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QUASICONFORMAL CONJUGACY CLASSES OF PARABOLIC ISOMETRIES OF COMPLEX HYPERBOLIC SPACE

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We investigate the quasiconformal conjugacy classes of parabolic isometries acting on complex hyperbolic space. Our main result is that a screw parabolic isometry cannot be quasiconformally conjugate to a translation. This implies that a cyclic group generated by a screw parabolic isometry is not quasiconformally stable in its deformation space.

We are interested in the quasiconformal deformation theory of a complex hyperbolic quasi-Fuchsian group. We mainly focus on the case that the group is a cyclic group generated by a parabolic isometry.

We recall the definition of quasiconformal stability from Kleinian group theory (see, for instance, [Bers 1970; Kapovich 2008; Marden 1974; Maskit 1988]). Let Γ be a finitely generated discrete subgroup of the orientation-preserving isometry group $\mathrm{Isom}(\mathbb{H}^{n+1})$ acting on real hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$. Such a group Γ is called a Kleinian group. A representation $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^{n+1})$ is said to be a *deformation* if it is a discrete, faithful and type-preserving representation. The Kleinian group Γ is said to be *quasiconformally stable* if any deformation $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^{n+1})$ sufficiently near the identity deformation is obtained by a quasiconformal conjugation. That is, there is a quasiconformal mapping of the boundary at infinity, $\phi : \partial\mathbb{H}^{n+1} \to \partial\mathbb{H}^{n+1}$, such that $\rho(g) = \phi \circ g \circ \phi^{-1}$ for any $g \in \Gamma$.

In $\mathbb{H}^2$ and $\mathbb{H}^3$, a geometrically finite Kleinian group is quasiconformally stable [Bers 1970; Marden 1974]. This is one of the fundamental results in the deformation theory of Kleinian groups. However, there is a nonelementary geometrically finite Kleinian group of hyperbolic 4-space which is not quasiconformally stable [Kim 2011]. This is mainly due to the presence of screw parabolic isometries in hyperbolic 4-space.

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Hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ has the natural boundary at infinity $\hat{\mathbb{R}}^n$. Every isometry of $\mathbb{H}^{n+1}$ extends continuously to a Möbius transformation of $\hat{\mathbb{R}}^n$ which is a finite composition of reflections in codimension-1 spheres or hyperplanes, and vice versa. On the boundary at infinity $\hat{\mathbb{R}}^n$, a parabolic isometry is Möbius conjugate to $x \mapsto Ax + e_1$ with $A \in \mathrm{SO}(n)$, $A(e_1) = e_1$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. If $A = I$, then it is called a *strictly* parabolic isometry or a *translation*; otherwise it is a *screw* parabolic isometry. There are no screw parabolic isometries if $n < 3$. This means that there is only one conformal, and hence quasiconformal, conjugacy class of parabolic isometries in lower dimensions. In $\mathbb{H}^4$, screw parabolic isometries are not quasiconformally conjugate to translations. Furthermore, there are infinitely many distinct quasiconformal conjugacy classes of screw parabolic isometries. Let Γ be a cyclic group generated by a translation. Then we can deform Γ into a cyclic group Γ' generated by a screw parabolic isometry such that Γ is arbitrary close to Γ'. Hence, the cyclic group Γ is not quasiconformally stable in its deformation space. We can generalize this to a nonelementary Kleinian group of $\mathbb{H}^4$ (see [Kim 2011] for details). On the other hand, it is known that a convex cocompact (i.e., geometrically finite without parabolic isometries) Kleinian group is quasiconformally stable in any dimension [Izeki 2000].

Now, we consider the case of complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$. A complex hyperbolic quasi-Fuchsian group is a discrete, faithful, type-preserving and geometrically finite representation of the fundamental group of a surface in the group $\mathrm{PU}(2, 1)$ of holomorphic isometries acting on complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ [Goldman 1999; Parker and Platis 2010; Schwartz 2007]. It is the complex counterpart of a Kleinian group of real hyperbolic space. The deformation space is the set of all such groups factored by the conjugation action of the holomorphic isometry group $\mathrm{PU}(2, 1)$. Naturally, we can ask if a complex hyperbolic quasi-Fuchsian group is quasiconformally stable in its deformation space (see [Parker and Platis 2010] for more related questions). To that end, we consider a cyclic group generated by a parabolic isometry of $\mathbb{H}^2_{\mathbb{C}}$.

The boundary at infinity of complex hyperbolic space can be identified with the one-point compactification of the Heisenberg group $\mathscr{H}$: $\partial\mathbb{H}^2_{\mathbb{C}} = \mathscr{H} \cup \{\infty\}$. A holomorphic isometry of $\mathbb{H}^2_{\mathbb{C}}$ extends continuously to an extended Heisenberg group automorphism of $\partial\mathbb{H}^2_{\mathbb{C}}$, and vice versa. On $\partial\mathbb{H}^2_{\mathbb{C}}$, a parabolic isometry of $\mathbb{H}^2_{\mathbb{C}}$ is conjugate to either a Heisenberg translation or the composition of a vertical translation and a rotation by an element of $\mathrm{PU}(2, 1)$. We call the latter a screw parabolic isometry.

A Heisenberg translation can be conjugate to either a horizontal translation or a vertical translation by an element of $\mathrm{PU}(2, 1)$. We can conjugate a horizontal translation (or a vertical translation) further by an element of $\mathrm{PU}(2, 1)$ so that the translation length is 1 with respect to the Cygan norm of the Heisenberg group.

Therefore, we have the following classification of conformal classes of parabolic isometries up to the conjugation action of PU(2, 1):

(1)
- a horizontal translation $T_{(1,0)}$,
- a vertical translation $T_{(0,1)}$,
- a 1-parameter family of screw parabolic isometries $\{A_\theta : \theta \in (0, 2\pi)\}$,

where

$$T_{(\zeta,\nu)} = \begin{pmatrix} 1 & -\sqrt{2}\bar{\zeta} & -|\zeta|^2 + i\nu \\ 0 & 1 & \sqrt{2}\zeta \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(2,1) \tag{2}$$

for $\zeta \in \mathbb{C}$, $\nu \in \mathbb{R}$, and

$$A_\theta = \begin{pmatrix} 1 & 0 & i \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(2,1) \tag{3}$$

for $\theta \in (0, 2\pi)$ (see Section 1B for details).

Miner [1994] showed that a horizontal translation and a vertical translation are not quasiconformally conjugate. That is, no quasiconformal mapping of the Heisenberg group conjugates a horizontal translation to a vertical one. We prove here that a screw parabolic isometry is not quasiconformally conjugate to a translation, as follows:

Theorem 3.3. *Let $T_{(0,1)}(z, t) = (z, t + 1)$ be a vertical translation and $A(z, t) = (e^{i\theta} z, t + 1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism of the Heisenberg group $\mathcal{H}$. Then A is not quasiconformally conjugate to $T_{(0,1)}$.*

Theorem 3.7. *Let $T_{(1,0)}(z, t) = (z + 1, t + 2\,\mathrm{Im}\,\bar{z})$ be a horizontal translation and $A_\theta(z, t) = (e^{i\theta} z, t + 1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism of the Heisenberg group $\mathcal{H}$. Then A_θ is not quasiconformally conjugate to $T_{(1,0)}$.*

A screw parabolic isometry is called *rational* if some iteration of it becomes a translation; otherwise, it is called *irrational*. For a rational screw parabolic isometry A, the *order* of A is the smallest positive integer n such that A^n becomes a translation. For the 1-parameter family of screw parabolic isometries from (1), we prove that a rational screw parabolic isometry cannot be quasiconformally conjugate to an irrational screw parabolic isometry in Corollary 3.4, that two distinct rational screw parabolic isometries are quasiconformally conjugate only if they have the same order in Corollary 3.5, and that two distinct irrational screw parabolic isometries are not quasiconformally conjugate to each other in Proposition 3.6. In summary, together with the result of [ibid.], we have the following distinct quasiconformal conjugacy classes of parabolic isometries of $\mathbb{H}^2_{\mathbb{C}}$ (compare with the list (1)):

- a horizontal translation $T_{(1,0)}$;
- a vertical translation $T_{(0,1)}$;
- a subfamily of irrational screw parabolic isometries $\{A_\vartheta : \vartheta \in (0, 2\pi) \text{ irrational}\}$;
- a subfamily of rational screw parabolic isometries $\{A_{2\pi i/n} : n = 2, 3, \ldots\}$.

Let $\Gamma < \mathrm{PU}(2, 1)$ be a cyclic group generated by a vertical translation. Then we can deform Γ into a cyclic group Γ' generated by a screw parabolic isometry such that Γ is arbitrary close to Γ' with respect to the l^2 norm of $\mathrm{PU}(2, 1)$. Applying Theorem 3.3, this shows that Γ is not quasiconformally stable in its deformation space. Thus, we have:

Theorem. *Let* $\Gamma < \mathrm{PU}(2, 1)$ *be a cyclic group generated by a vertical translation. Then it is not quasiconformally stable in its deformation space.*

This paper is organized as follows. In Section 1, we recall some basic facts related to complex hyperbolic geometry, the Heisenberg group and the theory of quasiconformal mappings. In Section 2, we construct a family of horizontal curves in a cylindrical region and compute the modulus of the curve family. This curve family will be used to prove Theorem 3.3 in Section 3. We will also prove Theorem 3.7 in Section 3.

1. Preliminaries

Throughout this section, we use [Goldman 1999] as references for the basic definitions of complex hyperbolic geometry and [Korányi and Reimann 1985; 1995] for the theory of quasiconformal mappings.

1A. ***Complex hyperbolic space.*** Let $\mathbb{C}^{2,1}$ be the complex vector space $\mathbb{C}^3$ with the Hermitian form of signature (2, 1) given by

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle = \boldsymbol{w}^* J \boldsymbol{z} = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1, \tag{4}$$

where the Hermitian matrix is

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consider the following subspaces of $\mathbb{C}^{2,1}$:

$$\begin{aligned} V_- &= \{\boldsymbol{z} \in \mathbb{C}^{2,1} : \langle \boldsymbol{z}, \boldsymbol{z} \rangle < 0\}, \\ V_0 &= \{\boldsymbol{z} \in \mathbb{C}^{2,1} - \{0\} : \langle \boldsymbol{z}, \boldsymbol{z} \rangle = 0\}. \end{aligned} \tag{5}$$

Let $\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \to \mathbb{CP}^2$ be the canonical projection onto complex projective space. Then complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ is defined to be $\mathbb{P}V_-$ and the boundary at infinity $\partial\mathbb{H}^2_{\mathbb{C}}$ to be $\mathbb{P}V_0$. We define the Siegel domain model of complex hyperbolic space by considering the section defined by $z_3 = 1$. For any $z = (z_1, z_2) \in \mathbb{C}^2$, we lift the point z to $\boldsymbol{z} = (z_1, z_2, 1) \in \mathbb{C}^{2,1}$, called the *standard lift of* z. Then $\langle \boldsymbol{z}, \boldsymbol{z}\rangle = z_1 + z_2\bar{z}_2 + \bar{z}_1$. Hence the Siegel domain model of complex hyperbolic space is defined by

$$\mathbb{H}^2_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\,\mathrm{Re}(z_1) + |z_2|^2 < 0\}. \tag{6}$$

The boundary is the one-point compactification of the paraboloid defined by $\{(z_1, z_2) \in \mathbb{C}^2 : 2\,\mathrm{Re}(z_1) + |z_2|^2 = 0\}$. The standard lift of ∞ is $(1, 0, 0) \in \mathbb{C}^{2,1}$.

The *Bergman metric* ρ on $\mathbb{H}^2_{\mathbb{C}}$ is defined by

$$\cosh^2\left(\frac{\rho(z, w)}{2}\right) = \frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle\langle \boldsymbol{w}, \boldsymbol{z}\rangle}{\langle \boldsymbol{z}, \boldsymbol{z}\rangle\langle \boldsymbol{w}, \boldsymbol{w}\rangle}, \tag{7}$$

where $\boldsymbol{z}$ and $\boldsymbol{w}$ are the standard lifts of z and $w \in \mathbb{H}^2_{\mathbb{C}}$. Let $\mathrm{SU}(2, 1)$ be the group of unitary matrices which preserve the given Hermitian form with determinant 1. Then the group of holomorphic isometries of $\mathbb{H}^2_{\mathbb{C}}$ is $\mathrm{PU}(2, 1) = \mathrm{SU}(2, 1)/\{I, \omega I, \omega^2 I\}$, where $\omega = (-1 + i\sqrt{3})/2$ is a cube root of unity.

Let $z = (z_1, z_2) \in \partial\mathbb{H}^2_{\mathbb{C}}$ be a finite point with standard lift $\boldsymbol{z} = (z_1, z_2, 1)$ satisfying

$$2\,\mathrm{Re}(z_1) + |z_2|^2 = 0. \tag{8}$$

We write $\zeta = z_2/\sqrt{2} \in \mathbb{C}$. Then (8) implies that $2\,\mathrm{Re}(z_1) = -2|\zeta|^2$. We can also write $z_1 = -|\zeta|^2 + i\nu$ for $\nu \in \mathbb{R}$. Thus,

$$\boldsymbol{z} = \begin{pmatrix} -|\zeta|^2 + i\nu \\ \sqrt{2}\zeta \\ 1 \end{pmatrix} \tag{9}$$

for $\zeta \in \mathbb{C}$ and $\nu \in \mathbb{R}$. Thus, we identify the boundary $\partial\mathbb{H}^2_{\mathbb{C}}$ with the one-point compactification of $\mathbb{C} \times \mathbb{R}$. Furthermore, an element $T_{(\zeta,\nu)} \in \mathrm{SU}(2, 1)$ of (2) is the unique unipotent upper triangular matrix which sends $(0, 0) \in \mathbb{C} \times \mathbb{R}$ to the finite point $(\zeta, \nu) \in \mathbb{C} \times \mathbb{R}$. The group structure of the unipotent upper triangular matrices induces a group multiplication on $\mathbb{C} \times \mathbb{R}$, which is the Heisenberg group structure.

1B. ***Heisenberg group.*** The Heisenberg group $\mathscr{H}$ can be described as the set of pairs $(z, t) \in \mathbb{C} \times \mathbb{R}$ with the group multiplication

$$(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}\, z_1\bar{z}_2). \tag{10}$$

The *Cygan* norm on $\mathscr{H}$ is defined by $|(z, t)| = (|z|^4 + t^2)^{1/4}$, and the Cygan metric d is given by

$$d((z_1, t_1), (z_2, t_2)) = |(z_1, t_1)^{-1} \cdot (z_2, t_2)|. \tag{11}$$

The Heisenberg group $\mathcal{H}$ acts on itself by left translation: $T_{(z_0,t_0)}(z,t)=(z_0,t_0)\cdot(z,t)$ for $(z_0,t_0)\in\mathcal{H}$. A Heisenberg translation of the form $T_{(0,t)}$ for $t\in\mathbb{R}$ is called a *vertical translation*. The unitary group $U(1)$ acts by *rotations*: $(z,t)\mapsto(\lambda z,t)$ for a unit $\lambda\in\mathbb{C}-\{1\}$. *Real dilation* is defined by $(z,t)\mapsto(rz,r^2t)$ for $r\in\mathbb{R}_+-\{1\}$. A parabolic Heisenberg group automorphism is either a Heisenberg translation or the composition of a vertical translation and a rotation. We call the latter type *screw parabolic*. A screw parabolic automorphism $A(z,t)=(e^{i\theta}z,t+s)$, for $\theta\in(0,2\pi)$, $s\in\mathbb{R}$, is said to be *rational* if some iteration of it becomes a Heisenberg translation. Otherwise, it is said to be *irrational*. The *Heisenberg similarity group* is generated by Heisenberg translations, rotations, and real dilations.

It is known to many people that there are two conformal conjugacy classes of Heisenberg translations. More precisely, we can conjugate a Heisenberg translation by a holomorphic isometry of $\mathbb{H}^2_{\mathbb{C}}$ to obtain a horizontal translation or a vertical translation in the following way. Let T be a nonvertical translation. We may conjugate T by a Heisenberg automorphism $m(z,t)=(\lambda e^{i\theta}z,\lambda^2t)$ for $\lambda\in\mathbb{R}_+$, $\theta\in[0,2\pi)$, such that

$$m\circ T\circ m^{-1}=T_{(r,s)}, \tag{12}$$

where $T_{(r,s)}(z,t)=(z+r,t+s+2r\,\mathrm{Im}\,\bar{z})$ for some real numbers r and s with $r\neq 0$. For a computation, we note that for $w\in\mathbb{C}$, $(w,t)(r,s)(-w,-t)=(r,s+4r\,\mathrm{Im}\,w)$ and hence $s+4r\,\mathrm{Im}\,w=0$ if $\mathrm{Im}\,w=-s/4r$. We conjugate both sides of (12) by a Heisenberg translation $T_{(w,t)}$ with $\mathrm{Im}\,w=-s/4r$ as follows:

$$T_{(w,t)}mTm^{-1}T^{-1}_{(w,t)} \;=\; T_{(w,t)}T_{(r,s)}T^{-1}_{(w,t)} \;=\; T_{(r,0)}. \tag{13}$$

Conjugating both sides of (13) by a dilation $L(z,t)=(Lz,L^2t)$ for some $L\in\mathbb{R}_+$, we have

$$LT_{(w,t)}mTm^{-1}T^{-1}_{(w,t)}L^{-1} \;=\; LT_{(r,0)}L^{-1} \;=\; T_{(1,0)}, \tag{14}$$

where $T_{(1,0)}(z,t)=(z+1,t+2\,\mathrm{Im}\,\bar{z})$. Thus, any nonvertical translation T is conjugate to $T_{(1,0)}$ by a Heisenberg automorphism.

A screw parabolic isometry can be conjugated to $A_\theta(z,t)=(e^{i\theta}z,t+1)$, with $\theta\in(0,2\pi)$, by an element of $\mathrm{SU}(2,1)$. In addition, two distinct normalized screw parabolic isometries are not $\mathrm{SU}(2,1)$-conjugate to each other. Therefore, we have the following classification of conformal classes of parabolic isometries up to the conjugation action of the holomorphic isometries of $\mathbb{H}^2_{\mathbb{C}}$:

- a horizontal translation $T_{(1,0)}(z,t)=(z+1,t+2\,\mathrm{Im}\,\bar{z})$;
- a vertical translation $T_{(0,1)}(z,t)=(z,t+1)$;

- a 1-parameter family of screw parabolic isometries

$$\{A_\theta(z,t) = (e^{i\theta}z, t+1) : \theta \in (0, 2\pi)\}.$$

1C. ***Quasiconformal mappings.*** Let $\phi : \mathcal{H} \to \mathcal{H}$ be a homeomorphism. We define

$$M(p,r) = \sup_{\{q:d(p,q)=r\}} d(\phi p, \phi q) \quad \text{and} \quad m(p,r) = \inf_{\{q:d(p,q)=r\}} d(\phi p, \phi q) \tag{15}$$

for $p \in \mathcal{H}$ and $r > 0$.

Definition 1.1. A homeomorphism $\phi : \mathcal{H} \to \mathcal{H}$ is called K-quasiconformal if the function

$$H(p) = \limsup_{r\to 0} \frac{M(p,r)}{m(p,r)} \tag{16}$$

for $p \in \mathcal{H}$ is uniformly bounded by K.

We also need to use the Carnot–Carathéodory metric d_{cc} for our proof with quasiconformal mappings. A smooth curve $\gamma : [0,1] \to \mathcal{H}$ is *horizontal* if, for all $t \in [0,1]$, its tangent vector $\dot{\gamma}(t)$ lies in the subspace of the tangent space spanned by the vector fields $X = \partial/\partial x + 2y\,\partial/\partial t$ and $Y = \partial/\partial y - 2x\,\partial/\partial t$ for $(x, y, t) \in \mathbb{C} \times \mathbb{R}$. We define a quadratic form g on the planes generated by vector fields X and Y such that X and Y are orthonormal. Then the Carnot–Carathéodory length of γ is given by

$$l(\gamma) = \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}\,dt \tag{17}$$

and the Carnot–Carathéodory distance d_{cc} between two points $p, q \in \mathcal{H}$ is the infimum of the Carnot–Carathéodory lengths of all horizontal curves connecting p to q.

Let Γ be a family of piecewise-C^1 horizontal curves. Denote by Σ_Γ the collection of nonnegative Borel measurable functions $\sigma : \mathcal{H} \to \mathbb{R}$ such that $\int_\gamma \sigma \geq 1$ for all $\gamma \in \Gamma$. These are the so-called *admissible functions*. Then we define the *modulus* of Γ by

$$\mathrm{M}(\Gamma) = \inf_{\sigma\in\Sigma_\Gamma} \int_{\mathcal{H}} \sigma^4\,d\mathrm{vol}. \tag{18}$$

We now relate the modulus of a curve family to a quasiconformal mapping.

Theorem 1.2 [Korányi and Reimann 1995]. *If a homeomorphism $\phi : \mathcal{H} \to \mathcal{H}$ is K-quasiconformal, then*

$$\frac{1}{K^2}\mathrm{M}(\Gamma) \leq \mathrm{M}(\phi\Gamma) \leq K^2\mathrm{M}(\Gamma) \tag{19}$$

for any curve family Γ.

The Cygan metric d and the Carnot–Carathéodory metric d_{cc} give us the same classes of quasiconformal mappings since they are bi-Lipschitz related:

Theorem 1.3 [Basmajian and Miner 1998]. *For any* $p, q \in \mathcal{H}$,

$$d(p,q) \le d_{cc}(p,q) \le \sqrt{\pi}\, d(p,q).$$

Finally, we need the following property of quasiconformal mappings.

Proposition 1.4 [Korányi and Reimann 1995]. *There exists a constant C such that for any K-quasiconformal mapping $\phi : \mathcal{H} \to \mathcal{H}$,*

$$\frac{M(p,r)}{m(p,r)} \le e^{KC}$$

for any $p \in \mathcal{H}$ and $r > 0$.

2. The modulus of a cylinder

We construct here a family of piecewise smooth horizontal curves in a cylindrical region and compute its modulus. Let $\alpha_0 : [0,1] \to \mathcal{H}$ be a piecewise smooth horizontal curve defined by $\alpha_0(t) = \alpha^1(t) * \alpha^2(t) * \alpha^3(t) * \alpha^4(t)$ (see Figure 1), where

$$\begin{aligned}
\alpha^1(t) &= (2ti, 0), & 0 \le t \le \tfrac{1}{4},\\
\alpha^2(t) &= \big(2t - \tfrac{1}{2} + \tfrac{1}{2}i,\ 2t - \tfrac{1}{2}\big), & \tfrac{1}{4} \le t \le \tfrac{1}{2},\\
\alpha^3(t) &= \big(\tfrac{1}{2} + \big(\tfrac{3}{2} - 2t\big)i,\ 2t - \tfrac{1}{2}\big), & \tfrac{1}{2} \le t \le \tfrac{3}{4},\\
\alpha^4(t) &= (2 - 2t, 1), & \tfrac{3}{4} \le t \le 1.
\end{aligned} \tag{20}$$

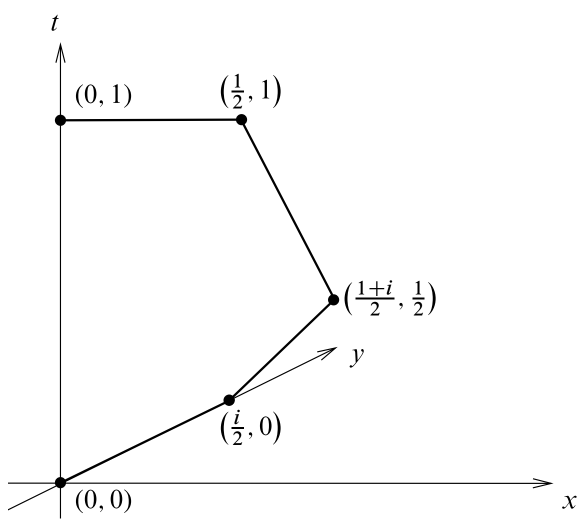

Figure 1. A piecewise smooth horizontal curve $\alpha_0(t)$.

Note that $\alpha_0(0) = (0, 0)$, $\alpha_0(1) = (0, 1)$, and $|\dot{\alpha}^i(t)| = 2$. The Carnot–Carathéodory length of α_0 is

$$l(\alpha_0) = \sum_{i=1}^{4} \int_0^{1/4} |\dot{\alpha}^i(t)|\, dt = 2.$$

Translating α_0 by $T_{(z,0)}$, for $(z, 0) \in \mathcal{H}$, produces a piecewise smooth horizontal curve α_z, given by

$$\alpha_z(t) = T_{(z,0)}\alpha_0(t), \quad 0 \le t \le 1, \tag{21}$$

from $\alpha_z(0) = (z, 0)$ to $\alpha_z(1) = (z, 1)$. Let $\alpha_z^i(t) = T_{(z,0)}\alpha^i(t)$. Then $\alpha_z(t) = \alpha_z^1(t) * \alpha_z^2(t) * \alpha_z^3(t) * \alpha_z^4(t)$, where

$$\alpha_z^1(t) = (x, y + 2t, -4xt), \qquad 0 \le t \le \tfrac{1}{4}, \tag{22}$$

$$\alpha_z^2(t) = \big(x + 2t - \tfrac{1}{2}, y + \tfrac{1}{2}, 2t - \tfrac{1}{2} - x + 4ty - y\big), \qquad \tfrac{1}{4} \le t \le \tfrac{1}{2}, \tag{23}$$

$$\alpha_z^3(t) = \big(x + \tfrac{1}{2}, y + \tfrac{3}{2} - 2t, 2t - \tfrac{1}{2} - 3x + 4tx + y\big), \qquad \tfrac{1}{2} \le t \le \tfrac{3}{4}, \tag{24}$$

$$\alpha_z^4(t) = (x + 2 - 2t, y, 1 + 2(2 - 2t)y), \qquad \tfrac{3}{4} \le t \le 1. \tag{25}$$

Since Heisenberg translations are isometries with respect to the Carnot–Carathéodory metric, all curves α_z have Carnot–Carathéodory length 2. Define a family of curves $\Gamma_{r,R}$ for $0 < r < R$ by

$$\Gamma_{r,R} = \{\alpha_z : r \le |z| \le R\}. \tag{26}$$

This family of curves defines a mapping α from the cylindrical region

$$D = \{(x, y) \in \mathbb{C} : r^2 < x^2 + y^2 < R^2\} \times [0, 1]$$

to $\mathcal{H}$, given by

$$\alpha(x, y, t) = \alpha_{x+yi}(t). \tag{27}$$

Let $D_i = \{(x, y) : r^2 < x^2 + y^2 < R^2\} \times [(i-1)/4, i/4]$, $i = 1, 2, 3, 4$, so that $D = \bigcup_{i=1}^4 D_i$. Then the Jacobian determinant of α is given by

$$|\mathrm{J}\alpha(x, y, t)| = \begin{cases} 4|x| & \text{on } D_1, \\ 4|1+y| & \text{on } D_2, \\ 4|1+x| & \text{on } D_3, \\ 4|y| & \text{on } D_4. \end{cases} \tag{28}$$

Lemma 2.1. *For $1 < r < R$, we have the following lower bound for the modulus of the curve family*:

$$\mathrm{M}(\Gamma_{r,R}) \ge \tfrac{1}{256}(R^2 - r^2)\left(\frac{\pi}{2} - 2\arctan\frac{1}{\sqrt{r^2 - 1}}\right).$$

Proof. Let σ be an arbitrary admissible function in $\Sigma_{\Gamma_{\sqrt{r},\sqrt{R}}}$. By Hölder's inequality,

$$1 \le \int_{\alpha_z} \sigma \le \left(\int_0^1 \sigma^2(\alpha_z(t))\,dt\right)^{1/2} \left(\int_0^1 |\dot{\alpha}_z(t)|^2\,dt\right)^{1/2}. \tag{29}$$

Since $\int_0^1 |\dot{\alpha}_z(t)|^2\,dt = 2$,

$$\frac{1}{2} \le \int_0^1 \sigma^2(\alpha_z(t))\,dt = \sum_{i=1}^{4} \int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^2(\alpha_z^i(t))\,dt. \tag{30}$$

Applying Hölder's inequality to each term of the right-hand side, we have

$$\begin{aligned} &\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^2(\alpha_z^i(t))|\mathrm{J}\alpha|^{1/2} \frac{1}{|\mathrm{J}\alpha|^{1/2}}\,dt \\ &\qquad \le \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))|\mathrm{J}\alpha|\,dt\right)^{1/2} \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \frac{1}{|\mathrm{J}\alpha|}\,dt\right)^{1/2}. \end{aligned} \tag{31}$$

From (30), using the Jacobian determinant (28) and (31), we have

$$\begin{aligned} \frac{1}{2} &\le \sum_{i=1}^{4} \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{\frac{1}{2}} \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \frac{1}{|\mathrm{J}\alpha|}\,dt\right)^{\frac{1}{2}} \\ &\le \left(\sum_{i=1}^{4} \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{\frac{1}{2}}\right) \left(\sum_{i=1}^{4} \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \frac{1}{|\mathrm{J}\alpha|}\,dt\right)^{\frac{1}{2}}\right) \\ &\le \frac{1}{4}\left(\sum_{i=1}^{4} \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{\frac{1}{2}}\right) \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x+1|}} + \frac{1}{\sqrt{|y|}} + \frac{1}{\sqrt{|y+1|}}\right). \end{aligned} \tag{32}$$

Thus,

$$\begin{aligned} &2\left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x+1|}} + \frac{1}{\sqrt{|y|}} + \frac{1}{\sqrt{|y+1|}}\right)^{-1} \\ &\qquad \le \sum_{i=1}^{4} \left(\int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{1/2} \le 4\left(\sum_{i=1}^{4} \int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\,|\mathrm{J}\alpha|\,dt\right)^{1/2}. \end{aligned} \tag{33}$$

Equations (31), (32) and (33) only hold if $|\mathrm{J}\alpha| \neq 0$. However, when we estimate a lower bound of the modulus in (35), we will restrict the domain of the integration so that we may assume $|\mathrm{J}\alpha| \neq 0$.

Using the trivial inequality

$$4\int_{\alpha(D)} \sigma^4\,d\mathrm{vol} \ge \sum_{i=1}^{4} \int_{\alpha(D_i)} \sigma^4\,d\mathrm{vol} \tag{34}$$

and defining $U = \{x, y : r^2 \le x^2 + y^2 \le R^2,\ x \ge 1,\ y \ge 1\}$, we have

$$
\begin{aligned}
(35)\qquad \int_{\mathscr{H}} \sigma^4\, d\mathrm{vol} &\ge \int_{\alpha(D)} \sigma^4\, d\mathrm{vol} = \frac{1}{4}\sum_{i=1}^{4} \int_{\alpha(D_i)} \sigma^4\, d\mathrm{vol} \\
&\ge \sum_{i=1}^{4} \iint_U \int_{\frac{i-1}{4}}^{\frac{i}{4}} \sigma^4(\alpha_z^i(t))\, |\mathrm{J}\alpha|\, dt\, dx\, dy \\
&\ge \iint_U \frac{1}{4}\left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x+1|}} + \frac{1}{\sqrt{|y|}} + \frac{1}{\sqrt{|y+1|}}\right)^{-2} dx\, dy \\
&\ge \tfrac{1}{256}\, \mathrm{Area}(U);
\end{aligned}
$$

the third inequality follows from (33) and the fact that $d\mathrm{vol} = 4\, dx\, dy\, dt$; for the last inequality we argue as follows:

$$
\begin{aligned}
&\left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x+1|}} + \frac{1}{\sqrt{|y|}} + \frac{1}{\sqrt{|y+1|}}\right)^{-2} \\
&\quad \ge \left(\frac{\sqrt{|x(x+1)y(y+1)|}}{\sqrt{|(x+1)y(y+1)|}+\sqrt{|xy(y+1)|}+\sqrt{|x(x+1)(y+1)|}+\sqrt{|x(x+1)y|}}\right)^2 \\
&\quad \ge \left(\frac{\sqrt{|x(x+1)y(y+1)|}}{4\sqrt{|(x+1)y(y+1)| + |xy(y+1)| + |x(x+1)(y+1)| + |x(x+1)y|}}\right)^2 \\
&\quad \ge \frac{1}{16} \cdot \frac{|x(x+1)y(y+1)|}{|(x+1)y(y+1)| + |xy(y+1)| + |x(x+1)(y+1)| + |x(x+1)y|} \\
&\quad \ge \frac{1}{16} \cdot \frac{x(x+1)y(y+1)}{4x(x+1)y(y+1)} = \frac{1}{64} \quad \text{if } x \ge 1,\ y \ge 1.
\end{aligned}
$$

Since σ was arbitrary, we obtain (see Figure 2)

$$
\mathrm{M}(\Gamma_{r,R}) \ge \tfrac{1}{256}(R^2 - r^2)\left(\frac{\pi}{2} - 2\arctan\frac{1}{\sqrt{r^2-1}}\right) \qquad \square
$$

3. Parabolic quasiconformal conjugacy classes

Throughout this section, let $A(z,t) = A_\theta(z,t) = (e^{i\theta}z, t+1)$ be a screw parabolic automorphism of the Heisenberg group $\mathscr{H}$ for $\theta \in (0, 2\pi)$, and

$$
T_{(z_0,t_0)}(z,t) = (z + z_0, t + t_0 + 2\,\mathrm{Im}\, z_0\bar{z})
$$

be a Heisenberg translation for $(z_0, t_0) \in \mathscr{H}$. An injective map $\phi : \mathscr{H} \to \mathscr{H}$ is called *quasisymmetric* if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$
(36)\qquad d(x,y) \le t\, d(x,z) \implies d(\phi x, \phi y) \le \eta(t)\, d(\phi x, \phi z)
$$

for $x, y, z \in \mathscr{H}$, $t \in [0, \infty)$.

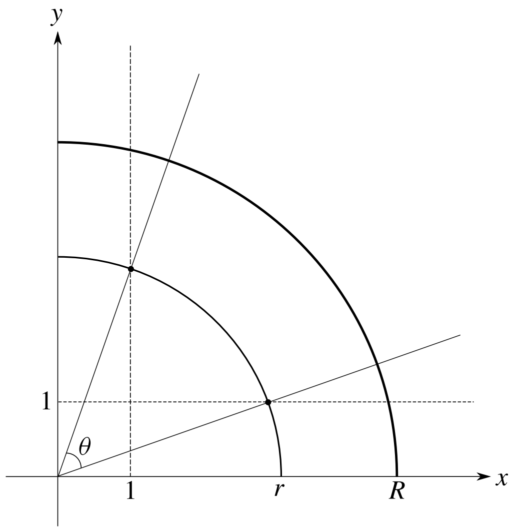

Figure 2. $\theta = \frac{\pi}{2} - 2\arctan\frac{1}{\sqrt{r^2-1}}$ (see previous page).

Theorem 3.1 [Heinonen and Holopainen 1997, Theorem 6.21]. *If $\phi : \mathcal{H} \to \mathcal{H}$ is quasiconformal, then it is quasisymmetric.*

Lemma 3.2. *Let $\phi : \mathcal{H} \to \mathcal{H}$ be a quasiconformal map that fixes all integer points $(0, n)$ on the vertical axis. Then there exist a nondecreasing function $c : [0, \infty) \to [0, \infty)$ and a constant $r_0 > 0$ satisfying*:

- $\lim_{r\to\infty} c(r) = \infty$,
- *for any $re^{i\theta} \in \mathbb{C}$ with $r > r_0$,*

$$|p(\phi(re^{i\theta}, 0))| \geq c(r), \tag{37}$$

where $p : \mathcal{H} \to \mathbb{C}$ is the vertical projection.

Proof. Throughout the proof, $[x]$ denotes the greatest integer less than or equal to x for any $x \in \mathbb{R}$ and $B(p, r)$ the ball of radius $r \geq 0$ centered at $p \in \mathcal{H}$.

We use the property that the quasiconformal map ϕ is quasisymmetric for a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ (Theorem 3.1). For any $re^{i\theta} \in \mathbb{C}$, $r > 0$,

$$\frac{d((0,0),(0,[r]^2))}{d((0,0),(re^{i\theta},0))} = \frac{[r]}{r} \leq 1$$

implies that

$$\frac{d(\phi(0,0),\phi(0,[r]^2))}{d(\phi(0,0),\phi(re^{i\theta},0))} = \frac{[r]}{d((0,0),\phi(re^{i\theta},0))} \leq \eta(1).$$

Thus we have

$$\frac{[r]}{\eta(1)} \le d((0,0), \phi(re^{i\theta},0)), \tag{38}$$

and hence $\phi(re^{i\theta},0)$ lies in the complement of the ball $B((0,0),[r]/\eta(1))$.

Similarly, for any $re^{i\theta} \in \mathbb{C}$ and any integer n,

$$\frac{d((0,n),(0,0))}{d((0,n),(re^{i\theta},0))} = \frac{\sqrt{|n|}}{(r^4+n^2)^{1/4}} \le 1 \tag{39}$$

implies

$$\frac{d(\phi(0,n),\phi(0,0))}{d(\phi(0,n),\phi(re^{i\theta},0))} = \frac{\sqrt{|n|}}{d((0,n),\phi(re^{i\theta},0))} \le \eta(1). \tag{40}$$

Thus,

$$\frac{\sqrt{|n|}}{\eta(1)} \le d((0,n),\phi(re^{i\theta},0)), \tag{41}$$

and hence $\phi(re^{i\theta},0)$ lies in the complement of the ball $B((0,n),\sqrt{|n|}/\eta(1))$. Since the integer n was arbitrary, the image $\phi(re^{i\theta},0)$ also lies in the complement of the set

$$\bigcup_{n\in\mathbb{Z}} B\left((0,n),\frac{\sqrt{|n|}}{\eta(1)}\right).$$

Therefore, together with (38), the image $\phi(re^{i\theta},0)$ should lie in the complement of

$$D_r = B\left((0,0),\frac{[r]}{\eta(1)}\right) \cup \bigcup_{n\in\mathbb{Z}} B\left((0,n),\frac{\sqrt{|n|}}{\eta(1)}\right).$$

Note that the t-intersects of the sphere of radius $[r]/\eta(1)$ centered at $(0,0)$ are $\pm(0,[r]^2/\eta^2(1))$. We put

$$n_r = \left[\frac{[r]^2}{\eta^2(1)}\right] \in \mathbb{N}.$$

Take a positive real number r_0 large enough that $n_{r_0} > \eta(1)$.

To finish the proof, we will show that for $r > r_0$, D_r contains an infinite cylinder

$$C_r = \{(z,t) \in \mathcal{H} : |z| \le c(r), t \in \mathbb{R}\},$$

where $c(r)$ is a positive function such that $\lim_{r\to\infty} c(r) = \infty$. Since D_r is symmetric with respect to the z-plane of $\mathcal{H}$, it suffices to show that the upper half of D_r, denoted by $\frac{1}{2}D_r$, contains a half cylinder $\frac{1}{2}C_r = \{(z,t) \in \mathcal{H} : |z| \le c(r), t \ge 0\}$.

Since

$$B\left((0,n),\frac{\sqrt{n_r}}{\eta(1)}\right) \subseteq B\left((0,n),\frac{\sqrt{n}}{\eta(1)}\right)$$

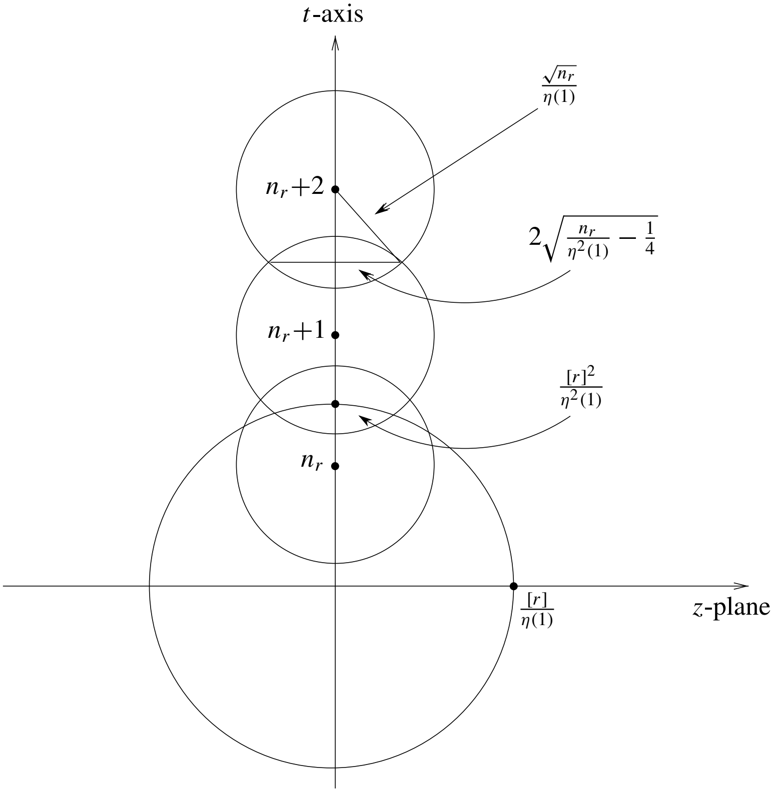

Figure 3. A set $\frac{1}{2}D'_r$.

for $n > n_r > 0$, the set $\frac{1}{2}D_r$ contains a proper subset $\frac{1}{2}D'_r$ (see Figure 3):

$$\tfrac{1}{2}D'_r = B\left((0,0), \frac{[r]}{\eta(1)}\right) \cup \bigcup_{n\geq n_r} B\left((0,n), \frac{\sqrt{n_r}}{\eta(1)}\right).$$

Take

$$c(r) = \min\left\{\sqrt{\frac{n_r}{\eta^2(1)}-\frac{1}{4}}, \frac{[r]}{\eta(1)}\right\}.$$

Then we see that $\frac{1}{2}D'_r$ contains the half cylinder $\frac{1}{2}C_r = \{(z,t)\in\mathscr{H} : |z| \leq c(r), t\geq 0\}$. Therefore, we have the lemma. □

Theorem 3.3. *Let $T_{(0,1)}(z,t) = (z, t+1)$ be a vertical translation and $A(z,t) = (e^{i\theta}z, t+1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism of the Heisenberg group $\mathscr{H}$. Then A is not quasiconformally conjugate to $T_{(0,1)}$.*

Proof. Suppose, to the contrary, that a K-quasiconformal map $\phi : \mathscr{H} \to \mathscr{H}$ exists such that

$$\phi \circ A \circ \phi^{-1} = T_{(0,1)}. \tag{42}$$

Let Γ_1 and Γ_2 be the cyclic groups generated by A and $T_{(0,1)}$, respectively. Then ϕ projects to a K-quasiconformal mapping, called ϕ again, between the quotients. That is,

$$\phi : \mathcal{H}/\Gamma_1 \to \mathcal{H}/\Gamma_2. \tag{43}$$

If ϕ does not fix $(0, 0)$, we compose (42) with a Heisenberg translation m that sends $\phi(0, 0)$ to $(0, 0)$, so that we have

$$m \circ \phi \circ A = m \circ T_{(0,1)} \circ \phi. \tag{44}$$

Because the vertical translation $T_{(0,1)}$ commutes with all Heisenberg translations,

$$m \circ \phi \circ A = T_{(0,1)} \circ m \circ \phi, \tag{45}$$

and since m is conformal, $m \circ \phi$ is also K-quasiconformal and fixes $(0, 0)$. Hence, without loss of generality we may assume that the quasiconformal mapping ϕ of (42) fixes $(0, 0)$.

Evaluating (42) at $(0, 0)$ shows that $\phi(0, 1) = (0, 1)$. By induction, ϕ fixes all integer points $\{(0, n)\}$ on the vertical axes. The global estimate of Proposition 1.4 implies that there exists a constant c_0 such that for any given integer r, there is some r' for which

$$B_{r'} \subseteq \phi(B_{\sqrt{r}}) \subseteq B_{c_0 r'}, \tag{46}$$

where B_t is a ball of radius t centered at the origin. Since the integer point $(0, r)$ is fixed by ϕ, the point $(0, r)$ also lies in $\phi(S_{\sqrt{r}})$, where S_t is the sphere of radius t centered at the origin. Hence, (46) implies that

$$r' \le \sqrt{r} \le c_0 r'. \tag{47}$$

We consider the curve family $\Gamma_{\sqrt{r},\sqrt{R}}$ from (26), where r and R are square integers satisfying $r_0 < \sqrt{r} < \sqrt{R}$ and r_0 is the constant from Lemma 3.2. We put $\Gamma = \Gamma_{\sqrt{r},\sqrt{R}}$ during this proof. All curves α_z in Γ have length $\sqrt{\pi}$ and are homotopic to the generator of $\pi_1(\mathcal{H}/\Gamma_1)$.

We now compute the modulus of the family $\phi\Gamma$ consisting of the images of curves in Γ under ϕ. For any $r > 0$, let l_r denote the Carnot–Carathéodory distance from $(r, 0)$ to $A(r, 0) = (e^{i\theta} r, 1)$. Since the Carnot–Carathéodory distance is larger than or equal to the Cygan distance (Theorem 1.3), we have

$$l_r \ge d\big((r, 0), (re^{i\theta}, 1)\big) = \Big(2^4 r^4 \sin^4 \frac{\theta}{2} + 1\Big)^{1/4}. \tag{48}$$

Since the Carnot–Carathéodory distance and the Cygan distance are invariant under Heisenberg translations, the length of any horizontal curve from (z, t) to $A(z, t) = (e^{i\theta} z, t + 1)$ is at least $l_{|z|}$. Note that $\phi\Gamma$ is the family of curves connecting

$\phi(p)$ to $A\phi(p)$, where p belongs to the annulus $\{(z,0) \in \mathcal{H} : \sqrt{r} \leq |z| \leq \sqrt{R}\}$. Using Lemma 3.2, we see that every curve $\gamma \in \phi\Gamma$ has length at least $l_{c(\sqrt{r})}$, where $c : [0,\infty) \to [0,\infty)$ is the function from the lemma.

We denote by D the support of the curve family $\phi\Gamma$:

$$D = \phi \circ \alpha\big(\big\{z \in \mathbb{C} : \sqrt{r} < |z| < \sqrt{R}\big\} \times [0,1]\big),$$

where α is the mapping of (27). Since each curve of Γ is contained in a fundamental domain for the action of the cyclic group $\langle A\rangle$, and the quasiconformal homeomorphism ϕ conjugates A to T (see (42)), D is also contained in a fundamental domain for the action of the cyclic group $\langle T\rangle$. Note that T is the vertical translation by 1. Thus, the intersection of a vertical line with D might have several components, but the total length is bounded by 1.

Now, let $\sigma = 1/l_{c(\sqrt{r})}$ be a constant function whose support is D. Then for any $\gamma \in \phi\Gamma$,

$$\int_\gamma \sigma = \frac{1}{l_{c(\sqrt{r})}} l(\gamma) \geq 1, \tag{49}$$

and hence σ is an admissible function of $\phi\Gamma$. Therefore,

$$\mathrm{M}(\phi\Gamma) \leq \int_{\mathcal{H}} \sigma^4 \, d\mathrm{vol} = \int_D \sigma^4 \, d\mathrm{vol} \leq \sigma^4 \int_{p(D)} 1 \, dx\, dy, \tag{50}$$

where $p : \mathcal{H} \to \mathbb{C}$ is the vertical projection.

Since the curves in Γ belong to the ball $B_{\sqrt{R+1}}$, $D \subseteq \phi(B_{\sqrt{R+1}})$. Again, Proposition 1.4 implies that

$$B_{\widetilde{R}} \subseteq \phi(B_{\sqrt{R+1}}) \subseteq B_{c_0\widetilde{R}} \tag{51}$$

for some $\widetilde{R} > 0$. Because the integer point $(0, R+1)$ is fixed by ϕ, $(0, R+1)$ lies in the image of the sphere $\phi(S_{\sqrt{R+1}})$ and hence $\widetilde{R} \leq \sqrt{R+1} \leq c_0\widetilde{R}$. In particular, we have $c_0\widetilde{R} \leq c_0\sqrt{R+1}$. Therefore, we have $p(D) \subseteq p(B_{c_0\sqrt{R+1}})$. From (50),

$$\begin{aligned}\sigma^4 \int_{p(D)} 1\, dx\, dy &\leq \frac{1}{l^4_{c(\sqrt{r})}} \int_{p(B_{c_0\sqrt{R+1}})} 1\, dx\, dy \\ &= \frac{\pi c_0^2 (R+1)}{l^4_{c(\sqrt{r})}} \leq \frac{\pi c_0^2(R+1)}{2^4 c^4(\sqrt{r}) \sin^4 \frac{\theta}{2} + 1}.\end{aligned} \tag{52}$$

Now we finish the proof by deriving a contradiction. Since ϕ is K-quasiconformal,

$$M(\Gamma) \leq K^2 M(\phi\Gamma). \tag{53}$$

Combining Lemma 2.1, (52), and (53), we have

$$\frac{1}{256}(R-r)\Big(\frac{\pi}{2}-2\arctan\frac{1}{\sqrt{r-1}}\Big)\le\frac{\pi c_0^2K^2(R+1)}{2^4c^4(\sqrt{r})\sin^4\frac{\theta}{2}+1}. \tag{54}$$

Because the square integers $r<R$ are arbitrary, we take $R=4r$. Lemma 3.2 implies that $c(\sqrt{r})\to\infty$ as $r\to\infty$, and hence we have a contradiction. □

For a positive real number $n\in\mathbb{R}_+-\{1\}$, we will denote simply by n the real dilation $(z,t)\mapsto(nz,n^2t)$ We will use the following normalization repeatedly:

$$\begin{aligned}(\sqrt{n})^{-1}A_\theta^n(\sqrt{n})(z,t)&=(\sqrt{n})^{-1}A_\theta^n(\sqrt{n}z,nt)\\&=(\sqrt{n})^{-1}(e^{ni\theta}\sqrt{n}z,nt+n)\\&=(e^{ni\theta}z,t+1)=A_{n\theta}(z,t),\end{aligned}\tag{55}$$

$$\begin{aligned}n^{-1}T_{(r,s)}n(z,t)&=n^{-1}T_{(r,s)}(nz,n^2t)\\&=n^{-1}(nz+r,n^2t+s+2rn\operatorname{Im}\bar{z})\\&=\Big(z+\frac{r}{n},t+\frac{s}{n^2}+\frac{2r}{n}\operatorname{Im}\bar{z}\Big)=T_{(r/n,s/n^2)}(z,t),\end{aligned}\tag{56}$$

where $n\in\mathbb{Z}$, $A_\theta(z,t)=(e^{i\theta}z,t+1)$ for $\theta\in[0,2\pi)$, and

$$T_{(r,s)}(z,t)=(z+r,t+s+2r\operatorname{Im}\bar{z})$$

for $r,s\in\mathbb{R}$.

Corollary 3.4. *A rational screw parabolic automorphism is not quasiconformally conjugate to an irrational screw parabolic automorphism.*

Proof. Let A_θ be a rational screw parabolic automorphism and A_ϑ be an irrational screw parabolic automorphism of $\mathscr{H}$. Suppose, to the contrary, that a K-quasiconformal map $\phi:\mathscr{H}\to\mathscr{H}$ exists such that $\phi\circ A_\vartheta\circ\phi^{-1}=A_\theta$. Then for any integer n,

$$\phi\circ A_\vartheta^n\circ\phi^{-1}=A_\theta^n. \tag{57}$$

Because A_θ is a rational screw parabolic automorphism, $A_\theta^{n_0}=T_{(0,n_0)}$ for some integer n_0. We conjugate both sides of (57) by a real dilation $\sqrt{n_0}$ and use (55) and (56) as follows:

$$\begin{aligned}(\sqrt{n_0})^{-1}\phi A_\vartheta^{n_0}\phi^{-1}\sqrt{n_0}&=(\sqrt{n_0})^{-1}T_{(0,n_0)}\sqrt{n_0},\\(\sqrt{n_0})^{-1}\phi(\sqrt{n_0}A_{n_0\vartheta}(\sqrt{n_0})^{-1})\phi^{-1}\sqrt{n_0}&=T_{(0,1)}.\end{aligned}\tag{58}$$

This implies that a screw parabolic $A_{n_0\vartheta}(z,t)=(e^{n_0\vartheta i}z,t+1)$ is conjugate to a vertical translation $T_{(0,1)}$ by a quasiconformal mapping $(\sqrt{n_0})^{-1}\phi\sqrt{n_0}$, which is a contradiction to Theorem 3.3. □

Applying the same idea as above, we also have:

Corollary 3.5. *If two rational screw parabolic automorphisms are quasiconformally conjugate, then they have the same order.*

Proposition 3.6. *Let A_θ and A_ϑ be two distinct irrational screw parabolic automorphisms for $\theta, \vartheta \in (0, 2\pi)$. Then A_θ and A_ϑ are not quasiconformally conjugate to each other.*

Proof. Using the normalization of (55), the proof follows the same idea of Proposition 4.15 of [Kim 2011]. □

We need the following theorem to prove that a screw parabolic automorphism is not quasiconformally conjugate to a horizontal translation.

Theorem F [Korányi and Reimann 1995]. *If $\{\varphi_n : G \to \hat{\mathcal{H}}\}$, for a proper subset $G \subset \mathcal{H}$, is a sequence of K-quasiconformal mappings such that every mapping φ_n omits two points a_n and b_n (depending on φ_n) with a distance at least l (l a fixed positive number independent of φ_n), then there exists a locally uniformly convergent subsequence converging to a K-quasiconformal mapping or to a constant.*

Theorem 3.7. *Let $T_{(1,0)}(z, t) = (z + 1, t + 2\operatorname{Im} \bar{z})$ be a horizontal translation and $A_\theta(z, t) = (e^{i\theta} z, t + 1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism of the Heisenberg group $\mathcal{H}$. Then A_θ is not quasiconformally conjugate to $T_{(1,0)}$.*

Proof. Suppose, to the contrary, that a K-quasiconformal map $\phi : \mathcal{H} \to \mathcal{H}$ exists such that

$$\phi \circ A_\theta \circ \phi^{-1} = T_{(1,0)}. \tag{59}$$

Then for any integer n, we also have

$$\phi \circ A_\theta^n \circ \phi^{-1} = T_{(1,0)}^n = T_{(n,0)}. \tag{60}$$

First consider the case that A_θ is a rational parabolic automorphism. Then there is a positive integer n_0 such that $A_\theta^{n_0} = T_{(0,n_0)}$. We conjugate both sides of (60) by a real dilation n as follows:

$$n^{-1}(\phi A_\theta^n \phi^{-1})n = n^{-1} T_{(n,0)} n = T_{(1,0)}. \tag{61}$$

In particular, when $n = n_0$,

$$n_0^{-1} \phi T_{(0,n_0)} \phi^{-1} n_0 = T_{(1,0)}. \tag{62}$$

Using that $T_{(0,n_0)} = \sqrt{n_0}\, T_{(0,1)} (\sqrt{n_0})^{-1}$, we rewrite the left-hand side of (62) as

$$(n_0^{-1} \phi \sqrt{n_0}) T_{(0,1)} ((\sqrt{n_0})^{-1} \phi^{-1} n_0) = T_{(1,0)}. \tag{63}$$

Because $(n_0)^{-1} \phi \sqrt{n_0}$ is also a K-quasiconformal mapping, (63) implies that the vertical translation $T_{(0,1)}$ is conjugate to the horizontal translation $T_{(1,0)}$ by the quasiconformal mapping $n_0^{-1}\phi\sqrt{n_0}$. This is a contradiction to Theorem 5.1 of [Miner 1994].

The second case is that A_θ is an irrational screw parabolic automorphism. Here we use the property that, under a mild condition, an infinite sequence of K-quasiconformal mappings is a normal family; see Theorem F.

It is possible that the quasiconformal mapping ϕ of (59) does not fix the origin $(0, 0)$. Hence, we conjugate both sides of (59) by a Heisenberg translation m which sends $\phi(0, 0)$ to $(0, 0)$ (m might be the identity map) so that we have

$$m\phi A_\theta \phi^{-1} m^{-1} = m \circ T_{(1,0)} \circ m^{-1}. \tag{64}$$

If $m \circ T_{(1,0)} \circ m^{-1}$ is a vertical translation, then we have proved the theorem. Otherwise, $mT_{(1,0)}m^{-1}$ is a nonvertical translation. Now we conjugate (64) by a rotation $\lambda : (z, t) \mapsto (\lambda z, t)$ for a unit $\lambda \in \mathbb{C}$ so that $\lambda m T_{(1,0)} m^{-1} \lambda = T_{(r,s)}$ for some real numbers $r \neq 0$ and s:

$$\lambda m\phi A_\theta \phi^{-1} m^{-1} \lambda^{-1} = \lambda m T_{(1,0)} m^{-1} \lambda^{-1} = T_{(r,s)}. \tag{65}$$

Let $\varphi = \lambda m\phi$. Then φ is a K-quasiconformal mapping, fixes the origin $(0, 0)$ and

$$\varphi \circ A_\theta \circ \varphi^{-1} = T_{(r,s)}. \tag{66}$$

(We note that if ϕ fixes $(0, 0)$, then m and λ are the identity map, $T_{(r,s)} = T_{(1,0)}$, and $\varphi = \phi$.)

Let n be any integer; then from (66), we have

$$\varphi A_\theta^n \varphi^{-1} = T_{(r,s)}^n = T_{(nr,ns)} \tag{67}$$

because r and s are real numbers. Evaluating (67) at $(0, 0)$ shows that

$$\varphi(0, n) = (nr, ns). \tag{68}$$

We conjugate both sides of (67) by a real dilation n and use equations (55) and (56) as follows:

$$\begin{gathered} n^{-1}\varphi A_\theta^n \varphi^{-1} n = n^{-1} T_{(nr,ns)} n, \\ n^{-1}\varphi(\sqrt{n} A_{n\theta} (\sqrt{n})^{-1})\varphi^{-1} n = T_{(r,s/n)}. \end{gathered} \tag{69}$$

Because A_θ is an irrational screw parabolic, there is a subsequence $\{A_{n_k\theta} : k \in \mathbb{N}\}$ which converges to the vertical translation $T_{(0,1)}$. For each $k \in \mathbb{N}$, let $\psi_k = n_k^{-1}\varphi\sqrt{n_k}$. Then each ψ_k is again K-quasiconformal, fixes $(0, 0)$, and

$$\psi_k A_{n_k\theta} \psi_k^{-1} = T_{(r,s/n_k)}. \tag{70}$$

To apply Theorem F, let $G = \mathscr{H} - \{(0, 0)\}$ and restrict each ψ_k on G. Thus, we have an infinite sequence of K-quasiconformal mappings, $\mathscr{F} = \{\psi_k : G \to \hat{\mathscr{H}} \mid k \in \mathbb{N}\}$. Note that each $\psi_k|_G$ omits $(0,0)$ and ∞ in $\hat{\mathcal{H}}$. Hence, the sequence $\mathcal{F}$ has a convergent subsequence whose limit, say ψ, is a K-quasiconformal homeomorphism for the following reason: for any integer m,

$$\psi_k(0,m) = n_k^{-1}\varphi_{\sqrt{n_k}}(0,m) = \left(mr, \frac{ms}{n_k}\right)$$

converges to $(mr, 0)$ as $k \to \infty$. Thus, $\psi(0,m) = (mr, 0)$ for any integer m, and hence ψ is not a constant function. We now extend ψ to $\mathcal{H}$ by defining $\psi(0,0) = (0,0)$. From (70), we have $\psi \circ T_{(0,1)} \circ \psi^{-1} = T_{(r,0)}$ which is a contradiction by Theorem 5.1 of [Miner 1994]. □

Corollary 3.8. *Let $T_{(1,0)}(z,t) = (z+1, t+2\,\mathrm{Im}\,\bar{z})$ be a horizontal translation and $A(z,t) = (e^{i\theta}z, t+1)$, for $\theta \in (0, 2\pi)$, be a screw parabolic automorphism in the Heisenberg group $\mathcal{H}$. Let Γ_1 and Γ_2 be the cyclic groups generated by $T_{(1,0)}$ and A, respectively. Then there exists no quasiconformal mapping between $\mathcal{H}/\Gamma_1$ and $\mathcal{H}/\Gamma_2$. In particular, Γ_1 is not quasiconformally conjugate to Γ_2.*

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