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REALIZATIONS OF THE THREE-POINT LIE ALGEBRA $\mathfrak{sl}(2, \mathcal{R}) \oplus (\Omega_{\mathcal{R}}/d\mathcal{R})$

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This paper is dedicated to Robert Wilson.

We describe the universal central extension of the three-point current algebra $\mathfrak{sl}(2, \mathcal{R})$, where $\mathcal{R} = \mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 + 4t]$, and construct realizations of it in terms of sums of partial differential operators.

1. Introduction

It is well known from the work of Kassel and Loday (see [Kassel and Loday 1982; 1984]) that if R is a commutative algebra and $\mathfrak{g}$ is a simple Lie algebra, both defined over the complex numbers, then the universal central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g} \otimes R$ is the vector space $(\mathfrak{g} \otimes R) \oplus \Omega^1_R/dR$, where Ω^1_R/dR is the space of Kähler differentials modulo exact forms (see [Kassel 1984]). The vector space $\hat{\mathfrak{g}}$ is made into a Lie algebra by defining

$$[x \otimes f, y \otimes g] := [xy] \otimes fg + (x, y)\overline{f\,dg}, \quad [x \otimes f, \omega] = 0$$

for $x, y \in \mathfrak{g}$, $f, g \in R$, $\omega \in \Omega^1_R/dR$, where $(\,-\,,\,-\,)$ denotes the Killing form on $\mathfrak{g}$. Here $\bar{a}$ denotes the image of $a \in \Omega^1_R$ in the quotient Ω^1_R/dR. A somewhat vague but natural question is whether there exist free field or Wakimoto-type realizations of these algebras. It is well known from the work of Wakimoto and of Feigin and Frenkel what the answer is when R is the ring of Laurent polynomials in one variable (see [Wakimoto 1986] and [Feigin and Frenkel 1990]). We find such a realization in the setting where $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 + 4t]$, and $\hat{\mathfrak{g}}$ is the three-point algebra.

In Kazhdan and Lusztig's explicit study [1991; 1993] of the tensor structure of modules for affine Lie algebras the ring of functions regular everywhere except at a finite number of points appears naturally. This algebra Bremner gave the name *n-point algebra*. In particular, in [Frenkel and Ben-Zvi 2001, Chapter 12], algebras of the form $\bigoplus_{i=1}^n \mathfrak{g}((t - x_i)) \oplus \mathbb{C}c$ appear in the description of the conformal blocks.

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These contain the n-point algebras $\mathfrak{g}\otimes\mathbb{C}[(t-x_1)^{-1},\dots,(t-x_N)^{-1}]\oplus\mathbb{C}c$ modulo part of the center Ω_R/dR. Bremner [1994a] explicitly described the universal central extension of such an algebra.

Consider now the Riemann sphere $\mathbb{C}\cup\{\infty\}$ with coordinate function s, and fix three distinct points a_1, a_2, a_3 on this Riemann sphere. Let R denote the ring of rational functions with poles only in the set $\{a_1, a_2, a_3\}$. It is known that the automorphism group $\mathrm{PGL}_2(\mathbb{C})$ of $\mathbb{C}(s)$ is simply 3-transitive, and R is a subring of $\mathbb{C}(s)$ that is isomorphic to the ring of rational functions with poles at $\{\infty, 0, 1, a\}$. Motivated by this isomorphism, one sets $a=a_4$ and here the *four-point ring* is $R=R_a=\mathbb{C}[s, s^{-1}, (s-1)^{-1}, (s-a)^{-1}]$, where $a\in\mathbb{C}\backslash\{0, 1\}$. Let $S:=S_b=\mathbb{C}[t, t^{-1}, u]$, where $u^2=t^2-2bt+1$ with b a complex number not equal to ±1. Then Bremner has shown us that $R_a\cong S_b$. As the latter, being $\mathbb{Z}_2$-graded, is a cousin to super Lie algebras, it is thus more immediately amendable to the theatrics of conformal field theory. Moreover, Bremner has given an explicit description of the universal central extension of $\mathfrak{g}\otimes R$ in terms of ultraspherical (Gegenbauer) polynomials where R is the four-point algebra (see [Bremner 1995]). In [Cox 2008] a realization was given for the four-point algebra where the center acts nontrivially.

In his study of the elliptic affine Lie algebras $\mathfrak{sl}(2, R)\oplus(\Omega_R/dR)$ where $R=\mathbb{C}[x, x^{-1}, y\mid y^2=4x^3-g_2x-g_3]$, Bremner [1994b] has also explicitly described the universal central extension of this algebra in terms of Pollaczek polynomials. Essentially the same algebras appear in [Fialowski and Schlichenmaier 2007; 2005]. Together with Bueno and Futorny, the first author described free-field-type realizations of the elliptic Lie algebra where $R=\mathbb{C}[t, t^{-1}, u\mid u^2=t^3-2bt^2-t]$, $b\neq\pm1$ (see [Bueno et al. 2009]).

Below, we study the three-point algebra case where R denotes the ring of rational functions with poles only in the set $\{a_1, a_2, a_3\}$. This algebra is isomorphic to $\mathbb{C}[s, s^{-1}, (s-1)^{-1}]$. Schlichenmaier [2003a] has a slightly different description of the three-point algebra as $\mathbb{C}[(z^2-a^2)^k, z(z^2-a^2)^k\mid k\in\mathbb{Z}]$, where $a\neq0$. We show that $R\cong\mathbb{C}[t, t^{-1}, u\mid u^2=t^2+4t]$, thus resembling S_b above. Our main result, Theorem 5.1, provides a natural free field realization in terms of a β-γ-system and the oscillator algebra of the three-point affine Lie algebra when $\mathfrak{g}=\mathfrak{sl}(2, \mathbb{C})$. Just as in the case of intermediate Wakimoto modules defined in [Cox and Futorny 2006], there are two different realizations depending on two different normal orderings. Besides Bremner's article mentioned above, other work on the universal central extension of three-point algebras can be found in [Benkart and Terwilliger 2007]. Previous related work on highest-weight modules of $\mathfrak{sl}(2, R)$ can be found in [Jakobsen and Kac 1985].

The three-point algebra is perhaps the simplest nontrivial example of a Krichever–Novikov algebra beyond an affine Kac–Moody algebra (see [Krichever and Novikov 1987a; 1987b; 1989]). A fair amount of interesting and fundamental work has been done by Krichever, Novikov, Schlichenmaier, and Sheinman on the representation theory of Krichever–Novikov algebras. In particular, Wess–Zumino–Witten–Novikov theory and analogues of the Knizhnik–Zamolodchikov (KZ) equations are developed for these algebras (see the survey article [Sheinman 2005], and, for example, [Schlichenmaier and Sheinman 1996; 1999; Sheinman 2003; Schlichenmaier 2003a; 2003b]).

The initial motivation for the use of Wakimoto's realization was to prove a conjecture of Kac and Kazhdan on the characters of certain irreducible representations of affine Kac–Moody algebras at the critical level (see [Wakimoto 1986] and [Frenkel 2005]). Another motivation for constructing free field realizations is that they are used to provide integral solutions to the KZ equations (see for example [Schechtman and Varchenko 1990] and [Etingof et al. 1998] and their references). A third is that they are used in determining the center of a certain completion of the enveloping algebra of an affine Lie algebra at the critical level, which is an important ingredient in the geometric Langland's correspondence [Frenkel 2007]. Yet a fourth is that free field realizations of an affine Lie algebra appear naturally in the context of the generalized AKNS hierarchies [Feigin and Frenkel 1999].

2. The three-point ring

The three-point algebra has at least four incarnations.

Three-point algebras. Fix a nonzero $a\in\mathbb{C}$. Let

$$\begin{aligned}\mathcal{S}&:=\mathbb{C}[s,s^{-1},(s-1)^{-1}],\\ \mathcal{R}&:=\mathbb{C}[t,t^{-1},u\mid u^2=t^2+4t],\\ \mathcal{A}&:=\mathcal{A}_a=\mathbb{C}[(z^2-a^2)^k,z(z^2-a^2)^j\mid k,j\in\mathbb{Z}].\end{aligned}$$

Note that Bremner introduced the ring $\mathcal{S}$ and Schlichenmaier [2003a] introduced $\mathcal{A}$. Variants of $\mathcal{R}$ were introduced by Bremner for elliptic and three-point algebras.

Proposition 2.1. (1) *The rings $\mathcal{R}$ and $\mathcal{S}$ are isomorphic by $t\mapsto s^{-1}(s-1)^2$ and $u\mapsto s-s^{-1}$.*

(2) *The rings $\mathcal{R}$ and $\mathcal{A}$ are isomorphic.*

Proof. (1) Let $\bar f:\mathbb{C}[t,u]\to\mathcal{S}$ be the ring homomorphism defined by $\bar f(t)=s^{-1}(s-1)^2=s-2+s^{-1}$, $\bar f(u)=s-s^{-1}$.

We first check that

$$\bar f(u^2-(t^2+4t))=(s-s^{-1})^2-(s-2+s^{-1})^2-4(s-2+s^{-1})=0$$

and $\bar f(t)=s^{-1}(s-1)^2$ is invertible in $\mathcal{S}$. Hence the map $\bar f$ descends to a well-defined ring homomorphism $f:\mathcal{R}\to\mathcal{S}$. To show that it is onto, we essentially solve for s and s^{-1} in terms of t and u. The inverse ring homomorphism of f is $\phi : \mathcal{S} \to \mathcal{R}$, given by

$$\phi(s) = \frac{t+2+u}{2}, \quad \phi(s^{-1}) = \frac{t+2-u}{2}.$$

In particular, $\phi((s-1)^{-1}) = (t^{-1}u-1)/2$.

For part (2), observe $\mathcal{A} = \mathbb{C}[z, (z-a)^{-1}, (z+a)^{-1}]$, so, mapping z to $2as-a$, we get $\mathcal{A} \cong \mathbb{C}[s, s^{-1}, (s-1)^{-1}]$. Thus an isomorphism between $\mathcal{A}$ and $\mathcal{R}$ is implemented by the assignment

$$z \mapsto a(t+u), \quad (z+a)^{-1} \mapsto \frac{t+2-u}{4a}, \quad (z-a)^{-1} \mapsto \frac{t^{-1}u-1}{4a}. \qquad \square$$

The universal central extension of the current algebra $\mathfrak{g} \otimes \mathcal{A}$. Let R be a commutative algebra defined over $\mathbb{C}$. Consider the left R-module $F = R \otimes R$ with left action given by $f(g \otimes h) = fg \otimes h$ for $f, g, h \in R$, and let K be the submodule generated by the elements $1 \otimes fg - f \otimes g - g \otimes f$. Then $\Omega_R^1 = F/K$ is the module of *Kähler differentials*. The element $f \otimes g + K$ is traditionally denoted by $f\,dg$. The canonical map $d : R \to \Omega_R^1$ is given by $df = 1 \otimes f + K$. The *exact differentials* are the elements of the subspace dR. The coset of $f\,dg$ modulo dR is denoted by $\overline{f\,dg}$. As Kassel has shown, the universal central extension of the current algebra $\mathfrak{g} \otimes R$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra defined over $\mathbb{C}$, is the vector space $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes R) \oplus \Omega_R^1/dR$, with Lie bracket given by

$$[x \otimes f, Y \otimes g] = [xy] \otimes fg + (x, y)\overline{f\,dg}, \quad [x \otimes f, \omega] = 0, \quad [\omega, \omega'] = 0,$$

where $x, y \in \mathfrak{g}$, $\omega, \omega' \in \Omega_R^1/dR$, and (x, y) denotes the Killing form on $\mathfrak{g}$.

There are at least four incarnations of the three-point algebras, three of which are defined as $\mathfrak{g} \otimes R \oplus \Omega_R/dR$ where $R = \mathcal{S}, \mathcal{R}, \mathcal{A}$ given above. The fourth incarnation appears in [Benkart and Terwilliger 2007] and is given in terms of the tetrahedron algebra. We will only work with $R = \mathcal{R}$.

Proposition 2.2 ([Bremner 1994a]; see also [Bremner 1995]). *Let $\mathcal{R}$ be as above. The set*

$$\{\omega_0 := \overline{t^{-1}\,dt},\ \omega_1 := \overline{t^{-1}u\,dt}\}$$

is a basis of $\Omega_{\mathcal{R}}^1/d\mathcal{R}$.

Proof. The proof follows almost exactly along the lines of [Bremner 1995] and [Bremner 1994a]. We know by the Riemann–Roch theorem that the space $\Omega_{\mathcal{R}}/d\mathcal{R}$ of Kähler differentials modulo exact forms on the sphere with three punctures has dimension 2 (see [Bremner 1994a]). We have the following formulae:

$$d(t^k) = kt^{k-1}\,dt,$$

$$d(t^k u) = t^k\,du + kt^{k-1}u\,dt, \tag{2-1}$$

$$t^k u\,dt \equiv -\frac{k+3}{4k+6}\,t^{k+1}u\,dt \qquad \text{mod } d\mathcal{R}, \tag{2-2}$$

$$t^{k-1}\,dt \equiv \frac{1}{k}d(t^k) \equiv 0 \qquad \text{mod } d\mathcal{R} \text{ for } k \neq 0. \tag{2-3}$$

By Equations (2-1), (2-2), and (2-3), we conclude that $\Omega_{\mathcal{R}}/d\mathcal{R}$ is spanned by $\{\overline{t^{-1}\,dt}, \overline{t^{-1}u\,dt}\}$. □

Corollary 2.3. *In Ω^1_R/dR, one has*

$$\overline{t^k\,dt^l} = -k\delta_{l,-k}\,\omega_0, \tag{2-4}$$

$$\overline{t^k u\,d(t^l u)} = ((l+1)\delta_{k+l,-2} + (4l+2)\delta_{k+l,-1})\omega_0, \tag{2-5}$$

$$\overline{t^k\,d(t^l u)} = -k\delta_{k,-l}\,\omega_1. \tag{2-6}$$

Proof. Using (2-1) above, we obtain

$$\begin{aligned} t^k\,d(t^l u) &\equiv t^k(lt^{l-1}u\,dt + t^l\,du) \\ &\equiv lt^{l+k-1}u\,dt + t^{l+k}\,du \\ &\equiv lt^{l+k-1}u\,dt - (l+k)t^{l+k-1}u\,dt \\ &\equiv -kt^{l+k-1}u\,dt \\ &\equiv -k\delta_{l+k,0}t^{-1}u\,dt \quad \text{mod } d\mathcal{R} \end{aligned}$$

in $\Omega_{\mathcal{R}}/\mathcal{R}$.

Next we observe $u\,du = \frac{1}{2}d(u^2) = \frac{1}{2}d(t^2+4t) = (t+2)\,dt$, so in $\Omega_{\mathcal{R}}$,

$$t^k u\,du = (t^{k+1} + 2t^k)\,dt. \tag{2-7}$$

By (2-7) and (2-3),

$$\begin{aligned} t^k u\,d(t^l u) &= t^k u(lt^{l-1}u\,dt + t^l\,du) \quad \text{in } \Omega_{\mathcal{R}} \\ &= (lt^{l+k-1}u^2\,dt + t^{l+k}u\,du) \\ &= (lt^{l+k-1}(t^2+4t)\,dt + (t^{l+k+1} + 2t^{l+k})\,dt) \\ &= l(t^{k+l+1} + 4t^{k+l})\,dt + (t^{l+k+1} + 2t^{l+k})\,dt) \\ &= (l+1)t^{k+l+1}\,dt + (4l+2)t^{k+l}\,dt \\ &\equiv ((l+1)\delta_{k+l,-2} + (4l+2)\delta_{k+l,-1})t^{-1}\,dt \quad \text{mod } \mathcal{R}. \end{aligned}$$

This completes the proof of the corollary. □

Theorem 2.4. *The universal central extension of the algebra $\mathfrak{sl}(2,\mathbb{C})\otimes\mathcal{R}$ is isomorphic to the Lie algebra with generators $e_n, e_n^1, f_n, f_n^1, h_n, h_n^1$, $n\in\mathbb{Z}$, ω_0, ω_1, and relations given by*

$$[x_m, x_n] := [x_m, x_n^1] = [x_m^1, x_n^1] = 0 \quad \text{for } x = e, f, \tag{2-8}$$

$$[h_m, h_n] := -2m\delta_{m,-n}\omega_0 = (n-m)\delta_{m,-n}\omega_0, \tag{2-9}$$

$$\begin{aligned}[h_m^1, h_n^1] &:= 2((n+1)\delta_{m+n,-2} + (4n+2)\delta_{m+n,-1})\omega_0 \\ &= (n-m)(\delta_{m+n,-2} + 4\delta_{m+n,-1})\omega_0,\end{aligned} \tag{2-10}$$

$$[h_m, h_n^1] := -2m\delta_{m,-n}\omega_1, \tag{2-11}$$

$$[\omega_i, x_m] := [\omega_i, \omega_j] = 0 \quad \text{for } x = e, f, h \text{ and } i, j\in\{0,1\} \tag{2-12}$$

$$[e_m, f_n] := h_{m+n} - m\delta_{m,-n}\omega_0, \tag{2-13}$$

$$[e_m, f_n^1] := h_{m+n}^1 - m\delta_{m,-n}\omega_1 =: [e_m^1, f_n], \tag{2-14}$$

$$\begin{aligned}[e_m^1, f_n^1] &:= h_{m+n+2} + 4h_{m+n+1} + ((n+1)\delta_{m+n,-2} + (4n+2)\delta_{m+n,-1})\omega_0 \\ &= h_{m+n+2} + 4h_{m+n+1} + \tfrac{1}{2}(n-m)(\delta_{m+n,-2} + 4\delta_{m+n,-1})\omega_0,\end{aligned} \tag{2-15}$$

$$[h_m, e_n] := 2e_{m+n}, \tag{2-16}$$

$$[h_m, e_n^1] := 2e_{m+n}^1 =: [h_m^1, e_m], \tag{2-17}$$

$$[h_m^1, e_n^1] := 2e_{m+n+2} + 8e_{m+n+1}, \tag{2-18}$$

$$[h_m, f_n] := -2f_{m+n}, \tag{2-19}$$

$$[h_m, f_n^1] := -2f_{m+n}^1 =: [h_m^1, f_m], \tag{2-20}$$

$$[h_m^1, f_n^1] := -2f_{m+n+2} - 8f_{m+n+1}, \tag{2-21}$$

for all $m, n\in\mathbb{Z}$.

Proof. Let $\mathfrak{f}$ denote the free Lie algebra with generators $e_n, e_n^1, f_n, f_n^1, h_n, h_n^1$, $n\in\mathbb{Z}$, ω_0, ω_1, and relations given above in (2-8) through (2-21). The map

$$\phi : \mathfrak{f} \to (\mathfrak{sl}(2,\mathbb{C})\otimes\mathcal{R}) \oplus (\Omega_{\mathcal{R}}/d\mathcal{R})$$

given by

$$\begin{aligned}\phi(e_n) &:= e\otimes t^n, & \phi(e_n^1) &= e\otimes ut^n,\\ \phi(f_n) &:= f\otimes t^n, & \phi(f_n^1) &= f\otimes ut^n,\\ \phi(h_n) &:= h\otimes t^n, & \phi(h_n^1) &= h\otimes ut^n,\\ \phi(\omega_0) &:= \overline{t^{-1}\,dt}, & \phi(\omega_1) &= \overline{t^{-1}u\,dt},\end{aligned}$$

for $n\in\mathbb{Z}$, is a surjective Lie algebra homomorphism.

Consider the subalgebras $S_+ = \langle e_n, e_n^1 \mid n\in\mathbb{Z}\rangle$, $S_0 = \langle h_n, h_n^1, \omega_0, \omega_1 \mid n\in\mathbb{Z}\rangle$, and $S_- = \langle f_n, f_n^1 \mid n\in\mathbb{Z}\rangle$, and set $S = S_- + S_0 + S_+$. By (2-8) through (2-12), we have

$$S_+=\sum_{n\in\mathbb{Z}}\mathbb{C}e_n+\sum_{n\in\mathbb{Z}}\mathbb{C}e_n^1,\quad S_-=\sum_{n\in\mathbb{Z}}\mathbb{C}f_n+\sum_{n\in\mathbb{Z}}\mathbb{C}f_n^1,$$
$$S_0=\sum_{n\in\mathbb{Z}}\mathbb{C}h_n+\sum_{n\in\mathbb{Z}}\mathbb{C}h_n^1+\mathbb{C}\omega_0+\mathbb{C}\omega_1.$$

By (2-13) through (2-18), we see that

$$[e_n, S_+]=[e_n^1, S_+]=0,\quad [h_n, S_+]\subseteq S_+,\quad [h_n^1, S_+]\subseteq S_+,$$
$$[f_n, S_+]\subseteq S_0,\quad [f_n^1, S_+]\subseteq S_0,$$

and similarly $[x_n, S_-]=[x_n^1, S_-]\subseteq S$, $[x_n, S_0]=[x_n^1, S_0]\subseteq S$ for $x=e, f, h$. To sum up, we observe that $[x_n, S]\subseteq S$ and $[x_n^1, S]\subseteq S$ for $n\in\mathbb{Z}$, $x=h, e, f$. Thus $[S, S]\subset S$. Now, S contains the generators of $\mathfrak{f}$ and is a subalgebra. Hence $S=\mathfrak{f}$. Now it is clear that ϕ is a Lie algebra isomorphism. □

3. A triangular decomposition of the three-point loop algebras $\mathfrak{g}\otimes R$

From now on we identify R_a with $\mathcal{S}$ and set $R=\mathcal{S}$, which has a basis t^i, t^iu for $i\in\mathbb{Z}$. Let $p: R\to R$ be the automorphism given by $p(t)=t$ and $p(u)=-u$. Then one can decompose $R=R^0\oplus R^1$, where $R^0=\mathbb{C}[t^{\pm1}]=\{r\in R\mid p(r)=r\}$ and $R^1=\mathbb{C}[t^{\pm1}]u=\{r\in R\mid p(r)=-r\}$ are the eigenspaces of p. From now on, $\mathfrak{g}$ will denote a simple Lie algebra over $\mathbb{C}$ with triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, and then the *three-point loop algebra* $L(\mathfrak{g}):=\mathfrak{g}\otimes R$ has a corresponding $\mathbb{Z}/2\mathbb{Z}$-grading: $L(\mathfrak{g})^i:=\mathfrak{g}\otimes R^i$ for $i=0, 1$. However, the degree of t does not render $L(\mathfrak{g})$ a $\mathbb{Z}$-graded Lie algebra. This leads us to the following notion:

Suppose I is an additive subgroup of the rational numbers $\mathbb{Q}$ and $\mathcal{A}$ is a $\mathbb{C}$-algebra such that $\mathcal{A}=\bigoplus_{i\in I}\mathcal{A}_i$, and that there exists a fixed $l\in\mathbb{N}$ with

$$\mathcal{A}_i\mathcal{A}_j\subset\bigoplus_{|k-(i+j)|\le l}\mathcal{A}_k$$

for all $i, j\in\mathbb{Z}$. Then $\mathcal{A}$ is said to be an *l-quasigraded algebra*. For nonzero $x\in\mathcal{A}_i$, one says that x is *homogeneous of degree i* and one writes $\deg x=i$.

For example, R has the structure of a 1-quasigraded algebra, where $I=\frac{1}{2}\mathbb{Z}$ and $\deg t^i=i$, $\deg t^iu=i+\frac{1}{2}$.

A *weak triangular decomposition* of a Lie algebra $\mathfrak{l}$ is a triple $(\mathfrak{H}, \mathfrak{l}_+, \sigma)$ satisfying

(1) $\mathfrak{h}$ and $\mathfrak{l}_+$ are subalgebras of $\mathfrak{l}$,

(2) $\mathfrak{h}$ is abelian and $[\mathfrak{h}, \mathfrak{l}_+]\subset\mathfrak{l}_+$,

(3) σ is an antiautomorphism of $\mathfrak{l}$ of order 2 which is the identity on $\mathfrak{h}$, and

(4) $\mathfrak{l}=\mathfrak{l}_+\oplus\mathfrak{h}\oplus\sigma(\mathfrak{l}_+)$.

We will let $\sigma(\mathfrak{l}_+)$ be denoted by $\mathfrak{l}_-$.

Theorem 3.1 [Bremner 1995, Theorem 2.1]. *The three-point loop algebra $L(\mathfrak{g})$ is a* 1*-quasigraded Lie algebra where* $\deg(x \otimes f) = \deg f$ *for f homogeneous. Set $R_+ = \mathbb{C}(1+u) \oplus \mathbb{C}[t,u]t$ and $R_- = p(R_+)$. Then $L(\mathfrak{g})$ has a weak triangular decomposition given by*

$$L(\mathfrak{g})_\pm = \mathfrak{g} \otimes R_\pm, \quad \mathcal{H} := \mathfrak{h} \otimes \mathbb{C}.$$

Formal distributions. We need some more notation that will simplify some of the arguments later. This notation follows roughly [Kac 1998] and [Matsuo and Nagatomo 1999]: The *formal delta function* $\delta(z/w)$ is the formal distribution

$$\delta(z/w) = z^{-1} \sum_{n\in\mathbb{Z}} z^{-n} w^n = w^{-1} \sum_{n\in\mathbb{Z}} z^n w^{-n}.$$

For any sequence of elements $\{a_m\}_{m\in\mathbb{Z}}$ in the ring $\operatorname{End}(V)$, V a vector space, the formal distribution

$$a(z) := \sum_{m\in\mathbb{Z}} a_m z^{-m-1}$$

is called a *field* if for any $v \in V$, $a_m v = 0$ for $m \gg 0$. If $a(z)$ is a field, then we set

$$a(z)_- := \sum_{m\geq 0} a_m z^{-m-1} \quad \text{and} \quad a(z)_+ := \sum_{m<0} a_m z^{-m-1}. \tag{3-1}$$

The *normal ordered product* of two distributions $a(z)$ and $b(w)$ (and their coefficients) is defined by

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} {:}a_m b_n{:} z^{-m-1} w^{-n-1} = {:}a(z)b(w){:} = a(z)_+ b(w) + b(w) a(z)_-. \tag{3-2}$$

Now we should point out that while ${:}a^1(z_1) \cdots a^m(z_m){:}$ is always defined as a formal series, we will only define ${:}a(z)b(z){:} := \lim_{w\to z} {:}a(z)b(w){:}$ for certain pairs $(a(z), b(w))$.

Then one defines recursively

$${:}a^1(z_1) \cdots a^k(z_k){:} = {:}a^1(z_1)\big({:}a^2(z_2)({:} \cdots {:}a^{k-1}(z_{k-1}) a^k(z_k){:}) \cdots {:}\big){:},$$

while the normal ordered product

$${:}a^1(z) \cdots a^k(z){:} = \lim_{z_1, z_2, \ldots, z_k \to z} {:}a^1(z_1)\big({:}a^2(z_2)({:} \cdots {:}a^{k-1}(z_{k-1}) a^k(z_k){:}) \cdots\big){:}$$

will only be defined for certain k-tuples $(a^1, \ldots, a^k)$.

Let

$$\lfloor ab \rfloor = a(z)b(w) - {:}a(z)b(w){:} = [a(z)_-, b(w)], \tag{3-3}$$

(half of $[a(z), b(w)]$) denote the *contraction* of any two formal distributions $a(z)$ and $b(w)$. Note that the variables z, w are usually suppressed in this notation when no confusion will arise.

For $m = i - \frac{1}{2}$, $i \in \mathbb{Z} + \frac{1}{2}$ and $x \in \mathfrak{g}$, define $x_{m+\frac{1}{2}} = x \otimes t^{i-\frac{1}{2}} u = x^1_m$ and $x_m := x \otimes t^m$. Motivated by conformal field theory, we set

$$x^1(z) := \sum_{m \in \mathbb{Z}} x_{m+\frac{1}{2}} z^{-m-1}, \quad x(z) := \sum_{m \in \mathbb{Z}} x_m z^{-m-1}.$$

Then the relations in Theorem 2.4 can be rewritten as

$$(3\text{-}4) \quad [x(z), y(w)] = [x, y](w)\delta(z/w) - (x, y)\omega_0 \partial_w \delta(z/w),$$

$$(3\text{-}5) \quad [x^1(z), y^1(w)] = P\big([x, y](w)\delta(z/w) - (x, y)\omega_0 \partial_w \delta(z/w)\big) - \tfrac{1}{2}(x, y)(\partial P)\omega_0 \delta(z/w),$$

$$(3\text{-}6) \quad [x(z), y^1(w)] = [x, y]^1(w)\delta(z/w) - (x, y)\omega_1 \partial_w \delta(z/w) = [x^1(z), y(w)],$$

where $x, y \in \{e, f, h\}$.

4. Oscillator algebras

The β-γ system. In the physics literature, the following construction is often called the β-γ system, which corresponds to our a and a^* below. Let $\hat{\mathfrak{a}}$ be the infinite-dimensional oscillator algebra with generators $a_n, a^*_n, a^1_n, a^{1*}_n$, $n \in \mathbb{Z}$ together with $\mathbf{1}$, satisfying the relations

$$[a_n, a_m] = [a_m, a^1_n] = [a_m, a^{1*}_n] = [a^*_n, a^*_m] = [a^*_n, a^1_m] = [a^*_n, a^{1*}_m] = 0,$$
$$[a^1_n, a^1_m] = [a^{1*}_n, a^{1*}_m] = 0 = [\mathfrak{a}, \mathbf{1}],$$
$$[a_n, a^*_m] = \delta_{m+n,0}\mathbf{1} = [a^1_n, a^{1*}_m].$$

For $c = a, a^1$, and respectively $X = x, x^1$, with $r = 0$ or $r = 1$, we define $\mathbb{C}[\boldsymbol{x}] := \mathbb{C}[x_n, x^1_n \mid n \in \mathbb{Z}]$ and $\rho_r : \hat{\mathfrak{a}} \to \mathfrak{gl}(\mathbb{C}[\boldsymbol{x}])$ by

$$(4\text{-}1) \qquad \rho_r(c_m) := \begin{cases} \partial/\partial X_m & \text{if } m \geq 0 \text{ and } r = 0, \\ X_m & \text{otherwise,} \end{cases}$$

$$(4\text{-}2) \qquad \rho_r(c^*_m) := \begin{cases} X_{-m} & \text{if } m \leq 0, \text{ and } r = 0, \\ -\partial/\partial X_{-m} & \text{otherwise,} \end{cases}$$

and $\rho_r(\mathbf{1}) = 1$. These two representations can be constructed using induction: For $r = 0$, the representation ρ_0 is the $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$a_m|0\rangle = a^1_m|0\rangle = 0 \text{ for } m \geq 0 \quad \text{and} \quad a^*_m|0\rangle = a^{1*}_m|0\rangle = 0 \text{ for } m > 0.$$

For $r = 1$, the representation ρ_1 is the $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$a^*_m|0\rangle = a^{1*}_m|0\rangle = 0 \text{ for } m \in \mathbb{Z}.$$

If we define

$$\alpha(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad \alpha^*(z) := \sum_{n\in\mathbb{Z}} a_n^* z^{-n} \tag{4-3}$$

and

$$\alpha^1(z) := \sum_{n\in\mathbb{Z}} a_n^1 z^{-n-1}, \quad \alpha^{1*}(z) := \sum_{n\in\mathbb{Z}} a_n^{1*} z^{-n}, \tag{4-4}$$

then

$$[\alpha(z),\alpha(w)] = [\alpha^*(z),\alpha^*(w)] = [\alpha^1(z),\alpha^1(w)] = [\alpha^{1*}(z),\alpha^{1*}(w)] = 0,$$
$$[\alpha(z),\alpha^*(w)] = [\alpha^1(z),\alpha^{1*}(w)] = \mathbf{1}\delta(z/w).$$

Note that $\rho_1(\alpha(z))$ and $\rho_1(\alpha^1(z))$ are not fields, whereas $\rho_r(\alpha^*(z))$ and $\rho_r(\alpha^{1*}(z))$ are always fields. Corresponding to these two representations there are two possible normal orderings: For $r=0$ we use the usual normal ordering given by (3-1) and for $r=1$ we define the *natural normal ordering* to be

$$\begin{aligned} \alpha(z)_+ &= \alpha(z), & \alpha(z)_- &= 0, \\ \alpha^1(z)_+ &= \alpha^1(z), & \alpha^1(z)_- &= 0, \\ \alpha^*(z)_+ &= 0, & \alpha^*(z)_- &= \alpha^*(z), \\ \alpha^{1*}(z)_+ &= 0, & \alpha^{1*}(z)_- &= \alpha^{1*}(z). \end{aligned}$$

This means in particular that for $r=0$ we get

$$\begin{aligned} \lfloor \alpha\alpha^* \rfloor = \lfloor \alpha(z),\alpha^*(w) \rfloor &= \sum_{m\geq 0} \delta_{m+n,0} z^{-m-1} w^{-n} \\ &= \delta_-(z/w) = \iota_{z,w}\Big(\frac{1}{z-w}\Big), \end{aligned} \tag{4-5}$$

$$\lfloor \alpha^*\alpha \rfloor = -\sum_{m\geq 1} \delta_{m+n,0} z^{-m} w^{-n-1} = -\delta_+(w/z) = \iota_{z,w}\Big(\frac{1}{w-z}\Big), \tag{4-6}$$

(where $\iota_{z,w}$ denotes Taylor series expansion in the "region" $|z| > |w|$), and, for $r=1$,

$$\lfloor \alpha\alpha^* \rfloor = [\alpha(z)_-,\alpha^*(w)] = 0, \tag{4-7}$$

$$\lfloor \alpha^*\alpha \rfloor = [\alpha^*(z)_-,\alpha(w)] = -\sum_{m,n\in\mathbb{Z}} \delta_{m+n,0} z^{-m} w^{-n-1} = -\delta(w/z), \tag{4-8}$$

while similar results hold for $\alpha^1(z)$. Notice that in both cases we have

$$[\alpha(z),\alpha^*(w)] = \lfloor \alpha(z)\alpha^*(w) \rfloor - \lfloor \alpha^*(w)\alpha(z) \rfloor = \delta(z/w).$$

Recall that the singular part of the *operator product expansion*

$$\lfloor ab\rfloor=\sum_{j=0}^{N-1}\iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right)c^j(w)$$

completely determines the bracket of mutually local formal distributions $a(z)$ and $b(w)$. (See Theorem A.3 of the appendix). One writes

$$a(z)b(w)\sim\sum_{j=0}^{N-1}\frac{c^j(w)}{(z-w)^{j+1}}.$$

The three-point Heisenberg algebra. The Cartan subalgebra $\mathfrak{h}$ tensored with $\mathcal{R}$ generates a subalgebra of $\hat{\mathfrak{g}}$ which is an extension of an oscillator algebra. This extension motivates the following definition: The Lie algebra with generators b_m, b^1_m, $m\in\mathbb{Z}$, $\mathbf{1}_0, \mathbf{1}_1$, and relations

(4-9) $$[b_m,b_n]=(n-m)\,\delta_{m+n,0}\,\mathbf{1}_0=-2m\,\delta_{m+n,0}\,\mathbf{1}_0,$$

(4-10) $$\begin{aligned}[b^1_m,b^1_n]&=(n-m)(\delta_{m+n,-2}+4\delta_{m+n,-1})\mathbf{1}_0\\&=2((n+1)\delta_{m+n,-2}+(4n+2)\delta_{m+n,-1})\mathbf{1}_0,\end{aligned}$$

(4-11) $$[b^1_m,b_n]=(n-m)\delta_{m,-n}\mathbf{1}_1=-2m\delta_{m,-n}\mathbf{1}_1,$$

(4-12) $$[b_m,\mathbf{1}_0]=[b^1_m,\mathbf{1}_0]=[b_m,\mathbf{1}_1]=[b^1_m,\mathbf{1}_1]=0.$$

We will give it the appellation the *three-point (affine) Heisenberg algebra*, and denote it by $\hat{\mathfrak{h}}_3$.

If we introduce the formal distributions

(4-13) $$\beta(z):=\sum_{n\in\mathbb{Z}}b_nz^{-n-1},\quad \beta^1(z):=\sum_{n\in\mathbb{Z}}b^1_nz^{-n-1}=\sum_{n\in\mathbb{Z}}b_{n+\frac12}z^{-n-1},$$

(where $b_{n+\frac12}:=b^1_n$), then, using calculations done earlier for the three-point Lie algebra, we can see that the relations above can be rewritten in the form

$$\begin{aligned}[\beta(z),\beta(w)]&=2\mathbf{1}_0\,\partial_z\delta(z/w)=-2\partial_w\delta(z/w)\mathbf{1}_0,\\ [\beta^1(z),\beta^1(w)]&=-2\big((w^2+4w)\partial_w\delta(z/w)+(2+w)\delta(z/w)\big)\mathbf{1}_0,\\ [\beta^1(z),\beta(w)]&=2\partial_z\delta(z/w)\mathbf{1}_1=-2\partial_w\delta(z/w)\mathbf{1}_1.\end{aligned}$$

Set

$$\hat{\mathfrak{h}}_3^{\pm}:=\sum_{n\gtrless0}(\mathbb{C}b_n+\mathbb{C}b^1_n),\quad \hat{\mathfrak{h}}_3^0:=\mathbb{C}\mathbf{1}_0\oplus\mathbb{C}\mathbf{1}_1\oplus\mathbb{C}b_0\oplus\mathbb{C}b^1_0.$$

We introduce a Borel-type subalgebra

$$\hat{\mathfrak{b}}_3=\hat{\mathfrak{h}}_3^+\oplus\hat{\mathfrak{h}}_3^0.$$

From the defining relations above, one can see that $\hat{\mathfrak{b}}_3$ is a subalgebra.

Lemma 4.1. *Let $\mathcal{V} = \mathbb{C}\boldsymbol{v}_0 \oplus \mathbb{C}\boldsymbol{v}_1$ be a two-dimensional representation of $\hat{\mathfrak{h}}_3^+$ with $\hat{\mathfrak{h}}_3^+ \boldsymbol{v}_i = 0$ for $i = 0, 1$. Suppose $\lambda, \mu, \nu, \varkappa, \chi_1, \kappa_0 \in \mathbb{C}$ are such that*

$$\begin{aligned} b_0\boldsymbol{v}_0 &= \lambda \boldsymbol{v}_0, & b_0\boldsymbol{v}_1 &= \lambda \boldsymbol{v}_1, \\ b_0^1\boldsymbol{v}_0 &= \mu\boldsymbol{v}_0 + \nu\boldsymbol{v}_1, & b_0^1\boldsymbol{v}_1 &= \varkappa\boldsymbol{v}_0 + \mu\boldsymbol{v}_1, \\ \mathbf{1}_1\boldsymbol{v}_i &= \chi_1\boldsymbol{v}_i, & \mathbf{1}_0\boldsymbol{v}_i &= \kappa_0\boldsymbol{v}_i \quad \text{for } i = 0, 1. \end{aligned}$$

Then the above equations define a representation of $\hat{\mathfrak{b}}_3$ on $\mathcal{V}$.

Proof. Since b_m acts by scalar multiplication for $m, n \geq 0$, the first defining relation (4-9) is satisfied for $m, n \geq 0$. The second relation (4-10) is also satisfied as the right-hand side is zero if $m, n \geq 0$. If $n = 0$, then since b_0 acts by a scalar, the relation (4-11) leads to no condition on $\lambda, \mu, \nu, \varkappa, \chi_1, \kappa_0 \in \mathfrak{h}_4^0$. If $m \geq 0$ and $n > 0$, the third relation doesn't give us a condition on χ_1 as

$$0 = b_m^1 b_n \boldsymbol{v}_i - b_n b_m^1 \boldsymbol{v}_i = [b_m^1, b_n]\boldsymbol{v}_i = -2\delta_{m,-n} m\chi_1\boldsymbol{v}_i = 0.$$

If $m = n = 0$, the third relation however becomes

$$0 = \lambda b_0^1 \boldsymbol{v}_i - b_0^1 \lambda \boldsymbol{v}_i = b_0^1 b_0 \boldsymbol{v}_i - b_0 b_0^1 \boldsymbol{v}_i = [b_0^1, b_0]\boldsymbol{v}_i = -2\cdot 0\chi_1\boldsymbol{v}_i = 0,$$

so there is no condition on χ_1. □

Let B_0^1 denote the linear transformation on $\mathcal{V}$ that agrees with the action of b_0^1. If we define the notion of a $\hat{\mathfrak{b}}_3$-submodule as is done in [Sheinman 1995, Definition 1.2], then $\mathcal{V}$ above is an irreducible $\hat{\mathfrak{b}}_3$-module when $\varkappa\nu \neq 0$, that is, when $\det B_0^1 \neq \mu^2$. If one induces from $\mathcal{V}$, the resulting representation for the three-point affine algebra cannot be irreducible if $\mathcal{V}$ is not irreducible as a quasigraded module itself.

Let $\mathbb{C}[\boldsymbol{y}] := \mathbb{C}[y_{-n}, y_{-m}^1 \mid m, n \in \mathbb{N}^*]$. The following is a straightforward computation:

Lemma 4.2. *The linear map $\rho : \hat{\mathfrak{b}}_3 \to \mathrm{End}(\mathbb{C}[\boldsymbol{y}] \otimes \mathcal{V})$ defined by*

(4-14) $\rho(b_n) = y_n$ *for $n < 0$,*

(4-15) $\rho(b_n^1) = y_n^1 + \delta_{n,-1}\partial_{y_{-3}^1}\chi_0 - \delta_{n,-3}\partial_{y_{-1}^1}\chi_0$ *for $n < 0$,*

(4-16) $\rho(b_n) = -n(2\partial_{y_{-n}}\chi_0 + 2\partial_{y_{-n}^1}\chi_1)$ *for $n > 0$,*

(4-17) $\rho(b_n^1) = -2n\partial_{y_{-n}}\chi_1 + 2(n+2)\partial_{y_{-n-4}^1}\chi_0 - 4c(n+1)\partial_{y_{-n-2}^1}\chi_0 + 2n\partial_{y_{-n}^1}\chi_0$ *for $n > 0$,*

(4-18) $\rho(b_0) = \lambda$,

(4-19) $\rho(b_0^1) = 4\partial_{y_{-4}^1}\chi_0 - 2c\partial_{y_{-2}^1}\chi_0 + B_0^1$,

is a representation of $\hat{\mathfrak{b}}_3$.

5. Two realizations of the affine three-point algebra $\hat{\mathfrak{g}}$

Assume that $\chi_0\in\mathbb{C}$, and define $\mathcal{V}$ as in Lemma 4.1. The $\alpha(z)$, $\alpha^1(z)$, $\alpha^*(z)$, and $\alpha^{1*}(z)$ are generating series of oscillator algebra elements as in (4-3) and (4-4). Our main result is the following:

Theorem 5.1. *Fix $r\in\{0,1\}$, which then fixes the corresponding normal ordering convention defined in the previous section. Set $\hat{\mathfrak{g}}=(\mathfrak{sl}(2,\mathbb{C})\otimes\mathcal{R})\oplus\mathbb{C}\omega_0\oplus\mathbb{C}\omega_1$. Then, using* (4-1), (4-2) *and Lemma 4.2, the following defines a representation of the three-point algebra $\hat{\mathfrak{g}}$ on $\mathbb{C}[\boldsymbol{x}]\otimes\mathbb{C}[\boldsymbol{y}]\otimes\mathcal{V}$:*

$$\tau(\omega_1)=0,\qquad \tau(\omega_0)=\chi_0=\kappa_0+4\delta_{r,0},$$

$$\tau(f(z))=-\alpha(z),\qquad \tau(f^1(z))=-\alpha^1(z),$$

$$\tau(h(z))=2(:\alpha(z)\alpha^*(z):+:\alpha^1(z)\alpha^{1*}(z):)+\beta,$$

$$\tau(h^1(z))=2(:\alpha^1(z)\alpha^*(z):+(z^2+4z):\alpha(z)\alpha^{1*}(z):)+\beta^1(z),$$

$$\begin{aligned}\tau(e(z))=&:\alpha(z)(\alpha^*(z))^2:+(z^2+4z):\alpha(z)(\alpha^{1*}(z))^2:+2:\alpha^1(z)\alpha^*(z)\alpha^{1*}(z):\\&+\beta(z)\alpha^*(z)+\beta^1(z)\alpha^{1*}(z)+\chi_0\partial\alpha^*(z),\end{aligned}$$

$$\begin{aligned}\tau(e^1(z))=&\,\alpha^1(z)\alpha^*(z)\alpha^*(z)+(z^2+4z)(\alpha^1(z)(\alpha^{1*}(z))^2+2:\alpha(z)\alpha^*(z)\alpha^{1*}(z):)\\&+\beta^1(z)\alpha^*+(z^2+4z)\beta(z)\alpha^{1*}(z)+\chi_0((z^2+4z)\partial_z\alpha^{1*}(z)+(z+2)\alpha^{1*}(z)).\end{aligned}$$

Before we go through the proof, it will be fruitful to review Kac's λ-notation (see [Kac 1998, Section 2.2] and [Wakimoto 2001] for some of its properties), used in operator product expansions. If $a(z)$ and $b(w)$ are formal distributions, then

$$[a(z),b(w)]=\sum_{j=0}^{\infty}\frac{(a_{(j)}b)(w)}{(z-w)^{j+1}}$$

is transformed under the *formal Fourier transform*

$$F^{\lambda}_{z,w}a(z,w)=\operatorname{Res}_z e^{\lambda(z-w)}a(z,w)$$

into the sum

$$[a_\lambda b]=\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}a_{(j)}b.$$

Set

$$P(w)=w^2+4w.$$

So for example we have the following:

Lemma 5.2. *Given the definitions in the previous section, we have*

(1) $[\beta^1{}_\lambda \beta^1] = -(2(w^2+4w)\lambda + (2w+4))\kappa_0 = -(2P\lambda + \partial P)\kappa_0$,

(2) $[:\alpha\alpha^*:_\lambda :\alpha\alpha^*:] = -\delta_{r,0}\lambda$,

(3) $[:\alpha(\alpha^*)^2:_\lambda :\alpha(\alpha^*)^2:] = -4\delta_{r,0}:\alpha^*\partial\alpha^*: - 4\delta_{r,0}:(\alpha^*)^2:\lambda$.

Note that similar expressions hold for $\alpha^1(z)$ and $\alpha^{1*}(z)$ (the λ-notation suppresses the variables z and w, which are understood).

Proof. We'll prove (2) and (3). By Wick's theorem,

$$\begin{aligned}
&:\alpha(z)\alpha^*(z)::\alpha(w)\alpha^*(w): \\
&\qquad = :\alpha(z)\alpha^*(z)\alpha(w)\alpha^*(w): + \lfloor \alpha(z), \alpha^*(w)\rfloor :\alpha(w)\alpha^*(z): \\
&\qquad\quad + \lfloor \alpha^*(z), \alpha(w)\rfloor :\alpha(z)\alpha^*(w): + \lfloor \alpha(z), \alpha^*(w)\rfloor \lfloor \alpha^*(z), \alpha(w)\rfloor \\
&\qquad = :\alpha(z)\alpha(w)\alpha^*(z)\alpha^*(w): + :\alpha(w)\alpha^*(z):\iota_{z,w}\Big(\frac{1}{z-w}\Big) \\
&\qquad\quad + :\alpha(z)\alpha^*(w):\iota_{z,w}\Big(\frac{1}{w-z}\Big) - \delta_{r,0}\iota_{z,w}\Big(\frac{1}{z-w}\Big)^2
\end{aligned}$$

and

$$\begin{aligned}
&[:\alpha(z)\alpha^*(z)^2:, :\alpha(w)\alpha^*(w)^2:] \\
&\quad = 2:\alpha(z)\alpha^*(z)\alpha^*(w)^2:\delta(z/w) - 2:\alpha(w)\alpha^*(z)^2\alpha^*(w):\delta(z/w) \\
&\qquad - 4\delta_{r,0}:\alpha^*(z)\alpha^*(w):\partial_w\delta(z/w) \\
&\quad = -4\delta_{r,0}:\alpha^*(z)\partial_w(\alpha^*(w):\delta(z/w)) + 4\delta_{r,0}:\alpha^*(z)\partial_w\alpha^*(w):\delta(z/w) \\
&\quad = -4\delta_{r,0}:\partial_w(\alpha^*(w)\alpha^*(w):\delta(z/w)) + 4\delta_{r,0}:\alpha^*(w)\partial_w\alpha^*(w):\delta(z/w) \\
&\quad = -4\delta_{r,0}:\partial_w\alpha^*(w)\alpha^*(w):\delta(z/w) - 4\delta_{r,0}:\alpha^*(w)\alpha^*(w):\partial_w\delta(z/w). \qquad \square
\end{aligned}$$

Proof of Theorem 5.1. We need to check the following table is preserved under τ:

$[\cdot_\lambda\cdot]$	$f(w)$	$f^1(w)$	$h(w)$	$h^1(w)$	$e(w)$	$e^1(w)$
$f(z)$	0	0	$*$	$*$	$*$	$*$
$f^1(z)$		0	$*$	$*$	$*$	$*$
$h(z)$			$*$	$*$	$*$	$*$
$h^1(z)$				$*$	$*$	$*$
$e(z)$					0	0
$e^1(z)$						0

Here, $*$ indicates nonzero formal distributions that are obtained from the defining relations (3-4), (3-5), and (3-6). The proof is carried out using Wick's theorem, Taylor's theorem, and Lemma 5.2, as one can see below:

$$
\begin{aligned}
&[\tau(f)_\lambda\tau(f)]=0,\quad [\tau(f)_\lambda\tau(f^1)]=0,\quad [\tau(f^1)_\lambda\tau(f^1)]=0,\\
&[\tau(f)_\lambda\tau(h)]=-[\alpha_\lambda(2(\alpha\alpha^*+\alpha^1\alpha^{1*})+\beta)]=-2\alpha=2\tau(f),\\
&[\tau(f)_\lambda\tau(h^1)]=-[\alpha_\lambda(2(\alpha^1\alpha^*+P\alpha\alpha^{1*})+\beta^1)]=-2\alpha^1=2\tau(f^1),\\
&[\tau(f)_\lambda\tau(e)]=-\big[\alpha_\lambda(:\alpha(\alpha^*)^2:+P:\alpha(\alpha^{1*})^2:\\
&\qquad +2:\alpha^1\alpha^*\alpha^{1*}:+\beta\alpha^*+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*)\big]\\
&\qquad =-2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)-\beta-\chi_0\lambda=-\tau(h)-\chi_0\lambda,\\
&[\tau(f)_\lambda\tau(e^1)]=-\big[\alpha_\lambda(\alpha^1(\alpha^*)^2+P(\alpha^1(\alpha^{1*})^2+2:\alpha\alpha^*\alpha^{1*}:)\\
&\qquad +\beta^1\alpha^*+P\beta\alpha^{1*}+\chi_0(P\partial\alpha^{1*}+\tfrac{1}{2}\partial P\alpha^{1*}))\big]\\
&\qquad =-2(:\alpha^1\alpha^*:+P:\alpha\alpha^{1*}:)-\beta^1=-\tau(h^1),\\
&[\tau(f^1)_\lambda\tau(h)]=-[\alpha^1_\lambda(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)]=-2\alpha^1=2\tau(f^1),\\
&[\tau(f^1)_\lambda\tau(h^1)]=-[\alpha^1_\lambda(2(:\alpha^1\alpha^*:+P:\alpha\alpha^{1*}:)+\beta^1)]=-2P\alpha^1=2P\tau(f^1),\\
&[\tau(f^1)_\lambda\tau(e)]=-[\alpha^1_\lambda(:\alpha(\alpha^*)^2:+P:\alpha(\alpha^{1*})^2:\\
&\qquad +2:\alpha^1\alpha^*\alpha^{1*}:+\beta\alpha^*+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*)]\\
&\qquad =-(2P:\alpha\alpha^{1*}:+2:\alpha^1\alpha^*:+\beta^1)=-\tau(h^1),\\
&[\tau(f^1)_\lambda\tau(e^1)]=-\big[\alpha^1_\lambda(\alpha^1(\alpha^*)^2+P(\alpha^1(\alpha^{1*})^2+2:\alpha\alpha^*\alpha^{1*}:)\\
&\qquad +\beta^1\alpha^*+P\beta\alpha^{1*}+\chi_0(P\partial\alpha^{1*}+\tfrac{1}{2}(\partial P)\alpha^{1*}))\big]\\
&\qquad =-(P(2(:\alpha^1\alpha^{1*}:+:\alpha\alpha^*:)+\beta+\chi_0\lambda)+\tfrac{1}{2}\chi_0\partial P)\\
&\qquad =-(P\tau(h)+P\chi_0\lambda+\chi_0\tfrac{1}{2}\partial P).
\end{aligned}
$$

Note that $:a(z)b(z):$ and $:b(z)a(z):$ are usually not equal, but $:\alpha^1(w)\alpha^{*1}(w):=\,:\alpha^{1*}(w)\alpha^1(w):$ and $:\alpha(w)\alpha^*(w):=\,:\alpha^*(w)\alpha(w):$. Thus, we calculate

$$
\begin{aligned}
[\tau(h)_\lambda\tau(h)]&=[(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)_\lambda(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)]\\
&=4(-:\alpha\alpha^*:+:\alpha^*\alpha:-:\alpha^1\alpha^{1*}:+:\alpha^{1*}\alpha^1:)-8\delta_{r,0}\lambda+[\beta_\lambda\beta]\\
&=-2(4\delta_{r,0}+\kappa_0)\lambda,
\end{aligned}
$$

which can be put into the form of (3-4):

$$
\begin{aligned}
[\tau(h(z)),\tau(h(w))]&=-2(4\delta_{r,0}+\kappa_0)\partial_w\delta(z/w)\\
&=-2\chi_0\partial_w\delta(z/w)=\tau(-2\omega_0\partial_w\delta(z/w)).
\end{aligned}
$$

Next, we calculate

$$[\tau(h)_\lambda\tau(h^1)]=4\big((:\alpha^*\alpha^1:-:\alpha^1\alpha^*:)+P(-:\alpha\alpha^{1*}:+:\alpha^{1*}\alpha:)\big)+[\beta_\lambda\beta^1].$$

Since $[a_n, a_m^{1*}] = [a_n^1, a_m^*] = 0$, we have $[\tau(h(z)), \tau(h^1(w))] = [\beta(z), \beta^1(w)] = 0$. As $\tau(\omega_1) = 0$, relation (3-6) is satisfied.

We continue with

$$\begin{aligned}[\tau(h^1)_\lambda \tau(h^1)] &= [(2(:\alpha^1\alpha^*: + P:\alpha\alpha^{1*}:) + \beta^1)_\lambda (2(:\alpha^1\alpha^*: + P:\alpha\alpha^{1*}:) + \beta^1)] \\ &= -8\delta_{r,0}P\lambda - 4\delta_{r,0}\partial P - 2\kappa_0(P\lambda + \tfrac{1}{2}\partial P),\end{aligned}$$

yielding the relation

$$\begin{aligned}[\tau(h^1(z)), \tau(h^1(w))] &= -2(4\delta_{r,0} + \kappa_0)((w^2+4w)\partial_w\delta(z/w) + (w+2))\delta(z/w)) \\ &= \tau(-(h,h)\omega_0 P\partial_w\delta(z/w) - \tfrac{1}{2}(h,h)\partial P\omega_0\delta(z/w)).\end{aligned}$$

Next we calculate the h paired with the e:

$$\begin{aligned}[\tau(h)_\lambda \tau(e)] &= \big[(2(:\alpha\alpha^*: + :\alpha^1\alpha^{1*}:) + \beta)_\lambda \\ &\qquad (:\alpha(\alpha^*)^2: + P:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta\alpha^* + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*)\big] \\ &= 4:\alpha(\alpha^*)^2: - 2:\alpha(\alpha^*)^2: - 4\delta_{r,0}\alpha^*\lambda \\ &\qquad - 2P:\alpha(\alpha^{1*})^2: + 4:\alpha^*\alpha^1\alpha^{1*}: + 2\alpha^*\beta + 2\chi_0\alpha^*\lambda + 2\chi_0\partial\alpha^* \\ &\qquad\quad + 4P:\alpha(\alpha^{1*})^2: - 4\delta_{r,0}\alpha^*\lambda + 2\beta^1\alpha^{1*} - 2\lambda\alpha^*\kappa_0 \\ &= 2\tau(e)\end{aligned}$$

and

$$\begin{aligned}[\tau(h^1)_\lambda \tau(e)] &= 2:\alpha^1(\alpha^*)^2: + 2P:\alpha^1(\alpha^{1*})^2: + 4P:\alpha\alpha^*\alpha^{1*}: \\ &\qquad + 2\delta(z/w)\alpha^*\beta^1 + 2P\beta\alpha^{1*} + 2P\chi_0\partial\alpha^{1*} + \partial P\alpha^{1*}\chi_0 \\ &= 2\tau(e^1)\end{aligned}$$

Similarly, $[\tau(h)_\lambda \tau(e^1)] = 2\tau(e^1)$ and $[\tau(h^1)_\lambda \tau(e^1)] = 2\tau(e^1)$.

We prove the Serre relation for just one of the relations, $[\tau(e)_\lambda \tau(e^1)]$; the proof of the others, $[\tau(e)_\lambda \tau(e)]$ and $[\tau(e^1)_\lambda \tau(e^1)]$, is similar, as the reader can verify.

After expanding the definitions and collecting terms, we have

$$\begin{aligned}&[\tau(e)_\lambda \tau(e^1)] \\ &\quad = [:\alpha(\alpha^*)^2:_\lambda (:\alpha^1(\alpha^*)^2: + 2P:\alpha\alpha^*\alpha^{1*}: + \beta^1\alpha^*)] \\ &\qquad + [P:\alpha(\alpha^{1*})^2:_\lambda (:\alpha^1(\alpha^*)^2: + P(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}:) + \beta^1\alpha^*)] \\ &\qquad + \big[2:\alpha^1\alpha^*\alpha^{1*}:_\lambda (\alpha^1(\alpha^*)^2 + P(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}:) + P\beta\alpha^{1*} \\ &\qquad\quad + \chi_0((w^2+4w)\partial_w\alpha^{1*} + (w+2)\alpha^{1*}))\big] \\ &\qquad + [\beta\alpha^*{}_\lambda (2P:\alpha\alpha^*\alpha^{1*}: + \beta^1\alpha^* + P\beta\alpha^{1*})] \\ &\qquad + [\beta^1\alpha^{1*}{}_\lambda (:\alpha^1(\alpha^*)^2: + P\alpha^1(\alpha^{1*})^2 + \beta^1\alpha^* + P\beta\alpha^{1*})] \\ &\qquad + [\chi_0\partial\alpha^*{}_\lambda (2P:\alpha\alpha^*\alpha^{1*}:)]\end{aligned}$$

$$
\begin{aligned}
&= 2{:}\alpha^1\alpha^*(\alpha^*)^2{:} + 2P{:}\alpha(\alpha^*)^2\alpha^{1*}{:} - 4P{:}\alpha(\alpha^*)^2\alpha^{1*}{:} - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda \\
&\quad - 4\delta_{r,0}P{:}\partial(\alpha^*)\alpha^{1*}{:} + \beta^1(\alpha^*)^2 - 2P{:}\alpha(\alpha^*)^2\alpha^{1*}{:} + 2P\alpha^1\alpha^*(\alpha^{1*})^2 \\
&\quad - 4\delta_{r,0}P{:}\alpha^{1*}\alpha^*{:}\lambda - 4\delta_{r,0}\partial P{:}\alpha^{1*}\alpha^*{:} - 4\delta_{r,0}P{:}\partial\alpha^{1*}\alpha^*{:} \\
&\quad - 2P^2{:}\alpha(\alpha^{1*})^3{:} + 2P^2{:}\alpha(\alpha^{1*})^3{:} + P\beta^1(\alpha^{1*})^2 \\
&\quad - 2{:}\alpha^1\alpha^*(\alpha^*)^2{:} + 4P{:}\alpha^1\alpha^*(\alpha^{1*})^2{:} - 2P{:}\alpha^1\alpha^*(\alpha^{1*})^2{:} \\
&\quad - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\partial(\alpha^*)\alpha^{1*}{:} \\
&\quad + 4P{:}\alpha(\alpha^*)^2\alpha^{1*}{:} - 4P{:}\alpha^1\alpha^*(\alpha^{1*})^2{:} \\
&\quad - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\alpha^*\partial\alpha^{1*}{:} + 2P\beta{:}\alpha^*\alpha^{1*}{:} \\
&\quad + 2\chi_0(P{:}\partial\alpha^*\alpha^{1*}{:} + P{:}\alpha^*\partial\alpha^{1*}{:} + P{:}\alpha^*\alpha^{1*}{:}\lambda + \tfrac{1}{2}(\partial P){:}\alpha^*\alpha^{1*}{:}) \\
&\quad - 2P\beta\alpha^*\alpha^{1*} - 2\kappa_0P\alpha^*\alpha^{1*}\lambda - 2\kappa_0P\partial\alpha^*\alpha^{1*} \\
&\quad - \beta^1(\alpha^*)^2 - P\beta^1(\alpha^{1*})^2 - \kappa_0(2P\alpha^*\alpha^{1*}\lambda + 2P\alpha^*\partial\alpha^{1*} + \partial P\alpha^*\alpha^{1*}) \\
&\quad + 2\chi_0P\alpha^*\alpha^{1*}\lambda \\
&= -4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\partial(\alpha^*)\alpha^{1*}{:} + \chi_1(2{:}\alpha^*\partial\alpha^*{:} + {:}(\alpha^*)^2{:}\lambda) \\
&\quad - 4\delta_{r,0}P{:}\alpha^{1*}\alpha^*{:}\lambda - 4\delta_{r,0}\partial P{:}\alpha^{1*}\alpha^*{:} - 4\delta_{r,0}P{:}\partial\alpha^{1*}\alpha^*{:} \\
&\quad - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\partial(\alpha^*)\alpha^{1*}{:} - 4\delta_{r,0}P{:}\alpha^*\alpha^{1*}{:}\lambda - 4\delta_{r,0}P{:}\alpha^*\partial\alpha^{1*}{:} \\
&\quad + 2\chi_0(P{:}\partial\alpha^*\alpha^{1*}{:} + P{:}\alpha^*\partial\alpha^{1*}{:} + P{:}\alpha^*\alpha^{1*}{:}\lambda + \tfrac{1}{2}(\partial P){:}\alpha^*\alpha^{1*}{:}) \\
&\quad - \kappa_0P\alpha^*\alpha^{1*}\lambda - \kappa_0P\partial\alpha^*\alpha^{1*} \\
&\quad - \kappa_0(2P\alpha^*\alpha^{1*}\lambda + 2P\alpha^*\partial\alpha^{1*} + \partial P\alpha^*\alpha^{1*}) \\
&\quad - 2\chi_0P\alpha^*\alpha^{1*}\lambda \\
&= 0. \qquad \square
\end{aligned}
$$

6. Further comments

We plan to use the above construction to help elucidate the structure of these representations of a three-point algebra, describe the space of their intertwining operators, and eventually describe the center of a certain completion of the universal enveloping algebra for the three-point algebra.

Appendix

For the convenience of the reader we include the following results, which are useful for performing the computations necessary for proving our results:

Theorem A.1 (Wick's theorem [Bogoliubov and Shirkov 1980; Huang 1998; Kac 1998]). *Let $a^i(z)$ and $b^j(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\boldsymbol{x}]\otimes\mathbb{C}[\boldsymbol{y}])$, *satisfying:*

(1) $[\lfloor a^i(z)b^j(w)\rfloor, c^k(x)_\pm] = [\lfloor a^ib^j\rfloor, c^k(x)_\pm] = 0$ *for all* i, j, k *and* $c^k(x) = a^k(z)$ *or* $c^k(x) = b^k(w)$.

(2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ *for all* i *and* j.

(3) *The products*

$$\lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{i_s} \rfloor \, : a^1(z) \cdots a^M(z) b^1(w) \cdots b^N(w) :_{(i_1,\ldots,i_s;\, j_1,\ldots,j_s)}$$

have coefficients in $\operatorname{End}(\mathbb{C}[\boldsymbol{x}] \otimes \mathbb{C}[\boldsymbol{y}])$ *for all subsets* $\{i_1, \ldots, i_s\} \subset \{1, \ldots, M\}$, $\{j_1, \ldots, j_s\} \subset \{1, \cdots N\}$. *Here, the subscript* $(i_1, \ldots, i_s;\, j_1, \ldots, j_s)$ *means that those factors* $a^i(z)$, $b^j(w)$ *with indices* $i \in \{i_1, \ldots, i_s\}$, $j \in \{j_1, \ldots, j_s\}$ *are to be omitted from the product* $: a^1 \cdots a^M b^1 \cdots b^N :$, *and when* $s = 0$ *we do not omit any factors.*

Then

$$: a^1(z) \cdots a^M(z) :: b^1(w) \cdots b^N(w) :$$
$$= \sum_{s=0}^{\min(M,N)} \sum_{\substack{i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{j_s} \rfloor : a^1(z) \cdots a^M(z) b^1(w) \cdots b^N(w) :_{(i_1,\ldots,i_s;\, j_1,\ldots,j_s)}.$$

Theorem A.2 (Taylor's theorem [Kac 1998, Theorem 2.4.3]). *Let* $a(z)$ *be a formal distribution. Then, in the region* $|z - w| < |w|$,

$$a(z) = \sum_{j=0}^{\infty} \partial_w^{(j)} a(w)(z-w)^j. \tag{A-1}$$

Theorem A.3 [Kac 1998, Theorem 2.3.2]. *Set* $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_n, x_n^1 \mid n \in \mathbb{Z}]$ *and* $\mathbb{C}[\boldsymbol{y}] = C[y_m, y_m^1 \mid m \in \mathbb{N}^*]$. *Let* $a(z)$ *and* $b(z)$ *be formal distributions with coefficients in the associative algebra* $\operatorname{End}(\mathbb{C}[\boldsymbol{x}] \otimes \mathbb{C}[\boldsymbol{y}])$, *where we are using the usual normal ordering. The following are equivalent*:

(i)
$$[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w),$$

where $c^j(w) \in \operatorname{End}(\mathbb{C}[\boldsymbol{x}] \otimes \mathbb{C}[\boldsymbol{y}])[\![w, w^{-1}]\!]$.

(ii)
$$\lfloor ab \rfloor = \sum_{j=0}^{N-1} \iota_{z,w} \left(\frac{1}{(z-w)^{j+1}} \right) c^j(w).$$

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