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MULTI-BUMP BOUND STATE SOLUTIONS FOR THE QUASILINEAR SCHRÖDINGER EQUATION WITH CRITICAL FREQUENCY

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We study the existence of single- and multi-bump solutions of quasilinear Schrödinger equations

$$-\Delta u + \lambda V(x)u - \tfrac{1}{2}(\Delta|u|^2)u = |u|^{p-2}u,$$

the function V being a critical frequency in the sense that $\inf_{x\in\mathbb{R}^N} V(x) = 0$. We show that if the zero set of V has several isolated connected components $\Omega_1, \dots, \Omega_k$ such that the interior of Ω_i is not empty and $\partial\Omega_i$ is smooth, then for $\lambda > 0$ large, there exists, for any nonempty subset $J \subset \{1, 2, \dots, k\}$, a standing wave solution trapped in a neighborhood of $\bigcup_{j\in J} \Omega_j$.

1. Introduction and main results

Consider the following quasilinear Schrödinger equation:

$$-\Delta u + \lambda V(x)u - \tfrac{1}{2}(\Delta|u|^2)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1-1}$$

where $N \geq 3$, $\lambda > 0$ is a parameter, $4 < p < 2\cdot 2^*$, and 2^* is the critical Sobolev exponent.

We are interested in the ground state solutions for (1-1), i.e., the positive solutions with least energy. Solutions of this type are related to the existence of standing wave solutions for the following quasilinear Schrödinger equation:

$$i\partial_t w = -\hbar^2\Delta w + V(x)w - f(|w|^2)w - k\Delta h(|w|^2)h'(|w|^2)w \quad \text{in } \mathbb{R}^N, \tag{1-2}$$

where V is a given potential, $\hbar$ is the Planck constant, k is a real constant, and f, h are real functions. Such quasilinear equations appear naturally in mathematical physics, and have been derived as models of several physical phenomena corresponding to various types of h (see, for example, [Brizhik et al. 2003; Brihaye and Hartmann 2006; Brüll and Lange 1986; Hartmann and Zakrzewski 2003; Kurihura 1981], and the references therein).

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Due to its significant application in mathematical physics, the equation (1-2) with $k = 0$ (the semilinear case) has attracted much attention in recent years. Many authors have obtained existence results for one-bump or multi-bump bound state solutions under different assumptions on the potential function V. We refer the readers to [Ambrosetti et al. 1997; Ambrosetti et al. 2001; Bartsch and Wang 2000; Cingolani and Lazzo 2000; Cingolani and Nolasco 1998; del Pino and Felmer 1998; del Pino and Felmer 1997], and the references therein.

In the quasilinear case (that is, the equation (1-2) with $k \neq 0$) we observe that, due to the presence of the quasilinear term, there is a different critical exponent than in the semilinear case, as observed in [Liu et al. 2003]; the number $q = 2 \cdot 2^* = 4N/(N-2)$ behaves as a critical exponent for the quasilinear equation. There has been much recent work concerned with the quasilinear Schrödinger equations (1-1) and (1-2). For instance, in [Colin 2003], a change of variables was used to prove the existence of soliton wave solutions; see also the paper by Liu, Wang and Wang [2003], where a change of variables was also used. In [Colin and Jeanjean 2004], various existence results for standing wave solutions to (1-1) for special f and h are obtained. For the stability and instability results for a special case of (1-2), we also refer the reader to [Colin et al. 2010].

For more recent related work on the quasilinear Schrödinger equation with critical exponents, we refer the reader to, for instance, [Liu et al. 2013; 2012; do Ó et al. 2010a; 2010b, Lins and Silva 2009], and to the references therein.

The current paper is concerned with the existence of one-bump or multi-bump bound states for the following quasilinear equation with frequency V:

$$-\Delta u + \lambda V(x)u - \tfrac{1}{2}(\Delta |u|^2)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Our hypotheses on V are:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $V(x) \geq 0$ and $\liminf_{|x|\to\infty} V(x) > 0$;

(V_2) $\Omega := \operatorname{int} V^{-1}(0)$ is nonempty, bounded, has smooth boundary, and $\overline{\Omega} = V^{-1}(0)$;

(V_3) Ω consists of k components:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k,$$

and $\overline{\Omega}_i \cap \overline{\Omega}_j = \varnothing$ for all $i \neq j$.

For the proof of the main theorem, we follow the idea of Y. Ding and K. Tanaka [2003] to modify the nonlinearity and use the decay flow. We point out that, although this idea has been used before to deal with other problems, it is not at all trivial to adapt the procedure for our problem. The appearance of the quasilinear term $\Delta(|u|^2)u$ forces us to consider our problem in an Orlicz space, and more delicate estimates are also needed.

To state the main results, we first introduce some necessary notation. We denote $\lambda V(x)$ by $V_\lambda(x)$. Formally, we define the functional J_λ by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1+u^2)|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda(x)u^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,dx, \tag{1-3}$$

where $u \in X := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V_\lambda(x)u^2 < \infty\}$. Note that, under our assumptions, the functional J_λ is not well defined on X. We make the following change of variables, which was first used by Liu, Wang, and Wang [2003].

Let $v = h(u) = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2})$, so $dv = \sqrt{1+u^2}du$. Moreover, $h(u)$ satisfies

$$h(u) \sim \begin{cases} u & \text{if } |u| \ll 1, \\ \frac{1}{2}u|u| & \text{if } |u| \gg 1. \end{cases} \tag{1-4}$$

Since $h'(u) > 0$, $h(u)$ is strictly monotone and hence has an inverse function denoted by $u = f(v)$. Obviously,

$$f(v) \sim \begin{cases} v & \text{if } |u| \ll 1, \\ \sqrt{2/|v|}\,v & \text{if } |v| \gg 1, \end{cases} \qquad f'(v) = \frac{1}{\sqrt{1+f^2(v)}}. \tag{1-5}$$

Let $G(v) = f^2(v)$. Then

$$G(v) = f^2(v) \sim \begin{cases} v^2 & \text{if } |v| \ll 1, \\ 2|v| & \text{if } |v| \gg 1, \end{cases} \tag{1-6}$$

and $G(v)$ is convex, so there exists $C_0 > 0$ such that $G(2v) \le C_0 G(v)$,

$$G'(v) = \frac{2f(v)}{\sqrt{1+f^2(v)}}, \qquad G''(v) = \frac{2}{(1+f^2(v))^2} > 0. \tag{1-7}$$

Now we introduce the Orlicz space (see [Rao and Ren 1991])

$$E_G^\lambda = \left\{ v \;\middle|\; \int_{\mathbb{R}^N} V_\lambda G(v) < +\infty \right\}$$

equipped with the norm

$$|v|_G^\lambda := \inf_{\xi>0} \xi\left(1 + \int_{\mathbb{R}^N} V_\lambda G(\xi^{-1}v)\,dx\right).$$

Then E_G^λ is a Banach space (see [Liu et al. 2003]).

Let

$$H_G^\lambda := \left\{ v \in E_G^\lambda \;\middle|\; \int_{\mathbb{R}^N} |\nabla v|^2\,dx < \infty \right\},$$

equipped with the norm

$$\|v\|_\lambda = \|\nabla v\|_{L^2} + |v|_G^\lambda.$$

Using the change of variable, we define the functional Φ_λ on H_G^λ by

$$\Phi_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + V_\lambda f^2(v))\,dx - \frac{1}{p}\int_{\mathbb{R}^N}|f(v)|^p\,dx. \tag{1-8}$$

Then Φ_λ is Gâteaux differentiable, and the Gâteaux derivative $\Phi'_\lambda(v)$ has the form

$$\langle \Phi'_\lambda(v), w\rangle = \int_{\mathbb{R}^N}\nabla v\nabla w\,dx + \int_{\mathbb{R}^N}V_\lambda(x)f(v)f'(v)w\,dx - \int_{\mathbb{R}^N}|f(v)|^{p-2}f(v)f'(v)w\,dx. \tag{1-9}$$

Obviously, $v \in H_G^\lambda$ is a critical point of Φ_λ if and only if v is a solution of the following equation:

$$-\Delta v + V_\lambda f(v)f'(v) = |f(v)|^{p-2}f(v)f'(v), \quad x \in \mathbb{R}^N. \tag{1-10}$$

Moreover, one can easily check that v is solution of (1-10) if and only if $u = f(v)$ is a solution of (1-1).

We define the Nehari manifold N_λ by $N_\lambda = \{v \in H_G^\lambda \setminus \{0\} \mid \langle \Phi'_\lambda(v), v\rangle = 0\}$, and let

$$c_\lambda = \inf_{v\in N_\lambda}\Phi_\lambda(v).$$

We say that $u = f(v)$ is a least energy solution of (1-1) if $v \in N_\lambda$ is such that c_λ is achieved.

Note that under our assumptions, for λ large enough, the following Dirichlet problem is a kind of *limit* problem:

$$\begin{cases} -\Delta u - \frac{1}{2}(\Delta|u|^2)u = |u|^{p-2}u, \ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1-11}$$

where $\Omega = \mathrm{int}\{V^{-1}(0)\}$.

In fact, by a minor change of the arguments in Guo and Tang [2012], one can easily see that under the conditions (V_1), (V_2), and $4 < p < 2\cdot 2^*$, for λ large, c_λ is achieved by a critical point v_λ of Φ_λ such that $u_\lambda = f(v_\lambda)$ is a solution of (1-1). Furthermore, for any sequence $\lambda_n \to +\infty$, $\{v_{\lambda_n}\}$ has a subsequence converging to v such that $u = f(v)$ is a least energy solution of (1-11). Thus by assumption (V_3), there is Ω_{i_0} $(1 \le i_0 \le k)$ such that $u = f(v)$ is indeed a least energy solution defined on Ω_{i_0} and $u = f(v) = 0$ elsewhere. Thus it is natural to ask whether, for a given $j \in \{1, 2, \dots, k\}$, (1-1) has a family of solutions $\{u_\lambda\}$ which converges to a least energy solution in Ω_j and to 0 elsewhere. In this paper, we answer this question in the affirmative. Moreover, we can also construct multi-bump type solutions.

Our main results are:

Theorem 1.1. *Suppose* (V_1)–(V_3) *hold. Then for any* $\varepsilon > 0$ *and any nonempty subset* J *of* $\{1, 2, \dots, k\}$, *there exists* $\Lambda = \Lambda(\varepsilon) > 0$ *such that, for* $\lambda \geq \Lambda$, (1-1) *has a solution* u_λ *such that* $v_\lambda = h(u_\lambda)$ *satisfies*

$$\left|\Phi_\lambda(v_\lambda) - \sum_{j\in J} c(\Omega_j)\right| \leq \varepsilon, \tag{1-12}$$

$$\int_{\mathbb{R}^N\setminus\Omega_J} (|\nabla v_\lambda|^2 + V_\lambda f^2(v_\lambda))\, dx \leq \varepsilon, \tag{1-13}$$

where $\Omega_J = \bigcup_{j\in J} \Omega_j$. *Moreover, for any sequence* $\lambda_n \to \infty$, *we can extract a subsequence* $\{\lambda_{n_i}\}$ *such that* $v_{\lambda_{n_i}}$ *converges strongly in* H^1_G *to a function* v *that satisfies* $v(x) = 0$ *for* $x \notin \Omega_J$, *and* $u = f(v)|_{\Omega_j}$ *is a least energy solution of*

$$\begin{cases} -\Delta u - \frac{1}{2}(\Delta |u|^2)u = |u|^{p-2}u, \ u > 0 \quad \text{in } \Omega_j, \\ u = 0 \quad \text{in } \partial\Omega_j, \end{cases} \tag{1-14}$$

for $j \in J$. *Here* $c(\Omega_j)$ *in* (1-12) *is the least energy of* (1-14).

Corollary 1.2. *Under the same assumptions as in Theorem 1.1, there exists* $\Lambda > 0$ *such that for* $\lambda > \Lambda$, (1-1) *has at least* $2^k - 1$ *bound states.*

The paper is organized as follows. In Section 2, we give some estimates in Orlicz space. In Section 3, we modify the functional by penalizing the nonlinearity. In Section 4, we consider compactness for the modified functional. In Section 5, we give some asymptotic properties for some sequences and prove that, for λ large, the critical points of the modified functional are indeed critical points of the original one. Section 6 is devoted to the properties of the limit problem. In Section 7, we give a minimax argument. In Section 8, we prove the existence of critical points by a flow argument; the proofs of the main results are also delivered in this section.

In the following, without specific notification, all the integral variables are x, and for simplicity we omit dx in every integral.

2. Some estimates in the Orlicz space

We begin with a precise estimate between the Orlicz norm and some integrals in Orlicz space H^λ_G, namely:

Lemma 2.1 [Guo and Tang 2012]. *There exist constants* $C_1, C_2 > 0$ *such that, for any* $v \in H^\lambda_G$,

$$C_1 \min\{\|v\|_\lambda, \|v\|_\lambda^2\} \leq \int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V_\lambda f^2(v) \leq C_2 \max\{\|v\|_\lambda, \|v\|_\lambda^2\}. \tag{2-1}$$

Let Ω'_j $(1 \leq j \leq k)$ be bounded open subsets with smooth boundary such that

$\overline{\Omega'_i}$ and $\overline{\Omega'_j}$ are disjoint if $i \neq j$ and that $\overline{\Omega}_j \subset \Omega'_j$ for all j. Let K be one of the following sets:

$$\mathbb{R}^N, \quad \Omega'_j \ (j = 1, 2, \dots, k), \quad \text{or} \quad \mathbb{R}^N \setminus \bigcup_{j \in J} \Omega'_j \ (J \subset \{1, 2, \dots, k\}). \tag{2-2}$$

Lemma 2.2. *There exist* $\delta_0 > 0$, $\nu_0 > 0$ *such that, for* $\lambda \geq 1$,

$$\delta_0 \int_K \big(|\nabla v|^2 + V_\lambda f^2(v)\big) \leq \int_K \big(|\nabla v|^2 + V_\lambda f^2(v)\big) - \nu_0 \int_K f^2(v). \tag{2-3}$$

Proof. We follow similar arguments as in the proof of Proposition 3.1 in [Tang 2008], but with necessary modifications. We omit it. □

3. Penalization of the functional

To proceed, we introduce the cut-off function $l(t) : \mathbb{R} \to \mathbb{R}$ defined by

$$l(t) = \begin{cases} \min\{t^{(p-2)/2}, \nu_0\} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}$$

where ν_0 is as in Lemma 2.2. For a fixed nonempty subset $J \subset \{1, 2, \dots, k\}$, set

$$\Omega_J = \bigcup_{j \in J} \Omega_j, \quad \Omega'_J = \bigcup_{j \in J} \Omega'_j, \quad \chi_{\Omega'_J}(x) = \begin{cases} 1 & \text{for } x \in \Omega'_J, \\ 0 & \text{for } x \notin \Omega'_J, \end{cases}$$

and

$$w(x, \xi^2) = \chi_{\Omega'_J}(x)\xi^{p-2} + (1 - \chi_{\Omega'_J}(x)) l(\xi^2),$$
$$W(x, \xi^2) = \int_0^{\xi^2} w(x, t)\, dt.$$

We define $\Psi_\lambda : H_G^\lambda \to \mathbb{R}$ by

$$\Psi_\lambda(v) = \tfrac{1}{2} \int_{\mathbb{R}^N} \big(|\nabla v|^2 + V_\lambda f^2(v)\big) - \tfrac{1}{2} \int_{\mathbb{R}^N} W(x, f^2(v)).$$

Then one can check that $\Psi_\lambda \in C^2(H_G^\lambda, \mathbb{R})$ and that its critical points are solutions of

$$-\Delta v + V_\lambda f(v) f'(v) = w(x, f^2(v)) f(v) f'(v) \quad \text{in } \mathbb{R}^N.$$

Note that $l(t) = t^{(p-2)/2}$ for $t \in \big[0, \nu_0^{2/(p-2)}\big]$, hence a critical point v of Ψ_λ is a solution of (1-10) if and only if $|f(v)|^2 \leq \nu_0^{2/(p-2)}$ in $\mathbb{R}^N \backslash \Omega'_J$.

4. Compactness of the modified functional

Proposition 4.1. *For $\lambda \geq 1$, Ψ_λ satisfies the* $(\mathrm{PS})_c$ *condition for all $c \in \mathbb{R}$. That is, any sequence $\{v_n\} \subset H_G^\lambda$ satisfying*

$$\Psi_\lambda(v_n) \to c, \tag{4-1}$$

$$\Psi_\lambda'(v_n) \to 0 \text{ strongly in } (H_G^\lambda)^*, \tag{4-2}$$

has a strongly convergent subsequence in H_G^λ, where $(H_G^\lambda)^*$ is the dual space of H_G^λ.

To prove Proposition 4.1, we require the following lemma:

Lemma 4.2. *Suppose that $\{v_n\} \subset H_G^\lambda$ is a* $(\mathrm{PS})_c$ *sequence. Then there exist two positive constants, $m(c)$ and $M(c)$, which are independent of $\lambda \geq 1$, such that*

$$m(c) \leq \liminf_{n\to\infty} \|v_n\|_\lambda^2 \leq \limsup_{n\to\infty} \|v_n\|_\lambda^2 \leq M(c).$$

Proof. Let $w_n = f(v_n)/f'(v_n)$. It follows from (4-1) and (4-2) that

$$\Psi_\lambda(v_n) - \frac{1}{p}\Psi_\lambda'(v_n)w_n = c + o(1) + \varepsilon_n\|w_n\|_\lambda,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Thus

$$\int_{\mathbb{R}^N}\Big(\frac{1}{2} - \frac{1}{p}\Big(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\Big)\Big)|\nabla v_n|^2 + \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^N} V_\lambda f^2(v_n)$$
$$-\frac{1}{2}\int_{\mathbb{R}^N} W(x, f^2(v_n)) + \frac{1}{p}\int_{\mathbb{R}^N} w(x, f^2(v_n)) f^2(v_n)$$
$$= c + o(1) + \varepsilon_n\|u_n\|_\lambda.$$

Let $L(t) = \int_0^t l(t)\,dt$; we have

$$\frac{1}{2}\int_{\mathbb{R}^N} W(x, f^2(v_n)) - \frac{1}{p}\int_{\mathbb{R}^N} w(x, f^2(v_n)) f^2(v_n)$$
$$= \int_{\mathbb{R}^N\setminus\Omega_J'}\Big(\frac{1}{2}L(f^2(v_n)) - \frac{1}{p}l(f^2(v_n))f^2(v_n)\Big).$$

Note that for $t \in \big[\nu_0^{2/(p-2)}, \infty\big)$,

$$\frac{1}{2}L(t^2) - \frac{1}{p}l(t^2)t^2 = \frac{1}{2}\Big(\nu_0 t^2 - \frac{p-2}{p}\nu_0^{p/(p-2)}\Big) - \frac{1}{p}t^2$$
$$= \Big(\frac{1}{2} - \frac{1}{p}\Big)\big(\nu_0 t^2 - \nu_0^{p/(p-2)}\big) \leq \Big(\frac{1}{2} - \frac{1}{p}\Big)\nu_0 t^2,$$

and for $t \leq \nu_0^{2/(p-2)}$,

$$\frac{1}{2}L(t^2) - \frac{1}{p}l(t^2)t^2 = 0.$$

We obtain that

$$\Big(\frac{1}{2}-\frac{2}{p}\Big)\int_{\mathbb{R}^N}|\nabla v_n|^2+\Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\mathbb{R}^N}V_\lambda f^2(v_n)-\Big(\frac{1}{2}-\frac{1}{p}\Big)\nu_0\int_{\mathbb{R}^N}f^2(v_n)$$
$$\le c+o(1)+\varepsilon_n\|v_n\|_\lambda.$$

Since $4<p<4N/(N-2)$, we have

$$\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V_\lambda f^2(v_n)-\nu_0\int_{\mathbb{R}^N}f^2(v_n)\le\Big(\frac{p-4}{2p}\Big)^{-1}c+o(1)+o(\|v_n\|_\lambda).$$

By Lemma 2.2, we get

$$\delta_0\int_{\mathbb{R}^N}\big(|\nabla v_n|^2\,dx+V_\lambda f^2(v_n)\big)\le\Big(\frac{p-4}{2p}\Big)^{-1}c+o(1)+\varepsilon_n\|v_n\|_\lambda.$$

It follows from Lemma 2.1 that

$$C_1\min\{\|v_n\|_\lambda,\,\|v_n\|_\lambda^2\}\le\delta_0^{-1}\Big(\frac{p-4}{2p}\Big)^{-1}c+o(1)+o(\|v_n\|_\lambda).$$

Thus $\|v_n\|_\lambda$ is bounded as $n\to\infty$, and

$$\limsup_{n\to\infty}\|v_n\|_\lambda\le M(c):=\max\bigg\{\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}\delta_0^{-1}c,\sqrt{\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}\delta_0^{-1}c}\bigg\}.$$

On the other hand, since

$$\frac{1}{2}L(t^2)-\frac{1}{p}l(t^2)t^2\ge0\quad\text{for all }\ t\in\mathbb{R},$$

we have

$$c+o(1)+\varepsilon_n\|w_n\|_\lambda\le\Big(\frac{1}{2}-\frac{1}{p}\Big)C_2\max\{\|v_n\|_\lambda,\,\|v_n\|_\lambda^2\}.$$

Therefore

$$\liminf_{n\to\infty}\|v_n\|_\lambda^2\ge m(c):=\min\bigg\{\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}C_2^{-1}c,\sqrt{\Big(\frac{1}{2}-\frac{1}{p}\Big)^{-1}C_2^{-1}c}\bigg\}.$$

This completes the proof of Lemma 4.2. □

Proof of Proposition 4.1. By Lemma 4.2, we know that $\{v_n\}$ is bounded in H_G^λ and thus is bounded in $D^{1,2}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, so there exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that:

$$\nabla v_n\rightharpoonup\nabla v\quad\text{weakly in }L^2(\mathbb{R}^N),$$
$$v_n\to v\quad\text{a.e. in }\mathbb{R}^N,$$
$$f(v_n)\rightharpoonup f(v)\quad\text{weakly in }L^q(\mathbb{R}^N)\text{ for }2\le q\le2\cdot2^*,$$
$$f(v_n)\to f(v)\quad\text{strongly in }L^p_{\rm loc}(\mathbb{R}^N).$$

Moreover, by Proposition 2.8 of [Guo and Tang 2012], v is a critical point of Ψ_λ, that is, for any $\psi \in H_G^\lambda$,

$$\int_{\mathbb{R}^N} \big(\nabla v \nabla \psi + V_\lambda f(v) f'(v)\psi\big) = \int_{\mathbb{R}^N} w(x, f^2(v)) f(v) f'(v)\psi.$$

Next we show that $v_n \to v$ strongly in H_G^λ. Indeed, it follows from (4-1) and (4-2) that

$$\begin{aligned}
o(1) &= (\Psi_\lambda'(v_n) - \Psi_\lambda'(v))\left(\frac{f(v_n)}{f'(v_n)} - \frac{f(v)}{f'(v)}\right)\\
&= \Psi_\lambda'(v_n)\frac{f(v_n)}{f'(v_n)} - \Psi_\lambda'(v_n)\frac{f(v)}{f'(v)} - \Psi_\lambda'(v)\frac{f(v_n)}{f'(v_n)} - \Psi_\lambda'(v)\frac{f(v)}{f'(v)}\\
&= \int_{\mathbb{R}^N}\left(1+\frac{f^2(v_n)}{1+f^2(v_n)}\right)|\nabla v_n|^2 + \int_{\mathbb{R}^N} V_\lambda f^2(v_n) - \int_{\mathbb{R}^N} w(x, f^2(v_n)) f^2(v_n)\\
&\quad - \int_{\mathbb{R}^N}\left(1+\frac{f^2(v)}{1+f^2(v)}\right)\nabla v_n \nabla v - \int_{\mathbb{R}^N} V_\lambda \frac{f(v_n)}{\sqrt{1+f^2(v_n)}} f(v)\sqrt{1+f^2(v)}\\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v_n)) f(v_n) f'(v_n)\frac{f(v)}{f'(v)} - \int_{\mathbb{R}^N}\left(1+\frac{f^2(v_n)}{1+f^2(v_n)}\right)\nabla v_n \nabla v\\
&\quad - \int_{\mathbb{R}^N} V_\lambda \frac{f(v)}{\sqrt{1+f^2(v)}} f(v_n)\sqrt{1+f^2(v_n)}\\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v)) f(v) f'(v)\frac{f(v_n)}{f'(v_n)}\\
&\quad + \int_{\mathbb{R}^N}\left(1+\frac{f^2(v)}{1+f^2(v)}\right)|\nabla v|^2 + \int_{\mathbb{R}^N} V_\lambda f^2(v) - \int_{\mathbb{R}^N} w(x, f^2(v)) f^2(v)\\
&= \int_{\mathbb{R}^N}\left(1+\frac{f^2(v_n)}{1+f^2(v_n)}\right)(\nabla v_n - \nabla v)^2\\
&\quad + \int_{\mathbb{R}^N}\left(\frac{f^2(v_n)}{1+f^2(v_n)} - \frac{f^2(v)}{1+f^2(v)}\right)\nabla v(\nabla v_n - \nabla v) && \text{(I)}\\
&\quad + \int_{\mathbb{R}^N} V_\lambda\left(f^2(v_n) - \frac{f(v)}{\sqrt{1+f^2(v)}} f(v_n)\sqrt{1+f^2(v_n)}\right) && \text{(II)}\\
&\quad + \int_{\mathbb{R}^N} V_\lambda\left(f^2(v) - \frac{f(v_n)}{\sqrt{1+f^2(v_n)}} f(v)\sqrt{1+f^2(v)}\right) && \text{(III)}\\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v_n))\left(f(v_n) f'(v_n)\frac{f(v)}{f'(v)} - f^2(v_n)\right) && \text{(IV)}\\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v))\left(f(v) f'(v)\frac{f(v_n)}{f'(v_n)} - f^2(v)\right) && \text{(V)}\\
&=: \int_{\mathbb{R}^N}\left(1+\frac{f^2(v_n)}{1+f^2(v_n)}\right)(\nabla v_n - \nabla v)^2 + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.
\end{aligned}$$

In the following we shall estimate the above terms one by one. First of all, note that since $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^2(\mathbb{R}^N)$ and

$$\frac{f^2(v_n)}{1+f^2(v_n)} - \frac{f^2(v)}{1+f^2(v)}$$

is bounded, we have $\mathrm{I} = o(1)$ as $n \to \infty$. Moreover,

$$\begin{aligned}
\mathrm{II}+\mathrm{III} &= \int_{\mathbb{R}^N} V_\lambda\Big(f^2(v_n) - \frac{f(v)}{\sqrt{1+f^2(v)}} f(v_n)\sqrt{1+f^2(v_n)}\Big) \\
&\quad + \int_{\mathbb{R}^N} V_\lambda\Big(f^2(v) - \frac{f(v_n)}{\sqrt{1+f^2(v_n)}} f(v)\sqrt{1+f^2(v)}\Big) \\
&= \int_{\mathbb{R}^N} V_\lambda f(v_n)(f(v_n)-f(v)) + V_\lambda f(v_n) f(v)\Big(1 - \frac{\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v)}}\Big) \\
&\quad + \int_{\mathbb{R}^N} V_\lambda f(v)(f(v)-f(v_n)) + V_\lambda f(v) f(v_n)\Big(1 - \frac{\sqrt{1+f^2(v)}}{\sqrt{1+f^2(v_n)}}\Big) \\
&= \int_{\mathbb{R}^N} V_\lambda (f(v_n)-f(v))^2 \\
&\quad + \int_{\mathbb{R}^N} V_\lambda \frac{f(v_n)f(v)}{\sqrt{1+f^2(v)}\big(\sqrt{1+f^2(v_n)}+\sqrt{1+f^2(v)}\big)}(f^2(v)-f^2(v_n)) \\
&\quad + \int_{\mathbb{R}^N} V_\lambda \frac{f(v)f(v_n)}{\sqrt{1+f^2(v_n)}\big(\sqrt{1+f^2(v_n)}+\sqrt{1+f^2(v)}\big)}(f^2(v_n)-f^2(v)) \\
&= \int_{\mathbb{R}^N} V_\lambda (f(v_n)-f(v))^2 + o(1) \quad \text{as } n\to\infty.
\end{aligned}$$

In the last equality, we use the facts that $f^2(v_n) \rightharpoonup f^2(v)$ weakly and that the two terms

$$\frac{f(v)f(v_n)}{\sqrt{1+f^2(v)}\big(\sqrt{1+f^2(v_n)}+\sqrt{1+f^2(v)}\big)}$$

and

$$\frac{f(v)f(v_n)}{\sqrt{1+f^2(v_n)}\big(\sqrt{1+f^2(v_n)}+\sqrt{1+f^2(v)}\big)}$$

are bounded. For the last two terms, we have

$$\begin{aligned}
\mathrm{IV}+\mathrm{V} &= \int_{\mathbb{R}^N} w(x, f^2(v_n)) f(v_n)\Big(\frac{f(v_n)}{f'(v)} f(v) - f(v_n)\Big) \\
&\quad + \int_{\mathbb{R}^N} w(x, f^2(v)) f(v)\Big(\frac{f(v)}{f'(v_n)} f(v_n) - f(v)\Big) \\
&= \int_{\Omega'_J} |f(v_n)|^{p-2} f(v_n)\Big(\frac{f'(v_n)}{f'(v)} f(v) - f(v_n)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v_n))f(v_n)\left(\frac{f(v_n')}{f'(v)}f(v)-f(v_n)\right)\\
&+\int_{\Omega'_J}|f(v)|^{p-2}f(v)\left(\frac{f'(v)}{f'(v_n)}f(v_n)-f(v)\right)\\
&+\int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v))f(v)\left(\frac{f'(v)}{f'(v_n)}f(v_n)-f(v)\right)\\
=&\int_{\Omega'_J}(f(v_n)-f(v))(f^{p-1}(v)-f^{p-1}(v_n))\\
&+\int_{\Omega'_J}|f(v_n)|^{p-2}f(v_n)\left(\frac{f'(v_n)}{f'(v)}-1\right)f(v) \qquad (\mathrm{I}_1)\\
&+\int_{\Omega'_J}|f(v)|^{p-2}f(v)\left(\frac{f'(v)}{f'(v_n)}-1\right)f(v_n) \qquad (\mathrm{I}_2)\\
&+\int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v_n))f(v_n)\left(\frac{f(v_n')}{f'(v)}f(v)-f(v_n)\right) \qquad (\mathrm{I}_3)\\
&+\int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v))f(v)\left(\frac{f'(v)}{f'(v_n)}f(v_n)-f(v)\right) \qquad (\mathrm{I}_4)\\
=&\int_{\Omega'_J}(f(v_n)-f(v))(f^{p-1}(v)-f^{p-1}(v_n))+\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{I}_1&=\int_{\Omega'_J}|f(v_n)|^{p-2}f(v_n)\left(\frac{f'(v_n)}{f'(v)}-1\right)f(v)\\
&=\int_{\Omega'_J}|f(v_n)|^{p-2}f(v_n)\frac{\sqrt{1+f^2(v)}-\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v_n)}}f(v)\\
&=\int_{\Omega'_J}|f(v_n)|^{p-2}f(v)(f(v)-f(v_n))\frac{f(v_n)(f(v)+f(v_n))}{\sqrt{1+f^2(v_n)}\left(\sqrt{1+f^2(v)}+\sqrt{1+f^2(v_n)}\right)}\\
&\le C\left(\int_{\Omega'_J}f^p(v_n)\right)^{(p-2/p)}\left(\int_{\Omega'_J}f^p(v)\right)^{1/p}\left(\int_{\Omega'_J}(f(v)-f(v_n))^p\right)^{1/p}\\
&=o(1)\text{ as } n\to\infty \quad (\text{since } f(v_n)\to f(v)\text{ strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^N)).
\end{aligned}
$$

Similarly, we have $\mathrm{I}_2=o(1)$ as $n\to\infty$.

As for $\mathrm{I}_3+\mathrm{I}_4$, we have

$$
\mathrm{I}_3+\mathrm{I}_4=\int_{\mathbb{R}^N\setminus\Omega'_J} l(f^2(v_n))f(v_n)\left(\frac{f(v_n')}{f'(v)}f(v)-f(v_n)\right)
$$

$$
\begin{aligned}
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v)) f(v) \left(\frac{f'(v)}{f'(v_n)} f(v_n) - f(v) \right) \\
&= \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n)) f(v_n)(f(v) - f(v_n)) \\
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v)) f(v)(f(v_n) - f(v)) \\
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n)) f(v_n) \left(\frac{f'(v_n)}{f'(v)} - 1 \right) f(v) \\
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v)) f(v) \left(\frac{f'(v)}{f'(v_n)} - 1 \right) f(v_n).
\end{aligned}
$$

For the same reasons that we used in the above estimates for I_1, we can see that the last two terms in the above equalities go to zero as n goes to infinity.

Thus

$$
\begin{aligned}
\mathrm{I}_3 + \mathrm{I}_4 &= \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n)) f(v_n)(f(v) - f(v_n)) \\
&\quad + \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v)) f(v)(f(v_n) - f(v)) + o(1) \\
&= \int_{\mathbb{R}^N \setminus \Omega'_J} (l(f^2(v_n)) - l(f^2(v))) f(v)(f(v) - f(v_n)) \\
&\quad - \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n))(f(v) - f(v_n))^2 + o(1).
\end{aligned}
$$

On the other hand, since $f(v_n) \to f(v)$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $f(v_n) \rightharpoonup f(v)$ weakly in $L^q(\mathbb{R}^N)$ for $2 \le q \le 2 \cdot 2^*$, and $l(t) \le \nu_0$ for all $t \ge 0$, we have

$$
\int_{\Omega'_J} (f(v_n) - f(v))(f^{p-1}(v) - f^{p-1}(v_n)) = o(1)
$$

and

$$
\int_{\mathbb{R}^N \setminus \Omega'_J} (l(f^2(v_n)) - l(f^2(v))) f(v)(f(v) - f(v_n)) = o(1).
$$

At last, we obtain the following estimate:

$$
\begin{aligned}
o(1) &= \int_{\mathbb{R}^N} \left(1 + \frac{f^2(v_n)}{1 + f^2(v_n)} \right) |\nabla v_n - \nabla v|^2 \\
&\quad + \int_{\mathbb{R}^N} V_\lambda (f(v) - f(v_n))^2 - \int_{\mathbb{R}^N \setminus \Omega'_J} l(f^2(v_n))(f(v) - f(v_n))^2.
\end{aligned}
$$

On the other hand, we can write

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(f(v)-f(v_n))|^2 \\
&\quad = \int_{\mathbb{R}^N} \left| \frac{\nabla v}{\sqrt{1+f^2(v)}} - \frac{\nabla v_n}{\sqrt{1+f^2(v_n)}} \right|^2 \\
&\quad = \int_{\mathbb{R}^N} \frac{1}{\sqrt{1+f^2(v_n)}} \left| \nabla v_n - \nabla v + \left(1 - \frac{\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v)}}\right) \nabla v \right|^2 \\
&\quad = \int_{\mathbb{R}^N} \frac{|\nabla v - \nabla v_n|^2}{\sqrt{1+f^2(v_n)}} + 2\int_{\mathbb{R}^N} \frac{1}{\sqrt{1+f^2(v_n)}} \left(1 - \frac{\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v)}}\right) \nabla v (\nabla v - \nabla v_n) \\
&\qquad + \int_{\mathbb{R}^N} \frac{1}{\sqrt{1+f^2(v_n)}} \left(1 - \frac{\sqrt{1+f^2(v_n)}}{\sqrt{1+f^2(v)}}\right)^2 |\nabla v|^2.
\end{aligned}
$$

We claim that both of the last two terms in the above last equality are $o(1)$ as $n \to \infty$. In fact, the first term goes to zero because $\nabla v_n \rightharpoonup \nabla v$, while the second term goes to zero by the dominated convergence theorem.

Thus we have

$$
\int_{\mathbb{R}^N} |\nabla(f(v)-f(v_n))|^2 \le \int_{\mathbb{R}^N} |\nabla v - \nabla v_n|^2
$$

by Lemma 2.2 and the definition of $l(t)$, so we get

$$
\begin{aligned}
&\delta_0 \int_{\mathbb{R}^N} \bigl(|\nabla(f(v)-f(v_n))|^2 + V_\lambda (f(v)-f(v_n))^2 \bigr) \\
&\qquad < \int_{\mathbb{R}^N} \left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right) |\nabla v_n - \nabla v|^2 + \int_{\mathbb{R}^N} V_\lambda (f(v)-f(v_n))^2 \\
&\qquad\qquad - \nu_0 \int_{\mathbb{R}^N \setminus \Omega'_J} (f(v)-f(v_n))^2 = o(1).
\end{aligned}
$$

Obviously, $\int_{\mathbb{R}^N} V_\lambda \bigl(f(v_n)-f(v)\bigr)^2 \to 0$ as $n \to \infty$. Hence

$$
\begin{aligned}
\int_{\mathbb{R}^N} V_\lambda (f^2(v_n) - f^2(v)) &= \int_{\mathbb{R}^N} (f(v_n)-f(v))(f(v_n)+f(v)) \\
&\le C \left(\int_{\mathbb{R}^N} V_\lambda (f(v_n)-f(v))^2 \right)^{1/2}
\end{aligned}
$$

for some constant C. By Proposition 2.1(3) of [Liu et al. 2003], we have $v_n \to v$ strongly in H_G^λ. This completes the proof of Proposition 4.1. □

5. Some asymptotic behavior

We denote by $H_G^{0,1}(\Omega_j)$ the closure of $C_0^\infty(\Omega)$ under the norm of $H_G^1(\Omega)$.

Proposition 5.1. *Assume that the sequences* $\{v_n\} \subset H_G^1$ *and* $\{\lambda_n\} \subset [0, \infty)$ *satisfy*

$$\lambda_n \to \infty, \tag{5-1}$$

$$\Psi_{\lambda_n}(v_n) \to c, \tag{5-2}$$

$$\|\Psi'_{\lambda_n}(v_n)\|^*_{\lambda_n} \to 0. \tag{5-3}$$

Then there exists a subsequence of $\{v_n\}$ (*still denoted by* $\{v_n\}$) *such that*

$$v_n \rightharpoonup v \quad \text{weakly in } H_G^1$$

for some $v \in H_G^1$. *Moreover, we have*:

(i) $v \equiv 0$ *in* $\mathbb{R}^N \setminus \Omega_J$, *and* v *is a solution of*

$$\begin{cases} -\Delta v = |f(v)|^{p-2} f(v) f'(v), & \text{in } \Omega_j, \\ \quad v \in H_G^{0,1}(\Omega_j) & \text{for } j \in J. \end{cases} \tag{5-4}$$

(ii) v_n *converges to* v *in a stronger sense*, *namely*

$$v_n \to v \quad \text{strongly in } H_G^1 \text{ as } \lambda_n \to \infty.$$

(iii) *The functions* $\{v_n\}$ *satisfy*:

$$\int_{\mathbb{R}^N} V_{\lambda_n} f^2(v_n) \to 0,$$
$$\Psi_\lambda(v_n) \to \sum_{j\in J} I_{\Omega_j}(v),$$
$$\|v_n\|_{\lambda_n, \mathbb{R}^N \setminus \Omega'_J} \to 0,$$
$$\|v_n\|_{\lambda_n, \Omega'_j} \to \int_{\Omega_j} |\nabla v|^2 \quad \text{for } j \in J, \text{ as } n \to \infty.$$

Proof. By arguments similar to those used in the proof of Lemma 4.2, we have

$$m(c) \le \liminf_{n\to\infty} \|v_n\|^2_{\lambda_n} \le \limsup_{n\to\infty} \|v_n\|^2_{\lambda_n} \le M(c).$$

Thus $\{v_n\}$ is bounded in H_G^1. Hence there is a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that:

$$\nabla v_n \rightharpoonup \nabla v \text{ weakly in } L^2(\mathbb{R}^N),$$
$$v_n \rightharpoonup v \text{ weakly in } L^q(\mathbb{R}^N) \text{ for } 2 \le q \le 2\cdot 2^*,$$
$$v_n \to v \text{ a.e. in } \mathbb{R}^N,$$
$$f(v_n) \to f(v) \text{ strongly in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for } 2 \le q < 2\cdot 2^*,$$
$$f(v_n) \rightharpoonup f(v) \text{ weakly in } L^q(\mathbb{R}^N) \text{ for } 2 \le q \le 2\cdot 2^*.$$

(i) Let $C_m := \{x \in \mathbb{R}^N \mid V(x) \geq 1/m\}$. Then for n large, we have

$$\int_{C_m} f^2(v_n) \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V f^2(v_n) \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} (V_{\lambda_n} f^2(v_n) + |\nabla v_n|^2)$$
$$\leq \frac{m}{\lambda_n} C \max\{\|v_n\|_{\lambda_n}, \|v_n\|^2_{\lambda_n}\} \to 0 \quad \text{as } \lambda_n \to \infty.$$

Thus

$$0 \leq \int_{C_m} f^2(v) \leq \lim_{n\to\infty} \int_{C_m} f^2(v_n) = 0.$$

Hence $f(v) = 0$ on $\bigcup_{m=1}^{\infty} C_m = \mathbb{R}^N \setminus \bar{\Omega}$. Note that $\Psi'_{\lambda_n}(v_n) \to 0$ as $\lambda_n \to \infty$, so we have

$$\begin{aligned} o(1) &= \Psi'_{\lambda_n}(v_n) \cdot \frac{f(v)}{f'(v)} \\ &= \int_{\Omega_j} \Big(1 + \frac{f^2(v)}{1+f^2(v)}\Big) \nabla v_n \nabla v - \int_{\Omega_j} w(x, f^2(v_n)) f(v_n) f'(v_n) \frac{f(v)}{f'(v)} \\ &= \int_{\Omega_j} \Big(1 + \frac{f^2(v)}{1+f^2(v)}\Big) |\nabla v|^2 - \int_{\Omega_j} w(x, f^2(v)) f^2(v) + o(1); \end{aligned}$$

here we use the fact that $f(v_n) \to f(v)$ strongly in $L^q_{\text{loc}}(\mathbb{R}^N)$.

On the other hand, by Lemma 2.2, we have

$$\begin{aligned} \delta_0 \|v\|_{\lambda_n} &\leq \int_{\Omega_j} (|\nabla v|^2 - \nu_0 f^2(v)) \\ &\leq \int_{\Omega_j} \Big(1 + \frac{f^2(v)}{1+f^2(v)}\Big) |\nabla v|^2 - \int_{\Omega_j} w(x, f^2(v)) f^2(v) = 0 \end{aligned}$$

Note that $\|v\|_{\lambda_n}$ indeed does not dependent on λ_n. We have that $v \equiv 0$ in Ω_j for $j \in \{1, 2, \ldots, k\} \setminus J$, and this completes the proof of part (i).

(ii) Indeed, by a similar argument as in the proof of Proposition 4.1, for n large, we have

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^2 + \int_{\mathbb{R}^N} V_{\lambda_n} (f(v_n) - f(v))^2 - \nu_0 \int_{\mathbb{R}^N \setminus \Omega_J} (f(v) - f(v_n))^2 \\ &\geq \delta_0 \Big(\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^2 + \int_{\mathbb{R}^N} V_{\lambda_n} (f(v_n) - f(v))^2 \Big) \\ &\geq \delta_0 C \min\big\{ \|v_n - v\|_{H_G^{\lambda_n}}, \|v_n - v\|^2_{H_G^{\lambda_n}} \big\}. \end{aligned}$$

Hence $\|v_n - v\|_{H^1_G} \to 0$ as $n \to \infty$. This completes the proof of part (ii).

(iii) This is a direct consequence of parts (i) and (ii). In fact, from (ii) and (i), one can see that

$$\begin{aligned}\tfrac{1}{2}\int_{\mathbb{R}^N} V_{\lambda_n} f^2(v_n) &= \tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_j} V_{\lambda_n} f^2(v_n)\\ &= \int_{\mathbb{R}^N\setminus\Omega_j} V_{\lambda_n} f^2(v_n)(f(v_n)-f(v))^2 \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Thus we have

$$\lim_{n\to\infty}\tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_j} V_{\lambda_n} f^2(v_n) = \tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_j} V_{\lambda_n} f^2(v) = 0.$$

Obviously, we get

$$\Psi_{\lambda_n}(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 + V_{\lambda_n} f^2(v_n) - \frac{1}{p}\int_{\mathbb{R}^N} W(x, f^2(v_n)) \to \sum_{j\in J} I_{\Omega_j}(v),$$

where $I_{\Omega_j}(v) = \frac{1}{2}\int_{\Omega_j} |\nabla v|^2 - (1/p)\int_{\Omega_j} |f(v)|^p$. Furthermore,

$$\begin{aligned}\lim_{n\to\infty} \|f^2(v_n)\|_{H_G^{\lambda_n}(\mathbb{R}^N\setminus\Omega'_j)} &= 0,\\ \lim_{n\to\infty} \|\nabla v_n\| &= \int_{\Omega_j} |\nabla v|^2 \ \text{ for } j\in J.\end{aligned}$$

This completes the proof of Proposition 5.1. □

Proposition 5.2. *There exist constants $M > 0$, $\Lambda_0 > 0$ such that if v_λ is a critical point of Ψ_λ for $\lambda \geq \Lambda_0$, then $|f(v_\lambda)|^2 \leq v_0^{2/(p-2)}$ and $\Psi_\lambda(v) \leq M$. In particular, v_λ solves the problem* (1-10).

Proof. Let $B_r(x) = \{y\in\mathbb{R}^N \mid |x-y| < r\}$. Since v_λ is a critical point of Ψ_λ, we have

$$\begin{aligned}&-\Delta v_\lambda + V_\lambda f(v_\lambda) f'(v_\lambda)\\ &\qquad = \chi(\Omega_j)|f(v_\lambda)|^{p-1} f'(v_\lambda) + (1-\chi(\Omega_j)) l(x, f^2(v_\lambda)) f(v_\lambda) f'(v_\lambda).\end{aligned}$$

That is,

$$-\Delta v_\lambda + \big(V_\lambda - \chi(\Omega_j)|f(v_\lambda)|^{p-12} - (1-\chi(\Omega_j)) l(x, f^2(v_\lambda))\big)\frac{f(v_\lambda)f'(v_\lambda)}{v_\lambda} v_\lambda = 0.$$

Let

$$V_0 = \big(V_\lambda - \chi(\Omega_j)|f(v_\lambda)|^{p-12} - (1-\chi(\Omega_j)) l(x, f^2(v_\lambda))\big)\frac{f(v_\lambda)f'(v_\lambda)}{v_\lambda}.$$

Then our assumptions on V imply that V_0 belongs to K_N^{loc}, the local Kato class, and thus $|v_\lambda(x)|_{L^\infty}$ is bounded (see Theorem C1.2 of [Simon 1982]). It follows from Theorem 8.17 of [Gilbarg and Trudinger 1983] that

$$|v_\lambda(x)| \le C \int_{B(x,r)} |v_\lambda(y)|^p \, dy.$$

By Proposition 5.1, we see that for any sequence $\lambda_n \to \infty$, we can extract a subsequence of $\{\lambda_n\}$ (still denoted by $\{\lambda_n\}$) such that $v_{\lambda_n} \to v \in H_0^1(\Omega_j)$ strongly in $L^2(\mathbb{R}^N \setminus \Omega_j)$. Since the sequence $\{\lambda_n\}$ can be chosen arbitrarily, we conclude that

$$v_\lambda \to v \in H_0^1(\Omega_j) \quad \text{strongly as } \lambda \to \infty.$$

Now choose $r \in (0, \operatorname{dist}(\Omega_J, \mathbb{R}^N \setminus \Omega'_J))$; we have, uniformly in $x \in \mathbb{R}^N \setminus \Omega'_J$, that

$$\begin{aligned}
|v_\lambda(x)| &\le C(r) \int_{B_r(x)} |v_\lambda(x)|^p \\
&\le C(r)(\text{meas } B_r(x))^{1-q/2^*} \biggl(\int_{B(x,r)} |v_\lambda(x)|^{2^*} \biggr)^{p/2^*} \\
&\le C(r)(\text{meas } B_r(x))^{1-q/2^*} \biggl(\int_{B(x,r)} |\nabla v_\lambda(x)|^2 \biggr)^{p/2} \\
&\le C(r) \biggl(\biggl(\int_{\mathbb{R}^N \setminus \Omega'_J} |\nabla v_\lambda|^2 \biggr)^{1/2} + V_\lambda f^2(v_\lambda) \biggr) \\
&\le C \max\{\|v_\lambda\|, \|v_\lambda\|^{1/2}\} \\
&\to 0 \quad \text{uniformly in } x \in \mathbb{R}^N \setminus \Omega'_J,
\end{aligned}$$

which implies that $f(|v_\lambda|) \to 0$ uniformly in $x \in \mathbb{R}^N \setminus \Omega'_J$. This completes the proof of Proposition 5.2. □

Remark 5.3. The critical points of Ψ_λ are not necessarily positive. In fact, if we replace the function v by its positive part v^+ in the nonlinearity term $W(x, f^2(v))$ of Ψ_λ, and the new functional is denoted by Ψ_λ^+, then by arguments similar to those above, one can see that the new functional Ψ_λ^+ still satisfies properties analogous to all those proved for Ψ_λ in previous sections. As a consequence, the critical points of Ψ^+ are positive. In the following, for convenience we only consider Ψ_λ instead of Ψ_λ^+.

Remark 5.4. Proposition 4.1 shows that Ψ_λ satisfies the Palais–Smale condition. We can easily check that Ψ_λ has mountain pass geometry. Hence, a mountain pass argument shows that, for each $\lambda > 0$, Ψ_λ admits a nontrivial critical point u_λ. In fact, $\Psi_\lambda(u_\lambda) \le \max_{t>0} I_{\Omega_j}(t\omega_j)$ (see Section 6 for the definition of I_{Ω_j} and ω_j) and thus $\Psi_\lambda(u_\lambda) \le M$, where M is independent of λ. As a result, by Proposition 5.2, we deduce the existence of a positive solution to (1-10) and thus a positive solution to the original problem (1-1) for $\lambda > \Lambda$. However, it is not clear whether such solutions concentrate on the set Ω_J. The aim of the following parts of the paper is to focus on the solutions with such properties.

6. Limit problem

For $j \in J$ we define the following two functionals:

$$I_{\Omega_j}(v) = \frac{1}{2}\int_{\Omega_j} |\nabla v|^2 - \frac{1}{p}\int_{\Omega_j} |f(v)|^p \quad \text{for } v \in H_G^{1,0}(\Omega_j),$$

and

$$\Psi_{\lambda,\Omega'_j}(u) = \frac{1}{2}\int_{\Omega'_j} (|\nabla v|^2 + V_\lambda f^2(v)) - \frac{1}{p}\int_{\Omega'_j} |f(v)|^p \quad \text{for } v \in H_G^1(\Omega'_j). \tag{6-1}$$

By Lemma 2.2 of [Guo and Tang 2012] and the following inequality

$$\|f(v)\|_{L^p} \le \|f(v)\|_2^{\theta} \|f(v)\|_{L^{2\cdot 2^*}}^{1-\theta} \quad \text{for } 0 < \theta < 1,$$

following a standard argument (see [Tang 2008]), one can see that both I_{Ω_j} and $\Psi_{\Omega'_j}$ satisfy the mountain pass geometry conditions. That is:

(i) $I_{\Omega_j}(0) = \Psi_{\lambda,\Omega'_j}(0) = 0$.

(ii) There exist $\rho_0 > 0$ and $\rho_1 > 0$, independent of $\lambda \ge 0$, such that

$$\begin{aligned} \|v\|_{H_G^{1,0}(\Omega_j)} \le \rho_0 &\Longrightarrow I_{\Omega_j}(v) \ge 0, \\ \|v\|_{H_G^{1,0}(\Omega_j)} = \rho_0 &\Longrightarrow I_{\Omega_j}(v) \ge \rho_1, \end{aligned} \tag{6-2}$$

and

$$\begin{aligned} \|v\|_{H_G^1(\Omega'_j)} \le \rho_0 &\Longrightarrow \Psi_{\lambda,\Omega'_j}(v) \ge 0, \\ \|v\|_{H_G^1(\Omega'_j)} = \rho_0 &\Longrightarrow \Psi_{\lambda,\Omega'_j}(v) \ge \rho_1. \end{aligned} \tag{6-3}$$

Here we use the notation

$$\|v\|_{H_G^{1,0}(\Omega_j)} = \int_{\Omega_j} |\nabla v|^2 \quad \text{for } v \in H_G^{0,1}(\Omega_j).$$

(iii) There exists $\psi_j \in C_0^\infty(\Omega_j)$ such that

$$\|\psi_j(x)\|_{H_G^{\lambda}(\Omega_j)} = \|\psi_j(x)\|_{H_G^{\lambda,0}(\Omega_j)} \ge \rho_1,$$
$$\Psi_{\lambda,\Omega'_j}(\psi_j) = I_{\Omega_j}(\psi_j) < 0.$$

We define

$$\begin{aligned} c_j &= \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} I_{\Omega_j}(\gamma(t)), \\ c_{\lambda,j} &= \inf_{\gamma\in\Gamma_{\lambda,j}} \max_{t\in[0,1]} \Psi_{\lambda,\Omega'_j}(\gamma(t)), \end{aligned} \tag{6-4}$$

where

$$\Gamma_j = \{\gamma \in C([0,1], H_G^{0,1}(\Omega_j)) \mid \gamma(0) = 0,\ I_{\Omega_j}(\gamma(1)) < 0\},$$
$$\Gamma_{\lambda,j} = \{\gamma \in C([0,1], H_G^{1}(\Omega'_j)) \mid \gamma(0) = 0,\ \Psi_{\lambda,\Omega'_j}(\gamma(1)) < 0\}.$$

By Proposition 2.3 and Lemma 2.2 of [Guo and Tang 2012], it is standard to verify that Φ_{λ,Ω'_j} and I_{Ω_j} satisfy the Palais–Smale condition and that c_j, $c_{\lambda,j}$ are achieved by critical points. We denote the corresponding critical points by ω_j and $\omega_{\lambda,j}$ respectively.

Lemma 6.1. (i) $0 < \rho_1 \le c_{\lambda,j} \le c_j$ *for all* $\lambda \ge 0$.

(ii) c_j *and* $c_{\lambda,j}$ *are least energy levels for* I_{Ω_j} *and* Φ_{λ,Ω'_j}, *respectively, i.e.*,

$$c_j = \inf\{I_{\Omega_j}(v) \mid v \in H_G^{0,1}(\Omega_j)\setminus\{0\} \textit{ is a critical point of } I_{\Omega_j}\},$$
$$c_{\lambda,j} = \inf\{\Psi_{\lambda,\Omega'_j}(v) \mid v \in H_G^{1}(\Omega'_j)\setminus\{0\} \textit{ is a critical point of } \Psi_{\lambda,\Omega'_j}\}.$$

(iii) $c_j = \max_{r>0} I_{\Omega_j}(r\omega_j)$, $c_{\lambda,j} = \max_{r>0} \Phi_{\lambda,\Omega'_j}(r\omega_{\lambda,j})$.

(iv) $c_{\lambda,j} \to c_j$ *as* $\lambda \to \infty$.

Proof. By (6-3), it is easy to see that $c_{\lambda,j} \ge \rho_1$. On the other hand, for any $v \in H_G^{0,1}(\Omega_j)$, we may extend v to $\tilde{v} \in H_G^1(\Omega'_j)$ by

$$\tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega_j, \\ 0 & \text{if } x \in \Omega'_j \setminus \bar{\Omega}_j, \end{cases}$$

so we may regard $H_G^{0,1}(\Omega_j) \subset H_G^1(\Omega'_j)$. Thus we have $\Gamma_j \subset \Gamma_{\lambda,j}$ and

$$\begin{aligned} c_{\lambda,j} &= \inf_{\gamma\in\Gamma_{\lambda,j}} \max_{t\in[0,1]} \Psi_{\lambda,\Omega'_j}(\gamma(t)) \\ &\le \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} \Psi_{\lambda,\Omega'_j}(\gamma(t)) \qquad (6\text{-}5) \\ &= \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} I_{\Omega_j}(\gamma(t)) = c_j. \end{aligned}$$

This proves (i).

Note that, since $f(v)$ is monotone with respect to v, and so is $|f(v)|^p$ with respect to $|f(v)|$, the proofs of (ii) and (iii) are standard; see [Tang 2008].

Now we prove (iv). Using Proposition 5.1, we may extract a subsequence $\lambda_n \to \infty$ such that

$$\omega_{\lambda_n,j} \to v_0 \quad \text{strongly in } H_G^1(\Omega'_j),$$

where $v_0 \in H_G^{0,1}(\Omega_j)$ is a solution of (5-4) and

$$\Psi_{\lambda_n,\Omega'_j}(\omega_{\lambda_n,j}) \to I_{\Omega_j}(v_0).$$

By the definition of c_j, we have

$$\limsup_{\lambda\to\infty} c_{\lambda,j} = \limsup_{\lambda\to\infty} \Psi_{\lambda,\Omega'_j}(\omega_{\lambda,j}) \geq I_{\Omega_j}(u_0) \geq c_j.$$

Comparing with (6-5), we get (iv). This completes the proof of Lemma 6.1. □

7. Minimax arguments

Now we give a minimax argument for Φ_λ (see (1-8)).

We choose $R \geq 2$ such that

$$I_{\Omega_j}(R\omega_j) < 0. \tag{7-1}$$

Without loss of generality, we assume that $J = \{1, 2, \dots, l\}$ $(l \leq k)$. Set

$$\gamma_0(s_1, s_2, \dots, s_l) = \sum_{j=1}^{l} s_j R\omega_j \quad \text{for } (s_1, s_2, \dots, s_l) \in [0, 1]^l, \tag{7-2}$$

$$\Gamma_J = \left\{ \gamma \in C([0, 1]^l, H_G^1) \;\middle|\; \begin{array}{l} \gamma(s_1, s_2, \dots, s_l) = \gamma_0(s_1, s_2, \dots, s_l) \\ \text{for } (s_1, s_2, \dots, s_l) \in \partial([0, l]^l) \end{array} \right\}.$$

We define

$$b_{\lambda,J} = \inf_{\gamma\in\Gamma_J} \max_{(s_1,s_2,\dots,s_l)\in\partial([0,1]^l)} \Phi_\lambda(\gamma(s_1, s_2, \dots, s_l)).$$

Note that the projection $t \mapsto tR\omega_j$ belongs to Γ_j and satisfies

$$\max_{t\in[0,1]} I_{\Omega_j}(tR\omega_j) = c_j$$

for any $j \in J$. Hence $\gamma_0 \in \Gamma_J$, $\Gamma_J \neq \varnothing$, and $b_{\lambda,J}$ is well defined. We denote $c_J = \sum_{j=1}^{l} c_j$. Then we have:

Lemma 7.1. (i) $\sum_{j=1}^{l} c_{\lambda,j} \leq b_{\lambda,J} \leq c_J$ *for all* $\lambda \geq 0$.

(ii) $\Psi_\lambda(\gamma(s_1, s_2, \dots, s_l)) \leq c_J - \rho_1$ *for all* $\lambda \geq 0, \gamma \in \Gamma_J$ *and* $(s_1, s_2, \dots, s_l) \in \partial([0, 1]^l)$, *where* ρ_1 *is given in* (6-2), (6-3).

Proof. For any given $\gamma \in \Gamma_J$, let

$$T_j(s_1, \dots, s_l) = \frac{\int_{\Omega'_j} |f(\gamma(s_1, \dots, s_l))|^{p-1} f'(\gamma(s_1, \dots, s_l))\gamma(s_1, \dots, s_l)}{\int_{\Omega'_j} |\nabla\gamma(s_1, \dots, s_l)|^2 + V_\lambda f(\gamma(s_1, \dots, s_l)) f'(\gamma(s_1, \dots, s_l))\gamma(s_1, \dots, s_l)}$$

for $j = 1, 2, \dots, l$.

We define a map $\mathcal{T} : [0, 1]^l \to \mathbb{R}^l$ by

$$\mathcal{T}(\cdot) = (T_1(\cdot), \dots, T_l(\cdot)).$$

Thus for $(s_1, s_2, \dots, s_l) \in \partial([0,1]^l)$, we have

$$\mathcal{T}(s_1,\dots,s_l) = \left(\frac{\int_{\Omega_1'} |f(s_1R\omega_1)|^{p-1} f'(s_1R\omega_1)s_1R\omega_1}{\int_{\Omega_1'} |\nabla(s_1R\omega_1)|^2 + V_\lambda f(s_1R\omega_1)s_1R\omega_1}, \dots, \frac{\int_{\Omega_l'} |f(s_lR\omega_l)|^{p-1} f'(s_lR\omega_l)s_lR\omega_l}{\int_{\Omega_l'} |\nabla(s_lR\omega_l)|^2 + V_\lambda f(s_lR\omega_l)s_lR\omega_l}\right).$$

To proceed, we consider the function ρ defined by

$$\rho(\alpha) = \frac{\int_{\Omega_j} |f(\alpha v)|^{p-1} f'(\alpha v)\alpha v}{\alpha^2 \int_{\Omega_j} |\nabla v|^2 + \int_{\Omega_j} V_\lambda f(\alpha v) f'(\alpha v)\alpha v} = \frac{\rho_1(\alpha)}{\int_{\Omega_j} |\nabla v|^2 + \rho_2(\alpha)},$$

where

$$\rho_1(\alpha) = \int_{\Omega_j} \frac{f(\alpha v)|f(\alpha v)|^{p-1} v}{\alpha\sqrt{1+f^2(\alpha v)}}, \quad \rho_2(\alpha) = \int_{\Omega_j} V_\lambda \frac{f(\alpha v) v}{\alpha\sqrt{1+f^2(\alpha v)}}.$$

By the proof of Lemma 3.2 of [Guo and Tang 2012], we see that ρ_1 is monotone increasing and ρ_2 is monotone decreasing; as a result, we see that ρ is monotone with respect to α. On the other hand, we note that $I_{\Omega_j}(R\omega_j) < 0$, $j = 1, 2, \dots, l$, for the same reason as in the proof of Lemma 4.2 of [Tang 2008], so we obtain

$$\deg(\mathcal{T}, [0,1]^l, (1, 1, \dots, 1)) = 1.$$

Hence there exists $(s_1, s_2, \dots, s_l) \in [0,1]^l$ such that

$$T_j(s_1, s_2, \dots, s_l) = 1 \quad \text{for } j = 1, 2, \dots, l. \tag{7-3}$$

Now we prove (i).

Since $\gamma_0 \in \Gamma_J$, we have

$$\begin{aligned} b_{\lambda,J} &\le \max_{(s_1,s_2,\dots,s_l)\in[0,1]^l} \Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)) \\ &= \max_{(s_1,s_2,\dots,s_l)\in[0,1]^l} \sum_{j=1}^l I_{\Omega_j}(s_jR\omega_j) = \sum_{j=1}^l c_j = c_J. \end{aligned}$$

On the other hand, by (7-3), for any $\gamma \in \Gamma_J$, there exists $s_\gamma \in [0,1]^l$ such that

$$\frac{\int_{\Omega_j'} |f(\gamma(s_\gamma))|^{p-1} f'(\gamma(s_\gamma))\gamma(s_\gamma)}{\int_{\Omega_j'} |\nabla\gamma(s_\gamma)|^2 + V_\lambda f(\gamma s_\gamma)\gamma(s_\gamma)} = 1 \quad \text{for} \quad j = 1, 2, \dots, l.$$

This implies that $\Psi'_{\lambda,\Omega_j'}(\gamma(s_\gamma)) \cdot \gamma(s_\gamma) = 0$ for $j = 1, 2, \dots, l$. Thus, if we define $u(x) = \gamma(s_\gamma)(x)$, we have

$$\Psi_\lambda(u) = \Psi_{\lambda,\mathbb{R}^N\setminus\Omega_J'}(u) + \sum_{j=1}^l \Psi_{\lambda,\Omega_j'}(u),$$

where

$$\Psi_{\lambda,\mathbb{R}^N\setminus\Omega'_J}(u)=\tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'_J}(|\nabla v|^2+V_\lambda f^2(v))-\tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'_J}W(f^2(v)).$$

Since $W(f^2(v))\le \nu_0 f^2(v)$, we have

$$\begin{aligned}\Psi_{\lambda,\mathbb{R}^N\setminus\Omega'_J}(u)&=\tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'_J}(|\nabla v|^2+V_\lambda f^2(v))-\tfrac{1}{2}\int_{\mathbb{R}^N\setminus\Omega'_J}W(f^2(v))\\&\ge \tfrac{1}{2}\|u\|^2_{H^\lambda_G(\mathbb{R}^N\setminus\Omega'_j)}-\tfrac{1}{2}\|u\|^2_{L^2(\mathbb{R}^N\setminus\Omega'_j)}\\&\ge \frac{\delta_0}{2}\|u\|^2_{H^\lambda_G(\mathbb{R}^N\setminus\Omega'_j)}\ge 0.\end{aligned}$$

Thus

$$\begin{aligned}\Psi_\lambda(u)=\Psi_{\lambda,\mathbb{R}^N\setminus\Omega'_J}(u)+\sum_{j=1}^{l}\Psi_{\lambda,\Omega'_j}(u)\ge\sum_{j=1}^{l}\Psi_{\lambda,\Omega'_j}(u)\\ \ge\sum_{j=1}^{l}\inf\bigl\{\Psi_{\lambda,\Omega'_j}(v)\mid v\in H^1_G(\Omega'_j),\ \Psi'_{\lambda,\Omega'_j}(v)\cdot v=0\bigr\}=\sum_{j=1}^{l}c_{\lambda,j}.\end{aligned}$$

Since $\gamma\in\Gamma_J$ is arbitrary, we have $b_{\lambda,J}\ge c_{\lambda,J}$.

For (ii), by the definition of γ_0, for $(s_1,s_2,\dots,s_l)\in\partial([0,1]^l)$ we have

$$\Psi_\lambda(\gamma_0(s_1,s_2,\dots,s_l))=\sum_{j=1}^{l}I_{\Omega_j}(s_jR\omega_j),$$

and $I_{\Omega_j}(s_jR\omega_j)\le c_j$ for $j=1,2,\dots,l$. On the other hand, for some $j_0\in J$, either $s_{j_0}=1$ or $s_{j_0}=0$, and thus $I_{\Omega_{j_0}}(s_{j_0}R\omega_{j_0})\le 0$. Therefore

$$\Psi_\lambda(\gamma_0(s_1,s_2,\dots,s_l))\le\sum_{j\ne j_0}I_{\Omega_j}(s_jR\omega_j)\le c_J-\rho_1.$$

This completes the proof of Lemma 7.1. □

Corollary 7.2. *We have $b_{\lambda,J}\to c_J$ as $\lambda\to\infty$. Moreover, $b_{\lambda,J}$ is a critical value of Ψ_λ for large λ.*

Proof. From Lemma 6.1, we know that $c_{\lambda,j}\to c_j$ as $\lambda\to\infty$. It follows from Lemma 7.1 that $b_{\lambda,J}\to c_J$ as $\lambda\to\infty$. Thus, we may choose Λ large enough such that for all $\lambda\ge\Lambda$, we have $b_{\lambda,J}>c_J-\rho_1$. Since Ψ_λ satisfies the Palais–Smale condition, by the standard deformation argument we can see that $b_{\lambda,J}$ is a critical value of Ψ_λ for $\lambda\ge\Lambda$. □

8. Flow arguments and the proofs of the main results

Let

$$\Psi_\lambda^{c_J} = \{v \in H_G^\lambda \mid \Psi_\lambda(v) \le c_J\}.$$

We choose

$$0 < \mu < \tfrac{1}{3} \min_{j \in J} c_j, \tag{8-1}$$

and define

$$D_\lambda^\mu = \{v \in H_G^\lambda \mid \|v\|_{H_G^\lambda(\mathbb{R}^N \setminus \Omega'_J)} \le \mu,\ |\Psi_{\lambda,\Omega'_j}(v) - c_j| \le \mu \text{ for all } j \in J\}.$$

Note that ω_j is the least energy solution of (5-4), and

$$\Psi_{\lambda,\Omega'_j}(\omega_j) = \frac{1}{2}\int_{\Omega_j} |\nabla \omega_j|^2 - \int_{\Omega_j} |f(\omega_j)|^p = c_j.$$

Thus $D_\lambda^\mu \cap \Psi_\lambda^{c_J}$ contains all the functions of the following form:

$$\omega(x) = \begin{cases} \omega_j(x) & \text{if } x \in \Omega_j, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega_J. \end{cases}$$

Lemma 8.1. *There exists $\sigma_0 > 0$ and $\Lambda_0 \ge 0$, independent of λ, such that*

$$\|\Psi'_\lambda(u)\|_\lambda^* \ge \sigma_0 \quad \textit{for all } \lambda \ge \Lambda_0 \textit{ and for all } u \in (D_\lambda^{2\mu} \setminus D_\lambda^\mu) \cap \Psi_\lambda^{c_J}. \tag{8-2}$$

Proof. We prove it by contradiction. Suppose that there exist $\lambda_n \to \infty$ and $v_n \in (D_{\lambda_n}^{2\mu} \setminus D_{\lambda_n}^\mu) \cap \Psi_{\lambda_n}^{c_J}$ such that $\|\Psi'_{\lambda_n}(u)\|_{\lambda_n}^* \to 0$. Since $v_n \in D_{\lambda_n}^{2\mu}$, thus v_n is bounded in H_G^1, and it turns out that $\Psi_{\lambda_n}(v_n)$ stays bounded as $n \to \infty$. We may assume that (up to a subsequence)

$$\Psi_{\lambda_n}(v_n) \to c \le c_J.$$

Applying Proposition 5.1, we can extract a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that $v_n \to v$ in H_G^1 and such that the following hold:

$$\lim_{n\to\infty} \Psi_{\lambda_n}(v_n) = \sum_{j=1}^{l} I_{\Omega_j}(v) \le c_J, \tag{8-3}$$

$$\lim_{n\to\infty} \|v_n\|^2_{H_G^{\lambda_n}(\Omega'_j)} = \int_{\Omega_j} |\nabla v|^2 \quad \text{for all } j \in J, \tag{8-4}$$

$$\lim_{n\to\infty} \int_{\Omega'_j} |f(v_n)|^p = \int_{\Omega_j} |f(v)|^p, \tag{8-5}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega'_J} (|\nabla v_n|^2 + V_{\lambda_n} f^2(v_n)) = 0. \tag{8-6}$$

Since $c_J = \sum_{j=1}^{l} c_j$ and c_j is the least energy level for $I_{\Omega_j}(u)$, we have two possibilities:

(1) $I_{\Omega_j}(v|_{\Omega_j}) = c_j$ for all $j \in J$.

(2) $I_{\Omega_{j_0}}(v|_{\Omega_{j_0}}) = 0$, that is, $u|_{\Omega_{j_0}} = 0$ for some $j_0 \in J$.

In case (1), we have

$$\frac{1}{2}\int_{\Omega_j} |\nabla v|^2 - \frac{1}{2}\int_{\Omega_j} |f(v)|^p = c_j \text{ for all } j \in J$$

and it follows from (8-3), (8-4), and (8-6) that $v_n \in D_{\lambda_n}^{\mu}$ for large n, which contradicts the fact that to $v_n \in D_{\lambda_n}^{2\mu} \setminus D_{\lambda_n}^{\mu}$.

In case (2), it follows from (8-3) and (8-4) that

$$|\Psi_{H_G^{\lambda_n}(\Omega'_j)}(v_n) - c_{j_0}| \to c_{j_0} \geq 3\mu.$$

This also contradicts the fact that $v_n \in D_{\lambda_n}^{2\mu} \setminus D_{\lambda_n}^{\mu}$. This completes the proof. □

Proposition 8.2. *Let μ satisfy* (7-3) *and let Λ_0 be the constant given in Lemma 8.1. Then for $\lambda \geq \Lambda_0$, there exists a solution v_λ of* (1-1) *such that $v_\lambda \in D_\lambda^{\mu} \cap \Psi_\lambda^{c_J}$.*

Proof. Assume, to the contrary, that Ψ_λ has no critical points in $D_\lambda^{\mu} \cap \Psi_\lambda^{c_J}$. Since Ψ_λ satisfies the Palais–Smale condition, there exists a constant $d_\lambda > 0$ such that

$$\|\Psi'_\lambda(v)\|_\lambda^* \geq d_\lambda \quad \text{for all } v \in D_\lambda^{\mu} \cap \Psi_\lambda^{c_J},$$

where $\|\cdot\|_\lambda^*$ is the norm of the dual space of H_G^λ. By Lemma 8.1 we have

$$\|\Psi'_\lambda(v)\|_\lambda^* \geq \sigma_0 \quad \text{for all } v \in (D_\lambda^{2\mu} \setminus D_\lambda^{\mu}) \cap \Psi_\lambda^{c_J}.$$

Let $\varphi : H_G^\lambda \to \mathbb{R}$ be a Lipschitz continuous function such that

$$\varphi(v) = \begin{cases} 1 & \text{for } v \in D_\lambda^{3\mu/2}, \\ 0 & \text{for } v \notin D_\lambda^{2\mu}, \end{cases}$$

and $0 \leq \varphi(v) \leq 1$ for any $v \in H_G^\lambda$.

Since $\Psi_\lambda \in C^1(H_G^\lambda, \mathbb{R})$, we denote by $\mathcal{G} : H_G^{\lambda *} \to H_G^\lambda$ the pseudogradient field of Ψ, which satisfies

$$\|\mathcal{G}(u)\|_{H_G^\lambda} \leq 2\|\Psi'(u)\|_\lambda^*, \quad \langle \Psi'(u), \mathcal{G}(u)\rangle \geq (\|\Psi'(u)\|_\lambda^*)^2. \tag{8-7}$$

Now for $v \in \Psi_\lambda^{c_J}$, we define $\widetilde{W}(v) : \Psi_\lambda^{c_J} \to H_G^\lambda$ by

$$\widetilde{W}(v) = -\varphi(v)\frac{\mathcal{G}(u)}{\|\Psi'_\lambda(v)\|_\lambda^*}.$$

We consider the deformation $\eta : [0, \infty) \times \Psi_\lambda^{c_J} \to \Psi_\lambda^{c_J}$ defined by

$$\frac{d\eta}{dt} = \widetilde{W}(\eta(t, v)), \quad \eta(0, v) = v \in \Psi_\lambda^{c_J}.$$

Then $\eta(t, v)$ satisfies

$$\frac{d}{dt}\Psi_\lambda(\eta(t, v)) = -\varphi(\eta(t, v))\frac{\langle \Psi_\lambda'(\eta(t, v)), \mathscr{G}(\eta(t, v))\rangle}{\|\Psi_\lambda'(u)(\eta(t, v))\|_\lambda^*} \le 0, \tag{8-8}$$

$$\left\|\frac{d\eta}{dt}\right\|_\lambda \le 2 \quad \text{for all } t, v, \tag{8-9}$$

$$\eta(t, v) = v \quad \text{for all } t \ge 0 \text{ and } v \in \Psi_\lambda^{c_J} \setminus D_\lambda^{2\mu}. \tag{8-10}$$

Let $\gamma_0(s_1, s_2, \dots, s_l) \in \Gamma_J$ be the path defined in (7-2). We consider

$$\eta(t, \gamma_0(s_1, s_2, \dots, s_l))$$

for large t. Since for all $(s_1, s_2, \dots, s_l) \in \partial([0, 1]^l)$, $\gamma_0(s_1, s_2, \dots, s_l) \notin D_\lambda^{2\mu}$, we have by (8-10) that

$$\eta(t, \gamma_0(s_1, s_2, \dots, s_l)) = \gamma_0(s_1, s_2, \dots, s_l) \quad \text{for all } (s_1, s_2, \dots, s_l) \in \partial([0, 1]^l),$$

and $\eta(t, \gamma_0(s_1, s_2, \dots, s_l)) \in \Gamma_J$ for all $t \ge 0$.

Since $\operatorname{supp} \gamma_0(s_1, s_2, \dots, s_l)(x) \subset \overline{\Omega}_J$ for all $(s_1, s_2, \dots, s_l) \in \partial([0, 1]^l)$, it follows that $\Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)(x))$ and $\|\gamma_0(s_1, s_2, \dots, s_l)(x)\|_{H_G^\lambda(\Omega_j')}$ do not depend on $\lambda \ge 0$. On the other hand,

$$\Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)(x)) \le c_J \quad \text{for all } (s_1, s_2, \dots, s_l) \in [0, 1]^l,$$

and $\Psi_\lambda(\gamma_0(s_1, s_2, \dots, s_l)(x)) = c_J$ if and only if $s_j = 1/R$; that is,

$$\gamma_0(s_1, s_2, \dots, s_l)(x)|_{\Omega_j} = \omega_j$$

for all $j \in J$. Thus we have that

$$m_0 := \max\{\Psi_\lambda(v) \mid v \in \gamma_0([0, 1]^l) \setminus D_\lambda^\mu\} \tag{8-11}$$

is independent of λ, and $m_0 < c_J$.

By (8-9), one can see that for any $t > 0$,

$$\|\eta(0, \gamma_0(s_1, \dots, s_l)) - \eta(t, \gamma_0(s_1, \dots, s_l))\|_{H_G^\lambda} \le 2t.$$

Since $\Psi_{\lambda,\Omega_j'} \in C^2(H_G^\lambda)$ for all $j = 1, \dots, l$, by the same arguments as in Proposition 4.5 of [Tang 2008], we have that for a large number T, there exists a positive number μ_0, which is independent of λ, such that for all $j = 1, 2, \dots, l$ and $t \in [0, T]$,

$$\|\Psi_{\lambda,\Omega_j'}'(\eta(t, \gamma_0(s_1, \dots, s_l)))\|_{H_G^\lambda}^* \le \mu_0.$$

We claim that for large T,

$$\max_{(s_1,s_2,\dots,s_l)\in[0,1]^l} \Psi_\lambda(\eta(T,\gamma_0(s_1,s_2,\dots,s_l)(x))) \le \max\bigl\{m_0, c_J - \tfrac{1}{2}\tau_0\mu\bigr\}, \tag{8-12}$$

where $\tau_0 = \max\{\sigma_0, \sigma_0/\mu_0\}$, and m_0 is given in (8-11) .

In fact, if $\gamma_0(s_1, s_2, \dots, s_l)(x) \notin D^\mu_\lambda$, then by (8-11) we have

$$\Psi_\lambda(\eta(T,\gamma_0(s_1,s_2,\dots,s_l)(x))) \le m_0,$$

and thus (8-12) holds.

Now we consider the case when $\gamma_0(s_1, s_2, \dots, s_l)(x) \in D^\mu_\lambda$. Set

$$\tilde{d}_\lambda := \min\{d_\lambda, \sigma_0\}, \quad T = \frac{\sigma_0\mu}{4\tilde{d}_\lambda}, \quad \text{and} \quad \tilde{\eta}(t) := \eta(t, \gamma_0(s_1, s_2, \dots, s_l)).$$

We have two cases:

(1) $\tilde{\eta}(t) \in D^{3\mu/2}_\lambda$ for all $t \in [0, T]$.

(2) $\tilde{\eta}(t_0) \in \partial D^{3\mu/2}_\lambda$ for some $t_0 \in [0, T]$.

If (1) holds, then $\varphi(\tilde{\eta}(t)) = 1$ and $\|\Psi'_\lambda(\tilde{\eta}(t))\|^*_\lambda \ge \tilde{d}_\lambda$ for all $t \in [0, T]$. It follows from (8-8) that

$$\begin{aligned}\Psi_\lambda(\tilde{\eta}(T)) &= \Psi_\lambda(\gamma_0(s_1,s_2,\dots,s_l)) + \int_0^T \frac{d}{ds}\Psi_\lambda(\tilde{\eta}(t)) \\ &\le c_J - 2\int_0^T \tilde{d}_\lambda\, ds = c_J - 2\tilde{d}_\lambda T \le c_J - \tfrac{1}{2}\tau_0\mu.\end{aligned}$$

If (2) holds, there exists $0 \le t_1 < t_2 \le T$ such that

$$\tilde{\eta}(t_1) \in \partial D^\mu_\lambda, \tag{8-13}$$

$$\tilde{\eta}(t_2) \in \partial D^{3\mu/2}_\lambda, \tag{8-14}$$

$$\tilde{\eta}(t) \in D^{3\mu/2}_\lambda \setminus D^\mu_\lambda \quad \text{for all } t \in [t_1, t_2]. \tag{8-15}$$

By (8-14), either

$$\|\tilde{\eta}(t_2)\|_{H^\lambda_G(\mathbb{R}^N\setminus\Omega'_j)} = \frac{3\mu}{2}$$

or

$$|\Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - c_{j_0}| = \frac{3\mu}{2} \quad \text{for some } j_0 \in J.$$

We only address the latter case; the former can be proved in a similar way. By (8-14), we have

$$|\Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1)) - c_{j_0}| \le \mu,$$

and hence

$$|\Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - \Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1))| \ge |\Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - c_{j_0}| - |\Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1)) - c_{j_0}| \ge \tfrac{1}{2}\mu.$$

On the other hand, by the mean value theorem, there exists $t' \in (t_1, t_2)$ such that

$$|\Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_2)) - \Psi_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t_1))| = \left|\Psi'_{\lambda,\Omega'_{j_0}}(\tilde{\eta}(t')) \cdot \frac{d\tilde{\eta}}{dt}\right|(t_2 - t_1).$$

Thus we have

$$\begin{aligned}\Psi_\lambda(\tilde{\eta}(T)) &= \Psi_\lambda(\gamma_0(s_1, s_2, \ldots, s_l)(x)) - \int_0^T \varphi(\tilde{\eta}(s))) \frac{\langle \Psi'(\tilde{\eta}(s))), \mathcal{G}(\tilde{\eta}(s)))\rangle}{\|\Psi'_\lambda(\tilde{\eta}(s)))\|^*_\lambda} v \\ &\le c_J - \int_{t_1}^{t_2} \varphi(\tilde{\eta}(s)))\|\Psi'_\lambda(\tilde{\eta}(s))\|^*_\lambda \, ds \\ &= c_J - \sigma_0(t_2 - t_1) \le c_J - \tfrac{1}{2}\tau_0\mu.\end{aligned}$$

Thus (8-12) is proved. Recall that $\tilde{\eta}(T) = \eta(T, \gamma_0(s_1, s_2, \ldots, s_l)) \in \Gamma_J$. Hence

$$b_{\lambda,J} \le \Psi_\lambda(\tilde{\eta}(T)) \le \max\{m_0, c_J - \tfrac{1}{2}\tau_0\mu\}. \tag{8-16}$$

However, by Corollary 7.2, we have $b_{\lambda,J} \to c_J$ as $\lambda \to \infty$. This contradicts (8-16), and hence Ψ_λ has a critical point $v_\lambda \in D^\mu_\lambda$ for large λ, so by Proposition 5.2, v_λ is a solution of the problem (1-10). □

Proof of Theorem 1.1. Let v_λ be a solution to the problem (1-1) obtained in Proposition 8.2. For any given sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$, we can extract a subsequence (still denoted by $\{\lambda_n\}$). Arguing as in the proof of Proposition 5.1, we can extract a subsequence of $\{v_{\lambda_n}\}$ (still denoted $\{v_{\lambda_n}\}$) such that $v_{\lambda_n} \to v$ in H^1_G and

$$\lim_{n\to\infty} \Psi_{\lambda_n}(v_n) = c_j \quad \text{for all } j \in J, \tag{8-17}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N \setminus \Omega'_J} (|\nabla v_{\lambda_n}|^2 + V_{\lambda_n}|f(v_{\lambda_n})|^2) = 0. \tag{8-18}$$

Since the limits in (8-17) and (8-18) do not depend on the choice of sequence $\{\lambda_n\}$ ($\lambda_n \to \infty$), then both (1-12) and (1-13) hold, and the limit function $v(x)$ satisfies:

(1) $v(x) = 0$ for $x \in \mathbb{R}^N \setminus \Omega_J$.

(2) $v|_{\Omega_j}$ is a least energy solution of

$$\begin{cases} -\Delta v(x) = |f(v)|^{p-1} f(v), & x \in \Omega_j, \\ \qquad v(x) \in H^{0,1}_G(\Omega_j) \end{cases}$$

for $j \in J$.

This completes the proof of Theorem 1.1. □

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