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A CLASS OF NEUMANN PROBLEMS ARISING IN CONFORMAL GEOMETRY

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In this paper, we solve a class of Neumann problems on a manifold with totally geodesic smooth boundary. As a consequence, we also solve the prescribing k-curvature problem of the modified Schouten tensor on such manifolds; that is, if the initial k-curvature of the modified Schouten tensor is positive for $\tau > n-1$ or negative for $\tau < 1$, then there exists a conformal metric such that its k-curvature defined by the modified Schouten tensor equals some prescribed function and the boundary remains totally geodesic.

1. Introduction

Let (M^n, g), $n \geq 3$, be a compact, smooth Riemannian manifold. The *modified Schouten tensor*

$$A_g^\tau := \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{\tau R_g}{2(n-1)} \cdot g\right)$$

was introduced by Gursky and Viaclovsky [2003] and A. Li and Y.-Y. Li [2003] independently, where $\tau \in \mathbb{R}$ and Ric_g, R_g are the Ricci tensor and the scalar curvature of g, respectively. Clearly, A_g^0 is the Ricci tensor, A_g^{n-1} is the Einstein tensor and A_g^1 is just the Schouten tensor.

Denote by $\lambda(g^{-1}A_g^\tau)$ the eigenvalues of A_g^τ. The k-curvature (or σ_k curvature) of A_g^τ is defined as $\sigma_k(\lambda(g^{-1}A_g^\tau))$, where σ_k is the k-th elementary symmetric function defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{for all } \lambda \in \mathbb{R}^n,$$

for any $1 \leq k \leq n$. We will use $\sigma_k(A_g^\tau) := \sigma_k(\lambda(g^{-1}A_g^\tau))$ for convenience.

The prescribing k-curvature problem of the modified Schouten tensor A_g^τ in conformal geometry is to find a metric $\tilde{g}$ in the conformal class $[g]$ of g satisfying

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the equation

$$\sigma_k^{1/k}(A_{\tilde{g}}^{\tau}) = \varphi(x), \tag{1-1}$$

where φ is a given smooth function on M. If $\tau = 1 = k$ and φ is constant, (1-1) is just the Yamabe problem, which has been solved by Yamabe, Trudinger, Aubin and Schoen (see [Lee and Parker 1987]). When $\tau = 1$, $k \geq 2$ and φ is constant, then (1-1) is called k-Yamabe problem, which has attracted enormous interest [Chang et al. 2002; Ge and Wang 2006; Guan and Wang 2003a; 2003b; Gursky and Viaclovsky 2007; Li and Li 2003; 2005; Sheng et al. 2007; Trudinger and Wang 2009; 2010; Viaclovsky 2000], etc. There are many interesting works on the Yamabe problem and k-Yamabe problem on a manifold with boundary [Chen 2007; 2009; Escobar 1992b; 1992a; Han and Li 1999; 2000; He and Sheng 2011a; 2011b; 2013; Jin et al. 2007; Jin 2007], etc.

Note that (1-1) is a fully nonlinear partial differential equation for $k \geq 2$. In order to study this problem, we need the following conceptions. Let

$$\Gamma_k^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

Therefore, we have $\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \cdots \subset \Gamma_1^+$. For a 2-symmetric form B defined on (M^n, g), $B \in \Gamma_k^+$ means that the eigenvalues of B, say $\lambda(g^{-1}B)$, lie in Γ_k^+. Set $\Gamma_k^- = -\Gamma_k^+$.

According to [Caffarelli et al. 1985], (1-1) is an elliptic equation for $A_g^{\tau} \in \Gamma_k^+$ or $A_g^{\tau} \in \Gamma_k^-$. When $\tau < 1$, $A_g^{\tau} \in \Gamma_k^-$ and $\varphi < 0$, Gursky and Viaclovsky [2003] proved that there exists a unique conformal metric $\tilde{g} \in [g]$ satisfying (1-1) on a closed manifold. Li and Sheng [2005] studied the same problem by a parabolic argument. Using a similar argument, Sheng and Zhang [2007] studied the case of $\tau > n-1$, $A_g^{\tau} \in \Gamma_k^+$ and $\varphi > 0$. For the manifold with boundary, Li and Sheng [2011] considered a Dirichlet problem of (1-1) for $\tau > n-1$ and $A_g^{\tau} \in \Gamma_k^+$; He and Sheng [2013] discussed more general equations and obtained many useful local estimates for both $\tau < 1$ and $\tau > n-1$. In [Sheng and Yuan 2013], we investigated a Neumann problem of (1-1) by a conformal flow and proved:

Theorem 1.1 [Sheng and Yuan 2013]. *Let $(\overline{M}^n, g)$, $n \geq 3$, be a compact manifold with smooth totally geodesic boundary ∂M. If $A_g^{\tau} \in \Gamma_k^+$ and $\tau > n-1$, or $A_g^{\tau} \in \Gamma_k^-$ and $\tau < 1$, then there exists a smooth metric $\tilde{g} \in [g]$ satisfying (1-1) for φ constant and such that ∂M is still totally geodesic.*

In this paper, we are interested in solving a class of Neumann problems on the manifold with totally geodesic boundary.

Let $(\overline{M}, g)$ be a compact manifold with smooth boundary ∂M. Denote the second fundamental form and the mean curvature of ∂M by L and μ. Under the conformal change of metric $\tilde{g} = e^{2u}g$, the second fundamental form L with respect to its unit inward normal ν satisfies

$$\tilde{L}e^{-u} = -\frac{\partial u}{\partial \nu} g + L.$$

The boundary is called umbilic if $L = \mu g$, and then totally geodesic if $\mu \equiv 0$. Note that the umbilicity is conformally invariant. Then the mean curvature changes as

$$\tilde{\mu} = \left(-\frac{\partial u}{\partial \nu} + \mu\right) e^{-u}. \tag{1-2}$$

Under the same conformal change, the *modified Schouten tensor* changes according to the formula

$$A^{\tau}_{\tilde{g}} = \frac{\tau - 1}{n - 2} \Delta u g - \nabla^2 u + du \otimes du + \frac{\tau - 2}{2} |\nabla u|^2 g + A^{\tau}_{g}, \tag{1-3}$$

where the covariant derivatives and norms are taken with respect to the background metric g. Let the boundary ∂M be totally geodesic with respect to the metric g. In order to preserve the boundary being totally geodesic under the conformal change, $\tilde{\mu} \equiv 0$. Hence, the two partial differential equations corresponding to Theorem 1.1 are

$$\begin{cases} \sigma_k^{1/k}\left(\frac{\tau-1}{n-2}\Delta u g - \nabla^2 u + du \otimes du + \frac{\tau-2}{2}|\nabla u|^2 g + A^{\tau}_{g}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = e^{2u}\,\text{const.} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-4}$$

for $\tau > n - 1$, and

$$\begin{cases} \sigma_k^{1/k}\left(\nabla^2 u + \frac{1-\tau}{n-2}\Delta u g - du \otimes du + \frac{2-\tau}{2}|\nabla u|^2 g - A^{\tau}_{g}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = e^{2u}\,\text{const.} & \text{in } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-5}$$

for $\tau < 1$, respectively.

Now, we consider more general equations than (1-4) and (1-5). Let $\Gamma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin satisfying $\Gamma_n \subset \Gamma \subset \Gamma_1$, and $F : \mathbb{R}^n \to \mathbb{R}$ be a general smooth, symmetric, homogeneous function of degree one in Γ normalized by $F(e) = F(1, \dots, 1) = 1$. Moreover, $F = 0$ on $\partial\Gamma$ and satisfies the following structure conditions in Γ:

(C1) F is positive.

(C2) F is concave (i.e., $\partial^2 F/(\partial\lambda_i \partial\lambda_j)$ is negative semidefinite).

(C3) F is monotone (i.e., $\partial F/\partial\lambda_i$ is positive).

According to [Lin and Trudinger 1994; Trudinger 1990], for any $0 \le l < k \le n$, the elementary symmetric functions and their quotients $(\sigma_k/\sigma_l)^{1/(k-l)}$ with $\sigma_0 = 1$ satisfy all the properties and structure conditions above on Γ_k^+.

For some positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$, we study the equation

$$\begin{cases} F(g^{-1}V[u]) = \Phi(x, u) & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-6}$$

where for constant $\bar{\theta} := (\tau - 1)/(n-2) > 1$, $a, b \in C^\infty(\overline{M})$, and the smooth symmetric 2-tensor $S \in \Gamma$, the matrix $(V[u])$ is defined by

$$V[u] = \bar{\theta} \Delta u g - \nabla^2 u + a(x)\, du \otimes du + b(x)|\nabla u|^2 g + S. \tag{1-7}$$

We call a function $v \in C^2(\overline{M})$ *admissible* if $\lambda(g^{-1}V[v]) \in \Gamma$.

Assume S is the symmetric 2-tensor on M satisfying one of the following conditions:

(S1) $S(\nu, X) = 0$, for any $X \in T(\partial M)$.

(S2) $S = A_g^\tau$.

Theorem 1.2 (main result). *Let $(\overline{M}^n, g)$, $n \ge 3$, be a compact manifold with smooth totally geodesic boundary ∂M. Suppose $\bar{\theta} > 1$ and the positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfies*

$$\partial_z \Phi > 0, \qquad \lim_{z \to +\infty} \Phi(x, z) = +\infty, \qquad \lim_{z \to -\infty} \Phi(x, z) = 0. \tag{1-8}$$

Then for any functions $a, b \in C^\infty(\overline{M})$ and $S \in \Gamma$ satisfying (S1) *or* (S2), *there exists a function $u \in C^\infty(\overline{M})$ solving the equation (1-6).*

For the other elliptic branch (1-5), we consider the equation

$$\begin{cases} F(g^{-1}W[u]) = \Phi(x, u) & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases} \tag{1-9}$$

where for constant $\theta := (1 - \tau)/(n-2) > 0$, $a, b \in C^\infty(\overline{M})$, and the smooth symmetric 2-tensor $T \in \Gamma$, the matrix $(W[u])$ is defined by

$$W[u] = \nabla^2 u + \theta \Delta u g + a(x)\, du \otimes du + b(x)|\nabla u|^2 g + T. \tag{1-10}$$

Theorem 1.3. *Let $(\overline{M}^n, g)$, $n \ge 3$, be a compact manifold with smooth totally geodesic boundary ∂M. Suppose $\theta > 0$ and the positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfies (1-8). Then for any functions $a, b \in C^\infty(\overline{M})$ and $T \in \Gamma$ with* (S1) *or $T = -A_g^\tau$, there exists a function $u \in C^\infty(\overline{M})$ solving the equation (1-9).*

Applying Theorems 1.2 and 1.3 to the quotient of the elementary symmetric functions, i.e., $F = (\sigma_k/\sigma_l)^{1/(k-l)}$ on Γ_k^+, we have the following corollaries.

Corollary 1.4. *Let* $(\overline{M}^n, g)$, $n \geq 3$, *be a compact manifold with smooth totally geodesic boundary* ∂M. *If* $\tau > n-1$ *and* $A_g^\tau \in \Gamma_k^+$, *then for any smooth function* $\varphi > 0$, *there exists a smooth metric* $\tilde{g} \in [g]$ *preserving* ∂M *totally geodesic and satisfying*

$$\left(\frac{\sigma_k}{\sigma_l}\right)^{\frac{1}{k-l}}(A_{\tilde{g}}^\tau) = \varphi(x) \quad \text{in } M. \tag{1-11}$$

Corollary 1.5. *Let* $(\overline{M}^n, g)$, $n \geq 3$, *be a compact manifold with smooth totally geodesic boundary* ∂M. *If* $\tau < 1$ *and* $A_g^\tau \in \Gamma_k^-$, *then for any smooth function* $\varphi < 0$, *there exists a smooth metric* $\tilde{g} \in [g]$ *preserving* ∂M *totally geodesic and satisfying* (1-11).

Remark 1.6. By choosing $l = 0$ and φ constant in Corollaries 1.4 and 1.5, we can get Theorem 1.1 directly. Different from the results in [Li and Sheng 2011; Sheng et al. 2007], we need not subjoin any restriction on $a(x)$ and $b(x)$ in Theorems 1.2 and 1.3. Contrary to this fact, [Sheng et al. 2007] gives a counterexample to show that there is no regularity if $a(x) = 0$ and $b(x) > 0$ when $\tau = 1$ and $A_g^\tau \in \Gamma_k^-$.

This paper is organized as follows. We introduce some lemmas in Section 2. By use of these lemmas, we can get the a priori global C^0 estimate for (1-6) in Section 3. Then we obtain the a priori global gradient and Hessian derivatives estimates in Section 4 and Section 5 respectively. By the a priori estimates and the standard continuity method, we show Theorem 1.2 in Section 6. In the last section, we consider (1-9) by the similar arguments in Sections 3–6, and prove Theorem 1.3.

2. Preliminaries

In this section, we first recall some facts of the function F satisfying the structure conditions (C1)–(C3) in Γ.

Lemma 2.1 (see [Chen 2005; 2009]). *Let* Γ *be an open convex cone with vertex at the origin satisfying* $\Gamma_n^+ \subset \Gamma$, *and let* $e = (1, \dots, 1)$ *be the identity. Suppose that* F *is a homogeneous symmetric function of degree one normalized with* $F(e) = 1$, *and that* F *is concave in* Γ. *Then*:

(a) $\sum_i \lambda_i \partial F(\lambda)/\partial \lambda_i = F(\lambda)$, *for* $\lambda \in \Gamma$.

(b) $\sum_i \partial F(\lambda)/\partial \lambda_i \geq F(e) = 1$, *for* $\lambda \in \Gamma$.

To get the boundary estimates, we need some facts. For any point $x_0 \in \partial M$, we consider Fermi coordinates $\{x_i\}_{1 \leq i \leq n}$ around x_0, where $\partial/\partial x_n$ is the unit inner normal with respect to the background metric g. A half-ball centered at x_0 of radius r is defined by

$$\bar{B}_r^+ = \left\{x_n \geq 0, \left(\sum_{i=1}^n x_i^2\right) \leq r^2\right\}.$$

Denote the boundary of $\bar{B}_r^+$ on ∂M by $\Sigma_r = \left\{x_n = 0, \sum_i x_i^2 \leq r^2\right\}$.

Throughout this paper, the Greek letters $\alpha, \beta, \gamma, \ldots = 1, \ldots, n-1$ stand for the tangential direction indices, while the Latin letters $i, j, k, \ldots = 1, \ldots, n$ stand for the full indices. In Fermi coordinates $\{x_i\}_{1\leq i\leq n}$, the metric is expressed as $g = g_{\alpha\beta}\, dx_\alpha\, dx_\beta + (dx_n)^2$. Then the Christoffel symbols on the boundary satisfy

$$\Gamma^n_{\alpha\beta} = L_{\alpha\beta}, \quad \Gamma^\beta_{\alpha n} = -L_{\alpha\gamma} g^{\gamma\beta}, \quad \Gamma^n_{\alpha n} = 0, \quad \Gamma^n_{nn} = 0, \quad \Gamma^\gamma_{nn} = 0, \quad \Gamma^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\alpha\beta} \tag{2-1}$$

on the boundary, where we denote the tensors and covariant differentiation with respect to the induced metric $g_{\alpha\beta}$ on the boundary by a tilde (e.g., $\tilde{\Gamma}^\gamma_{\alpha\beta}$, $\mu_{\tilde{\alpha}\tilde{\beta}}$). When the boundary is totally geodesic, we have

$$\Gamma^n_{\alpha\beta} = 0, \quad \Gamma^\beta_{\alpha n} = 0, \quad \Gamma^n_{\alpha n} = 0. \tag{2-2}$$

Lemma 2.2 [Chen 2007; He and Sheng 2013]. *Suppose ∂M is totally geodesic and $u_n = 0$ on ∂M. Then we have on the boundary that*

$$u_{n\alpha} = 0 \quad \textit{and} \quad u_{\alpha\beta n} = 0. \tag{2-3}$$

Lemma 2.3 [He and Sheng 2013]. *Let $(\overline{M}, g)$ be a compact Riemannian manifold with boundary and dimension $n \geq 3$. Assume the boundary ∂M is totally geodesic. Then at any boundary point $P \in \partial M$, there exists a conformal metric $\bar{g} = e^{2\bar{u}} g_0$ such that* (i) *$\bar{u}_n = 0$ on ∂M and the boundary ∂M is still totally geodesic*, (ii) *$\bar{R}_{ij}(P) = 0$ for $1 \leq i,\ j \leq n$*, (iii) *$\bar{R}_{nn,n}(P) = 0$, $\bar{R}_{\alpha n,\beta}(P) = 0$, $1 \leq \alpha,\ \beta \leq n-1$, and* (iv) *$\bar{R}_{\alpha\beta,n}(P) = 0$, $1 \leq \alpha, \beta \leq n-1$.*

3. Ellipticity and the global C^0 estimates

We first sketch the ellipticity properties of operator F; see [Li and Sheng 2011] for details.

For any function h on $\overline{M}$, we define

$$\mathcal{P}[h] := F(V[h]) - \Phi(x, h).$$

Then any solution u of (1-6) satisfies $\mathcal{P}[u] = 0$. Denote $u_s = u + sv$, $s \in \mathbb{R}$. The linearized operator of (1-6) is

$$\begin{aligned} \mathcal{L}v &:= \frac{d}{ds}\mathcal{P}[u_s]\Big|_{s=0} \\ &= P^{ij} v_{ij} + 2aF^{ij} v_i u_j + 2b v_l u_l \mathcal{T} - \partial_z \Phi(x, u) v, \end{aligned} \tag{3-1}$$

where $F^{ij} := (\partial F/\partial V_{ij})(V[u])$, $\mathcal{T} = \mathrm{tr}(F^{ij}) = F^{ij} g_{ij}$ and

$$P^{ij} := \bar{\theta}\mathcal{T} g^{ij} - F^{ij} \geq (\bar{\theta} - 1)\mathcal{T} g^{ij}.$$

Since u is admissible, (F^{ij}) is positive definite [Caffarelli et al. 1985]. Denote $\varepsilon_0 := \bar{\theta} - 1 > 0$. Hence, (P^{ij}) is positive definite, too.

Note that the coefficient of the zero order term in (3-1) is negative when $\partial_z \Phi$ is positive on $\overline{M} \times \mathbb{R}$.

Lemma 3.1. *Equation (1-6) is elliptic at any admissible solution. If $\partial_z \Phi$ is positive on $\overline{M} \times \mathbb{R}$, then the linearized operator $\mathcal{L} : C^{2,\alpha}(\overline{M}) \to C^{\alpha}(\overline{M}) (0 < \alpha < 1)$ is invertible.*

Now, we use the compactness of the manifold to get the global C^0 estimates of (1-6).

Proposition 3.2. *Suppose $S \in \Gamma$ and the positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfies (1-8). Then for any admissible solution $u \in C^2(\overline{M})$ of (1-6), we have*

$$\sup_{\overline{M}} |u| \leq C_0,$$

where the constant C_0 depends only on S and Φ.

Proof. Suppose x_0 be the maximum point of u on $\overline{M}$. Denote $u_{\max} = u(x_0)$.

If $x_0 \in \partial M$, at this point we have $u_n(x_0) < 0$, which contradicts with the boundary condition $u_n|_{\partial M} \equiv 0$. Hence, x_0 must be an interior point of M. Then at this point we have

$$\nabla u = 0 \quad \text{and} \quad \nabla^2 u \geq 0. \tag{3-2}$$

Substituting (3-2) into (1-6), we have

$$\Phi(x_0, u_{\max}) \leq F(S)(x_0) \leq \max_{x \in \overline{M}} F(S) \leq C.$$

Now, by the condition $\partial_z \Phi > 0$ and $\lim_{z \to +\infty} \Phi(x, z) = +\infty$, we know that

$$\max_{x \in \overline{M}} u = u_{\max} \leq C.$$

By a similar argument, we can get the lower bound of u by considering its minimum point on $\overline{M}$ and using the other condition of Φ. □

4. Gradient estimates

In this section we first consider the boundary gradient estimates of (1-6), then derive the global estimates.

For any point $y_0 \in \partial M$, let $\bar{B}_r^+$ and $\bar{B}_{r/2}^+$ be any two half-balls centered at y_0 in the Fermi coordinates $\{x_i\}_{1\le i\le n}$. Choosing a cutoff function η depending only on r such that $0 \le \eta \le 1$, $\eta = 1$ in $\bar{B}_{r/2}^+$, $\eta = 0$ outside $\bar{B}_r^+$. Moreover,

$$|\nabla \eta| \le b_0 \frac{\eta^{1/2}}{r} \quad \text{and} \quad |\nabla^2 \eta| \le \frac{b_0}{r^2}, \tag{4-1}$$

for a universal constant b_0, where the covariant derivatives and the norms $|\cdot|$ are taken with respect to g. Since η only depends on r, we have

$$\frac{\partial \eta}{\partial n} = 0 \quad \text{on } \partial M. \tag{4-2}$$

We also need the function $\psi : \mathbb{R} \to \mathbb{R}$ defined in [Gursky and Viaclovsky 2003] by

$$\psi(s) = \alpha_1(\alpha_2 + s)^p, \quad -\delta_1 < s < \delta_2, \tag{4-3}$$

where the positive constants δ_1 and δ_2 are given, and the constants α_1, α_2 and p will be fixed as follows. We have

$$\psi' = p\alpha_1(\alpha_2+s)^{p-1} \quad \text{and} \quad \psi'' = p(p-1)\alpha_1(\alpha_2+s)^{p-2} = \frac{p-1}{\alpha_2+s}\psi'.$$

Let α_2 and p be positive constants satisfying $\alpha_2 > \delta_1$ and $p > 3$. Take

$$\alpha_1 = \frac{1}{p^2 \max\{(\alpha_2+s)^p\}};$$

then

$$\psi \le \frac{1}{p^2}, \quad \psi' > 0 \quad \text{and} \quad \psi'' - \psi'^2 = \frac{\psi'}{\alpha_2+s}(p-1-p\psi) \ge \frac{\psi' p}{2(\alpha_2+s)}. \tag{4-4}$$

Proposition 4.1. *Suppose u is a C^3 solution of (1-6) on $\bar{B}_r^+$. Then there is a positive constant C depending only on $n, k, \bar{\theta}, g, r$, $|S|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$ and C_0 such that*

$$\sup_{\bar{B}_{r/2}^+} |\nabla u|_g \le C.$$

Proof. Consider the auxiliary function

$$G := \tfrac{1}{2}\eta e^{\beta}|\nabla u|^2, \quad \beta := x_n + \psi(u),$$

where the function ψ defined by (4-3). Let x_0 be the maximum point of G on $\bar{B}_r^+$. Without loss of generality, we may assume $r = 1$ and $|\nabla u|\,(x_0) \gg 1$.

Suppose $x_0 \in \Sigma_r$. Then $G_n(x_0) \le 0$. However, by (4-2), the boundary condition $u_n = 0$ and Lemma 2.2, we have

$$G_n(x_0) = \tfrac{1}{2}e^{\psi}\Big((1+\psi' u_n)|\nabla u|^2 + 2u_n u_{nn} + 2\sum_{\alpha=1}^{n-1} u_\alpha u_{\alpha n}\Big)(x_0)$$
$$= \tfrac{1}{2}e^{\psi}|\nabla u|^2(x_0) > 0.$$

It is a contradiction. Hence x_0 must be an interior point of $\bar{B}_r^+$. Then at x_0, for $1 \le i \le n$, we have

$$0 = (\log G)_i, \quad 0 \ge (\log G)_{ij},$$

that is,

$$\frac{2u_s u_{si}}{|\nabla u|^2} = -\Big(\frac{\eta_i}{\eta} + \beta_i\Big), \tag{4-5}$$

and

$$0 \ge \Big(\frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2}\Big) + \beta_{ij} + \frac{2u_{sj}u_{si} + 2u_s u_{sij}}{|\nabla u|^2} - \frac{4u_s u_{si} u_l u_{lj}}{|\nabla u|^4}. \tag{4-6}$$

Substituting (4-5) into (4-6), we have

$$0 \ge \Big(\frac{\eta_{ij}}{\eta} - 2\frac{\eta_i\eta_j}{\eta^2}\Big) + (\beta_{ij} - \beta_i\beta_j) + \frac{2u_{sj}u_{si} + 2u_s u_{sij}}{|\nabla u|^2} - \frac{1}{\eta}(\eta_i\beta_j + \eta_j\beta_i). \tag{4-7}$$

By (4-7), we have

$$0 \ge P^{ij}\Big(\frac{\eta_{ij}}{\eta} - 2\frac{\eta_i\eta_j}{\eta^2}\Big) + P^{ij}(\beta_{ij} - \beta_i\beta_j) + \frac{2}{|\nabla u|^2}P^{ij}u_{si}u_{sj} + \frac{2}{|\nabla u|^2}u_s P^{ij}u_{sij} - \frac{2}{\eta}P^{ij}\eta_i\beta_j, \tag{4-8}$$

where $P^{ij} = \bar{\theta}\mathcal{T}g^{ij} - F^{ij}$ is positive definite. It follows from (4-1) and (4-8) that

$$0 \ge \frac{2}{|\nabla u|^2}u_s P^{ij}u_{sij} + P^{ij}(\beta_{ij} - \beta_i\beta_j) - \frac{2}{\eta}P^{ij}\eta_i\beta_j - \frac{C}{\eta}\mathcal{T}, \tag{4-9}$$

where the constant C depends only on n and b_0.

Differentiating (1-6), we have

$$\nabla_s\Phi = P^{ij}u_{ijs} + F^{ij}(a_s u_i u_j + 2a u_{is}u_j + S_{ij},_s) + (b_s|\nabla u|^2 + 2b u_{ls}u_l)\mathcal{T}. \tag{4-10}$$

Then by (4-10) and Ricci identities $u_{sij} = u_{ijs} + R_{isjp}u_p$, we obtain

$$\frac{2}{|\nabla u|^2}u_s P^{ij}u_{sij} \ge \frac{2}{|\nabla u|^2}u_s\nabla_s\Phi - \frac{2}{|\nabla u|^2}u_s F^{ij}(a_s u_i u_j + 2a u_{is}u_j) - \frac{2}{|\nabla u|^2}u_s(b_s|\nabla u|^2 + 2b u_{ls}u_s)\mathcal{T} - C(1 + \frac{1}{|\nabla u|})\mathcal{T}.$$

where the constant C depends only on n, g and $|\nabla S|$.

Since $\nabla_s \Phi = \Phi_x + \Phi_z u_s$, by (4-5) and the inequality above, we have

$$\begin{aligned}(4\text{-}11)\quad \frac{2}{|\nabla u|^2} u_s P^{ij} u_{sij} \geq 2\Phi_z + \frac{2}{|\nabla u|^2} u_s \Phi_x - \frac{2a_s u_s}{|\nabla u|^2} F^{ij} u_i u_j + 2a F^{ij} u_j \left(\frac{\eta_i}{\eta} + \beta_i\right) \\ - 2b_s u_s \mathcal{T} + 2b\left(\frac{\eta_s}{\eta} + \beta_s\right) u_s \mathcal{T} - C\left(1 + \frac{1}{|\nabla u|}\right)\mathcal{T} \\ \geq C^* + 2a F^{ij} u_j \beta_i + 2b u_s \beta_s \mathcal{T} - \frac{C}{\sqrt{\eta}}(1 + |\nabla u|)\mathcal{T},\end{aligned}$$

where the constant C^* depends only on $|\Phi_x|$, $|\Phi_z|$, C_0, and C depends on n, b_0, $|a|_{C^1}$, $|b|_{C^1}$ and $|\nabla S|$.

Then by (4-9) and (4-11), we obtain

$$\begin{aligned}(4\text{-}12)\quad 0 \geq C^* + 2a F^{ij} u_j \beta_i + 2b u_s \beta_s \mathcal{T} \\ + P^{ij}(\beta_{ij} - \beta_i \beta_j) - \frac{2\eta_i}{\eta} P^{ij} \beta_j - C \frac{1}{\sqrt{\eta}}(|\nabla u| + 1)\mathcal{T}.\end{aligned}$$

Since $\beta := x_n + \psi(u)$, we have

$$\beta_i = \delta_{in} + \psi' u_i, \quad \beta_{ij} = \psi'' u_i u_j + \psi' u_{ij}$$

and

$$\beta_{ij} - \beta_i \beta_j = \psi' u_{ij} + (\psi'' - \psi'^2) u_i u_j - \psi'(\delta_{in} u_j + \delta_{jn} u_i) - \delta_{in}\delta_{jn}.$$

Therefore, we have the inequalities

$$(4\text{-}13)\quad 2a F^{ij} u_j \beta_i = 2a F^{ij} u_j (\delta_{in} + \psi' u_i) \geq 2a\psi' F^{ij} u_i u_j - C|\nabla u|\mathcal{T},$$

$$(4\text{-}14)\quad 2b u_s \beta_s \mathcal{T} = 2b u_s (\delta_{sn} + \psi' u_s)\mathcal{T} \geq 2b\psi' |\nabla u|^2 \mathcal{T} - C|\nabla u|\mathcal{T},$$

$$(4\text{-}15)\quad -\frac{2\eta_i}{\eta} P^{ij} \beta_j = -\frac{2}{\eta} P^{ij} \eta_i (\delta_{jn} + \psi' u_j) \geq -\frac{C}{\sqrt{\eta}}(|\nabla u| + 1)\mathcal{T},$$

$$(4\text{-}16)\quad P^{ij}(\beta_{ij} - \beta_i \beta_j) \geq \psi' P^{ij} u_{ij} + (\psi'' - \psi'^2) P^{ij} u_i u_j - C(|\nabla u| + 1)\mathcal{T}.$$

Plugging (4-13)–(4-16) into (4-12), we have

$$\begin{aligned}(4\text{-}17)\quad 0 \geq C^* + \psi' P^{ij} u_{ij} + (\psi'' - \psi'^2) P^{ij} u_i u_j + 2a\psi' F^{ij} u_i u_j \\ + 2b\psi' |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u| + 1)\mathcal{T}.\end{aligned}$$

By Lemma 2.1, we know that $F^{ij} V_{ij} = F(V) = \Phi$. Then

$$\begin{aligned}(4\text{-}18)\quad \psi' P^{ij} u_{ij} &= \psi' F^{ij} V_{ij} - \psi' F^{ij}(a u_i u_j + b|\nabla u|^2 g_{ij} + S_{ij}) \\ &\geq \psi' \Phi - a\psi' F^{ij} u_i u_j - b\psi' |\nabla u|^2 \mathcal{T} - C\mathcal{T}.\end{aligned}$$

Substituting (4-18) into (4-17), we get

$$
\begin{aligned}
0 &\geq C^* + \psi'\Phi + (\psi'' - \psi'^2)P^{ij}u_iu_j + a\psi' F^{ij}u_iu_j \\
&\qquad\qquad + b\psi'|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T} \\
&= C^* + \psi'\Phi + (\psi'' - \psi'^2 - a\psi')P^{ij}u_iu_j \\
&\qquad\qquad + (a\bar{\theta} + b)\psi'|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}.
\end{aligned}
\tag{4-19}
$$

Claim 4.2. *If $-\delta_1 < u < \delta_2$, then there exist positive constants α_1, α_2 and p depending only on $\bar{\theta}$, δ_1, δ_2, $|a|_{L^\infty(\overline{M})}$ and $|b|_{L^\infty(\overline{M})}$, such that $\psi' > 0$, and*

$$
(\psi'' - \psi'^2 - |a|_{L^\infty}\psi')\varepsilon_0 - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\psi' \geq \varepsilon_1 > 0, \tag{4-20}
$$

for some constant ε_1 depending only on $\bar{\theta}$, δ_1 and δ_2.

Note that $\Phi > 0$. Then by Claim 4.2, we have

$$
0 \geq C^* + \varepsilon_1|\nabla u|^2\mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u|+1)\mathcal{T}.
$$

Multiplying η^2 both sides of the inequality above, we have

$$
\varepsilon_1\eta^2|\nabla u|^2\mathcal{T} \leq 2C|\nabla u|\mathcal{T} + C^*. \tag{4-21}
$$

By Lemma 2.1, $\mathcal{T} \geq 1$. Then (4-21) implies the gradient estimates.

Proof of Claim 4.2. Since $-\delta_1 \leq u \leq \delta_2$. By (4-4), for

$$
\frac{\delta_1 + \delta_2}{2} \leq \alpha_2 \leq \delta_2, \qquad p > \max\{3,\, 8|a|_{L^\infty}\delta_2\},
$$

we have $\alpha_1 = 1/(p^2(2\delta_2)^p)$, $\psi' > 0$, and

$$
\psi'' - \psi'^2 - a\psi' \geq \psi'\left(\frac{p}{4\delta_2} - |a|_{L^\infty}\right) \geq \frac{\psi' p}{8\delta_2}.
$$

Furthermore, we can choose

$$
p > \max\left\{3,\, 8|a|_{L^\infty}\delta_2,\, \frac{16}{\varepsilon_0}(\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\delta_2\right\},
$$

such that

$$
\begin{aligned}
&(\psi'' - \psi'^2 - |a|_{L^\infty}\psi')\varepsilon_0 - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\psi' \\
&\qquad \geq \psi'\left(\frac{p\varepsilon_0}{8\delta_2} - (\bar{\theta}|a|_{L^\infty} + |b|_{L^\infty})\right) \geq \frac{\psi' p\varepsilon_0}{16\delta_2} \geq \frac{\varepsilon_0(\delta_2 - \delta_1)^{p-1}}{2^{p+3}\delta_2} \geq \varepsilon_1 > 0. \qquad \square
\end{aligned}
$$

Remark 4.3. If $\bar{B}_r^+$ and $\bar{B}_{r/2}^+$ are replaced by two local geodesic open balls in the interior of M and $\beta = \psi(u)$ in the auxiliary function G, we can get the interior gradient estimates for (1-6) by the proof of Proposition 4.1.

Since $\overline{M}$ is a compact manifold, by Proposition 4.1 and Remark 4.3, we can derive the global gradient estimate of (1-6).

Proposition 4.4. *Let u be a C^3 solution of (1-6) on $\overline{M}$. Then there is a positive constant C_1 depending only on n, k, $\bar{\theta}$, g, a, b, Φ, S and C_0 such that*

$$\sup_{\overline{M}} |\nabla u|_g \leq C_1. \tag{4-22}$$

5. Estimates for the second derivatives

Lemma 5.1. *Let u be a C^4 solution of (1-6). Then there is a positive constant C' depending only on n, k, $\bar{\theta}$, g, $|S|_{C^1(\bar{B}_r^+)}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$ and C_1, such that*

$$u_{nnn} \geq -C' \quad \text{on } \partial M. \tag{5-1}$$

Proof. We consider this lemma for S satisfying condition (S1) or (S2), respectively.

(i) Suppose S satisfy (S1). Then $S_{\alpha n} = S(\partial/\partial x_\alpha, \partial/\partial x_n) = 0$ on the boundary ∂M. By the boundary condition $u_n = 0$ and the Lemma 2.2, we have $V[u]_{\alpha n} = S_{\alpha n} = 0$. Applying an argument of Lemma 13 in [Chen 2009], we know that

$$F^{\alpha n}(V[u]) = 0. \tag{5-2}$$

Also by Lemma 2.2, we calculate that

$$\begin{aligned} V[u]_{\alpha\beta,n} &= \bar{\theta} u_{nnn} g_{\alpha\beta} + \bar{\theta} u_{\gamma\gamma n} g_{\alpha\beta} - u_{\alpha\beta n} + 2a u_{\alpha n} u_\beta + a_n u_\alpha u_\beta \\ &\qquad + 2b u_{\alpha n} u_\alpha g_{\alpha\beta} + 2b u_{nn} u_n g_{\alpha\beta} + b_n |\nabla u|^2 g_{\alpha\beta} + S_{\alpha\beta,n} \\ &= \bar{\theta} u_{nnn} g_{\alpha\beta} + a_n u_\alpha u_\beta + b_n |\nabla u|^2 g_{\alpha\beta} + S_{\alpha\beta,n} \\ &\leq \bar{\theta} u_{nnn} g_{\alpha\beta} + C, \end{aligned} \tag{5-3}$$

where the constant C depends only on $|\nabla a|$, $|\nabla b|$, C_1, g and $|\nabla S|$.

Similarly, we have

$$\begin{aligned} V[u]_{nnn} &= \bar{\theta} u_{\gamma\gamma n} + \bar{\theta} u_{nnn} - u_{nnn} + a_n u_n^2 + 2a u_n u_{nn} + 2b u_{\alpha n} u_\alpha \\ &\qquad + 2b u_n u_{nn} + b_n |\nabla u|^2 + S_{nn,n} \\ &\leq \bar{\theta} u_{nnn} - u_{nnn} + C. \end{aligned} \tag{5-4}$$

By differentiating (1-6) along the normal direction the on boundary, using (5-2)–(5-4), we have

$$\begin{aligned}\nabla_n\Phi &= F^{nn}V[u]_{nnn} + F^{\alpha\beta}V[u]_{\alpha\beta n}\\ &\le F^{nn}(\bar\theta u_{nnn} - u_{nnn}) + \bar\theta u_{nnn}F^{\alpha\beta}g_{\alpha\beta} + C\mathcal{T}\\ &= -F^{nn}u_{nnn} + \bar\theta u_{nnn}\mathcal{T} + C\mathcal{T},\end{aligned}$$

where we have used $g_{\alpha n} = 0$ and $g_{nn} = 1$. Since $\mathcal{T} > 1$, we have

$$0 \le -F^{nn}u_{nnn} + (\bar\theta u_{nnn} + C)\mathcal{T}, \tag{5-5}$$

where C also depends on $|\nabla\Phi|$.

If $\bar\theta u_{nnn} + C > 0$, we get $u_{nnn} > -C/\bar\theta$, which implies (5-1). If $\bar\theta u_{nnn} + C < 0$, by $F^{nn} < \mathcal{T}$ we have

$$0 \le -F^{nn}u_{nnn} + (\bar\theta u_{nnn} + C)F^{nn} = ((\bar\theta - 1)u_{nnn} + C)F^{nn}.$$

Since $F^{nn} > 0$, we have

$$(\bar\theta - 1)u_{nnn} + C \ge 0. \tag{5-6}$$

Note that $\bar\theta - 1 = \varepsilon_0 > 0$; then (5-6) implies (5-1).

(ii) Suppose $S = A^\tau_g$. For any $x_0 \in \partial M$, using the metric $\bar g$ in Lemma 2.3, we consider a metric $\hat g = e^{2v}\bar g$ such that $u = \bar u + v$ is a solution of (1-6). Now,

$$\begin{aligned}V[u]_{ij} &= \bar\theta\triangle\bar u g_{ij} + \bar\theta\triangle v g_{ij} - \bar u_{ij} - v_{ij} + a(\bar u_i\bar u_j + \bar u_i v_j + v_i\bar u_j + v_i v_j)\\ &\quad + b(|\nabla\bar u|^2 + 2\langle\nabla\bar u, \nabla v\rangle + |\nabla v|^2)g_{ij} + (A^\tau_g)_{ij}.\end{aligned} \tag{5-7}$$

By (1-3), we have

$$(A^\tau_{\bar g})_{ij} = \bar\theta\triangle\bar u g_{ij} - \bar u_{ij} + \bar u_i\bar u_j + \frac{(n-2)\bar\theta - 1}{2}|\nabla\bar u|^2 g_{ij} + (A^\tau_g)_{ij}. \tag{5-8}$$

Substituting (5-8) into (5-7), we obtain

$$\begin{aligned}V[u]_{ij} &= \bar\theta\triangle v g_{ij} - v_{ij} + a(\bar u_i v_j + v_i\bar u_j + v_i v_j) + (a-1)\bar u_i\bar u_j\\ &\quad + b(2\langle\nabla\bar u, \nabla v\rangle + |\nabla v|^2)g_{ij} + \left(b - \frac{(n-2)\bar\theta - 1}{2}\right)|\nabla\bar u|^2 g_{ij} + (A^\tau_{\bar g})_{ij}.\end{aligned}$$

Since $\bar g = e^{2\bar u} g$, we have

$$V[u]_{ij} = \bar\theta \bar\Delta v \bar g_{ij} - \bar\nabla^2_{ij} v + \bar\theta \bar g^{sl}(\bar\Gamma^k_{sl}(\bar g) - \Gamma^k_{sl}(g)) v_k \bar g_{ij} - (\bar\Gamma^k_{ij}(\bar g) - \Gamma^k_{ij}(g)) v_k + a(\bar u_i v_j + v_i \bar u_j + v_i v_j) + (a-1)\bar u_i \bar u_j + b(2\langle \bar\nabla \bar u, \bar\nabla v\rangle_{\bar g} + |\bar\nabla v|^2_{\bar g})\bar g_{ij} + \left(b - \frac{(n-2)\bar\theta - 1}{2}\right)|\nabla \bar u|^2_{\bar g} \bar g_{ij} + (A^\tau_{\bar g})_{ij}. \tag{5-9}$$

Denote $\bar V[v]_{ij} := V[u]_{ij}$. Then (1-6) becomes

$$\begin{cases} F(\bar V[v]) = \Phi(x, \bar u + v) & \text{in } M, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial M. \end{cases} \tag{5-10}$$

By the boundary condition $u_n = 0$, $\bar u_n = 0$ and Lemma 2.2, we have

$$u_{n\alpha} = 0, \quad u_{\alpha\beta n} = 0, \quad \bar u_{n\alpha} = 0, \quad \bar u_{\alpha\beta n} = 0. \tag{5-11}$$

Therefore $v_n = 0$, $v_{n\alpha} = 0$ and $v_{\alpha\beta n} = 0$ on ∂M. Since $\bar g_{\alpha n} = e^{2\bar u} g_{\alpha n} = 0$, we have

$$\bar V[v]_{\alpha n} = -\bar\nabla^2_{\alpha n} v - (\bar\Gamma^\delta_{\alpha n}(\bar g) - \Gamma^\delta_{\alpha n}(g_0)) v_\delta + (A^\tau_{\bar g})_{\alpha n}.$$

It follows from (2-2) and the boundary condition $u_n = 0$ that

$$\bar\Gamma^\delta_{\alpha n}(\bar g) = \Gamma^\delta_{\alpha n}(g) = 0, \quad \bar\Gamma^n_{\alpha\beta} = \Gamma^n_{\alpha\beta} = 0, \quad \bar\Gamma^n_{nn} = \Gamma^n_{nn} = 0. \tag{5-12}$$

Then

$$\bar\nabla^2_{\alpha n} v = v_{\alpha n} = 0 \quad \text{and} \quad \bar\nabla_n \bar\nabla^2_{\alpha\beta} v = v_{\alpha\beta n} = 0. \tag{5-13}$$

By Lemma 2.3, we get

$$(A^\tau_{\bar g})_{\alpha n}(x_0) = -\frac{1}{n-2}\left(\bar R_{\alpha n} - \frac{\tau \bar R}{2(n-1)} \bar g_{\alpha n}\right) = 0.$$

Hence, $\bar V[v]_{\alpha n}(x_0) = 0$. Then

$$F^{\alpha n}(\bar V[v]) = 0.$$

Now differentiating (5-10) along the normal direction and taking its value at x_0, we have

$$\nabla_n \Phi(x, \bar u + v) = F^{nn} \bar V_{nnn} + F^{\alpha\beta} \bar V_{\alpha\beta n}. \tag{5-14}$$

Since $\bar g_{ij,n} = \bar g^{ij}_{,n} = 0$, by (5-11)–(5-13), we have

$$\bar V[v]_{\alpha\beta n} = \bar\theta v_{nnn} \bar g_{\alpha\beta} - (\bar\Gamma^\delta_{\alpha\beta}(\bar g) - \Gamma^\delta_{\alpha\beta}(g))_{,n} v_\delta + \bar\theta \bar g^{sl}(\bar\Gamma^\delta_{sl}(\bar g) - \Gamma^\delta_{sl}(g))_{,n} v_\delta \bar g_{\alpha\beta} + (A^\tau_{\bar g})_{\alpha\beta,n}.$$

Since ∂M is totally geodesic, using Fermi coordinates, we have on ∂M

$$\bar{\Gamma}^{\delta}_{\alpha\beta}(g)_{,n} = \Gamma^{\delta}_{\alpha\beta}(g)_{,n} = 0$$

(see [He and Sheng 2013]). By Lemma 2.3 again,

$$\bar{R}_n(x_0) = \bar{g}^{\alpha\beta}\bar{R}_{\alpha\beta,n}(x_0) + \bar{g}^{\alpha n}\bar{R}_{\alpha n,n}(x_0) + \bar{g}^{nn}\bar{R}_{nn,n}(x_0) = 0.$$

Therefore

$$(A^{\tau}_{\bar{g}})_{\alpha\beta,n}(x_0) = -\frac{1}{n-2}\left(\bar{R}_{\alpha\beta,n} - \frac{\tau\bar{R}_n}{2(n-1)}\bar{g}_{\alpha\beta}\right)(x_0) = 0.$$

Hence, we obtain

$$\bar{V}[v]_{\alpha\beta n}(x_0) = \bar{\theta}v_{nnn}\bar{g}_{\alpha\beta}. \tag{5-15}$$

Similarly, we have

$$\bar{V}[v]_{nnn}(x_0) = \bar{\theta}v_{nnn}\bar{g}_{nn}(x_0) - v_{nnn}(x_0). \tag{5-16}$$

Denote $\bar{\mathcal{T}} = F^{ij}(\bar{V}[v])\bar{g}_{ij} \geq 1$. Plugging (5-15) and (5-16) into (5-14), we obtain

$$0 \leq C + \bar{\theta}v_{nnn}(x_0)\bar{\mathcal{T}} - F^{nn}v_{nnn}(x_0) \leq (C + \bar{\theta}v_{nnn}(x_0))\bar{\mathcal{T}} - F^{nn}v_{nnn}(x_0). \tag{5-17}$$

If $C + \bar{\theta}v_{nnn}(x_0) \geq 0$, then we have $v_{nnn}(x_0) \geq -C/\bar{\theta}$, which implies that

$$u_{nnn}(x_0) \geq \bar{u}_{nnn}(x_0) - \frac{C}{\bar{\theta}} > -C'.$$

If $C + \bar{\theta}v_{nnn}(x_0) < 0$, then by (5-17) we have

$$0 \leq (C + (\bar{\theta}-1)v_{nnn}(x_0))F^{nn}.$$

Since $F^{nn} > 0$ and $\bar{\theta} > 1$, we have $v_{nnn}(x_0) \geq -C/(\bar{\theta}-1)$, which also implies the lower bound of $u_{nnn}(x_0)$. □

Proposition 5.2. *Let u be a C^4 solution of (1-6) on $\bar{B}^+_r$. Then there is a positive constant C_2 depending only on n, k, $\bar{\theta}$, r, g, $|S|_{C^2(\bar{B}^+_r)}$, $|\Phi|_{C^2(\bar{B}^+_r)\times[-C_0,C_0]}$, $|a|_{C^2(\bar{B}^+_r)}$, $|b|_{C^2(\bar{B}^+_r)}$, and C_1, such that*

$$\sup_{\bar{B}^+_{r/2}} |\nabla^2 u|_g \leq C_2. \tag{5-18}$$

Proof. We control the bound of Δu at first. Since $V[u] \in \Gamma \subset \Gamma_1$, we have

$$0 \leq \operatorname{tr}(V[u]) = (n\bar{\theta}-1)\Delta u + (a+nb)|\nabla u|^2 + \operatorname{tr} S,$$

which implies that Δu has a lower bound by Proposition 4.4. We may assume $\Delta u > 0$.

Consider the auxiliary function

$$G := \eta e^{x_n}(\triangle u + m|\nabla u|^2),$$

where η satisfies (4-1) and (4-2), and m is a larger constant to be fixed. We may assume $r = 1$, and

$$K := \triangle u + m|\nabla u|^2 \gg 1.$$

Step 1. We may assume G attains its maximum at an interior point $x_0 \in B_r^+$. If $x_0 \in \Sigma_r$, by Lemmas 2.2 and 5.1 we have

$$G_n(x_0) = K + u_{nnn} + u_{\gamma\gamma n} + 2mu_{\alpha n}u_\alpha + 2mu_{nn}u_n > K - C'.$$

If $K - C' \le 0$, we then get the bound of $\triangle u$. If $K - C' > 0$, it contradicts with the maximum of G at the boundary point x_0.

Step 2. We must get an upper bound for $\triangle u$. By step 1, the maximum point x_0 of G is an interior point in $\bar{B}_r^+$. Then at x_0 we have

$$G_i = 0 \quad \text{and} \quad G_{ij} \le 0,$$

that is,

$$u_{lli} + 2mu_l u_{li} = K_i = -\left(\frac{\eta_i}{\eta} + \delta_{in}\right)K, \tag{5-19}$$

and

$$0 \ge G_{ij} = \eta e^{x_n}\left\{\left(\frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2}\right)K + \left(\frac{\eta_i}{\eta} + \delta_{in}\right)K_j + K_{ij}\right\}.$$

Substituting (5-19) into the inequality above, by the definition of η in (4-1), we have

$$0 \ge G_{ij} = \eta e^{x_n}(K_{ij} + \Lambda_{ij}K),$$

where

$$\Lambda_{ij} = \frac{\eta_{ij}}{\eta} - 2\frac{\eta_i\eta_j}{\eta^2} - \frac{1}{\eta}(\eta_i\delta_{jn} + \eta_j\delta_{in}) - \delta_{in}\delta_{jn} \ge -\frac{C}{\eta}\delta_{ij},$$

and C depends only on b_0. Then we have

$$0 \ge e^{-x_n}P^{ij}G_{ij} \ge \eta P^{ij}K_{ij} - CK\mathcal{T}. \tag{5-20}$$

Note that

$$K_{ij} = u_{llij} + 2mu_{li}u_{lj} + 2mu_l u_{lij}. \tag{5-21}$$

By Ricci identities, we have

$$|u_{ijl} - u_{lij}| \le C \quad \text{and} \quad |u_{ijll} - u_{llij}| \le C(|\nabla^2 u| + 1).$$

Then we have

$$P^{ij}K_{ij} \geq P^{ij}u_{ijll} + 2mP^{ij}u_{li}u_{lj} + 2mu_lP^{ij}u_{ijl} - C(|\nabla^2 u| + 1)\mathcal{T}. \tag{5-22}$$

By (4-10), we have

$$\begin{aligned} 2mu_lP^{ij}u_{ijl} &= 2mu_l\nabla_l\Phi - F^{ij}(a_lu_iu_j + 2au_{il}u_j + S_{ij},_l) - (b_l|\nabla u|^2 + 2bu_{ls}u_s)\mathcal{T} \\ &\geq -C(|\nabla^2 u| + 1)\mathcal{T}, \end{aligned} \tag{5-23}$$

since $\nabla_{ll}\Phi = \Phi_{xx} + 2\Phi_{xz}u_l + \Phi_z u_{ll} \geq -C + \Phi_z\Delta u \geq -C(|\nabla^2 u| + 1)$. Differentiating the equation (1-6) twice, using the concavity of F, we have

$$\begin{aligned} P^{ij}u_{ijll} &\geq \nabla_{ll}\Phi - F^{ij}(a_{ll}u_iu_j + 4a_lu_{il}u_j + 2au_{ill}u_j + 2au_{il}u_{jl} + S_{ij},_{ll}) \\ &\qquad - (b_{ll}|\nabla u|^2 + 4b_lu_{ls}u_s + 2bu_{sll}u_s + 2b|\nabla^2 u|^2)\mathcal{T} \\ &\geq -2aF^{ij}u_{ill}u_j - 2aF^{ij}u_{il}u_{jl} - 2bu_{sll}u_l\mathcal{T} \\ &\qquad - 2b|\nabla^2 u|^2\mathcal{T} - C(|\nabla^2 u| + 1)\mathcal{T}. \end{aligned} \tag{5-24}$$

By Ricci identities again, and (5-19) and (5-24), we get

$$P^{ij}u_{ijll} \geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}. \tag{5-25}$$

Now, plugging (5-23) and (5-25) into (5-22), and choosing

$$m > \max\left\{2|a|_{L^\infty}, \frac{4}{\varepsilon_0}(\bar\theta|a|_{L^\infty} + |b|_{L^\infty})\right\},$$

we obtain

$$\begin{aligned} P^{ij}K_{ij} &\geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2\mathcal{T} + 2mP^{ij}u_{li}u_{lj} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &= 2(m + a)P^{ij}u_{li}u_{lj} - 2(a\bar\theta + b)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq 2\big((m - |a|_{L^\infty})\varepsilon_0 - (\bar\theta|a|_{L^\infty} + |b|_{L^\infty})\big)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq 2\left(\frac{m\varepsilon_0}{2} - (\bar\theta|a|_{L^\infty} + |b|_{L^\infty})\right)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\ &\geq \frac{m\varepsilon_0}{2}|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}. \end{aligned} \tag{5-26}$$

It follows from (5-20) and (5-26) that

$$\eta^2\frac{m\varepsilon_0}{2}|\nabla^2 u|^2\mathcal{T} \leq C(|\nabla^2 u| + 1)\mathcal{T},$$

which implies that $\eta|\nabla^2 u| \le C$.

Step 3. We get the Hessian bound of u. As in [Chen 2009], we consider the maximum of

$$\bar{G} = \eta(x)e^{x_n}(\nabla^2 u + m\,du\otimes du)$$

over the set $(x,\xi)\in(\bar{B}_r^+,\mathbb{S}^n)$. Let $\bar{G}$ attain its maximum at some point x_0 and the direction $\xi\in T_{x_0}\overline{M}\cap\mathbb{S}^n$. Denote $K_\xi = u_{\xi\xi}+mu_\xi^2$. We may assume $K_\xi \gg C' > 0$, where C' is the one in Lemma 5.1.

Now, we can also show that x_0 does not belong to the boundary. Suppose $x_0\in\Sigma_r$. If ξ is a tangential vector, without loss of generality we may assume $\xi = \partial/\partial x_1$. By Lemma 2.2, we have on the boundary that

$$\begin{aligned}(\eta e^{x_n}(u_{11}+mu_1^2))_n &= \eta e^{x_n}((u_{11}+mu_1^2)+u_{11n}+2mu_1u_{1n})\\ &\ge u_{11}+mu_1^2 = K_1 > 0\end{aligned}$$

Therefore, we get a contradiction. If ξ is in the normal direction, by Lemma 2.2 and Lemma 5.1, we also have

$$\begin{aligned}(\eta e^{x_n}(u_{nn}+mu_n^2))_n &= \eta e^{x_n}((u_{nn}+mu_n^2)+u_{nnn}+2mu_nu_{nn})\\ &\ge u_{nn}-C' = K_n - C' > 0.\end{aligned}$$

Thus x_0 must be an interior point. By similar calculations as before, we can get the Hessian bounds. We omit the details here. □

Remark 5.3. Let B_r and $B_{r/2}$ be two local geodesic balls in the interior of M, and $G = \eta(\Delta u + m|\nabla u|^2)$. The same calculations in steps 2 and 3 yield the interior Hessian estimates for (1-6).

Therefore we have the following global estimates.

Proposition 5.4. *Let u be a C^4 solution of (1-6) on $\overline{M}$. Then there is a positive constant C_2 depending only on n, k, $\bar\theta$, g, a, b, Φ, S and C_1, such that*

$$\sup_{\overline{M}} |\nabla^2 u|_g \le C_2.$$

6. Proof of Theorem 1.2

We use the continuity method to prove the existence of (1-6). Since the argument is standard (see [Li and Sheng 2011]), we only sketch it here.

For $t\in[0,1]$, consider the equation

$$F\big(g^{-1}(\bar\theta\Delta u g - \nabla^2 u + a(x)du\otimes du + b(x)|\nabla u|^2 g + S_t)\big) = \Phi_t(x,u), \tag{6-1$_t$}$$

where

$$S_t = tS + \frac{1-t}{F(e)}g \quad\text{and}\quad \Phi_t(x,u) = (1-t)e^{2u} + t\Phi(x,u).$$

Clearly, S_t and Φ_t satisfy the following conditions:

- $S_t \in \Gamma$ and $|S_t|_{C^4(\overline{M})} \le C$, where the constant C is independent of t.
- S_t satisfies (S1) or $S_t = tA_g^\tau$ when $t \neq 0$ and $S_0 = \dfrac{1}{F(e)} g$ as long as S satisfies (S1) or (S2).
- $\Phi_t(x, u) > 0$, $\partial_z \Phi_t > 0$, $\lim_{z\to+\infty} \Phi_t(x, z) \to +\infty$, and $\lim_{z\to-\infty} \Phi_t(x, z) \to 0$.
- $|\Phi_t|_{C^2(\overline{M}\times[-C,C])} \le C$, where C is independent of t.

It follows from Sections 3, 4 and 5 that for each t, the admissible solution of (6-1$_t$) has uniform a priori C^2 estimates (independent of t). Then we obtain the uniform $C^{2,\alpha}$ estimates by Evans–Krylov theory [Krylov 1985]. Define

$$I = \{t \in [0, 1] \mid (6\text{-}1_t) \text{ has admissible solution}\}.$$

Clearly, $u \equiv 0$ is the unique admissible solution of (6.1_0). Hence, $I \neq \varnothing$. By Lemma 3.1, $I \subset [0, 1]$ is open. By the uniform a priori $C^{2,\alpha}$ estimates and the standard degree theory, we conclude that I is also closed. Then for $t = 1$, (1-6) is solvable. □

7. Proof of Theorem 1.3

Before proving Theorem 1.3, we first calculate a priori estimates for (1-9).

Proposition 7.1. *Suppose $T \in \Gamma$ and the positive function $\Phi(x, z) \in C^\infty(\overline{M}) \times \mathbb{R}$ satisfy (1-8). Then there exists a constant C_0 only depending on T and Φ, such that any solution $u \in C^2(\overline{M})$ of (1-9) satisfies*

$$\sup_{\overline{M}} |u| \le C_0.$$

The proof is similar to that of Proposition 3.2. We omit it here.

Proposition 7.2. *Suppose u is a C^3 solution of (1-9) on $\bar{B}_r^+$. Then there is a positive constant C depending only on n, k, θ, g, r, $|T|_{C^1(\bar{B}_r^+)}$, $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$, $|a|_{C^1(\bar{B}_r^+)}$, $|b|_{C^1(\bar{B}_r^+)}$ and C_0, such that*

$$\sup_{\bar{B}_{r/2}^+} |\nabla u|_g \le C.$$

Proof. Consider the auxiliary functions

$$G := \tfrac{1}{2}\eta e^{\beta} |\nabla u|^2, \quad \beta := x_n + \psi(u).$$

Then G can not attain its maximum at a boundary point $x_0 \in \Sigma_r$ by the same arguments in the proof of Proposition 4.1. Since the maximum point x_0 is an interior point, we can also get (4-5)–(4-7). Now, the difference from the proof of Proposition 4.1 is that we replace the operator P^{ij} in (4-8) by the operator

$$Q^{ij} := F^{ij} + \theta \mathcal{T} g^{ij}. \tag{7-1}$$

Then by similar calculations as in (4-9)–(4-16), we obtain

$$\begin{aligned} 0 \geq C^* + \psi' Q^{ij} u_{ij} + (\psi'' - \psi'^2) Q^{ij} u_i u_j + 2a\psi' Q^{ij} u_i u_j \\ + 2b\psi' |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u| + 1)\mathcal{T}. \end{aligned} \tag{7-2}$$

Since

$$\begin{aligned} \psi' Q^{ij} u_{ij} &= \psi' F^{ij} W_{ij} - \psi' F^{ij}(a u_i u_j + b|\nabla u|^2 g_{ij} + T_{ij}) \\ &\geq \psi' \Phi - a\psi' F^{ij} u_i u_j - b\psi' |\nabla u|^2 - C\mathcal{T}. \end{aligned} \tag{7-3}$$

Substituting (7-3) into (7-2), we get

$$\begin{aligned} 0 &\geq C^* + \psi'\Phi + (\psi'' - \psi'^2) Q^{ij} u_i u_j + a\psi' F^{ij} u_i u_j \\ &\qquad + b\psi' |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u| + 1)\mathcal{T} \\ &= C^* + \psi'\Phi + (\psi'' - \psi'^2 + a\psi') F^{ij} u_i u_j \\ &\qquad + (\theta(\psi'' - \psi'^2) + b\psi')|\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u| + 1)\mathcal{T}. \end{aligned} \tag{7-4}$$

By the similar argument as in Claim 4.2, we know that there exist positive constants α_1, α_2 and p depending only on θ, C_0, $|a|_{L^\infty(\overline{M})}$ and $|b|_{L^\infty(\overline{M})}$, such that

$$\psi' > 0, \quad \psi'' - \psi'^2 - |a|_{L^\infty}\psi' > 0, \quad \theta(\psi'' - \psi'^2) - |b|\psi' \geq \varepsilon_2 > 0,$$

where the constant ε_2 only depends on α_1, α_2 and p. Then we have

$$0 \geq C^* + \varepsilon_2 |\nabla u|^2 \mathcal{T} - \frac{C}{\sqrt{\eta}}(|\nabla u| + 1)\mathcal{T}. \tag{7-5}$$

Then multiplying by η^2 both sides of the inequality above and $\mathcal{T} >1$, we have

$$\varepsilon_2 \eta^2 |\nabla u|^2 \mathcal{T} \leq C|\nabla u| \mathcal{T} + C^*,$$

which implies the gradient estimates. □

To get the boundary Hessian estimates, we first prove the following:

Lemma 7.3. *Let u be a C^4 solution of (1-9). Then there is a positive constant C' depending only on $n, k, \theta, g, |T|_{C^1(\bar{B}_r^+)}, |a|_{C^1(\bar{B}_r^+)}, |b|_{C^1(\bar{B}_r^+)}, |\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$ and C_1 such that on ∂M, we have*

$$u_{nnn} \geq -C'.$$

Proof. (i) Let T satisfy the condition (S1). Then $T_{\alpha n} = 0$ on the boundary. Hence $W[u]_{\alpha n} = T_{\alpha n} = 0$. Therefore $F^{\alpha n}(W[u]) = 0$. By the similar calculations in Lemma 5.1, we have

$$W[u]_{\alpha\beta,n} \leq \theta u_{nnn} g_{\alpha\beta} + C \tag{7-6}$$

and

$$W[u]_{nnn} \leq u_{nnn} + \theta u_{nnn} + C, \tag{7-7}$$

where the constants C depend on $n, k, g, |T|_{C^1(\bar{B}_r^+)}, |a|_{C^1(\bar{B}_r^+)}, |b|_{C^1(\bar{B}_r^+)}$ and C_1.

Now, differentiating (1-9) along the normal direction and taking the value on the boundary, we have

$$\begin{aligned}\nabla_n \Phi &= F^{nn} W[u]_{nnn} + F^{\alpha\beta} W[u]_{\alpha\beta n} \\ &\leq F^{nn}(u_{nnn} + \theta u_{nnn}) + \theta u_{nnn} F^{\alpha\beta} g_{\alpha\beta} + C\mathcal{T} \\ &= F^{nn} u_{nnn} + \theta u_{nnn}\mathcal{T} + C\mathcal{T},\end{aligned} \tag{7-8}$$

that is,

$$0 \leq F^{nn} u_{nnn} + \theta u_{nnn}\mathcal{T} + C\mathcal{T} = F^{nn} u_{nnn} + (\theta u_{nnn} + C)\mathcal{T}, \tag{7-9}$$

where the constant C also depends on $|\Phi|_{C^1(\bar{B}_r^+)\times[-C_0,C_0]}$.

If $\theta u_{nnn} + C \geq 0$, then we get $u_{nnn} \geq -C/\theta$. If $\theta u_{nnn} + C < 0$, by $F^{nn} < \mathcal{T}$ and (7-9), we have

$$0 \leq F^{nn} u_{nnn} + (\theta u_{nnn} + C)F^{nn} = ((\theta+1)u_{nnn} + C)F^{nn}.$$

Since $F^{nn} > 0$, we get

$$(\theta+1)u_{nnn} + C \geq 0.$$

Note $\theta > 0$. Then we obtain $u_{nnn} \geq -C'$ again.

(ii) Suppose $T = -A_g^\tau$. Using the metric $\bar{g}$ in Lemma 2.3, we consider a new metric $\breve{g} = e^{2w}\bar{g}$ such that $u = \bar{u} + w$ is a solution of (1-9). Then similar to the calculation in the proof of Lemma 5.1, we have

$$\begin{aligned}W[u]_{ij} = {}& \theta\bar{\Delta} w \bar{g}_{ij} + \bar{\nabla}^2_{ij} w + \bar{\theta}\bar{g}^{sl}(\bar{\Gamma}^k_{sl}(\bar{g}) - \Gamma^k_{sl}(g)) w_k \bar{g}_{ij} + (\bar{\Gamma}^k_{ij}(\bar{g}) - \Gamma^k_{ij}(g)) w_k \\ &+ (a-1)\bar{u}_i\bar{u}_j + a(\bar{u}_i w_j + w_i\bar{u}_j + w_i w_j) + b\big(2\langle\bar{\nabla}\bar{u}, \bar{\nabla} w\rangle_{\bar{g}} + |\bar{\nabla} w|^2_{\bar{g}}\big)\bar{g}_{ij} \\ &+ \left(b - \frac{1+(n-2)\theta}{2}\right)|\bar{\nabla} u|^2_{\bar{g}}\bar{g}_{ij} - (A^\tau_{\bar{g}})_{ij}.\end{aligned}$$

Denote $\overline{W}[w]_{ij} := W[u]_{ij}$. Now, (1-9) becomes

(7-10)
$$\begin{cases} F(\overline{W}[w]) = \Phi(x, \bar{u}+w) & \text{in } M, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial M. \end{cases}$$

By Lemma 2.3, we find $(A^{\tau}_{\bar{g}})_{\alpha n}(x_0) = 0$. Then we have $\overline{W}[w]_{\alpha n}(x_0) = 0$ by Lemma 2.2 and (5-11)–(5-13), which implies $F^{\alpha n}(\overline{W}[w]) = 0$. By Lemma 2.2 again, we obtain

$$\overline{W}[w]_{\alpha\beta n}(x_0) = \theta w_{nnn}\bar{g}_{\alpha\beta}(x_0),$$

and

$$\overline{W}[w]_{nnn}(x_0) = \theta w_{nnn}\bar{g}_{nn}(x_0) + w_{nnn}(x_0).$$

Then by differentiating (7-10) along the normal direction and taking its value at x_0, we have

$$\begin{aligned} 0 &\le F^{nn}\overline{W}_{nnn} + F^{\alpha\beta}\overline{W}_{\alpha\beta n} + C \\ &\le F^{nn}w_{nnn}(x_0) + (\theta w_{nnn}(x_0) + C)\overline{\mathcal{T}}. \end{aligned}$$

If $\theta w_{nnn}(x_0) + C \ge 0$, we have $u_{nnn}(x_0) \ge -C'$ immediately. Now consider $\theta w_{nnn}(x_0) + C < 0$. Since $\overline{\mathcal{T}} > F^{nn} > 0$, we have

$$0 < F^{nn}w_{nnn}(x_0) + (\theta w_{nnn}(x_0) + C)F^{nn} \le ((\theta+1)w_{nnn}(x_0) + C)F^{nn}.$$

Hence, we must have $w_{nnn}(x_0) \ge -C/(\theta+1)$. Therefore, $u_{nnn}(x_0) \ge -C'$. □

Proposition 7.4. *Let u be a C^4 solution of* (1-9) *on $\bar{B}^+_r$. Then there is a positive constant C_2 depending only on n, k, θ, g, r, $|T|_{C^2(\bar{B}^+_r)}$, $|\Phi|_{C^2(\bar{B}^+_r)\times[-C_0,C_0]}$, $|a|_{C^2(\bar{B}^+_r)}$, $|b|_{C^2(\bar{B}^+_r)}$ and C_1 such that*

$$\sup_{\bar{B}^+_{r/2}} |\nabla^2 u|_g \le C_2.$$

Proof. We first estimate the bound of Δu. By $W[u] \in \Gamma^+_k \subset \Gamma_1$, we have

$$0 \le \operatorname{tr}(W[u]) = (n\theta+1)\Delta u + (a+nb)|\nabla u|^2 + \operatorname{tr} T,$$

which implies that Δu has lower bound. Hence, we may assume $\Delta u > 0$.

Consider the same auxiliary function in Proposition 5.2

$$G := \eta e^{q x_n}(\Delta u + m|\nabla u|^2),$$

where η satisfies (4-1) and (4-2), m is a larger constant to be fixed. We may assume $r = 1$ and $K := \Delta u + m|\nabla u|^2 \gg 1$.

Step 1. We show the maximum of G must be attained at an interior point of $\bar{B}^+_r$. If the maximum point x_0 of G belong to Σ_r, then by Lemma 2.2, Lemma 7.3 and the same calculations in Proposition 5.2, we know that $G_n(x_0) > 0$. It is a contradiction.

Step 2. We must get an upper bound for Δu. Since the maximum point of G is an interior point of $\bar{B}_r^+$ by step 1. Then at the maximum point x_0, we can get similar inequalities as in (5-19)–(5-24) by replacing P^{ij} by Q^{ij}. Corresponding to (5-26), for $m > \max\{|a|_{L^\infty(\overline{M})}, (|b|_{L^\infty(\overline{M})} + \varepsilon_3)/\theta\}$, $\varepsilon_3 > 0$, we obtain

$$
\begin{aligned}
Q^{ij}K_{ij} \quad & \\
&\geq -2aF^{ij}u_{il}u_{jl} - 2b|\nabla^2 u|^2\mathcal{T} + 2mQ^{ij}u_{li}u_{lj} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&= 2(m-a)F^{ij}u_{li}u_{lj} + 2(m\theta - b)|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&\geq 2(m - |a|_{L^\infty})F^{ij}u_{li}u_{lj} + 2(m\theta - |b|_{L^\infty})|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T} \\
&\geq 2\varepsilon_3|\nabla^2 u|^2\mathcal{T} - \frac{C}{\eta^{1/2}}(|\nabla^2 u| + 1)\mathcal{T}.
\end{aligned} \tag{7-11}
$$

It follows from (5-20) for Q^{ij} and (7-11) that $2\eta^2\varepsilon_3|\nabla^2 u|^2\mathcal{T} \leq C(|\nabla^2 u| + 1)\mathcal{T}$, which implies that $\eta|\nabla^2 u| \leq C$. □

Step 3. By Lemma 7.3 and the same argument in the step 3 of the proof of Proposition 5.2, we can get the Hessian estimates of u.

Remark 7.5. We can also get the interior gradient and Hessian estimates for the solutions of (1-9) by the same arguments in Remarks 4.3 and 5.3.

Proof of Theorem 1.3. Since the operator Q^{ij} in (7-1) is positive, by the argument in Section 3, we know that (1-9) is elliptic at any admissible solutions and its linearized operator is invertible as $\partial_z\Phi > 0$. Combining Propositions 7.1, 7.2, 7.4 and Remark 7.5, we can obtain

$$
|u|_{C^2(\overline{M})} \leq C, \tag{7-12}
$$

where the constant C depends only on $n, k, \theta, g, S, \Phi, a$ and b. By the global a priori C^2 estimates (7-12), we can prove Theorem 1.3 by a same argument in Section 6. □

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