Pacific Journal of Mathematics

A CLASS OF NEUMANN PROBLEMS ARISING IN CONFORMAL GEOMETRY

WEIMIN SHENG AND LI-XIA YUAN

Volume 270 No. 1 July 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 1, 2014
dx.doi.org/10.2140/pjm.2014.270.211

A CLASS OF NEUMANN PROBLEMS ARISING IN CONFORMAL GEOMETRY

WEIMIN SHENG AND LI-XIA YUAN

In this paper, we solve a class of Neumann problems on a manifold with totally geodesic smooth boundary. As a consequence, we also solve the prescribing k-curvature problem of the modified Schouten tensor on such manifolds; that is, if the initial k-curvature of the modified Schouten tensor is positive for $\tau > n-1$ or negative for $\tau < 1$, then there exists a conformal metric such that its k-curvature defined by the modified Schouten tensor equals some prescribed function and the boundary remains totally geodesic.

1. Introduction

Let (M^n, g), $n \geq 3$, be a compact, smooth Riemannian manifold. The *modified Schouten tensor*

$$A_g^\tau := \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{\tau R_g}{2(n-1)} \cdot g\right)$$

was introduced by Gursky and Viaclovsky [2003] and A. Li and Y.-Y. Li [2003] independently, where $\tau \in \mathbb{R}$ and Ric_g, R_g are the Ricci tensor and the scalar curvature of g, respectively. Clearly, A_g^0 is the Ricci tensor, A_g^{n-1} is the Einstein tensor and A_g^1 is just the Schouten tensor.

Denote by $\lambda(g^{-1}A_g^\tau)$ the eigenvalues of A_g^τ. The k-curvature (or σ_k curvature) of A_g^τ is defined as $\sigma_k(\lambda(g^{-1}A_g^\tau))$, where σ_k is the k-th elementary symmetric function defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{for all } \lambda \in \mathbb{R}^n,$$

for any $1 \leq k \leq n$. We will use $\sigma_k(A_g^\tau) := \sigma_k(\lambda(g^{-1}A_g^\tau))$ for convenience.

The prescribing k-curvature problem of the modified Schouten tensor A_g^τ in conformal geometry is to find a metric $\tilde{g}$ in the conformal class $[g]$ of g satisfying

This work was supported by NSFC 11131007 and Zhejiang NSFC LY14A010019.
MSC2010: primary 53C21; secondary 35J65.
Keywords: k-curvature, modified Schouten tensor, Neumann problem, umbilic boundary.

the equation

$$\sigma_k^{1/k}(A_{\tilde{g}}^{\tau}) = \varphi(x), \tag{1-1}$$

where φ is a given smooth function on M. If $\tau = 1 = k$ and φ is constant, (1-1) is just the Yamabe problem, which has been solved by Yamabe, Trudinger, Aubin and Schoen (see [Lee and Parker 1987]). When $\tau = 1$, $k \geq 2$ and φ is constant, then (1-1) is called k-Yamabe problem, which has attracted enormous interest [Chang et al. 2002; Ge and Wang 2006; Guan and Wang 2003a; 2003b; Gursky and Viaclovsky 2007; Li and Li 2003; 2005; Sheng et al. 2007; Trudinger and Wang 2009; 2010; Viaclovsky 2000], etc. There are many interesting works on the Yamabe problem and k-Yamabe problem on a manifold with boundary [Chen 2007; 2009; Escobar 1992b; 1992a; Han and Li 1999; 2000; He and Sheng 2011a; 2011b; 2013; Jin et al. 2007; Jin 2007], etc.

Note that (1-1) is a fully nonlinear partial differential equation for $k \geq 2$. In order to study this problem, we need the following conceptions. Let

$$\Gamma_k^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

Therefore, we have $\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+$. For a 2-symmetric form B defined on (M^n, g), $B \in \Gamma_k^+$ means that the eigenvalues of B, say $\lambda(g^{-1}B)$, lie in Γ_k^+. Set $\Gamma_k^- = -\Gamma_k^+$.

According to [Caffarelli et al. 1985], (1-1) is an elliptic equation for $A_g^\tau \in \Gamma_k^+$ or $A_g^\tau \in \Gamma_k^-$. When $\tau < 1$, $A_g^\tau \in \Gamma_k^-$ and $\varphi < 0$, Gursky and Viaclovsky [2003] proved that there exists a unique conformal metric $\tilde{g} \in [g]$ satisfying (1-1) on a closed manifold. Li and Sheng [2005] studied the same problem by a parabolic argument. Using a similar argument, Sheng and Zhang [2007] studied the case of $\tau > n - 1$, $A_g^\tau \in \Gamma_k^+$ and $\varphi > 0$. For the manifold with boundary, Li and Sheng [2011] considered a Dirichlet problem of (1-1) for $\tau > n - 1$ and $A_g^\tau \in \Gamma_k^+$; He and Sheng [2013] discussed more general equations and obtained many useful local estimates for both $\tau < 1$ and $\tau > n - 1$. In [Sheng and Yuan 2013], we investigated a Neumann problem of (1-1) by a conformal flow and proved:

Theorem 1.1 [Sheng and Yuan 2013]. *Let $(\overline{M}^n, g)$, $n \geq 3$, be a compact manifold with smooth totally geodesic boundary ∂M. If $A_g^\tau \in \Gamma_k^+$ and $\tau > n - 1$, or $A_g^\tau \in \Gamma_k^-$ and $\tau < 1$, then there exists a smooth metric $\tilde{g} \in [g]$ satisfying* (1-1) *for φ constant and such that ∂M is still totally geodesic.*

In this paper, we are interested in solving a class of Neumann problems on the manifold with totally geodesic boundary.

Let $(\overline{M}, g)$ be a compact manifold with smooth boundary ∂M. Denote the second fundamental form and the mean curvature of ∂M by L and μ. Under the conformal change of metric $\tilde{g} = e^{2u} g$, the second fundamental form L with respect to its unit inward normal ν satisfies

$$\tilde{L}e^{-u} = -\frac{\partial u}{\partial \nu} g + L.$$

The boundary is called umbilic if $L = \mu g$, and then totally geodesic if $\mu \equiv 0$. Note that the umbilicity is conformally invariant. Then the mean curvature changes as

$$\tilde{\mu} = \left(-\frac{\partial u}{\partial \nu} + \mu\right) e^{-u}. \tag{1-2}$$

Under the same conformal change, the *modified Schouten tensor* changes according to the formula

$$A_{\tilde{g}}^{\tau} = \frac{\tau - 1}{n-2} \Delta u g - \nabla^2 u + du \otimes du + \frac{\tau - 2}{2} |\nabla u|^2 g + A_g^{\tau}, \tag{1-3}$$

where the covariant derivatives and norms are taken with respect to the background metric g. Let the boundary ∂M be totally geodesic with respect to the metric g. In order to preserve the boundary being totally geodesic under the conformal change, $\tilde{\mu} \equiv 0$. Hence, the two partial differential equations corresponding to Theorem 1.1 are

$$\begin{cases} \sigma_k^{1/k}\left(\frac{\tau-1}{n-2}\Delta u g - \nabla^2 u + du \otimes du + \frac{\tau-2}{2}|\nabla u|^2 g + A_g^{\tau}\right) \\ \qquad\qquad = e^{2u}\,\text{const.} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-4}$$

for $\tau > n-1$, and

$$\begin{cases} \sigma_k^{1/k}\left(\nabla^2 u + \frac{1-\tau}{n-2}\Delta u g - du \otimes du + \frac{2-\tau}{2}|\nabla u|^2 g - A_g^{\tau}\right) \\ \qquad\qquad = e^{2u}\,\text{const.} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-5}$$

for $\tau < 1$, respectively.

Now, we consider more general equations than (1-4) and (1-5). Let $\Gamma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$, and $F : \mathbb{R}^n \to \mathbb{R}$ be a general smooth, symmetric, homogeneous function of degree one in Γ normalized by $F(e) = F(1, \dots, 1) = 1$. Moreover, $F = 0$ on $\partial\Gamma$ and satisfies the following structure conditions in Γ:

(C1) F is positive.

(C2) F is concave (i.e., $\partial^2 F/(\partial\lambda_i \partial\lambda_j)$ is negative semidefinite).

(C3) F is monotone (i.e., $\partial F/\partial\lambda_i$ is positive).

According to [Lin and Trudinger 1994; Trudinger 1990], for any $0 \leq l < k \leq n$, the elementary symmetric functions and their quotients $(\sigma_k/\sigma_l)^{1/(k-l)}$ with $\sigma_0 = 1$ satisfy all the properties and structure conditions above on Γ_k^+.

For some positive function $\Phi(x,z) \in C^\infty(\overline{M}) \times \mathbb{R}$, we study the equation

$$\begin{cases} F(g^{-1}V[u]) = \Phi(x,u) & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-6}$$

where for constant $\bar{\theta} := (\tau - 1)/(n-2) > 1$, $a, b \in C^\infty(\overline{M})$, and the smooth symmetric 2-tensor $S \in \Gamma$, the matrix $(V[u])$ is defined by

$$V[u] = \bar{\theta} \Delta u g - \nabla^2 u + a(x)\, du \otimes du + b(x)|\nabla u|^2 g + S. \tag{1-7}$$

We call a function $v \in C^2(\overline{M})$ *admissible* if $\lambda(g^{-1}V[v]) \in \Gamma$.

Assume S is the symmetric 2-tensor on M satisfying one of the following conditions:

(S1) $S(\nu, X) = 0$, for any $X \in T(\partial M)$.

(S2) $S = A_g^\tau$.

Theorem 1.2 (main result). *Let $(\overline{M}^n, g)$, $n \geq 3$, be a compact manifold with smooth totally geodesic boundary ∂M. Suppose $\bar{\theta} > 1$ and the positive function $\Phi(x,z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfies*

$$\partial_z \Phi > 0, \qquad \lim_{z \to +\infty} \Phi(x,z) = +\infty, \qquad \lim_{z \to -\infty} \Phi(x,z) = 0. \tag{1-8}$$

Then for any functions $a, b \in C^\infty(\overline{M})$ and $S \in \Gamma$ satisfying (S1) *or* (S2), *there exists a function $u \in C^\infty(\overline{M})$ solving the equation* (1-6).

For the other elliptic branch (1-5), we consider the equation

$$\begin{cases} F(g^{-1}W[u]) = \Phi(x,u) & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-9}$$

where for constant $\theta := (1-\tau)/(n-2) > 0$, $a, b \in C^\infty(\overline{M})$, and the smooth symmetric 2-tensor $T \in \Gamma$, the matrix $(W[u])$ is defined by

$$W[u] = \nabla^2 u + \theta \Delta u g + a(x)\, du \otimes du + b(x)|\nabla u|^2 g + T. \tag{1-10}$$

Theorem 1.3. *Let $(\overline{M}^n, g)$, $n \geq 3$, be a compact manifold with smooth totally geodesic boundary ∂M. Suppose $\theta > 0$ and the positive function $\Phi(x,z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfies* (1-8). *Then for any functions $a, b \in C^\infty(\overline{M})$ and $T \in \Gamma$ with* (S1) *or $T = -A_g^\tau$, there exists a function $u \in C^\infty(\overline{M})$ solving the equation* (1-9).

Applying Theorems 1.2 and 1.3 to the quotient of the elementary symmetric functions, i.e., $F = (\sigma_k/\sigma_l)^{1/(k-l)}$ on Γ_k^+, we have the following corollaries.

Corollary 1.4. *Let* $(\overline{M}^n, g)$, $n \geq 3$, *be a compact manifold with smooth totally geodesic boundary* ∂M. *If* $\tau > n-1$ *and* $A_g^\tau \in \Gamma_k^+$, *then for any smooth function* $\varphi > 0$, *there exists a smooth metric* $\tilde{g} \in [g]$ *preserving* ∂M *totally geodesic and satisfying*

$$\left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}} (A_{\tilde{g}}^\tau) = \varphi(x) \quad \text{in } M. \tag{1-11}$$

Corollary 1.5. *Let* $(\overline{M}^n, g)$, $n \geq 3$, *be a compact manifold with smooth totally geodesic boundary* ∂M. *If* $\tau < 1$ *and* $A_g^\tau \in \Gamma_k^-$, *then for any smooth function* $\varphi < 0$, *there exists a smooth metric* $\tilde{g} \in [g]$ *preserving* ∂M *totally geodesic and satisfying* (1-11).

Remark 1.6. By choosing $l = 0$ and φ constant in Corollaries 1.4 and 1.5, we can get Theorem 1.1 directly. Different from the results in [Li and Sheng 2011; Sheng et al. 2007], we need not subjoin any restriction on $a(x)$ and $b(x)$ in Theorems 1.2 and 1.3. Contrary to this fact, [Sheng et al. 2007] gives a counterexample to show that there is no regularity if $a(x) = 0$ and $b(x) > 0$ when $\tau = 1$ and $A_g^\tau \in \Gamma_k^-$.

This paper is organized as follows. We introduce some lemmas in Section 2. By use of these lemmas, we can get the a priori global C^0 estimate for (1-6) in Section 3. Then we obtain the a priori global gradient and Hessian derivatives estimates in Section 4 and Section 5 respectively. By the a priori estimates and the standard continuity method, we show Theorem 1.2 in Section 6. In the last section, we consider (1-9) by the similar arguments in Sections 3–6, and prove Theorem 1.3.

2. Preliminaries

In this section, we first recall some facts of the function F satisfying the structure conditions (C1)–(C3) in Γ.

Lemma 2.1 (see [Chen 2005; 2009]). *Let* Γ *be an open convex cone with vertex at the origin satisfying* $\Gamma_n^+ \subset \Gamma$, *and let* $e = (1, \dots, 1)$ *be the identity. Suppose that* F *is a homogeneous symmetric function of degree one normalized with* $F(e) = 1$, *and that* F *is concave in* Γ. *Then*:

(a) $\sum_i \lambda_i \partial F(\lambda)/\partial\lambda_i = F(\lambda)$, *for* $\lambda \in \Gamma$.

(b) $\sum_i \partial F(\lambda)/\partial\lambda_i \geq F(e) = 1$, *for* $\lambda \in \Gamma$.

To get the boundary estimates, we need some facts. For any point $x_0 \in \partial M$, we consider Fermi coordinates $\{x_i\}_{1\leq i\leq n}$ around x_0, where $\partial/\partial x_n$ is the unit inner normal with respect to the background metric g. A half-ball centered at x_0 of radius r is defined by

$$\bar{B}_r^+ = \left\{ x_n \geq 0, \left(\sum_{i=1}^n x_i^2 \right) \leq r^2 \right\}.$$

Denote the boundary of $\bar{B}_r^+$ on ∂M by $\Sigma_r = \left\{ x_n = 0, \sum_i x_i^2 \leq r^2 \right\}$.

Throughout this paper, the Greek letters $\alpha, \beta, \gamma, \ldots = 1, \ldots, n-1$ stand for the tangential direction indices, while the Latin letters $i, j, k, \ldots = 1, \ldots, n$ stand for the full indices. In Fermi coordinates $\{x_i\}_{1 \leq i \leq n}$, the metric is expressed as $g = g_{\alpha\beta}\, dx_\alpha\, dx_\beta + (dx_n)^2$. Then the Christoffel symbols on the boundary satisfy

$$\Gamma^n_{\alpha\beta} = L_{\alpha\beta}, \quad \Gamma^\beta_{\alpha n} = -L_{\alpha\gamma} g^{\gamma\beta}, \quad \Gamma^n_{\alpha n} = 0, \quad \Gamma^n_{nn} = 0, \quad \Gamma^\gamma_{nn} = 0, \quad \Gamma^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\alpha\beta} \tag{2-1}$$

on the boundary, where we denote the tensors and covariant differentiation with respect to the induced metric $g_{\alpha\beta}$ on the boundary by a tilde (e.g., $\tilde{\Gamma}^\gamma_{\alpha\beta}$, $\mu_{\tilde{\alpha}\tilde{\beta}}$). When the boundary is totally geodesic, we have

$$\Gamma^n_{\alpha\beta} = 0, \quad \Gamma^\beta_{\alpha n} = 0, \quad \Gamma^n_{\alpha n} = 0. \tag{2-2}$$

Lemma 2.2 [Chen 2007; He and Sheng 2013]. *Suppose ∂M is totally geodesic and $u_n = 0$ on ∂M. Then we have on the boundary that*

$$u_{n\alpha} = 0 \quad \text{and} \quad u_{\alpha\beta n} = 0. \tag{2-3}$$

Lemma 2.3 [He and Sheng 2013]. *Let $(\overline{M}, g)$ be a compact Riemannian manifold with boundary and dimension $n \geq 3$. Assume the boundary ∂M is totally geodesic. Then at any boundary point $P \in \partial M$, there exists a conformal metric $\bar{g} = e^{2\bar{u}} g_0$ such that* (i) *$\bar{u}_n = 0$ on ∂M and the boundary ∂M is still totally geodesic*, (ii) $\bar{R}_{ij}(P) = 0$ *for* $1 \leq i, j \leq n$, (iii) $\bar{R}_{nn,n}(P) = 0$, $\bar{R}_{\alpha n,\beta}(P) = 0$, $1 \leq \alpha, \beta \leq n-1$, *and* (iv) $\bar{R}_{\alpha\beta,n}(P) = 0$, $1 \leq \alpha, \beta \leq n-1$.

3. Ellipticity and the global C^0 estimates

We first sketch the ellipticity properties of operator F; see [Li and Sheng 2011] for details.

For any function h on $\overline{M}$, we define

$$\mathscr{P}[h] := F(V[h]) - \Phi(x, h).$$

Then any solution u of (1-6) satisfies $\mathscr{P}[u] = 0$. Denote $u_s = u + sv$, $s \in \mathbb{R}$. The linearized operator of (1-6) is

$$\begin{aligned} \mathscr{L}v &:= \frac{d}{ds}\mathscr{P}[u_s]\Big|_{s=0} \\ &= P^{ij} v_{ij} + 2aF^{ij} v_i u_j + 2b v_l u_l \mathscr{T} - \partial_z \Phi(x, u) v, \end{aligned} \tag{3-1}$$

where $F^{ij} := (\partial F/\partial V_{ij})(V[u])$, $\mathcal{T} = \mathrm{tr}(F^{ij}) = F^{ij} g_{ij}$ and

$$P^{ij} := \bar{\theta}\mathcal{T} g^{ij} - F^{ij} \geq (\bar{\theta} - 1)\mathcal{T} g^{ij}.$$

Since u is admissible, (F^{ij}) is positive definite [Caffarelli et al. 1985]. Denote $\varepsilon_0 := \bar{\theta} - 1 > 0$. Hence, (P^{ij}) is positive definite, too.

Note that the coefficient of the zero order term in (3-1) is negative when $\partial_z \Phi$ is positive on $\overline{M} \times \mathbb{R}$.

Lemma 3.1. *Equation* (1-6) *is elliptic at any admissible solution. If $\partial_z \Phi$ is positive on $\overline{M} \times \mathbb{R}$, then the linearized operator $\mathcal{L} : C^{2,\alpha}(\overline{M}) \to C^{\alpha}(\overline{M}) (0 < \alpha < 1)$ is invertible.*

Now, we use the compactness of the manifold to get the global C^0 estimates of (1-6).

Proposition 3.2. *Suppose $S \in \Gamma$ and the positive function $\Phi(x, z) \in C^{\infty}(\overline{M}) \times \mathbb{R}$ satisfies* (1-8). *Then for any admissible solution $u \in C^2(\overline{M})$ of* (1-6), *we have*

$$\sup_{\overline{M}} |u| \leq C_0,$$

where the constant C_0 depends only on S and Φ.

Proof. Suppose x_0 be the maximum point of u on $\overline{M}$. Denote $u_{\max} = u(x_0)$.

If $x_0 \in \partial M$, at this point we have $u_n(x_0) < 0$, which contradicts with the boundary condition $u_n|_{\partial M} \equiv 0$. Hence, x_0 must be an interior point of M. Then at this point we have

$$\nabla u = 0 \quad \text{and} \quad \nabla^2 u \geq 0. \tag{3-2}$$

Substituting (3-2) into (1-6), we have

$$\Phi(x_0, u_{\max}) \leq F(S)(x_0) \leq \max_{x \in \overline{M}} F(S) \leq C.$$

Now, by the condition $\partial_z \Phi > 0$ and $\lim_{z \to +\infty} \Phi(x, z) = +\infty$, we know that

$$\max_{x \in \overline{M}} u = u_{\max} \leq C.$$

By a similar argument, we can get the lower bound of u by considering its minimum point on $\overline{M}$ and using the other condition of Φ. □

4. Gradient estimates

In this section we first consider the boundary gradient estimates of (1-6), then derive the global estimates.

For any point $y_0 \in \partial M$, let $\bar{B}_r^+$ and $\bar{B}_{r/2}^+$ be any two half-balls centered at y_0 in the Fermi coordinates $\{x_i\}_{1\le i\le n}$. Choosing a cutoff function η depending only on r such that $0 \le \eta \le 1$, $\eta = 1$ in $\bar{B}_{r/2}^+$, $\eta = 0$ outside $\bar{B}_r^+$. Moreover,

$$|\nabla \eta| \le b_0 \frac{\eta^{1/2}}{r} \quad \text{and} \quad |\nabla^2 \eta| \le \frac{b_0}{r^2}, \tag{4-1}$$

for a universal constant b_0, where the covariant derivatives and the norms $|\cdot|$ are taken with respect to g. Since η only depends on r, we have

$$\frac{\partial \eta}{\partial n} = 0 \quad \text{on } \partial M. \tag{4-2}$$

We also need the function $\psi : \mathbb{R} \to \mathbb{R}$ defined in [Gursky and Viaclovsky 2003] by

$$\psi(s) = \alpha_1(\alpha_2 + s)^p, \quad -\delta_1 < s < \delta_2, \tag{4-3}$$

where the positive constants δ_1 and δ_2 are given, and the constants α_1, α_2 and p will be fixed as follows. We have

$$\psi' = p\alpha_1(\alpha_2 + s)^{p-1} \quad \text{and} \quad \psi'' = p(p-1)\alpha_1(\alpha_2 + s)^{p-2} = \frac{p-1}{\alpha_2 + s}\psi'.$$

Let α_2 and p be positive constants satisfying $\alpha_2 > \delta_1$ and $p > 3$. Take

$$\alpha_1 = \frac{1}{p^2 \max\{(\alpha_2 + s)^p\}};$$

then

$$\psi \le \frac{1}{p^2}, \quad \psi' > 0 \quad \text{and} \quad \psi'' - \psi'^2 = \frac{\psi'}{\alpha_2 + s}(p - 1 - p\psi) \ge \frac{\psi' p}{2(\alpha_2 + s)}. \tag{4-4}$$

Proposition 4.1. *Suppose u is a C^3 solution of* (1-6) *on $\bar{B}_r^+$. Then there is a positive constant C depending only on $n, k, \bar{\theta}, g, r, |S|_{C^1(\bar{B}_r^+)}, |\Phi|_{C^1(\bar{B}_r^+)\times[-C_0, C_0]}, |a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$ and C_0 such that*

$$\sup_{\bar{B}_{r/2}^+} |\nabla u|_g \le C.$$

Proof. Consider the auxiliary function

$$G := \tfrac{1}{2}\eta e^{\beta}|\nabla u|^2, \quad \beta := x_n + \psi(u),$$

where the function ψ defined by (4-3). Let x_0 be the maximum point of G on $\bar{B}_r^+$. Without loss of generality, we may assume $r = 1$ and $|\nabla u|\,(x_0) \gg 1$.

Suppose $x_0 \in \Sigma_r$. Then $G_n(x_0) \le 0$. However, by (4-2), the boundary condition $u_n = 0$ and Lemma 2.2, we have

$$\begin{aligned} G_n(x_0) &= \tfrac{1}{2} e^{\psi} \Big((1+\psi' u_n)|\nabla u|^2 + 2u_n u_{nn} + 2\sum_{\alpha=1}^{n-1} u_\alpha u_{\alpha n} \Big)(x_0) \\ &= \tfrac{1}{2} e^{\psi} |\nabla u|^2 (x_0) > 0. \end{aligned}$$

It is a contradiction. Hence x_0 must be an interior point of $\bar{B}_r^+$. Then at x_0, for $1 \le i \le n$, we have

$$0 = (\log G)_i, \quad 0 \ge (\log G)_{ij},$$

that is,

$$\frac{2u_s u_{si}}{|\nabla u|^2} = -\Big(\frac{\eta_i}{\eta} + \beta_i\Big), \tag{4-5}$$

and

$$0 \ge \Big(\frac{\eta_{ij}}{\eta} - \frac{\eta_i \eta_j}{\eta^2}\Big) + \beta_{ij} + \frac{2u_{sj}u_{si} + 2u_s u_{sij}}{|\nabla u|^2} - \frac{4u_s u_{si} u_l u_{lj}}{|\nabla u|^4}. \tag{4-6}$$

Substituting (4-5) into (4-6), we have

$$0 \ge \Big(\frac{\eta_{ij}}{\eta} - 2\frac{\eta_i \eta_j}{\eta^2}\Big) + (\beta_{ij} - \beta_i \beta_j) + \frac{2u_{sj}u_{si} + 2u_s u_{sij}}{|\nabla u|^2} - \frac{1}{\eta}(\eta_i \beta_j + \eta_j \beta_i). \tag{4-7}$$

By (4-7), we have

$$\begin{aligned} 0 \ge P^{ij}\Big(\frac{\eta_{ij}}{\eta} - 2\frac{\eta_i\eta_j}{\eta^2}\Big) &+ P^{ij}(\beta_{ij} - \beta_i\beta_j) \\ &+ \frac{2}{|\nabla u|^2} P^{ij} u_{si} u_{sj} + \frac{2}{|\nabla u|^2} u_s P^{ij} u_{sij} - \frac{2}{\eta} P^{ij} \eta_i \beta_j, \end{aligned} \tag{4-8}$$

where $P^{ij} = \bar{\theta}\mathcal{T} g^{ij} - F^{ij}$ is positive definite. It follows from (4-1) and (4-8) that

$$0 \ge \frac{2}{|\nabla u|^2} u_s P^{ij} u_{sij} + P^{ij}(\beta_{ij} - \beta_i\beta_j) - \frac{2}{\eta} P^{ij}\eta_i\beta_j - \frac{C}{\eta}\mathcal{T}, \tag{4-9}$$

where the constant C depends only on n and b_0.

Differentiating (1-6), we have

$$\nabla_s \Phi = P^{ij} u_{ijs} + F^{ij}(a_s u_i u_j + 2a u_{is} u_j + S_{ij},_s) + (b_s |\nabla u|^2 + 2b u_{ls} u_l)\mathcal{T}. \tag{4-10}$$

Then by (4-10) and Ricci identities $u_{sij} = u_{ijs} + R_{isjp} u_p$, we obtain

$$\begin{aligned} \frac{2}{|\nabla u|^2} u_s P^{ij} u_{sij} \ge \frac{2}{|\nabla u|^2} u_s \nabla_s \Phi &- \frac{2}{|\nabla u|^2} u_s F^{ij}(a_s u_i u_j + 2a u_{is} u_j) \\ &- \frac{2}{|\nabla u|^2} u_s (b_s |\nabla u|^2 + 2b u_{ls} u_s)\mathcal{T} - C\Big(1 + \frac{1}{|\nabla u|}\Big)\mathcal{T}. \end{aligned}$$

where the constant C depends only on n, g and $|\nabla S|$.

Since $\nabla_s\Phi = \Phi_x + \Phi_z u_s$, by (4-5) and the inequality above, we have

$$\begin{aligned}(4\text{-}11)\quad \frac{2}{|\nabla u|^2}u_s P^{ij}u_{sij} &\geq 2\Phi_z + \frac{2}{|\nabla u|^2}u_s\Phi_x - \frac{2a_s u_s}{|\nabla u|^2}F^{ij}u_iu_j + 2aF^{ij}u_j\left(\frac{\eta_i}{\eta}+\beta_i\right)\\ &\quad - 2b_su_s\mathcal{T} + 2b\left(\frac{\eta_s}{\eta}+\beta_s\right)u_s\mathcal{T} - C\left(1+\frac{1}{|\nabla u|}\right)\mathcal{T}\\ &\geq C^* + 2aF^{ij}u_j\beta_i + 2bu_s\beta_s\mathcal{T} - \frac{C}{\sqrt{\eta}}(1+|\nabla u|)\mathcal{T},\end{aligned}$$

where the constant C^* depends only on $|\Phi_x|$, $|\Phi_z|$, C_0, and C depends on n, b_0, $|a|_{C^1}$, $|b|_{C^1}$ and $|\nabla S|$.

Then by (4-9) and (4-11), we obtain

$$\begin{aligned}(4\text{-}12)\quad 0 &\geq C^* + 2aF^{ij}u_j\beta_i + 2bu_s\beta_s\mathcal{T}\\ &\quad + P^{ij}(\beta_{ij} - \beta_i\beta_j) - \frac{2\eta_i}{\eta}P^{ij}\beta_j - C\frac{1}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}.\end{aligned}$$

Since $\beta := x_n + \psi(u)$, we have

$$\beta_i = \delta_{in} + \psi' u_i, \qquad \beta_{ij} = \psi'' u_iu_j + \psi' u_{ij}$$

and

$$\beta_{ij} - \beta_i\beta_j = \psi' u_{ij} + (\psi'' - \psi'^2)u_iu_j - \psi'(\delta_{in}u_j + \delta_{jn}u_i) - \delta_{in}\delta_{jn}.$$

Therefore, we have the inequalities

$$(4\text{-}13)\quad 2aF^{ij}u_j\beta_i = 2aF^{ij}u_j(\delta_{in} + \psi' u_i) \geq 2a\psi' F^{ij}u_iu_j - C|\nabla u|\mathcal{T},$$

$$(4\text{-}14)\quad 2bu_s\beta_s\mathcal{T} = 2bu_s(\delta_{sn} + \psi' u_s)\mathcal{T} \geq 2b\psi'|\nabla u|^2\mathcal{T} - C|\nabla u|\mathcal{T},$$

$$(4\text{-}15)\quad -\frac{2\eta_i}{\eta}P^{ij}\beta_j = -\frac{2}{\eta}P^{ij}\eta_i(\delta_{jn} + \psi' u_j) \geq -\frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T},$$

$$(4\text{-}16)\quad P^{ij}(\beta_{ij} - \beta_i\beta_j) \geq \psi' P^{ij}u_{ij} + (\psi'' - \psi'^2)P^{ij}u_iu_j - C(|\nabla u|+1)\mathcal{T}.$$

Plugging (4-13)–(4-16) into (4-12), we have

$$\begin{aligned}(4\text{-}17)\quad 0 &\geq C^* + \psi' P^{ij}u_{ij} + (\psi'' - \psi'^2)P^{ij}u_iu_j + 2a\psi' F^{ij}u_iu_j\\ &\quad + 2b\psi'|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}.\end{aligned}$$

By Lemma 2.1, we know that $F^{ij}V_{ij} = F(V) = \Phi$. Then

$$\begin{aligned}(4\text{-}18)\quad \psi' P^{ij}u_{ij} &= \psi' F^{ij}V_{ij} - \psi' F^{ij}(au_iu_j + b|\nabla u|^2 g_{ij} + S_{ij})\\ &\geq \psi'\Phi - a\psi' F^{ij}u_iu_j - b\psi'|\nabla u|^2\mathcal{T} - C\mathcal{T}.\end{aligned}$$

Substituting (4-18) into (4-17), we get

$$
\begin{aligned}
(4\text{-}19)\quad 0 &\ge C^* + \psi'\Phi + (\psi'' - \psi'^2)P^{ij}u_i u_j + a\psi' F^{ij} u_i u_j \\
&\qquad + b\psi'|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T} \\
&= C^* + \psi'\Phi + (\psi'' - \psi'^2 - a\psi')P^{ij}u_i u_j \\
&\qquad + (a\bar{\theta} + b)\psi'|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}.
\end{aligned}
$$

Claim 4.2. *If $-\delta_1 < u < \delta_2$, then there exist positive constants α_1, α_2 and p depending only on $\bar{\theta}$, δ_1, δ_2, $|a|_{L^\infty(\bar{M})}$ and $|b|_{L^\infty(\bar{M})}$, such that $\psi' > 0$, and*

$$
(4\text{-}20)\qquad (\psi'' - \psi'^2 - |a|_{L^\infty}\psi')\varepsilon_0 - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\psi' \ge \varepsilon_1 > 0,
$$

for some constant ε_1 depending only on $\bar{\theta}$, δ_1 and δ_2.

Note that $\Phi > 0$. Then by Claim 4.2, we have

$$
0 \ge C^* + \varepsilon_1 |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u| + 1)\mathcal{T}.
$$

Multiplying η^2 both sides of the inequality above, we have

$$
(4\text{-}21)\qquad \varepsilon_1 \eta^2 |\nabla u|^2 \mathcal{T} \le 2C|\nabla u|\mathcal{T} + C^*.
$$

By Lemma 2.1, $\mathcal{T} \ge 1$. Then (4-21) implies the gradient estimates.

Proof of Claim 4.2. Since $-\delta_1 \le u \le \delta_2$. By (4-4), for

$$
\frac{\delta_1 + \delta_2}{2} \le \alpha_2 \le \delta_2, \qquad p > \max\{3, 8|a|_{L^\infty}\delta_2\},
$$

we have $\alpha_1 = 1/(p^2(2\delta_2)^p)$, $\psi' > 0$, and

$$
\psi'' - \psi'^2 - a\psi' \ge \psi'\left(\frac{p}{4\delta_2} - |a|_{L^\infty}\right) \ge \frac{\psi' p}{8\delta_2}.
$$

Furthermore, we can choose

$$
p > \max\left\{3, 8|a|_{L^\infty}\delta_2, \frac{16}{\varepsilon_0}(\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\delta_2\right\},
$$

such that

$$
\begin{aligned}
&(\psi'' - \psi'^2 - |a|_{L^\infty}\psi')\varepsilon_0 - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\psi' \\
&\qquad \ge \psi'\left(\frac{p\varepsilon_0}{8\delta_2} - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\right) \ge \frac{\psi' p \varepsilon_0}{16\delta_2} \ge \frac{\varepsilon_0(\delta_2 - \delta_1)^{p-1}}{2^{p+3}\delta_2} \ge \varepsilon_1 > 0. \quad \square
\end{aligned}
$$

Remark 4.3. If $\bar{B}_r^+$ and $\bar{B}_{r/2}^+$ are replaced by two local geodesic open balls in the interior of M and $\beta = \psi(u)$ in the auxiliary function G, we can get the interior gradient estimates for (1-6) by the proof of Proposition 4.1.

Since $\overline{M}$ is a compact manifold, by Proposition 4.1 and Remark 4.3, we can derive the global gradient estimate of (1-6).

Proposition 4.4. *Let u be a C^3 solution of* (1-6) *on $\overline{M}$. Then there is a positive constant C_1 depending only on n, k, $\bar{\theta}$, g, a, b, Φ, S and C_0 such that*

$$\sup_{\overline{M}} |\nabla u|_g \le C_1. \tag{4-22}$$

5. Estimates for the second derivatives

Lemma 5.1. *Let u be a C^4 solution of* (1-6). *Then there is a positive constant C' depending only on n, k, $\bar{\theta}$, g, $|S|_{C^1(\bar{B}_r^+)}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$ and C_1, such that*

$$u_{nnn} \ge -C' \quad \text{on } \partial M. \tag{5-1}$$

Proof. We consider this lemma for S satisfying condition (S1) or (S2), respectively.

(i) Suppose S satisfy (S1). Then $S_{\alpha n} = S(\partial/\partial x_\alpha, \partial/\partial x_n) = 0$ on the boundary ∂M. By the boundary condition $u_n = 0$ and the Lemma 2.2, we have $V[u]_{\alpha n} = S_{\alpha n} = 0$. Applying an argument of Lemma 13 in [Chen 2009], we know that

$$F^{\alpha n}(V[u]) = 0. \tag{5-2}$$

Also by Lemma 2.2, we calculate that

$$\begin{aligned} V[u]_{\alpha\beta,n} &= \bar{\theta} u_{nnn} g_{\alpha\beta} + \bar{\theta} u_{\gamma\gamma n} g_{\alpha\beta} - u_{\alpha\beta n} + 2a u_{\alpha n} u_\beta + a_n u_\alpha u_\beta \\ &\qquad + 2b u_{\alpha n} u_\alpha g_{\alpha\beta} + 2b u_{nn} u_n g_{\alpha\beta} + b_n |\nabla u|^2 g_{\alpha\beta} + S_{\alpha\beta,n} \\ &= \bar{\theta} u_{nnn} g_{\alpha\beta} + a_n u_\alpha u_\beta + b_n |\nabla u|^2 g_{\alpha\beta} + S_{\alpha\beta,n} \\ &\le \bar{\theta} u_{nnn} g_{\alpha\beta} + C, \end{aligned} \tag{5-3}$$

where the constant C depends only on $|\nabla a|$, $|\nabla b|$, C_1, g and $|\nabla S|$.

Similarly, we have

$$\begin{aligned} V[u]_{nnn} &= \bar{\theta} u_{\gamma\gamma n} + \bar{\theta} u_{nnn} - u_{nnn} + a_n u_n^2 + 2a u_n u_{nn} + 2b u_{\alpha n} u_\alpha \\ &\qquad + 2b u_n u_{nn} + b_n |\nabla u|^2 + S_{nn,n} \\ &\le \bar{\theta} u_{nnn} - u_{nnn} + C. \end{aligned} \tag{5-4}$$

By differentiating (1-6) along the normal direction the on boundary, using (5-2)–(5-4), we have

$$\begin{aligned}\nabla_n\Phi &= F^{nn}V[u]_{nnn}+F^{\alpha\beta}V[u]_{\alpha\beta n}\\ &\le F^{nn}(\bar\theta u_{nnn}-u_{nnn})+\bar\theta u_{nnn}F^{\alpha\beta}g_{\alpha\beta}+C\mathcal{T}\\ &=-F^{nn}u_{nnn}+\bar\theta u_{nnn}\mathcal{T}+C\mathcal{T},\end{aligned}$$

where we have used $g_{\alpha n}=0$ and $g_{nn}=1$. Since $\mathcal{T}>1$, we have

$$0\le -F^{nn}u_{nnn}+(\bar\theta u_{nnn}+C)\mathcal{T}, \tag{5-5}$$

where C also depends on $|\nabla\Phi|$.

If $\bar\theta u_{nnn}+C>0$, we get $u_{nnn}>-C/\bar\theta$, which implies (5-1). If $\bar\theta u_{nnn}+C<0$, by $F^{nn}<\mathcal{T}$ we have

$$0\le -F^{nn}u_{nnn}+(\bar\theta u_{nnn}+C)F^{nn}=((\bar\theta-1)u_{nnn}+C)F^{nn}.$$

Since $F^{nn}>0$, we have

$$(\bar\theta-1)u_{nnn}+C\ge 0. \tag{5-6}$$

Note that $\bar\theta-1=\varepsilon_0>0$; then (5-6) implies (5-1).

(ii) Suppose $S=A_g^\tau$. For any $x_0\in\partial M$, using the metric $\bar g$ in Lemma 2.3, we consider a metric $\hat g=e^{2v}\bar g$ such that $u=\bar u+v$ is a solution of (1-6). Now,

$$\begin{aligned}V[u]_{ij}&=\bar\theta\triangle\bar u g_{ij}+\bar\theta\triangle v g_{ij}-\bar u_{ij}-v_{ij}+a(\bar u_i\bar u_j+\bar u_iv_j+v_i\bar u_j+v_iv_j)\\ &\qquad+b(|\nabla\bar u|^2+2\langle\nabla\bar u,\nabla v\rangle+|\nabla v|^2)g_{ij}+(A_g^\tau)_{ij}.\end{aligned} \tag{5-7}$$

By (1-3), we have

$$(A_{\bar g}^\tau)_{ij}=\bar\theta\triangle\bar u g_{ij}-\bar u_{ij}+\bar u_i\bar u_j+\frac{(n-2)\bar\theta-1}{2}|\nabla\bar u|^2g_{ij}+(A_g^\tau)_{ij}. \tag{5-8}$$

Substituting (5-8) into (5-7), we obtain

$$\begin{aligned}V[u]_{ij}&=\bar\theta\triangle v g_{ij}-v_{ij}+a(\bar u_iv_j+v_i\bar u_j+v_iv_j)+(a-1)\bar u_i\bar u_j\\ &\qquad+b(2\langle\nabla\bar u,\nabla v\rangle+|\nabla v|^2)g_{ij}+\left(b-\frac{(n-2)\bar\theta-1}{2}\right)|\nabla\bar u|^2g_{ij}+(A_{\bar g}^\tau)_{ij}.\end{aligned}$$

Since $\bar{g} = e^{2\bar{u}} g$, we have

$$V[u]_{ij} = \bar{\theta}\bar{\triangle} v \bar{g}_{ij} - \bar{\nabla}^2_{ij} v + \bar{\theta}\bar{g}^{sl}(\bar{\Gamma}^k_{sl}(\bar{g}) - \Gamma^k_{sl}(g))v_k \bar{g}_{ij} - (\bar{\Gamma}^k_{ij}(\bar{g}) - \Gamma^k_{ij}(g))v_k + a(\bar{u}_i v_j + v_i \bar{u}_j + v_i v_j) + (a-1)\bar{u}_i\bar{u}_j + b(2\langle \bar{\nabla}\bar{u}, \bar{\nabla} v\rangle_{\bar{g}} + |\bar{\nabla} v|^2_{\bar{g}})\bar{g}_{ij} + \left(b - \frac{(n-2)\bar{\theta} - 1}{2}\right)|\nabla \bar{u}|^2_{\bar{g}} \bar{g}_{ij} + (A^{\tau}_{\bar{g}})_{ij}. \tag{5-9}$$

Denote $\bar{V}[v]_{ij} := V[u]_{ij}$. Then (1-6) becomes

$$\begin{cases} F(\bar{V}[v]) = \Phi(x, \bar{u} + v) & \text{in } M, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial M. \end{cases} \tag{5-10}$$

By the boundary condition $u_n = 0$, $\bar{u}_n = 0$ and Lemma 2.2, we have

$$u_{n\alpha} = 0, \quad u_{\alpha\beta n} = 0, \quad \bar{u}_{n\alpha} = 0, \quad \bar{u}_{\alpha\beta n} = 0. \tag{5-11}$$

Therefore $v_n = 0$, $v_{n\alpha} = 0$ and $v_{\alpha\beta n} = 0$ on ∂M. Since $\bar{g}_{\alpha n} = e^{2\bar{u}} g_{\alpha n} = 0$, we have

$$\bar{V}[v]_{\alpha n} = -\bar{\nabla}^2_{\alpha n} v - (\bar{\Gamma}^\delta_{\alpha n}(\bar{g}) - \Gamma^\delta_{\alpha n}(g_0))v_\delta + (A^\tau_{\bar{g}})_{\alpha n}.$$

It follows from (2-2) and the boundary condition $u_n = 0$ that

$$\bar{\Gamma}^\delta_{\alpha n}(\bar{g}) = \Gamma^\delta_{\alpha n}(g) = 0, \quad \bar{\Gamma}^n_{\alpha\beta} = \Gamma^n_{\alpha\beta} = 0, \quad \bar{\Gamma}^n_{nn} = \Gamma^n_{nn} = 0. \tag{5-12}$$

Then

$$\bar{\nabla}^2_{\alpha n} v = v_{\alpha n} = 0 \quad \text{and} \quad \bar{\nabla}_n \bar{\nabla}^2_{\alpha\beta} v = v_{\alpha\beta n} = 0. \tag{5-13}$$

By Lemma 2.3, we get

$$(A^\tau_{\bar{g}})_{\alpha n}(x_0) = -\frac{1}{n-2}\left(\bar{R}_{\alpha n} - \frac{\tau \bar{R}}{2(n-1)}\bar{g}_{\alpha n}\right) = 0.$$

Hence, $\bar{V}[v]_{\alpha n}(x_0) = 0$. Then

$$F^{\alpha n}(\bar{V}[v]) = 0.$$

Now differentiating (5-10) along the normal direction and taking its value at x_0, we have

$$\nabla_n \Phi(x, \bar{u} + v) = F^{nn}\bar{V}_{nnn} + F^{\alpha\beta}\bar{V}_{\alpha\beta n}. \tag{5-14}$$

Since $\bar{g}_{ij,n} = \bar{g}^{ij}_{,n} = 0$, by (5-11)–(5-13), we have

$$\bar{V}[v]_{\alpha\beta n} = \bar{\theta} v_{nnn}\bar{g}_{\alpha\beta} - (\bar{\Gamma}^\delta_{\alpha\beta}(\bar{g}) - \Gamma^\delta_{\alpha\beta}(g))_{,n} v_\delta + \bar{\theta}\bar{g}^{sl}(\bar{\Gamma}^\delta_{sl}(\bar{g}) - \Gamma^\delta_{sl}(g))_{,n} v_\delta \bar{g}_{\alpha\beta} + (A^\tau_{\bar{g}})_{\alpha\beta,n}.$$

Since ∂M is totally geodesic, using Fermi coordinates, we have on ∂M

$$\bar{\Gamma}^{\delta}_{\alpha\beta}(g)_{,n} = \Gamma^{\delta}_{\alpha\beta}(g)_{,n} = 0$$

(see [He and Sheng 2013]). By Lemma 2.3 again,

$$\bar{R}_n(x_0) = \bar{g}^{\alpha\beta}\bar{R}_{\alpha\beta,n}(x_0) + \bar{g}^{\alpha n}\bar{R}_{\alpha n,n}(x_0) + \bar{g}^{nn}\bar{R}_{nn,n}(x_0) = 0.$$

Therefore

$$(A^{\tau}_{\bar{g}})_{\alpha\beta,n}(x_0) = -\frac{1}{n-2}\left(\bar{R}_{\alpha\beta,n} - \frac{\tau\bar{R}_n}{2(n-1)}\bar{g}_{\alpha\beta}\right)(x_0) = 0.$$

Hence, we obtain

$$\bar{V}[v]_{\alpha\beta n}(x_0) = \bar{\theta} v_{nnn}\bar{g}_{\alpha\beta}. \tag{5-15}$$

Similarly, we have

$$\bar{V}[v]_{nnn}(x_0) = \bar{\theta} v_{nnn}\bar{g}_{nn}(x_0) - v_{nnn}(x_0). \tag{5-16}$$

Denote $\bar{\mathcal{T}} = F^{ij}(\bar{V}[v])\bar{g}_{ij} \geq 1$. Plugging (5-15) and (5-16) into (5-14), we obtain

$$0 \leq C + \bar{\theta} v_{nnn}(x_0)\bar{\mathcal{T}} - F^{nn}v_{nnn}(x_0) \leq (C + \bar{\theta} v_{nnn}(x_0))\bar{\mathcal{T}} - F^{nn}v_{nnn}(x_0). \tag{5-17}$$

If $C + \bar{\theta} v_{nnn}(x_0) \geq 0$, then we have $v_{nnn}(x_0) \geq -C/\bar{\theta}$, which implies that

$$u_{nnn}(x_0) \geq \bar{u}_{nnn}(x_0) - \frac{C}{\bar{\theta}} > -C'.$$

If $C + \bar{\theta} v_{nnn}(x_0) < 0$, then by (5-17) we have

$$0 \leq (C + (\bar{\theta} - 1)v_{nnn}(x_0))F^{nn}.$$

Since $F^{nn} > 0$ and $\bar{\theta} > 1$, we have $v_{nnn}(x_0) \geq -C/(\bar{\theta} - 1)$, which also implies the lower bound of $u_{nnn}(x_0)$. □

Proposition 5.2. *Let u be a C^4 solution of* (1-6) *on $\bar{B}^+_r$. Then there is a positive constant C_2 depending only on n, k, $\bar{\theta}$, r, g, $|S|_{C^2(\bar{B}^+_r)}$, $|\Phi|_{C^2(\bar{B}^+_r)\times[-C_0,C_0]}$, $|a|_{C^2(\bar{B}^+_r)}$, $|b|_{C^2(\bar{B}^+_r)}$, and C_1, such that*

$$\sup_{\bar{B}^+_{r/2}} |\nabla^2 u|_g \leq C_2. \tag{5-18}$$

Proof. We control the bound of $\triangle u$ at first. Since $V[u] \in \Gamma \subset \Gamma_1$, we have

$$0 \leq \operatorname{tr}(V[u]) = (n\bar{\theta} - 1)\triangle u + (a + nb)|\nabla u|^2 + \operatorname{tr} S,$$

which implies that $\triangle u$ has a lower bound by Proposition 4.4. We may assume $\triangle u > 0$.

Consider the auxiliary function

$$G := \eta e^{x_n}(\triangle u + m|\nabla u|^2),$$

where η satisfies (4-1) and (4-2), and m is a larger constant to be fixed. We may assume $r = 1$, and

$$K := \triangle u + m|\nabla u|^2 \gg 1.$$

Step 1. We may assume G attains its maximum at an interior point $x_0 \in B_r^+$. If $x_0 \in \Sigma_r$, by Lemmas 2.2 and 5.1 we have

$$G_n(x_0) = K + u_{nnn} + u_{\gamma\gamma n} + 2mu_{\alpha n}u_\alpha + 2mu_{nn}u_n > K - C'.$$

If $K - C' \leq 0$, we then get the bound of $\triangle u$. If $K - C' > 0$, it contradicts with the maximum of G at the boundary point x_0.

Step 2. We must get an upper bound for $\triangle u$. By step 1, the maximum point x_0 of G is an interior point in $\bar{B}_r^+$. Then at x_0 we have

$$G_i = 0 \quad \text{and} \quad G_{ij} \leq 0,$$

that is,

$$u_{lli} + 2mu_l u_{li} = K_i = -\left(\frac{\eta_i}{\eta} + \delta_{in}\right)K, \tag{5-19}$$

and

$$0 \geq G_{ij} = \eta e^{x_n}\left\{\left(\frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2}\right)K + \left(\frac{\eta_i}{\eta} + \delta_{in}\right)K_j + K_{ij}\right\}.$$

Substituting (5-19) into the inequality above, by the definition of η in (4-1), we have

$$0 \geq G_{ij} = \eta e^{x_n}(K_{ij} + \Lambda_{ij}K),$$

where

$$\Lambda_{ij} = \frac{\eta_{ij}}{\eta} - 2\frac{\eta_i\eta_j}{\eta^2} - \frac{1}{\eta}(\eta_i\delta_{jn} + \eta_j\delta_{in}) - \delta_{in}\delta_{jn} \geq -\frac{C}{\eta}\delta_{ij},$$

and C depends only on b_0. Then we have

$$0 \geq e^{-x_n}P^{ij}G_{ij} \geq \eta P^{ij}K_{ij} - CK\mathcal{T}. \tag{5-20}$$

Note that

$$K_{ij} = u_{llij} + 2mu_{li}u_{lj} + 2mu_l u_{lij}. \tag{5-21}$$

By Ricci identities, we have

$$|u_{ijl} - u_{lij}| \leq C \quad \text{and} \quad |u_{ijll} - u_{llij}| \leq C(|\nabla^2 u| + 1).$$

Then we have

$$P^{ij}K_{ij} \geq P^{ij}u_{ijll} + 2mP^{ij}u_{li}u_{lj} + 2mu_l P^{ij}u_{ijl} - C(|\nabla^2 u| + 1)\mathcal{T}. \tag{5-22}$$

By (4-10), we have

$$\begin{aligned} 2mu_l P^{ij}u_{ijl} &= 2mu_l \nabla_l \Phi - F^{ij}(a_l u_i u_j + 2au_{il}u_j + S_{ij},_l) - (b_l|\nabla u|^2 + 2bu_{ls}u_s)\mathcal{T} \\ &\geq -C(|\nabla^2 u| + 1)\mathcal{T}, \end{aligned} \tag{5-23}$$

since $\nabla_{ll}\Phi = \Phi_{xx} + 2\Phi_{xz}u_l + \Phi_z u_{ll} \geq -C + \Phi_z \triangle u \geq -C(|\nabla^2 u| + 1)$. Differentiating the equation (1-6) twice, using the concavity of F, we have

$$\begin{aligned} P^{ij}u_{ijll} \geq \nabla_{ll}\Phi - F^{ij}(a_{ll}u_i u_j + 4a_l u_{il}u_j + 2au_{ill}u_j + 2au_{il}u_{jl} + S_{ij},_{ll}) \\ - (b_{ll}|\nabla u|^2 + 4b_l u_{ls}u_s + 2bu_{sll}u_s + 2b|\nabla^2 u|^2)\mathcal{T} \\ \geq -2aF^{ij}u_{ill}u_j - 2aF^{ij}u_{il}u_{jl} - 2bu_{sll}u_l\mathcal{T} \\ - 2b|\nabla^2 u|^2\mathcal{T} - C(|\nabla^2 u| + 1)\mathcal{T}. \end{aligned} \tag{5-24}$$

By Ricci identities again, and (5-19) and (5-24), we get

$$P^{ij}u_{ijll} \geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}. \tag{5-25}$$

Now, plugging (5-23) and (5-25) into (5-22), and choosing

$$m > \max\left\{2|a|_{L^\infty}, \frac{4}{\varepsilon_0}(\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\right\},$$

we obtain

$$\begin{aligned} P^{ij}K_{ij} &\geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2\mathcal{T} + 2mP^{ij}u_{li}u_{lj} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &= 2(m + a)P^{ij}u_{li}u_{lj} - 2(a\bar{\theta} + b)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq 2\big((m - |a|_{L^\infty})\varepsilon_0 - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\big)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq 2\left(\frac{m\varepsilon_0}{2} - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\right)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq \frac{m\varepsilon_0}{2}|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}. \end{aligned} \tag{5-26}$$

It follows from (5-20) and (5-26) that

$$\eta^2 \frac{m\varepsilon_0}{2}|\nabla^2 u|^2\mathcal{T} \leq C(|\nabla^2 u| + 1)\mathcal{T},$$

which implies that $\eta|\nabla^2 u| \le C$.

Step 3. We get the Hessian bound of u. As in [Chen 2009], we consider the maximum of

$$\overline{G} = \eta(x)e^{x_n}(\nabla^2 u + m\, du \otimes du)$$

over the set $(x, \xi) \in (\overline{B}_r^+, \mathbb{S}^n)$. Let $\overline{G}$ attain its maximum at some point x_0 and the direction $\xi \in T_{x_0}\overline{M} \cap \mathbb{S}^n$. Denote $K_\xi = u_{\xi\xi} + mu_\xi^2$. We may assume $K_\xi \gg C' > 0$, where C' is the one in Lemma 5.1.

Now, we can also show that x_0 does not belong to the boundary. Suppose $x_0 \in \Sigma_r$. If ξ is a tangential vector, without loss of generality we may assume $\xi = \partial/\partial x_1$. By Lemma 2.2, we have on the boundary that

$$\begin{aligned}(\eta e^{x_n}(u_{11} + mu_1^2))_n &= \eta e^{x_n}((u_{11} + mu_1^2) + u_{11n} + 2mu_1u_{1n}) \\ &\ge u_{11} + mu_1^2 = K_1 > 0\end{aligned}$$

Therefore, we get a contradiction. If ξ is in the normal direction, by Lemma 2.2 and Lemma 5.1, we also have

$$\begin{aligned}(\eta e^{x_n}(u_{nn} + mu_n^2))_n &= \eta e^{x_n}((u_{nn} + mu_n^2) + u_{nnn} + 2mu_nu_{nn}) \\ &\ge u_{nn} - C' = K_n - C' > 0.\end{aligned}$$

Thus x_0 must be an interior point. By similar calculations as before, we can get the Hessian bounds. We omit the details here. □

Remark 5.3. Let B_r and $B_{r/2}$ be two local geodesic balls in the interior of M, and $G = \eta(\triangle u + m|\nabla u|^2)$. The same calculations in steps 2 and 3 yield the interior Hessian estimates for (1-6).

Therefore we have the following global estimates.

Proposition 5.4. *Let u be a C^4 solution of* (1-6) *on $\overline{M}$. Then there is a positive constant C_2 depending only on $n, k, \bar{\theta}, g, a, b, \Phi, S$ and C_1, such that*

$$\sup_{\overline{M}} |\nabla^2 u|_g \le C_2.$$

6. Proof of Theorem 1.2

We use the continuity method to prove the existence of (1-6). Since the argument is standard (see [Li and Sheng 2011]), we only sketch it here.

For $t \in [0, 1]$, consider the equation

$$F\big(g^{-1}(\bar{\theta}\triangle u g - \nabla^2 u + a(x)du \otimes du + b(x)|\nabla u|^2 g + S_t)\big) = \Phi_t(x, u), \tag{6-1$_t$}$$

where

$$S_t = tS + \frac{1-t}{F(e)}g \quad \text{and} \quad \Phi_t(x, u) = (1-t)e^{2u} + t\Phi(x, u).$$

Clearly, S_t and Φ_t satisfy the following conditions:

- $S_t \in \Gamma$ and $|S_t|_{C^4(\overline{M})} \le C$, where the constant C is independent of t.
- S_t satisfies (S1) or $S_t = tA_g^\tau$ when $t \neq 0$ and $S_0 = \dfrac{1}{F(e)} g$ as long as S satisfies (S1) or (S2).
- $\Phi_t(x, u) > 0$, $\partial_z \Phi_t > 0$, $\lim_{z\to+\infty} \Phi_t(x, z) \to +\infty$, and $\lim_{z\to-\infty} \Phi_t(x, z) \to 0$.
- $|\Phi_t|_{C^2(\overline{M}\times[-C,C])} \le C$, where C is independent of t.

It follows from Sections 3, 4 and 5 that for each t, the admissible solution of (6-1_t) has uniform a priori C^2 estimates (independent of t). Then we obtain the uniform $C^{2,\alpha}$ estimates by Evans–Krylov theory [Krylov 1985]. Define

$$I = \{t \in [0, 1] \mid (6\text{-}1_t) \text{ has admissible solution}\}.$$

Clearly, $u \equiv 0$ is the unique admissible solution of (6.1_0). Hence, $I \neq \varnothing$. By Lemma 3.1, $I \subset [0, 1]$ is open. By the uniform a priori $C^{2,\alpha}$ estimates and the standard degree theory, we conclude that I is also closed. Then for $t = 1$, (1-6) is solvable. □

7. Proof of Theorem 1.3

Before proving Theorem 1.3, we first calculate a priori estimates for (1-9).

Proposition 7.1. *Suppose $T \in \Gamma$ and the positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfy* (1-8). *Then there exists a constant C_0 only depending on T and Φ, such that any solution $u \in C^2(\overline{M})$ of* (1-9) *satisfies*

$$\sup_{\overline{M}} |u| \le C_0.$$

The proof is similar to that of Proposition 3.2. We omit it here.

Proposition 7.2. *Suppose u is a C^3 solution of* (1-9) *on $\bar{B}_r^+$. Then there is a positive constant C depending only on n, k, θ, g, r, $|T|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$ and C_0, such that*

$$\sup_{\bar{B}_{r/2}^+} |\nabla u|_g \le C.$$

Proof. Consider the auxiliary functions

$$G := \tfrac{1}{2}\eta e^\beta |\nabla u|^2, \quad \beta := x_n + \psi(u).$$

Then G can not attain its maximum at a boundary point $x_0 \in \Sigma_r$ by the same arguments in the proof of Proposition 4.1. Since the maximum point x_0 is an interior point, we can also get (4-5)–(4-7). Now, the difference from the proof of Proposition 4.1 is that we replace the operator P^{ij} in (4-8) by the operator

$$Q^{ij} := F^{ij} + \theta \mathcal{T} g^{ij}. \tag{7-1}$$

Then by similar calculations as in (4-9)–(4-16), we obtain

$$\begin{aligned} 0 \geq C^* + \psi' Q^{ij} u_{ij} + (\psi'' - \psi'^2) Q^{ij} u_i u_j + 2a\psi' Q^{ij} u_i u_j \\ + 2b\psi' |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}} (|\nabla u| + 1)\mathcal{T}. \end{aligned} \tag{7-2}$$

Since

$$\begin{aligned} \psi' Q^{ij} u_{ij} &= \psi' F^{ij} W_{ij} - \psi' F^{ij} (a u_i u_j + b|\nabla u|^2 g_{ij} + T_{ij}) \\ &\geq \psi' \Phi - a\psi' F^{ij} u_i u_j - b\psi' |\nabla u|^2 - C\mathcal{T}. \end{aligned} \tag{7-3}$$

Substituting (7-3) into (7-2), we get

$$\begin{aligned} 0 &\geq C^* + \psi' \Phi + (\psi'' - \psi'^2) Q^{ij} u_i u_j + a\psi' F^{ij} u_i u_j \\ &\qquad + b\psi' |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}} (|\nabla u| + 1)\mathcal{T} \\ &= C^* + \psi' \Phi + (\psi'' - \psi'^2 + a\psi') F^{ij} u_i u_j \\ &\qquad + (\theta(\psi'' - \psi'^2) + b\psi') |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}} (|\nabla u| + 1)\mathcal{T}. \end{aligned} \tag{7-4}$$

By the similar argument as in Claim 4.2, we know that there exist positive constants α_1, α_2 and p depending only on θ, C_0, $|a|_{L^\infty(\overline{M})}$ and $|b|_{L^\infty(\overline{M})}$, such that

$$\psi' > 0, \quad \psi'' - \psi'^2 - |a|_{L^\infty} \psi' > 0, \quad \theta(\psi'' - \psi'^2) - |b|\psi' \geq \varepsilon_2 > 0,$$

where the constant ε_2 only depends on α_1, α_2 and p. Then we have

$$0 \geq C^* + \varepsilon_2 |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}} (|\nabla u| + 1)\mathcal{T}. \tag{7-5}$$

Then multiplying by η^2 both sides of the inequality above and $\mathcal{T} > 1$, we have

$$\varepsilon_2 \eta^2 |\nabla u|^2 \mathcal{T} \leq C |\nabla u| \mathcal{T} + C^*,$$

which implies the gradient estimates. □

To get the boundary Hessian estimates, we first prove the following:

Lemma 7.3. *Let u be a C^4 solution of* (1-9)*. Then there is a positive constant C' depending only on n, k, θ, g, $|T|_{C^1(\bar{B}_r^+)}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$ and C_1 such that on ∂M, we have*

$$u_{nnn} \geq -C'.$$

Proof. (i) Let T satisfy the condition (S1). Then $T_{\alpha n} = 0$ on the boundary. Hence $W[u]_{\alpha n} = T_{\alpha n} = 0$. Therefore $F^{\alpha n}(W[u]) = 0$. By the similar calculations in Lemma 5.1, we have

$$W[u]_{\alpha\beta,n} \leq \theta u_{nnn} g_{\alpha\beta} + C \tag{7-6}$$

and

$$W[u]_{nnn} \leq u_{nnn} + \theta u_{nnn} + C, \tag{7-7}$$

where the constants C depend on n, k, g, $|T|_{C^1(\bar{B}_r^+)}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$ and C_1.

Now, differentiating (1-9) along the normal direction and taking the value on the boundary, we have

$$\begin{aligned}\nabla_n \Phi &= F^{nn} W[u]_{nnn} + F^{\alpha\beta} W[u]_{\alpha\beta n} \\ &\leq F^{nn}(u_{nnn} + \theta u_{nnn}) + \theta u_{nnn} F^{\alpha\beta} g_{\alpha\beta} + C\mathcal{T} \\ &= F^{nn} u_{nnn} + \theta u_{nnn}\mathcal{T} + C\mathcal{T},\end{aligned} \tag{7-8}$$

that is,

$$0 \leq F^{nn} u_{nnn} + \theta u_{nnn}\mathcal{T} + C\mathcal{T} = F^{nn} u_{nnn} + (\theta u_{nnn} + C)\mathcal{T}, \tag{7-9}$$

where the constant C also depends on $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$.

If $\theta u_{nnn} + C \geq 0$, then we get $u_{nnn} \geq -C/\theta$. If $\theta u_{nnn} + C < 0$, by $F^{nn} < \mathcal{T}$ and (7-9), we have

$$0 \leq F^{nn} u_{nnn} + (\theta u_{nnn} + C)F^{nn} = ((\theta+1)u_{nnn} + C)F^{nn}.$$

Since $F^{nn} > 0$, we get

$$(\theta+1)u_{nnn} + C \geq 0.$$

Note $\theta > 0$. Then we obtain $u_{nnn} \geq -C'$ again.

(ii) Suppose $T = -A_g^\tau$. Using the metric $\bar{g}$ in Lemma 2.3, we consider a new metric $\breve{g} = e^{2w}\bar{g}$ such that $u = \bar{u} + w$ is a solution of (1-9). Then similar to the calculation in the proof of Lemma 5.1, we have

$$\begin{aligned}W[u]_{ij} = \theta \bar{\Delta} w \bar{g}_{ij} &+ \bar{\nabla}^2_{ij} w + \theta \bar{g}^{sl}(\bar{\Gamma}^k_{sl}(\bar{g}) - \Gamma^k_{sl}(g)) w_k \bar{g}_{ij} + (\bar{\Gamma}^k_{ij}(\bar{g}) - \Gamma^k_{ij}(g)) w_k \\ &+ (a-1)\bar{u}_i\bar{u}_j + a(\bar{u}_i w_j + w_i \bar{u}_j + w_i w_j) + b\big(2\langle \bar{\nabla}\bar{u}, \bar{\nabla} w\rangle_{\bar{g}} + |\bar{\nabla} w|^2_{\bar{g}}\big)\bar{g}_{ij} \\ &+ \left(b - \frac{1+(n-2)\theta}{2}\right)|\bar{\nabla} u|^2_{\bar{g}} \bar{g}_{ij} - (A^\tau_{\bar{g}})_{ij}.\end{aligned}$$

Denote $\overline{W}[w]_{ij} := W[u]_{ij}$. Now, (1-9) becomes

$$\begin{cases} F(\overline{W}[w]) = \Phi(x, \bar{u}+w) & \text{in } M, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial M. \end{cases} \tag{7-10}$$

By Lemma 2.3, we find $(A_{\bar{g}}^{\tau})_{\alpha n}(x_0) = 0$. Then we have $\overline{W}[w]_{\alpha n}(x_0) = 0$ by Lemma 2.2 and (5-11)–(5-13), which implies $F^{\alpha n}(\overline{W}[w]) = 0$. By Lemma 2.2 again, we obtain

$$\overline{W}[w]_{\alpha\beta n}(x_0) = \theta w_{nnn} \bar{g}_{\alpha\beta}(x_0),$$

and

$$\overline{W}[w]_{nnn}(x_0) = \theta w_{nnn} \bar{g}_{nn}(x_0) + w_{nnn}(x_0).$$

Then by differentiating (7-10) along the normal direction and taking its value at x_0, we have

$$\begin{aligned} 0 &\le F^{nn} \overline{W}_{nnn} + F^{\alpha\beta} \overline{W}_{\alpha\beta n} + C \\ &\le F^{nn} w_{nnn}(x_0) + (\theta w_{nnn}(x_0) + C)\overline{\mathcal{T}}. \end{aligned}$$

If $\theta w_{nnn}(x_0) + C \ge 0$, we have $u_{nnn}(x_0) \ge -C'$ immediately. Now consider $\theta w_{nnn}(x_0) + C < 0$. Since $\overline{\mathcal{T}} > F^{nn} > 0$, we have

$$0 < F^{nn} w_{nnn}(x_0) + (\theta w_{nnn}(x_0) + C) F^{nn} \le ((\theta+1) w_{nnn}(x_0) + C) F^{nn}.$$

Hence, we must have $w_{nnn}(x_0) \ge -C/(\theta+1)$. Therefore, $u_{nnn}(x_0) \ge -C'$. □

Proposition 7.4. *Let u be a C^4 solution of* (1-9) *on $\bar{B}_r^+$. Then there is a positive constant C_2 depending only on n, k, θ, g, r, $|T|_{C^2(\bar{B}_r^+)}$, $|\Phi|_{C^2(\bar{B}_r^+)\times[-C_0, C_0]}$, $|a|_{C^2(\bar{B}_r^+)}$, $|b|_{C^2(\bar{B}_r^+)}$ and C_1 such that*

$$\sup_{\bar{B}_{r/2}^+} |\nabla^2 u|_g \le C_2.$$

Proof. We first estimate the bound of $\triangle u$. By $W[u] \in \Gamma_k^+ \subset \Gamma_1$, we have

$$0 \le \operatorname{tr}(W[u]) = (n\theta+1)\triangle u + (a+nb)|\nabla u|^2 + \operatorname{tr} T,$$

which implies that $\triangle u$ has lower bound. Hence, we may assume $\triangle u > 0$.

Consider the same auxiliary function in Proposition 5.2

$$G := \eta e^{q x_n} (\triangle u + m|\nabla u|^2),$$

where η satisfies (4-1) and (4-2), m is a larger constant to be fixed. We may assume $r = 1$ and $K := \triangle u + m|\nabla u|^2 \gg 1$.

Step 1. We show the maximum of G must be attained at an interior point of $\bar{B}_r^+$. If the maximum point x_0 of G belong to Σ_r, then by Lemma 2.2, Lemma 7.3 and the same calculations in Proposition 5.2, we know that $G_n(x_0) > 0$. It is a contradiction.

Step 2. We must get an upper bound for $\triangle u$. Since the maximum point of G is an interior point of $\bar{B}_r^+$ by step 1. Then at the maximum point x_0, we can get similar inequalities as in (5-19)–(5-24) by replacing P^{ij} by Q^{ij}. Corresponding to (5-26), for $m > \max\{|a|_{L^\infty(\bar{M})}, (|b|_{L^\infty(\bar{M})} + \varepsilon_3)/\theta\}$, $\varepsilon_3 > 0$, we obtain

$$
\begin{aligned}
\text{(7-11)}\quad & Q^{ij} K_{ij} \\
&\geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2 \mathcal{T} + 2mQ^{ij}u_{li}u_{lj} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&= 2(m-a)F^{ij}u_{li}u_{lj} + 2(m\theta - b)|\nabla^2 u|^2 \mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&\geq 2(m - |a|_{L^\infty})F^{ij}u_{li}u_{lj} + 2(m\theta - |b|_{L^\infty})|\nabla^2 u|^2 \mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&\geq 2\varepsilon_3 |\nabla^2 u|^2 \mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}.
\end{aligned}
$$

It follows from (5-20) for Q^{ij} and (7-11) that $2\eta^2\varepsilon_3|\nabla^2 u|^2\mathcal{T} \leq C(|\nabla^2 u| + 1)\mathcal{T}$, which implies that $\eta|\nabla^2 u| \leq C$. □

Step 3. By Lemma 7.3 and the same argument in the step 3 of the proof of Proposition 5.2, we can get the Hessian estimates of u.

Remark 7.5. We can also get the interior gradient and Hessian estimates for the solutions of (1-9) by the same arguments in Remarks 4.3 and 5.3.

Proof of Theorem 1.3. Since the operator Q^{ij} in (7-1) is positive, by the argument in Section 3, we know that (1-9) is elliptic at any admissible solutions and its linearized operator is invertible as $\partial_z \Phi > 0$. Combining Propositions 7.1, 7.2, 7.4 and Remark 7.5, we can obtain

$$
\text{(7-12)} \qquad |u|_{C^2(\bar{M})} \leq C,
$$

where the constant C depends only on $n, k, \theta, g, S, \Phi, a$ and b. By the global a priori C^2 estimates (7-12), we can prove Theorem 1.3 by a same argument in Section 6. □

References

[Caffarelli et al. 1985] L. Caffarelli, L. Nirenberg, and J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations, III: Functions of the eigenvalues of the Hessian", *Acta Math.* **155**:3-4 (1985), 261–301. MR 87f:35098 Zbl 0654.35031

[Chang et al. 2002] S.-Y. A. Chang, M. J. Gursky, and P. C. Yang, "An equation of Monge–Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature", *Ann. of Math.* (2) **155**:3 (2002), 709–787. MR 2003j:53048 Zbl 1031.53062

[Chen 2005] S.-y. S. Chen, "Local estimates for some fully nonlinear elliptic equations", *Int. Math. Res. Not.* **2005**:55 (2005), 3403–3425. MR 2006k:53051 Zbl 1159.35343

[Chen 2007] S.-y. S. Chen, "Boundary value problems for some fully nonlinear elliptic equations", *Calc. Var. Partial Differential Equations* **30**:1 (2007), 1–15. MR 2008e:35061 Zbl 1137.58010

[Chen 2009] S.-y. S. Chen, "Conformal deformation on manifolds with boundary", *Geom. Funct. Anal.* **19**:4 (2009), 1029–1064. MR 2010m:53048 Zbl 1185.53042

[Escobar 1992a] J. F. Escobar, "Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary", *Ann. of Math.* (2) **136**:1 (1992), 1–50. MR 93e:53046 Zbl 0766.53033

[Escobar 1992b] J. F. Escobar, "The Yamabe problem on manifolds with boundary", *J. Differential Geom.* **35**:1 (1992), 21–84. MR 93b:53030 Zbl 0771.53017

[Ge and Wang 2006] Y. Ge and G. Wang, "On a fully nonlinear Yamabe problem", *Ann. Sci. École Norm. Sup.* (4) **39**:4 (2006), 569–598. MR 2007k:53040 Zbl 1121.53027

[Guan and Wang 2003a] P. Guan and G. Wang, "A fully nonlinear conformal flow on locally conformally flat manifolds", *J. Reine Angew. Math.* **557** (2003), 219–238. MR 2004e:53101 Zbl 1033.53058

[Guan and Wang 2003b] P. Guan and G. Wang, "Local estimates for a class of fully nonlinear equations arising from conformal geometry", *Int. Math. Res. Not.* **2003**:26 (2003), 1413–1432. MR 2003m:53055 Zbl 1042.53021

[Gursky and Viaclovsky 2003] M. J. Gursky and J. A. Viaclovsky, "Fully nonlinear equations on Riemannian manifolds with negative curvature", *Indiana Univ. Math. J.* **52**:2 (2003), 399–419. MR 2004a:53039 Zbl 1036.53025

[Gursky and Viaclovsky 2007] M. J. Gursky and J. A. Viaclovsky, "Prescribing symmetric functions of the eigenvalues of the Ricci tensor", *Ann. of Math.* (2) **166**:2 (2007), 475–531. MR 2008k:53068 Zbl 1142.53027

[Han and Li 1999] Z.-C. Han and Y. Li, "The Yamabe problem on manifolds with boundary: existence and compactness results", *Duke Math. J.* **99**:3 (1999), 489–542. MR 2000j:53045 Zbl 0945.53023

[Han and Li 2000] Z.-C. Han and Y. Li, "The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature", *Comm. Anal. Geom.* **8**:4 (2000), 809–869. MR 2001m:53062 Zbl 0990.53033

[He and Sheng 2011a] Y. He and W. Sheng, "On existence of the prescribing k-curvature problem on manifolds with boundary", *Comm. Anal. Geom.* **19**:1 (2011), 53–77. MR 2012i:53025 Zbl 1237.53033

[He and Sheng 2011b] Y. He and W. Sheng, "Prescribing the symmetric function of the eigenvalues of the Schouten tensor", *Proc. Amer. Math. Soc.* **139**:3 (2011), 1127–1136. MR 2011m:53055 Zbl 1227.53054

[He and Sheng 2013] Y. He and W. Sheng, "Local estimates for elliptic equations arising in conformal geometry", *Int. Math. Res. Not.* **2013**:2 (2013), 258–290. MR 3010689

[Jin 2007] Q. Jin, "Local Hessian estimates for some conformally invariant fully nonlinear equations with boundary conditions", *Differential Integral Equations* **20**:2 (2007), 121–132. MR 2008b:53046 Zbl 1212.35121

[Jin et al. 2007] Q. Jin, A. Li, and Y. Li, "Estimates and existence results for a fully nonlinear Yamabe problem on manifolds with boundary", *Calc. Var. Partial Differential Equations* **28**:4 (2007), 509–543. MR 2008b:53045 Zbl 1153.35323

[Krylov 1985] N. V. Krylov, *Нелинейные эллиптические и параболические уравнения второго порядка*, Nauka, Moscow, 1985. Translated as *Nonlinear elliptic and parabolic equations of the second order*, Mathematics and its Applications (Soviet Series) **7**, D. Reidel, Dordrecht, 1987. MR 88d:35005 Zbl 0586.35002

[Lee and Parker 1987] J. M. Lee and T. H. Parker, "The Yamabe problem", *Bull. Amer. Math. Soc. (N.S.)* **17**:1 (1987), 37–91. MR 88f:53001 Zbl 0633.53062

[Li and Li 2003] A. Li and Y. Li, "On some conformally invariant fully nonlinear equations", *Comm. Pure Appl. Math.* **56**:10 (2003), 1416–1464. MR 2004e:35072 Zbl 1155.35353

[Li and Li 2005] A. Li and Y. Li, "On some conformally invariant fully nonlinear equations, II: Liouville, Harnack and Yamabe", *Acta Math.* **195** (2005), 117–154. MR 2007d:53053 Zbl 1216.35038

[Li and Sheng 2005] J. Li and W. Sheng, "Deforming metrics with negative curvature by a fully nonlinear flow", *Calc. Var. Partial Differential Equations* **23**:1 (2005), 33–50. MR 2005m:53121 Zbl 1069.53054

[Li and Sheng 2011] Q.-R. Li and W. Sheng, "Some Dirichlet problems arising from conformal geometry", *Pacific J. Math.* **251**:2 (2011), 337–359. MR 2012j:53038 Zbl 1230.53038

[Lin and Trudinger 1994] M. Lin and N. S. Trudinger, "On some inequalities for elementary symmetric functions", *Bull. Austral. Math. Soc.* **50**:2 (1994), 317–326. MR 95i:26036 Zbl 0855.26006

[Sheng and Yuan 2013] W. Sheng and L.-X. Yuan, "The k-Yamabe flow on manifolds with boundary", *Nonlinear Anal.* **82** (2013), 127–141. MR 3020900 Zbl 1263.53031

[Sheng and Zhang 2007] W. Sheng and Y. Zhang, "A class of fully nonlinear equations arising from conformal geometry", *Math. Z.* **255**:1 (2007), 17–34. MR 2007k:53042 Zbl 1133.53036

[Sheng et al. 2007] W. Sheng, N. S. Trudinger, and X.-J. Wang, "The Yamabe problem for higher order curvatures", *J. Differential Geom.* **77**:3 (2007), 515–553. MR 2008i:53048 Zbl 1133.53035

[Trudinger 1990] N. S. Trudinger, "The Dirichlet problem for the prescribed curvature equations", *Arch. Rational Mech. Anal.* **111**:2 (1990), 153–179. MR 91g:35118 Zbl 0721.35018

[Trudinger and Wang 2009] N. S. Trudinger and X.-J. Wang, "On Harnack inequalities and singularities of admissible metrics in the Yamabe problem", *Calc. Var. Partial Differential Equations* **35**:3 (2009), 317–338. MR 2010c:53056 Zbl 1163.53327

[Trudinger and Wang 2010] N. S. Trudinger and X.-J. Wang, "The intermediate case of the Yamabe problem for higher order curvatures", *Int. Math. Res. Not.* **2010** (2010), 2437–2458. MR 2011g:53074 Zbl 1194.53033

[Viaclovsky 2000] J. A. Viaclovsky, "Conformal geometry, contact geometry, and the calculus of variations", *Duke Math. J.* **101**:2 (2000), 283–316. MR 2001b:53038 Zbl 0990.53035

Received December 28, 2012. Revised February 11, 2014.

WEIMIN SHENG
DEPARTMENT OF MATHEMATICS
ZHEJIANG UNIVERSITY
38 ZHEDA RD
HANGZHOU 310027
CHINA
weimins@zju.edu.cn

LI-XIA YUAN
SCHOOL OF MATHEMATICAL SCIENCES
FUDAN UNIVERSITY
220 HANDAN RD
SHANGHAI 200433
CHINA
yuanlixia@fudan.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US $410/year for the electronic version, and $535/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 270 No. 1 July 2014

0030-8730(201407)270:1;1-4