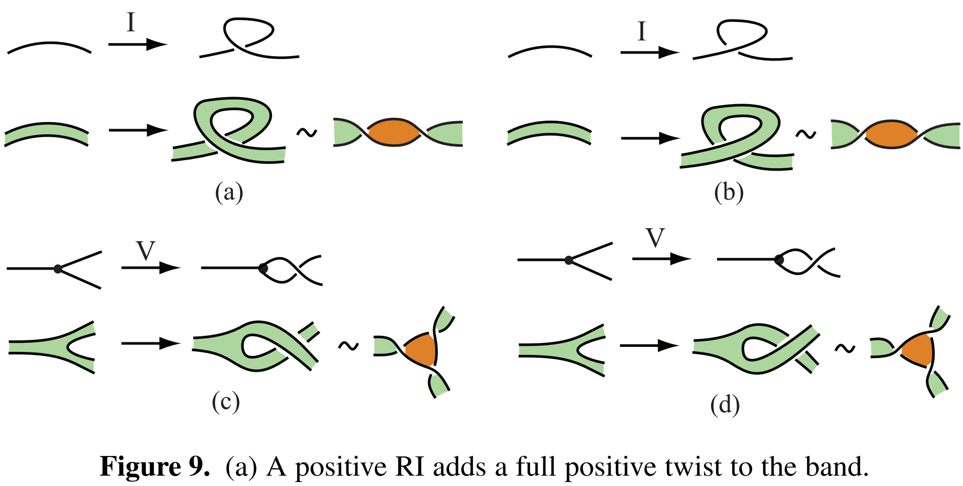


Figure 9. (a) A positive RI adds a full positive twist to the band. (b) A negative RI adds a full negative twist to the band. (c,d) RV adds a half twist on each of the three bands.

Since we proved earlier that the flat vertex graph can be moved to a planar embedding, we know that all of the crossings between edges will be eventually removed through Reidemeister moves. Thus this gives

$$a_1 = -[\text{cusps on } e_1] - 1 + 2\,\mathrm{cr}[e_1] + \mathrm{cr}[e_1, e_2] + \mathrm{cr}[e_1, e_3] - \mathrm{cr}[e_2, e_3].$$

This count is easily seen to be invariant under RII and RIII, since these do not change the signed crossing of the diagram. We show it is invariant under RIV at the end of the proof.

Next, we describe $\mathrm{tb}_1 + \mathrm{tb}_2 - \mathrm{tb}_3$ in terms of the number of cusps and the crossings between the edges. Recall that, for a cycle $\mathscr{C}$,

$$\mathrm{tb}(\mathscr{C}) = w(\mathscr{C}) - \tfrac{1}{2}\mathrm{cusps}(\mathscr{C}).$$

Thus,

$$\begin{aligned}\mathrm{tb}_1+\mathrm{tb}_2-\mathrm{tb}_3 &= w(\mathscr{C}_1)-\tfrac{1}{2}\mathrm{cusps}(\mathscr{C}_1)+w(\mathscr{C}_2)-\tfrac{1}{2}\mathrm{cusps}(\mathscr{C}_2)-w(\mathscr{C}_3)+\tfrac{1}{2}\mathrm{cusps}(\mathscr{C}_3)\\ &= \mathrm{cr}[e_1, e_2]+\mathrm{cr}[e_1]+\mathrm{cr}[e_2]-\tfrac{1}{2}\big([\text{cusps on } e_1]+[\text{cusps on } e_2]+2\big)\\ &\quad +\mathrm{cr}[e_1, e_3]+\mathrm{cr}[e_1]+\mathrm{cr}[e_3]-\tfrac{1}{2}\big([\text{cusps on } e_1]+[\text{cusps on } e_3]+2\big)\\ &\quad -\big(\mathrm{cr}[e_2, e_3]+\mathrm{cr}[e_2]+\mathrm{cr}[e_3]\big)+\tfrac{1}{2}\big([\text{cusps on } e_2]+[\text{cusps on } e_3]+2\big)\\ &= -[\text{cusps on } e_1]-1+2\,\mathrm{cr}[e_1]+\mathrm{cr}[e_1, e_2]+\mathrm{cr}[e_1, e_3]-\mathrm{cr}[e_2, e_3].\end{aligned}$$

Thus, $a_1 = \mathrm{tb}_1 + \mathrm{tb}_2 - \mathrm{tb}_3$.

Claim. *The sum* $2\mathrm{cr}[e_1] + \mathrm{cr}[e_1, e_2] + \mathrm{cr}[e_1, e_3] - \mathrm{cr}[e_2, e_3]$ *is unchanged under RIV.*

Proof of claim. Let $b_1 = 2\mathrm{cr}[e_1]+\mathrm{cr}[e_1, e_2]+\mathrm{cr}[e_1, e_3]-\mathrm{cr}[e_2, e_3]$. Let d represent the strand that is moved past the vertex. We distinguish two cases, (a) and (b), according to the number of crossings on each side of the vertex; see Figure 10. We check that the contributions to b_1 of the crossing before the move (left) is the same as the contribution to b_1 of the crossings after the move (right). The strand d can be part of e_1, e_2 or e_3. For both cases (a) and (b), the equality is shown step by step for $d = e_1$ and $d = e_3$. In a similar way b_1 is unchanged if $d = e_2$.

Case (a_1): If d is part of e_1, then $b_{1,\text{left}} = 2\mathrm{cr}[e_1]+\mathrm{cr}[e_1, e_2] = \mathrm{cr}[e_1]$, since the two crossings have opposite sign when seen in the cycle determined by e_1 and e_2; and $b_{1,\text{right}} = \mathrm{cr}[e_1, e_3] = \mathrm{cr}[e_1]$.

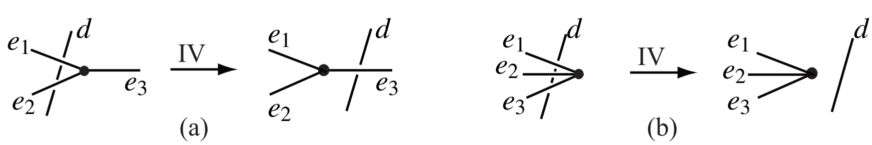


Figure 10. RIV changes crossings between different pairs of edges.

Case (a$_2$): If d is part of e_3, then $b_{1,\text{left}} = \text{cr}[e_1, e_3] - \text{cr}[e_2, e_3] = \text{cr}[e_2, e_3] - \text{cr}[e_3] = 0$, and $b_{1,\text{right}} = 0$.

Case (b$_1$): If d is part of e_1, then $b_{1,\text{left}} = 2\text{cr}[e_1] + \text{cr}[e_1, e_2] + \text{cr}[e_1, e_3] = 0$, and $b_{1,\text{right}} = 0$.

Case (b$_2$): If d is part of e_3, then $b_{1,\text{left}} = \text{cr}[e_1, e_3] - \text{cr}[e_2, e_3] = 0$, since both these crossings have sign opposite to $\text{cr}[e_3]$; and $b_{1,\text{right}} = 0$.

This completes the proof of the claim and the theorem. □

The combination of Theorem 12 and Theorem 14 gives a complete picture of the possible transverse push-offs of topologically planar Legendrian θ-graphs. In this case, the transverse push-off is completely described by the tb of the graph. So while this does not add to our ability to distinguish topologically planar Legendrian θ-graphs, it does add to our understanding of the interaction between a Legendrian graph and its transverse push-off.

It is worth noting that Theorem 14 also implies that the transverse push-off will either have one or three components. The possible transverse push-offs of a topologically planar Legendrian θ-graph are more restricted than it may first appear. Not all pretzel links will occur in this way. In Theorem 14, we found the pretzel coefficients as linear combinations with coefficients $+1$ or -1 of tb_1, tb_2 and tb_3. We note that the three pretzel coefficients have the same parity, restricting the number of components the transverse push-off can have. If exactly one of or all three of tb_1, tb_2 and tb_3 are odd, then all pretzel coefficients are odd and the pretzel curve is a knot. If none or exactly two of tb_1, tb_2 and tb_3 are odd, then all pretzel coefficients are even and the pretzel curve is a three component link. The pairwise linking between its components is equal to the number of full twists between the corresponding pair of strands in the pretzel presentation, i.e., $a_1/2$, $a_2/2$ and $a_3/2$.

4.3. ***The transverse push-off of θ_n-graphs.*** We give examples showing the boundary of the Legendrian ribbon associated to an θ_n-graph, $n > 3$, is not necessarily a pretzel-type link. Independent of n, each component of an n-pretzel type link is linked with at most two other components. The transverse push-offs of the graphs in Figure 11 have at least one component linking more than two other components of the link. The characterization as a pretzel curve of the topological type of the push-off is therefore exclusive to the case $n = 3$, that of θ-graphs.

For $n = 2k$, $k \geq 2$, let L_{2k} be the Legendrian θ_{2k}-graph whose front projection is the one in Figure 11, top left. The transverse push-off has the topological type of the link $L \cup L_k$ in Figure 11, top right. If k is odd, L has one component and it links all $k \geq 3$ components of L_k. If k is even, L has two components where each of the two components links all $k \geq 2$ components of L_k and the other component of L.

For $n = 2k+1$, $k \geq 3$, let L_{2k+1} be the Legendrian θ_{2k+1}-graph whose front projection is the one in Figure 11, middle left. The transverse push-off has the

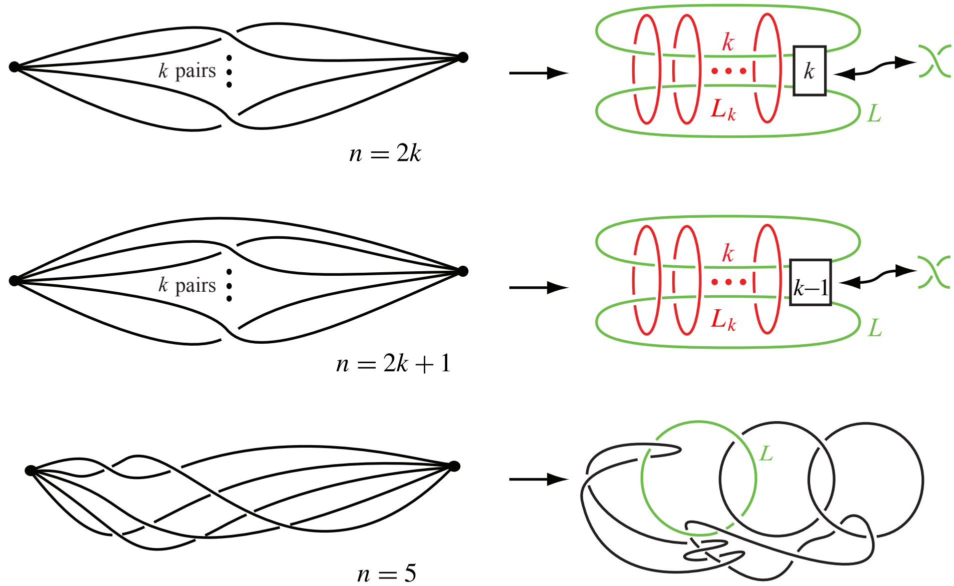


Figure 11. The θ_n-graphs on the left have the transverse push-offs shown on the right, which do not have the topological type of a pretzel-type curve.

topological type of the link $L \cup L_k$ in Figure 11, middle right. If k is even, L has one component and it links all $k \geq 3$ components of L_k. If k is odd, L has two components where each of the two components links all $k \geq 3$ components of L_k and the other component of L.

For $n = 5$, the link just discussed is a pretzel link and we give a different example in this case; see the bottom row of Figure 11. The highlighted component of the transverse push-off links three other components.

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A CLASS OF NEUMANN PROBLEMS ARISING IN CONFORMAL GEOMETRY

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In this paper, we solve a class of Neumann problems on a manifold with totally geodesic smooth boundary. As a consequence, we also solve the prescribing k-curvature problem of the modified Schouten tensor on such manifolds; that is, if the initial k-curvature of the modified Schouten tensor is positive for $\tau > n-1$ or negative for $\tau < 1$, then there exists a conformal metric such that its k-curvature defined by the modified Schouten tensor equals some prescribed function and the boundary remains totally geodesic.

1. Introduction

Let (M^n, g), $n \geq 3$, be a compact, smooth Riemannian manifold. The *modified Schouten tensor*

$$A_g^\tau := \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{\tau R_g}{2(n-1)} \cdot g\right)$$

was introduced by Gursky and Viaclovsky [2003] and A. Li and Y.-Y. Li [2003] independently, where $\tau \in \mathbb{R}$ and Ric_g, R_g are the Ricci tensor and the scalar curvature of g, respectively. Clearly, A_g^0 is the Ricci tensor, A_g^{n-1} is the Einstein tensor and A_g^1 is just the Schouten tensor.

Denote by $\lambda(g^{-1}A_g^\tau)$ the eigenvalues of A_g^τ. The k-curvature (or σ_k curvature) of A_g^τ is defined as $\sigma_k(\lambda(g^{-1}A_g^\tau))$, where σ_k is the k-th elementary symmetric function defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{for all } \lambda \in \mathbb{R}^n,$$

for any $1 \leq k \leq n$. We will use $\sigma_k(A_g^\tau) := \sigma_k(\lambda(g^{-1}A_g^\tau))$ for convenience.

The prescribing k-curvature problem of the modified Schouten tensor A_g^τ in conformal geometry is to find a metric $\tilde{g}$ in the conformal class $[g]$ of g satisfying

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the equation

$$\sigma_k^{1/k}(A_{\tilde g}^{\tau}) = \varphi(x), \tag{1-1}$$

where φ is a given smooth function on M. If $\tau = 1 = k$ and φ is constant, (1-1) is just the Yamabe problem, which has been solved by Yamabe, Trudinger, Aubin and Schoen (see [Lee and Parker 1987]). When $\tau = 1$, $k \geq 2$ and φ is constant, then (1-1) is called k-Yamabe problem, which has attracted enormous interest [Chang et al. 2002; Ge and Wang 2006; Guan and Wang 2003a; 2003b; Gursky and Viaclovsky 2007; Li and Li 2003; 2005; Sheng et al. 2007; Trudinger and Wang 2009; 2010; Viaclovsky 2000], etc. There are many interesting works on the Yamabe problem and k-Yamabe problem on a manifold with boundary [Chen 2007; 2009; Escobar 1992b; 1992a; Han and Li 1999; 2000; He and Sheng 2011a; 2011b; 2013; Jin et al. 2007; Jin 2007], etc.

Note that (1-1) is a fully nonlinear partial differential equation for $k \geq 2$. In order to study this problem, we need the following conceptions. Let

$$\Gamma_k^+ = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

Therefore, we have $\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+$. For a 2-symmetric form B defined on (M^n, g), $B \in \Gamma_k^+$ means that the eigenvalues of B, say $\lambda(g^{-1}B)$, lie in Γ_k^+. Set $\Gamma_k^- = -\Gamma_k^+$.

According to [Caffarelli et al. 1985], (1-1) is an elliptic equation for $A_g^\tau \in \Gamma_k^+$ or $A_g^\tau \in \Gamma_k^-$. When $\tau < 1$, $A_g^\tau \in \Gamma_k^-$ and $\varphi < 0$, Gursky and Viaclovsky [2003] proved that there exists a unique conformal metric $\tilde g \in [g]$ satisfying (1-1) on a closed manifold. Li and Sheng [2005] studied the same problem by a parabolic argument. Using a similar argument, Sheng and Zhang [2007] studied the case of $\tau > n-1$, $A_g^\tau \in \Gamma_k^+$ and $\varphi > 0$. For the manifold with boundary, Li and Sheng [2011] considered a Dirichlet problem of (1-1) for $\tau > n-1$ and $A_g^\tau \in \Gamma_k^+$; He and Sheng [2013] discussed more general equations and obtained many useful local estimates for both $\tau < 1$ and $\tau > n-1$. In [Sheng and Yuan 2013], we investigated a Neumann problem of (1-1) by a conformal flow and proved:

Theorem 1.1 [Sheng and Yuan 2013]. *Let $(\overline{M}^n, g)$, $n \geq 3$, be a compact manifold with smooth totally geodesic boundary ∂M. If $A_g^\tau \in \Gamma_k^+$ and $\tau > n-1$, or $A_g^\tau \in \Gamma_k^-$ and $\tau < 1$, then there exists a smooth metric $\tilde g \in [g]$ satisfying* (1-1) *for φ constant and such that ∂M is still totally geodesic.*

In this paper, we are interested in solving a class of Neumann problems on the manifold with totally geodesic boundary.

Let $(\overline{M}, g)$ be a compact manifold with smooth boundary ∂M. Denote the second fundamental form and the mean curvature of ∂M by L and μ. Under the conformal change of metric $\tilde g = e^{2u} g$, the second fundamental form L with respect to its unit inward normal ν satisfies

$$\tilde{L}e^{-u} = -\frac{\partial u}{\partial \nu} g + L.$$

The boundary is called umbilic if $L = \mu g$, and then totally geodesic if $\mu \equiv 0$. Note that the umbilicity is conformally invariant. Then the mean curvature changes as

$$\tilde{\mu} = \left(-\frac{\partial u}{\partial \nu} + \mu\right)e^{-u}. \tag{1-2}$$

Under the same conformal change, the *modified Schouten tensor* changes according to the formula

$$A^{\tau}_{\tilde{g}} = \frac{\tau - 1}{n - 2}\Delta u g - \nabla^2 u + du \otimes du + \frac{\tau - 2}{2}|\nabla u|^2 g + A^{\tau}_{g}, \tag{1-3}$$

where the covariant derivatives and norms are taken with respect to the background metric g. Let the boundary ∂M be totally geodesic with respect to the metric g. In order to preserve the boundary being totally geodesic under the conformal change, $\tilde{\mu} \equiv 0$. Hence, the two partial differential equations corresponding to Theorem 1.1 are

$$\begin{cases} \sigma_k^{1/k}\left(\frac{\tau-1}{n-2}\Delta u g - \nabla^2 u + du \otimes du + \frac{\tau-2}{2}|\nabla u|^2 g + A^{\tau}_{g}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = e^{2u}\,\text{const.} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-4}$$

for $\tau > n - 1$, and

$$\begin{cases} \sigma_k^{1/k}\left(\nabla^2 u + \frac{1-\tau}{n-2}\Delta u g - du \otimes du + \frac{2-\tau}{2}|\nabla u|^2 g - A^{\tau}_{g}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = e^{2u}\,\text{const.} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-5}$$

for $\tau < 1$, respectively.

Now, we consider more general equations than (1-4) and (1-5). Let $\Gamma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$, and $F : \mathbb{R}^n \to \mathbb{R}$ be a general smooth, symmetric, homogeneous function of degree one in Γ normalized by $F(e) = F(1, \dots, 1) = 1$. Moreover, $F = 0$ on $\partial\Gamma$ and satisfies the following structure conditions in Γ:

(C1) F is positive.

(C2) F is concave (i.e., $\partial^2 F/(\partial\lambda_i \partial\lambda_j)$ is negative semidefinite).

(C3) F is monotone (i.e., $\partial F/\partial\lambda_i$ is positive).

According to [Lin and Trudinger 1994; Trudinger 1990], for any $0 \le l < k \le n$, the elementary symmetric functions and their quotients $(\sigma_k/\sigma_l)^{1/(k-l)}$ with $\sigma_0 = 1$ satisfy all the properties and structure conditions above on Γ_k^+.

For some positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$, we study the equation

$$\begin{cases} F(g^{-1}V[u]) = \Phi(x, u) & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-6}$$

where for constant $\bar{\theta} := (\tau - 1)/(n-2) > 1$, $a, b \in C^\infty(\overline{M})$, and the smooth symmetric 2-tensor $S \in \Gamma$, the matrix $(V[u])$ is defined by

$$V[u] = \bar{\theta} \triangle u g - \nabla^2 u + a(x)\, du \otimes du + b(x)|\nabla u|^2 g + S. \tag{1-7}$$

We call a function $v \in C^2(\overline{M})$ *admissible* if $\lambda(g^{-1}V[v]) \in \Gamma$.

Assume S is the symmetric 2-tensor on M satisfying one of the following conditions:

(S1) $S(\nu, X) = 0$, for any $X \in T(\partial M)$.

(S2) $S = A_g^\tau$.

Theorem 1.2 (main result). *Let $(\overline{M}^n, g)$, $n \ge 3$, be a compact manifold with smooth totally geodesic boundary ∂M. Suppose $\bar{\theta} > 1$ and the positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfies*

$$\partial_z \Phi > 0, \qquad \lim_{z \to +\infty} \Phi(x, z) = +\infty, \qquad \lim_{z \to -\infty} \Phi(x, z) = 0. \tag{1-8}$$

Then for any functions $a, b \in C^\infty(\overline{M})$ and $S \in \Gamma$ satisfying (S1) *or* (S2), *there exists a function $u \in C^\infty(\overline{M})$ solving the equation* (1-6).

For the other elliptic branch (1-5), we consider the equation

$$\begin{cases} F(g^{-1}W[u]) = \Phi(x, u) & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-9}$$

where for constant $\theta := (1 - \tau)/(n-2) > 0$, $a, b \in C^\infty(\overline{M})$, and the smooth symmetric 2-tensor $T \in \Gamma$, the matrix $(W[u])$ is defined by

$$W[u] = \nabla^2 u + \theta \triangle u g + a(x)\, du \otimes du + b(x)|\nabla u|^2 g + T. \tag{1-10}$$

Theorem 1.3. *Let $(\overline{M}^n, g)$, $n \ge 3$, be a compact manifold with smooth totally geodesic boundary ∂M. Suppose $\theta > 0$ and the positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfies* (1-8). *Then for any functions $a, b \in C^\infty(\overline{M})$ and $T \in \Gamma$ with* (S1) *or $T = -A_g^\tau$, there exists a function $u \in C^\infty(\overline{M})$ solving the equation* (1-9).

Applying Theorems 1.2 and 1.3 to the quotient of the elementary symmetric functions, i.e., $F = (\sigma_k/\sigma_l)^{1/(k-l)}$ on Γ_k^+, we have the following corollaries.

Corollary 1.4. *Let $(\overline{M}^n, g)$, $n \geq 3$, be a compact manifold with smooth totally geodesic boundary ∂M. If $\tau > n-1$ and $A_g^\tau \in \Gamma_k^+$, then for any smooth function $\varphi > 0$, there exists a smooth metric $\tilde{g} \in [g]$ preserving ∂M totally geodesic and satisfying*

$$\left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}} (A_{\tilde{g}}^\tau) = \varphi(x) \quad \text{in } M. \tag{1-11}$$

Corollary 1.5. *Let $(\overline{M}^n, g)$, $n \geq 3$, be a compact manifold with smooth totally geodesic boundary ∂M. If $\tau < 1$ and $A_g^\tau \in \Gamma_k^-$, then for any smooth function $\varphi < 0$, there exists a smooth metric $\tilde{g} \in [g]$ preserving ∂M totally geodesic and satisfying* (1-11).

Remark 1.6. By choosing $l = 0$ and φ constant in Corollaries 1.4 and 1.5, we can get Theorem 1.1 directly. Different from the results in [Li and Sheng 2011; Sheng et al. 2007], we need not subjoin any restriction on $a(x)$ and $b(x)$ in Theorems 1.2 and 1.3. Contrary to this fact, [Sheng et al. 2007] gives a counterexample to show that there is no regularity if $a(x) = 0$ and $b(x) > 0$ when $\tau = 1$ and $A_g^\tau \in \Gamma_k^-$.

This paper is organized as follows. We introduce some lemmas in Section 2. By use of these lemmas, we can get the a priori global C^0 estimate for (1-6) in Section 3. Then we obtain the a priori global gradient and Hessian derivatives estimates in Section 4 and Section 5 respectively. By the a priori estimates and the standard continuity method, we show Theorem 1.2 in Section 6. In the last section, we consider (1-9) by the similar arguments in Sections 3–6, and prove Theorem 1.3.

2. Preliminaries

In this section, we first recall some facts of the function F satisfying the structure conditions (C1)–(C3) in Γ.

Lemma 2.1 (see [Chen 2005; 2009]). *Let Γ be an open convex cone with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma$, and let $e = (1, \dots, 1)$ be the identity. Suppose that F is a homogeneous symmetric function of degree one normalized with $F(e) = 1$, and that F is concave in Γ. Then*:

(a) $\sum_i \lambda_i \partial F(\lambda)/\partial \lambda_i = F(\lambda)$, *for* $\lambda \in \Gamma$.

(b) $\sum_i \partial F(\lambda)/\partial \lambda_i \geq F(e) = 1$, *for* $\lambda \in \Gamma$.

To get the boundary estimates, we need some facts. For any point $x_0 \in \partial M$, we consider Fermi coordinates $\{x_i\}_{1 \leq i \leq n}$ around x_0, where $\partial/\partial x_n$ is the unit inner normal with respect to the background metric g. A half-ball centered at x_0 of radius r is defined by

$$\bar{B}_r^+ = \left\{ x_n \geq 0, \left(\sum_{i=1}^{n} x_i^2 \right) \leq r^2 \right\}.$$

Denote the boundary of $\bar{B}_r^+$ on ∂M by $\Sigma_r = \left\{ x_n = 0, \sum_i x_i^2 \leq r^2 \right\}$.

Throughout this paper, the Greek letters $\alpha, \beta, \gamma, \ldots = 1, \ldots, n-1$ stand for the tangential direction indices, while the Latin letters $i, j, k, \ldots = 1, \ldots, n$ stand for the full indices. In Fermi coordinates $\{x_i\}_{1 \leq i \leq n}$, the metric is expressed as $g = g_{\alpha\beta}\, dx_\alpha\, dx_\beta + (dx_n)^2$. Then the Christoffel symbols on the boundary satisfy

$$\Gamma^n_{\alpha\beta} = L_{\alpha\beta}, \quad \Gamma^\beta_{\alpha n} = -L_{\alpha\gamma} g^{\gamma\beta}, \quad \Gamma^n_{\alpha n} = 0, \quad \Gamma^n_{nn} = 0, \quad \Gamma^\gamma_{nn} = 0, \quad \Gamma^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\alpha\beta} \tag{2-1}$$

on the boundary, where we denote the tensors and covariant differentiation with respect to the induced metric $g_{\alpha\beta}$ on the boundary by a tilde (e.g., $\tilde{\Gamma}^\gamma_{\alpha\beta}$, $\mu_{\tilde{\alpha}\tilde{\beta}}$). When the boundary is totally geodesic, we have

$$\Gamma^n_{\alpha\beta} = 0, \quad \Gamma^\beta_{\alpha n} = 0, \quad \Gamma^n_{\alpha n} = 0. \tag{2-2}$$

Lemma 2.2 [Chen 2007; He and Sheng 2013]. *Suppose ∂M is totally geodesic and $u_n = 0$ on ∂M. Then we have on the boundary that*

$$u_{n\alpha} = 0 \quad \text{and} \quad u_{\alpha\beta n} = 0. \tag{2-3}$$

Lemma 2.3 [He and Sheng 2013]. *Let $(\overline{M}, g)$ be a compact Riemannian manifold with boundary and dimension $n \geq 3$. Assume the boundary ∂M is totally geodesic. Then at any boundary point $P \in \partial M$, there exists a conformal metric $\bar{g} = e^{2\bar{u}} g_0$ such that* (i) *$\bar{u}_n = 0$ on ∂M and the boundary ∂M is still totally geodesic*, (ii) *$\bar{R}_{ij}(P) = 0$ for $1 \leq i, j \leq n$,* (iii) *$\bar{R}_{nn,n}(P) = 0$, $\bar{R}_{\alpha n,\beta}(P) = 0$, $1 \leq \alpha, \beta \leq n-1$, and* (iv) *$\bar{R}_{\alpha\beta,n}(P) = 0$, $1 \leq \alpha, \beta \leq n-1$.*

3. Ellipticity and the global C^0 estimates

We first sketch the ellipticity properties of operator F; see [Li and Sheng 2011] for details.

For any function h on $\overline{M}$, we define

$$\mathcal{P}[h] := F(V[h]) - \Phi(x, h).$$

Then any solution u of (1-6) satisfies $\mathcal{P}[u] = 0$. Denote $u_s = u + sv$, $s \in \mathbb{R}$. The linearized operator of (1-6) is

$$\begin{aligned} \mathcal{L}v &:= \frac{d}{ds}\mathcal{P}[u_s]\Big|_{s=0} \\ &= P^{ij} v_{ij} + 2aF^{ij} v_i u_j + 2b v_l u_l \mathcal{T} - \partial_z \Phi(x, u) v, \end{aligned} \tag{3-1}$$

where $F^{ij} := (\partial F/\partial V_{ij})(V[u])$, $\mathcal{T} = \operatorname{tr}(F^{ij}) = F^{ij} g_{ij}$ and

$$P^{ij} := \bar{\theta}\mathcal{T} g^{ij} - F^{ij} \geq (\bar{\theta} - 1)\mathcal{T} g^{ij}.$$

Since u is admissible, (F^{ij}) is positive definite [Caffarelli et al. 1985]. Denote $\varepsilon_0 := \bar{\theta} - 1 > 0$. Hence, (P^{ij}) is positive definite, too.

Note that the coefficient of the zero order term in (3-1) is negative when $\partial_z \Phi$ is positive on $\overline{M} \times \mathbb{R}$.

Lemma 3.1. *Equation* (1-6) *is elliptic at any admissible solution. If $\partial_z \Phi$ is positive on $\overline{M} \times \mathbb{R}$, then the linearized operator $\mathcal{L} : C^{2,\alpha}(\overline{M}) \to C^{\alpha}(\overline{M}) (0 < \alpha < 1)$ is invertible.*

Now, we use the compactness of the manifold to get the global C^0 estimates of (1-6).

Proposition 3.2. *Suppose $S \in \Gamma$ and the positive function $\Phi(x, z) \in C^{\infty}(\overline{M}) \times \mathbb{R}$ satisfies* (1-8). *Then for any admissible solution $u \in C^2(\overline{M})$ of* (1-6), *we have*

$$\sup_{\overline{M}} |u| \leq C_0,$$

where the constant C_0 depends only on S and Φ.

Proof. Suppose x_0 be the maximum point of u on $\overline{M}$. Denote $u_{\max} = u(x_0)$.

If $x_0 \in \partial M$, at this point we have $u_n(x_0) < 0$, which contradicts with the boundary condition $u_n|_{\partial M} \equiv 0$. Hence, x_0 must be an interior point of M. Then at this point we have

$$\nabla u = 0 \quad \text{and} \quad \nabla^2 u \geq 0. \tag{3-2}$$

Substituting (3-2) into (1-6), we have

$$\Phi(x_0, u_{\max}) \leq F(S)(x_0) \leq \max_{x \in \overline{M}} F(S) \leq C.$$

Now, by the condition $\partial_z \Phi > 0$ and $\lim_{z \to +\infty} \Phi(x, z) = +\infty$, we know that

$$\max_{x \in \overline{M}} u = u_{\max} \leq C.$$

By a similar argument, we can get the lower bound of u by considering its minimum point on $\overline{M}$ and using the other condition of Φ. $\square$

4. Gradient estimates

In this section we first consider the boundary gradient estimates of (1-6), then derive the global estimates.

For any point $y_0 \in \partial M$, let $\bar{B}_r^+$ and $\bar{B}_{r/2}^+$ be any two half-balls centered at y_0 in the Fermi coordinates $\{x_i\}_{1\le i\le n}$. Choosing a cutoff function η depending only on r such that $0 \le \eta \le 1$, $\eta = 1$ in $\bar{B}_{r/2}^+$, $\eta = 0$ outside $\bar{B}_r^+$. Moreover,

$$|\nabla\eta| \le b_0 \frac{\eta^{1/2}}{r} \quad \text{and} \quad |\nabla^2\eta| \le \frac{b_0}{r^2}, \tag{4-1}$$

for a universal constant b_0, where the covariant derivatives and the norms $|\cdot|$ are taken with respect to g. Since η only depends on r, we have

$$\frac{\partial\eta}{\partial n} = 0 \quad \text{on } \partial M. \tag{4-2}$$

We also need the function $\psi : \mathbb{R} \to \mathbb{R}$ defined in [Gursky and Viaclovsky 2003] by

$$\psi(s) = \alpha_1(\alpha_2 + s)^p, \quad -\delta_1 < s < \delta_2, \tag{4-3}$$

where the positive constants δ_1 and δ_2 are given, and the constants α_1, α_2 and p will be fixed as follows. We have

$$\psi' = p\alpha_1(\alpha_2+s)^{p-1} \quad \text{and} \quad \psi'' = p(p-1)\alpha_1(\alpha_2+s)^{p-2} = \frac{p-1}{\alpha_2+s}\psi'.$$

Let α_2 and p be positive constants satisfying $\alpha_2 > \delta_1$ and $p > 3$. Take

$$\alpha_1 = \frac{1}{p^2 \max\{(\alpha_2+s)^p\}};$$

then

$$\psi \le \frac{1}{p^2}, \quad \psi' > 0 \quad \text{and} \quad \psi'' - \psi'^2 = \frac{\psi'}{\alpha_2+s}(p-1-p\psi) \ge \frac{\psi' p}{2(\alpha_2+s)}. \tag{4-4}$$

Proposition 4.1. *Suppose u is a C^3 solution of* (1-6) *on $\bar{B}_r^+$. Then there is a positive constant C depending only on $n, k, \bar{\theta}, g, r, |S|_{C^1(\bar{B}_r^+)}, |\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}, |a|_{C^1(\bar{B}_r^+)}, |b|_{C^1(\bar{B}_r^+)}$ and C_0 such that*

$$\sup_{\bar{B}_{r/2}^+} |\nabla u|_g \le C.$$

Proof. Consider the auxiliary function

$$G := \tfrac{1}{2}\eta e^{\beta}|\nabla u|^2, \quad \beta := x_n + \psi(u),$$

where the function ψ defined by (4-3). Let x_0 be the maximum point of G on $\bar{B}_r^+$. Without loss of generality, we may assume $r = 1$ and $|\nabla u|\,(x_0) \gg 1$.

Suppose $x_0 \in \Sigma_r$. Then $G_n(x_0) \le 0$. However, by (4-2), the boundary condition $u_n = 0$ and Lemma 2.2, we have

$$G_n(x_0) = \tfrac{1}{2} e^{\psi}\Big((1+\psi' u_n)|\nabla u|^2 + 2u_n u_{nn} + 2\sum_{\alpha=1}^{n-1} u_\alpha u_{\alpha n}\Big)(x_0)$$
$$= \tfrac{1}{2} e^{\psi} |\nabla u|^2 (x_0) > 0.$$

It is a contradiction. Hence x_0 must be an interior point of $\bar{B}_r^+$. Then at x_0, for $1 \le i \le n$, we have

$$0 = (\log G)_i, \quad 0 \ge (\log G)_{ij},$$

that is,

$$\frac{2u_s u_{si}}{|\nabla u|^2} = -\Big(\frac{\eta_i}{\eta} + \beta_i\Big), \tag{4-5}$$

and

$$0 \ge \Big(\frac{\eta_{ij}}{\eta} - \frac{\eta_i \eta_j}{\eta^2}\Big) + \beta_{ij} + \frac{2u_{sj}u_{si} + 2u_s u_{sij}}{|\nabla u|^2} - \frac{4u_s u_{si} u_l u_{lj}}{|\nabla u|^4}. \tag{4-6}$$

Substituting (4-5) into (4-6), we have

$$0 \ge \Big(\frac{\eta_{ij}}{\eta} - 2\frac{\eta_i \eta_j}{\eta^2}\Big) + (\beta_{ij} - \beta_i\beta_j) + \frac{2u_{sj}u_{si} + 2u_s u_{sij}}{|\nabla u|^2} - \frac{1}{\eta}(\eta_i\beta_j + \eta_j\beta_i). \tag{4-7}$$

By (4-7), we have

$$\begin{aligned} 0 \ge P^{ij}\Big(\frac{\eta_{ij}}{\eta} - 2\frac{\eta_i \eta_j}{\eta^2}\Big) &+ P^{ij}(\beta_{ij} - \beta_i\beta_j) \\ &+ \frac{2}{|\nabla u|^2} P^{ij} u_{si} u_{sj} + \frac{2}{|\nabla u|^2} u_s P^{ij} u_{sij} - \frac{2}{\eta} P^{ij}\eta_i\beta_j, \end{aligned} \tag{4-8}$$

where $P^{ij} = \bar{\theta}\mathcal{T} g^{ij} - F^{ij}$ is positive definite. It follows from (4-1) and (4-8) that

$$0 \ge \frac{2}{|\nabla u|^2} u_s P^{ij} u_{sij} + P^{ij}(\beta_{ij} - \beta_i\beta_j) - \frac{2}{\eta} P^{ij}\eta_i\beta_j - \frac{C}{\eta}\mathcal{T}, \tag{4-9}$$

where the constant C depends only on n and b_0.

Differentiating (1-6), we have

$$\nabla_s \Phi = P^{ij} u_{ijs} + F^{ij}(a_s u_i u_j + 2a u_{is} u_j + S_{ij},_s) + (b_s|\nabla u|^2 + 2b u_{ls} u_l)\mathcal{T}. \tag{4-10}$$

Then by (4-10) and Ricci identities $u_{sij} = u_{ijs} + R_{isjp} u_p$, we obtain

$$\begin{aligned} \frac{2}{|\nabla u|^2} u_s P^{ij} u_{sij} \ge \frac{2}{|\nabla u|^2} u_s \nabla_s \Phi &- \frac{2}{|\nabla u|^2} u_s F^{ij}(a_s u_i u_j + 2a u_{is} u_j) \\ &- \frac{2}{|\nabla u|^2} u_s (b_s |\nabla u|^2 + 2b u_{ls} u_s)\mathcal{T} - C\Big(1 + \frac{1}{|\nabla u|}\Big)\mathcal{T}. \end{aligned}$$

where the constant C depends only on n, g and $|\nabla S|$.

Since $\nabla_s\Phi = \Phi_x + \Phi_z u_s$, by (4-5) and the inequality above, we have

$$\begin{aligned}\text{(4-11)}\quad \frac{2}{|\nabla u|^2}u_s P^{ij}u_{sij} &\geq 2\Phi_z + \frac{2}{|\nabla u|^2}u_s\Phi_x - \frac{2a_s u_s}{|\nabla u|^2}F^{ij}u_i u_j + 2aF^{ij}u_j\left(\frac{\eta_i}{\eta}+\beta_i\right)\\ &\quad - 2b_s u_s\mathcal{T} + 2b\left(\frac{\eta_s}{\eta}+\beta_s\right)u_s\mathcal{T} - C\left(1+\frac{1}{|\nabla u|}\right)\mathcal{T}\\ &\geq C^* + 2aF^{ij}u_j\beta_i + 2bu_s\beta_s\mathcal{T} - \frac{C}{\sqrt{\eta}}(1+|\nabla u|)\mathcal{T},\end{aligned}$$

where the constant C^* depends only on $|\Phi_x|$, $|\Phi_z|$, C_0, and C depends on n, b_0, $|a|_{C^1}$, $|b|_{C^1}$ and $|\nabla S|$.

Then by (4-9) and (4-11), we obtain

$$\begin{aligned}\text{(4-12)}\quad 0 &\geq C^* + 2aF^{ij}u_j\beta_i + 2bu_s\beta_s\mathcal{T}\\ &\quad + P^{ij}(\beta_{ij}-\beta_i\beta_j) - \frac{2\eta_i}{\eta}P^{ij}\beta_j - C\frac{1}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}.\end{aligned}$$

Since $\beta := x_n + \psi(u)$, we have

$$\beta_i = \delta_{in} + \psi' u_i, \quad \beta_{ij} = \psi'' u_i u_j + \psi' u_{ij}$$

and

$$\beta_{ij} - \beta_i\beta_j = \psi' u_{ij} + (\psi'' - \psi'^2)u_i u_j - \psi'(\delta_{in}u_j + \delta_{jn}u_i) - \delta_{in}\delta_{jn}.$$

Therefore, we have the inequalities

$$\text{(4-13)}\quad 2aF^{ij}u_j\beta_i = 2aF^{ij}u_j(\delta_{in}+\psi' u_i) \geq 2a\psi' F^{ij}u_i u_j - C|\nabla u|\mathcal{T},$$

$$\text{(4-14)}\quad 2bu_s\beta_s\mathcal{T} = 2bu_s(\delta_{sn}+\psi' u_s)\mathcal{T} \geq 2b\psi'|\nabla u|^2\mathcal{T} - C|\nabla u|\mathcal{T},$$

$$\text{(4-15)}\quad -\frac{2\eta_i}{\eta}P^{ij}\beta_j = -\frac{2}{\eta}P^{ij}\eta_i(\delta_{jn}+\psi' u_j) \geq -\frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T},$$

$$\text{(4-16)}\quad P^{ij}(\beta_{ij}-\beta_i\beta_j) \geq \psi' P^{ij}u_{ij} + (\psi''-\psi'^2)P^{ij}u_i u_j - C(|\nabla u|+1)\mathcal{T}.$$

Plugging (4-13)–(4-16) into (4-12), we have

$$\begin{aligned}\text{(4-17)}\quad 0 &\geq C^* + \psi' P^{ij}u_{ij} + (\psi''-\psi'^2)P^{ij}u_i u_j + 2a\psi' F^{ij}u_i u_j\\ &\quad + 2b\psi'|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}.\end{aligned}$$

By Lemma 2.1, we know that $F^{ij}V_{ij} = F(V) = \Phi$. Then

$$\begin{aligned}\text{(4-18)}\quad \psi' P^{ij}u_{ij} &= \psi' F^{ij}V_{ij} - \psi' F^{ij}(au_i u_j + b|\nabla u|^2 g_{ij} + S_{ij})\\ &\geq \psi'\Phi - a\psi' F^{ij}u_i u_j - b\psi'|\nabla u|^2\mathcal{T} - C\mathcal{T}.\end{aligned}$$

Substituting (4-18) into (4-17), we get

$$(4\text{-}19)\quad \begin{aligned} 0 &\geq C^* + \psi'\Phi + (\psi'' - \psi'^2)P^{ij}u_iu_j + a\psi' F^{ij}u_iu_j \\ &\qquad\qquad + b\psi'|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T} \\ &= C^* + \psi'\Phi + (\psi'' - \psi'^2 - a\psi')P^{ij}u_iu_j \\ &\qquad\qquad + (a\bar{\theta}+b)\psi'|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}. \end{aligned}$$

Claim 4.2. *If $-\delta_1 < u < \delta_2$, then there exist positive constants α_1, α_2 and p depending only on $\bar{\theta}$, δ_1, δ_2, $|a|_{L^\infty(\bar{M})}$ and $|b|_{L^\infty(\bar{M})}$, such that $\psi' > 0$, and*

$$(4\text{-}20)\qquad (\psi'' - \psi'^2 - |a|_{L^\infty}\psi')\varepsilon_0 - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\psi' \geq \varepsilon_1 > 0,$$

for some constant ε_1 depending only on $\bar{\theta}$, δ_1 and δ_2.

Note that $\Phi > 0$. Then by Claim 4.2, we have

$$0 \geq C^* + \varepsilon_1|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}.$$

Multiplying η^2 both sides of the inequality above, we have

$$(4\text{-}21)\qquad \varepsilon_1\eta^2|\nabla u|^2\mathcal{T} \leq 2C|\nabla u|\mathcal{T} + C^*.$$

By Lemma 2.1, $\mathcal{T} \geq 1$. Then (4-21) implies the gradient estimates.

Proof of Claim 4.2. Since $-\delta_1 \leq u \leq \delta_2$. By (4-4), for

$$\frac{\delta_1+\delta_2}{2} \leq \alpha_2 \leq \delta_2, \qquad p > \max\{3,\, 8|a|_{L^\infty}\delta_2\},$$

we have $\alpha_1 = 1/(p^2(2\delta_2)^p)$, $\psi' > 0$, and

$$\psi'' - \psi'^2 - a\psi' \geq \psi'\left(\frac{p}{4\delta_2} - |a|_{L^\infty}\right) \geq \frac{\psi' p}{8\delta_2}.$$

Furthermore, we can choose

$$p > \max\left\{3,\, 8|a|_{L^\infty}\delta_2,\, \frac{16}{\varepsilon_0}(\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\delta_2\right\},$$

such that

$$\begin{aligned} &(\psi'' - \psi'^2 - |a|_{L^\infty}\psi')\varepsilon_0 - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\psi' \\ &\qquad \geq \psi'\left(\frac{p\varepsilon_0}{8\delta_2} - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\right) \geq \frac{\psi' p\varepsilon_0}{16\delta_2} \geq \frac{\varepsilon_0(\delta_2-\delta_1)^{p-1}}{2^{p+3}\delta_2} \geq \varepsilon_1 > 0. \quad \square \end{aligned}$$

Remark 4.3. If $\bar{B}_r^+$ and $\bar{B}_{r/2}^+$ are replaced by two local geodesic open balls in the interior of M and $\beta = \psi(u)$ in the auxiliary function G, we can get the interior gradient estimates for (1-6) by the proof of Proposition 4.1.

Since $\overline{M}$ is a compact manifold, by Proposition 4.1 and Remark 4.3, we can derive the global gradient estimate of (1-6).

Proposition 4.4. *Let u be a C^3 solution of* (1-6) *on $\overline{M}$. Then there is a positive constant C_1 depending only on n, k, $\bar{\theta}$, g, a, b, Φ, S and C_0 such that*

$$\sup_{\overline{M}} |\nabla u|_g \le C_1. \tag{4-22}$$

5. Estimates for the second derivatives

Lemma 5.1. *Let u be a C^4 solution of* (1-6). *Then there is a positive constant C' depending only on n, k, $\bar{\theta}$, g, $|S|_{C^1(\bar{B}_r^+)}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$ and C_1, such that*

$$u_{nnn} \ge -C' \quad \text{on } \partial M. \tag{5-1}$$

Proof. We consider this lemma for S satisfying condition (S1) or (S2), respectively.

(i) Suppose S satisfy (S1). Then $S_{\alpha n} = S(\partial/\partial x_\alpha, \partial/\partial x_n) = 0$ on the boundary ∂M. By the boundary condition $u_n = 0$ and the Lemma 2.2, we have $V[u]_{\alpha n} = S_{\alpha n} = 0$. Applying an argument of Lemma 13 in [Chen 2009], we know that

$$F^{\alpha n}(V[u]) = 0. \tag{5-2}$$

Also by Lemma 2.2, we calculate that

$$\begin{aligned} V[u]_{\alpha\beta,n} &= \bar{\theta} u_{nnn} g_{\alpha\beta} + \bar{\theta} u_{\gamma\gamma n} g_{\alpha\beta} - u_{\alpha\beta n} + 2a u_{\alpha n} u_\beta + a_n u_\alpha u_\beta \\ &\qquad + 2b u_{\alpha n} u_\alpha g_{\alpha\beta} + 2b u_{nn} u_n g_{\alpha\beta} + b_n |\nabla u|^2 g_{\alpha\beta} + S_{\alpha\beta,n} \\ &= \bar{\theta} u_{nnn} g_{\alpha\beta} + a_n u_\alpha u_\beta + b_n |\nabla u|^2 g_{\alpha\beta} + S_{\alpha\beta,n} \\ &\le \bar{\theta} u_{nnn} g_{\alpha\beta} + C, \end{aligned} \tag{5-3}$$

where the constant C depends only on $|\nabla a|$, $|\nabla b|$, C_1, g and $|\nabla S|$.

Similarly, we have

$$\begin{aligned} V[u]_{nnn} &= \bar{\theta} u_{\gamma\gamma n} + \bar{\theta} u_{nnn} - u_{nnn} + a_n u_n^2 + 2a u_n u_{nn} + 2b u_{\alpha n} u_\alpha \\ &\qquad + 2b u_n u_{nn} + b_n |\nabla u|^2 + S_{nn,n} \\ &\le \bar{\theta} u_{nnn} - u_{nnn} + C. \end{aligned} \tag{5-4}$$

By differentiating (1-6) along the normal direction the on boundary, using (5-2)–(5-4), we have

$$\begin{aligned}\nabla_n \Phi &= F^{nn} V[u]_{nnn} + F^{\alpha\beta} V[u]_{\alpha\beta n} \\ &\le F^{nn}(\bar\theta u_{nnn} - u_{nnn}) + \bar\theta u_{nnn} F^{\alpha\beta} g_{\alpha\beta} + C\mathcal{T} \\ &= -F^{nn} u_{nnn} + \bar\theta u_{nnn}\mathcal{T} + C\mathcal{T},\end{aligned}$$

where we have used $g_{\alpha n} = 0$ and $g_{nn} = 1$. Since $\mathcal{T} > 1$, we have

$$0 \le -F^{nn} u_{nnn} + (\bar\theta u_{nnn} + C)\mathcal{T}, \tag{5-5}$$

where C also depends on $|\nabla\Phi|$.

If $\bar\theta u_{nnn} + C > 0$, we get $u_{nnn} > -C/\bar\theta$, which implies (5-1). If $\bar\theta u_{nnn} + C < 0$, by $F^{nn} < \mathcal{T}$ we have

$$0 \le -F^{nn} u_{nnn} + (\bar\theta u_{nnn} + C) F^{nn} = ((\bar\theta - 1) u_{nnn} + C) F^{nn}.$$

Since $F^{nn} > 0$, we have

$$(\bar\theta - 1) u_{nnn} + C \ge 0. \tag{5-6}$$

Note that $\bar\theta - 1 = \varepsilon_0 > 0$; then (5-6) implies (5-1).

(ii) Suppose $S = A^\tau_g$. For any $x_0 \in \partial M$, using the metric $\bar g$ in Lemma 2.3, we consider a metric $\hat g = e^{2v}\bar g$ such that $u = \bar u + v$ is a solution of (1-6). Now,

$$\begin{aligned}V[u]_{ij} = \bar\theta \triangle \bar u g_{ij} + \bar\theta \triangle v g_{ij} - \bar u_{ij} - v_{ij} + a(\bar u_i \bar u_j + \bar u_i v_j + v_i \bar u_j + v_i v_j) \\ + b(|\nabla \bar u|^2 + 2\langle \nabla \bar u, \nabla v\rangle + |\nabla v|^2) g_{ij} + (A^\tau_g)_{ij}.\end{aligned} \tag{5-7}$$

By (1-3), we have

$$(A^\tau_{\bar g})_{ij} = \bar\theta \triangle \bar u g_{ij} - \bar u_{ij} + \bar u_i \bar u_j + \frac{(n-2)\bar\theta - 1}{2} |\nabla \bar u|^2 g_{ij} + (A^\tau_g)_{ij}. \tag{5-8}$$

Substituting (5-8) into (5-7), we obtain

$$\begin{aligned}V[u]_{ij} = \bar\theta \triangle v g_{ij} - v_{ij} + a(\bar u_i v_j + v_i \bar u_j + v_i v_j) + (a-1)\bar u_i \bar u_j \\ + b(2\langle \nabla \bar u, \nabla v\rangle + |\nabla v|^2) g_{ij} + \left(b - \frac{(n-2)\bar\theta - 1}{2}\right) |\nabla \bar u|^2 g_{ij} + (A^\tau_{\bar g})_{ij}.\end{aligned}$$

Since $\bar{g}=e^{2\bar{u}}g$, we have

$$V[u]_{ij}=\bar{\theta}\bar{\triangle}v\bar{g}_{ij}-\bar{\nabla}^2_{ij}v+\bar{\theta}\bar{g}^{sl}(\bar{\Gamma}^k_{sl}(\bar{g})-\Gamma^k_{sl}(g))v_k\bar{g}_{ij}$$
$$-(\bar{\Gamma}^k_{ij}(\bar{g})-\Gamma^k_{ij}(g))v_k+a(\bar{u}_iv_j+v_i\bar{u}_j+v_iv_j)$$
$$+(a-1)\bar{u}_i\bar{u}_j+b(2\langle\bar{\nabla}\bar{u},\bar{\nabla}v\rangle_{\bar{g}}+|\bar{\nabla}v|^2_{\bar{g}})\bar{g}_{ij}$$
$$+\left(b-\frac{(n-2)\bar{\theta}-1}{2}\right)|\nabla\bar{u}|^2_{\bar{g}}\bar{g}_{ij}+(A^{\tau}_{\bar{g}})_{ij}. \tag{5-9}$$

Denote $\overline{V}[v]_{ij}:=V[u]_{ij}$. Then (1-6) becomes

$$\begin{cases} F(\overline{V}[v])=\Phi(x,\bar{u}+v) & \text{in } M,\\ \dfrac{\partial v}{\partial n}=0 & \text{on } \partial M. \end{cases} \tag{5-10}$$

By the boundary condition $u_n=0$, $\bar{u}_n=0$ and Lemma 2.2, we have

$$u_{n\alpha}=0,\quad u_{\alpha\beta n}=0,\quad \bar{u}_{n\alpha}=0,\quad \bar{u}_{\alpha\beta n}=0. \tag{5-11}$$

Therefore $v_n=0$, $v_{n\alpha}=0$ and $v_{\alpha\beta n}=0$ on ∂M. Since $\bar{g}_{\alpha n}=e^{2\bar{u}}g_{\alpha n}=0$, we have

$$\overline{V}[v]_{\alpha n}=-\overline{\nabla}^2_{\alpha n}v-(\overline{\Gamma}^{\delta}_{\alpha n}(\bar{g})-\Gamma^{\delta}_{\alpha n}(g_0))v_{\delta}+(A^{\tau}_{\bar{g}})_{\alpha n}.$$

It follows from (2-2) and the boundary condition $u_n=0$ that

$$\overline{\Gamma}^{\delta}_{\alpha n}(\bar{g})=\Gamma^{\delta}_{\alpha n}(g)=0,\quad \overline{\Gamma}^{n}_{\alpha\beta}=\Gamma^{n}_{\alpha\beta}=0,\quad \overline{\Gamma}^{n}_{nn}=\Gamma^{n}_{nn}=0. \tag{5-12}$$

Then

$$\overline{\nabla}^2_{\alpha n}v=v_{\alpha n}=0\quad\text{and}\quad \overline{\nabla}_n\overline{\nabla}^2_{\alpha\beta}v=v_{\alpha\beta n}=0. \tag{5-13}$$

By Lemma 2.3, we get

$$(A^{\tau}_{\bar{g}})_{\alpha n}(x_0)=-\frac{1}{n-2}\left(\overline{R}_{\alpha n}-\frac{\tau\overline{R}}{2(n-1)}\bar{g}_{\alpha n}\right)=0.$$

Hence, $\overline{V}[v]_{\alpha n}(x_0)=0$. Then

$$F^{\alpha n}(\overline{V}[v])=0.$$

Now differentiating (5-10) along the normal direction and taking its value at x_0, we have

$$\nabla_n\Phi(x,\bar{u}+v)=F^{nn}\overline{V}_{nnn}+F^{\alpha\beta}\overline{V}_{\alpha\beta n}. \tag{5-14}$$

Since $\bar{g}_{ij,n}=\bar{g}^{ij}_{,n}=0$, by (5-11)–(5-13), we have

$$\overline{V}[v]_{\alpha\beta n}$$
$$=\bar{\theta}v_{nnn}\bar{g}_{\alpha\beta}-(\overline{\Gamma}^{\delta}_{\alpha\beta}(\bar{g})-\Gamma^{\delta}_{\alpha\beta}(g))_{,n}v_{\delta}+\bar{\theta}\bar{g}^{sl}(\overline{\Gamma}^{\delta}_{sl}(\bar{g})-\Gamma^{\delta}_{sl}(g))_{,n}v_{\delta}\bar{g}_{\alpha\beta}+(A^{\tau}_{\bar{g}})_{\alpha\beta,n}.$$

Since ∂M is totally geodesic, using Fermi coordinates, we have on ∂M

$$\bar{\Gamma}^{\delta}_{\alpha\beta}(g)_{,n} = \Gamma^{\delta}_{\alpha\beta}(g)_{,n} = 0$$

(see [He and Sheng 2013]). By Lemma 2.3 again,

$$\bar{R}_n(x_0) = \bar{g}^{\alpha\beta}\bar{R}_{\alpha\beta,n}(x_0) + \bar{g}^{\alpha n}\bar{R}_{\alpha n,n}(x_0) + \bar{g}^{nn}\bar{R}_{nn,n}(x_0) = 0.$$

Therefore

$$(A^{\tau}_{\bar{g}})_{\alpha\beta,n}(x_0) = -\frac{1}{n-2}\left(\bar{R}_{\alpha\beta,n} - \frac{\tau\bar{R}_n}{2(n-1)}\bar{g}_{\alpha\beta}\right)(x_0) = 0.$$

Hence, we obtain

$$\bar{V}[v]_{\alpha\beta n}(x_0) = \bar{\theta} v_{nnn}\bar{g}_{\alpha\beta}. \tag{5-15}$$

Similarly, we have

$$\bar{V}[v]_{nnn}(x_0) = \bar{\theta} v_{nnn}\bar{g}_{nn}(x_0) - v_{nnn}(x_0). \tag{5-16}$$

Denote $\bar{\mathcal{T}} = F^{ij}(\bar{V}[v])\bar{g}_{ij} \geq 1$. Plugging (5-15) and (5-16) into (5-14), we obtain

$$0 \leq C + \bar{\theta} v_{nnn}(x_0)\bar{\mathcal{T}} - F^{nn}v_{nnn}(x_0) \leq (C + \bar{\theta} v_{nnn}(x_0))\bar{\mathcal{T}} - F^{nn}v_{nnn}(x_0). \tag{5-17}$$

If $C + \bar{\theta} v_{nnn}(x_0) \geq 0$, then we have $v_{nnn}(x_0) \geq -C/\bar{\theta}$, which implies that

$$u_{nnn}(x_0) \geq \bar{u}_{nnn}(x_0) - \frac{C}{\bar{\theta}} > -C'.$$

If $C + \bar{\theta} v_{nnn}(x_0) < 0$, then by (5-17) we have

$$0 \leq (C + (\bar{\theta} - 1)v_{nnn}(x_0))F^{nn}.$$

Since $F^{nn} > 0$ and $\bar{\theta} > 1$, we have $v_{nnn}(x_0) \geq -C/(\bar{\theta} - 1)$, which also implies the lower bound of $u_{nnn}(x_0)$. □

Proposition 5.2. *Let u be a C^4 solution of* (1-6) *on $\bar{B}^+_r$. Then there is a positive constant C_2 depending only on n, k, $\bar{\theta}$, r, g, $|S|_{C^2(\bar{B}^+_r)}$, $|\Phi|_{C^2(\bar{B}^+_r)\times[-C_0,C_0]}$, $|a|_{C^2(\bar{B}^+_r)}$, $|b|_{C^2(\bar{B}^+_r)}$, and C_1, such that*

$$\sup_{\bar{B}^+_{r/2}} |\nabla^2 u|_g \leq C_2. \tag{5-18}$$

Proof. We control the bound of $\triangle u$ at first. Since $V[u] \in \Gamma \subset \Gamma_1$, we have

$$0 \leq \operatorname{tr}(V[u]) = (n\bar{\theta} - 1)\triangle u + (a + nb)|\nabla u|^2 + \operatorname{tr} S,$$

which implies that $\triangle u$ has a lower bound by Proposition 4.4. We may assume $\triangle u > 0$.

Consider the auxiliary function

$$G := \eta e^{x_n}(\triangle u + m|\nabla u|^2),$$

where η satisfies (4-1) and (4-2), and m is a larger constant to be fixed. We may assume $r = 1$, and

$$K := \triangle u + m|\nabla u|^2 \gg 1.$$

Step 1. We may assume G attains its maximum at an interior point $x_0 \in B_r^+$. If $x_0 \in \Sigma_r$, by Lemmas 2.2 and 5.1 we have

$$G_n(x_0) = K + u_{nnn} + u_{\gamma\gamma n} + 2mu_{\alpha n}u_\alpha + 2mu_{nn}u_n > K - C'.$$

If $K - C' \le 0$, we then get the bound of $\triangle u$. If $K - C' > 0$, it contradicts with the maximum of G at the boundary point x_0.

Step 2. We must get an upper bound for $\triangle u$. By step 1, the maximum point x_0 of G is an interior point in $\bar{B}_r^+$. Then at x_0 we have

$$G_i = 0 \quad \text{and} \quad G_{ij} \le 0,$$

that is,

$$u_{lli} + 2mu_l u_{li} = K_i = -\left(\frac{\eta_i}{\eta} + \delta_{in}\right)K, \tag{5-19}$$

and

$$0 \ge G_{ij} = \eta e^{x_n}\left\{\left(\frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2}\right)K + \left(\frac{\eta_i}{\eta} + \delta_{in}\right)K_j + K_{ij}\right\}.$$

Substituting (5-19) into the inequality above, by the definition of η in (4-1), we have

$$0 \ge G_{ij} = \eta e^{x_n}(K_{ij} + \Lambda_{ij}K),$$

where

$$\Lambda_{ij} = \frac{\eta_{ij}}{\eta} - 2\frac{\eta_i\eta_j}{\eta^2} - \frac{1}{\eta}(\eta_i\delta_{jn} + \eta_j\delta_{in}) - \delta_{in}\delta_{jn} \ge -\frac{C}{\eta}\delta_{ij},$$

and C depends only on b_0. Then we have

$$0 \ge e^{-x_n}P^{ij}G_{ij} \ge \eta P^{ij}K_{ij} - CK\mathcal{T}. \tag{5-20}$$

Note that

$$K_{ij} = u_{llij} + 2mu_{li}u_{lj} + 2mu_l u_{lij}. \tag{5-21}$$

By Ricci identities, we have

$$|u_{ijl} - u_{lij}| \le C \quad \text{and} \quad |u_{ijll} - u_{llij}| \le C(|\nabla^2 u| + 1).$$

Then we have

$$P^{ij}K_{ij} \geq P^{ij}u_{ijll} + 2mP^{ij}u_{li}u_{lj} + 2mu_l P^{ij}u_{ijl} - C(|\nabla^2 u| + 1)\mathcal{T}. \tag{5-22}$$

By (4-10), we have

$$\begin{aligned} 2mu_l P^{ij}u_{ijl} &= 2mu_l \nabla_l \Phi - F^{ij}(a_l u_i u_j + 2au_{il}u_j + S_{ij,l}) - (b_l|\nabla u|^2 + 2bu_{ls}u_s)\mathcal{T} \\ &\geq -C(|\nabla^2 u| + 1)\mathcal{T}, \end{aligned} \tag{5-23}$$

since $\nabla_{ll}\Phi = \Phi_{xx} + 2\Phi_{xz}u_l + \Phi_z u_{ll} \geq -C + \Phi_z \triangle u \geq -C(|\nabla^2 u| + 1)$. Differentiating the equation (1-6) twice, using the concavity of F, we have

$$\begin{aligned} P^{ij}u_{ijll} \geq \nabla_{ll}\Phi &- F^{ij}(a_{ll}u_i u_j + 4a_l u_{il}u_j + 2au_{ill}u_j + 2au_{il}u_{jl} + S_{ij,ll}) \\ &- (b_{ll}|\nabla u|^2 + 4b_l u_{ls}u_s + 2bu_{sll}u_s + 2b|\nabla^2 u|^2)\mathcal{T} \\ \geq -2aF^{ij}&u_{ill}u_j - 2aF^{ij}u_{il}u_{jl} - 2bu_{sll}u_l\mathcal{T} \\ &- 2b|\nabla^2 u|^2\mathcal{T} - C(|\nabla^2 u| + 1)\mathcal{T}. \end{aligned} \tag{5-24}$$

By Ricci identities again, and (5-19) and (5-24), we get

$$P^{ij}u_{ijll} \geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}. \tag{5-25}$$

Now, plugging (5-23) and (5-25) into (5-22), and choosing

$$m > \max\left\{2|a|_{L^\infty}, \frac{4}{\varepsilon_0}(\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\right\},$$

we obtain

$$\begin{aligned} P^{ij}K_{ij} &\geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2\mathcal{T} + 2mP^{ij}u_{li}u_{lj} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &= 2(m+a)P^{ij}u_{li}u_{lj} - 2(a\bar{\theta} + b)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq 2\big((m - |a|_{L^\infty})\varepsilon_0 - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\big)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq 2\left(\frac{m\varepsilon_0}{2} - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\right)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq \frac{m\varepsilon_0}{2}|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}. \end{aligned} \tag{5-26}$$

It follows from (5-20) and (5-26) that

$$\eta^2 \frac{m\varepsilon_0}{2}|\nabla^2 u|^2\mathcal{T} \leq C(|\nabla^2 u| + 1)\mathcal{T},$$

which implies that $\eta|\nabla^2 u| \le C$.

Step 3. We get the Hessian bound of u. As in [Chen 2009], we consider the maximum of

$$\bar{G} = \eta(x)e^{x_n}(\nabla^2 u + m du \otimes du)$$

over the set $(x, \xi) \in (\bar{B}_r^+, \mathbb{S}^n)$. Let $\bar{G}$ attain its maximum at some point x_0 and the direction $\xi \in T_{x_0}\bar{M} \cap \mathbb{S}^n$. Denote $K_\xi = u_{\xi\xi} + m u_\xi^2$. We may assume $K_\xi \gg C' > 0$, where C' is the one in Lemma 5.1.

Now, we can also show that x_0 does not belong to the boundary. Suppose $x_0 \in \Sigma_r$. If ξ is a tangential vector, without loss of generality we may assume $\xi = \partial/\partial x_1$. By Lemma 2.2, we have on the boundary that

$$\begin{aligned}(\eta e^{x_n}(u_{11} + m u_1^2))_n &= \eta e^{x_n}((u_{11} + m u_1^2) + u_{11n} + 2m u_1 u_{1n}) \\ &\ge u_{11} + m u_1^2 = K_1 > 0\end{aligned}$$

Therefore, we get a contradiction. If ξ is in the normal direction, by Lemma 2.2 and Lemma 5.1, we also have

$$\begin{aligned}(\eta e^{x_n}(u_{nn} + m u_n^2))_n &= \eta e^{x_n}((u_{nn} + m u_n^2) + u_{nnn} + 2m u_n u_{nn}) \\ &\ge u_{nn} - C' = K_n - C' > 0.\end{aligned}$$

Thus x_0 must be an interior point. By similar calculations as before, we can get the Hessian bounds. We omit the details here. □

Remark 5.3. Let B_r and $B_{r/2}$ be two local geodesic balls in the interior of M, and $G = \eta(\triangle u + m|\nabla u|^2)$. The same calculations in steps 2 and 3 yield the interior Hessian estimates for (1-6).

Therefore we have the following global estimates.

Proposition 5.4. *Let u be a C^4 solution of* (1-6) *on $\bar{M}$. Then there is a positive constant C_2 depending only on n, k, $\bar{\theta}$, g, a, b, Φ, S and C_1, such that*

$$\sup_{\bar{M}} |\nabla^2 u|_g \le C_2.$$

6. Proof of Theorem 1.2

We use the continuity method to prove the existence of (1-6). Since the argument is standard (see [Li and Sheng 2011]), we only sketch it here.

For $t \in [0, 1]$, consider the equation

$$(6\text{-}1_t) \qquad F\big(g^{-1}(\bar{\theta}\triangle u g - \nabla^2 u + a(x) du \otimes du + b(x)|\nabla u|^2 g + S_t)\big) = \Phi_t(x, u),$$

where

$$S_t = tS + \frac{1-t}{F(e)} g \quad \text{and} \quad \Phi_t(x, u) = (1-t)e^{2u} + t\Phi(x, u).$$

Clearly, S_t and Φ_t satisfy the following conditions:

- $S_t \in \Gamma$ and $|S_t|_{C^4(\overline{M})} \le C$, where the constant C is independent of t.
- S_t satisfies (S1) or $S_t = tA_g^\tau$ when $t \ne 0$ and $S_0 = \dfrac{1}{F(e)} g$ as long as S satisfies (S1) or (S2).
- $\Phi_t(x, u) > 0$, $\partial_z \Phi_t > 0$, $\lim_{z\to+\infty} \Phi_t(x, z) \to +\infty$, and $\lim_{z\to-\infty} \Phi_t(x, z) \to 0$.
- $|\Phi_t|_{C^2(\overline{M}\times[-C,C])} \le C$, where C is independent of t.

It follows from Sections 3, 4 and 5 that for each t, the admissible solution of (6-1_t) has uniform a priori C^2 estimates (independent of t). Then we obtain the uniform $C^{2,\alpha}$ estimates by Evans–Krylov theory [Krylov 1985]. Define

$$I = \{t \in [0, 1] \mid (6\text{-}1_t) \text{ has admissible solution}\}.$$

Clearly, $u \equiv 0$ is the unique admissible solution of (6.1_0). Hence, $I \ne \varnothing$. By Lemma 3.1, $I \subset [0, 1]$ is open. By the uniform a priori $C^{2,\alpha}$ estimates and the standard degree theory, we conclude that I is also closed. Then for $t = 1$, (1-6) is solvable. □

7. Proof of Theorem 1.3

Before proving Theorem 1.3, we first calculate a priori estimates for (1-9).

Proposition 7.1. *Suppose $T \in \Gamma$ and the positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfy* (1-8). *Then there exists a constant C_0 only depending on T and Φ, such that any solution $u \in C^2(\overline{M})$ of* (1-9) *satisfies*

$$\sup_{\overline{M}} |u| \le C_0.$$

The proof is similar to that of Proposition 3.2. We omit it here.

Proposition 7.2. *Suppose u is a C^3 solution of* (1-9) *on $\bar{B}_r^+$. Then there is a positive constant C depending only on n, k, θ, g, r, $|T|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$ and C_0, such that*

$$\sup_{\bar{B}_{r/2}^+} |\nabla u|_g \le C.$$

Proof. Consider the auxiliary functions

$$G := \tfrac{1}{2}\eta e^\beta |\nabla u|^2, \quad \beta := x_n + \psi(u).$$

Then G can not attain its maximum at a boundary point $x_0 \in \Sigma_r$ by the same arguments in the proof of Proposition 4.1. Since the maximum point x_0 is an interior point, we can also get (4-5)–(4-7). Now, the difference from the proof of Proposition 4.1 is that we replace the operator P^{ij} in (4-8) by the operator

$$Q^{ij} := F^{ij} + \theta \mathcal{T} g^{ij}. \tag{7-1}$$

Then by similar calculations as in (4-9)–(4-16), we obtain

$$\begin{aligned} 0 \geq C^* + \psi' Q^{ij} u_{ij} + (\psi'' - \psi'^2) Q^{ij} u_i u_j + 2a\psi' Q^{ij} u_i u_j \\ + 2b\psi' |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}} (|\nabla u| + 1)\mathcal{T}. \end{aligned} \tag{7-2}$$

Since

$$\begin{aligned} \psi' Q^{ij} u_{ij} &= \psi' F^{ij} W_{ij} - \psi' F^{ij} (a u_i u_j + b |\nabla u|^2 g_{ij} + T_{ij}) \\ &\geq \psi' \Phi - a\psi' F^{ij} u_i u_j - b\psi' |\nabla u|^2 - C\mathcal{T}. \end{aligned} \tag{7-3}$$

Substituting (7-3) into (7-2), we get

$$\begin{aligned} 0 &\geq C^* + \psi' \Phi + (\psi'' - \psi'^2) Q^{ij} u_i u_j + a\psi' F^{ij} u_i u_j \\ &\qquad + b\psi' |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}} (|\nabla u| + 1)\mathcal{T} \\ &= C^* + \psi' \Phi + (\psi'' - \psi'^2 + a\psi') F^{ij} u_i u_j \\ &\qquad + (\theta(\psi'' - \psi'^2) + b\psi') |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}} (|\nabla u| + 1)\mathcal{T}. \end{aligned} \tag{7-4}$$

By the similar argument as in Claim 4.2, we know that there exist positive constants α_1, α_2 and p depending only on θ, C_0, $|a|_{L^\infty(\bar{M})}$ and $|b|_{L^\infty(\bar{M})}$, such that

$$\psi' > 0, \quad \psi'' - \psi'^2 - |a|_{L^\infty} \psi' > 0, \quad \theta(\psi'' - \psi'^2) - |b| \psi' \geq \varepsilon_2 > 0,$$

where the constant ε_2 only depends on α_1, α_2 and p. Then we have

$$0 \geq C^* + \varepsilon_2 |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}} (|\nabla u| + 1)\mathcal{T}. \tag{7-5}$$

Then multiplying by η^2 both sides of the inequality above and $\mathcal{T} > 1$, we have

$$\varepsilon_2 \eta^2 |\nabla u|^2 \mathcal{T} \leq C |\nabla u| \mathcal{T} + C^*,$$

which implies the gradient estimates. □

To get the boundary Hessian estimates, we first prove the following:

Lemma 7.3. *Let u be a C^4 solution of* (1-9). *Then there is a positive constant C' depending only on n, k, θ, g, $|T|_{C^1(\bar{B}_r^+)}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$ and C_1 such that on ∂M, we have*

$$u_{nnn} \geq -C'.$$

Proof. (i) Let T satisfy the condition (S1). Then $T_{\alpha n} = 0$ on the boundary. Hence $W[u]_{\alpha n} = T_{\alpha n} = 0$. Therefore $F^{\alpha n}(W[u]) = 0$. By the similar calculations in Lemma 5.1, we have

$$W[u]_{\alpha\beta,n} \leq \theta u_{nnn} g_{\alpha\beta} + C \tag{7-6}$$

and

$$W[u]_{nnn} \leq u_{nnn} + \theta u_{nnn} + C, \tag{7-7}$$

where the constants C depend on n, k, g, $|T|_{C^1(\bar{B}_r^+)}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$ and C_1.

Now, differentiating (1-9) along the normal direction and taking the value on the boundary, we have

$$\begin{aligned} \nabla_n \Phi &= F^{nn} W[u]_{nnn} + F^{\alpha\beta} W[u]_{\alpha\beta n} \\ &\leq F^{nn}(u_{nnn} + \theta u_{nnn}) + \theta u_{nnn} F^{\alpha\beta} g_{\alpha\beta} + C\mathcal{T} \\ &= F^{nn} u_{nnn} + \theta u_{nnn} \mathcal{T} + C\mathcal{T}, \end{aligned} \tag{7-8}$$

that is,

$$0 \leq F^{nn} u_{nnn} + \theta u_{nnn}\mathcal{T} + C\mathcal{T} = F^{nn} u_{nnn} + (\theta u_{nnn} + C)\mathcal{T}, \tag{7-9}$$

where the constant C also depends on $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$.

If $\theta u_{nnn} + C \geq 0$, then we get $u_{nnn} \geq -C/\theta$. If $\theta u_{nnn} + C < 0$, by $F^{nn} < \mathcal{T}$ and (7-9), we have

$$0 \leq F^{nn} u_{nnn} + (\theta u_{nnn} + C) F^{nn} = ((\theta + 1) u_{nnn} + C) F^{nn}.$$

Since $F^{nn} > 0$, we get

$$(\theta + 1) u_{nnn} + C \geq 0.$$

Note $\theta > 0$. Then we obtain $u_{nnn} \geq -C'$ again.

(ii) Suppose $T = -A_g^\tau$. Using the metric $\bar{g}$ in Lemma 2.3, we consider a new metric $\breve{g} = e^{2w}\bar{g}$ such that $u = \bar{u} + w$ is a solution of (1-9). Then similar to the calculation in the proof of Lemma 5.1, we have

$$\begin{aligned} W[u]_{ij} = {} & \theta \bar{\Delta} w \bar{g}_{ij} + \bar{\nabla}_{ij}^2 w + \bar{\theta} \bar{g}^{sl}(\bar{\Gamma}_{sl}^k(\bar{g}) - \Gamma_{sl}^k(g)) w_k \bar{g}_{ij} + (\bar{\Gamma}_{ij}^k(\bar{g}) - \Gamma_{ij}^k(g)) w_k \\ & + (a-1)\bar{u}_i \bar{u}_j + a(\bar{u}_i w_j + w_i \bar{u}_j + w_i w_j) + b\big(2\langle \bar{\nabla} \bar{u}, \bar{\nabla} w\rangle_{\bar{g}} + |\bar{\nabla} w|_{\bar{g}}^2\big)\bar{g}_{ij} \\ & + \left(b - \frac{1 + (n-2)\theta}{2}\right)|\bar{\nabla} u|_{\bar{g}}^2 \bar{g}_{ij} - (A_{\bar{g}}^\tau)_{ij}. \end{aligned}$$

Denote $\overline{W}[w]_{ij} := W[u]_{ij}$. Now, (1-9) becomes

$$\begin{cases} F(\overline{W}[w]) = \Phi(x, \bar{u} + w) & \text{in } M, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial M. \end{cases} \tag{7-10}$$

By Lemma 2.3, we find $(A^{\tau}_{\bar{g}})_{\alpha n}(x_0) = 0$. Then we have $\overline{W}[w]_{\alpha n}(x_0) = 0$ by Lemma 2.2 and (5-11)–(5-13), which implies $F^{\alpha n}(\overline{W}[w]) = 0$. By Lemma 2.2 again, we obtain

$$\overline{W}[w]_{\alpha\beta n}(x_0) = \theta w_{nnn} \bar{g}_{\alpha\beta}(x_0),$$

and

$$\overline{W}[w]_{nnn}(x_0) = \theta w_{nnn} \bar{g}_{nn}(x_0) + w_{nnn}(x_0).$$

Then by differentiating (7-10) along the normal direction and taking its value at x_0, we have

$$\begin{aligned} 0 &\le F^{nn} \overline{W}_{nnn} + F^{\alpha\beta} \overline{W}_{\alpha\beta n} + C \\ &\le F^{nn} w_{nnn}(x_0) + (\theta w_{nnn}(x_0) + C)\overline{\mathcal{T}}. \end{aligned}$$

If $\theta w_{nnn}(x_0) + C \ge 0$, we have $u_{nnn}(x_0) \ge -C'$ immediately. Now consider $\theta w_{nnn}(x_0) + C < 0$. Since $\overline{\mathcal{T}} > F^{nn} > 0$, we have

$$0 < F^{nn} w_{nnn}(x_0) + (\theta w_{nnn}(x_0) + C) F^{nn} \le ((\theta + 1) w_{nnn}(x_0) + C) F^{nn}.$$

Hence, we must have $w_{nnn}(x_0) \ge -C/(\theta + 1)$. Therefore, $u_{nnn}(x_0) \ge -C'$. □

Proposition 7.4. *Let u be a C^4 solution of* (1-9) *on $\bar{B}^+_r$. Then there is a positive constant C_2 depending only on n, k, θ, g, r, $|T|_{C^2(\bar{B}^+_r)}$, $|\Phi|_{C^2(\bar{B}^+_r) \times [-C_0, C_0]}$, $|a|_{C^2(\bar{B}^+_r)}$, $|b|_{C^2(\bar{B}^+_r)}$ and C_1 such that*

$$\sup_{\bar{B}^+_{r/2}} |\nabla^2 u|_g \le C_2.$$

Proof. We first estimate the bound of $\triangle u$. By $W[u] \in \Gamma^+_k \subset \Gamma_1$, we have

$$0 \le \operatorname{tr}(W[u]) = (n\theta + 1)\triangle u + (a + nb)|\nabla u|^2 + \operatorname{tr} T,$$

which implies that $\triangle u$ has lower bound. Hence, we may assume $\triangle u > 0$.

Consider the same auxiliary function in Proposition 5.2

$$G := \eta e^{q x_n} (\triangle u + m |\nabla u|^2),$$

where η satisfies (4-1) and (4-2), m is a larger constant to be fixed. We may assume $r = 1$ and $K := \triangle u + m|\nabla u|^2 \gg 1$.

Step 1. We show the maximum of G must be attained at an interior point of $\bar{B}^+_r$. If the maximum point x_0 of G belong to Σ_r, then by Lemma 2.2, Lemma 7.3 and the same calculations in Proposition 5.2, we know that $G_n(x_0) > 0$. It is a contradiction.

Step 2. We must get an upper bound for $\triangle u$. Since the maximum point of G is an interior point of $\bar{B}_r^+$ by step 1. Then at the maximum point x_0, we can get similar inequalities as in (5-19)–(5-24) by replacing P^{ij} by Q^{ij}. Corresponding to (5-26), for $m > \max\{|a|_{L^\infty(\overline{M})}, (|b|_{L^\infty(\overline{M})} + \varepsilon_3)/\theta\}$, $\varepsilon_3 > 0$, we obtain

$$
\begin{aligned}
\text{(7-11)}\quad & Q^{ij} K_{ij} \\
&\geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2\mathcal{T} + 2mQ^{ij}u_{li}u_{lj} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&= 2(m-a)F^{ij}u_{li}u_{lj} + 2(m\theta - b)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&\geq 2(m - |a|_{L^\infty})F^{ij}u_{li}u_{lj} + 2(m\theta - |b|_{L^\infty})|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&\geq 2\varepsilon_3|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}.
\end{aligned}
$$

It follows from (5-20) for Q^{ij} and (7-11) that $2\eta^2\varepsilon_3|\nabla^2 u|^2\mathcal{T} \leq C(|\nabla^2 u| + 1)\mathcal{T}$, which implies that $\eta|\nabla^2 u| \leq C$. □

Step 3. By Lemma 7.3 and the same argument in the step 3 of the proof of Proposition 5.2, we can get the Hessian estimates of u.

Remark 7.5. We can also get the interior gradient and Hessian estimates for the solutions of (1-9) by the same arguments in Remarks 4.3 and 5.3.

Proof of Theorem 1.3. Since the operator Q^{ij} in (7-1) is positive, by the argument in Section 3, we know that (1-9) is elliptic at any admissible solutions and its linearized operator is invertible as $\partial_z\Phi > 0$. Combining Propositions 7.1, 7.2, 7.4 and Remark 7.5, we can obtain

$$
\text{(7-12)}\qquad |u|_{C^2(\overline{M})} \leq C,
$$

where the constant C depends only on $n, k, \theta, g, S, \Phi, a$ and b. By the global a priori C^2 estimates (7-12), we can prove Theorem 1.3 by a same argument in Section 6. □

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RYSHKOV DOMAINS OF REDUCTIVE ALGEBRAIC GROUPS

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Dedicated to Professor Ichiro Satake on his 85th birthday

Let G be a connected reductive algebraic group defined over a number field k. In this paper, we introduce the Ryshkov domain R for the arithmetical minimum function m_Q defined from a height function associated to a maximal k-parabolic subgroup Q of G. The domain R is a $Q(\mathsf{k})$-invariant subset of the adele group $G(\mathbb{A})$. We show that a fundamental domain Ω for $Q(\mathsf{k})\backslash\mathsf{R}$ yields a fundamental domain for $G(\mathsf{k})\backslash G(\mathbb{A})$. We also see that any local maximum of m_Q is attained on the boundary of Ω.

Introduction

Let P_n be the cone of positive definite n by n real symmetric matrices, and let $\mathsf{m}(A)$ be the arithmetical minimum $\min_{0\neq x\in\mathbb{Z}^n} {}^t x A x$ of $A \in \mathsf{P}_n$. The function $f : A \mapsto \mathsf{m}(A)/(\det A)^{1/n}$ on P_n is called the Hermite invariant. Since the maximum of f gives the Hermite constant γ_n for dimension n, the determination of local maxima of f is a fundamental problem of lattice sphere packings in Euclidean spaces and the arithmetic theory of quadratic forms. Voronoi's theorem [1908, Théorème 17] states that f attains a local maximum at a point A if and only if A is perfect and eutactic. Moreover, perfect forms play an essential role in Voronoi's reduction theory of P_n with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ (see, e.g., [Martinet 2003] and [Schürmann 2009]). Ryshkov [1970] introduced a locally finite polyhedron $\mathsf{R}(\mathsf{m})$ in P_n defined by the condition $\mathsf{m}(A) \geq 1$. It is not difficult to show that A is perfect with $\mathsf{m}(A) = 1$ if and only if A is a vertex of the boundary of $\mathsf{R}(\mathsf{m})$. In particular, any local maximum of the Hermite invariant f is attained on the boundary of $\mathsf{R}(\mathsf{m})$. In this sense, we can say that the Ryshkov polyhedron $\mathsf{R}(\mathsf{m})$ is well matched with f.

Let G be a connected isotropic reductive algebraic group defined over a number field k, and let Q be a maximal k-parabolic subgroup of G. In previous papers [Watanabe 2000; 2003], we investigated a constant $\gamma(G, Q, \mathsf{k})$ as a generalization of Hermite's constant γ_n. Precisely, the constant $\gamma(G, Q, \mathsf{k})$ is defined to be

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the maximum of the function $\mathsf{m}_Q(g) = \min_{x \in Q(\mathsf{k})\backslash G(\mathsf{k})} H_Q(xg)$ on $G(\mathsf{k})\backslash G(\mathbb{A})^1$, where H_Q denotes the height function associated to Q. To prove the existence of the maximum of m_Q, we used Borel and Harish-Chandra's reduction theory for the adele group $G(\mathbb{A})$ with respect to $G(\mathsf{k})$. However, a Siegel set in $G(\mathbb{A})$ is not well matched with m_Q in a sense that one cannot obtain any information on locations of extreme points of m_Q in a Siegel set.

The purpose of this paper is to construct a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$ which is well matched with m_Q. We first consider an analog of the Ryshkov polyhedron. We set $X_Q(g) = \{x \in Q(\mathsf{k})\backslash G(\mathsf{k}) : m_Q(g) = H_Q(xg)\}$ for a given $g \in G(\mathbb{A})^1$. This is a finite subset of $Q(\mathsf{k})\backslash G(\mathsf{k})$ and is regarded as an analog of the set of minimal vectors of a positive definite real quadratic form. We define the domain $\mathsf{R}(\mathsf{m}_Q)$ as follows:

$$\mathsf{R}(\mathsf{m}_Q) = \{g \in G(\mathbb{A})^1 : \bar{e} \in X_Q(g)\},$$

where $\bar{e}$ denotes the trivial class $Q(\mathsf{k})$ in $Q(\mathsf{k})\backslash G(\mathsf{k})$. The set $\mathsf{R}(\mathsf{m}_Q)$ is a left $Q(\mathsf{k})$-invariant closed set with nonempty interior. The interior of $\mathsf{R}(\mathsf{m}_Q)$ is just a subset R_1 consisting of $g \in \mathsf{R}(\mathsf{m}_Q)$ such that $X_Q(g)$ is the one-point set $\{\bar{e}\}$. We denote by R_1^- the closure of R_1 in $G(\mathbb{A})^1$. Both R_1 and R_1^- are also left $Q(\mathsf{k})$-invariant. By Baer and Levi's theorem [1931, Satz 7], there exists an open fundamental domain Ω_Q of R_1^- with respect to $Q(\mathsf{k})$, that is, Ω_Q is a relatively open subset of R_1^- satisfying

- $Q(\mathsf{k})\Omega_Q^- = \mathsf{R}_1^-$, where Ω_Q^- denotes the closure of Ω_Q in R_1^-, and
- $\gamma\Omega_Q \cap \Omega_Q^- = \varnothing$ for any $\gamma \in Q(\mathsf{k}) \setminus \{e\}$.

Let Ω_Q° denote the interior of Ω_Q in $G(\mathbb{A})^1$. Then our main theorem is stated as follows:

Theorem. *The set Ω_Q° is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$. Any local maximum of m_Q is attained on the intersection of the boundary of Ω_Q° and the boundary of R_1^-.*

If we denote by r_G the k-rank of the commutator subgroup of G, then G has r_G standard maximal k-parabolic subgroups. Since Ω_Q depends on Q, we obtain r_G different kinds of fundamental domains of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$. The method to construct Ω_Q may be viewed as a generalization of the highest point method (see [Grenier 1988] and [Terras 1988, §4,4]). For example, let $\mathsf{k} = \mathbb{Q}$, $G = \mathrm{GL}_n$ and Q be a standard maximal $\mathbb{Q}$-parabolic subgroup such that $Q\backslash G$ is a projective space. Then our construction gives a fundamental domain Ω_Q whose Archimedean part is isomorphic with Grenier's fundamental domain. If we choose another standard maximal $\mathbb{Q}$-parabolic subgroup of GL_n as Q, then the

Archimedean part of Ω_Q yields a new kind of fundamental domain of P_n with respect to $\mathrm{GL}_n(\mathbb{Z})$ (see Example 3 in Section 7).

Notation. For a given ring $\mathfrak{A}$, the set of all n by k matrices with entries in $\mathfrak{A}$ is denoted by $\mathrm{M}_{n,k}(\mathfrak{A})$. We write $\mathrm{M}_n(\mathfrak{A})$ for $\mathrm{M}_{n,n}(\mathfrak{A})$. The transpose of a given matrix $a \in \mathrm{M}_{n,k}(\mathfrak{A})$ is denoted by ${}^t a$. In this paper, k denotes an algebraic number field of finite degree over $\mathbb{Q}$ and o the ring of integers of k. The sets of all infinite and finite places of k are denoted by p_∞ and p_f, respectively. For $\sigma \in \mathsf{p}_\infty \cup \mathsf{p}_f$, k_σ denotes the completion of k at σ. For $\sigma \in \mathsf{p}_f$, o_σ denotes the closure of o in k_σ. The étale $\mathbb{R}$-algebra $\mathsf{k}_\infty = \mathsf{k} \otimes_{\mathbb{Q}} \mathbb{R}$ is identified with $\prod_{\sigma\in\mathsf{p}_\infty} \mathsf{k}_\sigma$. Let $\mathbb{A}$ and $\mathbb{A}^\times$ denote the adele ring and the idèle group of k, respectively. The idèle norm of $\mathbb{A}^\times$ is denoted by $|\cdot|_{\mathbb{A}}$.

1. Height functions

Let G be a connected affine algebraic group defined over k. For any k-algebra $\mathfrak{A}$, $G(\mathfrak{A})$ stands for the set of $\mathfrak{A}$-rational points of G. Let $X^*(G)_{\mathsf{k}}$ be the free $\mathbb{Z}$-module consisting of all k-rational characters of G. For each $g \in G(\mathbb{A})$, we define the homomorphism $\vartheta_G(g) : X^*(G)_{\mathsf{k}} \to \mathbb{R}_{>0}$ by $\vartheta_G(g)(\chi) = |\chi(g)|_{\mathbb{A}}$ for $\chi \in X^*(G)_{\mathsf{k}}$. Then ϑ_G is a homomorphism from $G(\mathbb{A})$ into $\mathrm{Hom}_{\mathbb{Z}}(X^*(G)_{\mathsf{k}}, \mathbb{R}_{>0})$. We write $G(\mathbb{A})^1$ for the kernel of ϑ_G.

In the following, let G be a connected isotropic reductive group defined over k. We fix a maximal k-split torus S of G and a minimal k-parabolic subgroup P_0 of G containing S. Denote by Φ_{k} and Δ_{k} the relative root system of G with respect to S and the set of simple roots of Φ_{k} corresponding to P_0, respectively. Let M_0 be the centralizer of S in G. Then P_0 has a Levi decomposition $P_0 = M_0 U_0$, where U_0 is the unipotent radical of P_0. A k-parabolic subgroup of G containing P_0 is called a standard k-parabolic subgroup of G. Every standard k-parabolic subgroup R of G has a unique Levi subgroup M_R containing M_0. We denote by U_R the unipotent radical of R and by Z_R the greatest central k-split torus in M_R. Throughout this paper, we fix a maximal compact subgroup $K = \prod_{\sigma\in\mathsf{p}_\infty} K_\sigma \times \prod_{\sigma\in\mathsf{p}_f} K_\sigma$ of $G(\mathbb{A})$ satisfying the following property: for every standard k-parabolic subgroup R of G, $K \cap M_R(\mathbb{A})$ is a maximal compact subgroup of $M_R(\mathbb{A})$, and $M_R(\mathbb{A})$ possesses an Iwasawa decomposition $(M_R(\mathbb{A}) \cap U_0(\mathbb{A}))M_0(\mathbb{A})(K \cap M_R(\mathbb{A}))$.

Let Q be a standard proper maximal k-parabolic subgroup of G. There is only one simple root $\alpha_0 \in \Delta_{\mathsf{k}}$ such that the restriction of α_0 to Z_Q is nontrivial. Let n_Q be the positive integer such that $n_Q^{-1}\alpha_0|_{Z_Q}$ is a $\mathbb{Z}$-basis of $X^*(Z_Q/Z_G)_{\mathsf{k}}$. We write α_Q for $n_Q^{-1}\alpha_0|_{Z_Q}$ and $\widehat{\alpha}_Q$ for $\widehat{d}_Q n_Q^{-1}\alpha_0|_{Z_Q}$, where

$$\widehat{d}_Q = [X^*(Z_Q/Z_G)_{\mathsf{k}} : X^*(M_Q/Z_G)_{\mathsf{k}}].$$

Then $\widehat{\alpha}_Q$ is a $\mathbb{Z}$-basis of the submodule $X^*(M_Q/Z_G)_{\mathsf{k}}$ of $X^*(Z_Q/Z_G)_{\mathsf{k}}$. Define the map $z_Q : G(\mathbb{A}) \to Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A})$ by $z_Q(g) = Z_G(\mathbb{A})M_Q(\mathbb{A})^1 m$ if $g = umh$ with $u \in U_Q(\mathbb{A})$, $m \in M_Q(\mathbb{A})$ and $h \in K$. This is well defined and left $Z_G(\mathbb{A})Q(\mathbb{A})^1$-invariant. Since $Z_G(\mathbb{A})^1 = Z_G(\mathbb{A}) \cap G(\mathbb{A})^1 \subset M_Q(\mathbb{A})^1$, z_Q gives rise to a map from $Y_Q = Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$ to $M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1)$. Namely, we have the following commutative diagram, whose vertical arrows are natural maps:

$$\begin{array}{ccc} Y_Q & \xrightarrow{z_Q} & M_Q(\mathbb{A})^1\backslash(M_Q(\mathbb{A}) \cap G(\mathbb{A})^1) \\ \downarrow & & \downarrow \\ Z_G(\mathbb{A})Q(\mathbb{A})^1\backslash G(\mathbb{A}) & \xrightarrow{z_Q} & Z_G(\mathbb{A})M_Q(\mathbb{A})^1\backslash M_Q(\mathbb{A}). \end{array}$$

We define the height function $H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0}$ by $H_Q(g) = |\hat{\alpha}_Q(z_Q(g))|_{\mathbb{A}}^{-1}$ for $g \in G(\mathbb{A})$. We notice that the restriction of H_Q to $M_Q(\mathbb{A})$ is a homomorphism from $M_Q(\mathbb{A})$ onto $\mathbb{R}_{>0}$.

Example 1. Let G be a general linear group GL_n defined over the rational number field $\mathbb{Q}$, P_0 the group of upper triangular matrices in G and S the group of diagonal matrices in G. We fix an integer $k \in \{1, \dots, n-1\}$, and let

$$Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \mathrm{GL}_k(\mathbb{Q}),\ b \in \mathrm{M}_{k,n-k}(\mathbb{Q}),\ d \in \mathrm{GL}_{n-k}(\mathbb{Q}) \right\}.$$

Then Q is a standard maximal $\mathbb{Q}$-parabolic subgroup of G. The rational character $\hat{\alpha}_Q$ and the height H_Q are given by

$$\hat{\alpha}_Q\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = (\det a)^{(n-k)/r}(\det d)^{-k/r}$$

and

$$H_Q\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = |\det a|_{\mathbb{A}}^{-(n-k)/r}|\det d|_{\mathbb{A}}^{k/r},$$

where r denotes the greatest common divisor of k and $n-k$. The height H_Q has another expression. To explain this, let $\mathbb{Q}^n$ be an n-dimensional column vector space over $\mathbb{Q}$ with standard basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$. The maximal parabolic subgroup $Q(\mathbb{Q})$ stabilizes the subspace spanned by $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$. Let $V_{n,k}(\mathbb{Q}) = \bigwedge^k \mathbb{Q}^n$ be the k-th exterior product of $\mathbb{Q}^n$. We set $V_{n,k}(\mathbb{A}) = V_{n,k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}$ and $V_{n,k}(\mathbb{Q}_\sigma) = V_{n,k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\sigma$ for $\sigma \in \mathfrak{p}_\infty \cup \mathfrak{p}_f$. A $\mathbb{Q}$-basis of $V_{n,k}(\mathbb{Q})$ is formed by the elements $\boldsymbol{e}_I = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_k}$ with $I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, \dots, n\}$. For a unique infinite place $\infty \in \mathfrak{p}_\infty$, we define the local height $H_\infty : V_{n,k}(\mathbb{Q}_\infty) \to \mathbb{R}_{>0}$ by

$$H_\infty\Big(\sum_I a_I \boldsymbol{e}_I\Big) = \Big(\sum_I |a_I|_\infty^2\Big)^{1/2},$$

where $|\cdot|_\infty$ denotes the usual absolute value of $\mathbb{Q}_\infty = \mathbb{R}$. For each finite prime $p \in \mathsf{p}_f$, we define the local height $H_p : V_{n,k}(\mathbb{Q}_p) \to \mathbb{R}_{>0}$ by

$$H_p\left(\sum_I a_I \boldsymbol{e}_I\right) = \sup_I |a_I|_p,$$

where $|\cdot|_p$ denotes the p-adic absolute value of $\mathbb{Q}_p$ normalized so that $|p|_p = p^{-1}$. Then the global height $H_{n,k} : V_{n,k}(\mathbb{Q}) \to \mathbb{R}_{>0}$ is defined to be a product of all local heights, that is, $H_{n,k}(x) = \prod_{\sigma \in \mathsf{p}_\infty \cup \mathsf{p}_f} H_\sigma(x)$ for $x \in V_{n,k}(\mathbb{Q})$. This $H_{n,k}$ is immediately extended to the subset $\mathrm{GL}(V_{n,k}(\mathbb{A}))V_{n,k}(\mathbb{Q})$ of the adele space $V_{n,k}(\mathbb{A})$ by

$$H_{n,k}(Ax) = \prod_{\sigma \in \mathsf{p}_\infty \cup \mathsf{p}_f} H_\sigma(A_\sigma x)$$

for $A = (A_\sigma) \in \mathrm{GL}(V_{n,k}(\mathbb{A}))$ and $x \in V_{n,k}(\mathbb{Q})$. In particular, for $g \in G(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A})$, we can take the value $H_{n,k}(g\boldsymbol{e}_1 \wedge g\boldsymbol{e}_2 \wedge \cdots \wedge g\boldsymbol{e}_k)$. We choose a maximal compact subgroup K_∞ of $G(\mathbb{Q}_\infty)$ as $\{g \in G(\mathbb{Q}_\infty) : {}^tg^{-1} = g\}$. Let

$$K_f = \prod_{p \in \mathsf{p}_f} \mathrm{GL}_n(\mathbb{Z}_p) \quad \text{and} \quad K = K_\infty \times K_f.$$

Then, by elementary computations, we have

$$H_{n,k}(g\boldsymbol{e}_1 \wedge g\boldsymbol{e}_2 \wedge \cdots \wedge g\boldsymbol{e}_k) = |\det a|_{\mathbb{A}} \quad \text{if } g = h\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $h \in K$, $a \in \mathrm{GL}_k(\mathbb{A})$, $b \in \mathrm{M}_{k,n-k}(\mathbb{A})$ and $d \in \mathrm{GL}_{n-k}(\mathbb{A})$. Therefore, if $g \in G(\mathbb{A})^1$, that is, $|\det g|_{\mathbb{A}} = 1$, then

$$H_Q(g) = H_{n,k}\big(g^{-1}\boldsymbol{e}_1 \wedge g^{-1}\boldsymbol{e}_2 \wedge \cdots \wedge g^{-1}\boldsymbol{e}_k\big)^{n/r}.$$

2. Twisted height functions restricted to one parameter subgroups

Let $N_G(S)$ be the normalizer of S in G and $W_G = N_G(S)(\mathsf{k})/M_0(\mathsf{k})$ the Weyl group of G with respect to S. For a simple root $\alpha \in \Delta_{\mathsf{k}}$, $s_\alpha \in W_G$ denotes the simple reflection corresponding to α. Then $\{s_\alpha\}_{\alpha \in \Delta_{\mathsf{k}}}$ generates W_G. We denote by W_G^Q the subgroup of W_G generated by $\{s_\alpha\}_{\alpha \in \Delta_{\mathsf{k}} \setminus \{\alpha_0\}}$. For each $w \in W_G$, we use the same notation w for a representative of w in $N_G(S)(\mathsf{k})$. The following cell decomposition of $G(\mathsf{k})$ holds via Bruhat decomposition [Borel and Tits 1965, Proposition 4.10, Corollaire 5.20]:

$$G(\mathsf{k}) = \bigsqcup_{[w] \in W_G^Q \backslash W_G / W_G^Q} Q(\mathsf{k}) w Q(\mathsf{k}),$$

where $[w]$ stands for the class $W_G^Q w W_G^Q$ in $W_G^Q \backslash W_G / W_G^Q$.

The Weyl group W_G acts on $\boldsymbol{X}^*(S)_{\mathsf{k}}$ by $w \cdot \chi : t \mapsto \chi\big(w^{-1}tw\big)$ for $w \in W_G$ and $\chi \in \boldsymbol{X}^*(S)_{\mathsf{k}}$. We consider the restriction $\widehat{\alpha}_Q|_S$ of the rational character $\widehat{\alpha}_Q$ of M_Q to S.

Lemma 1. *The subgroup of W_G fixing $\widehat{\alpha}_Q|_S$ is equal to W_G^Q.*

Proof. Put $W' = \{w \in W_G \,:\, w \cdot \widehat{\alpha}_Q|_S = \widehat{\alpha}_Q|_S\}$. Since a representative of $w \in W_G^Q$ is contained in $M_Q(\mathsf{k})$, we have $\widehat{\alpha}_Q(w^{-1}tw) = \widehat{\alpha}_Q(w)^{-1}\widehat{\alpha}_Q(t)\widehat{\alpha}_Q(w) = \widehat{\alpha}_Q(t)$ for all $t \in S$. Hence W_G^Q is contained in W'. By [Humphreys 1990, §1.12 Theorem (a) and (c)] , W' is generated by a subset $W' \cap \{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}}$ of simple reflections. From $W_G^Q \subset W'$, it follows $\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}\setminus\{\alpha_0\}} \subset W' \cap \{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}} \subset \{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}}$. Since $\widehat{\alpha}_Q$ is nontrivial on S/Z_G, $W' \cap \{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}}$ must equal $\{s_\alpha\}_{\alpha\in\Delta_{\mathsf{k}}\setminus\{\alpha_0\}}$. Therefore W' coincides with W_G^Q. $\square$

Let $\boldsymbol{X}_*(S)_{\mathsf{k}}$ be the free $\mathbb{Z}$-module consisting of all k-rational cocharacters of S. A natural pairing

$$\langle\cdot,\cdot\rangle : \boldsymbol{X}^*(S)_{\mathsf{k}} \times \boldsymbol{X}_*(S)_{\mathsf{k}} \to \mathbb{Z}$$

defined as in [Borel 1991, §8.6] is a regular pairing over $\mathbb{Z}$.

Lemma 2. *Let w_1 and w_2 be elements of W_G such that $w_1^{-1}W_G^Q \neq w_2^{-1}W_G^Q$. Then there exist a cocharacter $\xi = \xi_{w_1,w_2} \in \boldsymbol{X}_*(S)_{\mathsf{k}}$ such that*

$$H_Q\big(w_1\xi(\lambda)w_1^{-1}\big) > H_Q\big(w_2\xi(\lambda)w_2^{-1}\big)$$

holds for all $\lambda \in \mathbb{A}^\times_{>1}$, where $\mathbb{A}^\times_{>1}$ denotes the set of $\lambda \in \mathbb{A}^\times$ satisfying $|\lambda|_{\mathbb{A}} > 1$.

Proof. Since $w_1^{-1}\cdot\widehat{\alpha}_Q|_S - w_2^{-1}\cdot\widehat{\alpha}_Q|_S \neq 0$ by Lemma 1, there is a $\xi \in \boldsymbol{X}_*(S)_{\mathsf{k}}$ such that $\langle w_1^{-1}\cdot\widehat{\alpha}_Q|_S - w_2^{-1}\cdot\widehat{\alpha}_Q|_S, \xi\rangle < 0$. The value $\ell = \langle w_1^{-1}\cdot\widehat{\alpha}_Q|_S - w_2^{-1}\cdot\widehat{\alpha}_Q|_S, \xi\rangle$ is a negative integer. We have

$$\widehat{\alpha}_Q(w_1\xi(\lambda)w_1^{-1})\cdot\widehat{\alpha}_Q(w_2\xi(\lambda)w_2^{-1})^{-1} = \lambda^\ell$$

for all $\lambda \in \boldsymbol{G}_m$. Therefore,

$$H_Q(w_1\xi(\lambda)w_1^{-1})H_Q(w_2\xi(\lambda)w_2^{-1})^{-1} = |\lambda|_{\mathbb{A}}^{-\ell} > 1$$

holds for all $\lambda \in \mathbb{A}^\times_{>1}$. $\square$

3. The Hermite function associated to Q and minimal points

We set $X_Q = Q(\mathsf{k})\backslash G(\mathsf{k})$, which is regarded as a subset of $Y_Q = Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$. Let $\pi_X : G(\mathsf{k}) \to X_Q$ be the natural quotient map. The symbol $\bar{e} = \pi_X(e) \in X_Q$ denotes the class of the unit element $e \in G(\mathsf{k})$. The Hermite function

$$\mathsf{m}_Q : G(\mathbb{A})^1 \to \mathbb{R}_{>0}$$

is defined to be

$$\mathsf{m}_Q(g) = \min_{x \in X_Q} H_Q(xg).$$

By definition, m_Q is a positive valued continuous function on $G(\mathsf{k})\backslash G(\mathbb{A})^1/K$.

For each $g \in G(\mathbb{A})^1$, we put

$$X_Q(g) = \{x \in X_Q : \mathsf{m}_Q(g) = H_Q(xg)\},$$

which is a finite subset of X_Q. Thus we can define the counting function $\mathsf{n}_Q(g) = \# X_Q(g)$.

Lemma 3. *For any $g \in G(\mathbb{A})^1$, $\gamma \in G(\mathsf{k})$ and $h \in K$, one has $X_Q(\gamma g h) = X_Q(g)\gamma^{-1}$. Especially, the counting function n_Q is left $G(\mathsf{k})$-invariant and right K-invariant.*

The following lemma is proved by the same method as in [Watanabe 2012, Proof of Proposition 4.1].

Lemma 4. *For $g \in G(\mathbb{A})^1$, there is a neighborhood $\mathcal{U}$ of g in $G(\mathbb{A})^1$ such that $X_Q(g') \subset X_Q(g)$ for all $g' \in \mathcal{U}$.*

Example 2. Let G be a general linear group GL_n defined over $\mathbb{Q}$. We keep notations used in Example 1. In this case, we can express m_Q in terms of some minimum of positive definite symmetric matrices. Since $\mathrm{GL}_n/\mathbb{Q}$ is of class number one, $G(\mathbb{A})^1 = \{g \in \mathrm{GL}_n(\mathbb{A}) : |\det g|_{\mathbb{A}} = 1\}$ has the following decomposition:

$$G(\mathbb{A})^1 = G(\mathbb{Q})(G(\mathbb{Q}_\infty)^1 \times K_f),$$

where $G(\mathbb{Q}_\infty)^1 = \{g \in \mathrm{GL}_n(\mathbb{Q}_\infty) : \det g = \pm 1\}$ and $K_f = \prod_{p \in \mathsf{p}_f} \mathrm{GL}_n(\mathbb{Z}_p)$. We fix $g = \delta(g_\infty \times g_f) \in G(\mathbb{A})^1$ with $\delta \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{Q}_\infty)^1$ and $g_f \in K_f$. From the left $G(\mathbb{Q})$-invariance and the right K-invariance of m_Q, it follows that

$$m_Q(g) = m_Q(g_\infty) = \min_{x \in X_Q} H_Q(x g_\infty) = \min_{\gamma \in G(\mathbb{Q})} H_Q(\gamma g_\infty).$$

Furthermore, since $G(\mathbb{Q}) = Q(\mathbb{Q})\,\mathrm{GL}_n(\mathbb{Z})$ and H_Q is left $Q(\mathbb{Q})$-invariant, we have

$$m_Q(g) = \min_{\gamma \in \mathrm{GL}_n(\mathbb{Z})} H_Q(\gamma g_\infty).$$

An elementary proof of the decomposition $G(\mathbb{Q}) = Q(\mathbb{Q})\,\mathrm{GL}_n(\mathbb{Z})$ is found in [Shimura 1994, Theorem 3]. By Example 1,

$$\begin{aligned} H_Q(\gamma g_\infty) &= H_{n,k}\big(g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_k\big)^{n/r} \\ &= H_\infty\big(g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_k\big)^{n/r} \prod_{p \in \mathsf{p}_f} H_p\big(\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge \gamma^{-1}\boldsymbol{e}_k\big)^{n/r} \\ &= H_\infty\big(g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_k\big)^{n/r}. \end{aligned}$$

Here we notice that $H_p(\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge \gamma^{-1}\boldsymbol{e}_k) = 1$ for all $p \in \mathsf{p}_f$ and $\gamma \in \mathrm{GL}_n(\mathbb{Z})$. For a given $\gamma \in \mathrm{GL}_n(\mathbb{Z})$, X_γ stands for the n by k matrix consisting of the first k columns of γ. Binet's formula (see [Bombieri and Gubler 2006, Proposition 2.8.8]) yields

$$H_\infty\big(g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_1 \wedge \cdots \wedge g_\infty^{-1}\gamma^{-1}\boldsymbol{e}_k\big) = \det\big({}^tX_{\gamma^{-1}}\,{}^tg_\infty^{-1}g_\infty^{-1}X_{\gamma^{-1}}\big)^{1/2}.$$

As a consequence, we obtain

$$m_Q(g) = \min_{X \in \mathrm{M}_{n,k}(\mathbb{Z})^*} \det\big({}^tX\,{}^tg_\infty^{-1}g_\infty^{-1}X\big)^{n/2r},$$

where $\mathrm{M}_{n,k}(\mathbb{Z})^*$ denotes the set of X_γ for all $\gamma \in \mathrm{GL}_n(\mathbb{Z})$. In the case of $k = 1$, $\mathrm{M}_{n,1}(\mathbb{Z})^*$ is just the set of primitive vectors of the lattice $\mathbb{Z}^n$, and hence $m_Q(g)$ coincides with the $n/2$ power of the arithmetical minimum of the positive definite symmetric matrix ${}^tg_\infty^{-1}g_\infty^{-1}$.

4. The Ryshkov domain of G associated to Q

We define the Ryshkov domain $\mathsf{R} = \mathsf{R}(\mathsf{m}_Q)$ of m_Q by

$$\mathsf{R} = \mathsf{R}(\mathsf{m}_Q) = \big\{g \in G(\mathbb{A})^1 : \mathsf{m}_Q(g)/H_Q(g) \geq 1\big\}.$$

Since $\mathsf{m}_Q(g) \leq H_Q(g)$ holds for all $g \in G(\mathbb{A})^1$, we have

$$\begin{aligned}\mathsf{R} &= \big\{g \in G(\mathbb{A})^1 : \mathsf{m}_Q(g) = H_Q(g)\big\} \\ &= \big\{g \in G(\mathbb{A})^1 : \bar{e} \in X_Q(g)\big\}.\end{aligned}$$

Since both H_Q and m_Q are continuous, R is a closed subset in $G(\mathbb{A})^1$.

Lemma 5. *One has $Q(\mathsf{k})\mathsf{R}K = \mathsf{R}$ and $G(\mathbb{A})^1 = G(\mathsf{k})\mathsf{R}$.*

Proof. The first assertion is obvious by the definition of H_Q. To prove the second assertion, we choose a minimal point $x \in X_Q(g)$ for a given $g \in G(\mathbb{A})^1$. There is a $\gamma \in G(\mathsf{k})$ such that $x = \pi_X(\gamma)$. Then $H_Q(xg) = H_Q(\gamma g) = \mathsf{m}_Q(g) = \mathsf{m}_Q(\gamma g)$ since m_Q is left $G(\mathsf{k})$-invariant. Therefore, $\gamma g \in \mathsf{R}$. □

Lemma 6. *Let C be an arbitrary subset of $G(\mathbb{A})^1$, and let $g \in G(\mathbb{A})^1$ and $\gamma \in G(\mathsf{k})$.*

(1) $\gamma g \in \mathsf{R}$ *if and only if* $\pi_X(\gamma) \in X_Q(g)$.

(2) $X_Q(g) = \pi_X(\{\gamma \in G(\mathsf{k}) : \gamma g \in \mathsf{R}\})$.

(3) $\gamma C \subset \mathsf{R}$ *if and only if* $\pi_X(\gamma) \in \bigcap_{g \in C} X_Q(g)$.

(4) $\bigcap_{g \in \mathsf{R}} X_Q(g) = \{\bar{e}\}$.

(5) $\gamma \mathsf{R} \subset \mathsf{R}$ *if and only if* $\gamma \in Q(\mathsf{k})$.

Proof. By definition, $\gamma g \in \mathsf{R}$ if and only if $\mathsf{m}_Q(\gamma g) = H_Q(\gamma g)$. This is equivalent to $\pi_X(\gamma) \in X_Q(g)$ because $\mathsf{m}_Q(\gamma g) = \mathsf{m}_Q(g)$. Both (2) and (3) follow from (1). For a point $x = \pi_X(\gamma) \in \bigcap_{g\in\mathsf{R}} X_Q(g)$, we have $\gamma Q(\mathsf{k})\mathsf{R} \subset \mathsf{R}$; in other words, $xQ(\mathsf{k}) \subset \bigcap_{g\in\mathsf{R}} X_Q(g)$. Since $xQ(\mathsf{k})$ is an infinite set for $x \neq \bar{e}$ by Bruhat decomposition, we must have $x = \bar{e}$. This shows (4). Item (5) follows from (3) and (4). ☐

Lemma 7. *Let $g_0 \in \mathsf{R}$ be an element such that $\mathsf{n}_Q(g_0) > 1$ and x_0 an arbitrary element in $X_Q(g_0)$. Then, any neighborhood $\mathcal{U}$ of g_0 in $G(\mathbb{A})^1$ contains a point g such that $X_Q(g) \subset X_Q(g_0)$ and $x_0 \notin X_Q(g)$.*

Proof. We may assume $\mathcal{U}$ satisfies $X_Q(g) \subset X_Q(g_0)$ for all $g \in \mathcal{U}$ by Lemma 4. Since $\mathsf{n}_Q(g_0) > 1$, there is an $x \in X_Q(g_0)$ such that $x \neq \bar{e}$. This x is of the form $\pi_X(w\gamma)$ with $w \in W_G \setminus W_G^Q$ and $\gamma \in Q(\mathsf{k})$. By Lemma 2, there is a cocharacter $\xi = \xi_{w,e} \in X_*(S)_{\mathsf{k}}$ such that $H_Q\big(w\xi(\lambda)w^{-1}\big) > H_Q(\xi(\lambda))$ holds for all $\lambda \in \mathbb{A}^\times_{>1}$. Let $\lambda \in \mathbb{A}^\times$ be an element sufficiently close to 1 so that $g_\lambda = \gamma^{-1}\xi(\lambda)\gamma g_0$ is contained in $\mathcal{U}$. We have

$$\begin{aligned} H_Q(g_\lambda) &= H_Q(\xi(\lambda)\gamma g_0) = H_Q(\xi(\lambda))H_Q(\gamma g_0) \\ &= H_Q(\xi(\lambda))H_Q(g_0) = H_Q(\xi(\lambda))\mathsf{m}_Q(g_0) \end{aligned}$$

and

$$\begin{aligned} H_Q(xg_\lambda) &= H_Q(w\xi(\lambda)\gamma g_0) = H_Q(w\xi(\lambda)w^{-1})H_Q(w\gamma g_0) \\ &= H_Q(w\xi(\lambda)w^{-1})\mathsf{m}_Q(g_0). \end{aligned}$$

If $x_0 = \bar{e}$, then we choose λ sufficiently close to 1 satisfying $\lambda^{-1} \in \mathbb{A}^\times_{>1}$. Since $X_Q(g_\lambda) \subset X_Q(g_0)$ and $\mathsf{m}_Q(g_\lambda) \le H_Q(xg_\lambda) < H_Q(g_\lambda)$, $X_Q(g_\lambda)$ does not contain $\bar{e}$. If $x_0 \neq \bar{e}$, then we choose x as x_0 and $\lambda \in \mathbb{A}^\times_{>1}$ sufficiently close to 1. Since $\mathsf{m}_Q(g_\lambda) \le H_Q(g_\lambda) < H_Q(x_0 g_\lambda)$, $X_Q(g_\lambda)$ does not contain x_0. ☐

Lemma 8. $\min_{g\in G(\mathbb{A})^1} \mathsf{n}_Q(g) = \min_{g\in\mathsf{R}} \mathsf{n}_Q(g) = 1$.

Proof. From Lemma 5 and the $G(\mathsf{k})$-invariance of n_Q, it follows that

$$\min_{g\in G(\mathbb{A})^1} \mathsf{n}_Q(g) = \min_{g\in\mathsf{R}} \mathsf{n}_Q(g).$$

If $g_0 \in \mathsf{R}$ satisfies $\min_{g\in\mathsf{R}} \mathsf{n}_Q(g) = \mathsf{n}_Q(g_0) > 1$, then by Lemmas 5 and 7, there exist a point $g_1 \in G(\mathbb{A})^1$ and $\gamma_1 \in G(\mathsf{k})$ such that $\mathsf{n}_Q(\gamma_1 g_1) = \mathsf{n}_Q(g_1) < \mathsf{n}_Q(g_0)$ and $\gamma_1 g_1 \in \mathsf{R}$. This is a contradiction. ☐

We define the subset R_1 of R by

$$\mathsf{R}_1 = \{g \in \mathsf{R} : \mathsf{n}_Q(g) = 1\} = \big\{g \in G(\mathbb{A})^1 : X_Q(g) = \{\bar{e}\}\big\}.$$

Lemma 9. *R_1 coincides with the interior R° of R in $G(\mathbb{A})^1$.*

Proof. For $g \in \mathsf{R}_1$, we choose a neighborhood $\mathcal{U}$ of g in $G(\mathbb{A})^1$ as in Lemma 4. Then $\mathcal{U} \subset \mathsf{R}_1$. Therefore, R_1 is open and is contained in R°. If there exists an element $g_0 \in \mathsf{R}^\circ$ such that $\mathsf{n}_Q(g_0) > 1$, then, by Lemma 7, R° contains an element g satisfying $\bar{e} \notin X_Q(g)$. This contradicts $g \in \mathsf{R}$. □

It is obvious that $G(\mathsf{k})\mathsf{R}_1 = \{g \in G(\mathbb{A})^1 : \mathsf{n}_Q(g) = 1\}$.

Lemma 10. *$G(\mathsf{k})\mathsf{R}_1$ is open and dense in $G(\mathbb{A})^1$.*

Proof. Since R_1 is open in $G(\mathbb{A})^1$, so is $G(\mathsf{k})\mathsf{R}_1$. We assume $G(\mathbb{A})^1 \setminus G(\mathsf{k})\mathsf{R}_1$ has an interior point g_0. Let $\mathcal{U}$ be a neighborhood of g_0 in $G(\mathbb{A})^1$ so that $\mathcal{U} \cap G(\mathsf{k})\mathsf{R}_1 = \varnothing$. By Lemma 5, we can take $\gamma_0 \in G(\mathsf{k})$ such that $\gamma_0 g_0 \in \mathsf{R}$. Since $\mathsf{n}_Q(\gamma_0 g_0) = \mathsf{n}_Q(g_0) > 1$, by Lemmas 5 and 7, there exist $g_1 \in \gamma_0 \mathcal{U}$ and $\gamma_1 \in G(\mathsf{k})$ such that $\mathsf{n}_Q(g_1) < \mathsf{n}_Q(g_0)$ and $\gamma_1 g_1 \in \mathsf{R}$. If $\mathsf{n}_Q(g_1) > 1$, then there exist $g_2 \in \gamma_1\gamma_0\mathcal{U}$ and $\gamma_2 \in G(\mathsf{k})$ such that $\mathsf{n}_Q(g_2) < \mathsf{n}_Q(g_1)$ and $\gamma_2 g_2 \in R$. This process terminates after finitely many iterations. At the last step, we obtain an element $g_\ell \in \gamma_{\ell-1} \cdots \gamma_0 \mathcal{U}$ such that $\mathsf{n}_Q(g_\ell) = 1$. Then $(\gamma_{\ell-1} \cdots \gamma_0)^{-1} g_\ell$ is contained in $\mathcal{U} \cap G(\mathsf{k})\mathsf{R}_1$. This contradicts $\mathcal{U} \cap G(\mathsf{k})\mathsf{R}_1 = \varnothing$. Therefore, $G(\mathbb{A})^1 \setminus G(\mathsf{k})\mathsf{R}_1$ is nowhere dense in $G(\mathbb{A})^1$. □

Lemma 11. *For $\gamma \in G(\mathsf{k})$, $\mathsf{R}_1 \cap \gamma\mathsf{R} \neq \varnothing$ if and only if $\gamma \in Q(\mathsf{k})$.*

Proof. If $\mathsf{R}_1 \cap \gamma\mathsf{R}$ has an element g, then $\pi_X(\gamma^{-1}) \in X_Q(g) = \{\bar{e}\}$ by Lemma 6. □

Lemma 12. *Let R_1^- be the closure of R_1. Then we have the following subdivision of $G(\mathbb{A})^1$:*

$$G(\mathbb{A})^1 = \bigcup_{\gamma Q(\mathsf{k}) \in G(\mathsf{k})/Q(\mathsf{k})} \gamma \mathsf{R}_1^-.$$

Proof. We fix an arbitrary $g \in G(\mathbb{A})^1$. By Lemma 10, there exists a sequence $\{g_n\} \subset G(\mathsf{k})\mathsf{R}_1$ such that $\lim_{n\to\infty} g_n = g$. We take a neighborhood $\mathcal{U}$ of g as in Lemma 4 and may assume that $\{g_n\} \subset \mathcal{U}$. Since $g_n \in G(\mathsf{k})\mathsf{R}_1$, $X_Q(g_n)$ consists of a single element $\pi_X(\gamma_n)$, where $\gamma_n \in G(\mathsf{k})$. From $g_n \in \mathcal{U}$, it follows that $\pi_X(\gamma_n) \in X_Q(g)$ for all n. Since $X_Q(g)$ is a finite set, we can take a subsequence $\{g_{n_j}\}$ such that $\pi_X(\gamma_{n_j}) = \pi_X(\gamma) \in X_Q(g)$ for all n_j. Then $\{g_{n_j}\} \subset \gamma^{-1}\mathsf{R}_1$, and g is contained in the closure of $\gamma^{-1}\mathsf{R}_1$. □

For $g \in G(\mathbb{A})^1$, we put

$$S_Q(g) = \pi_X(\{\gamma \in G(\mathsf{k}) : \gamma g \in \mathsf{R}_1^-\}).$$

By Lemmas 6 and 12, $S_Q(g)$ is a nonempty subset of $X_Q(g)$.

Lemma 13. *For $g_0 \in G(\mathbb{A})^1$, there is a neighborhood $\mathcal{U}$ of g_0 in $G(\mathbb{A})^1$ such that $S_Q(g) \subset S_Q(g_0)$ for all $g \in \mathcal{U}$.*

Proof. Let $\mathcal{U}$ be a neighborhood of g_0 such that $X_Q(g) \subset X_Q(g_0)$ for all $g \in \mathcal{U}$. Since $g_0 \notin \gamma^{-1}\mathsf{R}_1^-$ for any $\pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0)$, we can take a sufficiently small $\mathcal{U}$ so that $\mathcal{U} \cap \gamma^{-1}\mathsf{R}_1^- = \varnothing$ for all $\pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0)$. Then, for any $g \in \mathcal{U}$, $S_Q(g) \cap X_Q(g_0) \setminus S_Q(g_0)$ is empty; that is, $S_Q(g) \subset S_Q(g_0)$. □

Remark. We do not know whether $\mathsf{R}_1^- = \mathsf{R}$ holds or not in general. If it does, then $S_Q(g) = X_Q(g)$ holds for all g.

5. A fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$

Definition. Let T be a locally compact Hausdorff space and Γ be a discrete group acting on T from the left. Assume that the action of Γ on T is properly discontinuous. An open subset Ω of T is called an open fundamental domain of T with respect to Γ if Ω satisfies the following conditions:

(1) $T = \Gamma\Omega^-$, where Ω^- stands for the closure of Ω in T, and

(2) $\Omega \cap \gamma\Omega^- = \varnothing$ if $\gamma \in \Gamma \setminus \{e\}$.

A subset F of T is called a fundamental domain of T with respect to Γ if there is an open fundamental domain Ω as above such that $\Omega \subset \mathsf{F} \subset \Omega^-$.

By Baer and Levi's theorem [1931] (see also [van der Waerden 1935, §10]), an open fundamental domain of T with respect to Γ exists if the set of points stabilized by some nontrivial element of Γ is discrete in T. Thus there exists an open fundamental domain Ω_Q of R_1^- with respect to $Q(\mathsf{k})$. For a given subset A of R_1^-, A° and A^- denote the interior and the closure of A in $G(\mathbb{A})^1$, respectively. Since R_1^- is closed in $G(\mathbb{A})^1$, the closure of A in R_1^- coincides with A^-.

Lemma 14. *Let Ω_Q be an open fundamental domain of R_1^- with respect to $Q(\mathsf{k})$. Then one has $\Omega_Q^\circ = \Omega_Q \cap \mathsf{R}_1$ and $\Omega_Q^- = (\Omega_Q \cap \mathsf{R}_1)^-$.*

Proof. Since Ω_Q is an open set in R_1^- with respect to the relative topology, there is an open set $\mathcal{U}$ in $G(\mathbb{A})^1$ such that $\Omega_Q = \mathsf{R}_1^- \cap \mathcal{U}$. Therefore, $\Omega_Q \cap \mathsf{R}_1 = \mathcal{U} \cap \mathsf{R}_1$ is open in $G(\mathbb{A})^1$, and hence $\Omega_Q^\circ = \Omega_Q \cap \mathsf{R}_1$. Since R_1 is dense in R_1^- and Ω_Q is relatively open in R_1^-, the closure of $\Omega_Q \cap \mathsf{R}_1$ in R_1^- contains Ω_Q, that is, $\Omega_Q \subset (\Omega_Q \cap \mathsf{R}_1)^-$. Hence $\Omega_Q^- = (\Omega_Q \cap \mathsf{R}_1)^-$. □

Theorem 15. *Let Ω_Q be an open fundamental domain of R_1^- with respect to $Q(\mathsf{k})$. Then Ω_Q° is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$.*

Proof. From $\mathsf{R}_1^- = Q(\mathsf{k})\Omega_Q^-$ and Lemma 12, it follows $G(\mathbb{A})^1 = G(\mathsf{k})\Omega_Q^-$. For $\gamma \in G(\mathsf{k})$, we assume $\Omega_Q^\circ \cap \gamma\Omega_Q^- \neq \varnothing$. By Lemma 11, γ is contained in $Q(\mathsf{k})$. Since Ω_Q is an open fundamental domain of R_1^- with respect to $Q(\mathsf{k})$, γ must be equal to e. □

For a given subset A of $G(\mathbb{A})^1$, we denote by ∂A the boundary of A.

Lemma 16. *If $g_0 \in \mathsf{R}_1^-$ attains a local maximum of m_Q, then g_0 is in $\partial\mathsf{R}_1^-$.*

Proof. Suppose $g_0 \in \mathsf{R}_1$. Since R_1 is open, zg_0 is contained in R_1 if $z \in Z_Q(\mathbb{A})$ is sufficiently close to e. Then

$$\mathsf{m}_Q(zg_0) = H_Q(zg_0) = H_Q(z)H_Q(g_0) = H_Q(z)\mathsf{m}_Q(g_0).$$

Since $H_Q(z)$ can vary on the interval $(1-\epsilon, 1+\epsilon)$ for a sufficiently small $\epsilon > 0$, $\mathsf{m}_Q(g_0)$ is not a local maximum of m_Q. □

Since $(\Omega_Q^-)^\circ = \Omega_Q^\circ \subset \mathsf{R}_1$, the following theorem immediately follows from Lemma 16.

Theorem 17. *Let Ω_Q be the same as in Theorem 15. If $g_0 \in \Omega_Q^-$ attains a local maximum of m_Q, then g_0 is in $\partial\Omega_Q^- \cap \partial\mathsf{R}_1^-$.*

Remark. A point $g_0 \in G(\mathbb{A})^1$ is said to be extreme if g_0 attains a local maximum of m_Q. By Theorem 17, any extreme point is contained in $G(\mathsf{k})(\partial\Omega_Q^- \cap \partial\mathsf{R}_1^-)$. A candidate of the notion analogous to perfect quadratic forms is the following: a point $g \in G(\mathbb{A})^1$ is said to be Q-perfect if there is a neighborhood $\mathcal{U}$ of g such that

$$\mathcal{U} \cap \bigcap_{\pi_X(\delta)\in S_Q(g)} \delta^{-1}\mathsf{R}_1^- = \{g\}.$$

6. The case when G is of class number one

We put $K_f = \prod_{\sigma\in\mathsf{p}_f} K_\sigma$, $G_{\mathbb{A},\infty} = G(\mathsf{k}_\infty) \times K_f$, $G^1_{\mathbb{A},\infty} = G_{\mathbb{A},\infty} \cap G(\mathbb{A})^1$ and $G_\circ = G(\mathsf{k}) \cap G_{\mathbb{A},\infty}$. By identifying $G(\mathsf{k}_\infty)$ with the subgroup

$$\{(g_\sigma) \in G(\mathbb{A}) : g_\sigma = e \text{ for all } \sigma \in \mathsf{p}_f\}$$

of $G(\mathbb{A})$, we put $G(\mathsf{k}_\infty)^1 = G(\mathsf{k}_\infty) \cap G(\mathbb{A})^1$. The number $n_\mathsf{k}(G)$ of double cosets in $G(\mathbb{A})$ modulo $G(\mathsf{k})$ and $G_{\mathbb{A},\infty}$ is called the class number of G. For example, $n_\mathsf{k}(\mathrm{GL}_n)$ is equal to the class number of k. If G is almost k-simple, k-isotropic and simply connected, then $n_\mathsf{k}(G) = 1$ by the strong approximation theorem. In this section, we assume that $n_\mathsf{k}(G) = 1$. Then $G(\mathbb{A})^1 = G(\mathsf{k})G^1_{\mathbb{A},\infty}$. Let h_Q be the number of double cosets of $G(\mathsf{k})$ modulo $Q(\mathsf{k})$ and $G_\circ$. By [Borel 1963, Proposition 7.5], h_Q is equal to the class number of M_Q. Let $\{\xi_1 = e, \xi_2, \dots, \xi_{h_Q}\}$ be a complete system of representatives of $Q(\mathsf{k})\backslash G(\mathsf{k})/G_\circ$. For each ξ_i, we define

$$\mathsf{R}_{\xi_i,\infty} = \big\{g_\infty \in G(\mathsf{k}_\infty)^1 : \mathsf{m}_Q(g_\infty) = H_Q(\xi_i g_\infty)\big\}.$$

Since $G(\mathsf{k})$ is a disjoint union of $Q(\mathsf{k})\xi_i G_\circ$ for $i = 1, \dots, h_Q$, $\mathsf{m}_Q(g_\infty)$ equals

$$\min_{1\le i\le h_Q} \min_{\delta\in G_\circ} H_Q(\xi_i\delta g_\infty).$$

Lemma 18.
$$\mathsf{R} = \bigsqcup_{i=1}^{h_Q} Q(\mathsf{k})\xi_i(\mathsf{R}_{\xi_i,\infty} \times K_f).$$

Proof. For each i, $Q(\mathsf{k})\xi_i(\mathsf{R}_{\xi_i,\infty} \times K_f) \subset \mathsf{R}$ is trivial. Since

$$G(\mathbb{A})^1 = \bigsqcup_{i=1}^{h_Q} Q(\mathsf{k})\xi_i G^1_{\mathbb{A},\infty}$$

by [Borel 1963, §7], a given $g \in \mathsf{R}$ is represented as $g = \gamma\xi_i(g_\infty \times g_f)$ for some i, $\gamma \in Q(\mathsf{k})$ and $g_\infty \times g_f \in G^1_{\mathbb{A},\infty}$. Then $\mathsf{m}_Q(g) = H_Q(g)$ implies $\mathsf{m}_Q(g_\infty) = H_Q(\xi_i g_\infty)$. Therefore, $g_\infty \in \mathsf{R}_{\xi_i,\infty}$. □

We write Q_i for the conjugate $\xi_i^{-1}Q\xi_i$ of Q. This Q_i is a maximal k-parabolic subgroup of G. We put $Q_{i,\circ} = Q_i(\mathsf{k}) \cap G_{\mathbb{A},\infty}$.

Lemma 19. *If $g(\mathsf{R}_{\xi_i,\infty} \times K_f) \cap (\mathsf{R}_{\xi_i,\infty} \times K_f)$ is nonempty for $g \in Q_i(\mathsf{k})$, then $g \in Q_{i,\circ}$.*

Proof. If there is an $h \in \mathsf{R}_{\xi_i,\infty} \times K_f$ such that $gh \in \mathsf{R}_{\xi_i,\infty} \times K_f$, then

$$g \in (\mathsf{R}_{\xi_i,\infty} \times K_f)h^{-1} \subset G_{\mathbb{A},\infty}.$$ □

It is easy to prove that the group $Q_{i,\circ}$ stabilizes $\mathsf{R}_{\xi_i,\infty} \times K_f$ by left multiplication. We fix a complete system $\{\gamma_{ij}\}_j$ of representatives of $Q_i(\mathsf{k})/Q_{i,\circ}$. It follows from Lemma 19 that $\gamma_{ij}(\mathsf{R}_{\xi_i,\infty} \times K_f) \cap \gamma_{ik}(\mathsf{R}_{\xi_i,\infty} \times K_f) = \varnothing$ if $j \neq k$. Therefore, we obtain the following subdivision of R:

$$\mathsf{R} = \bigsqcup_{i=1}^{h_Q} \bigsqcup_{j} \xi_i\gamma_{ij}(\mathsf{R}_{\xi_i,\infty} \times K_f). \tag{1}$$

Let $\mathsf{R}^\circ_{\xi_i,\infty}$ be the interior of $\mathsf{R}_{\xi_i,\infty}$ and $\mathsf{R}^*_{\xi_i,\infty}$ the closure of $\mathsf{R}^\circ_{\xi_i,\infty}$ in $G(\mathsf{k}_\infty)^1$. Since the union of (1) is disjoint, it is obvious that

$$\mathsf{R}_1^- = \bigsqcup_{i=1}^{h_Q} \bigsqcup_{j} \xi_i\gamma_{ij}(\mathsf{R}^*_{\xi_i,\infty} \times K_f). \tag{2}$$

Proposition 20. *Let $\Omega_{i,\infty}$ be an open fundamental domain of $\mathsf{R}^*_{\xi_i,\infty}$ with respect to $Q_{i,\circ}$ for $i = 1, \ldots, h_Q$. Then the set*

$$\Omega = \bigsqcup_{i=1}^{h_Q} \xi_i(\Omega_{i,\infty} \times K_f)$$

gives an open fundamental domain of R_1^- with respect to $Q(\mathsf{k})$.

Proof. Let $\Omega^{-}_{i,\infty}$ denote the closure of $\Omega_{i,\infty}$ in $G(\mathsf{k}_\infty)^1$. For $g \in Q(\mathsf{k})$, we assume $\Omega \cap g\Omega^{-} \neq \varnothing$. Then, for some i, j,

$$\xi_i(\Omega_{i,\infty} \times K_f) \cap g\xi_j(\Omega^{-}_{j,\infty} \times K_f) \neq \varnothing. \tag{3}$$

There exist γ_{jk} and $\delta \in Q_{j,\circ}$ such that $\xi_j^{-1} g \xi_j = \gamma_{jk}\delta$. Then (3) is the same as

$$\xi_i(\Omega_{i,\infty} \times K_f) \cap \xi_j \gamma_{jk}(\delta\Omega^{-}_{j,\infty} \times K_f) \neq \varnothing.$$

By (1), we have $i = j$, $\gamma_{jk} = e$ and $\Omega_{j,\infty} \cap \delta\Omega^{-}_{j,\infty} \neq \varnothing$. Since $\Omega_{j,\infty}$ is an open fundamental domain of $\mathsf{R}^{*}_{\xi_j,\infty}$ with respect to $Q_{j,\circ}$, δ must be equal to e. Therefore, $\Omega \cap g\Omega^{-} \neq \varnothing$ implies $g = e$. Finally, $Q(\mathsf{k})\Omega^{-} = \mathsf{R}^{-}_1$ follows from (2) and $Q_{i,\circ}\Omega^{-}_{i,\infty} = \mathsf{R}^{*}_{\xi_i,\infty}$. □

By Theorem 17, we obtain the following.

Corollary 21. *If $g_0 \in \Omega^{-}$ attains a local maximum of m_Q, then g_0 is contained in the set*

$$\bigsqcup_{i=1}^{h_Q} \xi_i\big((\partial\Omega^{-}_{i,\infty} \cap \partial\mathsf{R}^{*}_{\xi_i,\infty}) \times K_f\big).$$

We consider the infinite part Ω_∞ of Ω given in Proposition 20, that is,

$$\Omega_\infty = \bigcup_{i=1}^{h_Q} \xi_i \Omega_{i,\infty}.$$

Let Ω°_∞ and Ω^{-}_∞ be the interior and the closure of Ω_∞ in $G(\mathsf{k}_\infty)^1$, respectively. The projection from $G(\mathbb{A})^1 = G(\mathsf{k})G^1_{\mathbb{A},\infty}$ to the infinite component $G(\mathsf{k}_\infty)^1$ gives an isomorphism $G(\mathsf{k})\backslash G(\mathbb{A})^1/K_f \cong G_\circ\backslash G(\mathsf{k}_\infty)^1$. Since Ω is a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$ by Theorem 15, we have $G_\circ\Omega^{-}_\infty = G(\mathsf{k}_\infty)^1$.

Corollary 22. *If $h_Q = 1$, then Ω_∞ is a fundamental domain of $G(\mathsf{k}_\infty)^1$ with respect to $G_\circ$.*

Proof. Since $\Omega_\infty = \Omega_{1,\infty}$ is a relatively open set in $\mathsf{R}^{*}_{e,\infty}$, we have $\Omega^{\circ}_\infty = \Omega_\infty \cap \mathsf{R}^{\circ}_{e,\infty}$. Thus the closure of Ω°_∞ coincides with Ω^{-}_∞. If $\Omega^{\circ}_\infty \cap g\Omega^{-}_\infty \neq \varnothing$ for $g \in G_\circ$, then $(\Omega^{\circ}_\infty \times K_f) \cap g(\Omega^{-}_\infty \times K_f) \neq \varnothing$ because $gK_f = K_f$. This implies $g = e$ since $\Omega^{\circ}_\infty \times K_f$ is an open fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathsf{k})$. □

7. Examples

Example 3. Let G be a general linear group GL_n defined over $\mathbb{Q}$. We continue an illustration given in Examples 1 and 2. We fix an integer $k \in \{1, \dots, n-1\}$, and let

$$Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,:\, a \in \mathrm{GL}_k(\mathbb{Q}),\ b \in \mathrm{M}_{k,n-k}(\mathbb{Q}),\ d \in \mathrm{GL}_{n-k}(\mathbb{Q}) \right\}.$$

Since $h_Q = 1$, we have $\xi_1 = e$ and $Q_1 = Q$.

Let P_n be the cone of positive definite n by n real symmetric matrices, and let P_n^1 be the intersection of P_n and $\mathrm{SL}_n(\mathbb{R})$. The group $G(\mathbb{Q}_\infty) = \mathrm{GL}_n(\mathbb{R})$ acts on P_n from the right by $(A, g) \mapsto A[g] = {}^t g A g$ for $(A, g) \in \mathsf{P}_n \times G(\mathbb{Q}_\infty)$. The maximal compact subgroup K_∞ of $G(\mathbb{Q}_\infty)$, defined as in Example 2, stabilizes the identity matrix $I_n \in \mathsf{P}_n$. The map $\pi : g \mapsto {}^t g^{-1} g^{-1}$ from $G(\mathbb{Q}_\infty)$ onto P_n gives an isomorphism between $G(\mathbb{Q}_\infty)/K_\infty$ and P_n. Since

$$G(\mathbb{Q}_\infty)^1 = \{g \in G(\mathbb{Q}_\infty) : \det g = \pm 1\},$$

we have $G(\mathbb{Q}_\infty)^1/K_\infty \cong \pi(G(\mathbb{Q}_\infty)^1) = \mathsf{P}_n^1$. An element $A \in \mathsf{P}_n$ is written as

$$A = \begin{pmatrix} I_k & 0 \\ {}^t u & I_{n-k} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} I_k & u \\ 0 & I_{n-k} \end{pmatrix},$$

where $v \in \mathsf{P}_k$, $w \in \mathsf{P}_{n-k}$ and $u \in \mathrm{M}_{k,n-k}(\mathbb{R})$. We write u_A, $A^{[k]}$ and $A_{[n-k]}$ for u, v and w, respectively.

By definition, $G_\mathbb{Z} = G(\mathbb{Q}) \cap G_{\mathbb{A},\infty}$ and $Q_\mathbb{Z} = Q(\mathbb{Q}) \cap G_{\mathbb{A},\infty}$ are just the groups $\mathrm{GL}_n(\mathbb{Z})$ and $Q(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ of unimodular integral matrices in $G(\mathbb{Q})$ and $Q(\mathbb{Q})$, respectively. As in Example 2, X_γ stands for the n by k matrix consisting of the first k-columns of $\gamma \in G_\mathbb{Z}$, and $\mathrm{M}_{n,k}(\mathbb{Z})^*$ stands for the set of X_γ for all $\gamma \in G_\mathbb{Z}$. We define the closed subset $\mathsf{F}_{n,k}$ of P_n as follows:

$$\mathsf{F}_{n,k} = \left\{ A \in \mathsf{P}_n : \det A^{[k]} \le \det({}^t X A X) \text{ for all } X \in \mathrm{M}_{n,k}(\mathbb{Z})^* \right\}.$$

In Example 2, we showed

$$H_Q(\gamma g) = \det\left({}^t X_{\gamma^{-1}} \pi(g) X_{\gamma^{-1}}\right)^{n/2r}$$

for any $\gamma \in G_\mathbb{Z}$ and $g \in G(\mathbb{Q}_\infty)^1$. Since $H_Q(g) = \left(\det \pi(g)^{[k]}\right)^{n/2r}$, we obtain

$$\mathsf{R}_{e,\infty}/K_\infty \cong \pi(\mathsf{R}_{e,\infty}) = \mathsf{F}_{n,k} \cap \mathrm{SL}_n(\mathbb{R}).$$

Therefore, $Q_\mathbb{Z} \backslash \mathsf{R}_{e,\infty}/K_\infty$ is isomorphic to $(\mathsf{F}_{n,k} \cap \mathrm{SL}_n(\mathbb{R}))/Q_\mathbb{Z}$. If $\gamma \in Q_\mathbb{Z}$ is of the form

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $a \in \mathrm{GL}_k(\mathbb{Z})$, $d \in \mathrm{GL}_{n-k}(\mathbb{Z})$ and $b \in \mathrm{M}_{k,n-k}(\mathbb{Z})$, then components of ${}^t\gamma A \gamma$ for $A \in \mathsf{P}_n$ are given by

$$u_{{}^t\gamma A\gamma} = a^{-1}(u_A d + b), \quad \left({}^t\gamma A \gamma\right)^{[k]} = {}^t a A^{[k]} a, \quad \left({}^t\gamma A\gamma\right)_{[n-k]} = {}^t d A_{[n-k]} d.$$

Let $\mathfrak{D}$ and $\mathfrak{E}$ be arbitrary fundamental domains for the quotients $\mathsf{P}_k/\mathrm{GL}_k(\mathbb{Z})$ and $\mathsf{P}_{n-k}/\mathrm{GL}_{n-k}(\mathbb{Z})$, respectively. We define the subset $\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E})$ of $\mathsf{F}_{n,k}$ as

$$\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E}) = \{A \in \mathsf{F}_{n,k} : A^{[k]} \in \mathfrak{D},\ A_{[n-k]} \in \mathfrak{E},\\ u_A = (u_{ij}),\ -\tfrac{1}{2} \le u_{ij} \le \tfrac{1}{2} \text{ for all } i,\, j, \text{ and } 0 \le u_{11}\}.$$

Since $\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E})$ is a fundamental domain of $\mathsf{F}_{n,k}$ with respect to $Q_{\mathbb{Z}}$, the inverse image $\pi^{-1}(\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E}) \cap \mathrm{SL}_n(\mathbb{R}))$ of $\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E}) \cap \mathrm{SL}_n(\mathbb{R})$ gives a fundamental domain of $\mathsf{R}_{e,\infty}$ with respect to $Q_{\mathbb{Z}}$. As a consequence of Theorem 15 and Proposition 20, the set

$$\pi^{-1}(\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E}) \cap \mathrm{SL}_n(\mathbb{R})) \times K_f$$

gives a fundamental domain of $G(\mathbb{A})^1$ with respect to $G(\mathbb{Q})$. Moreover, from Corollary 22, it follows that $\mathsf{F}_{n,k}(\mathfrak{D},\mathfrak{E})$ is a fundamental domain of P_n with respect to $\mathrm{GL}_n(\mathbb{Z})$.

In the case of $k = 1$, this gives an inductive construction of a fundamental domain Ω_n of P_n with respect to $\mathrm{GL}_n(\mathbb{Z})$ as follows. First, put $\Omega_2 = \mathsf{F}_{2,1}(\mathsf{P}_1, \mathsf{P}_1)$. By definition, Ω_2 is Minkowski's fundamental domain of P_2. Then we define inductively $\Omega_3 = \mathsf{F}_{3,1}(\mathsf{P}_1, \Omega_2), \dots, \Omega_n = \mathsf{F}_{n,1}(\mathsf{P}_1, \Omega_{n-1})$. The domain Ω_n coincides with Grenier's fundamental domain [1988].

Finally, we show that, in the case of $k = 1$, $\mathsf{R}_{e,\infty}/K_\infty$ corresponds to a face of the Ryshkov polyhedron $\mathsf{R}(\mathsf{m}) = \{A \in \mathsf{P}_n : \mathsf{m}(A) = \min_{0 \neq x \in \mathbb{Z}^n} {}^t x A x \ge 1\}$. For $A \in \mathsf{P}_n$, let $S(A)$ denote the set of minimal integral vectors of A:

$$S(A) = \{x \in \mathbb{Z}^n : \mathsf{m}(A) = {}^t x A x\}.$$

We take $\boldsymbol{e}_1 = {}^t(1, 0, \dots, 0) \in \mathbb{Z}^n$. It is obvious that the subset $\{A \in \mathsf{P}_n : \boldsymbol{e}_1 \in S(A)\}$ of P_n coincides with $\mathsf{F}_{n,1}$. As was shown in [Watanabe 2012, Lemma 1.5], $\mathscr{F}_{\{\boldsymbol{e}_1\}} = \mathsf{F}_{n,1} \cap \partial \mathsf{R}(\mathsf{m}) = \{A \in \mathsf{F}_{n,1} : \mathsf{m}(A) = 1\}$ is a face of $\mathsf{R}(\mathsf{m})$. It is easy to see that the map $A \mapsto \mathsf{m}(A)^{-1} A$ gives a bijection from $\mathsf{F}_{n,1} \cap \mathrm{SL}_n(\mathbb{R})$ onto $\mathscr{F}_{\{\boldsymbol{e}_1\}}$. Therefore, $\mathsf{R}_{e,\infty}/K_\infty \cong \pi(\mathsf{R}_{e,\infty})$ corresponds to $\mathscr{F}_{\{\boldsymbol{e}_1\}}$.

Example 4. Let k be a totally real number field of degree r and $n = 2m$ be an even integer. We consider a symplectic group

$$G(\mathsf{k}) = \mathrm{Sp}_n(\mathsf{k}) = \left\{ g \in \mathrm{GL}_{2m}(\mathsf{k}) \ :\ {}^t g \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \right\}.$$

For a fixed $k \in \{1, 2, \dots, m\}$, let Q denote the maximal parabolic subgroup of G given by

$$Q(\mathsf{k}) = U(\mathsf{k}) M(\mathsf{k}),$$

where

$$M(\mathsf{k}) = \left\{ \delta(a,b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & {}^t a^{-1} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{pmatrix} \ : \ \begin{array}{c} a \in \mathrm{GL}_k(\mathsf{k}), \\ b = (b_{ij}) \in \mathrm{Sp}_{2(m-k)}(\mathsf{k}) \end{array} \right\},$$

$$U(\mathsf{k}) = \left\{ \begin{pmatrix} I_k & * & * & * \\ 0 & I_{m-k} & * & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & * & I_{m-k} \end{pmatrix} \in G(\mathsf{k}) \right\}.$$

The module of k-rational characters $X^*(M)_{\mathsf{k}}$ of M is a free $\mathbb{Z}$-module of rank 1 generated by the character

$$\widehat{\alpha}_Q(\delta(a,b)) = \det a.$$

The height $H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0}$ is given by $H_Q(g) = |\det a|_{\mathbb{A}}^{-1}$ if $g = u\delta(a,b)h$ with $u \in U(\mathbb{A})$, $\delta(a,b) \in M(\mathbb{A})$ and $h \in K$.

We restrict ourselves to the case $k = m$. An element of $M(\mathbb{A})$ is denoted by

$$\delta(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \quad a \in \mathrm{GL}_m(\mathbb{A}).$$

Let

$$\mathsf{H}_m = \left\{ Z \in \mathrm{M}_m(\mathbb{C}) \ : \ {}^t Z = Z, \ \mathrm{Im} Z \in \mathsf{P}_m \right\}$$

be the Siegel upper half space and H_m^r the direct product of r copies of H_m. For $Z = (Z_\sigma)_{\sigma \in \mathsf{p}_\infty} \in \mathsf{H}_m^r$, $\mathrm{Re} Z$, $\mathrm{Im} Z$ and $\det Z$ stand for $(\mathrm{Re} Z_\sigma)_{\sigma \in \mathsf{p}_\infty}$, $(\mathrm{Im} Z_\sigma)_{\sigma \in \mathsf{p}_\infty}$ and $(\det Z_\sigma)_{\sigma \in \mathsf{p}_\infty}$, respectively. The group $G(\mathsf{k}_\infty)$ acts transitively on H_m^r by

$$g\langle Z \rangle = \left((a_\sigma Z_\sigma + b_\sigma)(c_\sigma Z_\sigma + d_\sigma)^{-1} \right)_{\sigma \in \mathsf{p}_\infty}$$

for $Z = (Z_\sigma) \in \mathsf{H}_m^r$ and

$$g = (g_\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}_{\sigma \in \mathsf{p}_\infty} \in G(\mathsf{k}_\infty).$$

The stabilizer K_∞ of $Z_0 = (\sqrt{-1} I_m, \ldots, \sqrt{-1} I_m) \in \mathsf{H}_m^r$ in $G(\mathsf{k}_\infty)$ is a maximal compact subgroup of $G(\mathsf{k}_\infty)$. We choose K as $K_\infty \times \prod_{\sigma \in \mathsf{p}_f} \mathrm{Sp}_n(\mathsf{o}_\sigma)$. The map $\pi : g_\infty \mapsto g\langle Z_0 \rangle$ from $G(\mathsf{k}_\infty)$ onto H_m^r gives an isomorphism $G(\mathsf{k}_\infty)/K_\infty \cong \mathsf{H}_m^r$, and hence $G(\mathsf{k}) \backslash G(\mathbb{A})/K \cong G_\circ \backslash \mathsf{H}_m^r$. Since $\mathrm{Im}\{(u\delta(a)h)\langle Z_0 \rangle\} = a\, {}^t a$ holds for $u \in U(\mathsf{k}_\infty)$, $a \in \mathrm{GL}_m(\mathsf{k}_\infty)$ and $h \in K_\infty$, we have

$$H_Q(g_\infty) = \mathrm{Nr}_{\mathsf{k}_\infty/\mathbb{R}}(\det \mathrm{Im}\{g_\infty \langle Z_0 \rangle\})^{-1/2} = \left(\prod_{\sigma \in \mathsf{p}_\infty} \det \mathrm{Im}\left\{ g_\sigma \left\langle \sqrt{-1} I_m \right\rangle \right\} \right)^{-1/2}$$

for any $g_\infty = (g_\sigma) \in G(\mathsf{k}_\infty)$, where $\mathrm{Nr}_{\mathsf{k}_\infty/\mathbb{R}}$ denotes the norm of k_∞ over $\mathbb{R}$.

The class number h_Q of $M \cong \mathrm{GL}_m$ defined over k is equal to the class number h_{k} of k. We assume $h_{\mathsf{k}} = 1$ for simplicity. Then $G(\mathsf{k}) = Q(\mathsf{k})G_{\mathsf{o}}$ and $G(\mathbb{A}) = Q(\mathsf{k})G_{\mathbb{A},\infty}$, and hence

$$\mathsf{m}_Q(g_\infty) = \min_{\gamma \in G_{\mathsf{o}}} H_Q(\gamma g_\infty).$$

Since

$$\mathrm{Nr}_{\mathsf{k}_\infty/\mathbb{R}}(\det \mathrm{Im}\{\gamma\langle Z\rangle\}) = \prod_{\sigma \in \mathsf{p}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))|^{-2}\mathrm{Nr}_{\mathsf{k}_\infty/\mathbb{R}}(\det \mathrm{Im} Z)$$

for $Z = (Z_\sigma) \in \mathsf{H}_m^r$ and

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_{\mathsf{o}} = \mathrm{Sp}_n(\mathsf{o}),$$

the condition $\mathsf{m}_Q(g_\infty) = H_Q(g_\infty)$ of g_∞ is equivalent with the following condition of $Z = g_\infty\langle Z_0\rangle$:

$$\prod_{\sigma \in \mathsf{p}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \qquad \text{for all} \quad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_{\mathsf{o}}.$$

Therefore, the domain $R_{e,\infty}$ modulo K_∞ is isomorphic to

$$\mathsf{F} = \left\{ (Z_\sigma) \in \mathsf{H}_m^r \; : \; \prod_{\sigma \in \mathsf{p}_\infty} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \text{ for all } \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_{\mathsf{o}} \right\}.$$

Let $\mathfrak{C}$ be an arbitrary fundamental domain of the additive group $\mathrm{M}_m(\mathsf{k}_\infty)$ with respect to $\mathrm{M}_m(\mathsf{o})$, and let $\mathfrak{D}$ be an arbitrary fundamental domain of P_m^r with respect to $\mathrm{GL}_m(\mathsf{o})$. It is easy to see that

$$\mathsf{F}(\mathfrak{C}, \mathfrak{D}) = \{Z \in \mathsf{F} : \mathrm{Re} Z \in \mathfrak{C},\ \mathrm{Im} Z \in \mathfrak{D}\}$$

is a fundamental domain of F with respect to Q_{o}. By Corollary 22, $\mathsf{F}(\mathfrak{C}, \mathfrak{D})$ is a fundamental domain of H_m^r with respect to G_{o}.

If $\mathsf{k} = \mathbb{Q}$ and $\mathfrak{D}$ is Minkowski's fundamental domain, then $\mathsf{F}(\mathfrak{C}, \mathfrak{D})$ coincides with Siegel's fundamental domain [1939].

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