Pacific Journal of Mathematics

Volume 270 No. 2 August 2014

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US $410/year for the electronic version, and $535/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.257

DISJOINTIFICATION INEQUALITIES IN SYMMETRIC QUASI-BANACH SPACES AND THEIR APPLICATIONS

SERGEY ASTASHKIN, FEDOR A. SUKOCHEV AND DMITRIY ZANIN

We demonstrate the relevance of the Prokhorov inequality to the study of Khintchine-type inequalities in symmetric function spaces. Our main result shows that the latter inequalities hold for a pair of quasi-Banach symmetric function spaces X and Y if and only if the Kruglov operator K acts from X to Y. We also obtain an extension of von Bahr–Esseen and Esseen–Janson L_p-estimates for sums of independent mean zero random variables to the class of quasi-Banach symmetric spaces. In particular, in contrast to the well-known Esseen–Janson theorem, we do not assume that the summands are equidistributed.

1. Introduction

The classical Khintchine inequality [1923] describes the span of independent centered $\{\pm 1\}$-valued Bernoulli random variables in quasi-Banach L_p-spaces. A particular case of the latter sequence is given by the Rademacher functions $r_n(t) := \operatorname{sgn}\sin(2^n\pi t)$, $t \in [0,1)$, $n \geq 1$. In this case, for all $p \in (0,\infty)$ the sequence $\{r_n\}_{n=1}^{\infty}$ in the L_p-spaces on the interval $(0,1)$ (equipped with Lebesgue measure m) is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of l_2. A famous extension of this inequality to a more general case of random variables was given later by Marcinkiewicz and Zygmund (see [1937, Theorem 13, p. 87] and [1938, Theorem 5, p. 109]): for every $1 \leq p < \infty$ there are constants $A_p > 0$ and $B_p > 0$ such that for any $n \in \mathbb{N}$ and for an arbitrary sequence of independent mean zero random variables $(f_k)_{k\in\mathbb{N}}$ from $L_p(0,1)$ we have

$$(1)\qquad A_p \left\| \left(\sum_{k=1}^{n} f_k^2 \right)^{1/2} \right\|_p \leq \left\| \sum_{k=1}^{n} f_k \right\|_p \leq B_p \left\| \left(\sum_{k=1}^{n} f_k^2 \right)^{1/2} \right\|_p .$$

In the special setting of Banach symmetric function spaces Johnson and Schechtman [1988] proved a far reaching generalization of the Marcinkiewicz–Zygmund inequality (1). More precisely, they established that if such a space X is either separable or has the Fatou property (for the relevant definitions see the following

MSC2010: primary 46E30; secondary 60G50, 46B09.
Keywords: Kruglov operator, Prokhorov inequality, quasi-Banach spaces.

section) and the lower Boyd index of X is strictly positive, then (1) holds (even for a more general case of martingale differences). Later on, Astashkin [2008] showed that inequality (1) holds in a Banach symmetric space X if and only if X satisfies the so-called Kruglov property. The latter property, introduced by Braverman [1994], has its origin in a remarkable result due to Rosenthal [1970] that for sequences $\{f_n\}_{n=1}^{\infty}$ of independent mean zero random variables in $L_p(0,1)$, $p \geq 2$, the mapping $f_n \to f_n(t-n+1)\chi_{[n-1,n)}(t)$, $t \in \mathbb{R}$, extends to an isomorphism between the closed linear span $[f_n]_{n=1}^{\infty}$ (taken in $L_p(0,1)$) and the closed linear span $[f_n(t-n+1)\chi_{[n-1,n)}]_{n=1}^{\infty}$ (taken in $L_p(0,\infty) \cap L_2(0,\infty)$). The main focus of the present paper is to establish optimal conditions on a *quasinormed* symmetric function space in which inequalities of the type (1) hold. Our techniques are centered around the so-called Kruglov operator, a natural generalization of the Kruglov property, which was introduced in [Astashkin and Sukochev 2005] (see also [Astashkin and Sukochev 2010]). The usage of this operator allows us to make a straightforward connection between sums of independent random variables and their disjoint translates. Another major ingredient of our approach consists in utilizing Prokhorov's famous inequality [1959] (see also Theorem 17 below) which allows us to treat the problem in the full generality.

Using our present method, we also provide a far-reaching extension of the well-known von Bahr–Esseen and Esseen–Janson L_p-estimates for sums of independent mean zero random variables (see [von Bahr and Esseen 1965] and [Esseen and Janson 1985]). We extend inequalities of such type to the class of quasi-Banach symmetric spaces, and, at the same time, we do not assume that the summands are equally distributed (which is in strong contrast with Esseen and Janson's approach [1985, Theorem 4]). Note that earlier, Braverman [1994, § II.2] generalized the von Bahr–Esseen inequality to (Banach) symmetric spaces with the Kruglov property.

2. Preliminaries

2.1. ***Quasi-Banach spaces.*** Let X be a linear space over the field of real numbers $\mathbb{R}$. A function $\|\cdot\|_X : X \to \mathbb{R}$ is called a *quasinorm* if the following conditions hold:

(a) $\|x+y\|_X \leq C(\|x\|_X + \|y\|_X)$ for every $x, y \in X$ and some constant $C > 0$.

(b) $\|cx\|_X = |c| \cdot \|x\|_X$ for every $x \in X$ and $c \in \mathbb{R}$.

(c) $\|x\|_X \geq 0$. Moreover, $\|x\|_X = 0$ if and only if $x = 0$.

The least of all constants C satisfying condition (a) above is called *the modulus of concavity* of the quasinorm $\|\cdot\|_X$ and is denoted by $C(X)$.

If X is a linear space over $\mathbb{R}$ and if $\|\cdot\|_X : X \to \mathbb{R}$ is a quasinorm, then $X = (X, \|\cdot\|_X)$ is called a *quasinormed space*. If every Cauchy sequence in a quasinormed space X converges, then X is called a *quasi-Banach* space.

For example, $L_p(0,1)$ and $L_p(0,\infty)$, $0 < p < 1$, are quasi-Banach spaces with modulus of concavity $C(p) = C(L_p) = 2^{1/p-1}$.

Recall that a quasinorm $\|\cdot\|_X$ in X is said to be a *p-norm*, $0 < p < 1$, if for any $x_1, x_2 \in X$ we have

$$\|x_1 + x_2\|_X^p \le \|x_1\|_X^p + \|x_2\|_X^p.$$

By the Aoki–Rolewicz theorem [Kalton et al. 1984], for any quasinorm $\|\cdot\|_X$ there exists $0 < p < 1$ such that $\|\cdot\|_X$ is a p-norm.

2.2. ***Symmetric function spaces.*** We are interested in those quasi-Banach spaces which consist of Lebesgue-measurable functions either on $(0,1)$ or on $(0,\infty)$.

For a Lebesgue-measurable, a.e. finite function x on $(0,1)$ (or $(0,\infty)$) we define its *distribution function* by

$$d_x(s) := m(\{t : x(t) > s\}), \quad s \in \mathbb{R},$$

where m stands for Lebesgue measure. Let $S(0,1)$ (respectively, $S(0,\infty)$) denote the space of all Lebesgue-measurable functions x on $(0,1)$ (respectively, on $(0,\infty)$ with $d_{|x|}(s) < \infty$ for sufficiently large s).

Two measurable functions x and y are called *equimeasurable* (written $x \sim y$) if their distribution functions d_x and d_y coincide. In particular, for every measurable function x, the function $|x|$ is equimeasurable with its *decreasing rearrangement* x^*, defined by the formula

$$x^*(t) := \inf\{\tau \ge 0 : d_{|x|}(\tau) < t\}, \quad t > 0.$$

If $x, y \ge 0$, then $x^* = y^*$ if and only if x and y are equimeasurable. We recall that a function x is said to be symmetrically distributed if x and $-x$ are equimeasurable.

As it is traditional in probability theory, we denote by ϕ_x the characteristic function of an element $x \in S(0,1)$; that is, $\phi_x(t) = \int_0^1 e^{itx(s)}\,ds$. Recall that functions $x, y \in S(0,1)$ are equimeasurable if and only if their characteristic functions ϕ_x and ϕ_y coincide.

Definition 1. Let $X \subset S(0,1)$ (or $X \subset S(0,\infty)$) be a quasi-Banach space.

(a) X is said to be a quasi-Banach function space if, from $x \in X$, $y \in S(0,1)$ (or $y \in S(0,\infty)$) and $|y| \le |x|$, it follows that $y \in X$ and $\|y\|_X \le \|x\|_X$.

(b) A quasi-Banach function space X is said to be symmetric if, for every $x \in X$ and any measurable function y, the assumption $y^* = x^*$ implies that $y \in X$ and $\|y\|_X = \|x\|_X$.

Without loss of generality, in what follows we assume that $\|\chi_{(0,1)}\|_X = 1$, where χ_E denotes the indicator function of a Lebesgue measurable set E.

The following assertion is well known in the Banach-space setting (see, for instance, [Lindenstrauss and Tzafriri 1979, Proposition 1.d.2]). For the reader's convenience, we provide a short proof.

Lemma 2. *Let X be a quasi-Banach function space. If $0 \le x$ and $y \in X$, then* $\|(xy)^{1/2}\|_X \le C(X)\|x\|_X^{1/2}\|y\|_X^{1/2}$.

Proof. It is easy to see that

$$(xy)^{1/2} \le \tfrac{1}{2}(\theta x + \theta^{-1} y), \quad \theta > 0,$$

and, therefore,

$$\|(xy)^{1/2}\|_X \le \frac{C(X)}{2}(\theta\|x\|_X + \theta^{-1}\|y\|_X).$$

Taking the infimum over all $\theta > 0$, we infer

$$\|(xy)^{1/2}\|_X \le C(X)\|x\|_X^{1/2}\|y\|_X^{1/2}. \qquad \square$$

Let X be a quasi-Banach symmetric function space and let $x_n \in X$, $n \in \mathbb{N}$, be such that $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$ and $x_n \to x$ almost everywhere. If, for every such sequence, we have $x \in X$ and $\|x\|_X \le \liminf_{n\to\infty} \|x_n\|_X$, then X is said to satisfy the *Fatou property*.

Suppose that X is a separable quasi-Banach symmetric space on $(0, 1)$. Denote by $\overline{X}$ the set of all $x \in S(0, 1)$ such that $\lim_{a\to+\infty} \|\,[|x|]_a\|_X < \infty$, where $[|x|]_a := |x|$ if $|x| < a$ and $[|x|]_a := 0$ if $|x| \ge a$. The set $\overline{X}$, equipped with the norm $\|x\|_{\overline{X}} := \lim_{a\to+\infty} \|\,[|x|]_a\|_X$, becomes a quasi-Banach symmetric space with the Fatou property. Moreover, X embeds isometrically into $\overline{X}$. It can be easily checked that for every quasi-Banach symmetric space X on $(0, 1)$ the continuous embedding $X \supset L_\infty(0, 1)$ holds. Then, the closure of $L_\infty(0, 1)$ in X, denoted by X_0, is a separable quasi-Banach symmetric space with the norm $\|\cdot\|_X$ whenever $X \ne L_\infty(0, 1)$.

If $\tau > 0$, the dilation operator σ_τ is defined by setting $\sigma_\tau x(s) = x(s/\tau)$, $s > 0$, in the case of the semiaxis. In the case of the interval $(0, 1)$, the operator σ_τ is defined by

$$\sigma_\tau x(s) := \begin{cases} x(s/\tau) & \text{if } s \le \min\{1, \tau\}, \\ 0 & \text{if } \tau < s \le 1. \end{cases}$$

Below we shall often consider the probability product space

$$(\Omega, \mathbb{P}) := \prod_{k=0}^{\infty} ((0, 1), m_k),$$

(m_k is the Lebesgue measure on $(0, 1)$, $k \ge 0$). Observe that in an arbitrary symmetric space the norms of any two elements with identical distribution coincide.

Hence, using a one-to-one measure-preserving transformation between measure spaces $(\Omega, \mathbb{P})$ and $((0,1), m)$, we will identify an arbitrary measurable function $x(\omega) = x(\omega_0, \omega_1, \dots, \omega_n, \dots)$ on $(\Omega, \mathbb{P})$ with the corresponding element from $S(0,1)$. Since a particular form of the measure-preserving transformation used in such identification is not important, we completely suppress it from the notations. Thus, we will view the set Ω as $(0,1)$ and any measurable function on $(\Omega, \mathbb{P})$ as a function from $S(0,1)$ and vice versa. A reader interested in more details of such identification is referred to [Astashkin and Sukochev 2010].

Let x_k, $k \geq 0$, be elements from $S(0,1)$ and let $y_k \in S(0,\infty)$, $k \geq 0$, be their disjoint copies; that is, $x_k \sim y_k$ for all $k \geq 0$, and $y_l y_m = 0$ if $l \neq m$. For the function $\sum_{k\geq 0} y_k$, which is frequently called the *disjoint sum* of x_k, $k \geq 0$, we shall use the suggestive notation $\bigoplus_{k\geq 0} x_k$. It is important to observe that the distribution function of a disjoint sum $\bigoplus_{k\geq 0} x_k$ does not depend on the particular choice of elements y_k, $k \geq 0$. In the special case when $\sum_{k=1}^{n} m(\mathrm{supp}(x_k)) \leq 1$, $n \in \mathbb{N}$, it is convenient to view the sum $\bigoplus_{k\geq 0} x_k$ as a measurable function on $(0,1)$.

The following useful construction was introduced in [Johnson et al. 1979] (see also [Lindenstrauss and Tzafriri 1979, 2.f]). If X is a quasi-Banach symmetric function space on $(0,1)$ and $0 < p \leq \infty$, then the set Z_X^p consists of all $f \in S(0,\infty)$ such that

$$\|f\|_{Z_X^p} := \|f^* \chi_{(0,1)}\|_X + \|\min\{f^*, f^*(1)\}\|_p < \infty.$$

It can be easily checked that the functional $\|\cdot\|_{Z_X^p}$ is a quasinorm on Z_X^p.

2.3. ***Kruglov operator and Kruglov property.*** The Kruglov property was introduced by Braverman [1994] when he compared sums of independent functions with sums of their disjoint copies in (Banach) symmetric spaces. Such terminology stems from related probabilistic constructions, due to Kruglov [1970], used in the study of infinitely divisible distributions (e.g., in analysis of the classical Levy–Khintchine formula).

Let $x \in S(0,1)$. By $\pi(x)$ we denote the random variable $\sum_{i=1}^{N} x_i$, where x_i, $i = 1, \dots, N$, are independent copies of x and N is a random variable having Poisson distribution with parameter 1 and independent with respect to the sequence $\{x_i\}$.

Definition 3. A quasi-Banach symmetric space X on $(0,1)$ is said to have the *Kruglov property* ($X \in \mathbb{K}$) if from $x \in X$ it follows that $\pi(x) \in X$.

Simplifying the situation, the Kruglov property holds for spaces sufficiently "remote" from the space $L_\infty(0,1)$. For example, if a symmetric Banach function space X contains $L_p(0,1)$ for some $p < \infty$, then X possesses the Kruglov property (see, e.g., [Braverman 1994, Theorem 1.2] or [Astashkin and Sukochev 2010]). For a more precise characterization of various classes of (Banach) symmetric function spaces possessing the Kruglov property, we refer the reader to [Astashkin and Sukochev 2005; 2007; 2010; Braverman 1994].

Now, we recall the definition of the Kruglov operator, which can be viewed as a natural generalization of the notion of the Kruglov property. Let $\{B_n\}_{n=0}^{\infty}$ be a fixed sequence of mutually disjoint measurable subsets of $(0,1)$ such that $m(B_n) = 1/(en!)$. Define the operator $K : S(0,1) \to S(0,1)$ by setting

$$Kx(\omega) := \sum_{n=1}^{\infty} \sum_{k=1}^{n} x(\omega_k)\chi_{B_n}(\omega_0).$$

It is not difficult to see that

$$\phi_{Kx}(t) = \phi_{\pi(x)}(t) = \exp(\phi_x(t) - 1), \quad t \in \mathbb{R}. \tag{2}$$

Therefore, by the definition of the Kruglov property, a quasi-Banach symmetric function space X has the Kruglov property if and only if the operator K acts boundedly in X. Though the following crucial theorem originated in [Astashkin and Sukochev 2005], the first explicit statement (with a proof) appeared in [Astashkin et al. 2011].

Theorem 4. *If a sequence $\{x_k\}_{k=1}^{n} \subset S(0,1)$, $n \in \mathbb{N}$, consists of disjointly supported functions, then the sequence $\{Kx_k\}_{k=1}^{n}$ consists of independent functions.*

We will need also the following assertion, which is an immediate consequence of [Astashkin and Sukochev 2010, Theorem 15].

Theorem 5. *If X is a separable quasi-Banach symmetric space on $(0,1)$ such that $K : \overline{X} \to \overline{X}$, then $K : X \to X$ and $\|K\|_{X\to X} = \|K\|_{\overline{X}\to\overline{X}}$.*

3. Disjointification inequalities for positive functions

We will use the following approximation to the function Kx, where x is an arbitrary measurable function on the interval $(0,1)$. For every $n \in \mathbb{N}$ define the operator $H_n : S(0,1) \to S(0,1)$ by the formula

$$H_n x(\omega) := \sum_{k=1}^{n} (\sigma_{1/n} x)(\omega_k). \tag{3}$$

The following result is well known (see the proof of Lemma 1.6 in [Braverman 1994] or of Theorem 22 in [Astashkin and Sukochev 2010]). However, we present its proof for the reader's convenience.

Lemma 6. *The sequence of functions $\{H_n x\}_{n=1}^{\infty}$ converges to the function Kx in distribution.*

Proof. It is not difficult to see that $\phi_{H_n x} = \phi^n_{\sigma_{1/n}x}$. On the other hand,

$$\phi_{\sigma_{1/n}x}(t) = \int_0^1 e^{it\sigma_{1/n}x(s)}\,ds = \left(1-\frac{1}{n}\right) + \frac{1}{n}\phi_x(t).$$

Therefore, by (2), we obtain

$$\phi_{H_n x} = \left(1+\frac{\phi_x - 1}{n}\right)^n \to \exp(\phi_x - 1) = \phi_{Kx}.$$

Since the convergence of distributions follows from the convergence of characteristic functions [Borovkov 1998, Theorem 6.2.1], the result follows. □

Theorem 7. *Let X and Y be quasi-Banach symmetric spaces on $(0,1)$ and let Y have the Fatou property. Suppose that there exists a positive constant $C > 0$ such that for every sequence of nonnegative independent functions $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, with $\sum_{k=1}^n m(\operatorname{supp}(x_k)) \le 1$, we have*

$$\left\|\sum_{k=1}^n x_k\right\|_Y \le C \cdot \left\|\bigoplus_{k=1}^n x_k\right\|_X. \tag{4}$$

Then the operator K maps X into Y and $\|K\|_{X\to Y} \le C$.

The assertion remains valid under the assumption that the inequality (4) *holds for $X = Y$, where X is a separable quasi-Banach symmetric space.*

Proof. For every $x \in X$, let us define $x_k(\omega) = (\sigma_{1/n}x)(\omega_k)$, $\omega \in \Omega$. It follows from the definition of disjoint sum that

$$\bigoplus_{k=1}^n x_k \sim x \quad \text{for every } n \in \mathbb{N}.$$

Therefore, applying (3) and (4), we obtain $\|H_n x\|_F \le C\|x\|_E$. Furthermore, by Lemma 6, the sequence $\{H_n x\}_{n\ge 1}$ converges to the function Kx in distribution when $n \to \infty$ and hence $(H_n x)^* \to (Kx)^*$ almost everywhere on $(0,1)$. Since Y has the Fatou property, it follows that $Kx \in Y$ and $\|Kx\|_Y \le C\|x\|_X$.

Suppose now that X is a separable quasi-Banach symmetric space such that (4) holds for every sequence of nonnegative independent functions $\{x_k\}_{k=1}^n \subset X$ such that $\sum_{k=1}^n m(\operatorname{supp}(x_k)) \le 1$, $n \in \mathbb{N}$. From the definition of the space $\overline{X}$ (see Section 2), it follows that a similar inequality with the same constant C holds also for every sequence of nonnegative independent functions $\{x_k\}_{k=1}^n \subset \overline{X}$ with $\sum_{k=1}^n m(\operatorname{supp}(x_k)) \le 1$, $n \in \mathbb{N}$. Therefore, since $\overline{X}$ has the Fatou property, by the first part of theorem, we conclude that $K : \overline{X} \to \overline{X}$ and $\|K\|_{\overline{X}\to\overline{X}} \le C$. An application of Theorem 5 completes the proof. □

Our next purpose is to establish the main result of this section (Theorem 16), which is in a sense converse to the assertion of the preceding theorem. The first step in its proof is Proposition 9 below. We also need some preparatory results.

Lemma 8. *For every positive $x \in S(0,1)$, we have $\sigma_{1/2}x^* \leq (Kx)^*$.*

Proof. Let B_n, $n \geq 1$, be the sets from the definition of the Kruglov operator K. Since the B_n are pairwise disjoint and

$$\sum_{n=1}^{\infty} m(B_n) = \frac{e-1}{e} > \frac{1}{2},$$

we may select a measurable set $B \subset \bigcup_{n\geq 1} B_n$ such that $m(B) = 1/2$. It is clear that $(Kx)(\omega) \geq x(\omega_1)\chi_B(\omega_0)$ for every $\omega \in \Omega$. Since the function $x(\omega_1)\chi_B(\omega_0)$ is equimeasurable with the function $\sigma_{1/2}x^*$, the assertion follows immediately. $\square$

Proposition 9. *Suppose that the operator K maps boundedly X into Y, where X and Y are quasi-Banach symmetric spaces on $(0,1)$. If $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, is a sequence of independent functions from X and if $\sum_{k=1}^n m(\operatorname{supp}(x_k)) \leq 1$, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \leq 2C(Y)\|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^n x_k\Big\|_X.$$

Proof. Without loss of generality, it may be assumed that $x_k \geq 0$, $1 \leq k \leq n$. Let $y_k \in S(0,1)$ be pairwise disjoint copies of x_k, $1 \leq k \leq n$. By Theorem 4, the sequence $\{Ky_k\}_{k=1}^n$ consists of independent functions. Observing that $K(\bigoplus_{k=1}^n x_k)$ is equimeasurable with $\sum_{k=1}^n Ky_k$, and the latter is equimeasurable with the function $\sum_{k=1}^n (Kx_k)^*(\omega_k)$, we arrive at

$$\Big\|\sum_{k=1}^n (Kx_k)^*(\omega_k)\Big\|_Y = \Big\|\sum_{k=1}^n Ky_k\Big\|_Y \leq \|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^n x_k\Big\|_X.$$

By Lemma 8, we have

$$\sum_{k=1}^n (\sigma_{1/2}x_k^*)(\omega_k) \leq \sum_{k=1}^n (Kx_k)^*(\omega_k),$$

and, therefore,

$$\Big\|\sum_{k=1}^n (\sigma_{1/2}x_k^*)(\omega_k)\Big\|_Y \leq \|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^n x_k\Big\|_X. \tag{5}$$

For an arbitrary $k \in \mathbb{N}$, let $x_k^{(1)}$ and $x_k^{(2)}$ be disjointly supported elements of $S(0,1)$ equimeasurable with the function $\sigma_{1/2}x_k^*$. A moment's reflection shows that the sum $x_k^{(1)} + x_k^{(2)}$ is equimeasurable with the function x_k^*, $k \in \mathbb{N}$. Hence, the function $\sum_{k=1}^{n} x_k$ is equimeasurable with the sum $y_0 + y_1$, where

$$y_i(\omega) := \sum_{k=1}^{n} x_k^{(i)}(\omega_k), \quad i = 0, 1,$$

which immediately implies

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_Y = \|y_0+y_1\|_Y \le C(Y)(\|y_0\|_Y+\|y_1\|_Y) \le 2C(Y)\Big\|\sum_{k=1}^{n} \sigma_{1/2} x_k^*(\omega_k)\Big\|_Y.$$

The assertion follows now from inequality (5). □

Our next objective is to omit the assumption $\sum_{k=1}^{n} m(\operatorname{supp}(x_k)) \le 1$. The main step is a disjointification inequality for bounded functions obtained below in Proposition 14. Let us start with some technical lemmas.

Lemma 10. *Let*

$$s_k := \sum_{n=k}^{\infty} \frac{1}{e \cdot n!}, \quad k \in \mathbb{N}.$$

Then $4ks_{k+1} \ge s_k$ *for every* $k \in \mathbb{N}$.

Proof. Clearly,

$$4ks_{k+1} \ge \frac{(k+1)^2}{k} s_{k+1} \ge \frac{(k+1)^2}{k} \cdot \frac{1}{e \cdot (k+1)!} = \frac{k+1}{k} \cdot \frac{1}{e \cdot k!}.$$

On the other hand, since $k! \cdot (k+1)^n \le (k+n)!$, we have that

$$\frac{k+1}{k} \cdot \frac{1}{e \cdot k!} = \frac{1}{e \cdot k!} \cdot \frac{1}{1 - 1/(k+1)}$$
$$= \frac{1}{e \cdot k!}\left(1 + \frac{1}{k+1} + \frac{1}{(k+1)^2} + \cdots\right) \ge \sum_{n=k}^{\infty} \frac{1}{e \cdot n!}. \qquad \square$$

By the definition of the Kruglov operator, the function $K\chi_{[0,1]}$ has the Poisson distribution with parameter 1. Let

$$\psi_0(t) := \int_0^t (K\chi_{[0,1]})^*(s)\, ds.$$

It is clear that $K : L_\infty(0,1) \to M_{\psi_0}$ and $\|K\|_{L_\infty \to M_{\psi_0}} = 1$. Here M_{ψ_0} is the Marcinkiewicz space consisting of all elements $x \in S(0,1)$ such that

$$\|x\|_{M_{\psi_0}} := \sup_{0<t\le 1} \frac{\int_0^t x^*(s)\, ds}{\psi_0(t)} < \infty.$$

Lemma 11. *The following inequality holds*:

$$\inf_{0<t<1-1/e} \frac{t\psi_0'(t)}{\psi_0(t)} \ge \frac{1}{4}.$$

Proof. Let s_k be as in Lemma 10. Since $\psi_0' = (K\chi_{[0,1]})^*$ is a Poisson random variable with parameter 1, it follows that

$$\psi_0'(t) = k \quad \text{for all } t \in (s_{k+1}, s_k),\ k \in \mathbb{N}.$$

Therefore,

$$\psi_0(s_{k+1}) = \int_0^{s_{k+1}} \psi_0'(t)\, dt = \sum_{n=k+1}^{\infty} \frac{n}{e \cdot n!} = \sum_{n=k}^{\infty} \frac{1}{e \cdot n!} = s_k, \quad k \in \mathbb{N}.$$

Now, let $0 < t < 1 - 1/e$. Then $t \in [s_{k+1}, s_k)$ for some $k \geq 1$, and so $\psi_0'(t) = k$. Since ψ_0 is concave, the function $t/\psi_0(t)$ increases. Therefore, by Lemma 10,

$$\frac{t\psi_0'(t)}{\psi_0(t)} = \frac{kt}{\psi_0(t)} \geq \frac{ks_{k+1}}{\psi_0(s_{k+1})} = \frac{ks_{k+1}}{s_k} \geq \frac{1}{4}. \qquad \square$$

Lemma 12. *If Y is a quasi-Banach symmetric space on $(0,1)$ such that the operator K maps $L_\infty(0,1)$ into Y, then $Y \supset M_{\psi_0}$ and*

$$\|x\|_Y \leq 8\, C(Y)\, \|x\|_{M_{\psi_0}} \cdot \|K\|_{L_\infty \to Y}, \quad x \in M_{\psi_0}.$$

Proof. It follows from Lemma 11 that

$$\begin{aligned}\|x\|_{M_{\psi_0}} &= \sup_{0<t\leq 1} \left(\frac{1}{\psi_0(t)} \int_0^t x^*(s)\, ds \right) \geq \sup_{0<t<1/2} \left(\frac{tx^*(t)}{\psi_0(t)} \right) \\ &\geq \inf_{0<t<1/2} \left(\frac{t\psi_0'(t)}{\psi_0(t)} \right) \cdot \sup_{0<t<1/2} \left(\frac{x^*(t)}{\psi_0'(t)} \right) \geq \frac{1}{4} \sup_{0<t<1/2} \left(\frac{x^*(t)}{\psi_0'(t)} \right).\end{aligned}$$

Therefore,

$$x^*(t) \leq 4\|x\|_{M_{\psi_0}} \psi_0'(t), \quad 0 < t \leq \tfrac{1}{2},$$

whence

$$x^*(t) \leq \sigma_2 x^*(t) \leq 4\|x\|_{M_{\psi_0}} \sigma_2 \psi_0'(t), \quad 0 < t \leq 1.$$

Combining the last inequality with the obvious equalities

$$\|K\|_{L_\infty \to Y} = \|K\chi_{[0,1]}\|_Y = \|\psi_0'\|_Y,$$

we obtain

$$\|x\|_Y \leq \|\sigma_2 x^*\|_Y \leq 4\|x\|_{M_{\psi_0}} \|\sigma_2 \psi_0'\|_Y \leq 8\, C(Y) \|x\|_{M_{\psi_0}} \|K\|_{L_\infty \to Y}. \qquad \square$$

In the following lemma, we use the classical notion of majorization. Let $0 \leq x, y \in L_1(0,1)$. We write $y \prec x$ if $\int_0^t y^*(s)\, ds \leq \int_0^t x^*(s)\, ds$ for all $t \in (0,1)$ and $\int_0^1 y^*(s)\, ds = \int_0^1 x^*(s)\, ds$.

Lemma 13. *Let $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^n$, $n \in \mathbb{N}$, be sequences of positive and independent functions from $L_1(0,1)$. If $y_k \prec x_k$ for each k, then*

$$\sum_{k=1}^n y_k \prec \sum_{k=1}^n x_k.$$

Proof. Define the functions $x, y \in L_1(0,1)$ by setting

$$x(\omega) := \sum_{k=1}^n x_k(\omega_k), \quad y(\omega) := \sum_{k=1}^n y_k(\omega_k).$$

It follows from the assumption that for every $1 \le k \le n$ there exists a bistochastic operator A_k (on $L_1(0,1)$) such that $A_k x_k = y_k$ [Bennett and Sharpley 1988, Proposition 3.2.9]. A moment's reflection shows that the operator $A := \bigotimes_{k=1}^n A_k$ is a bistochastic operator on $L_1(\Omega, \mathbb{P})$ (which we identify with $L_1(0,1)$) and that $Ax = \sum_{k=1}^n A_k x_k(\omega_k)$. Applying Proposition 3.2.4 of the same reference, we arrive at

$$y = \sum_{k=1}^n A_k x_k(\omega_k) = Ax \prec x.$$

Since $\sum_{k=1}^n x_k$ (respectively, $\sum_{k=1}^n y_k$) is equimeasurable with x (respectively, y), the assertion follows. □

Proposition 14. *If $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, is a sequence of bounded independent functions, then*

$$\Big\| \sum_{k=1}^n x_k \Big\|_{M_{\psi_0}} \le 2 \Big\| \bigoplus_{k=1}^n x_k \Big\|_{L_1 \cap L_\infty(0,\infty)}.$$

Proof. Without loss of generality, we can assume that $x_k \ge 0$ for $1 \le k \le n$. Suppose that

$$\Big\| \bigoplus_{k=1}^n x_k \Big\|_\infty = 1 \quad \text{and} \quad \|x_k\|_1 = \alpha_k.$$

If $\alpha = \sum_{k=1}^n \alpha_k > 1$, then $x_k \prec \alpha \chi_{[0,\alpha^{-1}\alpha_k]}$ for $1 \le k \le n$. Applying Lemma 13, we obtain

$$\sum_{k=1}^n x_k \prec \alpha \sum_{k=1}^n \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k).$$

From the definition of the norm of a Marcinkiewicz space, Proposition 9 and the equalities $\|K\|_{L_\infty \to M_{\psi_0}} = 1$ and $C(M_{\psi_0}) = 1$, we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{n} x_k\Big\|_{M_{\psi_0}} &\le \alpha\Big\|\sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k)\Big\|_{M_{\psi_0}}\\ &\le 2\alpha\Big\|\bigoplus_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}\Big\|_{\infty} = 2\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_{L_1(0,\infty)}. \qquad (6)\end{aligned}$$

If $\alpha=\sum_{k=1}^{n}\alpha_k<1$, then $x_k \prec \chi_{[0,\alpha_k]}$ for $1\le k\le n$. It follows from Lemma 13 that

$$\sum_{k=1}^{n} x_k \prec \sum_{k=1}^{n} \chi_{[0,\alpha_k]}(\omega_k).$$

Therefore, by Proposition 9, we have

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_{M_{\psi_0}} \le \Big\|\sum_{k=1}^{n} \chi_{[0,\alpha_k]}(\omega_k)\Big\|_{M_{\psi_0}} \le 2\Big\|\bigoplus_{k=1}^{n} \chi_{[0,\alpha_k]}\Big\|_{\infty} = 2.$$

Combining this estimate with inequality (6), we are done. □

The following statement is an immediate consequence of Proposition 14 and Lemma 12.

Corollary 15. *Let Y be a quasi-Banach symmetric space on $(0,1)$ such that the operator K maps $L_\infty(0,1)$ into Y. If $\{x_k\}_{k=1}^n$, $n\in\mathbb{N}$, is a sequence of bounded and independent functions, then*

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_{Y} \le 16\,C(Y)\,\|K\|_{L_\infty\to Y}\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_{L_1\cap L_\infty(0,\infty)}.$$

Now, we are ready to prove the main result of this section related to the comparison of sums of independent functions and their disjoint copies in quasi-Banach symmetric function spaces.

Theorem 16. *Let X and Y be quasi-Banach symmetric spaces on $(0,1)$ such that the operator K acts boundedly from X into Y. If $\{x_k\}_{k=1}^n\subset X$, $n\in\mathbb{N}$ is a sequence of independent functions, then*

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_{Y} \le 16\,C^2(Y)\,\|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_{Z_X^1}. \qquad (7)$$

Proof. Let us write x for $\bigoplus_{k=1}^{n} x_k$. Define the functions

$$x_{k,1}:=x_k\chi_{\{|x_k|>x^*(1)\}},\quad x_{k,2}:=x_k-x_{k,1},\quad 1\le k\le n.$$

The functions $x_{k,1}$, $1\le k\le n$, are independent, as are the functions $x_{k,2}$, $1\le k\le n$.

Moreover, it is easy to see that

$$\bigoplus_{k=1}^{n} |x_{k,1}| \sim x^*\chi_{(0,1)} \quad \text{and} \quad \bigoplus_{k=1}^{n} |x_{k,2}| \sim x^*\chi_{(1,\infty)}.$$

Since $L_\infty(0,1) \subset X$ and $\|x\|_X \le \|x\|_\infty$, $x \in L_\infty(0,1)$, it follows from the assumption of the theorem that $K : L_\infty(0,1) \to Y$ and $\|K\|_{L_\infty \to Y} \le \|K\|_{X\to Y}$. Therefore, applying Proposition 9 and Corollary 15, we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{n} x_k\Big\|_Y &\le C(Y)\bigg(\Big\|\sum_{k=1}^{n} x_{k,1}\Big\|_Y + \Big\|\sum_{k=1}^{n} x_{k,2}\Big\|_Y\bigg)\\ &\le 16\, C^2(Y)\|K\|_{X\to Y}\bigg(\Big\|\bigoplus_{k=1}^{n} x_{k,1}\Big\|_X + \Big\|\bigoplus_{k=1}^{n} x_{k,2}\Big\|_{L_1\cap L_\infty(0,\infty)}\bigg)\\ &\le 16\, C^2(Y)\|K\|_{X\to Y}(\|x^*\chi_{(0,1)}\|_X + \|\min\{x^*, x^*(1)\}\|_{L_1\cap L_\infty(0,\infty)}).\end{aligned}$$

□

4. Disjointification inequalities for symmetrically distributed (mean zero) functions

If we assume that the independent functions x_k, $1 \le k \le n$, in the statement of Theorem 16 are symmetrically distributed, then the disjointification inequality (7) can be significantly improved. In particular, we are able to extend estimates from [Astashkin and Sukochev 2007] for symmetric Banach function spaces to the quasi-Banach setting. Our main tool is the following remarkable inequality due to Prokhorov [1959], which we restate here using the direct sum notation.

Theorem 17. *If* $\{x_k\}_{k=1}^n$ $(n \in \mathbb{N})$ *is a sequence of bounded independent symmetrically distributed random variables on* $(0,1)$, *then for all* $t > 0$

$$m\bigg(\Big\{\sum_{k=1}^{n} x_k > t\Big\}\bigg) \le \exp\bigg(-\frac{t}{2\,\|\bigoplus_{k=1}^n x_k\|_\infty}\,\operatorname{arcsinh}\frac{t\,\|\bigoplus_{k=1}^n x_k\|_\infty}{2\,\|\bigoplus_{k=1}^n x_k\|_2^2}\bigg). \tag{8}$$

Let the function ψ_0 be as in the previous section.

Proposition 18. *If* $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, *is a sequence of bounded independent symmetrically distributed functions on* $(0,1)$, *then*

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_{M_{\psi_0}} \le C_{\text{abs}}\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_{L_2\cap L_\infty(0,\infty)},$$

for some absolute constant C_{abs}.

Proof. For every $m \geq 1$, we define a linear operator $A_m : L_2 \cap L_\infty(0,\infty) \to M_{\psi_0}$ by setting for $x \in L_2 \cap L_\infty(0,\infty)$

$$A_m x(\omega) := \sum_{k=1}^{m} x(k-1+\omega_{2k-1}) r(\omega_{2k}),$$

where $r(t) = 1$ if $0 \leq t \leq \frac{1}{2}$ and $r(t) = -1$ if $\frac{1}{2} < t \leq 1$. It is clear that

$$\|A_m\|_{L_2 \cap L_\infty \to M_{\psi_0}} \leq m, \quad m \in \mathbb{N}.$$

Our objective is to show that for every fixed $x \in L_2 \cap L_\infty(0,\infty)$ the orbit $\{A_m x\}_{m=1}^\infty$ is uniformly bounded in M_{ψ_0}. Provided we have done so, the uniform boundedness principle guarantees that the sequence $\{\|A_m\|_{L_2\cap L_\infty \to M_\psi}\}_{m=1}^\infty$ is uniformly bounded, and the assertion of the theorem would follow from this fact since the sum $\sum_{k=1}^n x_k$ for a given sequence $\{x_k\}_{k=1}^n$ of bounded independent symmetrically distributed functions on $(0,1)$ is equidistributed with the function $A_n z$, where

$$z := \bigoplus_{k=1}^{n} x_k.$$

Fix $x \in L_2 \cap L_\infty(0,\infty)$, and set

$$\alpha(x) := \|x\|_\infty + \sup_n \frac{\|x\chi_{[0,n]}\|_2^2}{\|x\chi_{[0,n]}\|_\infty}$$

(here, $0/0$ is set to be 0). Clearly, $\alpha(x) < \infty$ and our objective would be achieved if we show that

$$\|A_m x\|_{M_{\psi_0}} \leq 4e \cdot \alpha(x) \quad \text{for all } m \in \mathbb{N}. \tag{9}$$

Fix $m \in \mathbb{N}$. Since

$$\Big(\bigoplus_{k=1}^{m} x(k-1+\omega_{2k-1}) r(\omega_{2k})\Big)^* = (x\chi_{[0,m]})^*,$$

it follows from (8) that for every $t > 0$, we have

$$m(\{|A_m x| > t\alpha(x)\}) \leq \exp\Big(-\frac{t\alpha(x)}{2\|x\chi_{[0,m]}\|_\infty} \operatorname{arcsinh} \frac{t\alpha(x)\|x\chi_{[0,m]}\|_\infty}{2\|x\chi_{[0,m]}\|_2^2}\Big).$$

Combining this estimate with the obvious inequalities

$$\frac{t\alpha(x)}{2\|x\chi_{[0,m]}\|_\infty} \geq \frac{t}{2}, \quad \operatorname{arcsinh} \frac{t\alpha(x)\|x\chi_{[0,m]}\|_\infty}{2\|x\chi_{[0,m]}\|_2^2} \geq \operatorname{arcsinh} \frac{t}{2},$$

we arrive at

$$m(\{|A_m x| > t\alpha(x)\}) \leq \exp\Big(-\frac{t}{2} \operatorname{arcsinh} \frac{t}{2}\Big). \tag{10}$$

The right-hand side of the preceding inequality is in fact directly related to the distribution function of the function ψ_0'. Indeed, in the proof of Lemma 11 we have already pointed out that $\psi_0' := (K\chi_{[0,1]})^*$ is a Poisson random variable with parameter 1. A direct calculation yields the estimate

$$m(\{\psi_0' > t\}) \geq \exp(-1 - 2t \cdot \operatorname{arcsinh}(2t)), \quad t > 0,$$

which, in turn, implies

$$m(\{4\psi_0' > t\}) \geq \exp\Big(-1 - \frac{t}{2} \operatorname{arcsinh} \frac{t}{2}\Big), \quad t > 0.$$

Combining this with (10), we infer

$$m(\{|A_m x| > t\alpha(x)\}) \leq e \cdot m(\{4\psi_0' > t\}).$$

Since $\|\psi_0'\|_{M(\psi_0)} = 1$, from the preceding estimate and [Braverman 1994, Proposition 1.2], inequality (9) follows. $\square$

The corollary below follows from Proposition 18 and Lemma 12.

Corollary 19. *Let Y be a quasi-Banach symmetric space on $(0,1)$ such that the operator K maps $L_\infty(0,1)$ into Y. If $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, is a sequence of bounded independent symmetrically distributed functions, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \leq 8\, C_{\mathrm{abs}}\, C(Y)\, \|K\|_{L_\infty \to Y} \Big\|\bigoplus_{k=1}^n x_k\Big\|_{L_2 \cap L_\infty(0,\infty)}.$$

We need the following assertion proved by Braverman [1994, Proposition 1.11] in the Banach setting. The proof in the quasi-Banach setting is identical.

Lemma 20. *If a quasi-Banach symmetric space X on $(0,1)$ embeds into $L_1(0,1)$, then there exists a constant $C_0(X)$ such that*

$$\|x\|_X \leq C_0(X)\, \|x(\omega_1) - x(\omega_2)\|_X$$

for every mean zero function $x \in X$.

We are now ready to present the main result of this section.

Theorem 21. *Let X and Y be quasi-Banach symmetric spaces on $(0,1)$ such that $K : X \to Y$.*

(a) *If $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, is a sequence of independent symmetrically distributed functions, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \leq 8\, C_{\mathrm{abs}}\, C^2(Y)\, \|K\|_{X \to Y} \Big\|\bigoplus_{k=1}^n x_k\Big\|_{Z_X^2}. \tag{11}$$

(b) *If $X \subset L_1(0,1)$, then the inequality*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \le 16\, C_{\mathrm{abs}}\, C_0(Y)\, C^2(Y)\, C(X)\, \|K\|_{X\to Y} \Big\|\bigoplus_{k=1}^n x_k\Big\|_{Z_X^2} \tag{12}$$

holds for every sequence $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, of independent mean zero functions from X.

Proof. The proof of the first assertion is similar to the proof of Theorem 16, with the only difference being that the reference to Corollary 15 should be replaced with a reference to Corollary 19.

In the proof of the second assertion we use the standard symmetrization trick. Define the functions $y_k \in X$, $1 \le k \le n$, by setting

$$y_k(\omega) := x_k(\omega_{2k-1}) - x_k(\omega_{2k}).$$

By Lemma 20,

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \le C_0(Y) \Big\|\sum_{k=1}^n x_k(\omega_{2k-1}) - \sum_{k=1}^n x_k(\omega_{2k})\Big\|_Y = C_0(Y) \Big\|\sum_{k=1}^n y_k\Big\|_Y.$$

Evidently, y_k, $1 \le k \le n$, are independent and symmetrically distributed. Therefore, by (a), we obtain

$$\Big\|\sum_{k=1}^n y_k\Big\|_Y \le 8\, C_{\mathrm{abs}}\, C^2(Y)\, \|K\|_{X\to Y} \Big\|\bigoplus_{k=1}^n y_k\Big\|_{Z_X^2}.$$

Observing that for every $t > 0$, we have

$$m\bigg(\bigg\{\Big|\bigoplus_{k=1}^n y_k\Big| > t\bigg\}\bigg) \le 2m\bigg(\bigg\{s > 0 : \Big|\bigoplus_{k=1}^n x_k\Big| > t\bigg\}\bigg),$$

and appealing to the fact that Z_X^2 is a quasi-Banach symmetric space with modulus of concavity $C(X)$, we infer

$$\Big\|\bigoplus_{k=1}^n y_k\Big\|_{Z_X^2} \le 2\, C(X) \Big\|\bigoplus_{k=1}^n x_k\Big\|_{Z_X^2}.$$

Combining these inequalities, we conclude the proof. □

5. Khintchine inequality in quasi-Banach spaces

In this section, we provide an extension of the classical Khintchine inequality to general quasi-Banach symmetric function spaces. We begin with the formulation of the main results of this section.

Theorem 22. *Let X and Y be quasi-Banach symmetric spaces on the interval* $(0,1)$ *such that the operator K is bounded from X into Y. If $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, is a sequence of independent symmetrically distributed random variables from X, then*

$$\Big\| \sum_{k=1}^n x_k \Big\|_Y \le 512\, C_{\mathrm{abs}}\, C^6(X)\, C^2(Y)\, \|K\|_{X\to Y} \Big\| \Big(\sum_{k=1}^n x_k^2 \Big)^{1/2} \Big\|_X. \tag{13}$$

The next theorem shows that in the case when $X = Y$ the boundedness of the Kruglov operator is a necessary and sufficient condition for the inequalities of the type (13). In the Banach setting, an analogous result was earlier proved in [Astashkin 2008].

Theorem 23. *Let X be a quasi-Banach symmetric function space on* $(0,1)$ *which is separable or has the Fatou property. The following conditions are equivalent*:

(a) *There is a constant $C > 0$ such that the inequality*

$$\Big\| \sum_{k=1}^n x_k \Big\|_X \le C \Big\| \Big(\sum_{k=1}^n x_k^2 \Big)^{1/2} \Big\|_X$$

holds for every sequence $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, of independent symmetrically distributed functions.

(b) $K : X \to X$.

For the proof we will need a series of lemmas. The first two of them are well known; however, we present their short proofs for the reader's convenience.

Lemma 24. *Let X be a quasi-Banach symmetric space on* $(0,1)$. *If we set $p := \frac{1}{2}\log_2^{-1}(2\,C(X))$, then $X \subset L_p(0,1)$ and*

$$\|x\|_p \le 8\, C^3(X)\, \|x\|_X, \quad x \in X.$$

Proof. Define an increasing function ψ on $(0,1)$ by the formula $\psi(u) := \|\chi_{[0,u]}\|_X$, $0 < u < 1$. It follows from the definition of a quasinorm that

$$\psi(2u) \le 2\, C(X) \psi(u), \quad 0 < u \le 1,$$

whence

$$\psi(2^{-n}) \ge (2\, C(X))^{-n}, \quad n \ge 0.$$

If $u \in (0,1]$ is arbitrary, then $u \in [2^{-n-1}, 2^{-n}]$ for some $n \ge 0$. Hence,

$$\psi(u) \ge \psi(2^{-n-1}) \ge 2^{-(n+1)\log_2(2C(X))} \ge \frac{1}{2\, C(X)}\, u^{\log_2(2C(X))}.$$

If $x \in X$, then for every $0 < t \le 1$ we have

$$\|x\|_X \ge \|x^*(t)\chi_{[0,t]}\|_X \ge x^*(t) \frac{1}{2\, C(X)}\, t^{\log_2(2C(X))}.$$

Hence,

$$x^*(t) \le 2\,\|x\|_X\, C(X)\, t^{-\log_2(2C(X))}, \quad 0 < t \le 1.$$

The assertion follows immediately. □

Lemma 25. *If* $0 < p < 1$ *and* $x, y \in L_1(0,1)$ *are positive, then from* $y \prec x$ *it follows that* $\|y\|_p \ge \|x\|_p$.

Proof. Fix $\varepsilon > 0$. Passing to step-function approximation, we easily infer that there exist $n \in \mathbb{N}$ and a function

$$z := \sum_{k=1}^{n} \lambda_k x_k \qquad \text{with} \quad x_k \ge 0,\ x_k^* = x^* \ \text{and} \ \sum_{k=1}^{n} \lambda_k = 1,\ \lambda_k \ge 0,$$

such that $\|y - z\|_1 \le \varepsilon$. It follows now from the Minkowski inequality that

$$\|z\|_p = \Big\|\sum_{k=1}^{n} \lambda_k x_k\Big\|_p \ge \sum_{k=1}^{n} \lambda_k \|x_k\|_p = \|x\|_p.$$

Since $\varepsilon > 0$ is arbitrarily small and the quasinorm in $L_p(0,1)$, $0 < p < 1$, is continuous with respect to L_1-convergence, the proof is complete. □

Lemma 26. *Let* $0 < p < 1$ *and let* $\{y_k\}_{k=1}^n$, $n \in \mathbb{N}$, *be a sequence of positive bounded independent functions on* $(0,1)$. *We have*

$$\Big\|\bigoplus_{k=1}^{n} y_k\Big\|_1 \le 2^{1/p} \max\Big\{\sup_{1\le k\le n} \|y_k\|_\infty, \Big\|\sum_{k=1}^{n} y_k\Big\|_p\Big\}. \tag{14}$$

Proof. Without loss of generality, we can assume that

$$\sup_{1\le k\le n} \|y_k\|_\infty = 1, \quad \|y_k\|_1 = \alpha_k, \quad 1 \le k \le n.$$

Let $\alpha = \sum_{k=1}^{n} \alpha_k$. If $\alpha \le 1$, then the assertion is evident. If $\alpha \ge 1$, then

$$y_k \prec \alpha \chi_{[0,\alpha^{-1}\alpha_k]}, \quad 1 \le k \le n.$$

From Lemma 13 it follows that

$$\sum_{k=1}^{n} y_k \prec \alpha \sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k),$$

whence, according to Lemma 25, we have

$$\Big\|\sum_{k=1}^{n} y_k\Big\|_p \ge \alpha \Big\|\sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k)\Big\|_p.$$

Combining this inequality with [Johnson and Schechtman 1989, Lemma 3], we infer

$$2\,C(p)\Big\|\sum_{k=1}^{n} y_k\Big\|_p \geq \alpha\Big\|\bigoplus_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}\Big\|_p = \alpha.$$

Since $C(p) = 2^{1/p-1}$ for $0 < p < 1$, the assertion follows. □

Lemma 27. *Let X be a quasi-Banach symmetric space on* $(0,1)$. *If* $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, *is a sequence of bounded independent functions*, *then*

$$\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_2 \leq 32\,C^5(X)\max\Big\{\sup_{1\leq k\leq n}\|x_k\|_\infty,\ \Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X\Big\}.$$

Proof. If $p = \frac{1}{2}\log_2^{-1}(2\,C(X))$, then by Lemma 24 we have

$$8\,C^3(X)\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X \geq \Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_p = \Big\|\sum_{k=1}^{n} x_k^2\Big\|_{p/2}^{1/2}.$$

Clearly,

$$\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_2 = \Big\|\bigoplus_{k=1}^{n} x_k^2\Big\|_1^{1/2} \quad\text{and}\quad \|x_k\|_\infty = \|x_k^2\|_\infty^{1/2}.$$

Now, applying Lemma 26 to the functions $y_k = x_k^2$, $1 \leq k \leq n$, we obtain the result. □

Lemma 28. *Let X be a quasi-Banach symmetric space on* $(0,1)$. *If* $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, *is a sequence of independent functions and if* $x := \bigoplus_{k=1}^n x_k$, *then*

$$2C(X)\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X \geq x^*(1).$$

Proof. A simple argument shows that it is sufficient to consider the case when

$$\sum_{k=1}^{n} m(\mathrm{supp}(x_k)) = 1. \tag{15}$$

Since $|x_k| \geq x^*(1)\chi_{\mathrm{supp}(x_k)}$, $1 \leq k \leq n$, we have

$$\sum_{k=1}^{n} x_k^2 \geq (x^*(1))^2\sum_{k=1}^{n}\chi_{\mathrm{supp}(x_k)}.$$

Since the functions x_k, $1 \le k \le n$, are independent, the support of the function at the right-hand side of the inequality above has Lebesgue measure equal to

$$1 - \prod_{k=1}^{n} \big(1 - m(\mathrm{supp}(x_k))\big),$$

which is bigger than $\frac{1}{2}$ (thanks to the condition (15) and to the arithmetic-geometric mean inequality). Therefore, since $\|\chi_{(0,1)}\|_X = 1$, we obtain

$$\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X \ge x^*(1)\,\|\chi_{[0,1/2]}\|_X \ge \frac{x^*(1)}{2\,C(X)},$$

and the proof is complete. □

Let $1 \le p < \infty$ and let X be a quasi-Banach symmetric function space on $(0,1)$ or $(0,\infty)$. The *p-concavification* of X, $X^{1/p}$, is defined by

$$X^{1/p} := \{x \in S(0,1) \text{ (or } S(0,\infty)) : |x|^{1/p} \in X\}, \quad \|x\|_{X^{1/p}} := \|\,|x|^{1/p}\,\|_X^p.$$

Note that the space $X^{1/p}$, equipped with the quasinorm $\|\cdot\|_{X^{1/p}}$, is also a quasi-Banach symmetric function space (see, for instance, [Lindenstrauss and Tzafriri 1979]).

We are now ready to prove the main result of this section.

Proof of Theorem 22. Setting $x := \bigoplus_{k=1}^{n} x_k$, by Theorem 21, we have

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_Y \le 8\,C_{\mathrm{abs}}\,C^2(Y)\,\|K\|_{X\to Y}(\|x^*\chi_{(0,1)}\|_X + \|x^*\chi_{(1,\infty)}\|_2). \tag{16}$$

Arguing in the same way as in the proof of Theorem 16, we can define two sequences of independent functions $\{x_{k,1}\}$ and $\{x_{k,2}\}$ such that $x_{1k} + x_{2k} = x_k$, $|x_{k,1}| \le |x_k|$, $|x_{k,2}| \le |x_k|$, for $1 \le k \le n$, and the disjoint sums $\bigoplus_{k=1}^{n} |x_{k,1}|$ and $\bigoplus_{k=1}^{n} |x_{k,2}|$ are equimeasurable with the functions $x^*\chi_{(0,1)}$ and $x^*\chi_{(1,\infty)}$, respectively. Applying Lemma 27 to the sequence $\{x_{k,2}\}_{k=1}^{n}$, we obtain

$$\begin{aligned}\|x^*\chi_{(1,\infty)}\|_2 &= \Big\|\bigoplus_{k=1}^{n} x_{k,2}\Big\|_2 \\ &\le 32\,C^5(X)\max\Big\{\sup_{1\le k\le n}\|x_{k,2}\|_\infty,\ \Big\|\Big(\sum_{k=1}^{n} x_{k,2}^2\Big)^{1/2}\Big\|_X\Big\}.\end{aligned}$$

Note that $\|x_{k,2}\|_\infty \le x^*(1)$ for $1 \le k \le n$. Using Lemma 28, we obtain

$$\|x^*\chi_{(1,\infty)}\|_2 \le 64\,C^6(X)\,\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X. \tag{17}$$

On the other hand,

$$\|x^*\chi_{(0,1)}\|_X = \Big\|\bigoplus_{k=1}^n x_{k,1}\Big\|_X = \Big\|\bigoplus_{k=1}^n x_{k,1}^2\Big\|_{X^{1/2}}^{1/2},$$

and

$$\Big\|\Big(\sum_{k=1}^n x_k^2\Big)^{1/2}\Big\|_X \ge \Big\|\Big(\sum_{k=1}^n x_{k,1}^2\Big)^{1/2}\Big\|_X = \Big\|\sum_{k=1}^n x_{k,1}^2\Big\|_{X^{1/2}}^{1/2}.$$

Applying [Johnson and Schechtman 1989, Lemma 3] to the space $X^{1/2}$ and the functions $x_{k,1}^2$, we obtain

$$\|x^*\chi_{(0,1)}\|_X \le (2\,C(X^{1/2}))^{1/2}\Big\|\Big(\sum_{k=1}^n x_k^2\Big)^{1/2}\Big\|_X.$$

Since $C(X^{1/2}) \le 4C^2(X)$, the assertion follows now from the last inequality and inequalities (16) and (17). □

Lemma 29. *Let $x \in S(0,1)$, $x \ge 0$, and let $n \in \mathbb{N}$. If x_k, $k = 1, 2, \ldots, 2n$, are independent copies of the function $\sigma_{1/n}x$, then for all sufficiently large $n \in \mathbb{N}$ we have*

$$\Big(\sum_{k=1}^n x_{2k}\Big)^* \le \sigma_3\Big(\sum_{k=1}^{2n}(-1)^k x_k\Big)^*.$$

Proof. It is clear that the functions $x_{2k-1} - x_{2k}$, $1 \le k \le n$, are independent. Therefore,

$$\begin{aligned} m\Big(\Big\{\sum_{k=1}^n x_{2k} - x_{2k-1} > t\Big\}\Big) &\ge m\Big(\Big\{\sum_{k=1}^n x_{2k} > t,\ \sum_{k=1}^n x_{2k-1} = 0\Big\}\Big) \\ &= m\Big(\Big\{\sum_{k=1}^n x_{2k} > t\Big\}\Big)\cdot m\Big(\Big\{\sum_{k=1}^n x_{2k-1} = 0\Big\}\Big) \\ &= \Big(1-\frac{1}{n}\Big)^n m\Big(\Big\{\sum_{k=1}^n x_{2k} > t\Big\}\Big). \end{aligned}$$

Hence, for all sufficiently large $n \in \mathbb{N}$,

$$m\Big(\Big\{\Big|\sum_{k=1}^n x_{2k-1} - x_{2k}\Big| > t\Big\}\Big) \ge \frac{1}{3}\, m\Big(\Big\{\sum_{k=1}^n x_{2k} > t\Big\}\Big). \qquad \square$$

Proof of Theorem 23. We have to prove only the implication (a) $\Longrightarrow$ (b).

Let $x \in X$, $x \geq 0$, and $n \in \mathbb{N}$. Taking for x_k, $k = 1, 2, \dots, 2n$, independent copies of the function $\sigma_{1/n}x$, by Lemma 29 we have

$$\Big\|\sum_{k=1}^{n} x_{2k}\Big\|_X \leq \Big\|\sigma_3\Big(\sum_{k=1}^{2n}(-1)^k x_k\Big)\Big\|_X \leq 3\,C(X)^2\Big\|\sum_{k=1}^{2n}(-1)^k x_k\Big\|_X.$$

On the other hand, the functions $x_{2k-1} - x_{2k}$, $1 \leq k \leq n$, are independent and symmetrically distributed. Therefore, by the assumption, we have

$$\begin{aligned}\Big\|\sum_{k=1}^{2n}(-1)^k x_k\Big\|_X &\leq C\Big\|\Big(\sum_{k=1}^{n}(x_{2k-1} - x_{2k})^2\Big)^{1/2}\Big\|_X\\ &\leq C\Big\|\Big(\sum_{k=1}^{n}x_{2k-1}^2\Big)^{1/2} + \Big(\sum_{k=1}^{n}x_{2k}^2\Big)^{1/2}\Big\|_X\\ &\leq 2\,C\cdot C(X)\Big\|\Big(\sum_{k=1}^{n}x_{2k}^2\Big)^{1/2}\Big\|_X.\end{aligned}$$

Combining these inequalities, we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{n} x_{2k}\Big\| &\leq 6\,C\cdot C(X)^3\Big\|\Big(\sum_{k=1}^{n}x_{2k}^2\Big)^{1/2}\Big\|_X\\ &\leq 6\,C\cdot C(X)^3\Big\|\Big(\max_{1\leq k\leq n} x_{2k}\cdot\sum_{k=1}^{n}x_{2k}\Big)^{1/2}\Big\|_X.\end{aligned}$$

It follows now from Lemma 2 that

$$\Big\|\sum_{k=1}^{n} x_{2k}\Big\|_X \leq 6\,C\cdot C(X)^4\,\Big\|\max_{1\leq k\leq n} x_{2k}\Big\|_X^{1/2}\cdot\Big\|\sum_{k=1}^{n}x_{2k}\Big\|_X^{1/2}.$$

Hence,

$$\Big\|\sum_{k=1}^{n} x_{2k}\Big\|_X \leq 36\,C^2\,C(X)^8\Big\|\max_{1\leq k\leq n} x_{2k}\Big\|_X \leq 36\,C^2\,C(X)^8\Big\|\bigoplus_{k=1}^{n}x_{2k}\Big\|_X.$$

Appealing to the definition of x_k, $1 \leq k \leq 2n$, we obtain

$$\Big(\bigoplus_{k=1}^{n}x_{2k}\Big)^* = x^* \quad\text{and}\quad \Big(\sum_{k=1}^{n}x_{2k}\Big)^* = (H_n x)^*,$$

where the operator H_n is defined by (3).

Recall that, by Lemma 6, $(H_n x)^* \to (Kx)^*$ almost everywhere on $(0, 1)$. Therefore, if X has the Fatou property, it follows that $\|Kx\|_X \le 36C^2 C(X)^8 \|x\|_X$, and the proof in this case is complete. If X is separable, we can repeat almost verbatim the arguments used in the second part of the proof of Theorem 7. □

6. Von Bahr–Esseen type inequalities

We have the following remarkable theorem.

Theorem 30 [von Bahr and Esseen 1965, Theorem 2]. *If* $1 \le p \le 2$ *and* $\{f_k\}_{k=1}^n \subset L_p(0, 1)$, $n \in \mathbb{N}$, *is a sequence of independent mean zero functions*, *then*

$$\left\| \sum_{k=1}^n f_k \right\|_p \le \left(2 \sum_{k=1}^n \|f_k\|_p^p \right)^{1/p}. \tag{18}$$

In [Braverman 1994, § II,2], Theorem 30 is extended to Banach symmetric function spaces with the Kruglov property. Versions of disjointification inequalities obtained in Sections 3 and 4 for quasi-Banach symmetric spaces allow us to extend Braverman's result to the quasi-Banach setting. Moreover, we shall consider different quasinorms at the left- and right-hand sides of (18). Our proofs appear to be more straightforward (and simpler) than the proofs for the special case considered in [Braverman 1994].

Definition 31. Quasi-Banach symmetric function spaces X and Y (in this order) satisfy the von Bahr–Esseen r-estimate (written $(X, Y) \in (BE)_r$) if there exists a constant $B > 0$ such that

$$\left\| \sum_{k=1}^n f_k \right\|_Y \le B \left(\sum_{k=1}^n \|f_k\|_X^r \right)^{1/r} \tag{19}$$

for every sequence of independent symmetrically distributed functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$. If, in addition, $X = Y$, then we say that X satisfies the von Bahr–Esseen r-estimate (written $X \in (BE)_r$).

In view of this definition, we may restate Theorem 30 as $L_p(0, 1) \in (BE)_p$.

Remark 32. If $Y \subset L_1(0, 1)$, then an application of Lemma 20 yields the estimate (19) for all mean zero independent functions.

Clearly, $(X, Y) \in (BE)_r$ implies that $X \subset Y$. Taking Rademacher functions (see Section 1) as the f_k, it is easy to see that we always have $0 < r \le 2$. Finally, if X is p-normed, then $p \le r \le 2$.

Recall that a quasi-Banach lattice X satisfies an *upper r-estimate*, $r > 0$, if there is a constant $C > 0$ such that

$$\Big\| \sum_{k=1}^{n} x_k \Big\|_X \le C \Big(\sum_{k=1}^{n} \|x_k\|_X^r \Big)^{1/r}$$

for every sequence of mutually disjoint elements $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$.

Recall also that a quasi-Banach symmetric space $L_{r,\infty}$, $r > 0$, consists of all $x \in S(0,1)$ such that

$$\|x\|_{r,\infty} := \sup_{0<t\le 1} x^*(t)\, t^{1/r} < \infty.$$

Theorem 33. *Let $0 < r < 2$. For all quasi-Banach symmetric function spaces X and Y the following statements hold*:

(a) *If $K : X \to Y$ and X satisfies an upper r-estimate, then $(X, Y) \in (BE)_r$.*

(b) *If $K : Y \to Y$ and, for some $C > 0$ and for every sequence of mutually disjoint functions $\{f_k\}_{k=1}^n \subset X$ $(n \in \mathbb{N})$, we have*

$$\Big\| \sum_{k=1}^{n} f_k \Big\|_Y \le C \Big(\sum_{k=1}^{n} \|f_k\|_X^r \Big)^{1/r}, \tag{20}$$

then $(X, Y) \in (BE)_r$.

(c) *If $(X, Y) \in (BE)_r$, then* (20) *holds for every sequence of mutually disjoint functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$.*

The main part of the proof of Theorem 33 is given below in Lemma 36.

Let $0 < p < r < 2$ and let $r > 1$. Recall that $L_{r,\infty}$ satisfies an upper r-estimate (see, for example, [Braverman 1994, Theorem 1.12]) and that $K : L_{r,\infty} \to L_{r,\infty}$ by Theorem 1.3 of the same reference. Setting $X = L_{r,\infty}$ and $Y = L_p(0,1)$ and taking into account Remark 32, we obtain the well-known Esseen–Janson theorem (see [Esseen and Janson 1985, Theorem 4]). It is worth noting that, in contrast to the previous reference, we do not require that the functions f_k are equidistributed.

Lemma 34. *Let $r > 0$ and let X and Y be quasi-Banach symmetric function spaces. Suppose that there is a constant $C > 0$ such that for every sequence of mutually disjoint functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, inequality* (20) *holds. Then $X \subset L_{r,\infty}$.*

Proof. Fix $t \in (0,1]$ and let $n \in \mathbb{N}$ be such that $1/2 < nt \le 1$. Since $\chi_{(0,tn)} = \sum_{k=1}^n \chi_{(t(k-1),tk)}$, the functions $\varphi_X(t) := \|\chi_{(0,t)}\|_X$ and $\varphi_Y(t) := \|\chi_{(0,t)}\|_Y$ satisfy the estimate

$$\varphi_Y(tn) \le C \Big(\sum_{k=1}^{n} \|\chi_{(t(k-1),tk)}\|_X^r \Big)^{1/r} = C\, \varphi_X(t)\, n^{1/r},$$

by (20). Hence, we obtain that

$$\varphi_X(t) \geq C^{-1}\, \varphi_Y(tn)\, n^{-1/r} \geq C^{-1}\, \varphi_Y(1/2)\, t^{1/r} = C_1^{-1}\, t^{1/r},$$

whence for every $x \in X$

$$\|x\|_X \geq x^*(t)\, \|\chi_{(0,t)}\|_X = x^*(t)\varphi_X(t) \geq C_1^{-1}\, x^*(t)\, t^{1/r}, \quad 0 < t \leq 1.$$

Therefore, $\|x\|_{r,\infty} \leq C_1 \|x\|_X$ for all $x \in X$ and the proof is completed. □

Lemma 35. *Let X be a quasi-Banach symmetric function space on $(0,1)$ satisfying an upper r-estimate, $0 < r < 2$. There exists $C_X > 0$ such that for every sequence $\{x_k\}_{k=1}^{\infty} \subset X$ we have*

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z^2_{L_{r,\infty}}} \leq C_X \Big(\sum_{k=1}^{\infty} \|x_k\|_X^r \Big)^{1/r}.$$

Proof. By Lemma 34, we have $X \subset L_{r,\infty}$. Therefore, $x_k^* \leq \|x_k\|_{r,\infty}\, \xi_r$, where $\xi_r(t) = t^{-1/r}$, $0 < t \leq 1$, whence

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z^2_{L_{r,\infty}}} \leq C \Big\| \bigoplus_{k=1}^{\infty} \|x_k\|_{r,\infty}\, \xi_r \Big\|_{Z^2_{L_{r,\infty}}}.$$

Note that for any $a_k \geq 0$ we have

$$\bigoplus_{k=1}^{\infty} a_k \xi_r \sim \Big(\sum_{k=1}^{\infty} a_k^r \Big)^{1/r} \xi_r.$$

Hence,

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z^2_{L_{r,\infty}}} \leq C \Big(\sum_{k=1}^{\infty} \|x_k\|_{r,\infty}^r \Big)^{1/r} \|\xi_r\|_{Z^2_{L_{r,\infty}}} \leq C' \|\xi_r\|_{Z^2_{L_{r,\infty}}} \Big(\sum_{k=1}^{\infty} \|x_k\|_X^r \Big)^{1/r},$$

and the result follows. □

Lemma 36. *Let X be a quasi-Banach symmetric function space on $(0,1)$ satisfying an upper r-estimate, $0 < r < 2$. There exists a constant $B_X > 0$ such that for every sequence $\{x_k\}_{k=1}^{\infty} \subset X$ we have*

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z^2_X} \leq B_X \Big(\sum_{k=1}^{\infty} \|x_k\|_X^r \Big)^{1/r}.$$

Proof. By the definition of the quasinorm in Z^2_X, we have that

$$\|z\|_{Z^2_X} \leq \|z^* \chi_{(0,1)}\|_X + \|z\|_{Z^2_{L_{r,\infty}}}, \quad z \in Z^2_X. \tag{21}$$

Denote $\bigoplus_{k=1}^{\infty}|x_k|$ by x, for brevity. Without loss of generality, we can assume that x^* does not have any interval of constancy. Setting $y_k = x_k\chi_{\{|x_k|>x^*(1)\}}$, we have

$$\bigoplus_{k=1}^{\infty}|y_k| \sim x^*\chi_{(0,1)}.$$

Therefore, since X satisfies an upper r-estimate, we obtain

$$\|x^*\chi_{(0,1)}\|_X = \Big\|\bigoplus_{k=1}^{\infty} y_k\Big\|_X \le C\Big(\sum_{k=1}^{\infty}\|y_k\|_X^r\Big)^{1/r} \le C\Big(\sum_{k=1}^{\infty}\|x_k\|_X^r\Big)^{1/r}.$$

The assertion follows now from inequality (21) and the preceding lemma. □

Proof of Theorem 33. The first assertion follows from Theorem 21 and Lemma 36. The proof of the second assertion is identical.

Now, we prove the third assertion. Suppose that $(X,Y)\in(BE)_r$. Let the functions $f_k\in X$, $1\le k\le n$, be pairwise disjoint and let g_k, $1\le k\le n$, be their independent copies. Without loss of generality, we can assume that the f_k (and therefore the g_k as well) are symmetrically distributed. By [Johnson and Schechtman 1989, Theorem 1], we have

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_Y = \Big\|\sum_{k=1}^{n} f_k\Big\|_{Z_Y^2} \le C'\Big\|\sum_{k=1}^{n} g_k\Big\|_Y \le C'B\Big(\sum_{k=1}^{n}\|f_k\|_X^r\Big)^{1/r},$$

which is (20) with $C = C'B$. □

If $X = Y$, then estimate (20) means that X satisfies an upper r-estimate and we obtain the following corollary.

Corollary 37. *Let $0<r<2$ and let X be a quasi-Banach symmetric function space such that $K: X\to X$. Then $X\in(BE)_r$ if and only if X satisfies an upper r-estimate.*

In the Banach-space setting this result may be found in [Braverman 1994, Theorem 2.3].

For $r=2$, we have the following result.

Theorem 38. *Let X and Y be quasi-Banach symmetric function spaces.*

(a) *Suppose that $X\supset L_2(0,1)$. If $K: X\to Y$ and X satisfies an upper 2-estimate, or if $K: Y\to Y$ and for some $C>0$ and for every sequence of mutually disjoint functions $\{f_k\}_{k=1}^n\subset X$, $n\in\mathbb{N}$, we have*

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_Y \le C\Big(\sum_{k=1}^{n}\|f_k\|_X^2\Big)^{1/2}, \tag{22}$$

then $(X,Y)\in(BE)_2$.

(b) *If* $(X, Y) \in (BE)_2$, *then* $X \supset L_2(0,1)$ *and inequality* (22) *holds for some* $C > 0$ *and for every sequence of mutually disjoint functions* $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$.

Proof. (a) The proof is identical to that of the preceding theorem, substituting the reference to Lemma 36 with the reference to the following assertion.

Lemma 39. *Let a quasi-Banach symmetric space* X *satisfy an upper* 2*-estimate and let* $X \supset L_2(0,1)$. *There exists a constant* $B_X > 0$ *such that for every sequence* $\{x_k\}_{k=1}^\infty \subset X$ *we have*

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z_X^2} \le B_X \Big(\sum_{k=1}^{\infty} \|x_k\|_X^2 \Big)^{1/2}.$$

(b) Inequality (22) can be proved in exactly the same way as in Theorem 33. Therefore, it remains to show that $X \subset L_2(0,1)$.

Let $f \in X$ be symmetrically distributed and let $\{f_k\}_{k=1}^\infty$ be a sequence of its independent copies. By assumption, $(X, Y) \in (BE)_2$ and, therefore,

$$\Big\| n^{-1/2} \sum_{k=1}^{n} f_k \Big\|_Y \le C \Big(n^{-1} \sum_{k=1}^{n} \|f_k\|_X^2 \Big)^{1/2} = C\|f\|_X, \quad n = 1, 2, \ldots.$$

By Lemma 24, there exists $p > 0$ such that $Y \subset L_p(0,1)$. Hence, by the previous inequality, we have

$$\sup_{n \ge 1} \int_0^1 \Big| n^{-1/2} \sum_{k=1}^{n} f_k(t) \Big|^p \, dt < \infty.$$

Applying [Esseen and Janson 1985, Theorem 2], we obtain that $f \in L_2(0,1)$. Since both X and $L_2(0,1)$ are symmetric, the assertion follows. ☐

Corollary 40. *Let* X *be a quasi-Banach symmetric space such that* $K : X \to X$. *Then* $X \in (BE)_2$ *if and only if* X *satisfies an upper* 2*-estimate and* $X \subset L_2(0,1)$.

This assertion was proved by Braverman [1994, Theorem 2.4] in the Banach setting.

Remark 41. Though the condition $K : X \to X$ is essential in both Theorem 33 and Theorem 38, it is not necessary. For example, $\operatorname{Exp} L_2 \in (BE)_2$ [Braverman 1994, Theorem 2.9], but $K : \operatorname{Exp} L_2 \not\to \operatorname{Exp} L_2$.

References

[Astashkin 2008] S. V. Astashkin, “Независимые функции в симметричных пространствах и свойство Круглова”, *Mat. Sb.* **199**:7 (2008), 3–20. Translated as “Independent functions in symmetric spaces and the Kruglov property” in *Sb. Math.* **199**:7 (2008), 945–963. MR 2010b:46055 Zbl 1280.46015

[Astashkin and Sukochev 2005] S. V. Astashkin and F. A. Sukochev, "Series of independent random variables in rearrangement invariant spaces: an operator approach", *Israel J. Math.* **145** (2005), 125–156. MR 2006d:60012 Zbl 1084.46020

[Astashkin and Sukochev 2007] S. V. Astashkin and F. A. Sukochev, "Ряды независимых функций с нулевым средним в симметричных пространствах со свойством Круглова", *Zap. Nauchn. Sem.* (*POMI*) **345**:35 (2007), 25–50. Translated as "Series of zero-mean independent functions in symmetric spaces with the Kruglov property" in *J. Math. Sci.* (*NY*) **148**:6 (2008), 795–809. MR 2010j:60103

[Astashkin and Sukochev 2010] S. V. Astashkin and F. A. Sukochev, "Независимые функции и геометрия Банаховых пространств", *Uspekhi Mat. Nauk* **65**:6(396) (2010), 3–86. Translated as "Independent functions and the geometry of Banach spaces" in *Russian Math. Surveys* **65**:6 (2010), 1003–1081. MR 2012b:46002 Zbl 1219.46025

[Astashkin et al. 2011] S. V. Astashkin, F. A. Sukochev, and C. P. Wong, "Disjointification of martingale differences and conditionally independent random variables with some applications", *Stud. Math.* **205**:2 (2011), 171–200. MR 2012h:60134 Zbl 1238.46009

[von Bahr and Esseen 1965] B. von Bahr and C.-G. Esseen, "Inequalities for the rth absolute moment of a sum of random variables, $1 \leq r \leq 2$", *Ann. Math. Statist* **36** (1965), 299–303. MR 30 #645 Zbl 0134.36902

[Bennett and Sharpley 1988] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics **129**, Academic Press, Boston, 1988. MR 89e:46001 Zbl 0647.46057

[Borovkov 1998] A. A. Borovkov, *Probability theory*, Gordon and Breach, Amsterdam, 1998. MR 2000f:60001 Zbl 0918.60003

[Braverman 1994] M. S. Braverman, *Independent random variables and rearrangement invariant spaces*, London Mathematical Society Lecture Note Series **194**, Cambridge University Press, 1994. MR 95j:60027 Zbl 0817.46031

[Esseen and Janson 1985] C.-G. Esseen and S. Janson, "On moment conditions for normed sums of independent variables and martingale differences", *Stochastic Process. Appl.* **19**:1 (1985), 173–182. MR 86m:60120 Zbl 0554.60050

[Johnson and Schechtman 1988] W. B. Johnson and G. Schechtman, "Martingale inequalities in rearrangement invariant function spaces", *Israel J. Math.* **64**:3 (1988), 267–275. MR 90g:60048 Zbl 0672.60049

[Johnson and Schechtman 1989] W. B. Johnson and G. Schechtman, "Sums of independent random variables in rearrangement invariant function spaces", *Ann. Probab.* **17**:2 (1989), 789–808. MR 90h:60045 Zbl 0674.60051

[Johnson et al. 1979] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces*, Memoirs of the American Mathematical Society **19**:217, American Mathematical Society, Providence, RI, 1979. MR 82j:46025 Zbl 0421.46023

[Kalton et al. 1984] N. J. Kalton, N. T. Peck, and J. W. Roberts, *An F-space sampler*, London Mathematical Society Lecture Note Series **89**, Cambridge University Press, 1984. MR 87c:46002 Zbl 0556.46002

[Khintchine 1923] A. Khintchine, "Über dyadische Brüche", *Math. Z.* **18**:1 (1923), 109–116. MR 1544623 JFM 49.0132.01

[Kruglov 1970] V. M. Kruglov, "Замечание к теории безгранично делимых законов", *Teor. Verojatnost. i Primenen* **15**:2 (1970), 330–336. Translated as "A remark on the theory of infinitely divisible laws" in *Theory Prob. Appl.* **15**:2 (1970), 319–324. MR 42 #6899 Zbl 0301.60014

[Lindenstrauss and Tzafriri 1979] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces, II: Function spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete **97**, Springer, Berlin, 1979. MR 81c:46001 Zbl 0403.46022

[Marcinkiewicz and Zygmund 1937] J. Marcinkiewicz and A. Zygmund, "Sur les fonctions indépendantes", *Fundam. Math.* **29** (1937), 60–90. Zbl 0016.40901

[Marcinkiewicz and Zygmund 1938] J. Marcinkiewicz and A. Zygmund, "Quelques théorèmes sur les fonctions indépendantes", *Stud. Math.* **7** (1938), 104–120. Zbl 0018.07504

[Prokhorov 1959] Y. V. Prokhorov, "An extremal problem in probability theory", *Teor. Verojatnost. i Primenen* **4**:2 (1959), 211–214. In Russian; translated in *Theory Prob. Appl.* **4**:2 (1959), 201–203. MR 22 #12587 Zbl 0093.15102

[Rosenthal 1970] H. P. Rosenthal, "On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables", *Israel J. Math.* **8** (1970), 273–303. MR 42 #6602 Zbl 0213.19303

Received December 9, 2012.

SERGEY ASTASHKIN
SAMARA STATE UNIVERSITY
PAVLOVA 1
SAMARA
443011
RUSSIA
astashkn@ssu.samara.ru

FEDOR A. SUKOCHEV
SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES
SYDNEY NSW 2052
AUSTRALIA
f.sukochev@unsw.edu.au

DMITRIY ZANIN
SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES
SYDNEY NSW 2052
AUSTRALIA
d.zanin@unsw.edu.au

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.287

HAMILTONIAN EVOLUTIONS OF TWISTED POLYGONS IN PARABOLIC MANIFOLDS: THE LAGRANGIAN GRASSMANNIAN

GLORIA MARÍ BEFFA

We show that the moduli space of twisted polygons in G/P, where G is semisimple and P parabolic, and where $\mathfrak{g}$ has two coordinated gradations has a natural Poisson bracket that is directly linked to G-invariant evolutions of polygons. This structure is obtained by reducing the quotient twisted bracket on G^N (as defined by M. Semenov-Tian-Shansky) to the moduli space G^N/P^N. We prove that any Hamiltonian evolution with respect to this bracket is induced on G^N/P^N by an invariant evolution of polygons. We describe in detail the Lagrangian Grassmannian case ($G = \mathrm{Sp}(2n)$) and we describe a submanifold of Lagrangian subspaces where the reduced bracket becomes a decoupled system of Volterra Hamiltonian structures. We also describe a very simple evolution of polygons whose invariants evolve following a decoupled system of Volterra equations.

1. Introduction

The difference geometry of lattices, although a relatively young subject, has been known to be related to completely integrable systems almost from its conception. Indeed, parallel to the well-known fact that the sine-Gordon equation describes surfaces with constant negative Gauss curvature, the work of Bobenko and others on difference geometry of lattices (see for instance [Bobenko and Suris 2008]) consistently relates certain types of 2-lattices to completely integrable lattice systems. While in the continuous case the sine-Gordon equation appears as the Codazzi–Mainardi equation of the surface in appropriately chosen coordinates, in the lattice case they are described as the compatibility condition of special types of lattices, with the different properties of lattices playing the role of specially chosen coordinates.

More recently, a flurry of work on the pentagram map, its generalizations and related subjects (see, for example, [Ovsienko et al. 2010; Marí Beffa 2013; Khesin and Soloviev 2013], although the bibliography on this subject is quite vast) has

Supported in part by NSF grant DMS #0804541 and the Simons Foundation.
MSC2010: 39AXX.
Keywords: discrete Hamiltonian systems, discrete Lagrangian Grassmannian, Hamiltonian evolutions of polygons in parabolic manifolds, discrete Poisson reduction.

clearly pointed at a relation between dynamics of polygons, rather than 2-lattices, and completely integrable systems. Indeed the pentagram map, a map defined on projective planar polygons (both twisted and closed), was proven to be completely integrable and a discretization of the Boussinesq equation when written in terms of the discrete projective invariants of the polygons [Ovsienko et al. 2010; 2013; Soloviev 2013].

A plethora of work in the continuous case takes us in this same direction when one works with curves rather than surfaces. Most, if not all, well-known completely integrable PDEs have been realized as systems induced on differential invariants by a flow of curves in some homogeneous manifold. For example, the KdV equation is induced on the Schwarzian derivative of a flow in $\mathbb{RP}^1$ solution of the so-called Schwarzian-KdV equation (this is a classical result that one can check by hand). Similarly Adler–Gel'fand–Dikii flows are induced on projective differential invariants of flows in $\mathbb{RP}^n$ by some known evolutions (see [Marí Beffa 1999]). Likewise the literature shows realizations of mKdV [Terng and Thorbergsson 2001], NLS [Terng and Uhlenbeck 2006] Sawada–Kotera [Chou and Qu 2002], modified Sawada–Kotera [Chou and Qu 2003], vector sine-Gordon [Wang 2002] and most other well-known systems as flows of curves in several different manifolds. This list is by no means exhaustive and many equations are realized as a curve flow in more than one geometry; see, for example, [Calini et al. 2009; Chou and Qu 2002; 2003].

Inspired by the recent developments in discrete maps, we studied in [Mansfield et al. 2013] the relation between evolutions of twisted polygons in homogeneous manifolds and completely integrable lattice systems on the geometric invariants of the flow. In particular we found an evolution of projective planar polygons that when written in terms of projective curvatures becomes a modified Volterra lattice. We also found realizations of the Toda lattice as evolution of polygons in the centro-affine plane; an integrable discretization of the Toda lattice induced by a centro-affine map; and a realization of a Volterra-type equation as evolution of polygons on the homogeneous 2-sphere. In [Marí Beffa and Wang 2013] we proved that one can obtain a Hamiltonian structure on the moduli space of twisted polygons in $\mathbb{RP}^n$ through the reduction of a twisted Poisson bracket on lattices defined by Semenov-Tian-Shansky [1985], and that *any* Hamiltonian with respect to the reduced bracket was induced on invariants by an evolution of polygons in $\mathbb{RP}^n$, with the gradient of the Hamiltonian defining the evolution in a direct and simple fashion. The reduced bracket was a Hamiltonian structure for an integrable discretization of W_n-algebras, and this discretization was induced on projective invariants by a rather simple polygon evolution. We also found a second structure for the system via reduction of the right bracket, a structure that was not originally Poisson.

This paper can be viewed as a second part to [Marí Beffa and Wang 2013]. Here we consider the case of polygons in G/P, with G semisimple, P parabolic, and $\mathfrak{g}$ a $|1|$-graded algebra (and Lie algebra with parabolic gradation $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, $\mathfrak{p} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$). These include many of the well-known nonaffine geometries (conformal n-sphere, $\mathbb{RP}^n$, Grassmannians, Lagrangian Grassmannians, pure spinors and other flag manifolds). We assume that the parabolic gradation is coordinated with a second gradation of the form $\mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$, with $\mathfrak{h}$ commutative. We prove that the twisted Poisson bracket of Semenov-Tian-Shansky defined and associated to this second gradation can also be reduced to the moduli space of polygons in G/P resulting on a natural Poisson structure on the space of polygon invariants. We also prove that there are simple ways to connect the reduced Hamiltonian structure to evolutions of polygons with evolutions inducing Hamiltonian systems on the invariants of the flow. In particular, we prove that any Hamiltonian evolution is induced on invariants by an evolution of polygons in G/P. This result is valid also in more general settings, and we discuss this fact in our last section.

We study in detail the example of the Lagrangian Grassmannian, that is, polygons of Lagrangian subspaces in $\mathbb{R}^{2n}$, $M = \mathrm{Sp}(2n)/P$, with P a parabolic subgroup. We find an appropriate discrete moving frame along twisted polygons, and we define the *Schwarzian difference* of Lagrangian planes (a discrete analogue of the Schwarzian derivative defined in [Ovsienko 1993].) The frame provides us with a complete description of the invariants and produces a generating set that includes the eigenvalues of the Schwarzian difference. We then apply our general theorem to find a Hamiltonian structure on the space of invariants associated to our moving frame. We show that the reduced Poisson bracket can be reduced once more to the space of polygons for which the nonschwarzian invariants are equal to the identity, and we show that this reduction decouples into a system of n second Hamiltonian structures for the Volterra chain [Khanizadeh et al. 2013]. Using this information we define evolutions of Lagrangian planes inducing the Volterra chain on the eigenvalues of the Schwarzian difference of the flow. The continuous analogue of this study can be found in [Marí Beffa 2007].

Section 2 includes background definitions and results that will be used in the paper, both in the subject of discrete moving frames and on Poisson Lie groups and Semenov-Tian-Shansky's bracket. Section 3 proves the existence of the Poisson bracket on the moduli space (as represented by the discrete invariants) and its relation to the Sklyanin bracket (Theorem 3.4). In Section 4 we describe in detail the direct relation between polygon evolutions and reduced Hamiltonians; in particular we prove that any Hamiltonian is induced on invariants by a polygon evolution and we give the direct connection between both (Theorem 4.2). We study the Lagrangian Grassmannian in Section 5 while Section 6 summarizes the paper and discusses generalizations to other homogeneous manifolds and some open problems. Recall that in the projective case [Marí Beffa and Wang 2013] the Poisson structures obtained in the planar case were not preserved by the pentagram map (the bihamiltonian nature of the map is still an open problem, as far as we know). This was also pointed out by Marshall [2010].

2. Background and definitions

As a starting point we will give a brief description of discrete moving frames and their associated invariants. The description is taken from [Mansfield et al. 2013] and can also be found in [Marí Beffa and Wang 2013], but we include it here for completeness.

Discrete moving frames. Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra (it can be real or complex). Let M be a manifold and let $G \times M \to M$ be the action of the group G on M.

Definition 2.1 (twisted N-gon). *A twisted N-gon* in M is a map $\phi : \mathbb{Z} \to M$ such that $\phi(p+N) = g \cdot \phi(p)$ for some fixed $g \in G$ and for all $p \in \mathbb{Z}$. (The dot notation represents the action of G on M.) The element $g \in G$ is called *the monodromy* of the polygon. We will denote a twisted N-gon by its image $x = (x_s)$, where $x_s = \phi(s)$.

The main reason to work with twisted polygons is our desire to work with periodic invariants (in order to have a finite number of them). One could restrict further to closed polygons, but since the solution of a periodic discrete equation is, in general, twisted, restricting to closed polygons creates additional technical problems we would like to avoid here. We will denote by P_N the space of twisted N-gons in M. Clearly $P_N \cong M^N$, and since G acts on M, it also acts on P_N with the diagonal action $g \cdot (x_s) = (g \cdot x_s)$.

Definition 2.2 (discrete moving frame). Let G^N denote the Cartesian product of N copies of the group G. Elements of G^N will be denoted by (g_s). Allow G to act on the right of G^N using the inverse diagonal action $g \cdot (g_s) = (g_s g^{-1})$ (respectively left, using the diagonal action $g \cdot (g_s) = (g g_s)$). We say a map

$$\rho : P_N \to G^N$$

is a right (respectively left) *discrete moving frame* if ρ is equivariant with respect to the diagonal action of G on P_N and the right inverse (respectively left) diagonal action of G on G^N. Whenever $\rho(x) \in G^N$, we will denote by ρ_s its s-th component; that is $\rho = (\rho_s)$, where $\rho_s(x) \in G$ for all s, $x = (x_s)$. Clearly, if $\rho = (\rho_s)$ is a right moving frame, then $\rho^{-1} = (\rho_s^{-1})$ is a left moving frame, and vice versa. Thus, a moving frame associates an element of the group to each vertex of the polygon in an equivariant fashion. In our examples the moving frame will be invariant under the shift $\tau x_s = x_{s+1}$, but this need not be the case in general.

Proposition 2.3 [Mansfield et al. 2013]. *Let $\mathscr{C}$ be a collection $\mathscr{C}_1, \dots, \mathscr{C}_N$ of local cross-sections to the orbit of G through $x_1, \dots, x_N$. Let $\rho = (\rho_s) \in G$ be uniquely determined by the condition*

$$\rho_s \cdot (x_r) \in \mathscr{C}_s \tag{1}$$

for any s. Then $\rho = (\rho_s((x_r))) \in G^N$ is a right moving frame along the N-gon (x_r).

Discrete moving frames carry the invariant information of the polygon, as we see next.

Definition 2.4 (discrete invariant). Let $I : P_N \to \mathbb{R}$ be a function defined on N-gons. We say that I is a scalar *discrete invariant* if

$$I((g \cdot x_s)) = I((x_s)) \tag{2}$$

for any $g \in G$ and any $x = (x_s) \in P_N$.

We will naturally refer to vector invariants when considering vectors whose components are scalar invariants. Although not necessary, for simplicity of notation we will assume from now on that $G \subset \mathrm{GL}(n, \mathbb{R})$. Nevertheless, results are also true for some exceptional Lie algebras, as we will see later.

Definition 2.5 (Maurer–Cartan matrix). Let ρ be a right (respectively left) discrete moving frame evaluated along a twisted N-gon. The element of the group

$$K_s = \rho_{s+1}\rho_s^{-1} \quad \text{(respectively} \quad \rho_s^{-1}\rho_{s+1})$$

is called the right (respectively left) *s-Maurer–Cartan matrix* for ρ. We will call the equation $\rho_{s+1} = K_s\rho_s$ the right *s-Serret–Frenet equation* (respectively $\rho_{s+1} = \rho_s K_s$ is the left one). The element $K = (K_s) \in G^N$ is called the right (respectively left) *Maurer–Cartan matrix* for ρ.

One can directly check that if $K = (K_s)$ is a Maurer–Cartan matrix for the right frame ρ, then (K_s^{-1}) is a left one for the left frame $\rho^{-1} = (\rho_s^{-1})$, and vice versa. The entries of a Maurer–Cartan matrix are functional generators of *all discrete invariants* of polygons, as it was shown in [Mansfield et al. 2013]. This fact is an immediate consequence of the following *recursion formulas*: Let's denote by $\rho_r \cdot x_s = I_s^r$ the so-called *basic invariants*. One can check directly from the definitions that if K is a right Maurer–Cartan matrix, then

$$K_r \cdot I_s^r = I_s^{r+1} \tag{3}$$

for any r, s. The basic invariants *with r fixed* generate other invariants since from (2), if I is an invariant,

$$I((\rho_r \cdot x_s)) = I((x_s)) = I(I_r^s).$$

From this and (3), one concludes that the entries of K_s are generators also (see [Mansfield et al. 2013]).

Assume next that $M = G/H$, with G acting on M via left multiplication on representatives of the class. Let us denote by $o \in M$ the class of H.

The following theorem, which can be found in [Mansfield et al. 2013], describes how to write a general invariant polygon evolution in terms of moving frames. Denote by $\Phi_g : G/H \to G/H$ the map defined by the action of $g \in G$ on G/H, that is, $\Phi_g(x) = g \cdot x$.

Theorem 2.6. *Let ρ be a right moving frame, and for simplicity assume that $\rho_s \cdot x_s = o$ for all s. Any G-invariant evolution can be written as*

$$(x_s)_t = d\Phi_{\rho_s^{-1}}(o)(\boldsymbol{v}_s), \tag{4}$$

where $\boldsymbol{v}_s(x) \in T_{x_s}M$ is an invariant vector.

Notice that if, in general, $\rho_s \cdot x_s = y_s \neq o$, one can easily change ρ_s to $\hat{\rho}_s = g_s \rho_s$, where $g_s \cdot y_s = o$ (the action is transitive). The frame $\hat{\rho}_s$ will also be a right moving frame and thus one can always find a frame with the condition given by the theorem. This fact will greatly simplify both notations and calculations.

If a family of polygons $x(t)$ is evolving according to (4), there is a simple process to describe the evolution induced on the Maurer–Cartan matrices and hence on a generating set of invariants. It is described in the following theorem, which can also be found in [Mansfield et al. 2013], slightly modified.

Before our next theorem, let us settle some notation and choices. Assume

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \tag{5}$$

where $\mathfrak{m}$ is a linear complement to $\mathfrak{h}$. Consider $\varsigma : G/H \to G$ to be a section of G/H such that $\varsigma(o) = e \in G$ and $\mathfrak{m}$ is the tangent to the image of ς. Let ρ be a right moving frame coordinated with ς. That is, assume $\rho_s \cdot x_s = o$ so that $\rho_s = \rho_s^H \varsigma(x_s)^{-1}$, for some $\rho_s^H \in H$.

Let K_s be a right Maurer–Cartan matrix and define $N_s = (\rho_s)_t \rho_s^{-1} \in \mathfrak{g}$ to describe the time evolution of the frame.

Theorem 2.7. *Assume $x(t)$ is a flow of polygons solution of* (4). *Then*

$$(K_s)_t = N_{s+1} K_s - K_s N_s \tag{6}$$

and, if $N_s = N_s^{\mathfrak{h}} + N_s^{\mathfrak{m}}$ splits according to (5), *then*

$$N_s^{\mathfrak{m}} = -d\varsigma(o)\boldsymbol{v}_s. \tag{7}$$

In most examples Equation (6) and condition (7) completely determine N and the evolution of K, *even if we do not know the moving frame explicitly*. This will be clear later in our Lagrangian example.

Finally, in [Marí Beffa and Wang 2013], we proved the following theorem, which is true for any homogeneous manifold. Assume we have a nondegenerate twisted polygon $x = (x_s)$ in a manifold $M = G/H$ with associated right moving frame ρ such that $\rho_s \cdot x_s = o$ for all s. By nondegenerate we mean a polygon for which a moving frame can be constructed, but we can also think of generic cases. (It was shown in [Boutin 2002] that generically a moving frame always exists for N large enough.) Let us assume that the subgroup H^N acts naturally on G^N via the gauge transformation

$$(g_s) \to (h_{s+1} g_s h_s^{-1})$$

(assuming $h_{s+N} = h_s$ for all s).

Theorem 2.8. *In a neighborhood of a nondegenerate polygon, the right Maurer–Cartan matrices K associated to right moving frames ρ describe a section of the quotient G^N/H^N. That is, let $x \in G^N/H^N$ be a nondegenerate twisted polygon, $\mathcal{U}$ with $x \in \mathcal{U}$ an open set of G^N/H^N containing nondegenerate twisted polygons, and let $\mathcal{K}$ be the set of all the Maurer–Cartan matrices in G^N associated to right moving frames for elements in $\mathcal{U}$ and determined by a fixed transverse section as in Proposition 2.3. Then the map*

$$\mathcal{K} \to G^N/H^N, \quad (K_s) \to [(K_s)] \tag{8}$$

is a section of the quotient, a local isomorphism.

For more details, see [Mansfield et al. 2013].

Semenov-Tian-Shansky's twisted Poisson brackets. In this section we will assume that $\mathfrak{g}$ is semisimple and that $\langle \cdot, \cdot \rangle$ is a nondegenerate inner product in $\mathfrak{g}$ that allows us to identify $\mathfrak{g}$ and $\mathfrak{g}^*$ (a multiple of the one generated by the Killing form). Denote by $E_{i,j}$ the matrix with 0s everywhere except for the (i, j) entry, where it has a 1. Since we are assuming that $G \subset \mathrm{GL}(n, \mathbb{R})$, we can assume that, for example, the inner product is the trace of the product of matrices, so that $E^*_{i,j} = E_{j,i}$. The following definitions and descriptions are due to Drinfeld [1983].

Definition 2.9 (Poisson–Lie group). A Poisson–Lie group is a Lie group equipped with a Poisson bracket such that the multiplication map $G \times G \to G$ is a Poisson map, where we consider the manifold $G \times G$ with the product Poisson bracket.

Definition 2.10 (Lie bialgebra). Let $\mathfrak{g}$ be a Lie algebra such that $\mathfrak{g}^*$ also has a Lie algebra structure given by a bracket $[\cdot, \cdot]_*$. Let $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ be the dual map to the dual Lie bracket, that is,

$$\langle \delta(v), (\xi \wedge \eta) \rangle = \langle [\xi, \eta]_*, v \rangle$$

for all $\xi, \eta \in \mathfrak{g}^*$, $v \in \mathfrak{g}$. Assume that δ is a one-cocycle, that is

$$\delta([v, w]) = [v \otimes \mathbf{1} + \mathbf{1} \otimes v, \delta(w)] - [w \otimes \mathbf{1} + \mathbf{1} \otimes w, \delta(v)]$$

for all $v, w \in \mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{g}^*)$ is called a *Lie bialgebra*.

If G is a Lie–Poisson group, the linearization of the Poisson bracket at the identity defines a Lie bracket in $\mathfrak{g}^*$. The map δ is called the *cobracket*. The inverse result (any Lie bialgebra corresponds to a Lie–Poisson group) is also true for connected and simply connected Lie groups, as shown in [Drinfeld 1983].

Definition 2.11 (admissible subgroup). Let M be a Poisson manifold, G a Poisson–Lie group and $G \times M \to M$ a Poisson action. A subgroup $H \subset G$ is called *admissible* if the space $C^\infty(M)^H$ of H-invariant functions on M is a Poisson subalgebra of $C^\infty(M)$.

The following proposition describes admissible subgroups.

Proposition 2.12 [Semenov-Tian-Shansky 1985]. *Let* $(\mathfrak{g}, \mathfrak{g}^*)$ *be the tangent Lie bialgebra of a Poisson Lie group* G. *A Lie subgroup* $H \subset G$ *with Lie algebra* $\mathfrak{h} \subset \mathfrak{g}$ *is admissible if* $\mathfrak{h}^0 \subset \mathfrak{g}^*$ *is a Lie subalgebra*, *where* $\mathfrak{h}^0$ *is the annihilator of* $\mathfrak{h}$.

We will now describe the Poisson brackets that will be at the center of our study.

Definition 2.13 (factorizable Lie bialgebras and R-matrices). A Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is called *factorizable* if the following two conditions hold:

(a) $\mathfrak{g}$ is equipped with an invariant bilinear form $\langle \cdot, \cdot \rangle$ so that $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ via $\xi \in \mathfrak{g}^* \to v_\xi \in g$ with $\xi(\cdot) = \langle v_\xi, \cdot \rangle$

(b) the Lie bracket on $\mathfrak{g}^* \cong \mathfrak{g}$ is given by

$$[\xi, \eta]_* = \frac{1}{2}([R(\xi), \eta] + [\xi, R(\eta)]), \tag{9}$$

where $R \in \mathrm{End}(\mathfrak{g})$ is a skew-symmetric operator satisfying the *modified classical Yang–Baxter equation*

$$[R(\xi), R(\eta)] = R([R(\xi), \eta] + [\xi, R(\eta)]) - [\xi, \eta].$$

R is called a *classical R-matrix*. Let r be the 2-tensor image of R under the identification $\mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \mathrm{End}(\mathfrak{g})$. That is,

$$r(\xi \wedge \eta) = \langle \xi, R(\eta) \rangle. \tag{10}$$

The tensor r is often referred to as the R-matrix also.

The simplest example of an R-matrix is as follows: assume that $\mathfrak{g}$ has a splitting of the form $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_-$, where $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are subalgebras dual of each other and where $\mathfrak{h}_0$ is commutative (for example, $\mathfrak{h}_0$ could be the Cartan subalgebra). Then it is well-known that the map $R : \mathfrak{g} \to \mathfrak{g}$ given by

$$R(\xi_+ + \xi_0 + \xi_-) = \tfrac{1}{2}(\xi_+ - \xi_-) \tag{11}$$

defines a classical R-matrix.

Given a Poisson Lie group G and its associated factorizable Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, we can define an induced Poisson structure on G^N, as explained in [Semenov-Tian-Shansky 1985]. Indeed, we equip $\mathfrak{g}^N = \bigoplus_N \mathfrak{g}$ with a nondegenerate inner product given by

$$\langle X, Y\rangle = \sum_{k=1}^{N} \langle X_k, Y_k\rangle$$

and we extend $R \in \operatorname{End}(\mathfrak{g})$ to $R \in \operatorname{End}(\mathfrak{g}^N)$ using $R((X_s)) = (R(X_s))$. Then G^N is a Poisson Lie-group (with the product Poisson structure) and $(\mathfrak{g}^N, \mathfrak{g}_R^N)$ is its factorizable Lie bialgebra, where $\mathfrak{g}_R$ denotes $\mathfrak{g}$ with Lie bracket (9). Note that we are abusing notation, using $\langle\,\cdot\,,\cdot\,\rangle$ and R to denote both the inner product and the R-matrix in $\mathfrak{g}$ and $\mathfrak{g}^N$. We will point out the difference only when it is not clear from the context and notation.

Definition 2.14 (left and right gradients). Let $\mathcal{F} : G^N \to \mathbb{R}$ be a differentiable function. We define the *left gradient* of $\mathcal{F}$ at $L = (L_s) \in G^N$ as the element of $\mathfrak{g}^N$ denoted by $\nabla\mathcal{F}(L) = (\nabla_s\mathcal{F}(L))$, with $\nabla_s\mathcal{F}(L)$ satisfying

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{F}((\exp(\epsilon\xi_s)L_s)) = \langle\nabla_s\mathcal{F}(L), \xi_s\rangle$$

for all s and any $\xi = (\xi_s) \in \mathfrak{g}^N$.

Analogously, we define the *right gradient* of $\mathcal{F}$ at L as the element of $\mathfrak{g}^N$ denoted by $\nabla'\mathcal{F}(L) = (\nabla'_s\mathcal{F}(L))$, with $\nabla'_s\mathcal{F}(L)$ satisfying

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{F}((L_s\exp(\epsilon\xi_s))) = \langle\nabla'_s\mathcal{F}(L), \xi_s\rangle$$

for all s and any $\xi = (\xi_s) \in \mathfrak{g}^N$. Clearly

$$\nabla'_s\mathcal{F}(L) = L_s^{-1}\nabla_s\mathcal{F}(L)L_s. \tag{12}$$

If r is given as in (10) for some R-matrix R, the Poisson structure in G^N given by the formula

$$\{\mathcal{F}, \mathcal{G}\}_S(L) = \sum_{s=1}^{N} \hat{r}(\nabla_s\mathcal{F} \wedge \nabla_s\mathcal{G}) - \sum_{s=1}^{N} \hat{r}(\nabla'_s\mathcal{F} \wedge \nabla'_s\mathcal{G}) \tag{13}$$

is called the *Sklyanin bracket*. Now, given a factorizable Lie bialgebra, Semenov-Tian-Shansky [1985] defined what is called a *twisted Poisson structure on G^N*. Here we will give the definition of this structure, and we refer the reader to the same reference for explanations on how to obtain it and to [Frenkel et al. 1998, Theorem 1] for the explicit formula.

Let $\mathcal{F}, \mathcal{G} : G^N \to \mathbb{R}$ be two functions. Let τ be the shift operator $\tau(X_s) = X_{s+1}$. We define the τ-twisted Poisson bracket as

$$\begin{aligned}(14)\quad \{\mathcal{F}, \mathcal{G}\}(L) = &\sum_{s=1}^{N} r(\nabla_s \mathcal{F} \wedge \nabla_s \mathcal{G}) + \sum_{s=1}^{N} r(\nabla'_s \mathcal{F} \wedge \nabla'_s \mathcal{G}) \\ &- \sum_{s=1}^{N} (\tau \otimes \mathrm{id})(r)(\nabla'_s \mathcal{F} \otimes \nabla_s \mathcal{G}) + \sum_{s=1}^{N} (\tau \otimes \mathrm{id})(r)(\nabla'_s \mathcal{G} \otimes \nabla_s \mathcal{F}).\end{aligned}$$

In [Frenkel et al. 1998; Semenov-Tian-Shansky 1985] it was proved that not only is this a Poisson bracket but the gauge action of G^N on itself, that is, the action $G^N \times G^N \to G^N$ given by

$$(L_s) \to (g_{s+1} L_s g_s^{-1}), \tag{15}$$

is a Poisson map and the gauge orbits are Poisson submanifolds. This is the relevant bracket to our study of polygon evolutions.

3. A Hamiltonian bracket on the moduli space of twisted polygons in parabolic manifolds

Let G be a semisimple group and $\mathfrak{g}$ its Lie algebra. Assume $\mathfrak{g}$ has a gradation of the form

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \tag{16}$$

where $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are dual to each other with respect to an adjoint-invariant inner product. Let G_i be the subgroup of G with Lie algebra $\mathfrak{g}_i$, and $P \subset G$ the parabolic subgroup of G with Lie algebra $\mathfrak{p} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$.

Consider the space of polygons in the homogeneous manifold $M = G/P$. In this section we will show that under some assumptions, (14), defined in G^N, can be reduced to the quotient G^N/P^N to define a Poisson structure on the space of Maurer–Cartan matrices associated to polygons in M, and hence on the space of invariants as shown in Theorem 2.8.

Before we go into our main theorem, we will recall some known facts about the action of G on G/P when $\mathfrak{g}$ is a $|1|$-graded algebra as in (16). The following descriptions can be found, for example, in [Ochiai 1970]. Let G_1 and G_{-1} be the connected Lie subgroups of G corresponding to $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, respectively. We define G_0 to be the normalizer of $\mathfrak{g}_0$ in P, that is, $G_0 = \{a \in P | \mathrm{Ad}(a)(\mathfrak{g}_0) = \mathfrak{g}_0\}$.

Proposition 3.1. *The exponential mappings* $\exp : \mathfrak{g}_1 \to G_1$ *and* $\exp : \mathfrak{g}_{-1} \to G_{-1}$ *are bijective. Furthermore*, G_0 *is also the normalizer of* $\mathfrak{g}_{-1}$ *in* P *and* P *is the semidirect product of* G_0 *and* G_1.

The subgroup G_0 is called the *linear isotropy subgroup* of the semisimple homogeneous space G/P and it is clearly locally bijective, via the exponential map, to $\mathfrak{g}_0$. Perhaps a more important description for this paper is the following well-known result. It can be obtained from [Ochiai 1970], although here it is simplified for a clearer exposition.

Proposition 3.2. *Let* $G \times M \to M$ *be the action of* G *on* $M = G/P$ *given by left multiplication on class representatives. Let* G_i *and* $\mathfrak{g}_i$ *be given as above*, $i = 1, 0, -1$. *Then the infinitesimal action of* $\mathfrak{g}_{-1}$ *is constant in* x, *the one of* $\mathfrak{g}_0$ *is linear in* x *and the one of* $\mathfrak{g}_1$ *is quadratic in* x.

Next, assume that $\mathfrak{g}$ can be endowed with two different splittings: the original parabolic gradation (16), and a splitting of the form

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+, \tag{17}$$

where $\mathfrak{h}_0$ is commutative and $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are dual to each other. Assume also that this splitting can be chosen so that $\mathfrak{g}_1 \subset \mathfrak{g}_+$, $\mathfrak{g}_{-1} \subset \mathfrak{g}_-$ and $\mathfrak{h} \subset \mathfrak{g}_0$, while $\mathfrak{g}_0$ will have, in general, intersection with all $\mathfrak{g}_+$, $\mathfrak{h}$ and $\mathfrak{g}_-$.

Remark 3.3. This assumption is not too restrictive. For example, in the complex case, given a simple Lie algebra (a semisimple one will be the sum of its simple terms) one can always find two gradations related as above, a pair per root with weight equal to 1. One way to find the gradations is as follows:[1] Let $\mathfrak{h}$ be a choice of Cartan subalgebra and $\Delta = \{\alpha_r\}_{r=1}^{\ell}$ a simple root system associated to $\mathfrak{h}$. Let Φ^+ be the set of positive roots and Φ^- the set of negative roots. Let $\lambda \in \Delta$ have weight 1 and let $\Phi_1 = \{\alpha \in \Phi^+ \text{with } \lambda \text{ in its linear expansion}\}$ and Φ_{-1} the negative analogue. Define $\mathfrak{g}_1 = \bigoplus_{\alpha \in \Phi_1} \mathfrak{g}_\alpha$ and $\mathfrak{g}_{-1} = \bigoplus_{\alpha \in \Phi_{-1}} \mathfrak{g}_\alpha$. Let $\mathfrak{g}_0$ be the sum of the root spaces associated to the remaining roots (the ones that do not contain λ). We see that $\mathfrak{g}_1$ is commutative since λ cannot appear in any linear expansion with a coefficient higher than one, and if $\alpha, \beta \in \Phi_1$, then the coefficient of $\alpha + \beta$ would be 2. Likewise with Φ_{-1}. The second gradation is simply given by $\mathfrak{g}_+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, $\mathfrak{g}_- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$ and $\mathfrak{h}$, ensuring that $\mathfrak{g}_1 \subset \mathfrak{g}_+$ and $\mathfrak{g}_{-1} \subset \mathfrak{g}_-$.

Not all algebras have such roots. The ones that do are: A_r (with r different choices of roots), B_r (one choice), C_r (one choice), D_r (3 choices), E_6 (two

[1] The author is very grateful to Professor Georgia Benkart for the description and discussions on this matter.

choices), E_7 (one choice). In the case of a simple real Lie algebra, Kobayashi and Nagano [1964; 1965] described all semisimple real Lie algebras with gradations (16) as direct sums of simple ones belonging to the following list:

(1) $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$ with $\mathfrak{g}_0 = \mathfrak{sl}(p, \mathbb{R}) \oplus \mathfrak{sl}(q, \mathbb{R}) \oplus \mathbb{R}$;

(2) $\mathfrak{g} = \mathfrak{so}(n, n)$ with $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$;

(3) $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$ with $\mathfrak{g}_0 = \mathfrak{so}(p+q) \oplus \mathbb{R}$;

(4) $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$ with $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$;

(5) $\mathfrak{g} = E_6^1$ with $\mathfrak{g}_0 = \mathfrak{so}(5, 5) \oplus \mathbb{R}$;

(6) $\mathfrak{g} = E_7^1$ with $\mathfrak{g}_0 = E_6^1 \oplus \mathbb{R}$.

Using their representations, one can see that the standard finest gradation inherited from $\mathfrak{gl}(n, \mathbb{R})$ with $n = p + q$ will work as gradation (17) for (1); case (2) is very similar to (4), which we will describe in detail in our last section, while cases (3), (5) and (6) are not clear to us. As the reader can see, some of the exceptional cases satisfy our assumptions.

We are now ready for our main theorem.

Theorem 3.4. *Assume G and $\mathfrak{g}$ are as above. The twisted Poisson structure* (14) *defined on G^N, with r associated to* (17) *as in* (11), *is locally reducible to the quotient G^N/P^N, and the reduced bracket coincides with the reduction of the Sklyanin bracket* (13) *with tensor*

$$\hat{r}(\xi, \eta) = \langle \xi_{-1}, \eta_1 \rangle,$$

where ξ_{-1} and η_1 correspond to the parabolic gradation (16) *defining M.*

Notice that $\hat{r}$ is not an R-matrix and hence the Sklyanin bracket is not Poisson before reduction.

Proof. The proof is similar to the one for $\mathbb{RP}^n$ that appeared in [Marí Beffa and Wang 2013], with some differences. From Theorem 2.8, the quotient is locally a manifold, and as explained in [Semenov-Tian-Shansky 1985] the gauge action is a Poisson action for the twisted bracket, whose symplectic leaves are gauge orbits. Therefore, using the same reasoning as the one used in [Marí Beffa and Wang 2013, Theorem 5.5] we conclude that the bracket can be reduced whenever P is admissible (see Definition 2.11). According to Proposition 2.12, this is true whenever $\mathfrak{p}^0 = \mathfrak{g}_1$ is a Lie subalgebra of $\mathfrak{g}^*$, and this is the only condition we need to check to prove the first part of the theorem.

The Lie bracket in $\mathfrak{g}^*$ is defined by the linearization of the twisted Poisson bracket at the identity $e \in G$. That is,

$$[d_e\phi, d_e\varphi]_* = d_e\{\phi, \varphi\} \in \mathfrak{g}^*.$$

Since $\mathfrak{p}^0 = \mathfrak{g}_1$, we will look for functions φ_s^i such that $d_e\varphi_s^i$ generate $\mathfrak{g}_1$.

First of all, we can locally identify M with the section represented by G_{-1} through a map $x \mapsto \ell(x) \in G_{-1}$. Let $\varphi : U \subset M \to \mathbb{R}^n$ be local coordinates around o defined as follows: choose coordinates for G_{-1} given by the exponential map composed with linear coordinates in $\mathfrak{g}_{-1}$, and define $\varphi(\ell(x)) = \varphi(x)$. Assume φ^i are the components of φ; that is, if w_i are generators of $\mathfrak{g}_{-1}$, then $\ell(x) = \Pi_i \exp(\alpha_i w_i)$ and $\varphi^i(x) = \alpha_i$, $i = 1, \dots, n$ (recall that G_{-1} is commutative). Now, let $L \in G^N$ be close enough to $e \in G^N$ so that $L = (L_s)$ can be factored as $L_s = L_{-1}^s L_0^s L_1^s$ with $L_i^s \in G_i$, according to the gradation (16). We choose $x_s \in U \subset M$ such that $L_{-1}^s = \ell(x_s)$, and define $\varphi^i(L) = (\varphi^i(L_s)) = (\varphi^i(L_{-1}^s)) = (\varphi^i(\ell(x_s))) = (\varphi^i(x_s))$. Since $d_e\varphi^i$ is in the dual of the tangent to M at the identity (which we can identify with $\mathfrak{g}_{-1}$, with dual equal to $\mathfrak{g}_1$) and φ are coordinates, the elements $d_e\varphi^i$, $i = 1, \dots, n$, must generate $\mathfrak{g}_1 = \mathfrak{p}^0$. Now we only need to check that if $\{\cdot, \cdot\}$ is the quotient bracket in (14), then

$$[d_e\varphi^i, d_e\varphi^j]_* = d_e\{\varphi^i, \varphi^j\} \in \mathfrak{p}^0 = \mathfrak{g}_1.$$

This will imply that P is admissible.

Identify M^N with the section represented by G_{-1}^N via the map

$$(x_s) \to (\ell(x_s)) \in G_{-1}^N.$$

Then the action of G^N on M^N is uniquely determined by the relation

$$g_s \ell(x_s) = \ell(g_s \cdot x_s) p_s \tag{18}$$

for some $p_s \in P$. Let $\xi_s \in \mathfrak{g}$ and $V_s = \exp(\epsilon \xi_s)$. As before, assume $L_s = L_{-1}^s L_0^s L_1^s$ with $L_{-1}^s = \ell(x_s)$ for some $x_s \in M$. Let $V_s = \exp(\epsilon \xi_s)$. Using (18), we obtain

$$\varphi(V_s L_s) = \varphi(V_s L_{-1}^s) = \varphi(V_s \ell(x_s)) = \varphi(\ell(V_s \cdot x_s)) = \varphi(V_s \cdot x_s).$$

(1) If $\xi_s \in \mathfrak{g}_{-1}$ and given that the infinitesimal action of $\mathfrak{g}_{-1}$ on M is constant, we have that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi(V_s L_s) = \langle \nabla_s \varphi(L_s), \xi_s \rangle \tag{19}$$

is constant in L_s. That is to say, if $\nabla_s\varphi(L_s)$ splits according to the parabolic gradation (16), then its $\mathfrak{g}_1$ component is constant for any L_s and for all s.

(2) If $\xi \in \mathfrak{g}_0$, then $\varphi(V_s L_s)$ is again $\varphi(V_s \cdot x_s)$ as above. The infinitesimal action is now linear, and hence $\nabla_s\varphi(L_s)$ has a $\mathfrak{g}_0$-component that is linear in $L_{-1}^s = \ell(x_s)$, for all s. This will vanish at $x_s = 0$, or what is the same, at $L_s = e$.

(3) If $\xi \in \mathfrak{g}_1$, the infinitesimal action will be quadratic, and hence $\nabla_s\varphi(L_s)$ will have a $\mathfrak{g}_{-1}$ component that is quadratic in $L_{-1}^s = \ell(x_s)$, for all s. Thus, it vanishes at $x_s = 0$ or $L_s = e$.

We now calculate $d_e\{\varphi^i,\varphi^j\}$ where $\{\cdot,\cdot\}$ is the twisted bracket (14) with the r-matrix given by (11). We want to show that $d_e\{\varphi^i,\varphi^j\}\in\mathfrak{p}^0$ and so we need to show that $\frac{d}{d\epsilon}\big|_{\epsilon=0}\{\varphi_s^i,\varphi_s^j\}(e^{\epsilon\xi})=0$ whenever $\xi\in\mathfrak{p}=\mathfrak{g}_1\oplus\mathfrak{g}_0$.

Given that $(d_e\varphi_s^i)_{-1}=0$ and $(\nabla\varphi_s^i(L))_{-1}$ is quadratic in $\ell(x_s)$ (with $\ell(o)=e$) we can conclude that $\frac{d}{d\epsilon}\big|_{\epsilon=0}(\nabla\varphi_s^i(e^{\epsilon\xi}))_{-1}=0$.

Also, $\nabla'\varphi_s^i(e^{\epsilon\xi})=e^{-\epsilon\xi}\nabla\varphi_s^i(e^{\epsilon\xi})e^{\epsilon\xi}$, and therefore

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\nabla'\varphi_s^i(e^{\epsilon\xi})=[d_e\varphi_s^i,\xi_s]+\frac{d}{d\epsilon}\Big|_{\epsilon=0}\nabla\varphi_s^i(e^{\epsilon\xi}).$$

Since $d_e\varphi_s^i\in\mathfrak{g}_1$, whenever $\xi\in\mathfrak{p}$ we have that $[d_e\varphi_s^i,\xi_s]\in\mathfrak{p}$ and hence

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\big(\nabla'\varphi_s^i(e^{\epsilon\xi})\big)_{-1}=0.$$

Furthermore, $(d_e\varphi_s^i)_0=0$ also. Finally, we split

$$\langle\nabla_+\varphi_s^i,\nabla_-\varphi_s^i\rangle=\langle\nabla_1\varphi_s^i,\nabla_{-1}\varphi_s^i\rangle+\langle\nabla_+^0\varphi_s^i,\nabla_-^0\varphi_s^i\rangle,$$

where $\nabla_+^0\varphi_s^i$ and $\nabla_-^0\varphi_s^i$ are the components of $\nabla_+\varphi_s^i$ and $\nabla_+\varphi_s^i$ in $\mathfrak{g}_0$. Substituting this splitting in the definition of the twisted bracket and going over each one of its terms, we get that they all vanish, $\frac{d}{d\epsilon}\big|_{\epsilon=0}\{\varphi_s^i,\varphi_s^j\}(e^{\epsilon\xi})=0$, and hence $\mathfrak{p}^0$ is a subalgebra of $\mathfrak{g}^*$.

We now look at the second assertion of the theorem. The reduced bracket is calculated as follows: let $f,h:\mathcal{K}\to\mathbb{R}$ be two functions on the quotient space $\mathcal{K}=U^N/P^N$, where P^N is acting on the open set $U^N\subset M^N$ by gauge transformations. Consider two extensions of f,h to U^N, call them $\mathcal{F}$ and $\mathcal{H}$, constant on the gauge leaves of P. That means

$$\mathcal{F}(p_{s+1}K_sp_s^{-1})=\mathcal{F}(K_s)=f(\boldsymbol{k}_s)$$

for any $p_s\in P$, where $\boldsymbol{k}_s$ are coordinates for K_s (i.e., a generating system of invariants defined by K_s). Choosing $p_s=\exp(\epsilon\xi_s)$, $\xi_s\in\mathfrak{p}$ and differentiating, we get that

$$\sum_{s=1}^N\langle-\nabla_s'\mathcal{F}+\tau^{-1}\nabla_s\mathcal{F},\xi_s\rangle=0.$$

That is,

$$-\nabla'\mathcal{F}+\tau^{-1}\nabla\mathcal{F}\in(\mathfrak{p}^0)^N=\mathfrak{g}_1^N. \tag{20}$$

Likewise for $\mathcal{H}$. The reduced bracket is then defined as

$$\{f,h\}_{\text{inv}}(\boldsymbol{k})=\{\mathcal{F},\mathcal{H}\}(K). \tag{21}$$

We now use this description to finish the proof. Since $\tau\nabla'\mathcal{F}-\nabla\mathcal{F}\in(\mathfrak{p}^0)^N=\mathfrak{g}_1^N$ and $\mathfrak{g}_-\subset\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$, we have that $\tau(\nabla'\mathcal{F})_-=(\nabla\mathcal{F})_-$, and from this the reduced Poisson bracket can be expressed as

$$\begin{aligned}\{f,h\}(\boldsymbol{k}) &= \tfrac{1}{2}\big(\langle(\nabla\mathcal{F})_-,(\nabla\mathcal{H})_+\rangle-\langle(\nabla\mathcal{F})_+,(\nabla\mathcal{H})_-\rangle\\ &\quad+\langle(\nabla'\mathcal{F})_-,(\nabla'\mathcal{H})_+\rangle-\langle(\nabla'\mathcal{F})_+,(\nabla'\mathcal{H})_-\rangle\big)\\ &\quad-\langle\tau(\nabla'\mathcal{F})_-,(\nabla\mathcal{H})_+\rangle+\langle\tau(\nabla'\mathcal{H})_-,(\nabla\mathcal{F})_+\rangle\\ &= \tfrac{1}{2}\big(-\langle(\nabla\mathcal{F})_-,(\nabla\mathcal{H})_+\rangle+\langle(\nabla\mathcal{F})_+,(\nabla\mathcal{H})_-\rangle\\ &\quad+\langle(\nabla'\mathcal{F})_-,(\nabla'\mathcal{H})_+\rangle-\langle(\nabla'\mathcal{F})_+,(\nabla'\mathcal{H})_-\rangle\big)\\ &\quad-\tfrac{1}{2}\big(-\langle(\nabla\mathcal{F})_-,(\nabla\mathcal{H})_+\rangle+\langle(\nabla\mathcal{F})_+,(\nabla\mathcal{H})_-\rangle\\ &\quad+\langle(\nabla'\mathcal{F})_-,(\nabla'\mathcal{H})_+\rangle-\langle(\nabla'\mathcal{F})_+,(\nabla'\mathcal{H})_-\rangle\big)\\ &= \tfrac{1}{2}\langle(\nabla\mathcal{H})_-,(\nabla\mathcal{F})_+-\tau(\nabla'\mathcal{F})_+\rangle-\tfrac{1}{2}\langle(\nabla\mathcal{F})_-,(\nabla\mathcal{H})_+-\tau(\nabla'\mathcal{H})_+\rangle.\end{aligned}$$

Since $\mathfrak{g}_1\subset\mathfrak{g}_+$, this is equal to

$$\tfrac{1}{2}\langle(\nabla\mathcal{H})_{-1},(\nabla\mathcal{F})_1-\tau(\nabla'\mathcal{F})_1\rangle-\tfrac{1}{2}\langle(\nabla\mathcal{F})_{-1},(\nabla\mathcal{H})_1-\tau(\nabla'\mathcal{H})_1\rangle,$$

and from this we can go back to

$$\begin{aligned}-\tfrac{1}{2}\big(&-\langle(\nabla\mathcal{F})_{-1},(\nabla\mathcal{H})_1\rangle+\langle(\nabla\mathcal{F})_1,(\nabla\mathcal{H})_{-1}\rangle\\ &+\langle(\nabla'\mathcal{F})_{-1},(\nabla'\mathcal{H})_1\rangle-\langle(\nabla'\mathcal{F})_1,(\nabla'\mathcal{H})_{-1}\rangle\big),\end{aligned}$$

which coincides with the evaluation of (13) defined by the parabolic gradation (16) on the extensions $\mathcal{F}$ and $\mathcal{H}$. □

4. Polygon evolutions inducing a Hamiltonian evolution on invariants

In this section we will study which invariant evolutions of polygons induce an evolution on $\boldsymbol{k}$ which is Hamiltonian with respect to the reduced bracket we described in our previous section. In particular, we will link the invariant vector $\boldsymbol{v}_s$ describing the evolution (4), to the gradient of the Hamiltonian f determining the evolution of the invariants. The relation is simple and straightforward and we will show that any Hamiltonian flow on the invariants is induced by a polygon evolution.

First of all, recall that if (x_s) evolves under (4), then the evolution of the Maurer–Cartan invariants is given by (6), where $N=\rho_t\rho^{-1}\in\mathfrak{g}^N$ satisfies the condition

$$(N_s)_{-1}=-d\varsigma(o)\boldsymbol{v}_s.$$

Lemma 4.1. *Let h be a function of the invariants $\boldsymbol{k}$, and let $\mathcal{H}$ be an extension of h constant on the gauge orbits of P. Assume that, for a fixed function f,*

$$\sum_{s=1}^{N} \langle \nabla_s \mathcal{H}, (K_s)_t K_s^{-1} \rangle = \{f, h\}_{\mathrm{inv}}(\boldsymbol{k}) \quad \textit{for any function } h.$$

Then $K_t K^{-1}$ *defines a* $\{\cdot,\cdot\}_{\mathrm{inv}}$*-Hamiltonian evolution on the coordinates* $\boldsymbol{k}$, *with Hamiltonian* f.

Proof. Let $\boldsymbol{k} = (\boldsymbol{k}_s)$ and $\boldsymbol{k}_s = (\boldsymbol{k}_s^i)$ be coordinates for K (we write $K_s = K_s(\boldsymbol{k})$), and assume x evolves according to (4). The evolution induced on K_s (through $\boldsymbol{k}$) is given by the relation

$$(K_s)_t K_s^{-1} = \sum_{r=1}^{N} \sum_{i=1}^{n} (\boldsymbol{k}_r^i)_t \frac{\partial K_s}{\partial \boldsymbol{k}_r^i} K_s^{-1}.$$

On the other hand, let h be a function of $\boldsymbol{k}$ and $\mathcal{H}$ an extension constant on the leaves of P. If $Z_s = K_s(\boldsymbol{k}_r^i(\epsilon)) K_s^{-1}(\boldsymbol{k}_r^i)$, with $\boldsymbol{k}_r^i(0) = \boldsymbol{k}_r^i$ and $\frac{d}{d\epsilon}\big|_{\epsilon=0} \boldsymbol{k}_r^i(0) = v_r^i$, we have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}(Z_s K_s) = \sum_s \Big\langle \nabla_s \mathcal{H}(K), \frac{d}{d\epsilon}\Big|_{\epsilon=0} Z_s \Big\rangle \tag{22}$$

on the one side, while on the other side

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}(Z_s K_s) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}(K_s(\boldsymbol{k}_r^i(\epsilon))) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} h(\boldsymbol{k}_r^i(\epsilon)) = \sum_{r=1}^{N} \sum_{i=1}^{n} v_r^i \frac{\partial h}{\partial \boldsymbol{k}_r^i}.$$

We further see that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} Z_s = \sum_{r=1}^{N} \sum_{i=1}^{n} v_r^i \frac{\partial K}{\partial \boldsymbol{k}_r^i} K^{-1}.$$

Comparing the two sides of (22), which must be equal for any values of v_r^i, we arrive at

$$\frac{\partial h}{\partial \boldsymbol{k}_r^i} = \sum_s \Big\langle \nabla_s \mathcal{H}(K), \frac{\partial K_s}{\partial \boldsymbol{k}_r^i} K_s^{-1} \Big\rangle.$$

Finally, assume that

$$\sum_{s=1}^{N} \langle \nabla_s \mathcal{H}, (K_s)_t K_s^{-1} \rangle = \{f, h\}_{\mathrm{inv}}(\boldsymbol{k})$$

for any h. Then

$$\sum_{s=1}^{N} \sum_{r=1}^{N} \sum_{i=1}^{n} \Big\langle (\boldsymbol{k}_r^i)_t \frac{\partial K_s}{\partial \boldsymbol{k}_r^i} K_s^{-1}, \nabla_s \mathcal{H} \Big\rangle = \sum_s \sum_{r,i} (\boldsymbol{k}_r^i)_t \frac{\partial h}{\partial k_r^i} = \{f, h\}_{\mathrm{inv}}(\boldsymbol{k})$$

for any h, and hence, by definition, $\boldsymbol{k}$ evolves via a Hamiltonian evolution, with Hamiltonian function f. $\square$

Theorem 4.2. *Let (x_s) evolve using an evolution of the form* (4), *for some invariant vector $\boldsymbol{v}_s$, and let ς be a section such that $\rho_s = \rho_s^P \varsigma(x_s)^{-1}$, $\rho_s^P \in P$, for any s. Assume that there exits a function $f(\boldsymbol{k})$, with extension $\mathcal{F}$ constant on the gauge orbits of $\mathcal{P}$, and such that*

$$d\varsigma(o)\boldsymbol{v}_s = \tau^{-1}(\nabla_s \mathcal{F})_{-1}. \tag{23}$$

The evolution induced on $\boldsymbol{k}$ by (4) *is Hamiltonian with respect to $\{\cdot,\cdot\}_{\text{inv}}$, with Hamiltonian f.*

Proof. Using (6) and (21), we have that, on the one hand

$$\sum_{s=1}^{N} \langle (K_s)_t K_s^{-1}, \nabla_s \mathcal{H}\rangle = \sum_{s=1}^{N} \langle N_{s+1} - K_s N_s K_s^{-1}, \nabla_s \mathcal{H}\rangle, \tag{24}$$

and on the other hand

$$\{f,h\}_{\text{inv}}(\boldsymbol{k}) = \frac{1}{2}\sum_{s=1}^{N} \langle (\nabla_s \mathcal{H})_{-1}, (\nabla_s \mathcal{F} - \tau \nabla_s' \mathcal{F})_1\rangle - \langle (\nabla_s \mathcal{F})_{-1}, (\nabla_s \mathcal{H} - \tau \nabla_s' \mathcal{H})_1\rangle.$$

Now, since $\nabla_s \mathcal{F} - \tau \nabla_s' \mathcal{F} \in \mathfrak{g}_1$, and $\mathfrak{g}_{-1}$ is the dual of $\mathfrak{g}_1$, we have

$$\begin{aligned}\langle (\nabla_s \mathcal{H})_{-1}, (\nabla_s \mathcal{F} - \tau \nabla_s' \mathcal{F})_1\rangle &= \langle \nabla_s \mathcal{H}, \nabla_s \mathcal{F} - \tau \nabla_s' \mathcal{F}\rangle \\ &= \langle \nabla_s \mathcal{H}, \nabla_s \mathcal{F}\rangle - \langle \nabla_s \mathcal{H}, \tau \nabla_s' \mathcal{F}\rangle.\end{aligned}$$

Also, since $\langle\cdot,\cdot\rangle$ is invariant under the adjoint action and under the shift operator, $\sum_s \langle \nabla_s \mathcal{H}, \nabla_s \mathcal{F}\rangle = \sum_s \langle \tau \nabla_s' \mathcal{H}, \tau \nabla_s' \mathcal{F}\rangle$. Substituting this in our calculations we get

$$\begin{aligned}\sum_s \langle (\nabla_s \mathcal{H})_{-1}, (\nabla_s \mathcal{F} - \tau \nabla_s' \mathcal{F})_1\rangle &= \sum_s \langle \tau \nabla_s' \mathcal{H}, \tau \nabla_s' \mathcal{F}\rangle - \langle \nabla_s \mathcal{H}, \tau \nabla_s' \mathcal{F}\rangle \\ &= \sum_s \langle \tau \nabla_s' \mathcal{H} - \nabla_s \mathcal{H}, \tau \nabla_s' \mathcal{F}\rangle \\ &= \sum_s \langle (\tau \nabla_s' \mathcal{H} - \nabla_s \mathcal{H})_1, (\tau \nabla_s' \mathcal{F})_{-1}\rangle \\ &= \sum_s \langle (\tau \nabla_s' \mathcal{H} - \nabla_s \mathcal{H})_1, (\nabla_s \mathcal{F})_{-1}\rangle,\end{aligned}$$

where we have used that $(\nabla_s \mathcal{F})_{-1} = (\tau \nabla_s' \mathcal{F})_{-1}$ since $\nabla_s \mathcal{F} - \tau \nabla_s' \mathcal{F} \in \mathfrak{g}_1$. Therefore

$$\{f,h\}_{\text{inv}}(\boldsymbol{k}) = -\sum_{s=1}^{N} \langle (\nabla_s \mathcal{F})_{-1}, (\nabla_s \mathcal{H} - \tau \nabla_s' \mathcal{H})_1.$$

Back to (24). Since

$$\sum_s \langle K_s N_s K_s^{-1}, \nabla_s \mathcal{H}\rangle = \sum_s \langle N_s, K_s^{-1} \nabla_s \mathcal{H} K_s\rangle = \sum_s \langle \tau N_s, \tau(K_s^{-1} \nabla_s \mathcal{H} K_s)\rangle,$$

we have

$$
\begin{aligned}
\sum_{s=1}^{N}\langle N_{s+1}-K_sN_sK_s^{-1},\nabla_s\mathcal{H}\rangle &= \sum_{s=1}^{N}\langle \tau N_s,\nabla_s\mathcal{H}-\tau(K_s^{-1}\nabla_s\mathcal{H}K_s)\rangle\\
&= \sum_{s=1}^{N}\langle \tau N_s,\nabla_s\mathcal{H}-\tau\nabla_s'\mathcal{H}\rangle\\
&= \sum_{s=1}^{N}\langle (\tau N_s)_{-1},(\nabla_s\mathcal{H}-\tau\nabla_s'\mathcal{H})_1\rangle.
\end{aligned}
$$

But, if $\tau\varsigma(o)\boldsymbol{v}_s=\nabla_{-1}\mathcal{F}$ (and since $(N_s)_{-1}=-d\varsigma(o)\boldsymbol{v}_s$ by (7)), we get $(\tau N_s)_{-1}=-(\nabla\mathcal{F})_{-1}$, and hence

$$
\sum_{s=1}^{N}\langle (K_s)_tK_s^{-1},\nabla_s\mathcal{H}\rangle = -\sum_{s=1}^{N}\langle (\nabla_s\mathcal{F})_{-1},(\nabla_s\mathcal{H}-\tau\nabla_s'\mathcal{H})_1\rangle=\{f,h\}_{\mathrm{inv}}(\boldsymbol{k}).
$$

Using our previous lemma, we conclude the proof. □

Remark 4.3. In all examples we can think of the values of $(N_s)_{-1}=-d\varsigma(o)\boldsymbol{v}_s$ and condition (4) as determining N_s uniquely. This means that if x_s induces an evolution on $\boldsymbol{k}$ which is Hamiltonian with respect to $\{\cdot,\cdot\}_{\mathrm{inv}}$ with Hamiltonian f, then necessarily $\tau(N_s)_{-1}=-(\nabla_s\mathcal{F})_{-1}$, since this choice induces the same evolution and N_s is unique given those determining values. Hence, assuming that N is uniquely determined by $(N_s)_{-1}$, $s=1,\dots,N$, and (4), the converse of the theorem is also true.

5. The Lagrangian Grassmannian example: the Lagrangian Schwarzian difference and Volterra evolutions

In this section we apply the previous construction to the case of the Lagrangian Grassmannian. In this example $G=\mathrm{Sp}(2n)$ and the parabolic gradation of the algebra is given by

$$
\begin{pmatrix}\mathbf{0}&\mathbf{0}\\ Z&\mathbf{0}\end{pmatrix}\in\mathfrak{g}_1,\quad \begin{pmatrix}A&\mathbf{0}\\ \mathbf{0}&-A^T\end{pmatrix}\in\mathfrak{g}_0,\quad \begin{pmatrix}\mathbf{0}&Y\\ \mathbf{0}&\mathbf{0}\end{pmatrix}\in\mathfrak{g}_{-1},\tag{25}
$$

where $\mathbf{0}$ is the zero $n\times n$ block, Z and Y are symmetric matrices, and A is a general $n\times n$ matrix. Here $\mathfrak{p}=\mathfrak{g}_0\oplus\mathfrak{g}_1$. The associated local factorization of the group is given by

$$
g=\begin{pmatrix}I&\mathbf{0}\\ \hat{S}&I\end{pmatrix}\begin{pmatrix}\Theta&\mathbf{0}\\ \mathbf{0}&\Theta^{-T}\end{pmatrix}\begin{pmatrix}I&S\\ \mathbf{0}&I\end{pmatrix}\in G_1G_0G_{-1},\tag{26}
$$

with $\Theta \in \mathrm{GL}(n, \mathbb{R})$, $\widehat{S}$ and S symmetric. Also, $P = G_1 G_0$ and G/P is the Lagrangian Grassmannian. For a more geometric definition, consider n vectors defining a given n-dimensional subspace x of $\mathbb{R}^{2n}$. We can find n such vectors so that when placed as columns in a matrix, the matrix will look like

$$\begin{pmatrix} u \\ I \end{pmatrix}.$$

If the subspace is Lagrangian, u will be symmetric. We identify this subspace with the matrix

$$\varsigma(x) = \begin{pmatrix} I & u \\ \mathbf{0} & I \end{pmatrix} \in G_{-1}, \tag{27}$$

which defines a section of the quotient G/P.

The second gradation (17) is given by

$$\begin{pmatrix} A_L & \mathbf{0} \\ C & -A_L^T \end{pmatrix} \in \mathfrak{g}_+, \quad \begin{pmatrix} d & \mathbf{0} \\ \mathbf{0} & -d \end{pmatrix} \in \mathfrak{h}, \quad \begin{pmatrix} A_U & B \\ \mathbf{0} & -A_U^T \end{pmatrix} \in \mathfrak{g}_-, \tag{28}$$

where A_L is strictly lower triangular, A_U is strictly upper triangular, and d is diagonal. One can readily see that $\mathfrak{g}_+$, $\mathfrak{g}_-$ and $\mathfrak{h}$ are all subalgebras of $\mathfrak{g}$. Also clearly $\mathfrak{h}$ is commutative, $\mathfrak{g}_1 \subset \mathfrak{g}_+$, $\mathfrak{g}_{-1} \subset \mathfrak{g}_-$ and $\mathfrak{h} \subset \mathfrak{g}_0$, so that we can apply Theorem 3.4 to obtain a Poisson bracket on the moduli space of Lagrangian Grassmannian polygons. This structure is, in general, very complicated. What we want to do in this section is to show that some of the invariants of Lagrangian polygons behave in familiar and interesting ways under selected evolutions. For this we will go into details, constructing explicitly the invariants and their evolutions. We will then restrict the reduced bracket further to a submanifold generated by these special invariants.

A moving frame along Lagrangian Grassmannian polygons. Let g be factored as in (26). If we identify M with symmetric matrices u, using the section (27), and given that the action of $\mathrm{Sp}(2n)$ on M is determined by (18), we can write the action explicitly as

$$g \cdot u = \Theta(u + S)(\Theta^{-T} + \widehat{S}\Theta(u + S))^{-1}. \tag{29}$$

Assume we factor our right moving frame $\rho = (\rho_s) \in \mathrm{Sp}(2n)^N$ according to (26) as

$$\rho_s = \begin{pmatrix} I & \mathbf{0} \\ \widehat{S}_s & I \end{pmatrix} \begin{pmatrix} \Theta_s & \mathbf{0} \\ \mathbf{0} & \Theta_s^{-T} \end{pmatrix} \begin{pmatrix} I & S_s \\ \mathbf{0} & I \end{pmatrix}.$$

If we define transverse sections as in (1) through the normalizations

$$\rho_s \cdot u_s = \mathbf{0}, \quad \rho_s \cdot u_{s-1} = -I, \quad \rho_s \cdot u_{s+1} = I,$$

we obtain the equations

$$
\begin{aligned}
u_s + S_s &= \mathbf{0},\\
\Theta_s(u_{s+1} + S_s) &= \Theta_s^{-T} + \widehat{S}_s\Theta_s(u_{s+1} + S_s),\\
\Theta_s(u_{s-1} + S_s) &= -(\Theta_s^{-T} + \widehat{S}_s\Theta_s(u_{s-1} + S_s)).
\end{aligned}
$$

These can be solved for

$$S_s = -u_s, \quad \widehat{S}_s = I - \Theta_s^{-T}(u_{s+1} - u_s)^{-1}\Theta_s^{-1}$$

and

$$\Theta_s^T\Theta_s = \tfrac{1}{2}\big((u_{s+1} - u_s)^{-1} + (u_s - u_{s-1})^{-1}\big) = U_s^{-1}. \tag{30}$$

Equation (30) determines Θ completely up to an orthogonal factor, assuming that $U_s^{-1} = \frac{1}{2}\big((u_{s+1} - u_s)^{-1} + (u_s - u_{s-1})^{-1}\big)$ is positive definite. In fact, we have $\Theta_s = \theta_s U_s^{-1/2}$, where $\theta_s \in \mathrm{O}(n)$ and $U_s^{-1/2}$ is a square root of a symmetric matrix as defined in [Ovsienko 1993], unique up to the action of the orthogonal group. That is, $U_s^{-T/2}U_s^{-1/2} = U_s^{-1}$.

To determine the last factor θ_s, and with it the rest of the moving frame, we need to choose one more normalization, thus completing the definition of the transverse section. Let's choose $\rho_s \cdot u_{s+2}$ to be diagonal. After substituting all the values we have already found we get

$$\rho_s \cdot u_{s+2} = \big(I + \Theta^{-T}((u_{s+2} - u_s)^{-1} - (u_{s+1} - u_s)^{-1})\Theta^{-1}\big)^{-1}.$$

Definition 5.1 (Lagrangian Schwarzian difference). Given a generic polygon of Lagrangian planes u_s, we define $S(u) = (S_s(u))$ to be

$$
\begin{aligned}
S_s(u) &= U_s^{-1/2}\big(U_s^{-1} + (u_{s+2} - u_s)^{-1} - (u_{s+1} - u_s)^{-1}\big)^{-1}U_s^{-T/2}\\
&= U_s^{-1/2}\big[(u_{s+2} - u_s)^{-1} - \tfrac{1}{2}(u_{s+1} - u_s)^{-1} + \tfrac{1}{2}(u_s - u_{s-1})^{-1}\big]^{-1}U_s^{-T/2}
\end{aligned}
$$

and we call it *the Lagrangian Schwarzian difference of* u, where U_s is as in (30).

This definition is the discrete analogue to the Lagrangian Schwarzian derivative defined in [Ovsienko 1993] for curves of Lagrangian planes. In fact, if we denote $u(s+k) = \gamma(x + k\epsilon)$, a long but standard calculation shows that the continuous limit of $S_s(u)$ is indeed a multiple of the Lagrangian Schwarzian derivative defined in the same reference. Now, in order to diagonalize $\rho_s \cdot u_{s+2}$ we need to choose θ_s to be an element of the orthogonal group that diagonalizes the symmetric matrix $S_s(u)$. If we call

$$\widetilde{D}_s = \theta_s S_s(u)\theta_s^T, \tag{31}$$

then $\rho_s \cdot u_{s+2} = \widetilde{D}$ will be diagonal. These normalization choices describe transverse sections as in (1), and they determine the moving frame ρ uniquely. From now on we will denote $D_s = I - \widetilde{D}_s^{-1}$, and hence $I^s_{s+2} = (I + D_s)^{-1}$.

Maurer–Cartan invariants and their evolutions. Once we have determined a moving frame, we would like to describe the right Maurer–Cartan matrix associated to it. To do this we will use the recursion equations (3)

$$K_s \cdot I^s_k = I^{s+1}_k.$$

Using the choices $I^s_s = \mathbf{0}$, $I^s_{s+1} = I$, $I^s_{s-1} = -I$ and $I^s_{s+2} = (I + D_s)^{-1}$, we select the equations

$$K_s \cdot \mathbf{0} = -I, \quad K_s \cdot I = \mathbf{0}, \quad K_s \cdot I^s_{s+2} = I, \tag{32}$$

as those determining K. Assume that K_s factorizes as

$$K_s = \begin{pmatrix} I & \mathbf{0} \\ K_{s,1} & I \end{pmatrix} \begin{pmatrix} K_{s,0} & \mathbf{0} \\ \mathbf{0} & (K_{s,0})^{-T} \end{pmatrix} \begin{pmatrix} I & K_{s,-1} \\ \mathbf{0} & I \end{pmatrix}.$$

Straightforward calculations using formula (29) show that the three recursion equations (32) determine the values

$$K_{s,-1} = -I, \quad K^T_{s,0} K_{s,0} = -\tfrac{1}{2} D_s^{-1}, \quad K_{s,1} = K^{-T}_{s,0} K^{-1}_{s,0} - I.$$

Assuming that D_s is negative definite, we obtain the solutions

$$K_{s,-1} = -I, \quad K_{s,0} = \widehat{K}_{s,0} \widehat{D}_s, \quad K_{s,1} = -(I + 2\widehat{K}_{s,0} D_s \widehat{K}^T_{s,0}), \tag{33}$$

where

$$\widehat{D}_s = \frac{1}{\sqrt{2}} (-D_s)^{-1/2}, \quad \widehat{K}_{s,0} \in \mathrm{O}(n). \tag{34}$$

Remark 5.2. the negative definite condition imposed on D_s can be removed by merely choosing different normalizations. Indeed, if we choose arbitrary values for I^s_r, the relations between the different invariants determined by equations (32) become

$$K^T_{s,0}((I^s_{s+1})^{-1} - (I^s_{s-1})^{-1}) K_{s,0} = -(I^s_{s+1})^{-2} D_s^{-1}.$$

Thus, if D_s is positive definite, we could choose $I^s_{s+1} = -I$ and $I^s_{s-1} = I$ instead. We can also change the sign of the different entries in I, depending on the sign of the different eigenvalues of $S(u)$. For simplicity we will keep the choices above.

The following theorem summarizes our findings.

Theorem 5.3. *There exists a right moving frame along polygons of Lagrangian subspaces such that its associated Maurer–Cartan matrix is of the form*

$$K_s = \begin{pmatrix} I & \mathbf{0} \\ -(I + 2\widehat{K}_{s,0} D_s \widehat{K}_{s,0}^T) & I \end{pmatrix} \begin{pmatrix} \widehat{K}_{s,0} \widehat{D}_s & \mathbf{0} \\ \mathbf{0} & \widehat{K}_{s,0}^T \widehat{D}_s^{-1} \end{pmatrix} \begin{pmatrix} I & -I \\ \mathbf{0} & I \end{pmatrix} \tag{35}$$

where $\widehat{D}_s$ *is given as in* (34). *The entries of* D_s *and* $\widehat{K}_{s,0}$ *functionally generate all invariants of Lagrangian polygons.*

Next we turn to the study of invariant evolutions of Lagrangian polygons (that is, those for which $\mathrm{Sp}(2n)$ takes solutions to solutions) and the equations they induce on the invariants. Assume $(u_s(t))$, with $u_s(t)$ symmetric, represents a family of polygons of Lagrangian planes, and assume it is a solution of an invariant evolution. According to Theorem 2.6, the equation can be written in terms of our moving frame. Since the linearization at o of the action (29) is given by

$$v \mapsto \Theta v \Theta^T$$

and having in mind that the G_0 factor of ρ_s^{-1} is $U_s^{1/2}\theta_s^T$, from (4) we conclude that any invariant evolution can be written as

$$(u_s)_t = U_s^{1/2}\theta_s^T \boldsymbol{v}_s \theta_s U_s^{T/2} \tag{36}$$

for symmetric matrices $\boldsymbol{v}_s$ depending on the entries of (D_r) and $(\widehat{K}_{r,0})$, and where θ_s diagonalizes the Lagrangian Schwarzian difference of the flow. From now on we will assume that $D_s = \mathrm{diag}(d_i^s)$, with $d_i^s \neq d_j^s$ for all $i \neq j$.

Theorem 5.4. *Let* $\boldsymbol{v}_s$ *be diagonal, and assume the initial condition* $u_s(0)$ *satisfies* $\widehat{K}_{s,0} = I$. *Then* $\widehat{K}_{s,0} = I$ *is preserved by the flow* (36) *and whenever* $\boldsymbol{v}_s = -\frac{1}{2}(1+\tau^{-1})D_s\nabla_s f$ *for some Hamiltonian function* f, D_s *satisfies a Hamiltonian equation with respect to the Poisson structure*

$$\mathcal{P} = \sum_s D_s\big(D_s\tau^{-1} - D_s\tau + 2(\tau^{-1}-\tau) + \tau^{-1}D_s - \tau D_s + \tau^{-1}D_s\tau^{-1} - \tau D_s\tau\big)D_s. \tag{37}$$

with Hamiltonian f. *Assume* $\boldsymbol{v}_s = -I$ (*and hence* $\nabla_s f = D_s^{-1}$). *Then, as polygons evolve following*

$$(u_s)_t = 2((u_s - u_{s+1})^{-1} - (u_s - u_{s-1})^{-1})^{-1},$$

the eigenvalues of the Lagrangian Schwarzian difference evolve following a decoupled system of Volterra equations.

Proof. From now on, and to avoid cluttering, we will drop the subindex s and will only use it if needed to avoid confusion. Thus, N_{s+1} will become τN, N_s will become N, $\widehat{D}_s$ will become $\widehat{D}$, and so on.

We will use Theorem 2.7. Consider the section $\varsigma : M \to G_{-1}$ given by

$$\varsigma(u) = \begin{pmatrix} I & u \\ \mathbf{0} & I \end{pmatrix}.$$

It satisfies $\rho = \rho^P \varsigma(u)^{-1}$ with $\rho^P \in P$. Theorem 2.7 tells us that $N = \rho_t \rho^{-1}$ must be of the form

$$N = \begin{pmatrix} N_0 & -\boldsymbol{v} \\ N_1 & -N_0^T \end{pmatrix}$$

for some $N_0 \in \mathfrak{gl}(n)$, N_1 symmetric and for some symmetric matrix $\boldsymbol{v}$ depending on the invariants. A straightforward calculation shows that if K is as in (35), and if $\widehat{K}_0 = I$, then

$$KNK^{-1} = \begin{pmatrix} I & \mathbf{0} \\ K_1 & I \end{pmatrix} \begin{pmatrix} \widehat{D}(N_0 - N_1)\widehat{D}^{-1} & \widehat{D}(N_0 + N_0^T - N_1 - \boldsymbol{v})\widehat{D} \\ \widehat{D}^{-1} N_1 \widehat{D}^{-1} & \widehat{D}^{-1}(N_1 - N_0^T)\widehat{D} \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ -K_1 & I \end{pmatrix},$$

where $K_1 = -(I + 2D)$. To simplify formulas we will conjugate (4) by $\left(\begin{smallmatrix} I & 0 \\ -K_1 & I \end{smallmatrix}\right)$. Direct, although longer, calculations show that if $\widehat{K}_0 = I$

$$\begin{pmatrix} I & \mathbf{0} \\ -K_1 & I \end{pmatrix} K_t K^{-1} \begin{pmatrix} I & \mathbf{0} \\ K_1 & I \end{pmatrix} = \begin{pmatrix} (\widehat{K}_0)_t - \frac{1}{2} D^{-1} D_t & \mathbf{0} \\ 2(-D_t + D(\widehat{K}_0)_t - (\widehat{K}_0)_t D) & (\widehat{K}_0)_t + \frac{1}{2} D^{-1} D_t \end{pmatrix},$$

where we have used the relationship

$$\widehat{D}^{-1}(\widehat{D})_t = -\tfrac{1}{2} D^{-1} D_t.$$

Also

$$\begin{pmatrix} I & \mathbf{0} \\ -K_1 & I \end{pmatrix} \begin{pmatrix} \tau N_0 & -\tau \boldsymbol{v} \\ \tau N_1 & -\tau N_0^T \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ K_1 & I \end{pmatrix} =$$

$$\begin{pmatrix} \tau N_0 + \tau \boldsymbol{v}(I + 2D) & -\tau \boldsymbol{v} \\ \tau N_1 + \tau N_0 + \tau N_0^T + 2D\tau N_0 + 2T N_0^T D + (I+2D)^2 \tau \boldsymbol{v} & -\tau N_0^T - (I+2D)\tau \boldsymbol{v} \end{pmatrix}.$$

Substituting these values in the conjugation of 2.7 by $\left(\begin{smallmatrix} I & 0 \\ -K_1 & I \end{smallmatrix}\right)$, and equating the different entries, we arrive at the equations

$$\begin{aligned}
\tau \boldsymbol{v} &= \widehat{D}(\boldsymbol{v} + N_1 - N_0 - N_0^T)\widehat{D}, \\
D^{-1} D_t &= \widehat{D}(N_0 - N_1)\widehat{D}^{-1} + \widehat{D}^{-1}(N_0^T - N_1)\widehat{D} - 2(I + 2D)\tau \boldsymbol{v} - \tau N_0 - \tau N_0^T, \\
2(\widehat{K}_0)_t &= \tau N_0^T - \tau N_0 + \widehat{D}(N_0 - N_1)\widehat{D}^{-1} + \widehat{D}^{-1}(N_1 - N_0^T)\widehat{D}, \\
2(D(\widehat{K}_0)_t - (\widehat{K}_0)_t D - D_t) &= \tau N_1 + \tau N_0 + \tau N_0^T + 2(D\tau N_0 + \tau N_0^T D) \\
&\qquad + (I + 2D)^2 \tau \boldsymbol{v} - \widehat{D}^{-1} N_1 \widehat{D}^{-1}.
\end{aligned}$$

The last three equations result in a compatibility condition that can be obtained as follows: we use the second equation and we multiply once on the left and once on the right by D, thus obtaining two equations. We substitute the sum of these two equations in place of $2D_t$ in the last equation, and use the third equation to substitute $(\widehat{K}_0)_t$. After some straightforward work we obtain

$$\tau N_1 + \tau N_0 + \tau N_0^T = \widehat{D}^{-1}(N_0 + N_0^T - N_1)\widehat{D}^{-1} - \tau \boldsymbol{v}(I - 4D^2).$$

We now use the first equation, and we get

$$\tau N_1 = -(\tau(D\tau\boldsymbol{v}) + \tau\boldsymbol{v} + D\boldsymbol{v}). \tag{38}$$

From this, if $\boldsymbol{v}$ is diagonal, so is N_1. The first equation implies that $N_0 + N_0^T$ is also diagonal. Back to the second equation, we see that $\widehat{D}N_0\widehat{D}^{-1} + \widehat{D}^{-1}N_0^T\widehat{D}$ is once more diagonal, which with our assumption $d_i \neq d_j$ implies that N_0 is diagonal. If N_0 and N_1 are diagonal, the third equation tells us that $(\widehat{K}_0)_t = 0$, proving the first assertion of the theorem.

We can now find D_t. Using (38) and the first equation we have

$$2N_0 = (D\tau - \tau^{-1}D)\boldsymbol{v},$$

and substituting everything into the second equation we get

$$D^{-1}D_t = (D - \tau D\tau + \tau^{-1}D - D\tau + 2 - 2\tau)\boldsymbol{v}. \tag{39}$$

Finally, if we substitute $\boldsymbol{v} = \frac{1}{2}(1 + \tau^{-1})D\nabla f$, we have

$$D_t = \tfrac{1}{2}D\big(D\tau^{-1} - D\tau + 2(\tau^{-1} - \tau) + \tau^{-1}D - \tau D + \tau^{-1}D\tau^{-1} - \tau D\tau\big)D\nabla f, \tag{40}$$

which is a Hamiltonian equation with respect to the Hamiltonian structure

$$\mathcal{P} = D\big(D\tau^{-1} - D\tau + 2(\tau^{-1} - \tau) + \tau^{-1}D - \tau D + \tau^{-1}D\tau^{-1} - \tau D\tau\big)D.$$

This is the second Hamiltonian structure for the Volterra equation

$$D_t = D(\tau - \tau^{-1})D$$

(see [Khanizadeh et al. 2013]), which we obtain whenever $\boldsymbol{v} = I$ and $\nabla f = D^{-1}$. Adopting subindices again and using (36) and (30), if $\boldsymbol{v}_s = I$, the corresponding evolution for u_s is given by

$$(u_s)_t = U_s^{-1/2}U_s^{-T/2} = U_s^{-1} = 2\big((u_{s+1} - u_s)^{-1} + (u_s - u_{s-1})^{-1}\big)^{-1}$$

concluding the proof of the theorem. □

Of course, we did not guess the relation

$$\boldsymbol{v} = \tfrac{1}{2}(1 + \tau^{-1})D\nabla f; \tag{41}$$

it was given to us by the general reduction process and the compatibility condition (23). We will describe the reduction next for our particular example and see how we arrived to relation (41).

The reduced bracket and the double reduction to D. Before we prove that the reduced bracket found in the previous section is further reducible to D, we will describe the reduced bracket itself in a little more detail. Once more we are dropping the subindex to avoid cluttering. As explained in (21), to calculate the reduction one considers two functions f and h defined on the invariants (coordinates) generating D and $\widehat{K}_0$. Let us denote these invariants by $d = (d_i)$ and $k_{i,j}$, $i < j$. Let us also denote by $\delta_1 f$ the diagonal matrix $\delta_1 f = \mathrm{diag}\partial f/\partial d_i$, and by $\delta_0 f$ a skew symmetric matrix generated by $\partial f/\partial k_{i,j}$. (The precise form of $\delta_0 f$ will become clear along the process, therefore we will postpone the description until relevant.)

Set

$$\nabla\mathcal{F} = \begin{pmatrix} F_0 & F_{-1} \\ F_1 & -F_0^T \end{pmatrix} \tag{42}$$

and let us split F_0 as

$$F_0 = F_0^\sigma + F_0^{\sigma k} + F_0^d, \tag{43}$$

where σ indicates the symmetric diagonal free components, σk is the skew-symmetric component, and d the diagonal. Likewise for F_{-1} and F_1 (clearly $F_{-1}^{\sigma k} = F_1^{\sigma k} = 0$). Let us denote by $K(D, \widehat{K}_0)$ the family of matrices (35), and consider the element of $\mathrm{Sp}(2n)^N$

$$Z(\epsilon) = K(D+\epsilon V, \widehat{K}_0)K(D, \widehat{K}_0)^{-1},$$

where V is an arbitrary diagonal matrix. Let us call v the diagonal of V, written as a vector. On the one hand, direct calculations show that when $\widehat{K}_0 = I$,

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} Z(\epsilon) = \begin{pmatrix} -\frac{1}{2}D^{-1}V & \mathbf{0} \\ D^{-1}V & \frac{1}{2}D^{-1}V \end{pmatrix}.$$

On the other hand $Z(\epsilon)K = K(D+\epsilon V, \widehat{K}_0)$ and since $\mathcal{F}$ is an extension of f

$$F(Z(\epsilon)K) = F(K(D+\epsilon V, \widehat{K}_0)) = f(d+\epsilon v, k_{i,j}).$$

Differentiating with respect to ϵ,

$$\left\langle \nabla F, \begin{pmatrix} -\frac{1}{2}D^{-1}V & \mathbf{0} \\ D^{-1}V & \frac{1}{2}D^{-1}V \end{pmatrix} \right\rangle = \langle D^{-1}(F_{-1}^d - F_0^d), V\rangle = \langle \delta_1 f, V\rangle.$$

This is true for any value of V, and therefore

$$F_{-1}^d - F_0^d = D\delta_1 f. \tag{44}$$

Likewise, we can choose $Z(\epsilon)$ such that

$$F(Z(\epsilon)K) = F(D, \widehat{K}_0(\epsilon)) = f(d, k_{i,j}(\epsilon))$$

with $k_{i,j}(0) = k_{i,j}$ and $\frac{d}{d\epsilon}\big|_{\epsilon=0} k_{i,j}(\epsilon) = w_{i,j}$. Indeed $Z(\epsilon) = K(D, \widehat{K}_0(\epsilon))K^{-1}$, with $\widehat{K}_0(\epsilon)$ chosen so that $\widehat{K}_0(0) = \widehat{K}_0$ and $\widehat{K}_0^{-1}\frac{d}{d\epsilon}\big|_{\epsilon=0}\widehat{K}_0 = W$, $W = (w_{i,j})$. (One can be more specific and create $Z(\epsilon)$ using the exponential function, but any such family can be used and the precise form is not relevant.)

Differentiating with respect to ϵ, we have

$$\left\langle \nabla F, \begin{pmatrix} W & \mathbf{0} \\ \mathbf{0} & W \end{pmatrix} \right\rangle = \langle 2F_0, W\rangle = \langle \delta_0 f, W\rangle,$$

and hence $F_0^- = \frac{1}{2}\delta_0 f$. We can now determine how the matrix $\delta_0 f$ is created: it is defined as the skew-symmetric matrix such that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} f(d, k_{i,j}(\epsilon)) = \langle \delta_0 f, W\rangle$$

with $W = (w_{i,j})$ (that is, $(\delta_0 f)_{i,j} = \partial f/\partial k_{j,i}$). With this in mind, we proceed to our last theorem.

Theorem 5.5. *The reduced Poisson bracket* (21) *can be further reduced to the submanifold $\widehat{K}_0 = I$. When using the coordinates given by the invariants D, the resulting bracket is a decoupled system of Hamiltonian structures for the Volterra equation as in* (37).

Proof. More precisely, what we will show is that if f is independent of $k_{i,j}$ and h is independent of d_i, at $\widehat{K}_0 = I$ their reduced bracket vanishes, while the reduced bracket of two functions that only depend on D is given by the second Volterra structure. Once again, and as explained in (21), if f and h are two functions depending on D and $\widehat{K}_0$, their reduced bracket is defined as

$$\{f, h\}_{\text{inv}} = \tfrac{1}{2}\langle(\nabla\mathcal{H})_{-1}, (\nabla\mathcal{F})_1 - \tau(\nabla'\mathcal{F})_1\rangle - \tfrac{1}{2}\langle(\nabla\mathcal{F})_{-1}, (\nabla\mathcal{H})_1 - \tau(\nabla'\mathcal{H})_1\rangle,$$

where $\mathcal{F}$ and $\mathcal{H}$ are two extensions satisfying (20). Assume $\nabla\mathcal{F}$ is given as in (42) with the splitting (43). After conjugating $\nabla'\mathcal{F}$ back to $\nabla\mathcal{F}$ and simplifying, condition (20) results in two equations at $\widehat{K}_0 = I$, namely

$$\tau^{-1}F_0 = \widehat{D}^{-1}(F_0 - F_{-1}K_1)\widehat{D} + \widehat{D}(F_1 + K_1F_0 + F_0^TK_1 - K_1F_{-1}K_1)\widehat{D}, \tag{45}$$

$$\tau^{-1}F_{-1} = \widehat{D}(F_{-1} - F_0^T)\widehat{D}^{-1} - \tau^{-1}F_0. \tag{46}$$

Case 1: *Assume that f depends only on D.* Then $F_0^{\sigma k} = \frac{1}{2}\delta_0 f = \mathbf{0}$. Using (45),

$$\mathbf{0} = \tau^{-1}(F_0 - F_0^T) = \widehat{D}^{-1}(F_0 - F_{-1}K_1)\widehat{D} - \widehat{D}(F_0^T - K_1F_{-1})\widehat{D}^{-1}.$$

Using that $K_1 = I - \widehat{D}^{-2}$, we get

$$\mathbf{0} = \widehat{D}^{-1}(F_0 - F_{-1})\widehat{D} - \widehat{D}(F_0^T - F_{-1})\widehat{D}^{-1},$$

which, under the condition $d_i \neq d_j$, implies

$$F_0^\sigma = F_{-1}^\sigma.$$

Substituting this in (45) we get

$$2\tau F_1^\sigma = \widehat{D}^{-1} F_{-1}^\sigma \widehat{D}^{-1} - \widehat{D}^{-1} F_{-1}^\sigma \widehat{D}^{-1} = \mathbf{0}$$

and, therefore,

$$F_{-1}^\sigma = F_0^\sigma = \mathbf{0}.$$

This implies that F_1 and F_0 are both diagonal. If we now look at (45) we clearly see that F_1 is also diagonal. These diagonals can easily be found from (45)-(46) and the relation $F_{-1}^d = D\delta_1 f + F_0^d$. They are

$$F_{-1}^d = \tfrac{1}{2}(\tau + I)D\delta_1 f \tag{47}$$

$$F_0^d = \tfrac{1}{2}(\tau - I)D\delta_1 f \tag{48}$$

$$F_1^d = \big(D\tau^{-1}D + D^2 - \tfrac{1}{2}\tau D + \tfrac{3}{2}D\big)\delta_1 f. \tag{49}$$

Case 2: *Assume that f does not depend on D*. Then $F_{-1}^d = F_0^d + D\delta_1 f = F_0^d$, and equating the diagonals in (46) we get $F_{-1}^d = F_0^d = \mathbf{0}$. Doing the same with (45), we have that $F_1^d = \mathbf{0}$ also.

Now, assume that f depends on D, while h doesn't. Then, both $(\nabla\mathcal{F})_{-1}$ and $(\nabla\mathcal{F} - \tau\nabla'\mathcal{F})_1$, whose only nonconstant block is given by

$$F_1 - \tau(\widehat{D}(F_1 + K_1 F_0 + F_0^T K_1 - K_1 F_{-1} K_1)\widehat{D}),$$

are diagonal, while $(\nabla\mathcal{H})_{-1}$ and $(\nabla\mathcal{H} - \tau\nabla'\mathcal{H})_1$ are diagonal free. Therefore as one can readily see from the formula in (21),

$$\{f, h\}_{\text{inv}} = 0.$$

If both f and h depend on D only, we can substitute (47)-(48)-(49) in (21) to find the double reduction. It is given by

$$\begin{aligned}&\{f, h\}_{\text{inv}}(D)\\ &= \tfrac{1}{2}\big\langle\delta_1 h, D\big((\tau^{-1} - \tau)D + D(\tau^{-1} - \tau) + 2(\tau^{-1} - \tau) + \tau^{-1}D\tau^{-1} - \tau D\tau\big)D\delta_1 f\big\rangle,\end{aligned}$$

which is a decoupled system of second Hamiltonian structures for the Volterra lattice, as stated. $\square$

We can now see where the relation $\boldsymbol{v}_s = \frac{1}{2}(I + \tau^{-1})D_s\delta_s f$ came from: $F_{-1} = \frac{1}{2}(\tau + 1)D\delta_1 f$ and $\tau \boldsymbol{v}_s = F_{-1}$, according to (23).

6. Conclusion and further study

In this paper we have shown that, if G is semisimple and $\mathfrak{g}$ is a $|1|$-graded Lie algebra with a parabolic gradation compatible with a second grading of the form (17), then the moduli space of polygons in G/P is endowed with a natural Poisson bracket that can be linked to invariantizations of polygon evolutions. As an example we described in detail the case of polygons of Lagrangian subspaces in $\mathbb{R}^{2n}$. We show that under some nondegeneracy conditions the Poisson bracket can be restricted further to a certain submanifold of Lagrangian planes, and that on this submanifold the eigenvalues of the Lagrangian Schwarzian difference evolve following a Hamiltonian evolution, one that becomes a decoupled system of Volterra equations when a proper Hamiltonian is chosen.

In the continuous case, the existence of a Poisson structure on the moduli space of curves is guaranteed not only for the case of $|1|$-graded Lie algebras but also for general homogeneous manifolds of the form G/H with G semisimple [Marí Beffa 2010] and for semidirect products [Marí Beffa 2006]. It is well possible that the same is true for the discrete counterpart, but the discrete case is more difficult to study. The main obstacle is the need to rely on R-matrices to define the Poisson Lie group at the beginning of our construction. If we consider a general case G/H, with $\hat{\mathfrak{h}}$ being the Lie algebra of H, then to be able to use these Poisson structures we will need to coordinate $\hat{\mathfrak{h}}$ with a gradation of the form (17), with $\hat{\mathfrak{h}}^0 \subset \mathfrak{g}_+$: on the one hand (17) is used to define the R-matrix, and on the other hand $\hat{\mathfrak{h}}$ is used for the reduction itself, so both need to be coordinated throughout calculations. Not only that, if m is a linear complement for $\hat{\mathfrak{h}}$, so $\mathfrak{g} = \hat{\mathfrak{h}} \oplus \mathfrak{m}$, in order for the proof of Theorem 3.4 to go through, one can check that we would need the condition $\mathfrak{m}^* \cap \mathfrak{m} = \mathbf{0}$. At first sight, this seems to not be always possible since choosing $\hat{\mathfrak{h}} = \mathfrak{g}_1$ (instead of $\hat{\mathfrak{h}} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$) provides a simple counterexample. Furthermore, in the general case, the action of G on G/H will also determine whether or not the bracket reduces. Indeed, the fact that the infinitesimal action was either constant, linear or quadratic, depending on which subgroup of G we were using, was fundamental to the admissibility of $\mathfrak{p}$ (we need the action to vanish at zero, and the derivative of the infinitesimal action of $\hat{\mathfrak{h}}^0$ to also vanish at zero). Hence, one will have to decide which actions qualify and which ones don't. Thus, although a more general theorem is true for those other cases that satisfy these three conditions, it would not be as general as the theorem for curve evolutions. Surprisingly, the case of the homogeneous 2-sphere SO(3)/SO(2) does not satisfy these three conditions (one can check that if $\mathfrak{m}$ is a linear complement $\mathfrak{m}^* \cap \mathfrak{m} \neq \mathbf{0}$), but nevertheless in [Mansfield et al. 2013], we described polygon evolutions on the 2-sphere $\mathrm{SO}(3)/\mathrm{SO}(2)$ inducing an equation of Volterra type on the discrete curvature of the polygon. Thus, perhaps a somehow different approach is needed to increase the generality. Work in that direction is under way.

A different and very interesting question is how one can get a second Hamiltonian structure, a companion for the reduction, to be used for integrability of difference evolutions. This point is not at all clear: in the continuous case it is know that it comes from a reduction of a second Hamiltonian structure (see [Marí Beffa 2010]), but it is also known that this second structure is not always reducible (a counterexample can be found in [Marí Beffa 2007]). No such natural second structure seems to exist in the discrete case and the situation becomes more murky. In [Marí Beffa and Wang 2013] we showed that in the case of $\mathbb{RP}^n$, even though the *right* bracket (the portion of the Sklyanin bracket involving right gradients only) is not Poisson, when reduced to G^N/P^N the result is Poisson and a second Hamiltonian structure for integrable discretizations of W_n-algebras. It seems to be a similar situation as having the Sklyanin bracket reduce to a Poisson bracket, even though it is not a Poisson bracket with our choice of parabolic gradation. Thus, perhaps there is an underlying bracket that coordinates with the right bracket to give the same result, but what bracket this might be is unknown to us.

References

[Bobenko and Suris 2008] A. I. Bobenko and Y. B. Suris, *Discrete differential geometry: integrable structure*, Graduate Studies in Mathematics **98**, American Mathematical Society, Providence, RI, 2008. MR 2010f:37125 Zbl 1158.53001

[Boutin 2002] M. Boutin, "Joint invariant signatures for curve recognition", pp. 37–52 in *Inverse problems, image analysis, and medical imaging* (New Orleans, LA, 2001), edited by M. Z. Nashed and O. Scherzer, Contemp. Math. **313**, American Mathematical Society, Providence, RI, 2002. MR 2003m:53023 Zbl 1063.68085

[Calini et al. 2009] A. Calini, T. Ivey, and G. Marí Beffa, "Remarks on KdV-type flows on star-shaped curves", *Phys. D* **238**:8 (2009), 788–797. MR 2010m:37127 Zbl 1218.37102

[Chou and Qu 2002] K.-S. Chou and C. Qu, "Integrable equations arising from motions of plane curves", *Phys. D* **162**:1-2 (2002), 9–33. MR 2003c:37106 Zbl 0987.35139

[Chou and Qu 2003] K.-S. Chou and C. Qu, "Integrable equations arising from motions of plane curves, II", *J. Nonlinear Sci.* **13**:5 (2003), 487–517. MR 2005d:37155 Zbl 1045.35063

[Drinfeld 1983] V. G. Drinfeld, "Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang–Baxter equations", *Dokl. Akad. Nauk SSSR* **268**:2 (1983), 285–287. In Russian; translated in *Sov. Math. Dokl.* **27** (1983), 68–71. MR 84i:58044 Zbl 0526.58017

[Frenkel et al. 1998] E. Frenkel, N. Reshetikhin, and M. A. Semenov-Tian-Shansky, "Drinfeld–Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras, I: The case of Virasoro algebra", *Comm. Math. Phys.* **192**:3 (1998), 605–629. MR 2000b:17031 Zbl 0916.17020

[Khanizadeh et al. 2013] F. Khanizadeh, A. V. Mikhailov, and J. P. Wang, "Darboux transformations and recursion operators for differential-difference equations", *Theor. Math. Phys.* **177**:3 (2013), 1606–1654. arXiv 1305.0588

[Khesin and Soloviev 2013] B. Khesin and F. Soloviev, “Integrability of higher pentagram maps”, *Math. Ann.* **357**:3 (2013), 1005–1047. MR 3118623 Zbl 1280.37056

[Kobayashi and Nagano 1964] S. Kobayashi and T. Nagano, “On filtered Lie algebras and geometric structures, I”, *J. Math. Mech.* **13**:5 (1964), 875–907. MR 29 #5961 Zbl 0142.19504

[Kobayashi and Nagano 1965] S. Kobayashi and T. Nagano, “On filtered Lie algebras and geometric structures, II”, *J. Math. Mech.* **14**:3 (1965), 513–521. MR 32 #2512 Zbl 0163.28103

[Mansfield et al. 2013] E. Mansfield, G. Marí Beffa, and J. P. Wang, “Discrete moving frames and discrete integrable systems”, *Found. Comput. Math.* **13**:4 (2013), 545–582. MR 3085678 Zbl 1279.14045

[Marí Beffa 1999] G. Marí Beffa, “The theory of differential invariants and KdV Hamiltonian evolutions”, *Bull. Soc. Math. France* **127**:3 (1999), 363–391. MR 2001m:37142 Zbl 1053.37046

[Marí Beffa 2006] G. Marí Beffa, “Poisson geometry of differential invariants of curves in some nonsemisimple homogeneous spaces”, *Proc. Amer. Math. Soc.* **134**:3 (2006), 779–791. MR 2006f:53122 Zbl 1083.37053

[Marí Beffa 2007] G. Marí Beffa, “On completely integrable geometric evolutions of curves of Lagrangian planes”, *Proc. Roy. Soc. Edinburgh Sect. A* **137**:1 (2007), 111–131. MR 2008i:37144 Zbl 1130.37032

[Marí Beffa 2010] G. Marí Beffa, “Bi-Hamiltonian flows and their realizations as curves in real semisimple homogeneous manifolds”, *Pacific J. Math.* **247**:1 (2010), 163–188. MR 2012m:37118 Zbl 1213.37096

[Marí Beffa 2013] G. Marí Beffa, “On generalizations of the pentagram map: discretizations of AGD flows”, *J. Nonlinear Sci.* **23**:2 (2013), 303–334. MR 3041627 Zbl 06175210

[Marí Beffa and Wang 2013] G. Marí Beffa and J. P. Wang, “Hamiltonian evolutions of twisted polygons in $\mathbb{RP}^n$”, *Nonlinearity* **26**:9 (2013), 2515–2551. MR 3093293 Zbl 06214273

[Marshall 2010] I. Marshall, “Poisson reduction of the space of polygons”, preprint, 2010. arXiv 1007.1952v1

[Ochiai 1970] T. Ochiai, “Geometry associated with semisimple flat homogeneous spaces”, *Trans. Amer. Math. Soc.* **152** (1970), 159–193. MR 44 #2160 Zbl 0205.26004

[Ovsienko 1993] V. Ovsienko, “Lagrange Schwarzian derivative and symplectic Sturm theory”, *Ann. Fac. Sci. Toulouse Math.* (6) **2**:1 (1993), 73–96. MR 94g:58079 Zbl 0780.34004

[Ovsienko et al. 2010] V. Ovsienko, R. E. Schwartz, and S. Tabachnikov, “The pentagram map: a discrete integrable system”, *Comm. Math. Phys.* **299**:2 (2010), 409–446. MR 2012a:37140 Zbl 1209.37063

[Ovsienko et al. 2013] V. Ovsienko, R. E. Schwartz, and S. Tabachnikov, “Liouville–Arnold integrability of the pentagram map on closed polygons”, *Duke Math. J.* **162**:12 (2013), 2149–2196. MR 3102478 Zbl 06218376

[Semenov-Tian-Shansky 1985] M. A. Semenov-Tian-Shansky, “Dressing transformations and Poisson group actions”, *Publ. Res. Inst. Math. Sci.* **21**:6 (1985), 1237–1260. MR 88b:58057 Zbl 0673.58019

[Soloviev 2013] F. Soloviev, “Integrability of the pentagram map”, *Duke Math. J.* **162**:15 (2013), 2815–2853. MR 3161305 Zbl 1282.14061

[Terng and Thorbergsson 2001] C.-L. Terng and G. Thorbergsson, “Completely integrable curve flows on adjoint orbits”, *Results Math.* **40**:1-4 (2001), 286–309. MR 2002k:37141 Zbl 1023.37041

[Terng and Uhlenbeck 2006] C.-L. Terng and K. Uhlenbeck, “Schrödinger flows on Grassmannians”, pp. 235–256 in *Integrable systems, geometry, and topology*, edited by C.-L. Terng, AMS/IP Stud. Adv. Math. **36**, American Mathematical Society, Providence, RI, 2006. MR 2007a:37079 Zbl 1110.37056

[Wang 2002] J. P. Wang, "Generalized Hasimoto transformation and vector sine-Gordon equation", pp. 276–283 in *Symmetry and perturbation theory* (Cala Gonone, 2002), edited by S. Abenda et al., World Scientific, River Edge, NJ, 2002. MR 2004c:37172

Received May 20, 2013. Revised April 10, 2014.

GLORIA MARÍ BEFFA
MATHEMATICS DEPARTMENT
UNIVERSITY OF WISCONSIN
480 LINCOLN DRIVE
MADISON, WI 53706
UNITED STATES
maribeff@math.wisc.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.319

ON SCHWARZ–CHRISTOFFEL MAPPINGS

MARTIN CHUAQUI AND CHRISTIAN POMMERENKE

We extend previous work on Schwarz–Christoffel mappings, including the special cases when the image is a convex polygon or its complement. We center our analysis on the relationship between the pre-Schwarzian of such mappings and Blaschke products. For arbitrary Schwarz–Christoffel mappings, we resolve an open question from earlier work of Chuaqui, Duren and Osgood that relates the degrees of the associated Blaschke products with the number of convex and concave vertices of the polygon. In addition, we obtain a sharp sufficient condition in terms of the exterior angles for the injectivity of a mapping given by the Schwarz–Christoffel formula, and study the geometric interplay between the location of the zeros of the Blaschke products and the separation of the prevertices.

1. Introduction

The purpose of this paper is to provide further information about Schwarz–Christoffel mappings that adds to the results obtained in [Chuaqui et al. 2011; 2012]. We refer the reader to [Bhowmik et al. 2009] for related interesting work on concave functions.

Let f be a Schwarz–Christoffel mapping of the unit disk $\mathbb{D}$ onto the interior of an $(n+1)$-gon. In other words, f is a conformal map onto a domain in the extended complex plane whose boundary consists of finitely many line segments, rays or lines. In [Chuaqui et al. 2012], it is shown that the pre-Schwarzian of f has the form

$$\frac{f''}{f'} = \frac{2B_1/B_2}{1 - zB_1/B_2} \tag{1-1}$$

for some finite Blaschke products B_1, B_2 without common zeros, with respective degrees d_1, d_2 satisfying $d_1 + d_2 = n$. The polygon is convex if and only if $d_2 = 0$ (see also [Chuaqui et al. 2011]). The representation for f''/f' is obtained from a

---

Chuaqui was partially supported by Fondecyt Grant #1110321.
MSC2000: primary 30C20, 30C35; secondary 30C45.
Keywords: Schwarz–Christoffel mapping, prevertices, convex, concave, univalent mapping, Blaschke product.

well-known formula:

$$\frac{f''}{f'} = -2\sum_{k=1}^{n+1}\frac{\beta_k}{z-z_k}, \tag{1-2}$$

where each z_k is a prevertex and $2\pi\beta_k$ is the exterior angle at z_k (that is, π minus the interior angle); we have $\sum_{k=1}^{n+1}\beta_k = 1$. The formula (1-2) remains valid for polygons with one vertex at infinity. (The angle at infinity between two sides is, by definition, -1 times the angle determined, at their crossing in the plane, by the lines containing the sides.)

As a consequence of (1-1) and (1-2), the prevertices are shown to be the roots of the equation

$$\frac{zB_1(z)}{B_2(z)} = 1. \tag{1-3}$$

It is interesting that (1-3) corresponds to a polynomial equation of degree $n+1$ for which all roots are simple and lie on $|z| = 1$. This is a particular feature of the pair of Blaschke products B_1, B_2 arising from Schwarz–Christoffel mappings. Note that the topological degree of zB_1/B_2 on $\partial\mathbb{D}$ is $1+d_1-d_2$, so that zB_1/B_2 must be traversing in the negative sense at many of the prevertices. In fact, as the proof of Theorem 2 shows, at a prevertex z_k, zB_1/B_2 is traversing $\partial\mathbb{D}$ in the positive or negative sense according to whether $f(z_k)$ is a convex or a concave vertex. It is also interesting to observe that when $d_2 = 0$, any solution of (1-1) will result in a univalent mapping because $1+\mathrm{Re}\{zf''/f'\} \geq 0$. In this paper we answer the natural question of finding a geometric interpretation for the degree d_2, and show that this integer coincides with the number of concave vertices of the polygon.

In Section 2, we also address the case of Schwarz–Christoffel mappings f onto the exterior of an $(n+2)$-gon, with the normalization $f(0) = \infty$. In [Chuaqui et al. 2012], we showed that the pre-Schwarzian of such a mapping is given by

$$z\frac{f''}{f'} = \frac{2}{z^2(B_1/B_2)-1}, \tag{1-4}$$

for finite Blaschke products B_1, B_2 without common zeros, with degrees d_1, d_2, respectively, for which $d_1+d_2 = n$. The polygon is convex if and only if $d_2 = 0$ and, as before, we show in this paper that d_2 is equal to the number of concave vertices of the polygon.

Another issue we address in this paper is the question of when a solution of (1-2), or equivalently, of

$$f'(z) = \prod_{k=1}^{n+1}(z-z_k)^{-2\beta_k}, \quad \sum_{k=1}^{n+1}\beta_k = 1,$$

does indeed correspond to a *univalent* mapping. In Theorem 4 below we obtain the sharp sufficient condition $\sum_{k=1}^{n+1}|\beta_k|\le 2$ for univalence. The result is optimal in the sense that there are nonunivalent solutions of (1-2) for which $\sum_{k=1}^{n+1}|\beta_k|$ differs from 2 by an arbitrarily small amount.

In Section 3 we obtain results on the separation of the prevertices of convex or concave Schwarz–Christoffel mappings, expressed in terms of the location of the zeros $a_1,\dots,a_n$ of the Blaschke product B_1 that appears in (1-1) or (1-4) (recall that, in this case, $d_2=0$). The results are sharp, and show, for example, that the prevertices tend to be uniformly separated on $\partial\mathbb{D}$ when all $|a_k|$ are very small. Finally, in Section 4 we derive some necessary conditions for the location of the zeros of the Blaschke products B_1, B_2 in (1-1) and (1-4) for arbitrary polygonal mappings.

2. Blaschke products and univalence

In [Chuaqui et al. 2011] we revisit the classical theme of convex mappings. The starting point is the observation that such mappings correspond exactly to the solutions of

$$\frac{f''}{f'}=\frac{2h}{1-zh},$$

for some function h analytic in $\mathbb{D}$ and bounded by 1. The image $f(\mathbb{D})$ is the interior of a polygon if and only if h is a finite Blaschke product. We can express h in terms of $p=f''/f'$ as

$$h=\frac{p}{2+zp},$$

and draw the following result.

Theorem 1. *Let h be analytic in $\mathbb{D}$ with $|h(z)|\le 1$ everywhere. Then there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite Blaschke products converging to h locally uniformly in $\mathbb{D}$.*

Proof. Let f be the convex mapping corresponding to h as above, and let Ω_n be a sequence of convex polygons converging to $f(\mathbb{D})$ in the sense of Carathéodory. Properly normalized Schwarz–Christoffel mappings f_n of $\mathbb{D}$ onto Ω_n will converge locally uniformly to f. Each mapping f_n satisfies (1-1) for a certain finite Blaschke product $B_1=B_{1,n}$ and $B_2=1$. The theorem now follows by expressing $B_{1,n}$ in terms of the pre-Schwarzian of f_n. □

Next, we give an answer to an important issue left unresolved in [Chuaqui et al. 2012], namely the connection between the degrees d_1, d_2 and the number of convex and concave vertices of the polygon.

Theorem 2. *Let f map $\mathbb{D}$ onto the interior of an $(n+1)$-gon, and let B_1, B_2 be the corresponding Blaschke products in the representation* (1-1). *Then d_2 is equal to the number of concave vertices, while d_1+1 is equal to the number of convex vertices.*

Proof. Let

$$\varphi(t) = \arg\left\{e^{it}\frac{B_1}{B_2}(e^{it})\right\}, \tag{2-1}$$

with a well-defined branch of the argument once its value has been assigned at one given vertex. In any case,

$$\varphi'(t) = 1 + e^{it}\left(\frac{B_1'}{B_1}(e^{it}) - \frac{B_2'}{B_2}(e^{it})\right) = 1 + |B_1'(e^{it})| - |B_2'(e^{it})|. \tag{2-2}$$

On the other hand, we see from (1-1) and (1-2) that

$$\frac{B_1/B_2}{zB_1/B_2 - 1} = \sum_{k=1}^{n+1} \frac{\beta_k}{z - z_k};$$

hence

$$\beta_k = \lim_{z\to z_k}(z - z_k)\frac{B_1/B_2}{zB_1/B_2 - 1} = \frac{B_1/B_2}{(zB_1/B_2)'}(z_k) = \frac{1}{\varphi'(t_k)}, \tag{2-3}$$

where we have written $z_k = e^{it_k}$. We say that z_k is convex or concave according to whether the polygon is convex or concave at $f(z_k)$. We conclude that $\varphi'(t_k)$ is positive at convex prevertices and negative at concave prevertices. Furthermore, the points z_k are the solutions of (1-3). Hence we see from (2-1) that, for all k,

$$\varphi(t_k) = 2\pi j_k,\ j_k \in \mathbb{Z}, \quad \text{and} \quad \varphi(t) \neq 2\pi j,\ t \neq t_k.$$

It follows now that

$$\int_{t_k}^{t_{k+1}} \varphi'(t) = \begin{cases} 2\pi & \text{when } z_k,\, z_{k+1} \text{ are both convex,} \\ 0 & \text{when } z_k,\, z_{k+1} \text{ are one convex and one concave,} \\ -2\pi & \text{when } z_k,\, z_{k+1} \text{ are both concave.} \end{cases} \tag{2-4}$$

Let a be the number of consecutive convex prevertices, b the number of instances a vertex of one type is followed by one of the other type, and c the number of consecutive concave prevertices. Then $a + b + c = n + 1$, and we see by (2-4) that

$$\int_0^{2\pi} \varphi'(t)\,dt = 2\pi(1 + d_1 - d_2) = 2\pi(a - c).$$

Hence we have

$$1 + d_1 + d_2 = n + 1 = a + b + c, \quad 1 + d_1 - d_2 = a - c.$$

We conclude that

$$1 + d_1 = a + \frac{b}{2}, \quad d_2 = c + \frac{b}{2}.$$

To obtain the theorem, we claim that $c + (b/2)$ is equal to the number of concave vertices (or prevertices). To see this, let $z_k, \ldots, z_l$ be any maximal chain of consecutive concave prevertices. Hence z_{k-1} and z_{l+1} are convex prevertices. The collection $z_k, \dots, z_l$ of concave prevertices contributes $l-k$ to the count of c and contributes 2 to the count of b. Thus its contribution in the count of $c+(b/2)$ is exactly the number of points in the chain. This proves the claim, and completes the proof of the theorem. $\square$

Similar results hold for mappings f onto the exterior of an $(n{+}2)$-gon, having the important normalization $f(0)=\infty$. For such mappings we have that

$$\frac{f''}{f'} = 2\left(\sum_{k=1}^{n+2} \frac{\beta_k}{z-z_k} - \frac{1}{z}\right),$$

where, as before, z_k are the prevertices and $2\pi\beta_k$ are the exterior angles, which satisfy $-1<\beta_k<1$ and $\sum_{k=1}^{n+2}\beta_k = 1$. In [Chuaqui et al. 2012], this was shown to lead to

$$z\frac{f''}{f'} = \frac{2}{z^2(B_1/B_2)-1},$$

for Blaschke products B_1, B_2 of degree d_1, d_2 satisfying $d_1+d_2=n$. Again, the case $d_2=0$ corresponds exactly to when the polygon is convex. The prevertices appear as the solutions of the equation $z^2B_1 = B_2$, yet no further information was provided in connection with the degrees of the Blaschke products. With a similar argument as in the proof of Theorem 2, one can show:

Theorem 3. *Let f map $\mathbb{D}$ onto the exterior of an $(n{+}2)$-gon, and let B_1, B_2 be the corresponding Blaschke products in the representation* (1-4). *Then d_2 is equal to the number of concave vertices, while d_1+2 is equal to the number of convex vertices.*

Next, we address the question of the univalence of solutions of (1-2).

Theorem 4. *Let $0 \le t_1 < \cdots < t_{n+1} < 2\pi$, $z_k = e^{it_k}$, $\beta_k \in \mathbb{R}$, $k=1,\dots,n{+}1$, and let*

$$\sum_{k=1}^{n+1}\beta_k = 1, \quad \sum_{k=1}^{n+1}|\beta_k| \le 2\,. \tag{2-5}$$

Then the function f defined by

$$f'(z) = a\prod_{k=1}^{n+1}(z-z_k)^{-2\beta_k}, \quad a\in\mathbb{C},\ a\neq 0, \tag{2-6}$$

is univalent in $\mathbb{D}$.

Observe that there exist polygons with $\sum|\beta_k|$ arbitrarily large for which f remains univalent. For example, one can consider a polygon inscribed between two disjoint logarithmic spirals. On the other hand, once $\sum_{k=1}^{n+1}|\beta_k|$ is allowed to exceed 2, then the sum of exterior angles at concave vertices will be larger than π in absolute value, thus making it possible for the image $f(\mathbb{D})$ to intersect itself.

Proof of Theorem 4. Let f be given as in (2-6) and suppose that (2-5) holds. Then f is locally injective in $\mathbb{D}$, and we will show that f is univalent there, and, in fact, that it is close-to-convex. Among the various equivalent formulations of this geometric property (see, e.g., [Duren 1983, p. 48]), we will show for $0 \le \theta_1 < \theta_2 < 2\pi$ that

$$I = I(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1 + z\frac{f''}{f'}(z)\right\} d\theta > -\pi, \quad z = re^{i\theta}.$$

To prove this, observe that

$$\mathrm{Re}\left\{1 + z\frac{f''}{f'}(z)\right\} = \sum_k \beta_k \,\mathrm{Re}\left\{\frac{z_k + z}{z_k - z}\right\} = \sum_k \beta_k \frac{1 - r^2}{|z_k - z|^2}.$$

In trying to obtain a lower bound for I we can discard the terms with $\beta_k > 0$. For the other terms, we have that

$$\int_{\theta_1}^{\theta_2} \frac{1 - r^2}{|z_k - re^{i\theta}|^2}\, d\theta < 2\pi$$

because of the properties of the Poisson kernel. Hence

$$I > 2\pi \sum_{\beta_k < 0} \beta_k \ge -\pi,$$

as desired. □

Example. Avkhadiev and Wirths [2002; 2005; 2007] initiated the study of the so-called concave mappings, that is, univalent mappings of the disk $\mathbb{D}$ onto the complement of a convex set. As an example of Theorem 4 we can consider a convex polygon P with $\infty \in P$ and the conformal mapping of $\mathbb{D}$ onto the complement of P. Let $\pi\lambda$ be the angle of $f(\mathbb{D})$ at ∞, with $1 \le \lambda \le 2$. It follows from [Avkhadiev and Wirths 2005] that

$$f'(z) = a(z - z_{n+1})^{-\lambda-1} \prod_{k=1}^{n} (z - z_k)^{\gamma_k}, \qquad \sum_{k=1}^{n} \gamma_k = \lambda - 1,$$

that is, $\beta_k = \frac{1}{2}\gamma_k$ for $k = 1, \dots, n$, and $\beta_{n+1} = \frac{1}{2}(1 + \lambda)$. Therefore

$$\sum_{k=1}^{n+1} |\beta_k| = \tfrac{1}{2}(\lambda + 1) + \tfrac{1}{2}(\lambda - 1) = \lambda \in [1, 2].$$

Next, we establish the following variant of Theorem 4:

Theorem 5. *Let f be defined by* (2-6) *with* $\sum_{k=1}^{n+1} \beta_k = 1$, *and suppose that*

(2-7) $$\frac{\mathrm{Im}\{f(z)\}}{\mathrm{Im}\{z\}} > 0 \quad \textit{for } |z| \le 1,\ \mathrm{Im}\{z\} \ne 0.$$

Let $\theta_\pm$ be the interior angles of the polygon $f(\partial\mathbb{D})$ at $f(\pm 1)$. If

$$\sum_{k=1}^{n+1} |\beta_k| \le 3 + \frac{1}{\pi}\max(\theta_+ - \pi, 0) + \frac{1}{\pi}\max(\theta_- - \pi, 0), \tag{2-8}$$

then f is univalent.

The expression on the right-hand side of (2-8) lies in $[3, 5]$, and it is easy to see that any value in this range can be achieved. Therefore, Theorem 5 gives a better result than Theorem 4 under the stronger assumption (2-7). The condition (2-7) implies, in particular, that $f(\mathbb{D})$ is symmetric with respect to $\mathbb{R}$.

Proof of Theorem 5. By (2-7), the polygon $P = f(\partial\mathbb{D})$ is symmetric with respect to $\mathbb{R}$. Hence $m := (n+1)/2 \in \mathbb{N}$. We may assume that $z_1 = 1$, $z_2 = -1$ in (2-6). Then

$$\beta_1 = \frac{1}{2} - \frac{\theta_+}{2\pi}, \quad \beta_2 = \frac{1}{2} - \frac{\theta_-}{2\pi},$$

which satisfy $|\beta_1|, |\beta_2| \le \frac{1}{2}$. It follows that

$$\frac{1}{\pi}\max(\theta_+ - \pi, 0) = \max(-2\beta_1, 0) = |\beta_1| - \beta_1,$$
$$\frac{1}{\pi}\max(\theta_- - \pi, 0) = \max(-2\beta_2, 0) = |\beta_2| - \beta_2.$$

Let $\varphi_\pm$ be the conformal mappings of $\mathbb{D}$ onto the semidiscs $\{z \in \mathbb{D} : \operatorname{Im} z \gtrless 0\}$ such that $\varphi_\pm(1) = 1$, $\varphi_\pm(-1) = -1$ and $\varphi_\pm(\pm i) = \pm i$. Then

$$P_\pm = f(\varphi_\pm(\partial\mathbb{D})) = f(\partial\mathbb{D} \cap \{\operatorname{Im} z \gtrless 0\}) \cup [f(-1), f(+1)] \tag{2-9}$$

are the upper and lower parts of P union $[f(-1), f(1)]$. We may also assume that β_k, $k = 3, \dots, m+1$, belong to the vertices of P that lie in P_+.

Consider the upper polygon P_+. The values γ_k of P_+ corresponding to the β_k are

$$\gamma_k = \begin{cases} \frac{1}{4} + \frac{1}{2}\beta_k \ge 0 & k = 1, 2, \\ \beta_k & k = 3, \dots, m+1, \end{cases}$$

(for which $\sum_{k=1}^{m+1} \gamma_k = 1$). In light of the symmetry with respect to $\mathbb{R}$, we get

$$2\sum_{k=1}^{m+1} |\gamma_k| = 1 + \beta_1 + \beta_2 + \sum_{k=3}^{n+1} |\beta_k| = 1 + \sum_{k=1}^{n+1} |\beta_k| - (|\beta_1| - \beta_1) - (|\beta_2| - \beta_2).$$

Using (2-8), we conclude that

$$2\sum_{k=1}^{m+1} |\gamma_k| \le 4,$$

and it follows from Theorem 4 that $f \circ \varphi_+$ is univalent in $\mathbb{D}$. The same holds for $f \circ \varphi_-$.

By (2-9) we have

$$f(\mathbb{D}) = (f \circ \varphi_+)(\mathbb{D}) \cup f((-1, 1)) \cup (f \circ \varphi_-)(\mathbb{D}),$$

which are disjoint unions by (2-7). Hence f is univalent in $\mathbb{D}$. □

3. Separation of prevertices

Let f be a Schwarz–Christoffel mapping taking $\mathbb{D}$ onto a convex $(n+1)$-gon. Recall that

$$\frac{f''}{f'}(z) = \frac{2B(z)}{1 - zB(z)},$$

where $B(z)$ is a Blaschke product of degree n. We write

$$B(z) = c \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z},$$

where $|c| = 1$ and all $|a_k| < 1$. The prevertices $z_1, \dots, z_{n+1}$ correspond to the roots of the equation $zB(z) = 1$, and after rotating f, we may assume that $c = 1$. Recall also that, in this case, any choice of Blaschke product $B = B(z)$ will result in a univalent mapping f. The separation of consecutive prevertices z_k, z_{k+1} is to be understood as $\arg\{\bar{z}_k z_{k+1}\} \in (0, 2\pi)$.

Theorem 6. *Suppose that* $|a_k| \le r < 1$ *for all* k. *Then*:

(i) *The minimum separation in argument of consecutive prevertices is given by* 2θ, *where* θ *is the unique root in* $(0, \pi/2)$ *of the equation*

$$\pi = \theta + 2n \arctan\left\{\frac{1+r}{1-r} \tan \frac{\theta}{2}\right\}. \tag{3-1}$$

The result is sharp. The optimal configuration occurs when $a_1 = \dots = a_n = rc$ *for some root of the equation* $c^{n+1} = -1$, *and the lower bound is attained for the prevertices* $e^{i\theta}c$, $e^{-i\theta}c$. *The distance between any other pair of consecutive prevertices will be larger.*

(ii) *The maximum separation in argument of consecutive prevertices is given by* 2ψ, *where* ψ *is the unique root in* $(0, \pi)$ *of the equation*

$$\pi = \psi + 2n \arctan\left\{\frac{1-r}{1+r} \tan \frac{\psi}{2}\right\}. \tag{3-2}$$

The result is sharp. The optimal configuration occurs when $a_1 = \dots = a_n = rd$ *for some root of the equation* $d^{n+1} = (-1)^n$, *and the upper bound is attained for the prevertices* $-e^{i\psi}d$, $-e^{-i\psi}d$. *The distance between any other pair of consecutive prevertices will be larger.*

Proof. We must estimate the distance between two consecutive roots $a = e^{i\alpha}$, $b = e^{i\beta}$ of the equation $zB(z) = 1$. Because $zB(z)$ traces the boundary $\partial\mathbb{D}$ for $z \in \partial\mathbb{D}$ in a monotonic way, we must have from (2-2) and (2-4) that

$$\int_\alpha^\beta (1 + |B'(e^{it})|)\, dt = 2\pi, \tag{3-3}$$

with

$$|B'(e^{it})| = \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|e^{it} - a_k|^2}.$$

(Equation (3-3) shows that $0 < \beta - \alpha < 2\pi$.) We claim that for α, β fixed, the contribution of any single summand

$$\int_\alpha^\beta \frac{1 - |a_k|^2}{|e^{it} - a_k|^2}\, dt \tag{3-4}$$

will be maximal if $|a_k| = r$ and $a_k/|a_k|$ is the midpoint c of the shorter arc joining a and b. Let $r_k = |a_k|$ and write $a_k = r_k e^{it_k}$. Then

$$\int_\alpha^\beta \frac{1 - |a_k|^2}{|e^{it} - a_k|^2}\, dt = \int_{\alpha - t_k}^{\beta - t_k} \frac{1 - r_k^2}{|1 - r_k e^{it}|^2}\, dt.$$

For $r_k \le r$ given, this integral is maximal when $1 \in \partial\mathbb{D}$ is the midpoint of the shorter arc between $e^{i(\alpha - t_k)}$ and $e^{i(\beta - t_k)}$. The integral is then equal to

$$\int_{-\theta}^{\theta} \frac{1 - r_k^2}{|1 - r_k e^{it}|^2}\, dt = 4 \arctan\left\{ \frac{1+r_k}{1-r_k} \tan \frac{\theta}{2} \right\},$$

where $2\theta = \beta - \alpha$, and becomes maximal if $r_k = r$. In other words,

$$\int_\alpha^\beta \frac{1 - |a_k|^2}{|e^{it} - a_k|^2}\, dt \le 4 \arctan\left\{ \frac{1+r}{1-r} \tan \frac{\theta}{2} \right\}, \tag{3-5}$$

which proves our claim for the contribution of any single term, and therefore the minimum separation between consecutive roots will occur if this holds for all $k = 1, \dots, n$. Equation (3-1) follows. The analysis shows that for the extremal configuration, all $a_k = rc$ are equal and that $e^{\pm i\theta} c$ are roots of the equation $zB(z) = 1$. Because $B(c) = c$, then $cB(c) = c^{n+1}$, and since $zB(z)$ traces the arc between the two roots in symmetric fashion with respect to the midpoint, we conclude that $c^{n+1} = -1$. This proves part (i).

For part (ii), we observe that, for $r_k = |a_k|$ fixed, (3-4) will be minimal provided $-1 \in \partial\mathbb{D}$ is the midpoint of the shorter arc between $e^{i(\alpha - t_k)}$ and $e^{i(\beta - t_k)}$. The integral is then equal to

$$\int_{\pi - \psi}^{\pi + \psi} \frac{1 - r_k^2}{|1 - r_k e^{it}|^2}\, dt = 4 \arctan\left\{ \frac{1-r_k}{1+r_k} \tan \frac{\psi}{2} \right\},$$

where $2\psi = \beta - \alpha$, and becomes minimal when $r_k = r$. Thus,

$$\int_\alpha^\beta \frac{1-|a_k|^2}{|e^{it}-a_k|^2}\,dt \ge 4\arctan\left\{\frac{1-r}{1+r}\tan\frac{\theta}{2}\right\}. \tag{3-6}$$

Therefore, the maximum separation between consecutive roots will occur if, for all $k = 1, \dots, n$, we have that $|a_k| = r$ and a_k/r is equal to the midpoint of the longer arc between a and b. From this, (3-2) follows. As before, the analysis of the extremal configuration gives $a_k = rd$ for all k. The points $-e^{\pm i\psi}d$ are roots of $zB(z) = 1$, which by symmetry as before implies that $d^{n+1} = (-1)^n$. □

Corollary 7. *Suppose that $|a_k| \le \epsilon$ for all k. Then the maximum separation 2ψ and minimum separation 2θ between consecutive prevertices satisfy*

$$\frac{\pi}{1+(1+2\epsilon)n} + O(\epsilon^2) \le \theta \le \psi \le \frac{\pi}{1+(1-2\epsilon)n} + O(\epsilon^2), \quad \epsilon \to 0. \tag{3-7}$$

Proof. For fixed $x \in [0, \pi/2]$, let $F(\delta) = \arctan((1+\delta)\tan x)$. Then $F(0) = x$, $F'(0) = \sin x \cos x = \frac{1}{2}\sin 2x$ and $F''(0) = -2\sin^3 x \cos x$, hence

$$F(\delta) = x + \tfrac{1}{2}\sin 2x\delta + O(\delta^2), \quad \delta \to 0.$$

Using that $(1+r)/(1-r) = 1 + 2r + O(r^2)$, $r \to 0$, and that $\sin 2x \le 2x$, we see from (3-1) that the minimum separation θ satisfies

$$\pi \le \theta + 2n\left(\frac{\theta}{2} + \epsilon\sin(\theta) + O(\epsilon^2)\right) \le (1+(1+2\epsilon)n)\theta + O(\epsilon^2).$$

This implies the lower bound in (3-7). A similar analysis applies to the maximum separation ψ, and the upper bound in (3-7) obtains. □

Suppose now that f is a Schwarz–Christoffel mapping taking $\mathbb{D}$ onto the complement of a bounded convex $(n+2)$-gon, with the normalization $f(0) = \infty$. In this situation, we know that

$$z\frac{f''}{f'}(z) = \frac{2}{z^2B(z) - 1},$$

where $B(z)$ again is a Blaschke product of degree n. We write

$$B(z) = c\prod_{k=1}^n \frac{z - a_k}{1 - \bar{a}_k z},$$

where $|c| = 1$ and all $|a_k| < 1$. The prevertices $z_1, \dots, z_{n+2}$ are now given by the roots of the equation $z^2B(z) = 1$, and after a rotation of f, we may assume that $c = 1$. The following result is obtained in a way similar to Theorem 6, and the proof will be omitted.

Theorem 8. *Suppose that $|a_k| \le r < 1$ for all k. Then*:

(i) *The minimum separation in argument of consecutive prevertices is given by* 2θ, *where* θ *is the unique root in* $(0, \pi/2)$ *of the equation*

$$\pi = 2\theta + 2n \arctan\left\{\frac{1+r}{1-r} \tan\frac{\theta}{2}\right\}.$$

The result is sharp. The optimal configuration occurs when $\alpha_1 = \cdots = \alpha_n = rc$ *for some root of the equation* $c^{n+2} = -1$, *and the lower bound is attained for the prevertices* $e^{i\theta}c$, $e^{-i\theta}c$. *The distance between any other pair of consecutive prevertices will be larger.*

(ii) *The maximum separation in argument of consecutive prevertices is given by* 2ψ, *where* ψ *is the unique root in* $(0, \pi/2)$ *of the equation*

$$\pi = 2\psi + 2n \arctan\left\{\frac{1-r}{1+r} \tan\frac{\psi}{2}\right\}.$$

The result is sharp. The optimal configuration occurs when $\alpha_1 = \cdots = \alpha_n = rd$ *for some root of the equation* $d^{n+2} = (-1)^{n+1}$, *and the upper bound is attained for the prevertices* $-e^{i\psi}d$, $-e^{-i\psi}d$. *The distance between any other pair of consecutive prevertices will be larger.*

A statement similar to Corollary 7 can be made in this case. If $|a_k| \le \epsilon$ then the maximum and minimum separation between prevertices satisfy

$$\text{(3-8)} \qquad \frac{\pi}{2+(1+2\epsilon)n} + O(\epsilon^2) \le \theta \le \psi \le \frac{\pi}{2+(1-2\epsilon)n} + O(\epsilon^2), \quad \epsilon \to 0.$$

We finish this section with some remarks on the separation of prevertices for arbitrary polygonal mappings. Suppose f is a mapping of the form given by (1-1), where after rotation, we may assume expressions for B_1, B_2 given by

$$B_1(z) = \prod_{k=1}^{d_1} \frac{z-a_k}{1-\bar{a}_k z}, \quad B_2(z) = \prod_{k=1}^{d_2} \frac{z-b_k}{1-\bar{b}_k z}.$$

Then

$$\varphi'(t) = 1 + |B_1'(e^{it})| - |B_2'(e^{it})| = 1 + \sum_{k=1}^{d_1} \frac{1-|a_k|^2}{|e^{it}-a_k|^2} - \sum_{k=1}^{d_2} \frac{1-|b_k|^2}{|e^{it}-b_k|^2}.$$

Let $a = e^{i\alpha}$, $b = e^{i\beta}$ be consecutive *convex* prevertices, with separation $\beta - \alpha = 2\delta$, and let r be the radius of the smallest centered subdisk that contains the zeros of B_1, B_2. We deduce from (2-4) and the estimates (3-5), (3-6), that

$$\text{(3-9)} \qquad \delta + 2d_1 \arctan\frac{x}{\lambda} - 2d_2 \arctan(\lambda x) \le \pi \le \delta + 2d_1 \arctan(\lambda x) - 2d_2 \arctan\frac{x}{\lambda},$$

where $\lambda = \frac{1+r}{1-r}$ and $x = \tan\frac{\delta}{2}$. Thus, for example, with given d_1, d_2, a relatively small separation 2δ can only occur if r is rather close to 1. Because the univalence of f is no longer guaranteed when B_1, B_2 are chosen arbitrarily, it seems of interest to determine under which circumstances the inequalities (3-9) remain sharp. We provide here a simple example where one can show sharpness in the right-hand side of (3-9) when $d_1 = d_2 = 1$.

Example. Consider the Blaschke products B_1, B_2 given by

$$B_1(z) = \frac{z+r}{1+rz}, \quad B_2(z) = \frac{z-r}{1-rz}, \quad r \in (0, 1),$$

and let f be defined, up to an affine change, by

$$\frac{f''}{f'} = \frac{2B_1/B_2}{1-zB_1/B_2} = \frac{2(z+r)(1-rz)}{(z-r)(1+rz)-z(z+r)(1-rz)}.$$

In analyzing the roots of $zB_1 = B_2$, we observe that $z_3 = 1$ is one immediate solution. The other solutions are the roots of

$$rz^2 + (r^2 + 2r - 1)z + r = 0,$$

which are given by

$$z_{1,2} = \frac{(1-2r-r^2) \pm \sqrt{-(1-r^2)(r^2+4r-1)}}{2r}.$$

For $r > r_0 = \sqrt{5} - 2 = 0.236\ldots$, the discriminant is negative and $|z_{1,2}| = 1$, with $z_1 = z_2 (= -1)$ only for $r = 1$. In the partial fraction decomposition

$$\frac{f''}{f'} = -2\left(\frac{\beta_1}{z-z_1} + \frac{\beta_2}{z-z_2} + \frac{\beta_3}{z-z_3}\right),$$

we must have $\beta_1 = \beta_2$ because of symmetry, while $\beta_1 + \beta_2 + \beta_3 = 1$ by equating coefficients with the above representation for f''/f'. Recall (2-3), which relates the exterior angles $2\pi\beta_k$ with the boundary function $\varphi(t)$. Here

$$\varphi'(t) = 1 + \frac{1-r^2}{|1+re^{it}|^2} - \frac{1-r^2}{|1-re^{it}|^2};$$

hence

$$\varphi'(0) = 1 + \frac{1-r}{1+r} - \frac{1+r}{1-r}.$$

One readily verifies that $\varphi'(0) \le -2$ precisely when $r \ge r_1 = (1+\sqrt{13})/(5+\sqrt{13}) = 0.535\ldots$, in which case $\beta_3 \in \left(-\frac{1}{2}, 0\right)$ and $\beta_1 = \beta_2 \in \left(\frac{1}{2}, \frac{3}{4}\right)$. Thus, for $r \ge r_1$, we deduce from Theorem 4 that f is *univalent*, and $z_{1,2}$ are convex prevertices, while z_3 is a concave prevertex. The convex vertices $f(z_{1,2})$ are at infinity, and the image $f(\mathbb{D})$ corresponds to a half-plane minus a symmetric slit ending at the concave vertex when $r = r_1$, or a wedge when $r > r_1$.

Finally, to show sharpness in the right-hand side of (3-9), observe that for $r \ge r_1$ the conjugate points $z_1 = \bar{z}_2$ have negative real part, and thus their separation 2θ will correspond to the root $\theta \in (0, \pi/2)$ of the equation

$$\theta + 2\arctan(\lambda x) - 2\arctan\frac{x}{\lambda} = \pi.$$

For consecutive *concave* prevertices, we deduce in similar fashion that

$$\begin{aligned}(3\text{-}10)\quad \delta + 2d_1\arctan\frac{x}{\lambda} - 2d_2\arctan(\lambda x) &\le -\pi \\ &\le \delta + 2d_1\arctan(\lambda x) - 2d_2\arctan\frac{x}{\lambda},\end{aligned}$$

once again forcing r to be very close to 1 if a small separation is to happen.

A similar analysis can be carried through to obtain information about the separation between consecutive convex or concave prevertices in the case of exterior mappings. The resulting inequalities are analogous to (3-9) and (3-10), with the single term δ replaced by 2δ. The proof will be omitted.

4. Location of zeros

In this section we study the location of the zeros of the Blaschke products appearing in the representation formulas (1-1) and (1-4) of Schwarz–Christoffel mappings. Convex or concave mappings impose no restriction on the location of the zeros, since in the absence of the Blaschke product B_2, (1-1) and (1-4) will always render univalent mappings. It is probably an ambitious task to determine conditions on B_1 and B_2 that are both necessary and sufficient for all mappings of the form (1-1) and (1-4) to be univalent. Nevertheless, some necessary conditions can be established. We deal first with the case of mappings arising from (1-1). Because $1 + \mathrm{Re}\{zf''/f'\}$ will be positive or negative according to whether $|zB_1| < |B_2|$ or $|zB_1| > |B_2|$, it follows readily from the radius of convexity for the class $\mathcal{S}$ that we must have

$$|zB_1(z)| < |B_2(z)|, \quad |z| \le 2 - \sqrt{3}.$$

In particular, all zeros of B_2 must lie in the region $|z| > 2 - \sqrt{3}$.

Theorem 9. *Let f be given by* (1-1), *with d_1, d_2 the degrees of the Blaschke products B_1, B_2, respectively, and suppose that $d_2 \ge 1$. Suppose that all the zeros of B_1, B_2 are contained in the subdisk $|z| \le r < 1$. Then*

$$(4\text{-}1)\qquad r \ge \max\left\{\frac{\sqrt{4d_1d_2+9}+3-2d_2}{\sqrt{4d_1d_2+9}+3+2d_2}, \frac{2d_2-1-\sqrt{1+4d_1d_2}}{2d_2+1+\sqrt{1+4d_1d_2}}\right\}.$$

In particular, if $d_2 = 1$ then

$$(4\text{-}2)\qquad r \ge \frac{\sqrt{4n+5}+1}{\sqrt{4n+5}+5} \ge \frac{1}{2}.$$

The estimate (4-2) *is sharp for the Koebe function.*

Proof. Recall the boundary function $\varphi(t)$ in (2-2). At a concave prevertex e^{it_0}, the exterior angle $2\pi\beta_0$ lies in $[-\pi, 0)$, and hence β_0 lies in $\bigl[-\frac{1}{2}, 0\bigr)$. It follows from (2-3) that

$$\varphi'(t_0) \le -2.$$

If we write

$$B_1(z) = c_1 \prod_{k=1}^{d_1} \frac{z-a_k}{1-\bar{a}_k z}, \qquad B_2(z) = c_2 \prod_{k=1}^{d_2} \frac{z-b_k}{1-\bar{b}_k z},$$

then

$$\varphi'(t) = 1 + \sum_{k=1}^{d_1} \frac{1-|a_k|^2}{|e^{it}-a_k|^2} - \sum_{k=1}^{d_2} \frac{1-|b_k|^2}{|e^{it}-b_k|^2}.$$

After evaluating at $t = t_0$, a simple estimate gives

$$1 + d_1 \frac{1-r}{1+r} - d_2 \frac{1+r}{1-r} \le -2.$$

With $s = (1+r)/(1-r)$, we obtain

$$d_2 s^2 - 3s - d_1 \ge 0,$$

which implies (4-1). If $d_2 = 1$, then $d_1 = n-1$, which proves the first estimate in (4-2).

In order to obtain the second estimate, we observe that at a convex prevertex e^{it_1}, the exterior angle $2\pi\beta_1$ is positive, and therefore $\varphi'(t_1) > 0$. This now gives

$$1 + d_1 \frac{1+r}{1-r} - d_2 \frac{1-r}{1+r} > 0,$$

and the second estimate follows.

To show sharpness, we consider the Koebe function $k(z) = z/(1-z)^2$. Then $k'(z) = (1+z)/(1-z)^3$, and thus

$$\frac{k''}{k'}(z) = \frac{1}{z+1} - \frac{3}{z-1},$$

which is consistent with a polygonal mapping onto a 2-gon with a concave vertex with exterior angle $-\pi$ at $k(-1) = -\frac{1}{4}$, and a convex vertex with exterior angle 3π at $k(1) = \infty$. A calculation gives

$$\frac{k''}{k'}(z) = \frac{1/B_2(z)}{1 - z/B_2(z)},$$

with $B_2(z) = \bigl(z + \frac{1}{2}\bigr)/\bigl(1 + \frac{1}{2}z\bigr)$. Then $r = \frac{1}{2}$, which coincides with the lower bound in (4-2) with $n = 1$. $\square$

Remark. The first estimate in (4-1) is the better one when $d_1 \gg d_2$, while the second will provide better information for $d_2 \gg d_1$.

The final theorem describes the analogous situation for mappings of the form (1-4) onto the complement of polygons. The corresponding boundary function is now given by

$$\varphi(t) = \arg\left\{e^{2it}\frac{B_1}{B_2}(e^{it})\right\},$$

for which

$$\varphi'(t) = 2 + |B_1'(e^{it})| - |B_2'(e^{it})|.$$

Since the proof is based on an almost identical analysis, it will be omitted.

Theorem 10. *Let f be given by* (1-4), *with d_1, d_2 the degrees of the Blaschke products B_1, B_2, respectively, and suppose that $d_2 \geq 1$. Suppose that all the zeros of B_1, B_2 are contained in the subdisk $|z| \leq r < 1$. Then*

$$r \geq \max\left\{\frac{\sqrt{d_1d_2+4}+2-d_2}{\sqrt{d_1d_2+4}+2+d_2}, \frac{d_2-1-\sqrt{1+d_1d_2}}{d_2+1+\sqrt{1+d_1d_2}}\right\}. \tag{4-3}$$

Acknowledgement

We thank the referee for a very careful reading of the manuscript and several valuable suggestions.

References

[Avkhadiev and Wirths 2002] F. G. Avkhadiev and K.-J. Wirths, "Convex holes produce lower bounds for coefficients", *Complex Var. Theory Appl.* **47**:7 (2002), 553–563. MR 2003f:30013 Zbl 1028.30010

[Avkhadiev and Wirths 2005] F. G. Avkhadiev and K.-J. Wirths, "Concave schlicht functions with bounded opening angle at infinity", *Lobachevskii J. Math.* **17** (2005), 3–10. MR 2005m:30013 Zbl 1071.30005

[Avkhadiev and Wirths 2007] F. G. Avkhadiev and K.-J. Wirths, "A proof of the Livingston conjecture", *Forum Math.* **19**:1 (2007), 149–157. MR 2007k:30027 Zbl 1109.30021

[Bhowmik et al. 2009] B. Bhowmik, S. Ponnusamy, and K.-J. Wirths, "Вогнутые функции, произведения Бляшке и полигональные отображения", *Sibirsk. Mat. Zh.* **50**:4 (2009), 772–779. Translated as "Concave functions, Blaschke products, and polygonal mappings" in *Sib. Math. J.* **50**:4 (2009), 609–615. MR 2011a:30123 Zbl 1224.30035

[Chuaqui et al. 2011] M. Chuaqui, P. L. Duren, and B. Osgood, "Schwarzian derivatives of convex mappings", *Ann. Acad. Sci. Fenn. Math.* **36**:2 (2011), 449–460. MR 2012j:30024 Zbl 1239.30003

[Chuaqui et al. 2012] M. Chuaqui, P. L. Duren, and B. Osgood, "Concave conformal mappings and pre-vertices of Schwarz–Christoffel mappings", *Proc. Amer. Math. Soc.* **140**:10 (2012), 3495–3505. MR 2929018 Zbl 1283.30048

[Duren 1983] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer, New York, 1983. MR 85j:30034 Zbl 0514.30001

Received May 13, 2013. Revised November 29, 2013.

MARTIN CHUAQUI
FACULTAD DE MATEMÁTICAS
PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
CASILLA 306
22 SANTIAGO
CHILE
mchuaqui@mat.puc.cl

CHRISTIAN POMMERENKE
INSTITUT FÜR MATHEMATIK MA 8-1
TECHNISCHE UNIVERSITÄT
D-10623 BERLIN
GERMANY
pommeren@math.tu-berlin.de

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.335

VANISHING VISCOSITY IN THE PLANE FOR NONDECAYING VELOCITY AND VORTICITY, II

ELAINE COZZI

We consider solutions to the two-dimensional incompressible Navier–Stokes and Euler equations for which velocity and vorticity are bounded in the plane. We show that for every $T > 0$, the Navier–Stokes velocity converges in $L^\infty([0, T]; L^\infty(\mathbb{R}^2))$ as viscosity approaches 0 to the Euler velocity generated from the same initial data. This improves our earlier results to the effect that the vanishing viscosity limit holds on a sufficiently short time interval, or for all time under the assumption of decay of the velocity vector field at infinity.

1. Introduction

In this paper, we study the vanishing viscosity limit of solutions to the two-dimensional incompressible Navier–Stokes equations. Recall that the Navier–Stokes equations modeling incompressible viscous fluid flow on $\mathbb{R}^n$ are given by

$$\text{(NS)} \qquad \begin{cases} \partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu = -\nabla p_\nu, \\ \operatorname{div} u_\nu = 0, \\ u_\nu|_{t=0} = u_\nu^0. \end{cases}$$

When $\nu = 0$, the Navier–Stokes equations reduce to the Euler equations modeling incompressible inviscid fluid flow on $\mathbb{R}^n$:

$$\text{(E)} \qquad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u^0. \end{cases}$$

There are a number of results addressing the vanishing viscosity limit of solutions of (NS) on $\mathbb{R}^n$ under various assumptions on the initial data (see, for example, [Constantin 1986; Masmoudi 2007; Kelliher 2004; Chemin 1996; Kato 1972; Swann 1971]). Here we focus our attention on solutions to (NS) and (E) in the plane with bounded velocity and vorticity which do not necessarily decay at infinity. We show that such solutions to (NS) converge to solutions of (E) with the same initial

MSC2010: 76D05.
Keywords: fluid mechanics, inviscid limit.

data in the L^∞-norm, where convergence is uniform over any finite time interval. This result builds upon and is a continuation of work in [Cozzi 2009; 2010]. For this reason, we will often refer to these articles for background information and useful estimates.

The existence and uniqueness of solutions to (NS) without any decay assumptions on the initial velocity is considered by Giga, Inui, and Matsui in [Giga et al. 1999]. The authors establish the short-time existence and uniqueness of mild solutions v_ν to (NS) in the space $C([0, T_0]; BUC(\mathbb{R}^n))$ when initial velocity is in $BUC(\mathbb{R}^n)$ and $n \geq 2$. Here $BUC(\mathbb{R}^n)$ denotes the space of bounded uniformly continuous functions on $\mathbb{R}^n$. In [Giga et al. 2001], Giga, Matsui, and Sawada prove that when $n = 2$, the unique solution can be extended globally in time. Existence and uniqueness of solutions to (E) with bounded velocity and vorticity with $n = 2$ is due to Serfati [1995]. We briefly discuss these results in Section 2.

In this paper we prove that solutions u_ν to (NS) of [Giga et al. 2001] converge uniformly on $\mathbb{R}^2$ to Serfati solutions to (E) as viscosity approaches 0, where convergence is uniform over any finite time interval (see Theorem 3). To establish the result, we apply Littlewood–Paley theory and Bony's paradifferential calculus [1981] and follow the general strategy of [Cozzi 2009; 2010]. Specifically, we consider low and high frequencies of the difference between the solutions to (NS) and (E) separately. We first show that for fixed t and for any positive integer n,

$$\|u_\nu(t) - u(t)\|_{L^\infty} \leq n\|u_\nu(t) - u(t)\|_{B^0_{\infty,\infty}} + 2^{-n}\|\omega_\nu(t) - \omega(t)\|_{L^\infty}, \tag{1.3}$$

where $\omega_\nu = \operatorname{curl} u_\nu$ and $\omega = \operatorname{curl} u$. (See [Cozzi 2009] for a definition of the Besov space $B^0_{\infty,\infty}$.) Letting n be a function of ν such that n approaches ∞ as ν approaches 0, we show that the right-hand side of (1.3) approaches 0 as n approaches ∞. Since the second term on the right in (1.3) can be bounded above by $2^{-n}(\|\omega_\nu(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty})$, we have essentially reduced the problem to proving that the vanishing viscosity limit holds in the $B^0_{\infty,\infty}$-norm. Since L^∞ embeds continuously into $B^0_{\infty,\infty}$, we expect this problem to be easier than proving that the vanishing viscosity limit holds in the L^∞-norm; however, we must establish a rate of convergence sufficiently fast to combat the growth of the factor of n in front of the Besov norm.

Working in the Besov space $B^0_{\infty,\infty}$ has several advantages over working in L^∞. Recall that for two-dimensional fluids we can express the Euler velocity gradient in terms of its vorticity by the relation $\nabla u = \nabla\nabla^\perp\Delta^{-1}\omega$. We can also express the Euler pressure in terms of velocity by the equality $p(t) = \sum_{i,j=1}^2 R_i R_j u_i u_j(t)$, where R_i denotes the Riesz operator (similar relations hold for the Navier–Stokes velocity, vorticity, and pressure). The main mathematical obstacle when studying solutions to fluid equations in L^∞ is the lack of boundedness of the Calderon–Zygmund operators $\nabla\nabla^{\perp}\Delta^{-1}$ and $R_j R_i$ on L^∞. However, if we let Δ_j denote the Littlewood–Paley operator which projects in frequency space onto an annulus with inner and outer radius of order 2^j, then for any $j \geq 0$, $f \in \mathcal{S}'$, and Calderon–Zygmund operator A, we have

$$\|\Delta_j A f\|_{L^\infty} \leq \|\Delta_j f\|_{L^\infty}. \tag{1.4}$$

Therefore, when proving estimates in the $B^0_{\infty,\infty}$-norm, we can localize the frequencies of (NS) and (E) by applying the Littlewood–Paley operator Δ_j to the equations. We can then estimate the difference $\Delta_j(u_\nu - u)$ in the L^∞-norm using (1.4). The presence of the Littlewood–Paley operator thus facilitates estimates for velocity gradients and pressure terms.

In [Cozzi 2009] we proved that when u, u_ν, ω and ω_ν belong to $L^\infty_{\text{loc}}(\mathbb{R}^+; L^\infty(\mathbb{R}^2))$, there exists $T > 0$ such that

$$\|u_\nu - u\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \to 0 \text{ as } \nu \to 0. \tag{1.5}$$

To show (1.5), we reduced the problem to showing that the vanishing viscosity limit holds in the homogeneous $\dot{B}^0_{\infty,\infty}$-norm, but we were only able to show convergence in this norm for short time. In this paper, we show that (1.5) holds for every $T > 0$ by showing that the vanishing viscosity limit holds in the inhomogeneous $B^0_{\infty,\infty}$-norm on *any* finite time interval $[0, T]$.

We remark that this improvement of our previous result is not a consequence of using the inhomogeneous norm in place of the homogeneous norm. In fact, we are able to prove the same convergence result regardless of which Besov norm we use (the proof using the inhomogeneous norm is cleaner). Rather, in this paper we are able to improve upon the results in [Cozzi 2009] because we change our approach when estimating the commutator resulting from an application of the Littlewood–Paley operator to the nonlinear terms in (NS) and (E). Our approach here is similar to those in [Vishik 1999; Bahouri and Chemin 1994; Taniuchi et al. 2010]. As a result of our methods, we are able to prove the estimate

$$\|(u_\nu - u)(t)\|_{B^0_{\infty,\infty}} \leq C(T)2^{-n\alpha} + \int_0^t C(2^{-p} + p\|(u_\nu - u)(s)\|_{B^0_{\infty,\infty}}) \tag{1.6}$$

for any $p \in [2, \infty)$. By choosing p as a logarithmic function of $\|u_\nu - u\|_{B^0_{\infty,\infty}}$, we are able to apply Osgood's lemma, yielding a rate of convergence. In [Cozzi 2009], our methods only allow us to prove an estimate similar to (1.6) with n in place of p. Since n is a function of viscosity, we must apply Gronwall's lemma and introduce a factor of e^{nt} on the right hand side, which prevents us from proving that the inviscid limit holds on any finite time interval.

The paper is organized as follows. In Section 2, we review properties of nondecaying solutions to the fluid equations. In Section 3 and Section 4, we state and prove the main result; we devote Section 4 entirely to showing that the vanishing viscosity limit holds in the $B^0_{\infty,\infty}$-norm.

For background information on Littlewood–Paley theory, Bony's paraproduct decomposition, Besov spaces, and technical lemmas used throughout the paper, we refer the reader to Section 2 of [Cozzi 2009].

2. Existence and uniqueness of nondecaying solutions to the fluid equations

In this section, we briefly summarize what is known about nondecaying solutions to (NS) and (E). We begin with the mild solutions to (NS) established in [Giga et al. 1999]. By a mild solution to (NS), we mean a solution u_ν of the integral equation

$$u_\nu(t,x) = e^{t\nu\Delta}u^0_\nu - \int_0^t e^{(t-s)\nu\Delta}\mathbf{P}(u_\nu\cdot\nabla u_\nu)(s)\,ds. \tag{2.1}$$

In (2.1), $e^{\tau\nu\Delta}$ denotes convolution with the Gauss kernel; that is, for $f\in S'$, $e^{\tau\nu\Delta}f = G_{\tau\nu}*f$, where $G_{\tau\nu}(x) = 1/(4\pi\tau\nu)\exp(-|x|^2/(4\tau\nu))$. Also, $\mathbf{P}$ denotes the Helmholtz projection operator with i, j component given by $\delta_{ij}+R_iR_j$, where $R_l = (-\Delta)^{-1/2}\partial_l$ is the Riesz operator. Giga, Inui, and Matsui proved the following result regarding mild solutions in $\mathbb{R}^n$, $n\ge 2$:

Theorem 1 [Giga et al. 1999]. *Let BUC denote the space of bounded uniformly continuous functions, and assume u^0_ν belongs to $BUC(\mathbb{R}^n)$ for fixed $n\ge 2$. There exists $T_0>0$ and a unique solution to* (2.1) *in the space $C([0,T_0];BUC(\mathbb{R}^n))$ with initial data u^0_ν. Moreover, if we assume* $\operatorname{div} u^0_\nu = 0$, *and if we define $p_\nu(t) = \sum_{i,j=1}^2 R_iR_ju_{\nu i}u_{\nu j}(t)$ for each $t\in[0,T_0]$, then u_ν belongs to $C^\infty([0,T_0]\times\mathbb{R}^n)$ and solves* (NS).

Remark 2.2. For the main theorem of this paper, we assume that u^0 and ω^0 are bounded on $\mathbb{R}^2$ and that $\operatorname{div} u^0 = 0$. These assumptions imply that u^0 belongs to $C^\alpha(\mathbb{R}^2)$ for every $\alpha<1$ and is therefore in $BUC(\mathbb{R}^2)$ (see, for example, Lemma 4 of [Cozzi 2009]).

Giga, Matsui, and Sawada [2001] showed that when $n=2$, the solution to (NS) established in Theorem 1 can be extended to a global-in-time smooth solution. Sawada and Taniuchi [2007] showed that if u^0_ν and ω^0_ν belong to $L^\infty(\mathbb{R}^2)$, then the following exponential estimate holds:

$$\|u_\nu(t)\|_{L^\infty} \le C\|u^0_\nu\|_{L^\infty}e^{Ct\|\omega^0_\nu\|_{L^\infty}}. \tag{2.3}$$

For ideal incompressible fluids, we have the following result:

Theorem 2 [Serfati 1995]. *Let u^0 and ω^0 belong to $L^\infty(\mathbb{R}^2)$, and let $c\in\mathbb{R}$. For every $T>0$, there exists a unique solution (u,p) to* (E) *in the space*

$$L^\infty([0,T];L^\infty(\mathbb{R}^2))\times L^\infty([0,T];C(\mathbb{R}^2))$$

with $\omega \in L^\infty([0,T]; L^\infty(\mathbb{R}^2))$, $p(0) = c$, *and with* $p(t,x)/|x| \to 0$ *as* $|x| \to \infty$.

Serfati also proved an estimate analogous to (2.3) for his solutions:

$$\|u(t)\|_{L^\infty} \le C\|u^0\|_{L^\infty} e^{C_1\|\omega^0\|_{L^\infty} t}. \tag{2.4}$$

Finally, we recall that we have a uniform bound in time on the L^∞-norms of the vorticities corresponding to the solutions of (NS) and (E). For fixed $\nu \ge 0$, we have that

$$\|\omega_\nu(t)\|_{L^\infty} \le \|\omega_\nu^0\|_{L^\infty} \tag{2.5}$$

for all $t \ge 0$. One can prove this bound by applying the maximum principle to the vorticity formulations of (NS) and (E). We refer the reader to Lemma 3.1 of [Sawada and Taniuchi 2007] for a detailed proof.

3. Statement and proof of the main result

We are now prepared to state the main theorem:

Theorem 3. *Let u_ν be the unique solution to* (NS) *and u the unique solution to* (E), *both with initial data u^0 and ω^0 belonging to $L^\infty(\mathbb{R}^2)$, and with p_ν and p satisfying the conditions of Theorems 1 and 2, respectively. Let M be defined by* (3.2) *below and let $T > 0$ be fixed. Then there exists a constant $C_{M,T}$, increasing with both M and T, such that the following estimate holds for any fixed $\alpha \in (0,1)$:*

$$\|u_\nu - u\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \le C_{M,T}\big(2 - \log(\sqrt{\nu})^{\alpha e^{-C_{M,T}}}\big)(\sqrt{\nu})^{\alpha e^{-C_{M,T}}}. \tag{3.1}$$

Proof. Throughout the proof of Theorem 3, we let M denote a constant, dependent on T, which satisfies

$$M \ge 1 + \sup_{t\in[0,T]} (\|u_\nu(t)\|_{L^\infty} + \|u(t)\|_{L^\infty} + \|\omega_\nu(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty}). \tag{3.2}$$

We note that the value of M will change throughout the proof but will always satisfy (3.2). The existence results in Section 2 imply that M will be finite for any $T > 0$.

Let u be the unique Serfati solution to (E), and let u_ν be the unique solution to (NS) given by [Giga et al. 2001]. We fix n to be a positive integer and we fix $T > 0$. We will eventually choose $n = -\frac{1}{2}\log_2 \nu$ so that as ν approaches 0, n approaches ∞.

We begin with the following inequality:

$$\|u_\nu - u\|_{L^\infty([0,T];L^\infty)} \le \sum_{j=-1}^{n} \|\Delta_j(u_\nu - u)\|_{L^\infty([0,T];L^\infty)} + \sum_{j=n+1}^{\infty} \|\Delta_j(u_\nu - u)\|_{L^\infty([0,T];L^\infty)}. \tag{3.3}$$

We can estimate the second term on the right-hand side of (3.3) using Bernstein's lemma and the estimate

$$\|\Delta_j \nabla u\|_{L^\infty} \le \|\Delta_j \omega\|_{L^\infty} \quad \text{for} \quad j \ge 0. \tag{3.4}$$

(Both (3.4) and Bernstein's lemma can be found in Section 2 of [Cozzi 2009].) We obtain the inequality

$$\sum_{j=n+1}^{\infty} \|\Delta_j(u_\nu - u)\|_{L^\infty([0,T];L^\infty)} \le \sum_{j=n+1}^{\infty} 2^{-j} \|\Delta_j(\nabla u_\nu - \nabla u)\|_{L^\infty([0,T];L^\infty)} \le M2^{-n}. \tag{3.5}$$

To estimate the first term on the right-hand side of (3.3), we use the definition of $B^0_{\infty,\infty}$ to observe that

$$\sum_{j=-1}^{n} \|\Delta_j(u_\nu - u)\|_{L^\infty([0,T];L^\infty)} \le Cn\|u_\nu - u\|_{L^\infty([0,T];B^0_{\infty,\infty})}. \tag{3.6}$$

After substituting (3.6) and (3.5) into (3.3), we conclude that

$$\|u_\nu - u\|_{L^\infty([0,T];L^\infty)} \le Cn\|u_\nu - u\|_{L^\infty([0,T];B^0_{\infty,\infty})} + M2^{-n}. \tag{3.7}$$

We must estimate the difference of u_ν and u in the $B^0_{\infty,\infty}$-norm. We temporarily assume that the following estimate holds for all $\alpha \in (0, 1)$:

$$\|u_\nu - u\|_{L^\infty([0,T];B^0_{\infty,\infty})} \le C_{M,T}\big(2 - \log 2^{-n\alpha e^{-C_{M,T}}}\big)2^{-n\alpha e^{-C_{M,T}}}. \tag{3.8}$$

Assuming that (3.8) holds, we see from (3.7) and (3.8) that

$$\|u_\nu - u\|_{L^\infty([0,T];L^\infty)} \le C_{M,T}\big(2 - \log 2^{-n\alpha e^{-C_{M,T}}}\big)2^{-n\alpha e^{-C_{M,T}}}.$$

The estimate (3.1) follows after setting $\nu = 2^{-2n}$. Therefore, to complete the proof of Theorem 3, it remains to prove (3.8).

4. Proof of (3.8)

Let $u_n = S_n u$, $\omega_n = S_n \omega(u)$, $\bar{u}_n = u_\nu - u_n$, and $\bar{\omega}_n = \omega_\nu - \omega_n$. Throughout most of the proof of (3.8), the time t is fixed and suppressed in the calculations.

Fix $p \in (1, \infty)$ (to be chosen later). We apply Bernstein's lemma and (3.4) to establish the estimate

$$\|u_\nu - u\|_{B^0_{\infty,\infty}} \leq \sup_{-1\leq l\leq 2} \|\Delta_l(u_\nu - u)\|_{L^\infty} + \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)\|_{L^\infty} + \sup_{l>p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)\|_{L^\infty}. \tag{4.1}$$

The separation of frequencies at $l = 2$ will simplify estimates in what follows.

We will first consider the difference $\sup_{3\leq l\leq p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)\|_{L^\infty}$. We will eventually need to estimate the viscosity term $\nu\|\Delta\omega\|_{L^\infty}$. To facilitate this estimate, we smooth out the Euler vorticity and write

$$\begin{aligned}\sup_{3\leq l\leq p} 2^{-l}\|\Delta_l(\omega_\nu-\omega)\|_{L^\infty} &\leq \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n\|_{L^\infty} + \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l(\omega_n-\omega)\|_{L^\infty}\\ &\leq \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n\|_{L^\infty} + \sup_{l\geq n} 2^{-l}\|\Delta_l(\omega_n-\omega)\|_{L^\infty}\\ &\leq \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n\|_{L^\infty} + M2^{-n},\end{aligned} \tag{4.2}$$

where we used properties of the Fourier support of ω_n to get the second inequality. We now estimate $\sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n\|_{L^\infty}$. We note that ω_ν and ω_n satisfy

$$\partial_t\omega_\nu + u_\nu\cdot\nabla\omega_\nu - \nu\Delta\omega_\nu = 0 \tag{4.3}$$

and

$$\partial_t\omega_n + u_n\cdot\nabla\omega_n = \nabla\cdot\tau_n(u,\omega), \tag{4.4}$$

where $\tau_n(u,\omega) = (u - u_n)(\omega - \omega_n) - r_n(u,\omega)$ and

$$r_n(u,\omega) = \int \check{\psi}_0(y)(u(x - 2^{-n}y) - u(x))(\omega(x - 2^{-n}y) - \omega(x))\,dy.$$

Here ψ_0 denotes the Fourier multiplier associated with the Littlewood-Paley operator Δ_{-1}. Equation (4.4) was utilized by Constantin and Wu [1996] and by Constantin, E, and Titi in a proof of Onsager's conjecture in [Constantin et al. 1994]. We subtract (4.4) from (4.3) and, for fixed l, we apply the Littlewood–Paley operator Δ_l to the difference of the two equations. After adding $(S_{l-2}u_\nu)\cdot\nabla\Delta_l\bar{\omega}_n$ to both sides of the resulting equation, we obtain

$$\begin{aligned}(4.5)\quad \partial_t \Delta_l \bar{\omega}_n + (S_{l-2}u_\nu)\cdot\nabla\Delta_l\bar{\omega}_n - \nu\Delta\Delta_l\bar{\omega}_n &= (S_{l-2}u_\nu)\cdot\nabla\Delta_l\bar{\omega}_n - \Delta_l(u_\nu\cdot\nabla\bar{\omega}_n)\\ &\quad - \Delta_l(\bar{u}_n\cdot\nabla\omega_n) + \nu\Delta\Delta_l\omega_n - \Delta_l\nabla\cdot\tau_n(u,\omega).\end{aligned}$$

Borrowing notation from [Taniuchi et al. 2010], we define

$$(4.6)\qquad I^{l,k} = (S_{l-2}u_\nu^k)\partial_k\Delta_l\bar{\omega}_n - \partial_k\Delta_l(u_\nu^k\bar{\omega}_n) \quad\text{and}\quad J^{l,k} = -\partial_k\Delta_l(\bar{u}_n^k\omega_n).$$

From (4.5), we see that

$$\begin{aligned}(4.7)\quad \partial_t \Delta_l \bar{\omega}_n + (S_{l-2}u_\nu)\cdot\nabla\Delta_l\bar{\omega}_n - \nu\Delta\Delta_l\bar{\omega}_n &= \sum_{k=1}^{2}(I^{l,k}+J^{l,k}) + \nu\Delta\Delta_l\omega_n - \Delta_l\nabla\cdot\tau_n(u,\omega).\end{aligned}$$

Since $S_{l-2}u_\nu$ belongs to $L^1_{\text{loc}}(\mathbb{R}^+;\text{Lip}(\mathbb{R}^2))$ and is divergence-free, we can apply the following lemma for the transport diffusion equation from [Hmidi 2005].

Lemma 4. *Let $p\in[1,\infty]$, and let u be a divergence-free vector field belonging to $L^1_{\text{loc}}(\mathbb{R}^+;\text{Lip}(\mathbb{R}^d))$. Moreover, assume the function f belongs to $L^1_{\text{loc}}(\mathbb{R}^+;L^p(\mathbb{R}^d))$ and the function a^0 belongs to $L^p(\mathbb{R}^d)$. Then any solution a to the problem*

$$\begin{cases}\partial_t a + u\cdot\nabla a - \nu\Delta a = f,\\ a|_{t=0} = a^0,\end{cases}$$

satisfies the estimate

$$\|a(t)\|_{L^p} \le \|a^0\|_{L^p} + \int_0^t \|f(s)\|_{L^p}\,ds.$$

An application of Lemma 4 to (4.7) yields

$$\begin{aligned}(4.8)\quad \|\Delta_l\bar{\omega}_n(t)\|_{L^\infty} &\le \|\Delta_l\bar{\omega}_n(0)\|_{L^\infty} + \int_0^t\biggl(\sum_{k=1}^{2}\bigl(\|I^{l,k}(s)\|_{L^\infty} + \|J^{l,k}(s)\|_{L^\infty}\bigr)\biggr)ds\\ &\quad + \int_0^t(\nu\|\Delta\Delta_l\omega_n(s)\|_{L^\infty} + \|\Delta_l\nabla\cdot\tau_n(u,\omega)(s)\|_{L^\infty})\,ds.\end{aligned}$$

Our goal is to establish an upper bound for $\sup_{3\le l\le p} 2^{-l}\|\Delta_l\bar{\omega}_n(t)\|_{L^\infty}$. In what follows, we will estimate each term on the right-hand side of (4.8), multiply by 2^{-l}, and take the supremum over l satisfying $3\le l\le p$. Estimates for the last two terms on the right-hand side of (4.8) follow from work in [Cozzi 2009]. Indeed, in that paper we used boundedness of the Euler vorticity and membership of the Euler velocity in $C^\alpha(\mathbb{R}^2)$ for any $\alpha \in (0,1)$ to show that for such α,

$$\sup_{l\geq 0} 2^{-l}\|\Delta_l \nabla\cdot\tau_n(u,\omega)\|_{L^\infty} \leq \|\nabla\cdot\tau_n(u,\omega)\|_{L^\infty} \leq M2^{-n\alpha}. \tag{4.9}$$

We also showed there, using Bernstein's lemma and properties of the Fourier support of ω_n, that

$$\sup_{l\geq 0} 2^{-l}\nu\|\Delta_l\Delta\omega_n\|_{L^\infty} \leq 2^n\nu\|\omega_n\|_{L^\infty} \leq M2^{-n}, \tag{4.10}$$

where we set $\nu = 2^{-2n}$. To estimate the initial data, we used the Fourier support of $\omega_n^0 = S_n\omega^0$ to write

$$\sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n(0)\|_{L^\infty} \leq \sup_{l\geq n} 2^{-l}\|\Delta_l\bar{\omega}_n(0)\|_{L^\infty} \leq M2^{-n}. \tag{4.11}$$

Multiplying (4.8) by 2^{-l}, taking the supremum of (4.8) over l satisfying $3\leq l\leq p$, and applying the estimates (4.9), (4.10), and (4.11) gives

$$\begin{aligned}\sup_{3\leq l\leq p} & 2^{-l}\|\Delta_l\bar{\omega}_n(t)\|_{L^\infty}\\ &\leq M(t+1)2^{-n\alpha} + \sup_{3\leq l\leq p} 2^{-l}\int_0^t\Bigl(\sum_{k=1}^2\bigl(\|I^{l,k}(s)\|_{L^\infty} + \|J^{l,k}(s)\|_{L^\infty}\bigr)\Bigr)\,ds.\end{aligned} \tag{4.12}$$

It remains to estimate $I^{l,k}$ and $J^{l,k}$. We begin with $J^{l,k}$. We again borrow notation from [Taniuchi et al. 2010] and use Bony's paraproduct decomposition to write

$$\begin{aligned}J^{l,k} &= -\partial_k\Delta_l\sum_{\substack{|j-l|\leq 3\\ j\geq 1}} S_{j-2}\bar{u}_n^k\Delta_j\omega_n\\ &\quad -\partial_k\Delta_l\sum_{\substack{|j-l|\leq 3\\ j\geq 1}} \Delta_j\bar{u}_n^k S_{j-2}\omega_n\\ &\quad -\partial_k\Delta_l\sum_{\substack{|j-j'|\leq 1\\ \max\{j,j'\}\geq l-3}} \Delta_j\bar{u}_n^k\Delta_{j'}\omega_n\\ &= J_1^{l,k} + J_2^{l,k} + J_3^{l,k}.\end{aligned} \tag{4.13}$$

We estimate $J_1^{l,k}$. Several applications of Bernstein's lemma give

$$\begin{aligned}\|J_1^{l,k}\|_{L^\infty} &\leq 2^l\sum_{\substack{|j-l|\leq 3\\ j\geq 1}} \|S_{j-2}\bar{u}_n\|_{L^\infty}\|\Delta_j\omega_n\|_{L^\infty}\\ &\leq 2^l\sum_{\substack{|j-l|\leq 3\\ j\geq 1}}\|\Delta_j\omega\|_{L^\infty}\sum_{k\leq j-2}\|\Delta_k\bar{u}_n\|_{L^\infty}.\end{aligned} \tag{4.14}$$

Multiplying by 2^{-l} and taking the supremum over l satisfying $3 \le l \le p$, we conclude that

$$\sup_{3\le l\le p} 2^{-l}\|J_1^{l,k}\|_{L^\infty} \le Mp\|\bar{u}_n\|_{B^0_{\infty,\infty}}. \tag{4.15}$$

We now estimate $J_2^{l,k}$. We write

$$\begin{aligned}\|J_2^{l,k}\|_{L^\infty} &\le 2^l \sum_{\substack{|j-l|\le 3\\ j\ge 1}} \|\Delta_l(\Delta_j\bar{u}_n S_{j-2}\omega_n)\|_{L^\infty}\\ &\le 2^l \sum_{\substack{|j-l|\le 3\\ j\ge 1}} \|\Delta_j\bar{u}_n\|_{L^\infty}\|S_{j-2}\omega_n\|_{L^\infty}\end{aligned} \tag{4.16}$$

so that

$$\sup_{3\le l\le p} 2^{-l}\|J_2^{l,k}\|_{L^\infty} \le M\|\bar{u}_n\|_{B^0_{\infty,\infty}}. \tag{4.17}$$

To estimate $J_3^{l,k}$, we use properties of Littlewood–Paley operators to observe that

$$\begin{aligned}\|J_3^{l,k}\|_{L^\infty} &\le 2^l \sum_{\substack{|j-j'|\le 1\\ \max\{j,j'\}\ge l-3}} \|\Delta_j\bar{u}_n\|_{L^\infty}\|\Delta_{j'}\omega_n\|_{L^\infty}\\ &\le C2^l \sum_{j\ge l-3} \|\Delta_j\bar{u}_n\|_{L^\infty}\|\Delta_j\omega_n\|_{L^\infty} \le C2^l\|\omega\|_{L^\infty}\|\bar{u}_n\|_{B^0_{\infty,1}}.\end{aligned} \tag{4.18}$$

We estimate the $B^0_{\infty,1}$-norm of $\bar{u}_n$ as follows: We bound the low frequencies using the definition of $B^0_{\infty,\infty}$, and we estimate the high frequencies using Bernstein's lemma, (3.4), and boundedness of vorticity. We have the series of estimates

$$\|\bar{u}_n\|_{B^0_{\infty,1}} \le \sum_{j=-1}^{p} \|\Delta_j\bar{u}_n\|_{L^\infty} + \sum_{j>p} 2^{-j}\|\Delta_j\bar{\omega}_n\|_{L^\infty} \le Cp\|\bar{u}_n\|_{B^0_{\infty,\infty}} + M2^{-p}. \tag{4.19}$$

Substituting this estimate into (4.18), multiplying by 2^{-l} and taking the supremum over l between 3 and p yields the estimate

$$\sup_{3\le l\le p} 2^{-l}\|J_3^{l,k}\|_{L^\infty} \le M(2^{-p} + p\|\bar{u}_n\|_{B^0_{\infty,\infty}}). \tag{4.20}$$

Combining the estimates for (4.15), (4.17), and (4.20), we conclude that

$$\sup_{3\le l\le p} 2^{-l}\sum_{k=1}^{2}\|J^{l,k}\|_{L^\infty} \le M(2^{-p} + p\|\bar{u}_n\|_{B^0_{\infty,\infty}}). \tag{4.21}$$

We now estimate $I^{l,k}$ for l satisfying $3 \le l \le p$. We apply Theorem 6.1 of [Vishik 1999] to write

$$\begin{aligned}\sum_{k=1}^{2} \|I^{l,k}\|_{L^\infty} &\le C \sum_{|j-l|\le 3} \|S_{j-2}\nabla\bar{\omega}_n\|_{L^\infty}\|\Delta_j u_\nu\|_{L^\infty} \\ &\quad + \sum_{|j-l|\le 3} \|S_{j-2}\nabla u_\nu\|_{L^\infty}\|\Delta_j\bar{\omega}_n\|_{L^\infty} \\ &\quad + C2^l \sum_{\substack{j\ge l-3\\ |j-j'|\le 1}} 2^{-j}\|\Delta_j\nabla u_\nu\|_{L^\infty}\|\Delta_{j'}\bar{\omega}_n\|_{L^\infty} \\ &= X_1^l + X_2^l + X_3^l.\end{aligned}$$

To estimate X_1^l, keeping in mind that $l \ge 3$, we use Bernstein's lemma and (3.4) to write

$$\sum_{|j-l|\le 3} \|S_{j-2}\nabla\bar{\omega}_n\|_{L^\infty}\|\Delta_j u_\nu\|_{L^\infty} \le C2^l \sum_{|j-l|\le 3} \|S_{j-2}\bar{u}_n\|_{L^\infty}\|\Delta_j\omega_\nu\|_{L^\infty}.$$

The remainder of the estimate for X_1^l is identical to that for $J_1^{l,k}$. Multiplying by 2^{-l} and taking the supremum over l between 3 and p, we conclude that

$$\sup_{3\le l\le p} 2^{-l}X_1^l \le Mp\|\bar{u}_n\|_{B^0_{\infty,\infty}}. \tag{4.22}$$

To estimate X_2^l for $3 \le l \le p$, we again apply Bernstein's lemma and (3.4) to write

$$\begin{aligned}X_2^l &= \sum_{|j-l|\le 3} \|S_{j-2}\nabla u_\nu\|_{L^\infty}\|\Delta_j\bar{\omega}_n\|_{L^\infty} \\ &\le C2^l \sum_{|j-l|\le 3} (\|u_\nu\|_{L^\infty} + (j-1)\|\omega_\nu\|_{L^\infty})\|\Delta_j\bar{u}_n\|_{L^\infty} \\ &\le Ml2^l \sum_{|j-l|\le 3} \|\Delta_j\bar{u}_n\|_{L^\infty}.\end{aligned} \tag{4.23}$$

To get the first inequality above, we bounded the term $\|S_{j-2}\nabla u_\nu\|_{L^\infty}$ above by the sum resulting from the S_{j-2} operator. We then applied (3.4). After multiplying (4.23) by 2^{-l} and taking the supremum over l satisfying $3 \le l \le p$, we find that

$$\sup_{3\le l\le p} 2^{-l}X_2^l \le Mp\|\bar{u}_n\|_{B^0_{\infty,\infty}}. \tag{4.24}$$

The estimate for X_3^l is similar to that for $J_3^{l,k}$. For l satisfying $3 \le l \le p$, we write

$$X_3^l = C2^l \sum_{\substack{j \ge l-3 \\ |j-j'| \le 1}} 2^{-j} \|\Delta_j \nabla u_\nu\|_{L^\infty} \|\Delta_{j'} \bar{\omega}_n\|_{L^\infty} \le C2^l \sum_{j \ge l-3} \|\Delta_j \omega_\nu\|_{L^\infty} \|\Delta_j \bar{u}_n\|_{L^\infty}, \tag{4.25}$$

where we used Bernstein's lemma and (3.4) to get the last inequality. We now use the same argument as in (4.18) and (4.19) to conclude that

$$\sup_{3 \le l \le p} 2^{-l} X_3^l \le M(2^{-p} + p\|\bar{u}_n\|_{B^0_{\infty,\infty}}). \tag{4.26}$$

Combining the above estimates for X_1^l, X_2^l, and X_3^l, we have

$$\sup_{3 \le l \le p} 2^{-l} \sum_{k=1}^{2} \|I^{l,k}\|_{L^\infty} \le M(2^{-p} + p\|\bar{u}_n\|_{B^0_{\infty,\infty}}). \tag{4.27}$$

Applying the estimates (4.21) and (4.27) to (4.12), we conclude that

$$\sup_{3 \le l \le p} 2^{-l} \|\Delta_l \bar{\omega}_n(t)\|_{L^\infty} \le C(t+1)2^{-n\alpha} + M \int_0^t (2^{-p} + p\|W(s)\|_{\dot{B}^0_{\infty,\infty}})\, ds \tag{4.28}$$

for any $\alpha \in (0, 1)$. We substitute (4.28) into (4.2). This gives

$$\sup_{3 \le l \le p} 2^{-l} \|\Delta_l(\omega_\nu - \omega)(t)\|_{L^\infty} \le C(t+1)2^{-n\alpha} + M \int_0^t (2^{-p} + p\|\bar{u}_n(s)\|_{\dot{B}^0_{\infty,\infty}})\, ds. \tag{4.29}$$

Inspection of (4.1) reveals that we must still estimate $\sup_{-1 \le l \le 2} \|\Delta_l(u_\nu - u)(t)\|_{L^\infty}$ and $\sup_{l>p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)(t)\|_{L^\infty}$. These two terms are more straightforward. We estimate the term $\sup_{l>p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)(t)\|_{L^\infty}$ by observing that

$$\sup_{l>p} 2^{-l} \|\Delta_l(\omega_\nu - \omega)(t)\|_{L^\infty} \le M2^{-p}. \tag{4.30}$$

To estimate $\sup_{-1 \le l \le 2} \|\Delta_l(u_\nu - u)(t)\|_{L^\infty}$, we use the velocity formulation. Setting $\bar{p} = p_\nu - p$ and $\bar{u} = u_\nu - u$, we subtract (E) from (NS). This gives

$$\partial_t \bar{u} + u_\nu \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u - \nu \Delta \bar{u} = -\nabla \bar{p} + \nu \Delta u_\nu. \tag{4.31}$$

We apply Δ_l to (4.31) for $-1 \le l \le 2$. This gives

$$\partial_t \Delta_l \bar{u} + (\Delta_l u_\nu) \cdot \nabla \Delta_l \bar{u} - \nu \Delta_l \Delta \bar{u} = (\Delta_l u_\nu) \cdot \nabla \Delta_l \bar{u} - \Delta_l(u_\nu \cdot \nabla \bar{u}) - \Delta_l(\bar{u} \cdot \nabla u) - \Delta_l \nabla \bar{p} + \nu \Delta_l \Delta u_\nu. \tag{4.32}$$

Again by Lemma 4, we have

$$\begin{aligned}(4.33)\quad &\|\Delta_l\bar u(t)\|_{L^\infty}\\ &\le\int_0^t\big(\|(\Delta_l u_\nu\cdot\nabla\Delta_l\bar u)(s)\|_{L^\infty}+\|\Delta_l(u_\nu\cdot\nabla\bar u)(s)\|_{L^\infty}\\ &\qquad+\|\Delta_l(\bar u\cdot\nabla u)(s)\|_{L^\infty}+\|\Delta_l\nabla\bar p(s)\|_{L^\infty}+\nu\|\Delta_l\Delta u_\nu(s)\|_{L^\infty}\big)\,ds.\end{aligned}$$

We have the following straightforward estimates, all which follow from Bernstein's lemma and the divergence-free property of the velocity:

$$(4.34)\qquad \begin{aligned}&\|(\Delta_l u_\nu)\cdot\nabla\Delta_l\bar u\|_{L^\infty}\le C\|u_\nu\|_{L^\infty}2^l\|\Delta_l\bar u\|_{L^\infty}\le M2^l\|\bar u\|_{L^\infty},\\ &\|\Delta_l(u_\nu\cdot\nabla\bar u)\|_{L^\infty}\le C2^l\|u_\nu\|_{L^\infty}\|\bar u\|_{L^\infty}\le M2^l\|\bar u\|_{L^\infty},\\ &\|\Delta_l(\bar u\cdot\nabla u)\|_{L^\infty}\le 2^l\|\bar u\|_{L^\infty}\|u\|_{L^\infty}\le M2^l\|\bar u\|_{L^\infty},\\ &\nu\|\Delta_l\Delta u_\nu\|_{L^\infty}\le C\nu 2^{2l}\|u_\nu\|_{L^\infty}\le M\nu 2^{2l}.\end{aligned}$$

To estimate the pressure, we follow an argument in [Taniuchi et al. 2010]. For $0\le l\le 2$, if φ_l is the Fourier multiplier associated with Δ_l, then

$$\begin{aligned}(4.35)\qquad \|\Delta_l\nabla\bar p\|_{L^\infty}&=\Big\|\sum_{i,i'=1}^{2}R_iR_{i'}\nabla\Delta_l\big(\bar u^iu^{i'}+u^i_\nu\bar u^{i'}\big)\Big\|_{L^\infty}\\ &\le\|R_iR_{i'}\nabla\check\varphi_l\|_{L^1}\big\|\bar u^iu^{i'}+u^i_\nu\bar u^{i'}\big\|_{L^\infty}\le M2^l\|\bar u\|_{L^\infty},\end{aligned}$$

where we applied the estimates $\|R_iR_{i'}\nabla\check\varphi_l\|_{L^1}\le\|R_iR_{i'}\nabla\check\varphi_l\|_{\mathcal H^1}\le\|\nabla\check\varphi_l\|_{\mathcal H^1}\le C2^l$ to get the last inequality. For the case $l=-1$, we apply the same series of estimates as in (4.35) with $\check\psi_0$ in place of $\check\varphi_l$.

Substituting the estimates (4.34) and (4.35) into (4.33) and taking the supremum over $-1\le l\le 2$ yields

$$(4.36)\qquad \sup_{-1\le l\le 2}\|\Delta_l\bar u(t)\|_{L^\infty}\le M\int_0^t(\|\bar u\|_{L^\infty}+2^{-2n}),$$

where we used the equality $\nu=2^{-2n}$. We now apply the embedding $B^0_{\infty,1}\hookrightarrow L^\infty$, along with (4.19), to conclude that

$$(4.37)\qquad \sup_{-1\le l\le 2}\|\Delta_l\bar u(t)\|_{L^\infty}\le Mt2^{-2n}+M\int_0^t(p\|\bar u(s)\|_{B^0_{\infty,\infty}}+2^{-p})\,ds.$$

We substitute the estimates (4.37), (4.29), and (4.30) into (4.1). We conclude that

$$\begin{aligned}(4.38)\qquad \sup_{l\ge -1}\|\Delta_l\bar u(t)\|_{L^\infty}&\le M(T+1)2^{-n\alpha}+M2^{-p}\\ &\quad+\int_0^t M(2^{-p}+p\|\bar u(s)\|_{B^0_{\infty,\infty}})\,ds.\end{aligned}$$

To complete the proof of (3.8), we will apply Osgood's lemma to (4.38). We first note that by the embedding $L^\infty \hookrightarrow B^0_{\infty,\infty}$,

$$\|\bar{u}(t)\|_{B^0_{\infty,\infty}} \le \|\bar{u}(t)\|_{L^\infty} \le \|u_\nu(t)\|_{L^\infty} + \|u(t)\|_{L^\infty} \le M$$

for all $t \in [0, T]$. For each $t \in [0, T]$, set

$$\delta(t) = \frac{\int_0^t \|\bar{u}(s)\|_{B^0_{\infty,\infty}} ds}{MT} \le 1, \tag{4.39}$$

and set $p = 2 - \log \delta(t)$. Then (4.38) reduces to

$$\|\bar{u}(t)\|_{B^0_{\infty,\infty}} \le M(T+1)2^{-n\alpha} + M(T+1)\delta(t) + M^2 T(2 - \log_2 \delta(t))\delta(t). \tag{4.40}$$

Integrating both sides over $[0, t]$ and dividing both sides by MT yields the inequality

$$\delta(t) \le (T+1)2^{-n\alpha} + \left(\frac{T+1}{T} + M\right) \int_0^t (2 - \log_2 \delta(s))\delta(s)\, ds. \tag{4.41}$$

We are now in a position to use Osgood's lemma (see [Chemin and Lerner 1995]):

Lemma 5 (Osgood's lemma). *Let ρ be a measurable positive function, let γ be a locally integrable positive function, and let μ be a continuous increasing function. Assume that for some number $\beta > 0$, the function ρ satisfies*

$$\rho(t) \le \beta + \int_{t_0}^t \gamma(s)\mu(\rho(s))\, ds.$$

Then $-\phi(\rho(t)) + \phi(\beta) \le \int_{t_0}^t \gamma(s)\, ds$, *where* $\phi(x) = \int_x^1 \frac{1}{\mu(r)}\, dr$.

We set $\mu(r) = r(2 - \log r)$, $\rho(t) = \delta(t)$, $\beta = (T+1)2^{-n\alpha}$, and

$$\gamma(t) = \frac{T+1}{T} + M := C_0(M, T),$$

and we apply Osgood's lemma to obtain, for any $t \le T$,

$$-\log(2 - \log \delta(t)) + \log(2 - \log((T+1)2^{-n\alpha})) \le C_0(M, T)t.$$

Taking the exponential twice gives

$$\delta(t) \le e^{2 - 2e^{-C_0(M,T)t}} ((T+1)2^{-n\alpha})^{e^{-C_0(M,T)t}}. \tag{4.42}$$

The inequality (3.8) follows after substituting (4.42) into (4.40) and letting $\nu = 2^{-2n}$. □

Acknowledgement

This work was supported by the National Science Foundation under grant DMS-1049698.

References

[Bahouri and Chemin 1994] H. Bahouri and J.-Y. Chemin, "Équations de transport relatives á des champs de vecteurs non-lipschitziens et mécanique des fluides", *Arch. Rational Mech. Anal.* **127**:2 (1994), 159–181. MR 95g:35164 Zbl 0821.76012

[Bony 1981] J.-M. Bony, "Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires", *Ann. Sci. École Norm. Sup.* (4) **14**:2 (1981), 209–246. MR 84h:35177 Zbl 0495.35024

[Chemin 1996] J.-Y. Chemin, "A remark on the inviscid limit for two-dimensional incompressible fluids", *Comm. Partial Differential Equations* **21**:11-12 (1996), 1771–1779. MR 98c:35127 Zbl 0876.35087

[Chemin and Lerner 1995] J.-Y. Chemin and N. Lerner, "Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes", *J. Differential Equations* **121**:2 (1995), 314–328. MR 96h:35153 Zbl 0878.35089

[Constantin 1986] P. Constantin, "Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations", *Comm. Math. Phys.* **104**:2 (1986), 311–326. MR 87f:35200 Zbl 0655.76041

[Constantin and Wu 1996] P. Constantin and J. Wu, "The inviscid limit for non-smooth vorticity", *Indiana Univ. Math. J.* **45**:1 (1996), 67–81. MR 97g:35129 Zbl 0859.76015

[Constantin et al. 1994] P. Constantin, W. E, and E. S. Titi, "Onsager's conjecture on the energy conservation for solutions of Euler's equation", *Comm. Math. Phys.* **165**:1 (1994), 207–209. MR 96e:76025 Zbl 0818.35085

[Cozzi 2009] E. Cozzi, "Vanishing viscosity in the plane for nondecaying velocity and vorticity", *SIAM J. Math. Anal.* **41**:2 (2009), 495–510. MR 2011a:76029 Zbl 05696712

[Cozzi 2010] E. Cozzi, "A finite time result for vanishing viscosity in the plane with nondecaying vorticity", *Commun. Math. Sci.* **8**:4 (2010), 851–862. MR 2012a:35226 Zbl 05843215

[Giga et al. 1999] Y. Giga, K. Inui, and S. Matsui, "On the Cauchy problem for the Navier–Stokes equations with nondecaying initial data", pp. 27–68 in *Advances in fluid dynamics*, edited by P. Maremonti, Quad. Mat. **4**, Dept. Math., Seconda Univ. Napoli, Caserta, 1999. MR 2001g:35210 Zbl 0961.35110

[Giga et al. 2001] Y. Giga, S. Matsui, and O. Sawada, "Global existence of two-dimensional Navier–Stokes flow with nondecaying initial velocity", *J. Math. Fluid Mech.* **3**:3 (2001), 302–315. MR 2002h:35236 Zbl 0992.35066

[Hmidi 2005] T. Hmidi, "Régularité höldérienne des poches de tourbillon visqueuses", *J. Math. Pures Appl.* (9) **84**:11 (2005), 1455–1495. MR 2006i:35278 Zbl 1095.35024

[Kato 1972] T. Kato, "Nonstationary flows of viscous and ideal fluids in $\mathbf{R}^3$", *J. Functional Analysis* **9** (1972), 296–305. MR 58 #1753 Zbl 0229.76018

[Kelliher 2004] J. P. Kelliher, "The inviscid limit for two-dimensional incompressible fluids with unbounded vorticity", *Math. Res. Lett.* **11**:4 (2004), 519–528. MR 2005g:76007 Zbl 1112.76007

[Masmoudi 2007] N. Masmoudi, "Remarks about the inviscid limit of the Navier–Stokes system", *Comm. Math. Phys.* **270**:3 (2007), 777–788. MR 2008c:35234 Zbl 1118.35030

[Sawada and Taniuchi 2007] O. Sawada and Y. Taniuchi, "A remark on L^∞ solutions to the 2-D Navier–Stokes equations", *J. Math. Fluid Mech.* **9**:4 (2007), 533–542. MR 2009f:35255 Zbl 1132.35437

[Serfati 1995] P. Serfati, "Solutions C^∞ en temps, n-log Lipschitz bornées en espace et équation d'Euler", *C. R. Acad. Sci. Paris Sér. I Math.* **320**:5 (1995), 555–558. MR 96c:35147 Zbl 0835.76012

[Swann 1971] H. S. G. Swann, "The convergence with vanishing viscosity of nonstationary Navier–Stokes flow to ideal flow in R_3", *Trans. Amer. Math. Soc.* **157** (1971), 373–397. MR 43 #3662 Zbl 0218.76023

[Taniuchi et al. 2010] Y. Taniuchi, T. Tashiro, and T. Yoneda, "On the two-dimensional Euler equations with spatially almost periodic initial data", *J. Math. Fluid Mech.* **12**:4 (2010), 594–612. MR 2012c:35337 Zbl 1270.35357

[Vishik 1999] M. Vishik, "Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type", *Ann. Sci. École Norm. Sup.* (4) **32**:6 (1999), 769–812. MR 2000i:76008 Zbl 0938.35128

Received May 8, 2013. Revised October 31, 2013.

ELAINE COZZI
DEPARTMENT OF MATHEMATICS
OREGON STATE UNIVERSITY
368 KIDDER HALL
CORVALLIS, OR 97331
UNITED STATES
cozzie@math.oregonstate.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.351

AFFINE QUANTUM SCHUR ALGEBRAS AND AFFINE HECKE ALGEBRAS

QIANG FU

Let F be the Schur functor from the category of finite-dimensional $\mathscr{H}_\triangle(r)_\mathbb{C}$-modules to that of finite-dimensional $\mathscr{S}_\triangle(n,r)_\mathbb{C}$-modules, where $\mathscr{H}_\triangle(r)_\mathbb{C}$ is the extended affine Hecke algebra of type A over $\mathbb{C}$ and $\mathscr{S}_\triangle(n,r)_\mathbb{C}$ is the affine quantum Schur algebras over $\mathbb{C}$. The Drinfeld polynomials associated with $\mathsf{F}(V)$, where V is an irreducible $\mathscr{H}_\triangle(r)_\mathbb{C}$-module, have been previously determined when $n > r$. Here we generalize these results to the case $n \leqslant r$. As an application, we recover the classification of finite-dimensional irreducible $\mathscr{S}_\triangle(n,r)_\mathbb{C}$-modules proved by Deng, Du and Fu using a different method. As another application, we generalize a result of Green to the affine case.

1. Introduction

Finite-dimensional irreducible modules for quantum affine algebras were classified by Chari and Pressley [1991; 1994; 1995; 1997] in terms of Drinfeld polynomials. Finite-dimensional irreducible modules for $\mathscr{H}_\triangle(r)_\mathbb{C}$ were classified in [Zelevinsky 1980; Rogawski 1985], where $\mathscr{H}_\triangle(r)_\mathbb{C}$ is the extended affine Hecke algebra of type A over the complex field $\mathbb{C}$ with a non-root of unity. The category of finite-dimensional $\mathscr{H}_\triangle(r)_\mathbb{C}$-modules and the category of finite-dimensional $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$-modules which are of level r are related by a functor $\mathscr{F}$ defined in [Chari and Pressley 1996, §4.2]. Here $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$ is quantum affine $\mathfrak{sl}_n$ over $\mathbb{C}$. Chari and Pressley [loc. cit.] proved that $\mathscr{F}$ is an equivalence of categories if $n > r$. Furthermore the Drinfeld polynomials associated with $\mathscr{F}(V)$ were determined in [loc. cit., §7.6] in the case of $n > r$, where V is an irreducible $\mathscr{H}_\triangle(r)_\mathbb{C}$-module.

Let $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ be quantum affine $\mathfrak{gl}_n$ over $\mathbb{C}$. In [Frenkel and Mukhin 2002], finite-dimensional irreducible polynomial representations of $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ were classified. It was proved in [Deng, Du and Fu 2012, Theorem 3.8.1] that the natural algebra homomorphism ζ_r from $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ to the affine quantum Schur algebra $\mathscr{S}_\triangle(n,r)_\mathbb{C}$ is

Supported by the National Natural Science Foundation of China, the Program NCET, Fok Ying Tung Education Foundation and the Fundamental Research Funds for the Central Universities.
MSC2010: 17B37, 20C08, 20G43.
Keywords: affine quantum Schur algebras, affine Hecke algebras, Schur functor.

surjective. Every $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-module can be regarded as a $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$-module via ζ_r. Let F be the Schur functor from the category of finite-dimensional $\mathcal{H}_\triangle(r)_\mathbb{C}$-modules to the category of finite-dimensional $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules. It was proved in [Deng, Du and Fu 2012, Theorem 4.1.3 and Proposition 4.2.1] that F is an equivalence of categories in the case of $n \geqslant r$ and that $\mathsf{F}(V)|_{\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)}$ is isomorphic to $\mathcal{F}(V)$ for any $\mathcal{H}_\triangle(r)_\mathbb{C}$-module V. Furthermore, using [Chari and Pressley 1996, §7.6], the Drinfeld polynomials associated with $\mathsf{F}(V)$ were determined in [Deng, Du and Fu 2012, Theorem 4.4.2] in the case of $n > r$, where V is an irreducible $\mathcal{H}_\triangle(r)_\mathbb{C}$-module. We will generalize these results to the case of $n \leqslant r$ in Theorem 4.9. Using this result, we will prove in Corollary 4.10 the classification theorem of finite-dimensional irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules, which was established in [Deng, Du and Fu 2012, Theorem 4.6.8]. Finally, we will relate the parametrization of irreducible $\mathcal{S}_\triangle(N,r)_\mathbb{C}$-modules, via the functor G defined in (4.10.1), to the parametrization of irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules in Theorem 4.11. This result is the affine version of [Green 2007, (6.5f)].

2. Quantum affine $\mathfrak{gl}_n$

Let $v \in \mathbb{C}^*$ be a complex number which is not a root of unity, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Let $(c_{i,j})$ be the Cartan matrix of affine type A_{n-1}. We recall the Drinfeld's new realization of quantum affine $\mathfrak{gl}_n$ as follows.

Definition 2.1. The *quantum loop algebra* $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ (or *quantum affine* $\mathfrak{gl}_n$) is the $\mathbb{C}$-algebra generated by $\mathrm{x}^\pm_{i,s}$ $(1 \leqslant i < n,\ s \in \mathbb{Z})$, $\mathrm{k}_i^{\pm 1}$, and $\mathrm{g}_{i,t}$ $(1 \leqslant i \leqslant n,\ t \in \mathbb{Z}\setminus\{0\})$ with the following relations:

(QLA1) $\mathrm{k}_i\mathrm{k}_i^{-1} = 1 = \mathrm{k}_i^{-1}\mathrm{k}_i,\ [\mathrm{k}_i, \mathrm{k}_j] = 0,$

(QLA2) $\mathrm{k}_i\mathrm{x}^\pm_{j,s} = v^{\pm(\delta_{i,j}-\delta_{i,j+1})}\mathrm{x}^\pm_{j,s}\mathrm{k}_i,\ [\mathrm{k}_i, \mathrm{g}_{j,s}] = 0,$

(QLA3) $[\mathrm{g}_{i,s}, \mathrm{x}^\pm_{j,t}] = \begin{cases} 0 & \text{if } i \neq j,\ j+1, \\ \pm v^{-js}([s]/s)\mathrm{x}^\pm_{j,s+t} & \text{if } i = j, \\ \mp v^{-js}([s]/s)\mathrm{x}^\pm_{j,s+t} & \text{if } i = j+1, \end{cases}$

(QLA4) $[\mathrm{g}_{i,s}, \mathrm{g}_{j,t}] = 0,$

(QLA5) $[\mathrm{x}^+_{i,s}, \mathrm{x}^-_{j,t}] = \delta_{i,j}(\phi^+_{i,s+t} - \phi^-_{i,s+t})/(v - v^{-1}),$

(QLA6) $\mathrm{x}^\pm_{i,s}\mathrm{x}^\pm_{j,t} = \mathrm{x}^\pm_{j,t}\mathrm{x}^\pm_{i,s}$ for $|i-j|>1$, and $[\mathrm{x}^\pm_{i,s+1}, \mathrm{x}^\pm_{j,t}]_{v^{\pm c_{ij}}} = -[\mathrm{x}^\pm_{j,t+1}, \mathrm{x}^\pm_{i,s}]_{v^{\pm c_{ij}}},$

(QLA7) $[\mathrm{x}^\pm_{i,s}, [\mathrm{x}^\pm_{j,t}, \mathrm{x}^\pm_{i,p}]_v]_v = -[\mathrm{x}^\pm_{i,p}, [\mathrm{x}^\pm_{j,t}, \mathrm{x}^\pm_{i,s}]_v]_v$ for $|i-j| = 1,$

where $[x,y]_a = xy - ayx$, $[s] = (v^s - v^{-s})/(v - v^{-1})$, and the $\phi^\pm_{i,s}$ are defined via generating functions in the indeterminate u by

$$\Phi^\pm_i(u) := \widetilde{\mathrm{k}}_i^{\pm 1}\exp\Big(\pm(v - v^{-1})\sum_{m\geqslant 1}\mathrm{h}_{i,\pm m}u^{\pm m}\Big) = \sum_{s\geqslant 0}\phi^\pm_{i,\pm s}u^{\pm s}$$

with $\widetilde{\mathtt{k}}_i = \mathtt{k}_i/\mathtt{k}_{i+1}$ ($\mathtt{k}_{n+1} = \mathtt{k}_1$) and $\mathtt{h}_{i,\pm m} = v^{\pm(i-1)m}\mathtt{g}_{i,\pm m} - v^{\pm(i+1)m}\mathtt{g}_{i+1,\pm m}$ ($1 \leqslant i < n$).

The algebra $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ has another presentation which we now describe. Let $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ be the double Ringel–Hall algebra of the cyclic quiver $\triangle(n)$. By [Deng, Du and Fu 2012, Theorem 2.3.1], the algebra $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ has the following presentation.

Lemma 2.2. *The double Ringel–Hall algebra $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ of the cyclic quiver $\triangle(n)$ is the $\mathbb{C}$-algebra generated by E_i, F_i, K_i, K_i^{-1}, z_s^+, z_s^-, for $1 \leqslant i \leqslant n$, $s \in \mathbb{Z}^+$, and relations*:

(QGL1) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1,$

(QGL2) $K_iE_j = v^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i,\ K_iF_j = v^{-\delta_{i,j}+\delta_{i,j+1}}F_jK_i,$

(QGL3) $E_iF_j - F_jE_i = \delta_{i,j}(\widetilde{K}_i - \widetilde{K}_i^{-1})/(v - v^{-1})$, *where* $\widetilde{K}_i = K_iK_{i+1}^{-1}$,

(QGL4) $$\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1-c_{i,j} \\ a \end{bmatrix} E_i^aE_jE_i^b = 0 \text{ for } i \neq j,$$

(QGL5) $$\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1-c_{i,j} \\ a \end{bmatrix} F_i^aF_jF_i^b = 0 \text{ for } i \neq j,$$

(QGL6) $\mathrm{z}_s^+\mathrm{z}_t^+ = \mathrm{z}_t^+\mathrm{z}_s^+,\ \mathrm{z}_s^-\mathrm{z}_t^- = \mathrm{z}_t^-\mathrm{z}_s^-,\ \mathrm{z}_s^+\mathrm{z}_t^- = \mathrm{z}_t^-\mathrm{z}_s^+,$

(QGL7) $K_i\mathrm{z}_s^+ = \mathrm{z}_s^+K_i,\ K_i\mathrm{z}_s^- = \mathrm{z}_s^-K_i,$

(QGL8) $E_i\mathrm{z}_s^+ = \mathrm{z}_s^+E_i,\ E_i\mathrm{z}_s^- = \mathrm{z}_s^-E_i,\ F_i\mathrm{z}_s^- = \mathrm{z}_s^-F_i$, *and* $\mathrm{z}_s^+F_i = F_i\mathrm{z}_s^+$,

where $1 \leqslant i, j \leqslant n,\ s, t \in \mathbb{Z}^+$, *and*

$$\begin{bmatrix} c \\ a \end{bmatrix} = \prod_{s=1}^{a} \frac{v^{c-s+1} - v^{-c+s-1}}{v^s - v^{-s}} \quad \text{for } c \in \mathbb{Z}.$$

It is a Hopf algebra with comultiplication Δ, counit ε, and antipode σ defined by

$$\Delta(E_i) = E_i \otimes \widetilde{K}_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + \widetilde{K}_i^{-1} \otimes F_i,$$
$$\Delta(K_i^{\pm1}) = K_i^{\pm1} \otimes K_i^{\pm1}, \quad \Delta(\mathrm{z}_s^\pm) = \mathrm{z}_s^\pm \otimes 1 + 1 \otimes \mathrm{z}_s^\pm,$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0 = \varepsilon(\mathrm{z}_s^\pm), \quad \varepsilon(K_i) = 1,$$
$$\sigma(E_i) = -E_i\widetilde{K}_i^{-1}, \quad \sigma(F_i) = -\widetilde{K}_iF_i, \quad \sigma(K_i^{\pm1}) = K_i^{\mp1}, \quad \sigma(\mathrm{z}_s^\pm) = -\mathrm{z}_s^\pm,$$

where $1 \leqslant i \leqslant n$ and $s \in \mathbb{Z}^+$.

Let $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$ be the subalgebra of $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ generated by E_i, F_i, $\widetilde{K}_i$, $\widetilde{K}_i^{-1}$ for $i \in [1, n]$. Beck [1994] proved that $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$ is isomorphic to the subalgebra of $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ generated by all $\mathtt{x}_{i,s}^\pm$, $\widetilde{\mathtt{k}}_i^{\pm1}$, and $\mathtt{h}_{i,t}$. The following result extends Beck's isomorphism.

Lemma 2.3 [Deng, Du and Fu 2012, Proposition 4.4.1]. *There is a Hopf algebra isomorphism*

$$f : \mathfrak{D}_{\vartriangle,\mathbb{C}}(n) \to \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$$

such that

$$K_i^{\pm 1} \mapsto \mathtt{k}_i^{\pm 1}, \quad E_j \mapsto \mathtt{x}_{j,0}^+, \quad F_j \mapsto \mathtt{x}_{j,0}^- \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j < n),$$
$$E_n \mapsto v\mathscr{X}\widetilde{\mathtt{k}}_n, \quad F_n \mapsto v^{-1}\widetilde{\mathtt{k}}_n^{-1}\mathscr{Y}, \quad \mathtt{z}_s^{\pm} \mapsto \mp s v^{\pm s}\theta_{\pm s}\ (s \geqslant 1),$$

where

$$\theta_{\pm s} = \mp\frac{1}{[s]}(\mathtt{g}_{1,\pm s} + \cdots + \mathtt{g}_{n,\pm s}),$$
$$\mathscr{X} = [\mathtt{x}_{n-1,0}^-, [\mathtt{x}_{n-2,0}^-, \ldots, [\mathtt{x}_{2,0}^-, \mathtt{x}_{1,1}^-]_{v^{-1}} \cdots]_{v^{-1}}]_{v^{-1}},$$
$$\mathscr{Y} = [\cdots[[\mathtt{x}_{1,-1}^+, \mathtt{x}_{2,0}^+]_v, \mathtt{x}_{3,0}^+]_v, \ldots, \mathtt{x}_{n-1,0}^+]_v.$$

We now review the classification theorem of finite-dimensional irreducible polynomial $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-modules. We first need to introduce the elements $\mathscr{Q}_{i,s} \in \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$, which will be used to define pseudo-highest weight modules. For $1 \leqslant i \leqslant n$ and $s \in \mathbb{Z}$, define the elements $\mathscr{Q}_{i,s} \in \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ through the generating functions

$$\mathscr{Q}_i^{\pm}(u) := \exp\left(-\sum_{t\geqslant 1}\frac{1}{[t]}g_{i,\pm t}(vu)^{\pm t}\right) = \sum_{s\geqslant 0}\mathscr{Q}_{i,\pm s}u^{\pm s} \in \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)[[u,u^{-1}]].$$

For a representation V of $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$, a nonzero vector $w \in V$ is called a *pseudo-highest weight vector* if there exists some $Q_{i,s} \in \mathbb{C}$ such that

$$\mathtt{x}_{j,s}^+ w = 0, \quad \mathscr{Q}_{i,s}w = Q_{i,s}w, \quad \mathtt{k}_i w = v^{\lambda_i}w \tag{2.3.1}$$

for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$ and $s \in \mathbb{Z}$. The module V is called a *pseudo-highest weight module* if $V = \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)w$ for some pseudo-highest weight vector w. We also write the short form $\mathscr{Q}_i^{\pm}(u)w = Q_i^{\pm}(u)w$ for the relations $\mathscr{Q}_{i,s}w = Q_{i,s}w$ $(s \in \mathbb{Z})$, where

$$Q_i^{\pm}(u) = \sum_{s\geqslant 0} Q_{i,\pm s}u^{\pm s}.$$

Let V be a finite-dimensional polynomial representation of $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ of type 1. Then $V = \oplus_{\lambda\in\mathbb{N}^n} V_\lambda$, where

$$V_\lambda = \{x \in V \mid \mathtt{k}_j x = v^{\lambda_j}x,\ 1 \leqslant j \leqslant n\},$$

and, since all $\mathscr{Q}_{i,s}$ commute with the $\mathtt{k}_j$, each V_λ is a direct sum of generalized eigenspaces of the form

$$V_{\lambda,\gamma} = \{x \in V_\lambda \mid (\mathscr{Q}_{i,s} - \gamma_{i,s})^p x = 0 \quad \text{for some } p\ (1 \leqslant i \leqslant n,\ s \in \mathbb{Z})\}, \tag{2.3.2}$$

where $\gamma = (\gamma_{i,s})$ with $\gamma_{i,s} \in \mathbb{C}$. Let $\Gamma_i^{\pm}(u) = \sum_{s\geqslant 0}\gamma_{i,\pm s}u^{\pm s}$.

A finite-dimensional $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module V is called a *polynomial representation* if the restriction of V to $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$ is a polynomial representation of type 1 and, for every weight $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ of V, the formal power series $\Gamma_i^{\pm}(u)$ associated to the eigenvalues $(\gamma_{i,s})_{s\in\mathbb{Z}}$ defining the generalized eigenspaces $V_{\lambda,\gamma}$ as given in (2.3.2), are polynomials in $u^{\pm}$ of degree λ_i so that the zeroes of the functions $\Gamma_i^+(u)$ and $\Gamma_i^-(u)$ are the same.

Following [Frenkel and Mukhin 2002], an n-tuple of polynomials

$$\boldsymbol{Q} = (Q_1(u), \dots, Q_n(u))$$

with constant terms 1 is called *dominant* if, for each $1 \leqslant i \leqslant n-1$, the ratio $Q_i(v^{i-1}u)/Q_{i+1}(v^{i+1}u)$ is a polynomial. Let $\mathcal{Q}(n)$ be the set of dominant n-tuples of polynomials.

For $g(u) = \prod_{1\leqslant i\leqslant m} (1 - a_i u) \in \mathbb{C}[u]$ with constant term 1 and $a_i \in \mathbb{C}^*$, define

$$g^{\pm}(u) = \prod_{1\leqslant i\leqslant m} (1 - a_i^{\pm 1} u^{\pm 1}). \tag{2.3.3}$$

For $\boldsymbol{Q} = (Q_1(u), \dots, Q_n(u)) \in \mathcal{Q}(n)$, define $Q_{i,s} \in \mathbb{C}$, for $1 \leqslant i \leqslant n$ and $s \in \mathbb{Z}$, by the formula

$$Q_i^{\pm}(u) = \sum_{s\geqslant 0} Q_{i,\pm s} u^{\pm s},$$

where $Q_i^{\pm}(u)$ is defined using (2.3.3). Let $I(\boldsymbol{Q})$ be the left ideal of $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ generated by $\mathtt{x}^+_{j,s}$, $\mathcal{Q}_{i,s} - Q_{i,s}$, and $\mathtt{k}_i - v^{\lambda_i}$, for $1 \leqslant j \leqslant n-1$, $1 \leqslant i \leqslant n$, and $s \in \mathbb{Z}$, where $\lambda_i = \deg Q_i(u)$, and define

$$M(\boldsymbol{Q}) = \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)/I(\boldsymbol{Q}).$$

Then $M(\boldsymbol{Q})$ has a unique irreducible quotient, denoted by $L(\boldsymbol{Q})$. The polynomials $Q_i(u)$ are called *Drinfeld polynomials* associated with $L(\boldsymbol{Q})$.

Theorem 2.4 [Frenkel and Mukhin 2002]. *The* $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$*-modules* $L(\boldsymbol{Q})$ *with* $\boldsymbol{Q} \in \mathcal{Q}(n)$ *are all nonisomorphic finite-dimensional irreducible polynomial representations of* $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$.

If $\boldsymbol{Q}$, $\boldsymbol{Q}' \in \mathcal{Q}(n)$ satisfies $Q_j(v^{j-1}u)/Q_{j+1}(v^{j+1}u) = Q'_j(v^{j-1}u)/Q'_{j+1}(v^{j+1}u)$ and $\deg Q_j(u) - \deg Q_{j+1}(u) = \deg Q'_j(u) - \deg Q'_{j+1}(u)$ for $1 \leqslant j \leqslant n-1$, then $L(\boldsymbol{Q})|_{\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)} \cong L(\boldsymbol{Q}')|_{\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)}$, by [Deng, Du and Fu 2012, Lemma 4.7.1, Corollary 4.7.2]. Thus we can denote $L(\boldsymbol{Q})|_{\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)}$ by $\bar{L}(\boldsymbol{P})$, where $\boldsymbol{P} = (P_1(u), \dots, P_{n-1}(u))$ with $P_j(u) = Q_j(v^{j-1}u)/Q_{j+1}(v^{j+1}u)$.

Let $\mathcal{P}(n)$ be the set of $(n-1)$-tuples of polynomials with constant term 1. The following result is due to Chari and Pressley [1991; 1994; 1995].

Theorem 2.5. *The modules* $\bar{L}(\boldsymbol{P})$ *with* $\boldsymbol{P} \in \mathcal{P}(n)$ *are all nonisomorphic finite-dimensional irreducible* $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$*-modules of type* 1.

3. Affine quantum Schur algebras

In this section we collect some facts about extended affine Hecke algebras and affine quantum Schur algebras, which will be used in Section 4. The extended affine Hecke algebra $\mathcal{H}_\triangle(r)_\mathbb{C}$ is defined to be the algebra generated by

$$T_i, \quad X_j^{\pm 1} \quad (1 \leqslant i \leqslant r-1,\ 1 \leqslant j \leqslant r),$$

and relations

$$\begin{gathered}(T_i+1)(T_i-v^2)=0,\\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, \quad T_iT_j=T_jT_i\ (|i-j|>1),\\ X_iX_i^{-1}=1=X_i^{-1}X_i, \quad X_iX_j=X_jX_i,\\ T_iX_iT_i=v^2X_{i+1}, \quad X_jT_i=T_iX_j\ (j\neq i,i+1).\end{gathered}$$

Let $\mathfrak{S}_r$ be the symmetric group with generators $s_i := (i, i+1)$ for $1 \leqslant i \leqslant r-1$. Let $I(n,r) = \{(i_1,\ldots,i_r) \in \mathbb{Z}^r \mid 1 \leqslant i_k \leqslant n,\ \forall k\}$. The symmetric group $\mathfrak{S}_r$ acts on the set $I(n,r)$ by place permutation:

$$\boldsymbol{i}w = (i_{w(1)},\ldots,i_{w(r)}), \quad \text{for } \boldsymbol{i} \in I(n,r) \text{ and } w \in \mathfrak{S}_r.$$

Let $\Omega_\mathbb{C}$ be a vector space over $\mathbb{C}$ with basis $\{\omega_i \mid i \in \mathbb{Z}\}$. For $\boldsymbol{i} = (i_1,\ldots,i_r) \in \mathbb{Z}^r$, write

$$\omega_{\boldsymbol{i}} = \omega_{i_1} \otimes \omega_{i_2} \otimes \cdots \otimes \omega_{i_r} = \omega_{i_1}\omega_{i_2}\cdots\omega_{i_r} \in \Omega_\mathbb{C}^{\otimes r}.$$

The tensor space $\Omega_\mathbb{C}^{\otimes r}$ admits a right $\mathcal{H}_\triangle(r)_\mathbb{C}$-module structure defined by

$$\begin{cases}\omega_{\boldsymbol{i}}\cdot X_t^{-1} = \omega_{i_1}\cdots\omega_{i_{t-1}}\omega_{i_t+n}\omega_{i_{t+1}}\cdots\omega_{i_r} & \text{for all } \boldsymbol{i} \in \mathbb{Z}^r,\\ \omega_{\boldsymbol{i}}\cdot T_k = \begin{cases} v^2\omega_{\boldsymbol{i}} & \text{if } i_k = i_{k+1},\\ v\omega_{\boldsymbol{i}s_k} & \text{if } i_k < i_{k+1},\\ v\omega_{\boldsymbol{i}s_k} + (v^2-1)\omega_{\boldsymbol{i}} & \text{if } i_{k+1} < i_k,\end{cases} & \text{for all } \boldsymbol{i} \in I(n,r),\end{cases}$$

where $1 \leqslant k \leqslant r-1$ and $1 \leqslant t \leqslant r$.

The algebra

$$\mathcal{S}_\triangle(n,r)_\mathbb{C} := \mathrm{End}_{\mathcal{H}_\triangle(r)_\mathbb{C}}(\mathcal{T}_\triangle(n,r))$$

is called an affine q-Schur algebra, where $\mathcal{T}_\triangle(n,r) = \Omega_\mathbb{C}^{\otimes r}$. Let $\Omega_{n,\mathbb{C}}$ be the subspace of $\Omega_\mathbb{C}$ spanned by ω_i with $1 \leqslant i \leqslant n$ and $\mathcal{H}(r)_\mathbb{C}$ be the subalgebra of $\mathcal{H}_\triangle(r)_\mathbb{C}$ generated by T_k for $1 \leqslant k \leqslant r-1$. Then the algebra $\mathcal{S}(n,r)_\mathbb{C} := \mathrm{End}_{\mathcal{H}(r)_\mathbb{C}}(\mathcal{T}(n,r))$ is called a q-Schur algebra, where $\mathcal{T}(n,r) = \Omega_{n,\mathbb{C}}^{\otimes r}$.

The algebras $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ and $\mathcal{S}_\triangle(n,r)_\mathbb{C}$ are related by an algebra homomorphism ζ_r, which we now describe. For $i \in \mathbb{Z}$, let $\bar{\imath}$ denotes the corresponding integer modulo n.

The complex vector space $\Omega_{\mathbb{C}}$ is a natural $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$-module with the action

$$E_i \cdot \omega_s = \delta_{\overline{i+1},\bar{s}}\omega_{s-1}, \quad F_i \cdot \omega_s = \delta_{\bar{i},\bar{s}}\omega_{s+1}, \quad K_i^{\pm 1} \cdot \omega_s = v^{\pm\delta_{\bar{i},\bar{s}}}\omega_s,$$
$$\mathsf{z}_t^{+} \cdot \omega_s = \omega_{s-tn}, \quad \mathsf{z}_t^{-} \cdot \omega_s = \omega_{s+tn}. \tag{3.0.1}$$

The Hopf algebra structure induces a $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$-module $\Omega_{\mathbb{C}}^{\otimes r}$. By [Deng, Du and Fu 2012, Proposition 3.5.5], the actions of $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ and $\mathscr{H}_{\triangle}(r)_{\mathbb{C}}$ on $\Omega_{\mathbb{C}}^{\otimes r}$ are commute. We will identify $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ and $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ via the algebra isomorphism f defined in Lemma 2.3. Consequently, there is an algebra homomorphism

$$\zeta_r : \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n) = \mathfrak{D}_{\triangle,\mathbb{C}}(n) \to \mathscr{S}_{\triangle}(n,r)_{\mathbb{C}}.$$

It is proved in [Deng, Du and Fu 2012, Theorem 3.8.1] that ζ_r is surjective. Let $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$ be the subalgebra of $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ generated by E_i, F_i, K_j, K_j^{-1} for $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n$. The restriction of ζ_r to $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$ induces a surjective algebra homomorphism $\zeta_r : \mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n) \to \mathscr{S}(n,r)_{\mathbb{C}}$ (see [Jimbo 1986]). Every $\mathscr{S}_{\triangle}(n,r)_{\mathbb{C}}$-module (resp., $\mathscr{S}(n,r)_{\mathbb{C}}$-module) will be inflated into a $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module (resp., $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$-module) via ζ_r.

The following easy lemma relates $\Omega_{\mathbb{C}}^{\otimes r}$ with $\Omega_{n,\mathbb{C}}^{\otimes r}$.

Lemma 3.1 [Deng, Du and Fu 2012, Lemma 4.1.1]. *There is a* $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$-$\mathscr{H}_{\triangle}(r)_{\mathbb{C}}$-*bimodule isomorphism*

$$\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} \mathscr{H}_{\triangle}(r)_{\mathbb{C}} \xrightarrow{\sim} \Omega_{\mathbb{C}}^{\otimes r}, \quad x \otimes h \mapsto xh.$$

The irreducible $\mathscr{H}_{\triangle}(r)_{\mathbb{C}}$-modules were classified in [Zelevinsky 1980; Rogawski 1985], which we now describe. For $\boldsymbol{a} = (a_1, \ldots, a_r) \in (\mathbb{C}^*)^r$, let $M_{\boldsymbol{a}} = \mathscr{H}_{\triangle}(r)_{\mathbb{C}}/J_{\boldsymbol{a}}$, where $J_{\boldsymbol{a}}$ is the left ideal of $\mathscr{H}_{\triangle}(r)_{\mathbb{C}}$ generated by $X_j - a_j$ for $1 \leqslant j \leqslant r$.

A *segment* s with center $a \in \mathbb{C}^*$ is by definition an ordered sequence

$$\mathsf{s} = (av^{-k+1}, av^{-k+3}, \ldots, av^{k-1}) \in (\mathbb{C}^*)^k.$$

Here k is called the length of the segment, denoted by $|\mathsf{s}|$. If $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\}$ is an unordered collection of segments, define $\wp(\boldsymbol{s})$ to be the partition associated with the sequence $(|\mathsf{s}_1|, \ldots, |\mathsf{s}_p|)$. That is, $\wp(\boldsymbol{s}) = (|\mathsf{s}_{i_1}|, \ldots, |\mathsf{s}_{i_p}|)$ with $|\mathsf{s}_{i_1}| \geqslant \cdots \geqslant |\mathsf{s}_{i_p}|$, where $|\mathsf{s}_{i_1}|, \ldots, |\mathsf{s}_{i_p}|$ is a permutation of $|\mathsf{s}_1|, \ldots, |\mathsf{s}_p|$. We also call $|\boldsymbol{s}| := |\wp(\boldsymbol{s})|$ the length of $\boldsymbol{s}$.

Let $\mathscr{S}_r$ be the set of unordered collections of segments $\boldsymbol{s}$ with $|\boldsymbol{s}| = r$. Then $\mathscr{S}_r = \bigcup_{\mu \in \Lambda^+(r)} \mathscr{S}_{r,\mu}$, where $\mathscr{S}_{r,\mu} = \{\boldsymbol{s} \in \mathscr{S}_r \mid \wp(\boldsymbol{s}) = \mu\}$ and $\Lambda^+(r)$ is the set of partitions of r.

If $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ is reduced let $T_w = T_{i_1}T_{i_2}\cdots T_{i_m}$. For $p \geqslant 1$ let

$$\Lambda(p,r) = \Big\{\mu \in \mathbb{N}^p \Bigm| \sum_{1 \leqslant i \leqslant p} \mu_i = r\Big\} \tag{3.1.1}$$

For $\mu \in \Lambda(p, r)$ let $\mathfrak{S}_\mu$ be the corresponding standard Young subgroup of the symmetric group $\mathfrak{S}_r$, and let $\mathscr{D}_\mu = \{d \in \mathfrak{S}_r \mid \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_\mu\}$. For $\mu \in \Lambda(p, r)$ let

$$\mathscr{I}_\mu = \mathscr{H}(r)_{\mathbb{C}} y_\mu, \tag{3.1.2}$$

where

$$y_\mu = \sum_{w \in \mathfrak{S}_\mu} (-v^2)^{-\ell(w)} T_w \in \mathscr{H}(r)_{\mathbb{C}}.$$

For $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\} \in \mathscr{S}_{r,\mu}$, let $\boldsymbol{a}(\boldsymbol{s}) = (\mathsf{s}_1, \ldots, \mathsf{s}_p) \in (\mathbb{C}^*)^r$ be the r-tuple obtained by juxtaposing the segments in $\boldsymbol{s}$. Let $\iota : \mathscr{H}(r)_{\mathbb{C}} \to M_{\boldsymbol{a}(\boldsymbol{s})}$ be the natural $\mathscr{H}(r)_{\mathbb{C}}$-module isomorphism defined by sending h to $\bar{h}$. Let

$$\bar{\mathscr{I}}_\mu = \iota(\mathscr{I}_\mu) = \mathscr{H}(r)_{\mathbb{C}} \bar{y}_\mu = \mathscr{H}_\triangle(r)_{\mathbb{C}} \bar{y}_\mu.$$

Then,

$$\mathscr{H}(r)_{\mathbb{C}} y_\mu \cong E_\mu \oplus \Bigg(\bigoplus_{\substack{\nu \vdash r \\ \nu \rhd \lambda}} m_{\nu,\mu} E_\nu \Bigg), \tag{3.1.3}$$

where E_ν is the left cell module defined by the Kazhdan–Lusztig's C-basis [1979] associated with the left cell containing $w_{0,\nu}$.

Let $V_{\boldsymbol{s}}$ be the unique composition factor of the $\mathscr{H}_\triangle(r)_{\mathbb{C}}$-module $\mathscr{H}_\triangle(r)_{\mathbb{C}} \bar{y}_\mu$ such that the multiplicity of E_μ in $V_{\boldsymbol{s}}$ as an $\mathscr{H}(r)_{\mathbb{C}}$-module is nonzero.

The following classification theorem is due to [Zelevinsky 1980; Rogawski 1985].

Theorem 3.2. *The modules $V_{\boldsymbol{s}}$ with $\boldsymbol{s} \in \mathscr{S}_r$ are all nonisomorphic finite-dimensional irreducible $\mathscr{H}_\triangle(r)_{\mathbb{C}}$-modules.*

Let $\mathscr{S}_\triangle(n, r)_{\mathbb{C}}$-$\mathsf{mod}$ (resp., $\mathscr{H}_\triangle(r)_{\mathbb{C}}$-$\mathsf{mod}$) be the category of finite-dimensional $\mathscr{S}_\triangle(n, r)_{\mathbb{C}}$-modules (resp., $\mathscr{H}_\triangle(r)_{\mathbb{C}}$-modules). The categories $\mathscr{S}_\triangle(n, r)_{\mathbb{C}}$-$\mathsf{mod}$ and $\mathscr{H}_\triangle(r)_{\mathbb{C}}$-$\mathsf{mod}$ are related by the Schur functor F, which we now define. Using the $\mathscr{S}_\triangle(n, r)_{\mathbb{C}}$-$\mathscr{H}_\triangle(r)_{\mathbb{C}}$-bimodule $\Omega_{\mathbb{C}}^{\otimes r}$, we define a functor

$$\mathsf{F} = \mathsf{F}_{n,r} : \mathscr{H}_\triangle(r)_{\mathbb{C}}\text{-}\mathsf{mod} \to \mathscr{S}_\triangle(n, r)_{\mathbb{C}}\text{-}\mathsf{mod}, \quad V \mapsto \Omega_{\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}_\triangle(r)_{\mathbb{C}}} V. \tag{3.2.1}$$

Let

$$\mathscr{S}_r^{(n)} = \{\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\} \in \mathscr{S}_r,\ p \geqslant 1,\ |\mathsf{s}_i| \leqslant n,\ \forall i\}.$$

The following classification theorem is given in [Deng, Du and Fu 2012, Theorems 4.3.4 and 4.5.3].

Lemma 3.3. *For $\boldsymbol{s} \in \mathscr{S}_r$ we have $\mathsf{F}(V_{\boldsymbol{s}}) \neq 0$ if and only if $\boldsymbol{s} \in \mathscr{S}_r^{(n)}$. Furthermore, the set*

$$\{\mathsf{F}(V_{\boldsymbol{s}}) \mid \boldsymbol{s} \in \mathscr{S}_r^{(n)}\}$$

is a complete set of nonisomorphic finite-dimensional irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules.

The following result, which will be used in Theorem 4.9, is taken from [Chari and Pressley 1996, §7.6; Deng, Du and Fu 2012, Theorem 4.4.2 and Lemma 4.6.5].

Lemma 3.4. *Assume $n \geqslant r$. Let $\boldsymbol{s} = (av^{-r+1}, av^{-r+3}, \dots, av^{r-1})$ be a single segment and $\mu = \wp(\boldsymbol{s}) = (r)$. Then $V_{\boldsymbol{s}} = \bar{\mathcal{I}}_\mu$ and $\mathsf{F}(V_{\boldsymbol{s}}) \cong L(\boldsymbol{Q})$, where $\boldsymbol{Q} = (Q_1(u), \dots, Q_n(u))$ with*

$$Q_n(u) = (1 - av^{-n+1}u)^{\delta_{n,r}},$$
$$\frac{Q_i(uv^{i-1})}{Q_{i+1}(uv^{i+1})} = (1-au)^{\delta_{i,r}} \quad \textit{for } 1 \leqslant i \leqslant n-1.$$

4. Identification of irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules

In this section we will prove that $\mathsf{F}(\bar{\mathcal{I}}_{\wp(\boldsymbol{s})})$ is isomorphic to the tensor product of irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules for $\boldsymbol{s} \in \mathcal{S}_r^{(n)}$ and $\mathsf{F}(\bar{\mathcal{I}}_{\wp(\boldsymbol{s})}) = 0$ for $\boldsymbol{s} \notin \mathcal{S}_r^{(n)}$ in Proposition 4.6. Using this result, we will relate the parametrization of irreducible $\mathcal{H}_\triangle(r)_\mathbb{C}$-modules, via the functor F defined in (3.2.1), to the parametrization of finite-dimensional irreducible polynomial representations of $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ in Theorem 4.9. As applications, we will classify finite-dimensional irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules in Corollary 4.10, and generalize [Green 2007, (6.5f)] to the affine case.

To compute $\mathsf{F}(\bar{\mathcal{I}}_{\wp(\boldsymbol{s})})$, we need Proposition 4.3 of [Rogawski 1985], which we now describe. For $1 \leqslant j \leqslant p$, let $\mathcal{H}_{\mu,j}$ be the subalgebra of $\mathcal{H}(r)_\mathbb{C}$ generated by T_i with $s_i \in \mathfrak{S}_{\mu^{(j)}}$, where

$$\mu^{(j)} = (1^{\mu_{[1,j-1]}}, \mu_j, 1^{r-\mu_{[1,j]}}),$$

and $\mu_{[1,j]} = \mu_1 + \mu_2 + \cdots + \mu_j$. Since $\mathcal{H}_{\mu,j} \cong \mathcal{H}(\mu_j)_\mathbb{C}$ for $1 \leqslant j \leqslant p$ and $\Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ is a right $\mathcal{H}(\mu_j)_\mathbb{C}$-module, $\Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ can be also regarded as a right $\mathcal{H}_{\mu,j}$-module.

Recall the notation $\mathcal{I}_\mu$ defined in (3.1.2). For $\mu \in \Lambda(p,r)$ and $1 \leqslant j \leqslant p$ let

$$\mathcal{J}_\mu = \bigcap_{\substack{s_i \in \mathfrak{S}_\mu \\ 1 \leqslant i \leqslant r-1}} \mathcal{H}(r)_\mathbb{C} C_i, \quad \mathcal{J}_{\mu,j} = \bigcap_{\substack{s_i \in \mathfrak{S}_{\mu^{(j)}} \\ 1 \leqslant i \leqslant r-1}} \mathcal{H}_{\mu,j} C_i, \quad \text{and} \quad \mathcal{I}_{\mu,j} = \mathcal{H}_{\mu,j} y_{\mu^{(j)}}.$$

where $C_i = v^{-1}T_i - v$ and $y_{\mu^{(j)}} = \sum_{w \in \mathfrak{S}_{\mu^{(j)}}} (-v^2)^{-\ell(w)} T_w$. By Proposition 4.3 of [Rogawski 1985] we have:

Lemma 4.1. *We have $\mathcal{I}_\mu = \mathcal{J}_\mu$, $\mathcal{I}_{\mu,j} = \mathcal{J}_{\mu,j}$ for $\mu \in \Lambda(p,r)$ and $1 \leqslant j \leqslant p$.*

Lemma 4.2. *Assume I is a left ideal of $\mathcal{H}(r)_\mathbb{C}$. Then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} I \cong \Omega_{n,\mathbb{C}}^{\otimes r} I$.*

Proof. Since $\mathcal{H}(r)_{\mathbb{C}}$ is semisimple, there exist a left ideal J of $\mathcal{H}(r)_{\mathbb{C}}$ such that $\mathcal{H}(r)_{\mathbb{C}} = I \oplus J$. Then $\Omega_{n,\mathbb{C}}^{\otimes r} \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} \mathcal{H}(r)_{\mathbb{C}} \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \oplus \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} J$. Thus the natural linear map $f : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \to \Omega_{n,\mathbb{C}}^{\otimes r}$ defined by sending $w \otimes h$ to wh is injective. Consequently, $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \cong \mathrm{Im}(f) = \Omega_{n,\mathbb{C}}^{\otimes r} I$. □

By Lemmas 3.1, 4.1, and 4.2 we conclude that $\mathsf{F}(\bar{\mathcal{I}}_\mu) \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} \bar{\mathcal{I}}_\mu \cong \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu$, where $\mu = \wp(\boldsymbol{s})$ for some $\boldsymbol{s} \in \mathcal{S}_r$. We now compute $\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu$.

Lemma 4.3. *For $\mu \in \Lambda(p, r)$, we have*

$$\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu = \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{I}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{I}_{\mu,p}.$$

Proof. Since $\mathcal{I}_\mu = \bigcap_{1 \leqslant j \leqslant p} \mathcal{I}_{\mu^{(j)}}$ we have $\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu \subseteq \bigcap_{1 \leqslant j \leqslant p} \big(\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_{\mu^{(j)}}\big)$. Furthermore by Lemma 4.1 we have $\mathcal{I}_{\mu^{(j)}} = \mathcal{I}_{\mu^{(j)}} = \mathcal{X}_{\mu,j} \mathcal{I}_{\mu,j} = \mathcal{X}_{\mu,j} \mathcal{I}_{\mu,j}$ where $\mathcal{X}_{\mu,j} = \mathrm{span}\{T_w \mid w \in \mathcal{D}_{\mu^{(j)}}^{-1}\}$. This implies that

$$\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_{\mu^{(j)}} = \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_{\mu,j} = \Omega_{n,\mathbb{C}}^{\mu_1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\mu_{j-1}} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_j} \mathcal{I}_{\mu_j} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_{j+1}} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p}$$

for $1 \leqslant j \leqslant p$. Thus,

$$\begin{aligned}\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu &\subseteq \bigcap_{1 \leqslant j \leqslant p} \big(\Omega_{n,\mathbb{C}}^{\mu_1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\mu_{j-1}} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_j} \mathcal{I}_{\mu_j} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_{j+1}} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p}\big)\\ &= \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{I}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{I}_{\mu,p}.\end{aligned}$$

On the other hand, we assume $w_1 h_1 \otimes \cdots \otimes w_p h_p \in \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{I}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{I}_{\mu,p}$, where $w_j \in \Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ and $h_j \in \mathcal{I}_{\mu,j}$. Since $h_k h_l = h_l h_k$ for any k, l and $h_j \in \mathcal{I}_{\mu,j}$, we have $h_1 h_2 \cdots h_p = (h_1 \cdots h_{j-1} h_{j+1} \cdots h_p) h_j \in \mathcal{H}(r)_{\mathbb{C}} \mathcal{I}_{\mu,j} \subseteq \mathcal{H}(r)_{\mathbb{C}} C_i$ for $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant p$ with $s_i \in \mathfrak{S}_{\mu^{(j)}}$. This implies that $h_1 h_2 \cdots h_p \in \mathcal{I}_\mu$. It follows that $w_1 h_1 \otimes \cdots \otimes w_p h_p = (w_1 \otimes \cdots \otimes w_p) h_1 \cdots h_p \in \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu$. The assertion follows. □

For $\mu \in \Lambda(p, r)$ and $1 \leqslant j \leqslant p$, let $\widetilde{\mathcal{H}}_{\mu,j}$ be the subalgebra of $\mathcal{H}_\Delta(r)_{\mathbb{C}}$ generated by T_i and $X_{\mu_{[1,j-1]}+1}, \ldots, X_{\mu_{[1,j]}}$ with $s_i \in \mathfrak{S}_{\mu^{(j)}}$. Since $\widetilde{\mathcal{H}}_{\mu,j} \cong \mathcal{H}_\Delta(\mu_j)_{\mathbb{C}}$ and $\Omega_{\mathbb{C}}^{\otimes \mu_j}$ is a right $\mathcal{H}_\Delta(\mu_j)_{\mathbb{C}}$-module, one can regard $\Omega_{\mathbb{C}}^{\otimes \mu_j}$ as a right $\widetilde{\mathcal{H}}_{\mu,j}$-module.

For $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\} \in \mathcal{S}_{r,\mu}$, let $\boldsymbol{a} = (\mathsf{s}_1, \ldots, \mathsf{s}_p) \in (\mathbb{C}^*)^r$ be the r-tuple obtained by juxtaposing the segments in $\boldsymbol{s}$. For $1 \leqslant j \leqslant p$ let $\mathfrak{I}_{\mu,j}$ be the left ideal of $\widetilde{\mathcal{H}}_{\mu,j}$ generated by $X_k - a_k$ for $\mu_{[1,j-1]} + 1 \leqslant k \leqslant \mu_{[1,j]}$. Let $\iota_j : \mathcal{H}_{\mu,j} \to \widetilde{\mathcal{H}}_{\mu,j}/\mathfrak{I}_{\mu,j}$ be the natural $\mathcal{H}_{\mu,j}$-module isomorphism defined by sending h to $\bar{h}$. Let

$$\bar{\mathcal{I}}_{\mu,j} = \iota_j(\mathcal{I}_{\mu,j}) = \mathcal{H}_{\mu,j} \bar{y}_{\mu^{(j)}} = \widetilde{\mathcal{H}}_{\mu,j} \bar{y}_{\mu^{(j)}}.$$

By Lemma 4.3 we have the following corollary.

Corollary 4.4. *Maintain the notation above. There is a $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$-module isomorphism*

$$\varphi : (\Omega_{\mathbb{C}}^{\otimes \mu_1} \otimes_{\widetilde{\mathcal{H}}_{\mu,1}} \bar{\mathcal{I}}_{\mu,1}) \otimes \cdots \otimes (\Omega_{\mathbb{C}}^{\otimes \mu_p} \otimes_{\widetilde{\mathcal{H}}_{\mu,p}} \bar{\mathcal{I}}_{\mu,p}) \to \mathsf{F}(\bar{\mathcal{I}}_\mu)$$

such that $\varphi(w_1\otimes\bar{h}_1\otimes\cdots\otimes w_p\otimes\bar{h}_p)=w_1\otimes\cdots\otimes w_p\otimes\overline{h_1\cdots h_p}$ *for* $w_j\in\Omega_{n,\mathbb{C}}^{\otimes\mu_j}$ *and* $h_j\in\mathscr{I}_{\mu,j}$ *with* $1\leqslant j\leqslant p$.

Proof. Combining Lemmas 3.1, 4.1 with 4.2 yields $\mathsf{F}(\bar{\mathscr{I}}_\mu)\cong\Omega_{n,\mathbb{C}}^{\otimes r}\otimes_{\mathscr{H}(r)_\mathbb{C}}\bar{\mathscr{I}}_\mu\cong\Omega_{n,\mathbb{C}}^{\otimes r}\mathscr{I}_\mu$ and $\Omega_{\mathbb{C}}^{\otimes\mu_j}\otimes_{\widetilde{\mathscr{H}}_{\mu,j}}\bar{\mathscr{I}}_{\mu,j}\cong\Omega_{n,\mathbb{C}}^{\otimes\mu_j}\otimes_{\mathscr{H}_{\mu,j}}\bar{\mathscr{I}}_{\mu,j}\cong\Omega_{n,\mathbb{C}}^{\otimes\mu_j}\mathscr{I}_{\mu,j}$ for $1\leqslant j\leqslant p$. This, together with Lemma 4.3, implies the assertion. $\square$

We now prove that φ is in fact a $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$-module isomorphism.

Lemma 4.5. *The map* φ *is a* $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$*-module homomorphism.*

Proof. Let $u\in\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ and $w=w_1\otimes\bar{h}_1\otimes\cdots\otimes w_p\otimes\bar{h}_p\in(\Omega_{\mathbb{C}}^{\otimes\mu_1}\otimes_{\widetilde{\mathscr{H}}_{\mu,1}}\bar{\mathscr{I}}_{\mu,1})\otimes\cdots\otimes(\Omega_{\mathbb{C}}^{\otimes\mu_p}\otimes_{\widetilde{\mathscr{H}}_{\mu,p}}\bar{\mathscr{I}}_{\mu,p})$, where $w_i\in\Omega_{n,\mathbb{C}}^{\otimes\mu_i}$ and $h_i\in\mathscr{I}_{\mu,i}$ for $1\leqslant i\leqslant p$. Assume $\Delta^{(p-1)}(u)=\sum_{(u)}u_1\otimes\cdots\otimes u_p$, $u_iw_i=\sum_{k_i}w_{i,k_i}g_{i,k_i}$ and $g_{i,k_i}h_i=\sum_{j_i}g_{i,k_i,j_i}X_{j_i}$, where $w_{i,k_i}\in\Omega_{n,\mathbb{C}}^{\otimes\mu_i}$, $g_{i,k_i}\in\widetilde{\mathscr{H}}_{\mu,i}$, and $g_{i,k_i,j_i}\in\mathscr{H}_{\mu,i}$, $X_{j_i}\in\widetilde{\mathscr{H}}_{\mu,i}$. Then

$$g_{i,k_i}(\iota_i(h_i))=g_{i,k_i}\overline{h_i}=\sum_{j_i}a_{j_i}\overline{g_{i,k_i,j_i}}.$$

Hence,

$$\begin{aligned}
uw&=\sum_{(u)}u_1w_1\otimes\bar{h}_1\otimes\cdots\otimes u_pw_p\otimes\bar{h}_p\\
&=\sum_{(u)}\sum_{k_1,\dots,k_p}w_{1,k_1}\otimes g_{1,k_1}\bar{h}_1\otimes\cdots\otimes w_{p,k_p}\otimes g_{p,k_p}\bar{h}_p\\
&=\sum_{(u)}\sum_{\substack{k_1,\dots,k_p\\ j_1,\dots,j_p}}a_{j_1}\cdots a_{j_p}w_{1,k_1}\otimes\overline{g_{1,k_1,j_1}}\otimes\cdots\otimes w_{p,k_p}\otimes\overline{g_{p,k_p,j_p}}.
\end{aligned}$$

Since

$$g_{1,k_1}\cdots g_{p,k_p}\overline{h_1\cdots h_p}=\overline{g_{1,k_1}h_1\cdots g_{p,k_p}h_p}=\sum_{j_1,\dots,j_p}a_{j_1}\cdots a_{j_p}\overline{g_{1,k_1,j_1}\cdots g_{p,k_p,j_p}},$$

we conclude that

$$\begin{aligned}
\varphi(uw)&=\sum_{(u)}\sum_{\substack{k_1,\dots,k_p\\ j_1,\dots,j_p}}a_{j_1}\cdots a_{j_p}w_{1,k_1}\otimes\cdots\otimes w_{p,k_p}\otimes\overline{g_{1,k_1,j_1}\cdots g_{p,k_p,j_p}}\\
&=\sum_{(u)}\sum_{k_1,\dots,k_p}w_{1,k_1}\otimes\cdots\otimes w_{p,k_p}\otimes g_{1,k_1}\cdots g_{p,k_p}\overline{h_1\cdots h_p}\\
&=\sum_{(u)}u_1w_1\otimes\cdots\otimes u_pw_p\otimes\overline{h_1\cdots h_p}\\
&=u(w_1\otimes\cdots\otimes w_p\otimes\overline{h_1\cdots h_p})\\
&=u\varphi(w).
\end{aligned}$$

$\square$

We can now describe $\mathsf{F}(\bar{\mathscr{I}}_{\wp(s)})$ as follows.

Proposition 4.6. *Let* $\boldsymbol{s} = \{\mathsf{s}_1, \dots, \mathsf{s}_p\} \in \mathscr{S}_{r,\mu}$ *with* $\mathsf{s}_i = (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \dots, a_i v^{\mu_i-1})$. *Then* $\mathsf{F}(\bar{\mathscr{I}}_\mu) = 0$ *for* $\boldsymbol{s} \notin \mathscr{S}_r^{(n)}$ *and* $\mathsf{F}(\bar{\mathscr{I}}_\mu) \cong L(\boldsymbol{Q}_1) \otimes \cdots \otimes L(\boldsymbol{Q}_p)$ *for* $\boldsymbol{s} \in \mathscr{S}_r^{(n)}$, *where* $\boldsymbol{Q}_i = (Q_{i,1}(u), \dots, Q_{i,n}(u))$ *with* $Q_{i,n}(u) = (1 - a_i v^{-n+1} u)^{\delta_{\mu_i,n}}$ *and* $Q_{i,j}(uv^{j-1})/Q_{i,j+1}(uv^{j+1}) = (1 - a_i u)^{\delta_{j,\mu_i}}$ *for* $1 \leqslant i \leqslant p$ *and* $1 \leqslant j \leqslant n-1$.

Proof. Since $\bar{\mathscr{I}}_{\mu_i} \cong V_{\mathsf{s}_i}$ for $1 \leqslant i \leqslant p$, by Corollary 4.4 and Lemma 4.5 we conclude that $\mathsf{F}(\bar{\mathscr{I}}_\mu) = \mathsf{F}_{n,r}(\bar{\mathscr{I}}_\mu) \cong \mathsf{F}_{n,\mu_1}(V_{\mathsf{s}_1}) \otimes \cdots \otimes \mathsf{F}_{n,\mu_p}(V_{\mathsf{s}_p})$. If $\boldsymbol{s} \notin \mathscr{S}_r^{(n)}$, then there exists $1 \leqslant k \leqslant p$ such that $|\mathsf{s}_k| = \mu_k > n$. By Lemma 3.3 we have $\mathsf{F}_{n,\mu_k}(V_{\mathsf{s}_k}) = 0$ and hence $\mathsf{F}(\bar{\mathscr{I}}_\mu) = 0$. If $\boldsymbol{s} \in \mathscr{S}_r^{(n)}$, then by Lemma 3.4 we have $\mathsf{F}_{n,\mu_i}(V_{\mathsf{s}_i}) \cong L(\boldsymbol{Q}_i)$ for $1 \leqslant i \leqslant p$. Consequently, $\mathsf{F}(\bar{\mathscr{I}}_\mu) \cong L(\boldsymbol{Q}_1) \otimes \cdots \otimes L(\boldsymbol{Q}_p)$. □

We now turn to studying $\mathsf{F}(V_{\boldsymbol{s}})$ for $\boldsymbol{s} \in \mathscr{S}_r^{(n)}$. To compute $\mathsf{F}(V_{\boldsymbol{s}})$, we need to generalize [Chari and Pressley 1996, §7.2] to the case of $n \leqslant r$. Recall the notation $\Lambda(n,r)$ defined in (3.1.1). Let $\Lambda^+(n,r) = \Lambda(n,r) \cap \Lambda^+(r)$. For $\lambda \in \mathbb{N}^n$ let $L(\lambda)$ be the irreducible $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$-module with highest weight λ. For $1 \leqslant i \leqslant n$, let $\mathfrak{k}_i = \zeta_r(K_i)$ and

$$\begin{bmatrix} \mathfrak{k}_i; 0 \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{\mathfrak{k}_i v^{-s+1} - \mathfrak{k}_i^{-1} v^{s-1}}{v^s - v^{-s}}.$$

For $\mu \in \mathbb{N}^n$ let $\mathfrak{k}_\mu = \left[\begin{smallmatrix} \mathfrak{k}_1; 0 \\ \mu_1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} \mathfrak{k}_n; 0 \\ \mu_n \end{smallmatrix}\right]$. The following result is the generalization of [Chari and Pressley 1996, §7.2].

Lemma 4.7. *Let* $\mu \in \Lambda^+(r)$. *Then* $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu \neq 0$ *if and only if* $\mu' \in \Lambda(n,r)$, *where* μ' *is the dual partition of* μ. *Furthermore if* $\mu' \in \Lambda^+(n,r)$, *then* $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu \cong L(\mu')$.

Proof. We choose N such that $N > \max\{n,r\}$. Let $e = \sum_{\mu \in \Lambda(n,r)} \mathfrak{k}_\mu \in \mathcal{S}(N,r)_{\mathbb{C}}$. It is well known that for $\mu \in \Lambda^+(N,r)$, $eL(\mu) \neq 0$ if and only if $\mu \in \Lambda(n,r)$ (see [Green 2007, (6.5f)]). Furthermore by [Chari and Pressley 1996, §7.2; Deng, Du and Fu 2012, Lemma 4.3.3] we have $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu \cong e\big(\Omega_{N,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu\big) \cong e(L(\mu'))$. Thus $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu \neq 0$ if and only if $\mu' \in \Lambda(n,r)$. If $\mu' \in \Lambda^+(n,r)$, then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu \cong e(L(\mu')) \cong L(\mu')$. □

In the case of $n > r$, the Drinfeld polynomials associated with $\mathsf{F}(V_{\boldsymbol{s}})$ were calculated for $\boldsymbol{s} \in \mathscr{S}_r^{(n)}$ in [Chari and Pressley 1996, §7.6; Deng, Du and Fu 2012, Theorem 4.4.2]. We are now prepared to use Proposition 4.6 and Lemma 4.7 to generalize these results to the case of $n \leqslant r$ in Theorem 4.9.

Let $\mathfrak{Q}(n)_r = \{\boldsymbol{Q} \in \mathfrak{Q}(n) \mid \sum_{1 \leqslant i \leqslant n} \deg Q_i(u) = r\}$. For $\boldsymbol{s} = \{\mathsf{s}_1, \dots, \mathsf{s}_p\} \in \mathscr{S}_r^{(n)}$ with

$$\mathsf{s}_i = (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \dots, a_i v^{\mu_i-1}) \in (\mathbb{C}^*)^{\mu_i},$$

define $\boldsymbol{Q}_{s} = (Q_1(u), \ldots, Q_n(u))$ by setting $Q_n(u) = \prod_{\substack{1 \leqslant i \leqslant p \\ \mu_i = n}} (1 - a_i u v^{-n+1})$ and

$$Q_i(u) = P_i(uv^{-i+1}) P_{i+1}(uv^{-i+2}) \cdots P_{n-1}(uv^{n-2i}) Q_n(uv^{2(n-i)})$$

for $1 \leqslant i \leqslant n-1$, where

$$P_i(u) = \prod_{\substack{1 \leqslant j \leqslant p \\ \mu_j = i}} (1 - a_j u).$$

Then

$$\sum_{1 \leqslant i \leqslant n} \deg Q_i(u) = n \deg Q_n(u) + \sum_{1 \leqslant i \leqslant n-1} i \deg P_i(u) = \sum_{1 \leqslant i \leqslant p} \mu_i = r.$$

So $\boldsymbol{Q}_{s} \in \mathfrak{Q}(n)_r$. Consequently, we obtain a map $\partial_{n,r} : \mathscr{S}_r^{(n)} \to \mathfrak{Q}(n)_r$ defined by sending $\boldsymbol{s}$ to $\boldsymbol{Q}_{s}$.

Lemma 4.8. *The map $\partial_{n,r} : \mathscr{S}_r^{(n)} \to \mathfrak{Q}(n)_r$ is bijective.*

Proof. It is clear that $\partial_{n,r}$ is injective. Let $\boldsymbol{Q} = (Q_1(u), \ldots, Q_n(u)) \in \mathfrak{Q}(n)_r$ and let $\lambda \in \Lambda(n, r)$, with $\lambda_i = \deg Q_i(u)$. For $1 \leqslant j \leqslant n-1$ let

$$P_j(u) = \frac{Q_j(uv^{j-1})}{Q_{j+1}(uv^{j+1})}$$

and $\nu_j = \deg P_j(u) = \lambda_j - \lambda_{j+1}$. We write, for $1 \leqslant i \leqslant n-1$,

$$P_i(u) = (1 - a_{\nu_1 + \cdots + \nu_{i-1} + 1} u)(1 - a_{\nu_1 + \cdots + \nu_{i-1} + 2} u) \cdots (1 - a_{\nu_1 + \cdots + \nu_{i-1} + \nu_i} u),$$

and $Q_n(u) = (1 - b_1 u) \cdots (1 - b_{\lambda_n} u)$. Let $p' = \sum_{1 \leqslant i \leqslant n-1} \nu_i$ and $p = p' + \lambda_n$. Let $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\}$, where

$$\mathsf{s}_i = \begin{cases} (a_i v^{-\mu_i + 1}, a_i v^{-\mu_i + 3}, \ldots, a_i v^{\mu_i - 1}) & \text{for } 1 \leqslant i \leqslant p', \\ (b_{i-p'}, b_{i-p'} v^2, \ldots, b_{i-p'} v^{2(n-1)}) & \text{for } p' + 1 \leqslant i \leqslant p, \end{cases}$$

and $(\mu_1, \ldots, \mu_{p'}) = (1^{\nu_1}, \ldots, (n-1)^{\nu_{n-1}})$. Since

$$\sum_{1 \leqslant i \leqslant p} |\mathsf{s}_i| = \sum_{1 \leqslant j \leqslant p'} \mu_j + n\lambda_n = \sum_{1 \leqslant i \leqslant n-1} i\nu_i + n\lambda_n = \sum_{1 \leqslant i \leqslant n} \lambda_i = r,$$

we have $\boldsymbol{s} \in \mathscr{S}_r^{(n)}$. It is easy to see that $\partial_{n,r}(\boldsymbol{s}) = \boldsymbol{Q}$. Thus $\partial_{n,r}$ is surjective. □

Theorem 4.9. *For $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\} \in \mathscr{S}_r^{(n)}$ with $\mathsf{s}_i = (a_i v^{-\mu_i + 1}, a_i v^{-\mu_i + 3}, \ldots, a_i v^{\mu_i - 1})$, we have $\mathsf{F}(V_{\boldsymbol{s}}) \cong L(\boldsymbol{Q}_{s})$, where $\boldsymbol{Q}_{s} = \partial_{n,r}(\boldsymbol{s})$. In particular, we have $\mathsf{F}(V_{\boldsymbol{s}})|_{\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)} \cong \bar{L}(\boldsymbol{P})$, where*

$$P_i(u) = \prod_{\substack{1 \leqslant j \leqslant p \\ \mu_j = i}} (1 - a_j u) \quad \text{for } 1 \leqslant i \leqslant n-1.$$

Proof. Let $W = \mathsf{F}(\bar{\mathscr{I}}_\mu)$. By Proposition 4.6 we have $W \cong L(\boldsymbol{Q}_1) \otimes \cdots \otimes L(\boldsymbol{Q}_p)$, where $\boldsymbol{Q}_i = (Q_{i,1}(u), \ldots, Q_{i,n}(u))$ with $Q_{i,n}(u) = (1 - a_i v^{-n+1} u)^{\delta_{\mu_i,n}}$ and

$$P_{i,j}(u) := \frac{Q_{i,j}(uv^{j-1})}{Q_{i,j+1}(uv^{j+1})} = (1 - a_i u)^{\delta_{j,\mu_i}}$$

for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant n-1$. We will identify W with $L(\boldsymbol{Q}_1) \otimes \cdots \otimes L(\boldsymbol{Q}_p)$. Let $w = w_1 \otimes \cdots \otimes w_p \in W$, where w_i is the pseudo-highest weight vector in $L(\boldsymbol{Q}_i)$. Then by [Chari and Pressley 1996, §6.3; Frenkel and Mukhin 2002, Lemma 4.1] we conclude that w is the pseudo-highest weight vector in W such that $\mathtt{k}_i w = v^{\lambda_i} w$ and $\mathscr{Q}_i^{\pm}(u) w = Q_i^{\pm}(u) w$ for $1 \leqslant i \leqslant n$, where $\lambda_i = \deg Q_i^{+}(u)$,

$$\begin{aligned} Q_n^{\pm}(u) &= \prod_{1 \leqslant i \leqslant p} Q_{i,n}^{\pm}(u) = \prod_{1 \leqslant i \leqslant p} (1 - (a_i u)^{\pm 1} v^{\pm(-n+1)})^{\delta_{\mu_i,n}} \\ &= \prod_{\substack{1 \leqslant i \leqslant p \\ \mu_i = n}} (1 - (a_i u)^{\pm 1} v^{\pm(-n+1)}) \end{aligned}$$

and

$$\begin{aligned} P_j^{\pm}(u) &:= \frac{Q_j^{\pm}(v^{j-1}u)}{Q_{j+1}^{\pm}(v^{j+1}u)} = \prod_{1 \leqslant i \leqslant p} P_{i,j}^{\pm}(u) \\ &= \prod_{1 \leqslant i \leqslant p} (1 - (a_i u)^{\pm 1})^{\delta_{j,\mu_i}} = \prod_{\substack{1 \leqslant i \leqslant p \\ \mu_i = j}} (1 - (a_i u)^{\pm 1}) \end{aligned}$$

for $1 \leqslant j \leqslant n-1$. By definition we have $\boldsymbol{Q_s} = (Q_1^{+}(u), \ldots, Q_n^{+}(u))$. Since

$$\lambda_j = \deg Q_j^{+}(u) = \lambda_n + \sum_{j \leqslant s \leqslant n-1} \deg P_s^{+}(u) = \big|\{1 \leqslant i \leqslant p \mid \mu_i \geqslant j\}\big|$$

for $1 \leqslant j \leqslant n$, we have $\lambda = (\lambda_1, \ldots, \lambda_n) = \mu'$.

Let $L = \mathsf{F}(V_{\boldsymbol{s}})$. Since $V_{\boldsymbol{s}}$ is a semisimple $\mathscr{H}(r)_{\mathbb{C}}$-module, by Lemmas 3.1 and 4.7 we have $[L : L(\lambda)] = [L : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu] = [\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} V_{\boldsymbol{s}} : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu] = [V_{\boldsymbol{s}} : E_\mu] = 1$. Thus

$$\dim L_\lambda = 1. \tag{4.9.1}$$

Since $V_{\boldsymbol{s}}$ is the irreducible subquotient of $\bar{\mathscr{I}}_\mu$, there is a surjective $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module homomorphism $f : M \to L$, where M is a certain submodule of W. Since $1 = \dim L_\lambda \leqslant \dim M_\lambda \leqslant \dim W_\lambda = 1$, we conclude that $\dim M_\lambda = \dim W_\lambda = 1$. Hence $M_\lambda = W_\lambda = \mathrm{span}\{w\}$ and $L_\lambda = \mathrm{span}\{f(w)\}$. By (4.9.1) we have $f(w) \neq 0$. Since f is a $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module homomorphism, $f(w)$ is the pseudo-highest weight vector in L such that $\mathtt{k}_i f(w) = f(\mathtt{k}_i w) = v^{\lambda_i} f(w)$ and $\mathscr{Q}_i^{\pm}(u) f(w) = f(\mathscr{Q}_i^{\pm}(u) w) = Q_i^{\pm}(u) f(w)$ for $1 \leqslant i \leqslant n$. This implies that L is the irreducible quotient module of $M(\boldsymbol{Q_s})$ and hence $L \cong L(\boldsymbol{Q_s})$. $\square$

Combining Lemmas 3.3, 4.8 with 4.9 yields the following classification theorem of irreducible $\mathcal{S}_\triangle(n, r)_\mathbb{C}$-modules, which was proved as Theorem 4.6.8 in [Deng, Du and Fu 2012] using a different approach.

Corollary 4.10. *The set* $\{L(\boldsymbol{Q}) \mid \boldsymbol{Q} \in \mathfrak{Q}(n)_r\}$ *is a complete set of nonisomorphic finite-dimensional irreducible* $\mathcal{S}_\triangle(n, r)_\mathbb{C}$*-modules.*

Finally we will use Theorem 4.9 to generalize [Green 2007, (6.5f)] to the affine case in Theorem 4.11. Assume $N \geqslant n$. Let $e = \sum_{\lambda\in\Lambda(n,r)} \mathfrak{k}_\lambda \in \mathcal{S}_\triangle(N, r)_\mathbb{C}$. Then $e\mathcal{S}_\triangle(N, r)_\mathbb{C}e \cong \mathcal{S}_\triangle(n, r)_\mathbb{C}$. Consequently, the categories $e\mathcal{S}_\triangle(N, r)_\mathbb{C}e$-mod and $\mathcal{S}_\triangle(n, r)_\mathbb{C}$-mod may be identified. With this identification, we define a functor

$$\mathsf{G} = \mathsf{G}_{N,n,r} : \mathcal{S}_\triangle(N, r)_\mathbb{C}\text{-mod} \to \mathcal{S}_\triangle(n, r)_\mathbb{C}\text{-mod}, \quad V \mapsto eV. \tag{4.10.1}$$

Then by definition we have $\mathsf{G}_{N,n,r} \circ \mathsf{F}_{N,r} = \mathsf{F}_{n,r}$. For $\boldsymbol{Q} = (Q_1(u), \dots, Q_n(u)) \in \mathfrak{Q}(n)_r$ let $\widetilde{\boldsymbol{Q}} = (Q_1(u), \dots, Q_n(u), 1, \dots, 1) \in \mathfrak{Q}(N)_r$. Let $\widetilde{\mathfrak{Q}}(n)_r = \{\widetilde{\boldsymbol{Q}} \mid \boldsymbol{Q} \in \mathfrak{Q}(n)_r\} \subseteq \mathfrak{Q}(N)_r$. Clearly, by definition, we have

$$\partial_{N,r}(\boldsymbol{s}) = \widetilde{\partial_{n,r}(\boldsymbol{s})} \quad \text{for } \boldsymbol{s} \in \mathcal{S}_r^{(n)}. \tag{4.10.2}$$

Theorem 4.11. *Assume* $N \geqslant n$. *Then* $\mathsf{G}(L(\widetilde{\boldsymbol{Q}})) \cong L(\boldsymbol{Q})$ *for* $\boldsymbol{Q} \in \mathfrak{Q}(n)_r$. *In particular we have* $\dim L(\widetilde{\boldsymbol{Q}})_\alpha = \dim L(\boldsymbol{Q})_\alpha$ *for* $\alpha \in \Lambda(n, r)$. *Furthermore, for* $\boldsymbol{Q}' \in \mathfrak{Q}(N)_r$, $\mathsf{G}(L(\boldsymbol{Q}')) \neq 0$ *if and only if* $\boldsymbol{Q}' \in \widetilde{\mathfrak{Q}}(n)_r$.

Proof. If $\boldsymbol{Q} \in \mathfrak{Q}(n)_r$ then by Lemma 4.8 we conclude that there exist $\boldsymbol{s} \in \mathcal{S}_r^{(n)}$ such that $\boldsymbol{Q} = \partial_{n,r}(\boldsymbol{s})$. By Theorem 4.9 and (4.10.2) we have $L(\widetilde{\boldsymbol{Q}}) \cong \mathcal{T}_\triangle(N, r) \otimes_{\mathcal{H}_\triangle(r)_\mathbb{C}} V_{\boldsymbol{s}}$. So by [Deng, Du and Fu 2012, Lemma 4.3.3] and Theorem 4.9 we have

$$\mathsf{G}(L(\widetilde{\boldsymbol{Q}})) \cong (e\mathcal{T}_\triangle(N, r)) \otimes_{\mathcal{H}_\triangle(r)_\mathbb{C}} V_{\boldsymbol{s}} \cong \mathcal{T}_\triangle(n, r) \otimes_{\mathcal{H}_\triangle(r)_\mathbb{C}} V_{\boldsymbol{s}} \cong L(\boldsymbol{Q}).$$

By [Green 2007, (6.2g)], the set $\{\mathsf{G}(L(\boldsymbol{Q}')) \neq 0 \mid \boldsymbol{Q}' \in \mathfrak{Q}(N)_r\}$ forms a complete set of non-isomorphic irreducible $\mathcal{S}_\triangle(n, r)_\mathbb{C}$-modules. This together with Corollary 4.10 implies that $\{\mathsf{G}(L(\boldsymbol{Q}')) \neq 0 \mid \boldsymbol{Q}' \in \mathfrak{Q}(N)_r\} = \{\mathsf{G}(L(\widetilde{\boldsymbol{Q}})) \mid \boldsymbol{Q} \in \mathfrak{Q}(n)_r\}$. Consequently, $\mathsf{G}(L(\boldsymbol{Q}')) \neq 0$ if and only if $\boldsymbol{Q}' \in \widetilde{\mathfrak{Q}}(n)_r$. □

References

[Beck 1994] J. Beck, "Braid group action and quantum affine algebras", *Comm. Math. Phys.* **165**:3 (1994), 555–568. MR 95i:17011 Zbl 0807.17013

[Chari and Pressley 1991] V. Chari and A. Pressley, "Quantum affine algebras", *Comm. Math. Phys.* **142**:2 (1991), 261–283. MR 93d:17017 Zbl 0739.17004

[Chari and Pressley 1994] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, 1994. MR 95j:17010 Zbl 0839.17009

[Chari and Pressley 1995] V. Chari and A. Pressley, "Quantum affine algebras and their representations", pp. 59–78 in *Representations of groups* (Banff, AB, 1994), edited by B. N. Allison and G. H. Cliff, CMS Conf. Proc. **16**, Amer. Math. Soc., Providence, RI, 1995. MR 96j:17009 Zbl 0855.17009

[Chari and Pressley 1996] V. Chari and A. Pressley, "Quantum affine algebras and affine Hecke algebras", *Pacific J. Math.* **174**:2 (1996), 295–326. MR 97i:17011 Zbl 0881.17011

[Chari and Pressley 1997] V. Chari and A. Pressley, "Quantum affine algebras at roots of unity", *Represent. Theory* **1** (1997), 280–328. MR 98e:17018 Zbl 0891.17013

[Deng, Du and Fu 2012] B. Deng, J. Du, and Q. Fu, *A double Hall algebra approach to affine quantum Schur–Weyl theory*, London Mathematical Society Lecture Note Series **401**, Cambridge University Press, 2012. MR 3113018 Zbl 1269.20045

[Frenkel and Mukhin 2002] E. Frenkel and E. Mukhin, "The Hopf algebra Rep $U_q\widehat{\mathfrak{gl}}_\infty$", *Selecta Math.* (*N.S.*) **8**:4 (2002), 537–635. MR 2003k:17019 Zbl 1034.17009

[Green 2007] J. A. Green, *Polynomial representations of* GL_n, Lecture Notes in Mathematics **830**, Springer, Berlin, 2007. MR 2009b:20084 Zbl 1108.20044

[Jimbo 1986] M. Jimbo, "A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang–Baxter equation", *Lett. Math. Phys.* **11**:3 (1986), 247–252. MR 87k:17011 Zbl 0602.17005

[Kazhdan and Lusztig 1979] D. Kazhdan and G. Lusztig, "Representations of Coxeter groups and Hecke algebras", *Invent. Math.* **53**:2 (1979), 165–184. MR 81j:20066 Zbl 0499.20035

[Rogawski 1985] J. D. Rogawski, "On modules over the Hecke algebra of a p-adic group", *Invent. Math.* **79**:3 (1985), 443–465. MR 86j:22028 Zbl 0579.20037

[Zelevinsky 1980] A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, II: On irreducible representations of $\mathrm{GL}(n)$", *Ann. Sci. École Norm. Sup.* (4) **13**:2 (1980), 165–210. MR 83g:22012 Zbl 0441.22014

Received May 31, 2013.

QIANG FU
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI, 200092
CHINA

q.fu@tongji.edu.cn

q.fu@hotmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.367

ON THE CLASSIFICATION OF KILLING SUBMERSIONS AND THEIR ISOMETRIES

JOSÉ M. MANZANO

A Killing submersion is a Riemannian submersion from an orientable 3-manifold to an orientable surface whose fibers are the integral curves of a unit Killing vector field in the 3-manifold. We classify all Killing submersions over simply connected Riemannian surfaces and give explicit models for many Killing submersions, including those over simply connected constant Gaussian curvature surfaces. We also fully describe the isometries of the total space preserving the vertical direction. As a consequence, we prove that the only simply connected homogeneous 3-manifolds which admit a structure of Killing submersion are the $\mathbb{E}(\kappa,\tau)$-spaces, whose isometry group has dimension at least 4.

1. Introduction

Simply connected homogeneous Riemannian 3-manifolds with an isometry group of dimension 4 or 6 different from $\mathbb{H}^3$ can be represented by a 2-parameter family $\mathbb{E}(\kappa,\tau)$, where $\kappa,\tau\in\mathbb{R}$. They include $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^2\times\mathbb{R}$, $\mathbb{S}^2\times\mathbb{R}$, the Heisenberg group, the Berger spheres and the universal cover of the special linear group $\mathrm{SL}_2(\mathbb{R})$ endowed with a left-invariant metric (see [Daniel 2007; Daniel et al. 2009; Meeks and Pérez 2012]). The $\mathbb{E}(\kappa,\tau)$-spaces are 3-manifolds admitting a global unit Killing vector field whose integral curves are the fibers of a certain Riemannian submersion over the simply connected constant Gaussian curvature surface $\mathbb{M}^2(\kappa)$. In the Riemannian product 3-manifolds $M\times\mathbb{R}$, the projection over the first factor is a Riemannian submersion whose fibers are also the trajectories of a unit Killing vector field. In general, Riemannian submersions sharing this property will be called Killing submersions (see [Espinar and de Oliveira 2013; Rosenberg et al. 2010] and Definition 1.1 below).

Constant mean curvature surfaces in $\mathbb{E}(\kappa,\tau)$ and $M\times\mathbb{R}$ have been extensively studied during the last decade and many results have been recently extended to the

Research partially supported by the Spanish MCI research projects MTM2007-61775 and MTM2011-22547, and the Junta de Andalucía Grant P09-FQM-5088.
MSC2010: primary 53C15; secondary 53C30.
Keywords: unit Killing vector field, Riemannian submersions, homogeneous spaces.

Killing submersion setting (e.g., see [Dajczer and de Lira 2009, 2012; Espinar and de Oliveira 2013; Leandro and Rosenberg 2009; Rosenberg et al. 2010; Meroño and Ortiz 2014]). Nevertheless, apart from the aforementioned spaces, the theory of Killing submersions suffers from a lack of examples. It is necessary to mention that these 3-manifolds are well-understood at the level of differential topology (see [Besse 2008; Greub et al. 1976; Steenrod 1951]) since the projection defines principal bundles with totally geodesic fibers. Nevertheless, the objective of this paper is to classify them in the Riemannian category provided that the base is simply connected, and give explicit models depending on the base surface and a special geometric function, the so-called *bundle curvature*.

The bundle curvature has proved to be a very natural function in the surface theory of Killing submersions. For instance, a Calabi-type correspondence for surfaces which are graphs in the direction of the unit Killing field has been obtained recently [Lee and Manzano 2013], swapping the bundle curvature and the mean curvature of the graph. Note that Killing submersions also have dual Lorentzian counterparts when the Killing vector field is assumed to be timelike: they lead to interesting stationary spacetimes and are also related to Finsler metrics (see [Javaloyes et al. 2013]).

Let $\pi : \mathbb{E} \to M$ be a differentiable submersion from a Riemannian 3-manifold $\mathbb{E}$ onto a surface M. A vector $v \in T\mathbb{E}$ will be called *vertical* when $v \in \ker(\mathrm{d}\pi)$ and *horizontal* when $v \in \ker(\mathrm{d}\pi)^{\perp}$. The submersion π is Riemannian when it preserves the length of horizontal vectors.

Definition 1.1. The Riemannian submersion $\pi : \mathbb{E} \to M$, where $\mathbb{E}$ and M are connected and orientable, is called a Killing submersion if it admits a complete vertical unit Killing vector field.

As a matter of fact, any 3-manifold $\overline{M}$ admitting a unit Killing vector field ξ is locally isometric to the total space of a certain Killing submersion, so the definition is not as restrictive as it may seem.

The bundle curvature of a Killing submersion $\pi : \mathbb{E} \to M$ is defined (see Lemma 2.1) as the unique function $\tau \in \mathscr{C}^\infty(\mathbb{E})$ satisfying

$$\overline{\nabla}_X \xi = \tau X \wedge \xi \quad \text{for all } X \in \mathfrak{X}(\mathbb{E}),$$

where $\wedge$ is the cross product in $\mathbb{E}$, ξ is a vertical unit Killing vector field, and $\overline{\nabla}$ denotes the Levi-Civita connection in $\mathbb{E}$. The bundle curvature is constant along the fibers of π so it can be seen as a function $\tau \in \mathscr{C}^\infty(M)$ (see Propositions 3.3 and 4.6 for other geometric interpretations of τ). This gives rise to some natural questions: Given a Riemannian surface M and $\tau \in \mathscr{C}^\infty(M)$, does there exist a Killing submersion over M with bundle curvature τ? Is it unique? The main aim of Sections 2 and 4 will be to give affirmative answers to these questions when M

is simply connected. More specifically, we will classify Killing submersions up to isomorphism, in the following sense:

Definition 1.2. Let $\pi : \mathbb{E} \to M$ and $\pi' : \mathbb{E}' \to M'$ be two Killing submersions. A (local) isomorphism of Killing submersions from π to π' is a pair (f, h), where $h : M \to M'$ is an isometry and $f : \mathbb{E} \to \mathbb{E}'$ is a (local) isometry, such that $\pi' \circ f = h \circ \pi$.

Note that if (f, h) is an isomorphism of Killing submersions, then f maps fibers of π into fibers of π', and, if we consider a unit vertical Killing vector field ξ in $\mathbb{E}$, then $f_*\xi$ is also a unit vertical Killing vector field in $\mathbb{E}'$.

Given a simply connected Riemannian surface M and $\tau \in \mathscr{C}^\infty(M)$, we will show that there exists a Killing submersion over M with bundle curvature τ, and it is unique (up to isomorphism) if the total space $\mathbb{E}$ is also simply connected. In the process, it will turn out that the bundle curvature determines locally the geometry of the submersion, but the topology of $\mathbb{E}$ is also conditioned by the bundle curvature. More explicitly:

- If M is a topological disk, then the submersion is isomorphic to the projection $\pi_1 : M \times \mathbb{R} \to M$, $\pi_1(p, t) = p$, for some Riemannian metric on $M \times \mathbb{R}$ such that ∂_t is a unit vertical Killing vector field. In particular, the fibers of the submersion have infinite length.
- If, on the contrary, $M = (\mathbb{S}^2, g)$ for some Riemannian metric g, then we shall distinguish cases depending on whether the *total bundle curvature* $T = \int_M \tau$ vanishes or not:
  - If $T = 0$, then π is isomorphic to $\pi_1 : \mathbb{S}^2 \times \mathbb{R} \to (\mathbb{S}^2, g)$, $\pi_1(p, t) = p$, for some metric on $\mathbb{S}^2 \times \mathbb{R}$ such that ∂_t is a unit vertical Killing vector field, so the fibers have infinite length.
  - If $T \neq 0$, then π is isomorphic to $\pi_{\mathrm{Hopf}} : \mathbb{S}^3 \to (\mathbb{S}^2, g)$, $\pi_{\mathrm{Hopf}}(z, w) = (2z\bar{w}, |z|^2 - |w|^2)$, where $\mathbb{S}^3 \subset \mathbb{C}^2$ is endowed with a metric such that $(\pi/T)(iz, iw)$ is a unit vertical Killing vector field. In this case, the fibers have length $|2T|$.

When the total space is not simply connected, Killing submersions over M are also classified as the quotients of those listed above under a vertical translation (i.e., an element of the 1-parameter group of isometries associated to the unit Killing vector field).

Though this theoretical description is exhaustive, we will give explicit models for a wide class of Killing submersions. Firstly, for those over a disk with a conformal metric in terms of the conformal factor, the obtained examples will generalize the metrics for the $\mathbb{E}(\kappa, \tau)$-spaces in [Daniel 2007]. Secondly, we will obtain a general method to produce trivial Killing submersions (i.e., admitting a global smooth section) over any surface by isometrically embedding it in $\mathbb{R}^n$ for some $n \geq 3$. Finally, explicit models will also be obtained for Killing submersions over the round sphere $\mathbb{S}^2(\kappa)$ via the Hopf fibration (generalizing the metrics of the Berger spheres in [Torralbo 2012]).

The geometries of M and $\mathbb{E}$ of a Killing submersion $\pi : \mathbb{E} \to M$ are well-related, and geodesics or isometries are good samples of that. On the one hand, geodesics of $\mathbb{E}$ can be divided into three different types: vertical ones, horizontal ones (which are horizontal lifts of geodesics of M) and those which are neither vertical nor horizontal, each of which makes a constant angle with the vertical direction and whose projection is well-understood (see Proposition 3.6). In particular, M is complete if and only if $\mathbb{E}$ is complete. On the other hand, a beautiful classification result is obtained when we look for *Killing isometries* (i.e., isometries of $\mathbb{E}$ preserving the vertical direction). More explicitly, if $\mathbb{E}$ and M are simply connected, and $\tau \in \mathscr{C}^\infty(M)$ denotes the bundle curvature, then:

(a) Given a Killing isometry $f : \mathbb{E} \to \mathbb{E}$, there exists a unique isometry $h : M \to M$ such that $\pi \circ f = h \circ \pi$. Moreover, $\tau \circ h = \tau$ if f is orientation-preserving and $\tau \circ h = -\tau$ if it is orientation-reversing.

(b) Conversely, given an isometry $h : M \to M$ and $p_0, q_0 \in \mathbb{E}$ with $h(\pi(p_0)) = \pi(q_0)$, the following properties hold:
   - If $\tau \circ h = \tau$, then there is a unique orientation-preserving Killing isometry $f : \mathbb{E} \to \mathbb{E}$ with $\pi \circ f = h \circ \pi$ and $f(p_0) = q_0$.
   - If $\tau \circ h = -\tau$, then there is a unique orientation-reversing Killing isometry $f : \mathbb{E} \to \mathbb{E}$ with $\pi \circ f = h \circ \pi$ and $f(p_0) = q_0$.

This construction provides a surjective group morphism from the group of Killing isometries of $\mathbb{E}$ to the group of isometries of M which either preserve τ or map it to $-\tau$. Its kernel consists of isometries of $\mathbb{E}$ that leave the fibers invariant (i.e., vertical translations, and also symmetries with respect to a horizontal slice when $\tau = 0$). In particular, 1-parameter groups of isometries of M preserving τ give rise to 1-parameter groups of isometries in $\mathbb{E}$. Such groups have proven to be essential in surface theory, leading to many geometric features, e.g., they are related to holomorphic quadratic differentials (see [Abresch and Rosenberg 2005]) and conjugate constructions (see [Manzano and Torralbo 2012]).

Finally, note that simply connected homogeneous 3-manifolds are classified: they are all isometric to Lie groups endowed with left-invariant metrics except for $\mathbb{S}^2(\kappa)\times\mathbb{R}$, where $\kappa > 0$ (see [Meeks and Pérez 2012, Theorem 2.4]). In Section 5, we will characterize the homogeneous spaces $\mathbb{E}(\kappa, \tau)$ as the only simply connected homogeneous 3-manifolds admitting a Killing submersion structure (see Theorem 5.2). Hence, the only Killing submersions whose total space is isometric to a Lie group endowed with a left invariant metric are the $\mathbb{E}(\kappa, \tau)$-spaces, except for $\mathbb{S}^2(\kappa) \times \mathbb{R}$.

2. Uniqueness results

The bundle curvature. The next result can be found in [Espinar and de Oliveira 2013; Souam and Van der Veken 2012], but we will include the proof here for completeness.

Lemma 2.1. *Let $\pi : \mathbb{E} \to M$ be a Killing submersion. Then there exists a function $\tau \in \mathscr{C}^\infty(\mathbb{E})$ such that $\overline{\nabla}_X \xi = \tau X \wedge \xi$ for all $X \in \mathfrak{X}(\mathbb{E})$.*

The function τ will be called the bundle curvature *of the submersion.*

Proof. First of all, note that $\overline{\nabla}_\xi \xi = 0$. Indeed, given $X \in \mathfrak{X}(\mathbb{E})$, we have

$$\langle \overline{\nabla}_\xi \xi, X\rangle = -\langle \overline{\nabla}_X \xi, \xi\rangle = -\tfrac{1}{2} X\langle \xi, \xi\rangle = 0,$$

since ξ is Killing and unitary.

Let us now take $X \in \mathfrak{X}(\mathbb{E})$ linearly independent of ξ. On the one hand, it is clear that $\langle \overline{\nabla}_X \xi, \xi\rangle = 0$ and, on the other hand, $\langle \overline{\nabla}_X \xi, X\rangle = 0$ since ξ is Killing. Then there exists a unique function $\tau_X \in \mathscr{C}^\infty(\mathbb{E})$ such that $\overline{\nabla}_X \xi = \tau_X X \wedge \xi$, so it suffices to prove that τ_X does not depend on X. It is clear that τ_X only depends on the horizontal part of X so it will be enough to prove that $\tau_X = \tau_Y$ for all $X, Y \in \mathfrak{X}(\mathbb{E})$ horizontal. By using again that ξ is a Killing vector field, we get

$$\tau_Y \langle Y \wedge \xi, X\rangle = \langle \overline{\nabla}_Y \xi, X\rangle = -\langle \overline{\nabla}_X \xi, Y\rangle = -\tau_X \langle X \wedge \xi, Y\rangle = \tau_X \langle Y \wedge \xi, X\rangle,$$

so $\tau_X = \tau_Y$, where X and Y are linearly independent. Elsewhere, the identity $\tau_X = \tau_Y$ follows from the linearity of the connection. □

Observe that the function τ in the conditions of Lemma 2.1 is unique and its sign depends on the choice of orientation in $\mathbb{E}$. We will give now some consequences of this result in order to fix some notation.

Remark 2.2. (1) The condition $\overline{\nabla}_\xi \xi = 0$ implies that the fibers of the submersion are geodesics of $\mathbb{E}$, which will be called *vertical geodesics*.

(2) The elements of the 1-parameter group of isometries $\{\phi_t\}_{t\in\mathbb{R}}$ associated to the Killing vector field ξ will be called *vertical translations*.

Note that ϕ_t preserves the Killing field ξ and the orientation in $\mathbb{E}$. Thus, if we apply $d\phi_t$ to the identity in Lemma 2.1, we easily get $\tau = \tau \circ \phi_t$ for all $t \in \mathbb{R}$. This means that the bundle curvature is constant along the fibers and, hence, it may be considered as a function either in $\mathbb{E}$ or in the base M.

(3) More generally, let (f, h) be an isomorphism between two Killing submersions $\pi : \mathbb{E} \to M$ and $\pi' : \mathbb{E}' \to M'$ (see Definition 1.2) and define $\tau \in \mathscr{C}^\infty(\mathbb{E})$ and $\tau' \in \mathscr{C}^\infty(\mathbb{E}')$ as their bundle curvatures with respect to some orientations in $\mathbb{E}$ and $\mathbb{E}'$, respectively. Then $\tau \circ f = \tau$ when f preserves the orientation, and $\tau \circ f = -\tau'$ when f reverses the orientation.

In the product spaces $M \times \mathbb{R}$ the projection over the first factor is a Killing submersion, so its bundle curvature is $\tau \equiv 0$ (from Lemma 2.1 it is easy to deduce that $\tau \equiv 0$ in a Killing submersion if and only if the horizontal distribution in the total space is integrable). Given $\kappa, \tau \in \mathbb{R}$, there exists a Killing submersion $\pi : \mathbb{E}(\kappa, \tau) \to \mathbb{M}^2(\kappa)$ with constant bundle curvature τ. If $\kappa > 0$ and $\tau \neq 0$, the projection is the Hopf fibration and we obtain the Berger spheres; in the remaining cases the fibers have infinite length. We refer the reader to [Daniel 2007] for a description of these examples, although Berger spheres from a global point of view can be found in [Torralbo 2012].

Other examples derived from the aforementioned ones are their Riemannian quotients by a convenient vertical translation. Thus the length of the fibers will play an important role in the theory. Since fibers are geodesics, the following result follows from [Besse 2008, Theorem 9.56].

Lemma 2.3. *Let $\pi : \mathbb{E} \to M$ be a Killing submersion. Then all the fibers of π share the same (finite or infinite) length.*

Local representation of a Killing submersion. Given a surface M and $\tau \in \mathscr{C}^\infty(M)$, we are interested in finding all Killing submersions over M with bundle curvature τ. Let us begin by giving a useful technical tool that will simplify some arguments throughout the paper.

Proposition 2.4. *Let $\pi : \mathbb{E} \to M$ be a Killing submersion, and suppose that M is noncompact. Then π admits a global smooth section $F : M \to \mathbb{E}$. Hence,*

$$\Psi : M \times \mathbb{R} \to \mathbb{E}, \quad \Psi(p, t) = \phi_t(F(p)),$$

is a local diffeomorphism, where $\{\phi_t\}$ denotes the 1-parameter group of vertical translations. Moreover, Ψ is a global diffeomorphism if and only if the fibers of π have infinite length.

Proof. We can suppose that the fibers of π have finite length (otherwise, we take a quotient of π under a vertical translation ϕ_t for some $t > 0$). Then π is a codimension-one circle bundle over a noncompact surface and [Greub et al. 1976, Section VIII.5] yields the existence of a global smooth section. Moreover, Ψ is a local diffeomorphism since its differential is injective at every point.

Finally, note that Ψ is a global diffeomorphism if and only if it is injective, but $\Psi(p', t') = \Psi(p, t)$ implies $p = p'$ since $\Psi(p', t')$ and $\Psi(p, t)$ belong to the same fiber of π, so the last assertion in the statement holds. □

This result will be mostly used to ensure that there exists a smooth section $F : U \to \mathbb{E}$ for any coordinate chart (U, φ) in M, but it also implies that exceptional topologies for the total space may only arise when the base is compact. Note that, if the base is compact, then Proposition 2.4 no longer holds, as the Hopf fibration from $\mathbb{S}^3$ to $\mathbb{S}^2$ shows.

The following result will be the cornerstone of the subsequent development yielding a standard way of describing π in terms of M and τ.

Proposition 2.5. *Let $\pi : \mathbb{E} \to M$ be a Killing submersion. Let $U \subset M$ be an open set such that there is a conformal diffeomorphism $\varphi : U \to \Omega \subset \mathbb{R}^2$. Then*:

(a) *Given a smooth section $F_0 : U \to \pi^{-1}(U)$, the transformation*

$$f : \Omega \times \mathbb{R} \to \pi^{-1}(U), \quad (x, y, t) \mapsto \phi_t(F_0(\varphi^{-1}(x, y))), \tag{2-1}$$

is a local diffeomorphism and satisfies $\pi \circ f = \varphi \circ \pi_1$ in $\Omega \times \mathbb{R}$, where $\pi_1 : \Omega \times \mathbb{R} \to \Omega$ is the projection over the first factor.

(b) *Let us write the induced metric in Ω as $\mathrm{d}s^2_\lambda = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2)$ for some positive $\lambda \in \mathscr{C}^\infty(\Omega)$. Then there exist $a, b \in \mathscr{C}^\infty(\Omega)$ such that the metric in $\Omega \times \mathbb{R}$ which makes f a local isometry can be expressed as*

$$\mathrm{d}s^2 = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2) + (\mathrm{d}t - \lambda(a\,\mathrm{d}x + b\,\mathrm{d}y))^2. \tag{2-2}$$

(c) *$\pi_1 : (\Omega \times \mathbb{R}, \mathrm{d}s^2) \to (\Omega, \mathrm{d}s^2_\lambda)$ is a Killing submersion with unit Killing vector field ∂_t, and (f, φ^{-1}) is a local isomorphism from π_1 to π.*

Moreover, if the fibers of π have infinite length, then f is a global diffeomorphism.

Proof. We deduce from Proposition 2.4 that $\Psi : U \times \mathbb{R} \to \pi^{-1}(U)$ given by $\Psi(p, t) = \phi_t(F_0(p))$ is a local diffeomorphism, so $f = \Psi \circ (\varphi^{-1} \times \mathrm{id}_\mathbb{R})$ is also a local diffeomorphism, and it obviously satisfies the condition $\varphi \circ \pi_1 = \pi \circ f$, so (a) is proved. Note that Proposition 2.4 also ensures that f is a global diffeomorphism if the fibers of π have infinite length.

To prove (b), consider the unique Riemannian metric $\mathrm{d}s^2$ in $\varphi(U) \times \mathbb{R}$ making f a local isometry. The condition $\varphi \circ \pi_1 = \pi \circ f$ implies that π_1 is a Killing submersion. Vertical translations for π correspond (through f) to isometries of the form $(x, y, t) \mapsto (x, y, t + \mu)$, $\mu \in \mathbb{R}$, in $(\varphi(U) \times \mathbb{R}, \mathrm{d}s^2)$. In particular, ∂_t is a unit vertical Killing vector field in $(\varphi(U) \times \mathbb{R}, \mathrm{d}s^2)$.

Let $\{e_1, e_2\}$ be the orthonormal frame in $(\varphi(U), \mathrm{d}s^2_\lambda)$, where $e_1 = (1/\lambda)\,\partial_x$ and $e_2 = (1/\lambda)\,\partial_y$, and let $\{E_1, E_2\}$ be the horizontal lift of $\{e_1, e_2\}$ with respect to π_1 and $E_3 = \partial_t$. Since π_1 is the projection over the first two variables, there exist $a, b \in \mathscr{C}^\infty(\varphi(U))$ such that

$$\begin{cases} (E_1)_{(x,y,t)} = \dfrac{1}{\lambda(x, y)}\,\partial_x + a(x, y)\,\partial_t, \\ (E_2)_{(x,y,t)} = \dfrac{1}{\lambda(x, y)}\,\partial_y + b(x, y)\,\partial_t, \\ (E_3)_{(x,y,t)} = \partial_t. \end{cases} \tag{2-3}$$

Note that $\{E_1, E_2, E_3\}$ is an orthonormal frame in $(\varphi(U)\times\mathbb{R}, \mathrm{d}s^2)$ which can be supposed positively oriented after possibly swapping e_1 and e_2. Now it is straightforward to show that the global frame (2-3) is orthonormal for $\mathrm{d}s^2$ if and only if $\mathrm{d}s^2$ is the metric given by (2-2). $\square$

Regardless of the values of the functions $a, b \in \mathscr{C}^\infty(\Omega)$, the Riemannian metric given by (2-2) has the property that the projection over the first two variables is a Killing submersion over $(\Omega, \mathrm{d}s^2_\lambda)$.

Definition 2.6 (canonical example). Given an open set $\Omega\subset\mathbb{R}^2$ and $\lambda, a, b\in\mathscr{C}^\infty(\Omega)$ with $\lambda>0$, the Killing submersion

$$\pi_1 : (\Omega\times\mathbb{R}, \mathrm{d}s^2_{\lambda,a,b}) \mapsto (\Omega, \mathrm{d}s^2_\lambda), \quad \pi_1(x, y, z) = (x, y),$$
$$\mathrm{d}s^2_{\lambda,a,b} = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2) + (\mathrm{d}z - \lambda(a\,\mathrm{d}x + b\,\mathrm{d}y))^2,$$

will be called the canonical example associated to (λ, a, b).

Equation (2-3) defines a global orthonormal frame $\{E_1, E_2, E_3\}$ for $\mathrm{d}s^2_{\lambda,a,b}$, where E_1 and E_2 are horizontal, and E_3 is a unit vertical Killing field. It is easy to check that $[E_1, E_3] = [E_2, E_3] = 0$ and

$$[E_1, E_2] = \frac{\lambda_y}{\lambda^2}E_1 - \frac{\lambda_x}{\lambda^2}E_2 + \left(\frac{1}{\lambda^2}(b\lambda_x - a\lambda_y) + \frac{1}{\lambda}(b_x - a_y)\right)E_3.$$

Taking into account Lemma 2.1, we can compute the bundle curvature τ associated to this canonical example as

$$\begin{aligned}(2\text{-}4)\qquad 2\tau &= \langle\overline{\nabla}_{E_1}E_2, E_3\rangle - \langle\overline{\nabla}_{E_2}E_1, E_3\rangle = \langle[E_1, E_2], E_3\rangle\\ &= \frac{1}{\lambda^2}(b\lambda_x - a\lambda_y) + \frac{1}{\lambda}(b_x - a_y) = \frac{1}{\lambda^2}((\lambda b)_x - (\lambda a)_y).\end{aligned}$$

This divergence formula will come in handy in the sequel.

Lemma 2.7 (classification of canonical examples). *Let $\Omega\subseteq\mathbb{R}^2$ be a simply connected open set and $\lambda, a_0, a_1, b_0, b_1 \in \mathscr{C}^\infty(\Omega)$ such that $\lambda>0$. The following assertions are equivalent*:

(i) *There exists $d\in\mathscr{C}^\infty(\Omega)$ such that the pair $(f_d, \mathrm{id}_\Omega)$, where*

$$(2\text{-}5)\qquad \begin{aligned} f_d : (\Omega\times\mathbb{R}, \mathrm{d}s^2_{\lambda,a_0,b_0}) &\to (\Omega\times\mathbb{R}, \mathrm{d}s^2_{\lambda,a_1,b_1}),\\ (x, y, z) &\mapsto (x, y, z - d(x, y)),\end{aligned}$$

is an isomorphism of Killing submersions.

(ii) *There exists $d\in\mathscr{C}^\infty(\Omega)$ such that $d_x = \lambda(a_1 - a_0)$ and $d_y = \lambda(b_1 - b_0)$.*

(iii) *The bundle curvatures $\tau_0, \tau_1 \in \mathscr{C}^\infty(\Omega)$ of the two submersions coincide.*

Proof. It is easy to check that f_d is an isometry if and only if d satisfies (ii), so the equivalence between (i) and (ii) is proved. Since f_d preserves the orientation in $\Omega \times \mathbb{R}$, we get that (i) implies (iii) from Remark 2.2. Finally, to prove that (iii) implies (ii), observe that $\tau_0 = \tau_1$ means $(\lambda b_0)_x - (\lambda a_0)_y = (\lambda b_1)_x - (\lambda a_1)_y$ in view of (2-4). Equivalently, we have $(\lambda(a_1 - a_0))_y = (\lambda(b_1 - b_0))_x$, so (ii) follows from Poincaré's lemma and the fact that Ω is simply connected. □

We remark that condition (i) in the statement is equivalent to the fact that the canonical examples for (λ, a_0, b_0) and (λ, a_1, b_1) represent the same Killing submersion for different initial sections. The function d is, up to an additive constant, the vertical distance between such sections.

Lemma 2.7 is actually a local classification result for Killing submersions, since we have proved that all Killing submersions are locally equivalent to canonical examples. We will now give the general version.

Theorem 2.8 (uniqueness). *For $i \in \{0, 1\}$, let $\pi_i : \mathbb{E}_i \to M_i$ be a Killing submersion, M_i being simply connected, with bundle curvature $\tau_i \in \mathscr{C}^\infty(M_i)$ for a given orientation in $\mathbb{E}_i$. Suppose that the fibers of π_0 and the fibers of π_1 have the same length and there exists an isometry $h : M_0 \to M_1$. Let $p_0 \in \mathbb{E}_0$ and $p_1 \in \mathbb{E}_1$ be such that $h(\pi_0(p_0)) = \pi_1(p_1)$.*

(a) *If $\tau_1 \circ h = \tau_0$, then there exists a unique orientation-preserving isometry $f : \mathbb{E}_0 \to \mathbb{E}_1$ such that $\pi_1 \circ f = h \circ \pi_0$ and $f(p_0) = p_1$.*

(b) *If $\tau_1 \circ h = -\tau_0$, then there exists a unique orientation-reversing isometry $f : \mathbb{E}_0 \to \mathbb{E}_1$ such that $\pi_1 \circ f = h \circ \pi_0$ and $f(p_0) = p_1$.*

Proof. Let us first consider the case of M_i being a topological disk, so there exist conformal diffeomorphisms $\varphi_i : M_i \to \Omega$ such that $h \circ \varphi_0 = \varphi_1$, where $\Omega \subset \mathbb{R}^2$ is an open set. For $i \in \{0, 1\}$, Proposition 2.4 guarantees the existence of a global smooth section $F_i : M_i \to \mathbb{E}_i$ and a local diffeomorphism $f_i : \Omega \times \mathbb{R} \to \mathbb{E}_i$, given by $f_i(x, y, t) = \phi_t^i(F_i(\varphi_i^{-1}(x, y)))$ as in Proposition 2.5, where $\{\phi_t^i\}$ is the 1-parameter group of vertical translations associated to π_i. In other words, we obtain a commutative diagram as in Figure 1, where $\pi : \Omega \times \mathbb{R} \to \Omega$ is the projection over the first factor.

Observe that φ_0 and φ_1 induce the same metric $\lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2)$ on Ω, and f_0 and f_1 induce canonical metrics $\mathrm{d}s^2_{\lambda,a_0,b_0}$ and $\mathrm{d}s^2_{\lambda,a_1,b_1}$ on $\Omega \times \mathbb{R}$, respectively. Moreover, the condition $\tau_1 \circ h = \tau_0$ ensures that both canonical examples for (λ, b_0, a_0) and (λ, b_1, a_1) have the same bundle curvature, so Lemma 2.7 yields the existence of a isometry

$$\hat{f} : (\Omega \times \mathbb{R}, \mathrm{d}s^2_{\lambda,a_0,b_0}) \to (\Omega \times \mathbb{R}, \mathrm{d}s^2_{\lambda,a_1,b_1})$$

$$\begin{tikzcd}
\mathbb{E}_0 \arrow[rrr, "\pi_0"] \arrow[ddd, dotted, "f"'] & & & M_0 \arrow[dl, "\varphi_0"'] \arrow[ddd, "h"] \\
& \Omega\times\mathbb{R} \arrow[ul, "f_0"'] \arrow[r, "\pi"] \arrow[d, dotted, "\hat{f}"] & \Omega \arrow[d, equal] & \\
& \Omega\times\mathbb{R} \arrow[r, "\pi"] \arrow[dl, "f_1"] & \Omega & \\
\mathbb{E}_1 \arrow[rrr, "\pi_1"'] & & & M_1 \arrow[ul, "\varphi_1"]
\end{tikzcd}$$

Figure 1. Horizontal and vertical arrows represent Killing submersions and isometries, respectively. Diagonal ones relate the original diagram with the canonical examples.

of the form $\hat{f}(x, y, z) = f(x, y, z + d(x, y))$ for some $d \in \mathscr{C}^\infty(\Omega)$, so $\pi \circ \hat{f} = \pi$. If $\tau_1 \circ h = -\tau_0$, the canonical examples for (λ, b_0, a_0) and (λ, b_1, a_1) have opposite bundle curvatures, so it is easy to see that there exists an isometry of the form $\hat{f}(x, y, z) = f(x, y, -z - d(x, y))$ for some $d \in \mathscr{C}^\infty(\Omega)$.

In both cases, the isometry $\hat{f}$ induces an isometry from the quotient of

$$(\Omega \times \mathbb{R}, \mathrm{d}s^2_{\lambda,a_0,b_0})$$

by a vertical translation to the quotient of $(\Omega \times \mathbb{R}, \mathrm{d}s^2_{\lambda,a_1,b_1})$ by the same vertical translation. Adjusting the translation so that the length of the fibers of the quotient is the same as in $\mathbb{E}_0$ or $\mathbb{E}_1$, the isometry in the quotient provides an isometry $f : \mathbb{E}_0 \to \mathbb{E}_1$ such that $\pi_1 \circ f = h \circ \pi_0$. We get $f(p_0) = p_1$ by just composing f with a vertical translation.

Finally, suppose that M_0 and M_1 are topological 2-spheres. Let $U_0 = M_0 \setminus \{q_0\}$ for some $q_0 \neq \pi_0(p_0)$ and $U_1 = h(U_0) = M_1 \setminus \{h(q_0)\}$. Note that $h : U_0 \to U_1$ is an isometry in the conditions of the disk case so it lifts to an isometry $f : V_0 \to V_1$, where $V_i = \pi_i^{-1}(U_i)$ for $i \in \{0, 1\}$, satisfying $\pi_1 \circ f = h \circ \pi_0$ in V_0 and $f(p_0) = p_1$. Now, let $\widetilde{p}_0 \in V_0$ be such that $\pi_0(\widetilde{p}_0) \neq \pi_0(p_0)$, and $\widetilde{p}_1 = f(\widetilde{p}_0)$. Now take $\widetilde{q}_0 \in M_0$ such that $\widetilde{q}_0 \notin \{\pi_0(\widetilde{p}_0), q_0\}$, and $\widetilde{U}_0 = M_0 \setminus \{\widetilde{q}_0\}$, $\widetilde{U}_1 = h(\widetilde{U}_0) = M_1 \setminus \{h(\widetilde{q}_0)\}$. The same reasoning above gives an isometry $\widetilde{f} : \widetilde{V}_0 \to \widetilde{V}_1$, where $\widetilde{V}_i = \pi_i^{-1}(\widetilde{U}_i)$ for $i \in \{1, 2\}$, satisfying the condition $\pi_1 \circ \widetilde{f} = h \circ \pi_0$ in $\widetilde{V}_0$ and $\widetilde{f}(\widetilde{p}_0) = \widetilde{p}_1 = f(\widetilde{p}_0)$.

Since $V = V_0 \cap \widetilde{V}_0$ is connected, we have $f(\widetilde{p}_0) = \widetilde{f}(\widetilde{p}_0)$, and $(\mathrm{d}f)_{\widetilde{p}_0} = (\mathrm{d}\widetilde{f})_{\widetilde{p}_0}$ (because both f and $\widetilde{f}$ preserve the vertical direction and $\pi_1 \circ \widetilde{f} = h \circ \pi_0 = \pi_1 \circ f$ in V), and we conclude that $f = \widetilde{f}$ in V. As $V_0 \cup \widetilde{V}_0 = \mathbb{E}_0$, we deduce that f can be extended (by $\widetilde{f}$) to an isometry from $\mathbb{E}_0$ to $\mathbb{E}_1$, and it trivially satisfies the conditions in the statement. $\square$

Killing isometries. We will now specialize some of the results in the previous section to study isometries of the total space of a Killing submersion $\pi : \mathbb{E} \to M$ preserving the Killing submersion structure, i.e., those preserving the direction of a unit vertical Killing vector field ξ.

Definition 2.9. In the previous notation, the isometries of $\mathbb{E}$ satisfying $f_*\xi = \xi$ or $f_*\xi = -\xi$ will be called Killing isometries.

The definition does not depend on the choice of ξ. If $f_*\xi = \xi$ (resp. $f_*\xi = -\xi$), then f is said to preserve (resp. reverse) the orientation of the fibers. Note that preserving the orientation of the fibers is not related to preserving or reversing the orientation of the total space $\mathbb{E}$.

Lemma 2.10. *Let $\pi : \mathbb{E} \to M$ be a Killing submersion with bundle curvature $\tau \in \mathscr{C}^\infty(M)$, and let $f : \mathbb{E} \to \mathbb{E}$ be a Killing isometry. Then*:

(a) *There exists a unique isometry $h : M \to M$ such that $\pi \circ f = h \circ \pi$.*

(b) *If f preserves the orientation in $\mathbb{E}$, then $\tau \circ h = \tau$.*

(c) *If f reverses the orientation in $\mathbb{E}$, then $\tau \circ h = -\tau$.*

Proof. Item (a) follows from the fact that f maps fibers to fibers and from the fact that $\mathrm{d}\pi$ is an isometry when restricted to the horizontal distribution. Now, it is easy to see that (f, h) is an isomorphism of Killing submersions (see Definition 1.2), so (b) and (c) follow from Remark 2.2. ☐

In fact, the map $f \mapsto h$ defined by (a) of Lemma 2.10 can be easily proved to be a group morphism from the group of Killing isometries to the group of isometries of M with $\tau \circ h = \pm\tau$. Moreover, the normal subgroup of orientation-preserving isometries is mapped to those isometries of M which preserve τ. As an application of Theorem 2.8, we can prove that this morphism is surjective and its kernel consists of the vertical translations and, for $\tau \equiv 0$, also the symmetries with respect to a slice.

Corollary 2.11. *Let $\pi : \mathbb{E} \to M$ be a Killing submersion with bundle curvature $\tau \in \mathscr{C}^\infty(M)$ and suppose that M is simply connected. Let $h : M \to M$ be an isometry and take $p_0, q_0 \in \mathbb{E}$ such that $h(\pi(p_0)) = \pi(q_0)$.*

(a) *If $\tau \circ h = \tau$ in M, then there exists a unique orientation-preserving Killing isometry $f : \mathbb{E} \to \mathbb{E}$ such that $\pi \circ f = h \circ \pi$ and $f(p_0) = q_0$.*

(b) *If $\tau \circ h = -\tau$ in M, then there exists a unique orientation-reversing Killing isometry $f : \mathbb{E} \to \mathbb{E}$ such that $\pi \circ f = h \circ \pi$ and $f(p_0) = q_0$.*

As an immediate consequence, in the following two situations there do not exist Killing isometries reversing the orientation of the total space:

- If the bundle curvature is a nonzero constant.
- If M is a Riemannian 2-sphere and $\int_M \tau \neq 0$.

3. Curves in Killing submersions

The horizontal lift of a curve.

Definition 3.1. Let $\pi : \mathbb{E} \to M$ be a Killing submersion and $\alpha : [c, d] \to M$ a $\mathscr{C}^1$-curve. A horizontal (or Legendrian) lift of α is a $\mathscr{C}^1$-curve $\widetilde{\alpha} : [c, d] \to \mathbb{E}$ such that $\widetilde{\alpha}'$ is always horizontal and $\pi \circ \widetilde{\alpha} = \alpha$ in $[c, d]$.

This concept extends to piecewise $\mathscr{C}^1$-curves $\alpha : [c, d] \to M$, i.e., α such that there is a partition $c = t_0 < t_1 < \ldots < t_n = d$ so that $\alpha_{|[t_{i-1},t_i]}$ is $\mathscr{C}^1$ for all $i \in \{1, \ldots, n\}$. A horizontal lift of α is a continuous curve $\widetilde{\alpha} : [c, d] \to \mathbb{E}$ such that $\widetilde{\alpha}_{|[t_{i-1},t_i]}$ is a horizontal lift of $\alpha_{|[t_{i-1},t_i]}$ for all $i \in \{1, \ldots, n\}$.

Lemma 3.2. *Let $\alpha : [c, d] \to M$ be a piecewise $\mathscr{C}^1$-curve. Given $p_0 \in \mathbb{E}$ such that $\pi(p_0) = \alpha(c)$, there exists a unique horizontal lift $\widetilde{\alpha}$ of α such that $\widetilde{\alpha}(c) = p_0$.*

Proof. Let $c = t_0 < t_1 < \ldots < t_n = d$ be a partition such that $\alpha_{|[t_{i-1},t_i]}$ is a $\mathscr{C}^1$-curve. We can refine the partition so that $\alpha([t_{i-1}, t_i]) \subset U_i$ for some conformal chart (U_i, φ_i) of M for all i. Thus, we can assume that α is contained in such a chart (U, φ), so $\widetilde{\alpha}$ will be contained in $\pi^{-1}(U)$.

This allows us to work in the canonical example given in Definition 2.6 for $\Omega = \varphi(U)$ and some $\lambda, a, b \in \mathscr{C}^\infty(\varphi(U))$ with $\lambda > 0$. Writing in coordinates $\alpha(t) = (x(t), y(t)) \in \varphi(U)$, a horizontal lift of α must be of the form $\widetilde{\alpha}(t) = (x(t), y(t), z(t))$ for some $z : [c, d] \to \mathbb{R}$, and must satisfy $\langle \widetilde{\alpha}', \partial_z \rangle = 0$. This last condition can be developed as

$$z' = \lambda(x, y) \cdot (a(x, y)x' + b(x, y)y'). \tag{3-1}$$

Since $\pi(p_0) = \alpha(c)$, we have $p_0 = (x(c), y(c), z_0)$ for some $z_0 \in \mathbb{R}$. We deduce that there exists a unique $\mathscr{C}^1$-function $z(t)$ satisfying (3-1) with initial condition $z(c) = z_0$, so the horizontal lift exists and is unique. □

We can now give a geometric meaning of the bundle curvature in terms of the difference of heights of the endpoints of the horizontal lift of closed curves (see also [Daniel et al. 2009, Proposition 1.6.2]). Supposing that the fibers have infinite length will be necessary for the difference of heights to make sense.

Proposition 3.3. *Let $\pi : \mathbb{E} \to M$ be a Killing submersion whose fibers have infinite length. Given a simple piecewise $\mathscr{C}^1$-curve $\alpha : [c, d] \to M$ bounding an orientable relatively compact open set $G \subset M$ and a horizontal lift $\widetilde{\alpha}$ of α, we have*

$$\Big|\int_G \tau\Big| = \frac{h}{2},$$

where h is the length of the vertical segment joining $\widetilde{\alpha}(c)$ and $\widetilde{\alpha}(d)$.

Proof. Let us consider an atlas of M consisting of conformal charts. We will first suppose that $\overline{G}$ is contained in one of the charts (U, φ), so we can suppose that we are working in the canonical example given by Definition 2.6 for $\lambda, a, b \in \mathscr{C}^\infty(\varphi(U))$ with $\lambda > 0$. Moreover, (2-4) allows us to write τ as a divergence in $\varphi(G)$. The divergence theorem yields

$$\int_{\varphi(G)} 2\tau = \int_{\varphi(G)} \operatorname{div}\left(\frac{b}{\lambda}\,\partial_x - \frac{a}{\lambda}\,\partial_y\right) = \int_{\partial\varphi(G)} \left\langle \frac{b}{\lambda}\,\partial_x - \frac{a}{\lambda}\,\partial_y, \eta\right\rangle,$$

where η is the outer unit conormal to $\varphi(G)$ along its boundary. We write in coordinates $\alpha = (x, y)$ and $\widetilde{\alpha} = (x, y, z)$, and suppose α is parametrized by arc-length (i.e., $(x')^2 + (y')^2 = 1/\lambda^2$). Hence $\eta = -y'\,\partial_x + x'\,\partial_y$, up to a sign, so we deduce from (3-1) that

$$\left|\int_G 2\tau\right| = \left|\int_c^d \lambda\cdot(ax' + by')\right| = \left|\int_c^d z'\right| = |z(d) - z(c)|.$$

As $h = |z(d) - z(c)|$ in this model, we are done.

If $\overline{G}$ does not lie in a single chart, we can triangulate $\overline{G}$ by a finite number of triangles with piecewise $\mathscr{C}^1$ boundaries so each triangle is contained in a coordinate chart of the atlas (see, for instance, the proof of [Jost 2002, Theorem 2.3.A.1]) and α can be expressed as a finite sum of the boundaries of these triangles. As G is orientable, such boundaries can be oriented so that the interior ones cancel out in pairs. The argument above applied to each triangle together with the divergence theorem gives the desired result. □

Geodesics. Let $\pi : \mathbb{E} \to M$ be a Killing submersion. Given two vector fields $X, Y \in \mathfrak{X}(M)$, we can consider their horizontal lifts $\overline{X}, \overline{Y} \in \mathfrak{X}(\mathbb{E})$. Then the following equality holds (see [do Carmo 1992, pp. 185–187]):

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + [\overline{X}, \overline{Y}]^v, \tag{3-2}$$

where ∇ and $\overline{\nabla}$ are the Levi-Civita connections in M and $\mathbb{E}$, respectively, $\overline{\nabla_X Y}$ is the horizontal lift of $\nabla_X Y$ and $[\overline{X}, \overline{Y}]^v$ is the vertical part of $[\overline{X}, \overline{Y}]$.

From (3-2) we deduce that the horizontal lift of a geodesic in M is a geodesic in $\mathbb{E}$. Since not all geodesics are horizontal or vertical, we will need a slight improvement of this argument to classify them all.

Lemma 3.4. *Geodesics in $\mathbb{E}$ make a constant angle with a vertical Killing vector field ξ.*

Proof. Given a geodesic γ in $\mathbb{E}$, we can compute

$$\frac{d}{dt}\langle\gamma', \xi\rangle = \langle\overline{\nabla}_{\gamma'}\gamma', \xi\rangle + \langle\gamma', \overline{\nabla}_{\gamma'}\xi\rangle.$$

The first term on the right-hand side vanishes since γ is a geodesic, and the second one also vanishes because $\overline{\nabla}_{\gamma'}\xi = \tau\gamma' \wedge \xi$ (see Lemma 2.1). $\square$

Given a real number $\mu \in \mathbb{R}$ and a smooth curve $\alpha : [a,b] \to M$, we can consider the smooth curve

$$\gamma : [a,b] \to \mathbb{E}, \quad \gamma(t) = \phi_{\mu t}(\widetilde{\alpha}(t)), \tag{3-3}$$

where $\{\phi_t\}$ is the group of vertical translations associated to a unit vertical vector field ξ. The chain rule allows us to compute

$$\gamma'(t) = \mu\xi_{\gamma(t)} + (d\phi_{\mu t})_{\widetilde{\alpha}(t)}(\widetilde{\alpha}'(t)),$$

so γ makes a constant angle with ξ and will be our candidate to geodesic. Taking into account that $[\widetilde{\alpha}', \xi] = 0$ and (3-2), we get

$$\overline{\nabla}_{\gamma'}\gamma' = 2\mu\tau\widetilde{\alpha}' \wedge \xi + \overline{\nabla_{\alpha'}\alpha'}. \tag{3-4}$$

Let us suppose that α has unit speed and consider J the $\pm(\pi/2)$-rotation in TM (the sign will be chosen below). Then there exists a function $\kappa_g : [a,b] \to \mathbb{R}$, the geodesic curvature, such that $\nabla_{\alpha'}\alpha' = \kappa_g \cdot J\alpha'$. The horizontal lift of $J\alpha'$ is a horizontal and unitary vector field along $\widetilde{\alpha}$, orthogonal to $\widetilde{\alpha}'$. Hence, we can choose the sign of J so the horizontal lift of $J\alpha'$ is equal to $-\widetilde{\alpha}' \wedge \xi$. Now (3-4) implies that γ is a geodesic if and only if

$$\kappa_g(t) = 2\mu\,\tau(\alpha(t)). \tag{3-5}$$

Lemma 3.5. *Given $\mu \in \mathbb{R}$, $p \in M$ and $v \in T_pM$, there exist $\varepsilon > 0$ and a unique unit-speed smooth curve $\alpha :]{-\varepsilon}, \varepsilon[\to M$ such that $\alpha(0) = p$, $\alpha'(0) = v$ and satisfying* (3-5).

Moreover, if M is complete then α extends to the whole real line.

Proof. We will work in a conformal parametrization $\varphi : U \subset \mathbb{R}^2 \to M$ compatible with the orientation fixed above, where U is a neighborhood of p. Then we identify α with the coordinates $(x, y) = \varphi^{-1} \circ \alpha$. Since α has unit speed, there must exist a smooth function θ such that $x' = \lambda^{-1}\cos\theta$ and $y' = \lambda^{-1}\sin\theta$, where λ denotes the conformal factor. The geodesic curvature of α with respect to $J\alpha' = -y'\,\partial_x + x'\,\partial_y$ is given by

$$\kappa_g = \theta' + \frac{\lambda_x}{\lambda^2}\sin\theta - \frac{\lambda_y}{\lambda^2}\cos\theta.$$

Now, (3-5) becomes the first-order ODE system

$$\begin{cases} x' &= \dfrac{1}{\lambda(x,y)}\cos\theta, \\ y' &= \dfrac{1}{\lambda(x,y)}\sin\theta, \\ \theta' &= 2\mu\,\tau(x,y) - \dfrac{\lambda_x(x,y)}{\lambda(x,y)^2}\sin\theta + \dfrac{\lambda_y(x,y)}{\lambda^2(x,y)}\cos\theta. \end{cases} \tag{3-6}$$

The general theory of ODEs guarantees the existence of a unique smooth solution in a neighborhood of the origin when prescribing $\alpha(0)$, $\alpha'(0)$ (note that these initial data are equivalent to $x(0)$, $y(0)$ and $\theta(0)$). Observe that the solution can be extended as long as α is contained in U, so if M is complete and we take an atlas consisting of conformal parametrizations compatible with the orientation, then α extends to the whole real line. □

It is important to notice that the curve γ given by Lemma 3.5 satisfies $\|\gamma'\|^2 = 1+\mu^2$, so after a reparametrization by arc-length, we obtain $\langle\gamma', \xi\rangle = \mu/\sqrt{1+\mu^2}$. This last expression varies in $]-1,1[$ when $\mu \in \mathbb{R}$, so this construction covers all geodesics in $\mathbb{E}$, except for the vertical ones.

Proposition 3.6. *Given $p \in \mathbb{E}$, all geodesics in $\mathbb{E}$ passing through p are of one* (*and only one*) *of the following types*:

(1) *vertical geodesics* (*fibers of the submersion*),

(2) *horizontal lifts of geodesics in M passing through $\pi(p)$*,

(3) *of the form $\gamma(t) = \phi_{\mu t}(\tilde{\alpha}(t))$, where $\tilde{\alpha}$ is a horizontal lift of α in M such that $\alpha(0) = \pi(p)$ and satisfying* (3-5) *for some $\mu \neq 0$.*

In particular, *if M is complete*, *then so is $\mathbb{E}$.*

Remark 3.7. When the bundle curvature is constant, nonvertical geodesics project into curves with constant geodesic curvature. Moreover, the geodesic is horizontal if and only if its projection is also a geodesic. This gives an easy way to compute geodesics in the $\mathbb{E}(\kappa,\tau)$-spaces.

4. Existence results

When the base is simply connected, Theorem 2.8 gives a uniqueness result for Killing submersions; in this section we will investigate the existence problem and prove that we can fix beforehand any bundle curvature under the same assumption of simple connectedness.

Killing submersions over a disk. Given an open set $\Omega \subset \mathbb{R}^2$ and $\lambda, \tau \in \mathscr{C}^\infty(\Omega)$ with $\lambda > 0$, we wonder whether it is possible to solve for a and b in (2-4). An explicit way of doing so when Ω is star-shaped is given in the following lemma by just taking $\delta = 2\lambda^2\tau$.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}$ be open and star-shaped with respect to the origin, and $\delta \in \mathscr{C}^\infty(\Omega)$. Then $\eta \in \mathscr{C}^\infty(\Omega)$, given by*

$$\eta(x, y) = \int_0^1 s\,\delta(xs, ys)\,\mathrm{d}s,$$

satisfies the identity $\delta = (x\eta)_x + (y\eta)_y$.

Proof. It is a direct computation. □

Theorem 4.2. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set star-shaped with respect to the origin and $\lambda, \tau \in \mathscr{C}^\infty(\Omega)$ with $\lambda > 0$. If $\pi : \mathbb{E} \to (\Omega, \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2))$ is a Killing submersion with bundle curvature τ and $\mathbb{E}$ is simply connected, then it is isomorphic to the canonical example*

$$\pi_1 : (\Omega \times \mathbb{R}, \mathrm{d}s^2) \to (\Omega, \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2)), \quad \pi_1(x, y, z) = (x, y),$$
$$\mathrm{d}s^2 = \lambda(x, y)^2(\mathrm{d}x^2 + \mathrm{d}y^2) + (\mathrm{d}z + \eta(x, y)(y\,\mathrm{d}x - x\,\mathrm{d}y))^2,$$

where the function $\eta \in \mathscr{C}^\infty(\Omega)$ is given by

$$\eta(x, y) = 2\int_0^1 s\;\tau(xs, ys)\;\lambda(xs, ys)^2\,\mathrm{d}s. \tag{4-1}$$

Remark 4.3. Note that star-shapeness makes everything explicit but an existence and uniqueness theorem also holds in the (more general) simply connected case. It suffices to conformally parametrize such a simply connected domain by a disk and apply Theorem 4.2.

Remark 4.4. If we drop the condition that $\mathbb{E}$ is simply connected, it can be easily shown that any Killing submersion $\pi : \mathbb{E} \to \Omega$ is isomorphic to a Riemannian quotient of the Killing submersion constructed in Theorem 4.2 by a vertical translation. In particular, $\mathbb{E}$ is diffeomorphic to $\Omega \times \mathbb{S}^1$.

It is interesting to specialize Theorem 4.2 to the case $M = \mathbb{M}^2(\kappa)$, the complete simply connected surface with constant Gaussian curvature $\kappa \in \mathbb{R}$, to get models for all Killing submersions over $\mathbb{R}^2$, $\mathbb{H}^2(\kappa)$ and $\mathbb{S}^2(\kappa)$ minus a point. Given $\kappa \in \mathbb{R}$, we define $\lambda_\kappa \in \mathscr{C}^\infty(\Omega_\kappa)$ as

$$\lambda_\kappa(x, y) = \left(1 + \frac{\kappa}{4}(x^2 + y^2)\right)^{-1},$$

where

$$\Omega_k = \begin{cases} \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < -4/\kappa\} & \text{if } \kappa < 0, \\ \mathbb{R}^2 & \text{if } \kappa \geq 0. \end{cases}$$

Then the metric $\lambda_\kappa^2(\mathrm{d}x^2 + \mathrm{d}y^2)$ in Ω_κ has constant Gaussian curvature κ. If τ is constant, then $\eta = \tau\,\lambda_\kappa$ in (4-1), and we obtain the metrics of the spaces

$\mathbb{E}(\kappa, \tau) \equiv \Omega_\kappa \times \mathbb{R}$ given in [Daniel 2007, Section 2.3]:

$$\lambda_\kappa^2(\mathrm{d}x^2 + \mathrm{d}y^2) + (\mathrm{d}z + \tau\,\lambda_\kappa\,(y\,\mathrm{d}x - x\,\mathrm{d}y))^2.$$

Recall that we are not considering a whole fiber of a point in $\mathbb{S}^2(\kappa)$ for $\kappa > 0$. The global case will be treated in the next section.

Killing submersions over a 2-sphere. We can define Killing submersions over $\mathbb{S}^2$ as different as the Riemannian products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{S}^1$ (both with $\tau = 0$) or the Berger spheres and the lens spaces $L(n, 1)$ via the Hopf projection (see Remark 4.8 below). Throughout this section, we will suppose that the surface playing the role of base surface is $(\mathbb{S}^2, g)$ for some Riemannian metric g.

Unlike in the cases treated above, this surface is compact. Hence, given a Killing submersion $\pi : \mathbb{E} \to (\mathbb{S}^2, g)$ and its bundle curvature $\tau \in \mathscr{C}^\infty(M)$, the *total bundle curvature*

$$T = \int_M \tau$$

is well-defined and finite. This quantity will make the difference between the possible topologies of the total space.

The case $T = 0$.

Proposition 4.5. *Let $\pi : \mathbb{E} \to (\mathbb{S}^2, g)$ be a Killing submersion with total bundle curvature $T = 0$. Then the submersion admits a global smooth section.*

(a) *If the length of the fibers of π is infinite, then it is isomorphic to*

$$\pi_1 : (\mathbb{S}^2 \times \mathbb{R}, \mathrm{d}s^2) \to (\mathbb{S}^2, g), \quad \pi_1(p, t) = p,$$

for some Riemannian metric $\mathrm{d}s^2$ defined in $\mathbb{S}^2 \times \mathbb{R}$ and such that ∂_t is a unit vertical Killing vector field.

(b) *Otherwise, the Killing submersion is isomorphic to the Riemannian quotient of the example in (a) by some vertical translation.*

Proof. The condition $T = 0$ guarantees the existence of an equator $\Gamma \subset \mathbb{S}^2$ such that D_1 and D_2, the two open components of $\mathbb{S}^2 \setminus \Gamma$, satisfy

$$\int_{D_1} \tau = \int_{D_2} \tau = 0.$$

Let $\widetilde{\Gamma} \subset \mathbb{E}$ be any horizontal lift of Γ. If the fibers of π have infinite length, then Proposition 3.3 implies that $\widetilde{\Gamma}$ is a closed curve in $\mathbb{E}$. For $i \in \{1, 2\}$, as $\widetilde{\Gamma}$ lies in the boundary of $\pi^{-1}(\overline{D}_i)$ and projects one-to-one by π onto Γ, there exists a section $F_i : \overline{D}_i \to \mathbb{E}$ with $F_i(\Gamma) = \widetilde{\Gamma}$. Thus $F : \mathbb{S}^2 \to \mathbb{E}$ defined by $F = F_i$ in $\overline{D}_i$ is a global continuous section, and there is no loss of generality in supposing that F is smooth (just by perturbing it in a neighborhood of Γ). Then $\Psi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{E}$

given by $\Psi(p,t)=\phi_t(F(p))$ is a global diffeomorphism, where ϕ_t denotes the 1-parameter group of vertical isometries. The induced metric $\mathrm{d}s^2$ in $\mathbb{S}^2\times\mathbb{R}$ through Ψ satisfies the requirements of (a).

In the case that the length of the fibers of π is finite, we can work in the universal cover of $\pi^{-1}(\overline{D}_i)$, for $i\in\{1,2\}$, repeat the arguments above, and finally take a convenient quotient by a vertical translation. □

The rest of this section is devoted to obtaining explicit models for the metrics in $\mathbb{S}^2\times\mathbb{R}$ making the projection over the first factor a Killing submersion. This will show that Proposition 4.5 is sharp, but we will also obtain a quite general method for constructing Killing submersions.

Inspired by the canonical metrics in (2-2), let us consider arbitrary functions $a_1,\dots,a_n\in\mathscr{C}^\infty(\mathbb{R}^n)$ and the projection over the first n coordinates

$$\pi:(\mathbb{R}^{n+1},\mathrm{d}s^2)\to\mathbb{R}^n.$$

Here, we endow $\mathbb{R}^n$ with the usual metric and

$$\mathrm{d}s^2=\sum_{k=1}^n \mathrm{d}x_i^2+\left(\mathrm{d}t-\sum_{k=1}^n a_k\,\mathrm{d}x_k\right)^2, \tag{4-2}$$

where we denote by $(x_1,\dots,x_n,t)$ the usual coordinates of $\mathbb{R}^{n+1}$. Then π is a Riemannian submersion whose fibers are the integral curves of the unit Killing vector field ∂_t in $(\mathbb{R}^{n+1},\mathrm{d}s^2)$.

Given a smooth orientable surface Σ, we can isometrically embed it in $\mathbb{R}^n$ for some $n\in\mathbb{N}$ by the Nash embedding theorem [1956]. Then we shall consider the metric induced by (4-2) in $\Sigma\times\mathbb{R}\subset\mathbb{R}^{n+1}$. Obviously, π restricts to a Killing submersion $\Sigma\times\mathbb{R}\to\Sigma$. We will now compute its bundle curvature in terms of the functions a_k, but we will first need a convention for the orientation in $\Sigma\times\mathbb{R}$: if a local frame $\{e_1,e_2\}$ in Σ is positively oriented, then $\{E_1,E_2,\partial_t\}$ will be said positively oriented in $\Sigma\times\mathbb{R}$, where E_i is the horizontal lift of e_i for $i\in\{1,2\}$.

Proposition 4.6. *Let Σ be a smooth oriented surface isometrically embedded in $\mathbb{R}^n$. The Killing submersion $\Sigma\times\mathbb{R}\to\Sigma$ defined above has bundle curvature*

$$\tau=\tfrac{1}{2}\,\mathrm{div}_\Sigma(JT),$$

where $T=(\partial_t)^\top\in\mathfrak{X}(\Sigma)$ is the component of ∂_t tangent to $\Sigma\equiv\Sigma\times\{0\}\subset\mathbb{R}^{n+1}$ with respect to $\mathrm{d}s^2$, *and $J:\mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ is a $(\pi/2)$-rotation in $T\Sigma$.*

Proof. Let $X:\Omega\subset\mathbb{R}^2\to\Sigma$ be a local conformal parametrization of Σ with conformal factor $\lambda\in\mathscr{C}^\infty(\Omega)$, and such that $\{(1/\lambda)X_u,(1/\lambda)X_v\}$ is a positively oriented orthonormal frame of $T\Sigma$. Let $\{E_1,E_2\}\subset\mathfrak{X}(X(\Omega)\times\mathbb{R})$ be a horizontal lift of the frame $\{(1/\lambda)X_u,(1/\lambda)X_v\}$ which, together with $E_3=\partial_t$, is a positively oriented orthonormal frame in $X(\Omega)\times\mathbb{R}$. As in (2-4), we can compute τ from the identity $2\tau=\langle[E_1,E_2],E_3\rangle$. Note that there exist $f,g\in\mathscr{C}^\infty(X(\Omega))$ such that $E_1=(1/\lambda)X_u+f\,\partial_t$ and $E_2=(1/\lambda)X_v+g\,\partial_t$, so

$$\begin{aligned}[E_1,E_2]&=\left[\frac{1}{\lambda}X_u,\frac{1}{\lambda}X_v\right]+\left[\frac{1}{\lambda}X_u,g\,\partial_t\right]+\left[f\,\partial_t,\frac{1}{\lambda}X_u\right]+[f\,\partial_t,g\,\partial_t]\\&=\frac{1}{\lambda^3}(\lambda_vX_u-\lambda_uX_v)+\frac{1}{\lambda}(g_u-f_v)\,\partial_t.\end{aligned}$$

Moreover, since $0=\langle E_1,E_3\rangle=\langle(1/\lambda)X_u+f\,\partial_t,\partial_t\rangle$, we deduce that $\langle X_u,\partial_t\rangle=-\lambda f$ and, analogously, $\langle X_v,\partial_t\rangle=-\lambda g$. Hence,

$$2\tau=\langle[E_1,E_2],E_3\rangle=\frac{1}{\lambda^2}((\lambda g)_u-(\lambda f)_v)=\operatorname{div}_\Sigma(Y),$$

where $Y\in\mathfrak{X}(\Sigma)$ is the vector field $(g/\lambda)X_u-(f/\lambda)X_v$. From here, it is easy to check that $Y=JT$ and we are done. □

Remark 4.7. If Σ is compact, then $\int_\Sigma\tau=(1/2)\int_\Sigma\operatorname{div}(JT)=0$ as an application of the divergence theorem. Conversely, every function on a compact orientable surface Σ with zero integral is well-known to be the divergence of some vector field on Σ.

As a particular case, we may consider the round sphere

$$\mathbb{S}^2(\kappa)=\left\{(x,y,z)\in\mathbb{R}^3:x^2+y^2+z^2=\frac{1}{\kappa}\right\}\subset\mathbb{R}^3,$$

and endow $\mathbb{S}^2\times\mathbb{R}\subset\mathbb{R}^4$ with the metric given by (4-2) for $n=3$ and some $a_1,a_2,a_3\in\mathscr{C}^\infty(\mathbb{R}^3)$. The stereographic projection $X:\mathbb{R}^2\to\mathbb{S}^2(\kappa)\backslash\{(0,0,1/\sqrt{\kappa})\}$ defined by

$$X(u,v)=\left(\frac{2u}{\kappa(u^2+v^2)+1},\frac{2v}{\kappa(u^2+v^2)+1},\frac{1}{\sqrt{\kappa}}\frac{\kappa(u^2+v^2)-1}{\kappa(u^2+v^2)+1}\right)\tag{4-3}$$

allows us to work out the bundle curvature τ of the induced Killing submersion $\mathbb{S}^2(\kappa)\times\mathbb{R}\to\mathbb{S}^2(\kappa)$ as in the proof of Proposition 4.6. We get

$$2\tau=\sqrt{\kappa}((ya_3-za_2)_x+(za_1-xa_3)_y+(xa_2-ya_1)_z).$$

The case $T\neq0$. Let us consider the 3-sphere

$$\mathbb{S}^3=\{(z,w)\in\mathbb{C}^2:|z|^2+|w|^2=1\}\subset\mathbb{C}^2,$$

and $\mathbb{S}^2(\kappa)=\{(z,t):|z|^2+t^2=1/\kappa\}\subset\mathbb{C}\times\mathbb{R}$ for $\kappa>0$. The submersion

$$\pi_{\mathrm{Hopf}}:\mathbb{S}^3\to\mathbb{S}^2(\kappa),\quad (z,w)\mapsto\frac{1}{\sqrt{\kappa}}(2z\bar{w},|z|^2-|w|^2),\tag{4-4}$$

is known as the *Hopf projection*. The fiber passing through $(z,w)\in\mathbb{S}^3$ is given by $\{(e^{it}z, e^{it}w) : t \in \mathbb{R}\}$ and the orbit of a point under the 1-parameter group of diffeomorphisms

$$\phi_t(z, w) = (e^{it}z, e^{it}w), \quad t \in \mathbb{R},$$

coincides with its fiber with respect to the submersion.

Remark 4.8. Given a natural number $n \in \mathbb{N}$, we can consider the quotient of $\mathbb{S}^3$ under the group of diffeomorphisms $G_n = \{\phi_{2\pi k/n} : k \in \{1, \dots, n\}\}$, which is cyclic and has order n. The quotient $\mathbb{S}^3/G_n$ is known as the lens space $L(n, 1)$. The condition $\pi_{\mathrm{Hopf}} \circ \phi_t = \pi_{\mathrm{Hopf}}$ guarantees that π_{Hopf} induces a submersion $\pi_n : L(n, 1) \to \mathbb{S}^2(\kappa)$. Observe that, for any $n \in \mathbb{N}$, the space $L(n, 1)$ is orientable and its fundamental group is isomorphic to the cyclic group of order n, so two lens spaces $L(n, 1)$ and $L(m, 1)$ are not homeomorphic for $m \neq n$ (see [Saveliev 1999] for a more detailed description).

If we endow $\mathbb{S}^3$ with a metric making π_{Hopf} a Killing submersion, then the fibers of π_{Hopf} have finite length (they are compact) and it is easy to check that π_n is a Killing submersion when we consider the quotient metric, for all n. Moreover, the length of the fibers of π_{Hopf} in $\mathbb{S}^3$ is n times the length of the corresponding fibers of π_n in $L(n, 1)$.

Proposition 4.9. *Let $\pi : \mathbb{E} \to (\mathbb{S}^2, g)$ be a Killing submersion with total bundle curvature $T \neq 0$. Then there exists $n \in \mathbb{N}$ such that the length of the fibers is equal to $|2T|/n$.*

(a) *If $n = 1$, then $\pi : \mathbb{E} \to (\mathbb{S}^2, g)$ is isomorphic to the Hopf fibration*

$$\pi_{\mathrm{Hopf}} : (\mathbb{S}^3, \mathrm{d}s^2) \to (\mathbb{S}^2, g), \quad \pi_{\mathrm{Hopf}}(z, w) = (2z\bar{w}, |z|^2 - |w|^2),$$

for some Riemannian metric $\mathrm{d}s^2$ in $\mathbb{S}^3$ such that $\xi_{(z,w)} = (\pi/T)(iz, iw)$ is a unit Killing vector field.

(b) *If $n > 1$, then $\pi : \mathbb{E} \to (\mathbb{S}^2, g)$ is isomorphic to the Riemannian quotient of a submersion as in* (a) *by a vertical translation of length $|2T|/n$.*

Proof. As in the proof of Proposition 4.5, let us take a geodesic Γ which divides $\mathbb{S}^2$ in two hemispheres D_1 and D_2 such that

$$\int_{D_1} \tau = \int_{D_2} \tau = \frac{T}{2}.$$

We parametrize Γ as $\gamma : [a, b] \to \mathbb{S}^2$ and a horizontal lift $\widetilde{\Gamma}$ of Γ as $\widetilde{\gamma} : [a, b] \to \mathbb{E}$. The universal Riemannian covering space of $\pi^{-1}(\overline{D}_i)$, for $i \in \{1, 2\}$, will be denoted by $W_i \equiv \overline{D}_i \times \mathbb{R}$, and is a closed solid cylinder. The curve $\widetilde{\Gamma}$ can be lifted to both W_1 and W_2. Since the outer conormal vector fields to $\overline{D}_1$ and $\overline{D}_2$ along their boundary have opposite directions, the difference of heights between $\widetilde{\gamma}(a)$ and $\widetilde{\gamma}(b)$ when we consider them in W_1 or W_2 is equal to $|T|$, but they have opposite signs (see

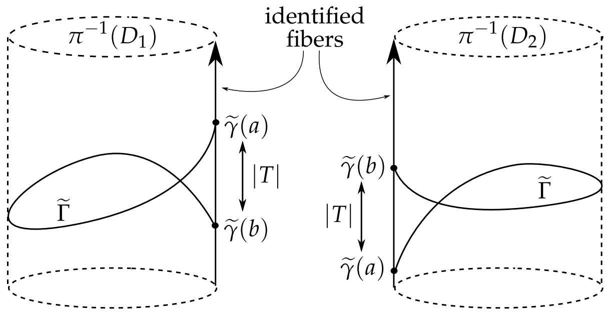

Figure 2. The curve $\widetilde{\Gamma}$ is represented in the solid cylinders W_1 and W_2 covering $\pi^{-1}(D_1)$ and $\pi^{-1}(D_2)$, respectively, and its endpoints lie on the vertical geodesic containing the vertical arrow representing a global vertical Killing vector field. After gluing along this geodesic, we conclude that the length of the fibers is an integer divisor of $|2T|$.

the proof of Proposition 3.3). In other words, we will arrive at $\widetilde{\gamma}(b)$ after traveling vertically from $\widetilde{\gamma}(a)$ a distance of $|T|$, and, if we continue from $\widetilde{\gamma}(b)$, we will arrive again at $\widetilde{\gamma}(a)$ after the same distance (see Figure 2). Thus, the length of the fibers is an integer divisor of $|2T|$. In particular, $\pi^{-1}(\overline{D}_1)$ and $\pi^{-1}(\overline{D}_2)$ are solid tori.

Now, observe that the curve $\widetilde{\Gamma}$ determines how $\pi^{-1}(\overline{D}_1)$ and $\pi^{-1}(\overline{D}_2)$ must be glued together, and $\widetilde{\Gamma}$ turns n times in the vertical direction, so we can work in a n-sheet vertical covering space of both tori where $\widetilde{\Gamma}$ will look like Figure 2 after identifying the top and bottom faces of the cylinders. This way of gluing the two tori along Γ provides a manifold diffeomorphic to $\mathbb{S}^3$, and the induced fibration is the Hopf fibration (see [Saveliev 1999]). By pulling the metric in $\mathbb{E}$ back via this diffeomorphism, (a) in the statement follows. Item (b) is also proved since we only need to undo the covering space procedure by taking a quotient with respect to a vertical translation of length $|2T|/n$. □

We can now combine the local existence given by Theorem 4.2 with Propositions 4.5 and 4.9 to obtain a description of *all* Killing submersions over a Riemannian 2-sphere.

Theorem 4.10. *Let g be a Riemannian metric on $\mathbb{S}^2$ and $\tau \in \mathscr{C}^\infty(\mathbb{S}^2)$. Up to isomorphism, there exists a unique Killing submersion over $(\mathbb{S}^2, g)$ with bundle curvature τ and whose total space is simply connected.*

Proof. The uniqueness is a consequence of Theorem 2.8 and the description of the length of the fibers in Propositions 4.5 and 4.9. We will now assume that $T = \int_{(\mathbb{S}^2,g)} \tau \neq 0$ (the case $T = 0$ is similar) and prove its existence.

Consider an equator $\Gamma \subset \mathbb{S}^2$ splitting $\mathbb{S}^2$ in two hemispheres D_1 and D_2. By applying Theorem 4.2 in a neighborhood of $\overline{D}_1$ and $\overline{D}_2$, we obtain Killing submersions π_1 and π_2 over such neighborhoods with the desired bundle curvature and noncompact fibers. The argument in Proposition 4.9 guarantees that, after taking the quotient by vertical translations of length $|2T|$, the two submersions can be glued together along $\pi^{-1}(\Gamma)$ to produce a (continuous) submersion $\pi : \mathbb{S}^3 \to \mathbb{S}^2$. In order to prove that π is smooth along $\pi^{-1}(\Gamma)$, observe that both π_1 and π_2 are defined in a neighborhood of Γ where they share the same bundle curvature. Thus they locally coincide by Theorem 2.8 in a neighborhood of each $p \in \pi^{-1}(\Gamma)$. $\square$

In the previous section, we showed a constructive method to obtain trivial Killing submersions in a global way. Now, we will do the same for Killing submersions with $T \neq 0$ for round spheres $\mathbb{S}^2(\kappa)$ as base surfaces, though the method can be also adapted to the case $T = 0$.

Let us consider the Hopf fibration given by (4-4) and the global frame in $\mathbb{S}^3 \subset \mathbb{C}^2$ defined by

$$(E_1)_{(z,w)} = (-\bar{w}, \bar{z}), \quad (E_2)_{(z,w)} = (-i\bar{w}, i\bar{z}), \quad (E_3)_{(z,w)} = (iz, iw).$$

This frame is orthonormal when we endow $\mathbb{S}^3$ with the round metric of curvature one. Let $\tau \in \mathscr{C}^\infty(\mathbb{S}^2(\kappa))$ be a function with integral $T \neq 0$. Note that τ induces a function in $\tilde{\tau} \in \mathscr{C}^\infty(\mathbb{R}^2)$ via the stereographic projection given by (4-3). Theorem 4.2 allows us to construct a Killing submersion over $\mathbb{S}^2(\kappa) \setminus \{(0, 0, 1/\sqrt{\kappa})\}$ with bundle curvature $\tilde{\tau}$. To do this, we calculate the associated function $\tilde{\eta} \in \mathscr{C}^\infty(\mathbb{R}^2)$ given by

$$\tilde{\eta}(x, y) = 2 \int_0^1 \frac{s \cdot \tilde{\tau}(sx, sy)}{(1 + (\kappa/4)s^2(x^2 + y^2))^2}\, \mathrm{d}s,$$

which extends smoothly to infinity since $\tilde{\tau}$ extends smoothly to infinity, and thus induces $\eta \in \mathscr{C}^\infty(\mathbb{S}^2(\kappa))$ by pulling back via the stereographic projection again. Hence this construction induces a Riemannian metric in $\mathbb{S}^3$ minus the fiber of $(0, 0, 1/\sqrt{\kappa})$ but can be extended to the whole $\mathbb{S}^3$. It can be shown that this metric in $\mathbb{S}^3$ is the determined by the fact that

$$\begin{aligned}
Y_1 &= \frac{\sqrt{\kappa}}{2} E_1 - \frac{\operatorname{Im}(zw)\big(\kappa T|w|^2 - 4\pi\eta(\pi_{\mathrm{Hopf}}(z, w))\big)}{2\pi\sqrt{\kappa}|w|^4} E_3,\\
Y_2 &= \frac{\sqrt{\kappa}}{2} E_2 + \frac{\operatorname{Im}(zw)\big(\kappa T|w|^2 - 4\pi\eta(\pi_{\mathrm{Hopf}}(z, w))\big)}{2\pi\sqrt{\kappa}|w|^4} E_3,\\
Y_3 &= \frac{\pi}{T} E_3,
\end{aligned}$$

defines a global orthonormal frame. If τ is constant, then $\kappa T = 4\pi\tau$ and

$$\eta(\pi_{\mathrm{Hopf}}(z, w)) = |w|^2\tau,$$

so the coefficients of E_3 in Y_1 and Y_2 vanish, and we get the metrics of the Berger spheres given by Torralbo [2012].

5. Characterization of homogeneous Killing submersions

Recall that a Riemannian manifold is said to be *homogeneous* when its isometry group acts transitively on the manifold. In this section, we will characterize the $\mathbb{E}(\kappa,\tau)$-spaces as the only simply connected homogeneous 3-manifolds admitting the structure of a Killing submersion.

In order to obtain this result, we will compute the Riemannian curvature of the total space $\mathbb{E}$ of a Killing submersion $\pi:\mathbb{E}\to M$ in terms of M and the bundle curvature τ. Since the computation is purely local, we will work in a canonical example (see Definition 2.6) associated to some functions $\lambda, a, b\in\mathscr{C}^\infty(\Omega)$ with $\lambda>0$ and $\Omega\subset\mathbb{R}^2$ (a different approach can be found in [Espinar and de Oliveira 2013]). The Koszul formula yields the Levi-Civita connection in the canonical orthonormal frame $\{E_1,E_2,E_3\}$ given by (2-3):

$$\begin{aligned}
&\overline{\nabla}_{E_1}E_1=-\frac{\lambda_y}{\lambda^2}E_2, && \overline{\nabla}_{E_1}E_2=\frac{\lambda_y}{\lambda^2}E_1+\tau E_3, && \overline{\nabla}_{E_1}E_3=-\tau E_2,\\
&\overline{\nabla}_{E_2}E_1=\frac{\lambda_x}{\lambda^2}E_2-\tau E_3, && \overline{\nabla}_{E_2}E_2=-\frac{\lambda_x}{\lambda^2}E_1, && \overline{\nabla}_{E_2}E_3=\tau E_1,\\
&\overline{\nabla}_{E_3}E_1=-\tau E_2, && \overline{\nabla}_{E_3}E_2=\tau E_1, && \overline{\nabla}_{E_3}E_3=0.
\end{aligned}\tag{5-1}$$

Since the Gaussian curvature K_M of M can be written in terms of the conformal factor as

$$K_M=-\frac{\Delta_0(\log\lambda)}{\lambda^2}=\frac{\lambda_x^2+\lambda_y^2}{\lambda^4}-\frac{\lambda_{xx}+\lambda_{yy}}{\lambda^3},$$

it is easy to work out any sectional curvature in $\mathbb{E}$.

Lemma 5.1. *Let $\pi:\mathbb{E}\to M$ be a Killing submersion and $p\in\mathbb{E}$. Given a linear plane $\Pi\subseteq T_p\mathbb{E}$ with normal vector $N\in T_p\mathbb{E}$, its sectional curvature is*

$$K(\Pi)=\nu^2(K_M-3\tau^2)+(1-\nu^2)\tau^2-2\nu\langle N\wedge\xi_p,(\overline{\nabla}\tau)_p\rangle,$$

where $\nu=\langle N,\xi_p\rangle$, ξ denotes the unit Killing vector field, K_M is the Gaussian curvature of M at $\pi(p)$, and τ is the bundle curvature at p.

The sectional curvature is $K_M-3\tau^2$ for horizontal planes (i.e., planes which are orthogonal to ξ) and τ^2 for vertical planes (i.e., planes containing the direction ξ). In particular, we deduce that hyperbolic 3-space, $\mathbb{H}^3$, does not admit a Killing submersion structure since $\mathbb{H}^3$ has constant sectional curvature of -1 and vertical planes in a Killing submersion always have nonnegative sectional curvature.

On the other hand, given $v \in T_p\mathbb{E}$ with $\|v\| = 1$, the Ricci curvature of v can be easily deduced from Lemma 5.1 as

$$\mathrm{Ric}(v) = (K_M - 2\tau^2) - \langle v, \xi_p\rangle^2 (K_M - 4\tau^2) + 2\langle v, \xi_p\rangle\langle v \wedge \xi_p, \overline{\nabla}\tau\rangle. \tag{5-2}$$

The scalar curvature is $\rho = 2(K_M - \tau^2)$.

Theorem 5.2. *Let $\pi : \mathbb{E} \to M$ be a Killing submersion. If $\mathbb{E}$ is homogeneous, then both the Gaussian curvature of M and the bundle curvature are constant. In particular, $\mathbb{E}$ is a $\mathbb{E}(\kappa, \tau)$-space or its quotient by a vertical translation.*

Proof. Given $p \in \mathbb{E}$ and $v \in T_p\mathbb{E}$ with $\|v\| = 1$, we can decompose $v = u + \sigma\xi_p$, where u is horizontal and $\sigma \in \mathbb{R}$. From (5-2), we get

$$\mathrm{Ric}(v) = (K_M - 2\tau^2) + \sigma \cdot \big(\langle u \wedge \xi_p, (\overline{\nabla}\tau)_p\rangle - (K_M - 4\tau^2)\sigma\big).$$

Let $U_p = \{v \in T_p\mathbb{E} : \|v\| = 1\}$ and $A_p = \{v \in U_p : \mathrm{Ric}(v) = K_M - 2\tau^2\}$. Observe that the vectors $v \in U_p$ satisfying $\sigma = 0$ form a great circle and the same happens for $\langle u \wedge \xi_p, (\overline{\nabla}\tau)_p\rangle - (K_M - 4\tau^2)\sigma = 0$ if $(\overline{\nabla}\tau)_p \neq 0$ or $K_M \neq 4\tau^2$. We deduce

$$A_p = \begin{cases} U_p & \text{if } K_M = 4\tau^2 \text{ and } (\overline{\nabla}\tau)_p = 0, \\ \text{a great circle} & \text{if } K_M \neq 4\tau^2 \text{ and } (\overline{\nabla}\tau)_p = 0, \\ \text{two great circles} & \text{if } (\overline{\nabla}\tau)_p \neq 0. \end{cases} \tag{5-3}$$

Let $f : \mathbb{E} \to \mathbb{E}$ be an isometry. Since any two great circles in a sphere intersect and $\mathrm{d}f_p$ maps great circles in U_p to great circles in $U_{f(p)}$, we deduce that $\mathrm{d}f_p(A_p)$ and $A_{f(p)}$ intersect. As a consequence, $K_M - 2\tau^2$ attains the same value at the points p and $f(p)$. If $\mathbb{E}$ is homogeneous, then this implies that $K_M - 2\tau^2$ is constant, but on the other hand, the scalar curvature $2(K_M - \tau^2)$ is also constant; hence both K_M and τ are constant. □

Remark 5.3. Given a 3-dimensional metric Lie group G (i.e., it is endowed with a left-invariant metric) with isometry group of dimension 3, it is homogeneous. We deduce that the set of points where a Killing vector field (i.e., a right-invariant vector field) is unitary has empty interior. Otherwise, this open subset would be locally isometric to a $\mathbb{E}(\kappa, \tau)$-space, and this is impossible (see [Meeks and Pérez 2012] for a detailed description of metric Lie groups).

Finally, let us mention that the condition $K_M = 4\tau^2$ does not imply that $\mathbb{E}$ has constant sectional curvature (unless τ is constant), but it says that horizontal and vertical planes have the same sectional curvature. Note that, if $(\nabla\tau)_p \neq 0$ and $K_M = 4\tau^2$ at some $p \in M$, then the set A_p in (5-3) consists of two *orthogonal* great circles in the unit sphere U_p.

Acknowledgements

I would like to express my gratitude to Hojoo Lee, Pablo Mira and Joaquín Pérez for suggesting some improvements in the preparation of this paper. I would also like to thank Luis Guijarro and Carlos Ivorra for showing me some useful references.

References

[Abresch and Rosenberg 2005] U. Abresch and H. Rosenberg, "Generalized Hopf differentials", *Mat. Contemp.* **28** (2005), 1–28. MR 2006h:53004 Zbl 1118.53036

[Besse 2008] A. L. Besse, *Einstein manifolds*, Springer, Berlin, 2008. MR 2008k:53084 Zbl 1147.53001

[do Carmo 1992] M. P. do Carmo, *Riemannian geometry*, Birkhäuser, Boston, 1992. MR 92i:53001 Zbl 0752.53001

[Dajczer and de Lira 2009] M. Dajczer and J. H. de Lira, "Killing graphs with prescribed mean curvature and Riemannian submersions", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:3 (2009), 763–775. MR 2010f:53093 Zbl 1169.53046

[Dajczer and de Lira 2012] M. Dajczer and J. H. de Lira, "Conformal Killing graphs with prescribed mean curvature", *J. Geom. Anal.* **22**:3 (2012), 780–799. MR 2927678 Zbl 1261.53058

[Daniel 2007] B. Daniel, "Isometric immersions into 3-dimensional homogeneous manifolds", *Comment. Math. Helv.* **82**:1 (2007), 87–131. MR 2008a:53058 Zbl 1123.53029

[Daniel et al. 2009] B. Daniel, L. Hauswirth, and P. Mira, "Homogeneous 3-manifolds", Lecture notes from the 4th KIAS Workshop on Differential Geometry, Korea Institute for Advanced Study, Seoul, 2009.

[Espinar and de Oliveira 2013] J. M. Espinar and I. S. de Oliveira, "Locally convex surfaces immersed in a Killing submersion", *Bull. Braz. Math. Soc.* (*N.S.*) **44**:1 (2013), 155–171. MR 3077638 Zbl 1270.53080

[Greub et al. 1976] W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology, III: Cohomology of principal bundles and homogeneous spaces*, Pure and Applied Mathematics **47**, Academic Press, New York, 1976. MR 53 #4110 Zbl 0372.57001

[Javaloyes et al. 2013] M. A. Javaloyes, L. Lichtenfelz, and P. Piccione, "Almost isometries of non-reversible metrics with applications to stationary spacetimes", preprint, 2013. arXiv 1205.4539

[Jost 2002] J. Jost, *Compact Riemann surfaces: an introduction to contemporary mathematics*, 2nd ed., Springer, Berlin, 2002. MR 2003h:32020 Zbl 1086.30038

[Leandro and Rosenberg 2009] C. Leandro and H. Rosenberg, "Removable singularities for sections of Riemannian submersions of prescribed mean curvature", *Bull. Sci. Math.* **133**:4 (2009), 445–452. MR 2011a:53105 Zbl 1172.53038

[Lee and Manzano 2013] H. Lee and J. M. Manzano, "Generalized Calabi's correspondence and complete spacelike surfaces", preprint, 2013. arXiv 1301.7241

[Manzano and Torralbo 2012] J. M. Manzano and F. Torralbo, "New examples of constant mean curvature surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$", preprint, 2012. To appear in *Michigan Math. J.* arXiv 1104.1259

[Meeks and Pérez 2012] W. H. Meeks, III and J. Pérez, "Constant mean curvature surfaces in metric Lie groups", pp. 25–110 in *Geometric analysis: partial differential equations and surfaces* (Granada, 2010), edited by J. Pérez and J. A. Gálvez, Contemp. Math. **570**, Amer. Math. Soc., Providence, RI, 2012. MR 2963596 Zbl 1267.53006

[Meroño and Ortiz 2014] M. A. Meroño and I. Ortiz, "First stability eigenvalue characterization of CMC Hopf tori into Riemannian Killing submersions", *J. Math. Anal. Appl.* **417**:1 (2014), 400–410. MR 3191435

[Nash 1956] J. Nash, "The imbedding problem for Riemannian manifolds", *Ann. of Math.* (2) **63** (1956), 20–63. MR 17,782b Zbl 0070.38603

[Rosenberg et al. 2010] H. Rosenberg, R. Souam, and E. Toubiana, "General curvature estimates for stable H-surfaces in 3-manifolds and applications", *J. Differential Geom.* **84**:3 (2010), 623–648. MR 2011g:53015 Zbl 1198.53062

[Saveliev 1999] N. Saveliev, *Lectures on the topology of* 3*-manifolds: an introduction to the Casson invariant*, de Gruyter, Berlin, 1999. MR 2001h:57024 Zbl 0932.57001

[Souam and Van der Veken 2012] R. Souam and J. Van der Veken, "Totally umbilical hypersurfaces of manifolds admitting a unit Killing field", *Trans. Amer. Math. Soc.* **364**:7 (2012), 3609–3626. MR 2901226 Zbl 1277.53021

[Steenrod 1951] N. Steenrod, *The topology of fibre bundles*, Princeton Mathematical Series **14**, Princeton University Press, Princeton, NJ, 1951. MR 12,522b Zbl 0054.07103

[Torralbo 2012] F. Torralbo, "Compact minimal surfaces in the Berger spheres", *Ann. Global Anal. Geom.* **41**:4 (2012), 391–405. MR 2897028 Zbl 1242.53076

Received June 6, 2013.

JOSÉ M. MANZANO
DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
UNIVERSIDAD DE GRANADA
18071 GRANADA
SPAIN
jmmanzano@ugr.es

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.393

LOCALLY LIPSCHITZ CONTRACTIBILITY OF ALEXANDROV SPACES AND ITS APPLICATIONS

AYATO MITSUISHI AND TAKAO YAMAGUCHI

We prove that any finite-dimensional Alexandrov space with a lower curvature bound is locally Lipschitz contractible. As an application, we obtain a sufficient condition for solving the Plateau problem in an Alexandrov space, as considered by Mese and Zulkowski.

1. Introduction

Alexandrov spaces appear naturally in the collapsing and convergence theory of Riemannian manifolds and play an important role in Riemannian geometry. In this paper, when we say simply an Alexandrov space, we mean an Alexandrov space *of curvature bounded from below locally* and *of finite dimension*. The fundamental properties of such spaces were well studied in [Burago et al. 1992]. Perelman [1991] carried out a remarkable study of topological structures for Alexandrov spaces, proving a topological stability theorem: if two compact Alexandrov spaces of the same dimension are very close in the Gromov–Hausdorff topology, then they are homeomorphic. See also [Kapovitch 2007]. This further implies that, for any point in an Alexandrov space, its small open ball is homeomorphic to its tangent cone. In particular, an open ball of small radius with respect to its center is contractible. It is expected by geometers that corresponding statements obtained by replacing homeomorphic by bi-Lipschitz homeomorphic could be proved. Until now, we did not know any Lipschitz structure of an Alexandrov space around singular points. The main purpose of this paper is to prove that any finite-dimensional Alexandrov space with a lower curvature bound is *strongly locally Lipschitz contractible* in the sense defined later. For short, SLLC denotes this property. The SLLC-condition is a strong version of the LLC-condition introduced in [Yamaguchi 1997] (see Remark 4.5).

We define strongly locally Lipschitz contractibility. We denote by $U(p, r)$ an open ball centered at p of radius r in a metric space.

MSC2010: 53C20, 53C21, 53C23.
Keywords: Alexandrov space, Lipschitz contractibility.

Definition 1.1. A metric space X is *strongly locally Lipschitz contractible*, or SLLC, if for every point $p \in X$, there exists $r > 0$ and a map

$$h : U(p, r) \times [0, 1] \to U(p, r)$$

such that h is a homotopy from $h(\,\cdot\,, 0) = \mathrm{id}_{U(p,r)}$ to $h(\,\cdot\,, 1) = p$ and h is Lipschitz (i.e., there exists $C, C' > 0$ such that

$$d(h(x, s), h(y, t)) \le C d(x, y) + C'|s - t|$$

for every $x, y \in U(p, r)$ and $s, t \in [0, 1]$) and such that for every $r' < r$, the image of h restricted to $U(p, r') \times [0, 1]$ is $U(p, r')$.

We call such a ball $U(p, r)$ a Lipschitz contractible ball and h a Lipschitz contraction on $U(p, r)$.

A main result in the present paper is the following.

Theorem 1.2. *Any finite-dimensional Alexandrov space is strongly locally Lipschitz contractible.*

In [Yamaguchi 1997], a weaker form of Theorem 1.2 was conjectured.

For metric spaces P and X and possibly empty subsets $Q \subset P$ and $A \subset X$, we denote by $f : (P, Q) \to (X, A)$ a map from P to X with $f(Q) \subset A$. Two maps f and g from (P, Q) to (X, A) are *homotopic* (resp. *Lipschitz homotopic*) if there exists a continuous (resp. Lipschitz) map

$$h : (P \times [0, 1], Q \times [0, 1]) \to (X, A)$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in P$. Then, we write $f \sim g$ (resp. $f \sim_{\mathrm{Lip}} g$). Let us denote by

$$[(P, Q), (X, A)] \quad \text{and} \quad [(P, Q), (X, A)]_{\mathrm{Lip}}$$

respectively the set of all homotopy classes of continuous maps from (P, Q) to (X, A) and the set of all Lipschitz homotopy classes of Lipschitz maps from (P, Q) to (X, A).

Let us consider a *Lipschitz simplicial complex*: a metric space which admits a triangulation such that each simplex is a bi-Lipschitz image of a simplex in a Euclidean space. For a precise definition, see Section 4.

Corollary 1.3. *Let P be a finite Lipschitz simplicial complex and Q a possibly empty subcomplex of P. Let X be an Alexandrov space and A an open subset of X. Then, the natural map from* $[(P, Q), (X, A)]_{\mathrm{Lip}}$ *to* $[(P, Q), (X, A)]$ *is bijective.*

For a metric space X, a point $x_0 \in X$, and $k \in \mathbb{N}$, we define *the k-th Lipschitz homotopy group* $\pi_k^{\mathrm{Lip}}(X, x_0)$ by setting $\pi_k^{\mathrm{Lip}}(X, x_0) = [(S^k, *), (X, x_0)]_{\mathrm{Lip}}$ as sets, where $* \in S^k$ is an arbitrary point; it is equipped with the group operation of the usual homotopy groups.

Corollary 1.4. *For an Alexandrov space X, a point $x_0 \in X$, and $k \in \mathbb{N}$, the natural map*

$$\pi_k^{\mathrm{Lip}}(X, x_0) \to \pi_k(X, x_0).$$

is an isomorphism of groups.

Application: the Plateau problem. The Plateau problem in an Alexandrov space was considered by Mese and Zulkowski [2010] as follows. Let $W^{1,2}(D^2, X)$ denote the $(1, 2)$-Sobolev space from D^2 to an Alexandrov space X, in the sense of the Sobolev space of a metric space target defined by Korevaar and Schoen [1993]. Giving a closed Jordan curve Γ in X, we set

$$\mathcal{F}_\Gamma := \big\{ u \in W^{1,2}(D^2, X) \cap C(D^2, X) \;\big|\; u|_{\partial D^2} \text{ parametrizes } \Gamma \text{ monotonically} \big\}.$$

Mese and Zulkowski defined the area $A(u)$ of a Sobolev map $u \in W^{1,2}(D^2, X)$. Under these settings, the Plateau problem is stated as follows:

Find a map $u \in W^{1,2}(D^2, X)$ such that $A(u) = \inf\{A(v) \mid v \in \mathcal{F}_\Gamma\}$.

Theorem 1.5 [Mese and Zulkowski 2010]. *Let X be a finite-dimensional compact Alexandrov space and Γ a closed Jordan curve in X. If $\mathcal{F}_\Gamma \neq \varnothing$, then there exists a solution to the Plateau problem.*

For Alexandrov spaces, no condition on Γ implying $\mathcal{F}_\Gamma \neq \varnothing$ was known. As an application of Theorem 1.2, we can obtain such a condition of Γ.

Corollary 1.6. *Let Γ be a rectifiable closed Jordan curve in an Alexandrov space X. If Γ is topologically contractible in X, then $\mathcal{F}_\Gamma \neq \varnothing$.*

Application: simplicial volume. Yamaguchi [1997, Theorem 0.5] proved, assuming an LLC-condition on an Alexandrov space, an inequality between Gromov's simplicial volume and the Hausdorff measure of the Alexandrov space. As an immediate consequence of Theorem 1.2, we obtain:

Corollary 1.7 [Gromov 1982; Yamaguchi 1997]. *Let X be a compact orientable n-dimensional Alexandrov space without boundary, of curvature $\geq \kappa$ for $\kappa < 0$. Then $\|X\| \leq n!\,(n-1)^n \sqrt{-\kappa}^{\,n}\, \mathcal{H}^n(X)$.*

Here, $\|X\|$ is Gromov's simplicial volume, which is the ℓ_1-norm of the fundamental class of X, and $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. For precise terminology, we refer to [Gromov 1982; Yamaguchi 1997].

Further, if we assume "a lower Ricci curvature bound" for X in the sense of [Bacher and Sturm 2010], then we obtain the following:

Theorem 1.8. *Let X be a compact orientable n-dimensional Alexandrov space without boundary. Let m be a locally finite Borel measure on X with full support that is absolutely continuous with respect to $\mathcal{H}^n$. If the metric measure space*

(X, m) *satisfies the reduced curvature-dimension condition* $\mathrm{CD}^*(K, N)$ *locally for* $K, N \in \mathbb{R}$ *with* $N \geq 1$ *and* $K < 0$, *then*

$$\|X\| \leq n! \sqrt{-(N-1)K}^{\,n} \mathcal{H}^n(X).$$

Theorem 1.8 is new even if X is a manifold, because a reference measure m can be freely chosen.

Organization. We review fundamental properties of Alexandrov spaces in Section 2. In particular, we recall the theory of the gradient flow of distance functions on an Alexandrov space, established in [Petrunin 1995] and [Perelman and Petrunin 1994]. In Section 3, we prove that the distance function from a metric sphere at each point in an Alexandrov space is regular on a much smaller concentric punctured ball. Then, using the gradient flow of the distance function, we prove Theorem 1.2. In Section 4, we recall precise terminology of the applications in the introduction, and prove Corollaries 1.3, 1.4 and 1.6. In Section 5, we note that our proof given in Section 3 also works for infinite-dimensional Alexandrov spaces whenever the space of directions is compact. In Section 6, we recall several notions of a lower Ricci curvature bound on a metric space together with a Borel measure and their relation. By using the Bishop–Gromov-type volume growth inequality, we prove Theorem 1.8.

2. Preliminaries

This section consists of a review of the definition of Alexandrov spaces and a somewhat detailed review of the gradient flow theory of semiconcave functions on Alexandrov spaces. For further details, we refer to [Burago et al. 1992; 2001] or [Petrunin 2007].

We recall the definition of Alexandrov spaces:

Definition 2.1 [Burago et al. 1992; 2001]. Let $\kappa \in \mathbb{R}$. We call a complete metric space X an *Alexandrov space of curvature* $\geq \kappa$ if it satisfies the following:

(1) X is a geodesic space; i.e., for every x and y in X, there is a curve $\gamma : [0, |x, y|] \to X$ such that $\gamma(0) = x$, $\gamma(|x, y|) = y$, and the length of γ equals $|x, y|$. Here, $|x, y|$ denotes the distance between x and y, written also as $|xy|$ or $d(x, y)$. We call such a curve γ a geodesic between x and y, and denote it by xy.

(2) X has curvature $\geq \kappa$; i.e., for every $p, q, r \in X$ (with $|p, q| + |q, r| + |r, p| < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$) and every x in a geodesic qr between q and r, taking a comparison triangle $\triangle \tilde{p}\tilde{q}\tilde{r} = \tilde{\triangle} pqr$ in a simply connected complete surface $\mathbb{M}_\kappa$ of constant curvature κ and a corresponding point $\tilde{x}$ in $\tilde{q}\tilde{r}$, we have

$$|p, x| \geq |\tilde{p}, \tilde{x}|.$$

We simply say that a complete metric space X is an *Alexandrov space* if it is a geodesic space, and for any $p \in X$, there exist a neighborhood U of p and $\kappa \in \mathbb{R}$ such that U has curvature $\geq \kappa$ in the sense that it satisfies condition (2); i.e., any triangle in U (whose sides are contained in U) is not thinner than its comparison triangle in $\mathbb{M}_\kappa$.

If X is compact, then it has a uniform lower curvature bound. Throughout the paper, we do not need a uniform lower curvature bound, since we are mainly interested in local properties. It is known that if X has a uniform lower curvature bound, say κ, then X has curvature $\geq \kappa$ [Burago et al. 1992].

Semiconcave functions. In this subsection, we refer to [Petrunin 2007; 1995].

Definition 2.2. Let I be an interval and $\lambda \in \mathbb{R}$. We say a function $f : I \to \mathbb{R}$ is λ-*concave* if the function

$$\bar{f}(t) = f(t) - \frac{\lambda}{2} t^2$$

is concave on I. That is, for any $t < t' < t''$ in I, we have

$$\frac{\bar{f}(t') - \bar{f}(t)}{t' - t} \geq \frac{\bar{f}(t'') - \bar{f}(t')}{t'' - t'}.$$

We say a function $f : I \to \mathbb{R}$ is λ-*concave in the barrier sense* if for any $t_0 \in \operatorname{int} I$, there exist a neighborhood I_0 of t_0 in I and a twice-differentiable function $g : I_0 \to \mathbb{R}$ such that

$$g(t_0) = f(t_0),\ g \geq f \text{ and } g'' \leq \lambda \text{ on } \operatorname{int} I.$$

Lemma 2.3 [Petrunin 1995]. *Let $f : I \to \mathbb{R}$ be a continuous function on an interval I and $\lambda \in \mathbb{R}$. Then the following are equivalent*:

(1) *f is λ-concave in the sense of Definition 2.2.*

(2) *For any $t_0 \in I$, there is $A \in \mathbb{R}$ such that*

$$f(t) \leq f(t_0) + A(t - t_0) + \frac{\lambda}{2}(t - t_0)^2$$

for any $t \in I$.

(3) *f is λ-concave in the barrier sense.*

Proof. By considering $f(t) - (\lambda/2)\, t^2$, we may assume that $\lambda = 0$.

Let us prove the implication (1) $\Rightarrow$ (2). Let us take $t_0 \in I$, not equal to the supremum of I. By the concavity of f, the value

$$A = \lim_{\varepsilon \to 0+} \frac{f(t_0 + \varepsilon) - f(t_0)}{\varepsilon}$$

is well-defined. And, the concavity of f implies

$$f(t) \le f(t_0) + A(t - t_0).$$

When $t_0 \in I$ is the supremum of I, we obtain the same inequality as above by replacing A with $\lim_{\varepsilon\to 0+}(f(t_0-\varepsilon) - f(t_0))/\varepsilon$.

The implication (2) $\Rightarrow$ (3) is trivial.

Let us assume that f satisfies (3), and take t_0 in the interior of I. Then there exists a twice-differentiable function $g : I \to \mathbb{R}$ such that

$$g(t_0) = f(t_0),\ g \ge f \text{ and } g'' \le 0.$$

Hence, for any $t' < t_0 < t$, we have

$$\frac{f(t) - f(t_0)}{t - t_0} \le \frac{g(t) - g(t_0)}{t - t_0} \le \frac{g(t_0) - g(t')}{t_0 - t'} \le \frac{f(t_0) - f(t')}{t_0 - t'}.$$

Therefore, f is concave. $\square$

Let X be a geodesic space and U be an open subset of X. Let $f : U \to \mathbb{R}$ be a function. We say that f is *λ-concave* on U if for every geodesic $\gamma : I \to U$, the function $f \circ \gamma : I \to \mathbb{R}$ is λ-concave on I. For a function $g : U \to \mathbb{R}$, we say that f is *g-concave* if for any $p \in U$ and $\varepsilon > 0$, there is an open neighborhood V of p in U, such that f is $(g(p)+\varepsilon)$-concave on V. We say that $f : U \to \mathbb{R}$ is *g-concave in the barrier sense* if for any $p \in U$ and $\varepsilon > 0$, there exists an open neighborhood V of p in U such that for every geodesic γ contained in V, $f \circ \gamma$ is $(g(p)+\varepsilon)$-concave in the barrier sense. By an argument similar to the proof of Lemma 2.3, f is g-concave if and only if f is g-concave in the barrier sense.

From now on, we fix an Alexandrov space X. We use results and notions on Alexandrov spaces obtained in [Burago et al. 1992], and we refer to [Burago et al. 2001] and [Petrunin 2007]. T_pX denotes the tangent cone of X at p and Σ_pX denotes the space of directions of X at p.

For any λ-concave function $f : U \to \mathbb{R}$ on an open subset U of X, $p \in U$, and $\delta > 0$, a function $f_\delta : \delta^{-1}U \to \mathbb{R}$ is defined as the same function $f_\delta = f$ on the same domain $\delta^{-1}U = U$ as sets. Since the metric of $\delta^{-1}U$ is the metric of U multiplied by δ^{-1}, f_δ is $\delta^2\lambda$-concave on $\delta^{-1}U$. In addition, if f is Lipschitz near p, then the blow-up $d_pf : T_pX \to \mathbb{R}$, that is, the limit with respect to some sequence $\delta_i \to 0$,

$$\lim_{i\to\infty} f_{\delta_i} : \lim_{i\to\infty} (\delta_i^{-1}U, p) \to \mathbb{R}$$

is 0-concave on T_pX. We call d_pf the *differential* of f at p. Note that the differential of a locally Lipschitz semiconcave function always exists and does not depend on the choice of sequence (δ_i). Actually, $d_p f(\xi)$ is calculated by

$$d_p f(\xi) = \lim_{t \to 0+} \frac{f(\exp_p(t\xi)) - f(p)}{t}$$

if $\xi \in \Sigma'_p$ is a geodesic direction, where $\exp_p(t\xi)$ denotes the geodesic starting from p with the direction ξ.

Distance functions as semiconcave functions. For any real number κ, let us define "trigonometric functions" sn_κ and cs_κ by the following ODE:

$$\begin{cases} \mathrm{sn}''_\kappa(t) + \kappa\, \mathrm{sn}_\kappa(t) = 0, & \mathrm{sn}_\kappa(0) = 0, \quad \mathrm{sn}'_\kappa(0) = 1; \\ \mathrm{cs}''_\kappa(t) + \kappa\, \mathrm{cs}_\kappa(t) = 0, & \mathrm{cs}_\kappa(0) = 1, \quad \mathrm{cs}'_\kappa(0) = 0. \end{cases}$$

They are explicitly represented as follows.

$$\mathrm{sn}_\kappa(t) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n+1)!} t^{2n+1} = \begin{cases} \dfrac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, t) & \text{if } \kappa > 0, \\ t & \text{if } \kappa = 0, \\ \dfrac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, t) & \text{if } \kappa < 0, \end{cases}$$

$$\mathrm{cs}_\kappa(t) = \mathrm{sn}'_\kappa(t) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n)!} t^{2n} = \begin{cases} \cos(\sqrt{\kappa}\, t) & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh(\sqrt{-\kappa}\, t) & \text{if } \kappa < 0. \end{cases}$$

These functions are elementary for the space form $\mathbb{M}_\kappa$ in the sense that they satisfy the following: Let us take any points $p, q, r \in \mathbb{M}_\kappa$ with $|pq| + |qr| + |rp| < 2 \operatorname{diam} \mathbb{M}_\kappa$, and set $\theta := \angle qpr$. Let γ be the geodesic pr with $\gamma(0) = p$ and $\gamma(|p, r|) = r$. We set $\ell(t) = |q, \gamma(t)|$. When $\kappa \neq 0$, the cosine formula states

$$\mathrm{cs}_\kappa(\ell(t)) = \mathrm{cs}_\kappa |pq| \, \mathrm{cs}_\kappa t + \kappa\, \mathrm{sn}_\kappa |pq| \, \mathrm{sn}_\kappa t \cos\theta.$$

Also, we have

$$(\mathrm{cs}_\kappa(\ell(t)))'' + \kappa\ \mathrm{cs}_\kappa(\ell(t)) = 0. \tag{2-1}$$

Lemma 2.4 [Perelman and Petrunin 1994]. *The distance function d_A from a closed subset A in an Alexandrov space X of curvature $\geq \kappa$ is $(\mathrm{cs}_\kappa(d_A)/\mathrm{sn}_\kappa(d_A))$-concave on $(X - A) \cap \{d_A < \pi/(2\sqrt{\kappa})\}$. Here, if $\kappa \leq 0$, then we consider $\pi/(2\sqrt{\kappa})$ as $+\infty$, and if $\kappa = 0$, then we consider $\mathrm{cs}_\kappa(d_A)/\mathrm{sn}_\kappa(d_A)$ as $1/d_A$.*

Proof. We consider the case that $\kappa \neq 0$. Let us take any geodesic γ contained in $(X - A) \cap \{d_A < \pi/(2\sqrt{\kappa})\}$. We take x on γ and reparametrize γ as $x = \gamma(0)$. We choose $w \in A$ such that $|Ax| = |wx|$. We set $\ell(t) := |A, \gamma(t)|$. Let us take a geodesic $\tilde{\gamma}$ and a point $\tilde{w}$ in the κ-plane $\mathbb{M}_\kappa$ such that $|\tilde{w}\tilde{\gamma}(0)| = |wx|$ and $\angle(\uparrow_{\tilde{x}}^{\tilde{w}}$

, $\tilde{\gamma}^+(0)) = \angle(\uparrow_x^w, \gamma^+(0))$. Let us set $\tilde{\ell}(t) := |\tilde{w}, \tilde{\gamma}(t)|$. By Alexandrov convexity, $\ell(t) \le \tilde{\ell}(t)$.

From (2-1), a standard calculation implies

$$\tilde{\ell}'' = \frac{\mathrm{cs}_\kappa(\tilde{\ell})}{\mathrm{sn}_\kappa(\tilde{\ell})}(1-(\tilde{\ell}')^2) \le \frac{\mathrm{cs}_\kappa(\tilde{\ell})}{\mathrm{sn}_\kappa(\tilde{\ell})}.$$

Therefore, ℓ is $(\mathrm{cs}_\kappa(\ell)/\mathrm{sn}_\kappa(\ell))$-concave. The proof is complete if $\kappa \ne 0$. When X has nonnegative curvature, taking a negative number κ as a lower curvature bound of X and letting κ tend to 0, we obtain $\mathrm{cs}_\kappa(d_A)/\mathrm{sn}_\kappa(d_A) \to 1/d_A$. □

Gradient flows. For vectors v, w in the tangent cone T_pX, setting $o = o_p$, the origin of T_pX, we define $|v| = |o, v|$ and

$$\langle v, w\rangle = \begin{cases} |v||w|\cos\angle vow & \text{if } |v|, |w| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5 [Perelman and Petrunin 1994; Petrunin 1995]. Let f be a λ-concave function on an open subset U of X. We say that a vector $g \in T_pX$ at $p \in U$ is a *gradient* of f at p if it satisfies

(1) $df_p(v) \le \langle v, g\rangle$ for all $v \in T_pX$;

(2) $df_p(g) = \langle g, g\rangle$.

We recall that a unique such g exists, which is denoted by $\nabla_p f = \nabla f(p)$.

We say that f is *regular at* p if $d_p f(v) > 0$ for some $v \in T_pX$, or equivalently, $|\nabla_p f| > 0$. Otherwise, f is said to be *critical at* p.

Definition 2.6 [Perelman and Petrunin 1994; Petrunin 1995]. Let $f : U \to \mathbb{R}$ be a semiconcave function on an open subset U of an Alexandrov space. A Lipschitz curve $\gamma : [0, a) \to X$ on an interval $[0, a)$ is said to be a *gradient curve on U for f* if for any $t \in [0, a)$ with $\gamma(t) \in U$,

$$\lim_{\varepsilon\to 0+} \frac{f\circ\gamma(t+\varepsilon) - f\circ\gamma(t)}{\varepsilon}$$

exists and is equal to $|\nabla f|^2(\gamma(t))$.

Note that if f is critical at $\gamma(t)$, the gradient curve γ for f satisfies $\gamma(t') = \gamma(t)$ for any $t' \ge t$.

The (multivalued) logarithm map $\log_p : X \to T_pX$ is defined for $x \ne p$ as $\log_p(x) = |px| \cdot \uparrow_p^x$, where $\uparrow_p^x$ is a direction of a geodesic px, and for $x = p$ as $\log_p(x) = o_p$. If γ is a gradient curve on U, then for t with $\gamma(t) \in U$, the forward direction

$$\gamma^+(t) := \lim_{\varepsilon\to 0+} \frac{\log_{\gamma(t)}(\gamma(t+\varepsilon))}{\varepsilon} \in T_{\gamma(t)}X$$

exists and is equal to the gradient $\nabla f(\gamma(t))$.

Proposition 2.7 [Kapovitch et al. 2010; Petrunin 2007; Petrunin 1995; Perelman and Petrunin 1994]. *Letting γ and η be gradient curves starting from $x = \gamma(0)$ and $y = \eta(0)$ in an open subset U for a λ-concave function $f : U \to \mathbb{R}$, we obtain*

$$|\gamma(s)\eta(s)| \leq e^{\lambda s}|xy|$$

for every $s \geq 0$.

This proposition implies a gradient curve starting at $x \in U$ is unique on its domain.

Theorem 2.8 [Petrunin 1995; 2007; Perelman and Petrunin 1994]. *For any open subset U of an Alexandrov space, a semiconcave function f on U, and $x \in U$, there exists a unique maximal gradient curve*

$$\gamma : [0, a) \to U$$

with $\gamma(0) = x$ for f, where γ is maximal, if for every gradient curve $\eta : [0, b) \to U$ for f with $\eta(0) = x$, we have $b \leq a$.

Definition 2.9 [Perelman and Petrunin 1994; Petrunin 1995]. Let U be an open subset of an Alexandrov space X and $f : U \to \mathbb{R}$ a semiconcave function. Let $\{[0, a_x)\}_{x \in U}$ be a family of intervals for $a_x > 0$. A map

$$\Phi : \bigcup_{x \in U} \{x\} \times [0, a_x) \to U$$

is a *gradient flow* of f on U (with respect to $\{[0, a_x)\}_{x \in U}$) if for every $x \in U$, $\Phi(x, 0) = x$ and the restriction

$$\Phi(x, \cdot\,) : [0, a_x) \to U$$

is a gradient curve of f on U.

A gradient flow Φ is *maximal* if each domain $[0, a_x)$ of the gradient curve is maximal.

By Theorem 2.8 and Proposition 2.7, a maximal gradient flow on U always exists and is unique.

Let Φ be the gradient flow of a semiconcave function on an open subset U. By a standard argument, we obtain

$$\Phi(x, s + t) = \Phi(\Phi(x, s), t)$$

for every $x \in U$ and $s, t \geq 0$, wherever the formula is defined.

3. Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. Let us fix a finite-dimensional Alexandrov space X. As we show in Section 5, the proof works for an infinite-dimensional Alexandrov space with an additional assumption.

We first prove the following. Consider the distance function $f = d(S(p, R), \cdot)$ from a metric sphere $S(p, R) = \{q \in X \mid |pq| = R\}$. We may assume that a neighborhood of p has curvature ≥ -1 by rescaling the metric of X if necessary. We denote by $B(p, R)$ the closed ball centered at p of radius R.

Proposition 3.1. *For any $p \in X$ and $\varepsilon > 0$, there exists $R > 0$ and $\delta_0 = \delta_0(\varepsilon, R) > 0$ such that the distance function*

$$f = d(S(p, R), \cdot)$$

from the metric sphere $S(p, R)$ satisfies

$$d_x f(\uparrow_x^p) > \cos\varepsilon \tag{3-1}$$

for every $x \in B(p, \delta_0 R) - \{p\}$. In particular, f is regular on $B(p, \delta_0 R) - \{p\}$.

Proposition 3.1 is key in our paper, which implies the important Lemma 3.3 later.

For a subset A of an Alexandrov space and $x \notin A$, we denote by A'_x the set of all directions of geodesics from x to A of length $|x, A|$.

Proof of Proposition 3.1. Since the tangent cone T_pX is isometric to the metric cone $K(\Sigma_p)$ over the space of directions Σ_p, there exists a positive constant R satisfying the following:

(3-2) For any $v \in \Sigma_p$, there is $q \in S(p, R)$ such that $\angle(v, \uparrow_p^q) \leq \varepsilon$.

From now on, we set $S := S(p, R)$. For any $x \in S(p, \delta R)$, fixing a direction $\uparrow_p^x \in x'_p$, let us take $q_1, q_2 \in S$ such that

$$|x, q_1| = |x, S| := \min_{q \in S} |x, q| \tag{3-3}$$

and

$$\angle xpq_2 = \angle(\uparrow_p^x, \uparrow_p^{q_2}) = \angle(\uparrow_p^x, S'_p) := \min_{v \in S'_p} \angle(\uparrow_p^x, v). \tag{3-4}$$

By the condition (3-2), we have

$$\tilde{\angle} xpq_2 \leq \angle xpq_2 \leq \varepsilon.$$

Then, by the law of sines, we obtain

$$\sin\tilde{\angle} pxq_2 = \frac{\sinh R}{\sinh |xq_2|} \sin\tilde{\angle} xpq_2 \leq \frac{\sinh R}{\sinh R(1-\delta)} \sin\varepsilon. \tag{3-5}$$

On the other hand, by the law of cosines, we obtain

$$\begin{aligned}\cosh |xq_2| &= \cosh \delta R \cosh R - \sinh \delta R \sinh R \cos \tilde{\angle} xpq_2 \\ &\leq \cosh \delta R \cosh R - \sinh \delta R \sinh R \cos \varepsilon\end{aligned}$$

and

$$\begin{aligned}-\sinh \delta R \sinh |xq_2| \cos \tilde{\angle} pxq_2 &= \cosh R - \cosh \delta R \cosh |xq_2| \\ &\geq \cosh R\{1 - \cosh^2 \delta R\} + \sinh R \sinh \delta R \cos \varepsilon.\end{aligned}$$

Therefore, if δ is smaller than some constant, then

$$-\cos \tilde{\angle} pxq_2 > 0. \tag{3-6}$$

By (3-5) and (3-6), we obtain

$$\tilde{\angle} pxq_2 \geq \pi - (1 + \tau(\delta))\varepsilon. \tag{3-7}$$

Next, let us consider the point q_1 taken as in (3-3). Then, it satisfies

$$\tilde{\angle} xpq_1 = \min_{q \in S} \tilde{\angle} xpq \leq \min_{q \in S} \angle xpq \leq \varepsilon.$$

By a similar argument with q_1 instead of q_2, we obtain

$$\tilde{\angle} pxq_1 \geq \pi - (1 + \tau(\delta))\varepsilon. \tag{3-8}$$

By the quadruple condition, with (3-7) and (3-8), we obtain

$$\tilde{\angle} q_1 x q_2 \leq 2\pi - \tilde{\angle} pxq_1 - \tilde{\angle} pxq_2 \leq (2 + \tau(\delta))\varepsilon.$$

If δ is small with respect to ε, then we obtain

$$|q_1 q_2| \leq 3R\varepsilon.$$

Therefore, we obtain

$$\tilde{\angle} q_1 p q_2 \leq 4\varepsilon. \tag{3-9}$$

For any $y \in px - \{p, x\}$, we set $q_3 = q_3(y) \in S$ to be such that

$$|y, q_3| = |y, S|.$$

By an argument similar to above, we obtain

$$\tilde{\angle} pyq_3 \geq \pi - (1 + \tau(|py|/R))\varepsilon > \pi - 2\varepsilon. \tag{3-10}$$

Then, we have

$$\tilde{\angle} xyq_3 < 2\varepsilon.$$

By the Gauss–Bonnet theorem, if y is near x, then

$$\tilde{\angle} yxq_3 > \pi - 3\varepsilon.$$

By the first variation formula, we obtain

$$df_x(\uparrow_x^p) = \lim_{xp \ni y \to x} \frac{|Sy| - |Sx|}{|xy|} \geq \liminf_{xp \ni y \to x} \frac{|q_3 y| - |q_3 x|}{|xy|} \geq \cos 3\varepsilon.$$

This completes the proof. □

We fix δ_0 as in the conclusion of Proposition 3.1, and fix $\delta \leq \delta_0$.

Lemma 3.2. *For any $x \in B(p, \delta R) - \{p\}$, we have*

$$\angle(\nabla_x f, \uparrow_x^p) < \varepsilon \text{ and } |\nabla_x f, \uparrow_x^p| < \sqrt{2}\varepsilon.$$

Proof. By Proposition 3.1, we have

$$df_x(\uparrow_x^p) > \cos \varepsilon.$$

By the definition of the gradient, we obtain

$$df_x(\uparrow_x^p) \leq |\nabla_x f| \cos \angle(\nabla_x f, \uparrow_x^p) \leq \cos \angle(\nabla f, \uparrow_x^p).$$

Therefore, we have $\angle(\nabla_x f, \uparrow_x^p) < \varepsilon$.

Since f is 1-Lipschitz, $|\nabla f| \leq 1$. And, by the above inequality,

$$|\nabla f|_x = \max_{\xi \in \Sigma_x} df_x(\xi) \geq df(\uparrow_x^p) > \cos \varepsilon.$$

Then, we obtain

$$|\nabla_x f, \uparrow_x^p|^2 < |\nabla f|^2 + 1 - 2|\nabla f| \cos \varepsilon \leq 2 \sin^2 \varepsilon.$$

Therefore, $|\nabla f, \uparrow_x^p| < \sqrt{2}\varepsilon$. □

Let us consider the gradient flow Φ_t of $f = d(S, \cdot)$.

Lemma 3.3. *For every $x \in B(p, \delta R)$,*

$$|\Phi_t(x), p| \leq |x, p| - \cos \varepsilon \cdot t,$$

whenever this formula is defined. In particular, for any $t \geq \delta R / \cos \varepsilon$, we have $\Phi_t(x) = p$.

Proof. Let us set $\gamma(t) = \Phi_t(x)$, the gradient curve for f starting from $\gamma(0) = x$. If $\gamma(t_0) \neq p$, then

$$\frac{d}{dt}\Big|_{t=t_0+} |\Phi_t(x), p| = -\langle \nabla_{\gamma(t_0)} f, \uparrow_{\gamma(t_0)}^p \rangle < -\cos \varepsilon.$$

Integrating this, we have

$$|\Phi_{t_0}(x), p| - |x, p| \le -\cos\varepsilon \cdot t_0.$$

This completes the proof. □

Finally, we estimate the Lipschitz constant of the flow Φ on $B(p, \delta R)$. Let us recall that f is λ-concave on $B(p, \delta R)$ for some λ. By Lemma 2.4, λ can be given as follows:

$$\frac{\cosh(f)}{\sinh(f)} \le \frac{\cosh R}{\sinh(R(1-\delta))} = \lambda.$$

By Proposition 2.7, for any $x, y \in B(p, \delta R)$,

$$|\Phi(x, t), \Phi(y, t)| \le e^{\lambda t}|xy|.$$

Since f is 1-Lipschitz, for $x \in B(p, \delta R)$ and $t' < t$, we have

$$|\Phi(x, t), \Phi(x, t')| \le \int_{t'}^{t} \left| \frac{d}{ds}^{+} \Phi(x, s) \right| ds = \int_{t'}^{t} |\nabla f|(\Phi(x, s))\, ds \le t - t'.$$

Therefore, we obtain the following:

Lemma 3.4. *For any $x, y \in B(p, \delta R)$ and $t \ge s \ge 0$,*

$$|\Phi(x, s), \Phi(y, t)| \le e^{\lambda s}|x, y| + t - s.$$

Note that, by Lemma 3.3, setting $\ell = \delta_0 R/\cos\varepsilon$, the term $e^{\lambda \ell}$ can be bounded from above by a constant arbitrary close to 1 if we choose δ_0 and R small enough.

By Lemma 3.4, we obtain a Lipschitz homotopy

$$\varphi : B(p, \delta_0 R) \times [0, 1] \to B(p, \delta_0 R)$$

with $\varphi(\,\cdot\,, 1) = p$, defined by $\varphi(x, t) = \Phi(x, \ell t)$ for $(x, t) \in B(p, \delta_0 R) \times [0, 1]$. This completes the proof of Theorem 1.2.

Remark 3.5. In the above argument, we employ the distance function from $S(p, R)$ to prove Theorem 1.2. Similarly, one can use the averaged distance function constructed in [Perelman 1993] and [Kapovitch 2005] to prove Theorem 1.2.

4. Proof of applications

Proof of Corollaries 1.3 and 1.4. Let V be a metric space, U a subset of V, and $p \in V$. We say that U is *Lipschitz contractible to p in V* if there exists a Lipschitz map

$$h : U \times [0, 1] \to V$$

such that

$$h(x, 0) = x \quad \text{and} \quad h(x, 1) = p$$

for any $x \in U$. We call such an h a *Lipschitz contraction* from U to p in V. We say that U is *Lipschitz contractible in* V if U is Lipschitz contractible to some point in V.

Lemma 4.1. *Let U be Lipschitz contractible in a metric space V. For any Lipschitz map $\varphi : S^{n-1} \to U$, there exists a Lipschitz map $\tilde{\varphi} : D^n \to V$ such that $\tilde{\varphi}|_{S^{n-1}} = \varphi$.*

Proof. By definition, there exist $p \in V$ and a Lipschitz map

$$h : U \times [0, 1] \to V$$

such that

$$h(x, 0) = x \text{ and } h(x, 1) = p$$

for any $x \in U$. We define a map

$$\varphi_1 : S^{n-1} \times [0, 1] \to V$$

by $\varphi_1 = h \circ (\varphi \times \mathrm{id})$. Then, φ_1 is Lipschitz with Lipschitz constant at most $\mathrm{Lip}\,(h) \cdot \max\{1, \mathrm{Lip}\,(\varphi)\}$. We define a map

$$\varphi_2 : D^n \times \{1\} \to V$$

by $\varphi_2(v, 1) = p$ for all $v \in D^n$. And we consider a space

$$Y = S^{n-1} \times [0, 1] \cup D^n \times \{1\}$$

equipped with a length metric with respect to a gluing $S^{n-1} \times \{1\} \ni (v, 1) \mapsto (v, 1) \in \partial D^n \times \{1\}$. Now we define a map $\varphi_3 : Y \to V$ by

$$\varphi_3 = \begin{cases} \varphi_1 & \text{on } S^{n-1} \times [0, 1], \\ \varphi_2 & \text{on } D^n \times \{1\}. \end{cases}$$

This is well-defined. Then, φ_3 is $\mathrm{Lip}\,(\varphi_1)$-Lipschitz. Indeed, for $x \in S^{n-1} \times [0, 1]$ and $y \in D^n \times \{1\}$, we have

$$|\varphi_3(x), \varphi_3(y)| = |\varphi_3(x), p|.$$

Let $\bar{x} \in S^{n-1} \times \{1\}$ be the foot of a perpendicular segment from x to $S^{n-1} \times \{1\}$. We note that $|x, \bar{x}| \le |x, y|$ and $\varphi_3(\bar{x}) = p$. Then, we obtain

$$|\varphi_3(x), p| = |\varphi_3(x), \varphi_3(\bar{x})| = |\varphi_1(x), \varphi_1(\bar{x})| \le \mathrm{Lip}\,(\varphi_1)|x, \bar{x}| \le \mathrm{Lip}\,(\varphi_1)|x, y|.$$

Obviously, there exists a bi-Lipschitz homeomorphism

$$f : D^n \to Y$$

with $f(0) = (0, 1) \in D^n \times \{1\}$ preserving the boundaries, in the sense that $f(v) = (v, 0) \in S^{n-1} \times \{0\}$ for any $v \in S^{n-1}$. Then, we obtain a Lipschitz map $\tilde{\varphi} := \varphi_3 \circ f$ satisfying the desired condition. $\square$

Definition 4.2. We say that a metric space Y is a *Lipschitz simplicial complex* if there exists a triangulation T of Y satisfying the following: For each simplex $S \in T$, there exists a bi-Lipschitz homeomorphism $\varphi_S : \triangle^{\dim S} \to S$. Here, the simplex $\triangle^{\dim S}$ is a standard simplex equipped with the Euclidean metric and S is given the restricted metric of Y. We say that such a triangulation T is a *Lipschitz triangulation* of Y. The dimension of Y is given by $\dim Y = \sup_{S \in T} \dim S$. We only deal with Y such that $\dim Y < \infty$.

A Lipschitz simplicial complex Y is called *finite* if it has a Lipschitz triangulation consisting of finitely many elements.

Note that a subdivision (for instance, the barycentric one) of a Lipschitz triangulation is also a Lipschitz triangulation.

Proposition 4.3. *Let X be an SLLC space, Y a Lipschitz simplicial complex, and $f : Y \to X$ a continuous map. Then, there exists a homotopy*

$$h : Y \times [0, 1] \to X$$

from $h_0 = f$ such that h_1 is Lipschitz on each simplex of Y.

Further, if f is Lipschitz on a subcomplex A of Y, then a homotopy h can be chosen that is relative to A, that is, satisfying $h(a, t) = a$ for any $a \in A$ and $t \in [0, 1]$.

Proof. If $\dim Y = 0$, then we set $h(x, t) = f(x)$ for $x \in Y$ and $t \in [0, 1]$. Then, h is the desired homotopy.

We assume that the assertion holds for $\dim Y \le k - 1$. First, we prove that for any $f : \triangle^k \to X$, there exists a homotopy

$$h : \triangle^k \times [0, 1] \to X$$

from $h_0 = f$ to a Lipschitz map h_1. Taking a subdivision if necessary, let us take a finite Lipschitz triangulation T of $\triangle^k$ satisfying the following: For any k-simplex $E \in T$, there exists an open subset U_E of X which is a Lipschitz contractible ball such that $f(E) \subset U_E$. For any simplex $F \in T$ of $\dim F \le k - 1$, we set

$$U_F = \bigcap_{F \subset E \in T} U_E.$$

This is an open subset of X. Let us denote by Z a $(k-1)$-skeleton of $\triangle^k$ with respect to T. By the inductive assumption, there exists a homotopy

$$h : Z \times [0, 1] \to X$$

from $h_0 = f|_Z$ such that for every simplex F of Z, the following hold:

- $h_1|_F$ is Lipschitz.
- $h(F \times [0, 1]) \subset U_F$.

- If $f|_F$ is Lipschitz, then $h_t|_F = f|_F$ for any t.

Let E be a k-simplex of $\triangle^k$ with respect to T. We denote by $h^{\partial E}$ the restriction of h to $\partial E \times [0, 1]$. Then, the image of $h^{\partial E}$ is contained in $\bigcup_{T \ni F \subset \partial E} U_F \subset U_E$. Since the pair $(E, \partial E)$ has the homotopy extension property, there exists a homotopy

$$h^E : E \times [0, 1] \to U_E$$

from $f|_E$ which is an extension of $h^{\partial E}$. Then, h_1^E is Lipschitz on ∂E. For another k-simplex E' of $\triangle^k$ with common face $E \cap E'$,

$$h_t^E = h_t^{E'}$$

on $E \cap E'$ for all t. Since U_E is a Lipschitz contractible ball, by Lemma 4.1 there is a homotopy

$$\bar{h}^E : E \times [0, 1] \to X$$

relative to ∂E from $\bar{h}_0^E = h_1^E$ to a Lipschitz map $\bar{h}_1^E : E \to X$. Let us define a homotopy $\hat{h}^E : E \to X$ by

$$\hat{h}^E(x, t) = \begin{cases} h^E(x, t) & \text{if } t \in [0, 1/2], \\ \bar{h}^E(x, t) & \text{if } t \in [1/2, 1]. \end{cases}$$

We define $\hat{h} : \triangle^k \times [0, 1] \to X$ by

$$\hat{h}(x, t) = \hat{h}^E(x, t)$$

for $x \in E \in T$. Then, $\hat{h}_0 = f$ and $\hat{h}_1$ is Lipschitz.

Next, we consider a continuous map $f : Y \to X$ from a Lipschitz simplicial complex Y with $\dim Y = k$. Let Z be a $(k-1)$-simplex of Y. By the inductive assumption, there exists a homotopy

$$h : Z \times [0, 1] \to X$$

from $h_0 = f|_Z$, and h_1 is Lipschitz on every simplex of Z. From now on, let us denote by E a k-skeleton of Y. By using the homotopy extension property for $(E, \partial E)$ and Lemma 4.1, we obtain a homotopy

$$h^E : E \times [0, 1] \to X$$

which is an extension of $h|_{\partial E \times [0,1]}$, with $h_0^E = f|_E$. Since $h_1^E|_{\partial E} = h_1|_{\partial E}$ is Lipschitz, there exists a homotopy

$$\bar{h}^E : E \times [0, 1] \to X$$

relative to ∂E from $\bar{h}_0^E = h_1^E$ to a Lipschitz map $\bar{h}_1^E$. We set $\bar{h}(x,t) = h(x,1)$ for $x \in Z$ and $t \in [0,1]$. And, we define a homotopy $\hat{h} : Y \times [0,1] \to X$ by

$$\hat{h}(x,t) = \begin{cases} h(x,2t) & \text{if } x \in Z \text{ and } t \in [0,1/2], \\ \bar{h}(x,2t-1) & \text{if } x \in Z \text{ and } t \in [1/2,1], \\ h^E(x,2t) & \text{if } x \in E \subset Y \text{ and } t \in [0,1/2], \\ \bar{h}^E(x,2t-1) & \text{if } x \in E \subset Y \text{ and } t \in [1/2,1]. \end{cases}$$

Then, $\hat{h}_0 = f$ and $\hat{h}_1$ is Lipschitz on every simplex. □

Corollary 4.4. *Let Y be a Lipschitz simplicial complex, X an SLLC space, and $f : Y \to X$ a continuous map. Let T be a Lipschitz triangulation of Y and $\{U_F \mid F \in T\}$ a family of open subsets of X satisfying the following properties*:

- *$f(F) \subset U_F$ for $F \in T$.*
- *$U_F \subset U_E$ for $F, E \in T$ with $F \subset E$.*

Then, there exists a homotopy $h : Y \times [0,1] \to X$ from $h_0 = f$ such that for every $F \in T$:

- *h_1 is Lipschitz on F.*
- *$h(F \times [0,1]) \subset U_F$.*
- *If f is Lipschitz on F, then $h_t = f$ on F for all t.*

For instance, fixing $\varepsilon > 0$ and setting U_F an ε-neighborhood of $f(F)$ for every $F \in T$, the family $\{U_F \mid F \in T\}$ satisfies the assumption of Corollary 4.4.

Proof of Corollary 4.4. If $\dim Y = 0$, the assertion is trivial. We assume that Corollary 4.4 holds when $\dim Y \le k-1$ for some $k \ge 1$. Let Y be a Lipschitz simplicial complex with $\dim Y = k$ and T a Lipschitz triangulation of Y. Let us take a family $\{U_F \mid F \in T\}$ of open subsets satisfying the assumption of Corollary 4.4. By inductive assumption, there exists a homotopy

$$h : Y^{(k-1)} \times [0,1] \to X$$

from $h_0 = f|_{Y^{(k-1)}}$, and h_1 is Lipschitz on each $F \in T$ of $\dim \le k-1$ and $h_t(F) \subset U_F$ for all t. Let us denote by E a k-simplex in T. By Proposition 4.3, there exists a homotopy

$$h^E : E \times [0,1] \to U_E$$

from $h_0^E = f|_E$ to a Lipschitz map h_1^E such that $h_t^E = h_t$ on ∂E for all t. Then, the concatenation map

$$\hat{h}(x,t) = \begin{cases} h(x,t) & \text{if } x \in Y^{(k-1)}, \\ h^E(x,t) & \text{if } x \in E, \end{cases}$$

is the desired homotopy. □

Remark 4.5. We note that Proposition 4.3 and Corollary 4.4 above can be also proved assuming X is just LLC instead of SLLC. Here, we say that a metric space X is locally Lipschitz contractible, for short LLC, if for any $p \in X$ and $\varepsilon > 0$, there exist $r \in (0, \varepsilon]$ and a Lipschitz contraction φ from $U(p, r)$ to p in $U(p, \epsilon)$. We also remark that Corollaries 1.3 and 1.4 are true if X is just LLC.

Let us start to prove Corollaries 1.3 and 1.4.

Proof of Corollaries 1.3 and 1.4. Let us take a finite Lipschitz simplicial complex pair (P, Q), with Q possibly empty. We prove Corollaries 1.3 and 1.4 assuming X to be SLLC. Let A be an open subset in X. Let us consider a continuous map $f : (P, Q) \to (X, A)$. By Corollary 4.4 and Theorem 1.2, we obtain a homotopy

$$\varphi : (P, Q) \times [0, 1] \to (X, A)$$

from $\varphi_0 = f$ to a Lipschitz map $\varphi_1 : (P, Q) \to (X, A)$. Here, we note that since A is open in X, the homotopy φ_t can be chosen so that $\varphi_t(Q) \subset A$. Then, we obtain a correspondence

$$C((P, Q), (X, A)) \ni f \mapsto \varphi_1 \in \mathrm{Lip}((P, Q), (X, A)), \tag{4-1}$$

where $C(*, **)$ (resp. $\mathrm{Lip}(*, **)$) denotes the set of all continuous (resp. Lipschitz) maps from $*$ to $**$.

Let us consider two homotopic continuous maps f and g from (P, Q) to (X, A). From the correspondence (4-1), we obtain Lipschitz maps f' and g' from (P, Q) to (X, A) which are homotopic to f and g, respectively. Connecting these homotopies, we obtain a homotopy

$$H : (P, Q) \times [0, 1] \to (X, A)$$

between $H(\cdot\,, 0) = f'$ and $H(\cdot\,, 1) = g'$. Now, we consider a Lipschitz simplicial complex $\tilde{P} = P \times [0, 1]$ and a subcomplex $\tilde{R} = P \times \{0, 1\}$. Then, the map H is Lipschitz on $\tilde{R}$. Hence, by Proposition 4.3, we obtain a homotopy

$$\tilde{H} : \tilde{P} \times [0, 1] \to X$$

relative to $\tilde{R}$ from $\tilde{H}(\cdot\,, 0) = H$ to a Lipschitz map $\tilde{H}(\cdot\,, 1)$. Then, $\tilde{H}(\cdot\,, 1)$ is a Lipschitz homotopy between f' and g'. Therefore, we conclude that the correspondence (4-1) sends a homotopy to a Lipschitz homotopy. This completes the proof of Corollary 1.3.

Let us consider a pointed n-sphere (S^n, p_0) and an Alexandrov space X with point $x_0 \in X$. Then, for any map $f : (S^n, p_0) \to (X, x_0)$, the restriction $f|_{\{p_0\}}$ is always Lipschitz. Hence, by an argument as above and Proposition 4.3, we obtain the conclusion of Corollary 1.4. $\square$

Plateau problem. We first recall the definition of the Sobolev space of a metric space target in order to state the setting of Plateau problem in an Alexandrov space as in the introduction, referring to [Korevaar and Schoen 1993] and [Mese and Zulkowski 2010]. For a complete metric space X and a domain Ω in a Riemannian manifold having compact closure, a function $u : \Omega \to X$ is said to be an L^2-*map* if u is Borel measurable and, for some (equivalently, any) point $p_0 \in X$, the integral

$$\int_\Omega |u(x), p_0|^2 \, d\mu$$

is finite, where μ is the Riemannian volume measure. The set of all L^2-maps from Ω to X is denoted by $L^2(\Omega, X)$. We recall the definition of the energy of $u \in L^2(\Omega, X)$: For any $\varepsilon > 0$, we set $\Omega_\varepsilon = \{x \in \Omega \mid d(\partial\Omega, x) > \varepsilon\}$, and define an approximate energy density $e^u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by

$$e^u_\varepsilon(x) = \frac{1}{\omega_n} \int_{S(x,\varepsilon)} \frac{d(u(x), u(y))^2}{\varepsilon^2} \frac{d\sigma}{\varepsilon^{n-1}}.$$

Here, $n = \dim \Omega$, $S(x, \varepsilon)$ is the metric sphere around x with radius ε and σ is the surface measure on it. By [Korevaar and Schoen 1993, 1.2(iii)], we obtain

$$\int_{\Omega_\varepsilon} e^u_\varepsilon(x) \, d\mu \le C\varepsilon^{-2}.$$

Let us take a Borel measure ν on the interval $(0, 2)$ satisfying

$$\nu \ge 0, \quad \nu((0, 2)) = 1, \quad \text{and} \quad \int_0^2 \lambda^{-2} \, d\nu(\lambda) < \infty.$$

An averaged approximate energy density ${}_\nu e^u_\varepsilon(x)$ is defined by

$${}_\nu e^u_\varepsilon(x) = \begin{cases} \displaystyle\int_0^2 e^u_{\lambda\varepsilon}(x) \, d\nu(\lambda) & \text{if } x \in \Omega_{2\varepsilon}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C_c(\Omega)$ be the set of all continuous function on Ω with compact support. We define a functional $E^u_\varepsilon : C_c(\Omega) \to \mathbb{R}$ by

$$E^u_\varepsilon(f) := \int_\Omega f(x)_\nu e^u_\varepsilon \, d\mu(x).$$

Then, the *energy* of u is defined by

$$E^u = \sup_{\substack{f \in C_c(\Omega) \\ 0 \le f \le 1}} \limsup_{\varepsilon \to 0} E^u_\varepsilon(f).$$

The $(1,2)$-Sobolev space is defined as

$$W^{1,2}(\Omega, X) = \{u \in L^2(\Omega, X) \mid E^u < \infty\}.$$

We start to prove Corollary 1.6.

Proof of Corollary 1.6. Let Γ be a rectifiable closed Jordan curve in an Alexandrov space X which is topologically contractible. Since Γ is rectifiable, we can take a Lipschitz monotonic parametrization

$$\gamma : S^1 \to \Gamma.$$

By the contractibility of Γ, there exists a continuous map

$$h : \Gamma \times [0,1] \to X$$

such that $h(\,\cdot\,,0) = \mathrm{id}_\Gamma$ and $h(\,\cdot\,,1) = p$ for some $p \in X$. We define a map $f : S^1 \times [0,1] \to X$ by $f(x,t) = h(\gamma(x),t)$. Further, we set $f(y,1) = p$ for $y \in D^2$. By taking a reparametrization of $f : S^1 \times [0,1] \cup D^2 \times \{1\} \to X$, we obtain a continuous map

$$g : D^2 \to X$$

such that $g|_{\partial D^2} = \gamma$.

By Proposition 4.3, there exists a homotopy

$$\tilde{h} : D^2 \times [0,1] \to X$$

relative to ∂D^2 such that $\tilde{h}(\,\cdot\,,0) = g$ and $\tilde{h}(\,\cdot\,,1)$ is Lipschitz. Thus, we obtain a Lipschitz map $\tilde{g} = \tilde{h}(\,\cdot\,,1)$ such that $\tilde{g}|_{\partial D^2} = \gamma$. By the definition of the energy, we obtain

$$E(\tilde{g}) \le \mathrm{Lip}(\tilde{g})^2 < \infty.$$

Here, $\mathrm{Lip}(\tilde{g})$ is the Lipschitz constant of $\tilde{g}$. Therefore, we conclude $\tilde{g} \in \mathcal{F}_\Gamma$. $\square$

5. A note on the infinite-dimensional case

It is known that the (Hausdorff) dimension of an Alexandrov space is a nonnegative integer or is infinite. There are only a few works on infinite-dimensional Alexandrov spaces. It is not known whether an infinite-dimensional Alexandrov space is locally contractible.

When we consider an Alexandrov space of *possibly infinite dimension*, we somewhat generalize Definition 2.1 as follows: A complete metric space X is called an *Alexandrov space* if it is a length metric space and satisfies the quadruple condition locally. Here, a complete metric space X is a *length* metric space if for every two points $p, q \in X$ and any $\varepsilon > 0$, there exists a point $r \in X$ satisfying $\max\{|pr|, |rq|\} < |pq|/2 + \varepsilon$. Since a length metric space has no geodesics in general, to define a notion of a lower curvature bound, we change the triangle comparison condition to the quadruple condition. Here, an open subset U of a length space X satisfies the *quadruple condition* modeled on the κ-plane $\mathbb{M}_\kappa$ if for any four distinct points p_0, p_1, p_2 and p_3 in U, we have

$$\tilde{\angle} p_1 p_0 p_2 + \tilde{\angle} p_2 p_0 p_3 + \tilde{\angle} p_3 p_0 p_1 \leq 2\pi,$$

where $\tilde{\angle} = \tilde{\angle}_\kappa$ denotes the comparison angle modeled on $\mathbb{M}_\kappa$.

By a standard argument, any geodesic triangle (if one exists) in an Alexandrov space of possibly infinite dimension satisfies the triangle comparison condition. It is known that finite-dimensional Alexandrov spaces are proper metric space; in particular, by Hopf–Rinow theorem, they are geodesic spaces.

Plaut proved that an Alexandrov space of infinite dimension is an "almost" geodesic space. Precisely:

Theorem 5.1 [Plaut 1996]. *Let X be an Alexandrov space of infinite dimension. For any $p \in X$, the subset $J_p \subset X$ defined by*

$$J_p = \bigcap_{\delta > 0} \{q \in X - \{p\} \mid \textit{there exists } x \in X - \{p, q\} \textit{ with } \tilde{\angle} pqx > \pi - \delta\}$$

is a dense G_δ-subset in X, and, for every $q \in J_p$, there exists a unique geodesic connecting p and q.

We now show that the compactness of the space of directions at some point implies Lipschitz contractibility around the point.

Proposition 5.2. *Let X be an Alexandrov space of infinite dimension. Suppose that there exists a point $p \in X$ such that the space of directions Σ_p at p is compact. Then, the following are true*:

(i) *The pointed Gromov–Hausdorff limit as $r \to \infty$ of the scaling space (rX, p) exists and is isometric to the cone over Σ_p.*

(ii) *Σ_p is a geodesic space.*

(iii) *X is proper.*

(iv) *There exists $R_0 > 0$, depending on p, such that for every $R \leq R_0$, $U(p, R)$ is Lipschitz contractible to p in itself.*

Proof. (i) Let $K = K(\Sigma_p)$ be the Euclidean cone over Σ_p and B be the unit ball around the origin o. Let J_p be the set defined in Theorem 5.1. For any $\varepsilon > 0$, we take a finite ε-net $\{v_\alpha\}_\alpha \subset B$. We may assume that every v_α is contained in $K(\Sigma'_p) - \{o\}$. That is, there exists $r > 0$ such that for every α, there is a geodesic γ_α starting from p having direction $v_\alpha/|v_\alpha|$, with length at least r. Let $x_\alpha \in B(p, r)$ be defined by $x_\alpha = \gamma_\alpha(r|v_\alpha|)$. Then, $\{x_\alpha\}_\alpha$ is an ε-net in $(1/r)B(p, r)$. Indeed, for any $x \in B(p, r) \cap J_p$, setting $v = \log_p(x) \in K(\Sigma_p)$, we have $(1/r)v \in B$. Then, there exists α such that $|v_\alpha, (1/r)v| \le \varepsilon$. Therefore, $|rv_\alpha, v| \le r\varepsilon$. We may assume that a lower curvature bound of X is less than or equal to 0. Then

$$\exp_p : B(o, r) \cap \mathrm{dom}(\exp_p) \to B(p, r)$$

is 1-Lipschitz, where $\mathrm{dom}(\exp_p)$ is the domain of $\exp_p$. Therefore, $|x_\alpha, x|_X \le r\varepsilon$.

Let us retake r to be small enough that

$$\left| \frac{|x_\alpha, x_\beta|}{r} - |v_\alpha, v_\beta| \right| \le \varepsilon.$$

Then, the map $v_\alpha \mapsto x_\alpha$ implies a $C\varepsilon$-approximation between B and $(1/r)B(p, r)$ for any small r. Here, C is a constant not depending on any other term. Therefore, the pointed spaces $((1/r)X, p)$ are Gromov–Hausdorff convergent to $(K(\Sigma_p), o)$ as $r \to 0$.

Clearly (ii) holds by (i) and (iii). We prove (iii). Let us consider any closed ball $B(p, r)$ centered at p. Let us take any sequence $\{x_i\} \subset B(p, r)$. We take $y_i \in B(p, r) \cap J_p$ such that $|x_i, y_i| \le 1/i$. Then, $v_i = \log_p(y_i) \in B(o, r) \subset T_pX$ is well-defined. By (i), T_pX is proper. Hence, there exists a convergent subsequence $\{v_{n(i)}\}_i$ of $\{v_i\}_i$. Since $\exp_p$ is Lipschitz, $\{x_{n(i)}\}$ is convergent.

We recall that the proof of Theorem 1.2 started from the assertion (3-2) in the proof of Proposition 3.1. The assertion (i) guarantees (3-2). Therefore, one can prove (iv) in the same way as the proof of Theorem 1.2. □

6. An estimation of simplicial volume of Alexandrov spaces

In this section, we consider an Alexandrov space having a lower *Ricci* curvature bound, and we prove an estimation of the simplicial volume of such a space as stated in Theorem 1.8. The original form of Theorem 1.8 was proved by Gromov [1982] when X is a Riemannian manifold with a lower Ricci curvature bound.

Gromov's original proof was depending on the well-known Bishop–Gromov volume inequality. For an Alexandrov space of curvature $\ge \kappa$ for some $\kappa \in \mathbb{R}$, its Hausdorff measure is known to satisfy the Bishop–Gromov-type volume growth estimate. The second author's proof of Corollary 1.7 was depending on this volume growth estimate [Yamaguchi 1997]. It is known that several natural generalized notions of a lower Ricci curvature bound induce a volume growth estimate. Among them, the local reduced curvature-dimension condition introduced by Bacher and Sturm [2010] can be used as a general condition implying the inequality in Theorem 1.8. For completeness, we recall the definitions of several generalized notions of lower Ricci curvature bound, and prove Theorem 1.8.

Several notions of lower Ricci curvature bound. We recall several generalized notions of a lower bound of Ricci curvature, defined on a pair consisting of a metric space and a Borel measure on it. For the theory, history and undefined terms of the following, we refer to [Sturm 2006a; 2006b; Bacher and Sturm 2010; Cavalletti and Sturm 2012; Ohta 2007] and their references.

In this section, we denote by M a complete separable metric space. By $\mathcal{P}_2(M)$ we denote the set of all Borel probability measures μ on M with finite second moment. A metric called the L_2-Wasserstein distance W_2 is defined on $\mathcal{P}_2(M)$. Let us fix a locally finite Borel measure m on M. Such a pair (M, m) is called a metric measure space. Let us denote by $\mathcal{P}_\infty(M, m)$ the subset of $\mathcal{P}_2(M)$ consisting of all measures which are absolutely continuous in m and have bounded support.

From now on, K and N denote real numbers with $N \geq 1$. For $\nu \in \mathcal{P}_\infty(M, m)$ with density $\rho = d\nu/dm$, its Rényi entropy with respect to m is given by

$$S_N(\nu|m) := -\int_M \rho^{1-1/N}\, dm = -\int_M \rho^{-1/N}\, d\nu.$$

For $t \in [0, 1]$, a function $\sigma_{K,N}^{(t)} : (0, \infty) \to [0, \infty)$ is defined as

$$\sigma_{K,N}^{(t)}(\theta) = \begin{cases} +\infty & \text{if } K\theta^2 \geq N\pi^2, \\ \dfrac{\mathrm{sn}_{K/N}(t\theta)}{\mathrm{sn}_{K/N}(\theta)} & \text{otherwise.} \end{cases}$$

And, we set $\tau_{K,N}^{(t)}(\theta) = t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}$.

Definition 6.1 [Bacher and Sturm 2010; Cavalletti and Sturm 2012; Sturm 2006b]. Let K and N be real numbers with $N \geq 1$. Let (M, m) be a metric measure space. We say that (M, m) satisfies the *reduced curvature-dimension condition* $\mathrm{CD}^*(K, N)$ *locally* — denoted by $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$ — if for any $p \in M$ there exists a neighborhood $M(p)$ such that for all $\nu_0, \nu_1 \in \mathcal{P}_\infty(M, m)$ supported on $M(p)$, denoting those densities by ρ_0, ρ_1 with respect to m, there exist an optimal coupling q of ν_0 and ν_1 and a geodesic $\Gamma : [0, 1] \to \mathcal{P}_\infty(M, m)$, parametrized proportionally to arclength, connecting $\nu_0 = \Gamma(0)$ and $\nu_1 = \Gamma(1)$, such that

$$\begin{aligned} S_{N'}(\Gamma(t)|m) \leq -\int_{M\times M} \big[&\sigma_{K,N'}^{(1-t)}(d(x_0, x_1))\rho_0^{-1/N'}(x_0) \\ &+ \sigma_{K,N'}^{(t)}(d(x_0, x_1))\rho_1^{-1/N'}(x_1)\big]\, dq(x_0, x_1) \end{aligned}$$

holds for all $t \in [0, 1]$ and all $N' \geq N$.

We say that (M, m) satisfies the *curvature-dimension condition* $\mathrm{CD}(K, N)$ *locally* — denoted by $\mathrm{CD}_{\mathrm{loc}}(K, N)$ — if it satisfies $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$ with $\sigma_{K,N'}^{(s)}$ replaced by $\tau_{K,N'}^{(s)}$ for each $s \in [0, 1]$ and $N' \geq N$.

The (global) conditions $\mathrm{CD}^*(K, N)$ and $\mathrm{CD}(K, N)$ are defined similarly, and imply corresponding local conditions.

From the inequality $\tau^{(t)}_{K,N}(\theta) \ge \sigma^{(t)}_{K,N}(\theta)$, $\mathrm{CD}(K, N)$ implies $\mathrm{CD}^*(K, N)$ (and $\mathrm{CD}_{\mathrm{loc}}(K, N)$ implies $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$). Further, it is known that the local CD-conditions are equivalent in the following sense:

When a mathematical condition $\varphi(K)$ is given for each $K \in \mathbb{R}$, we say that an mathematical object P satisfies $\varphi(K-)$ if P satisfies $\varphi(K')$ for all $K' < K$.

Theorem 6.2 [Bacher and Sturm 2010, Proposition 5.5]. *Let $K, N \in \mathbb{R}$ with $N \ge 1$ and let (M, m) be a metric measure space. Then, (M, m) satisfies $\mathrm{CD}^*_{\mathrm{loc}}(K-, N)$ if and only if it satisfies $\mathrm{CD}_{\mathrm{loc}}(K-, N)$.*

There is another notion of a lower Ricci curvature bound in metric measure spaces which is called the *measure contraction property*, denoted by $\mathrm{MCP}(K, N)$. Since we do not use its theory to prove Theorem 1.8 in this paper, we omit its definition. For the definition and theory, we refer to [Ohta 2007] and [Sturm 2006b].

A metric measure space (M, m) is called nonbranching if M is a geodesic space and is nonbranching in the sense that for any four points x, y, z_1, z_2 in M, if y is a common midpoint of x and z_1 and of x and z_2, then $z_1 = z_2$. It is known that a nonbranching metric measure space satisfying $\mathrm{CD}(K, N)$ satisfies $\mathrm{MCP}(K, N)$. Recently, Cavalletti and Sturm proved:

Theorem 6.3 [2012, Theorem 1.1]. *Let (M, m) be a nonbranching metric measure space. Let $K, N \in \mathbb{R}$ with $N \ge 1$. If (M, m) satisfies $\mathrm{CD}_{\mathrm{loc}}(K, N)$, then it satisfies $\mathrm{MCP}(K, N)$.*

Bishop–Gromov volume growth estimate. Let (M, m) be a metric measure space and $x \in \mathrm{supp}(m)$. We set

$$v_x(r) := m(B(x, r)).$$

For $K, N \in \mathbb{R}$ with $N > 1$, we define

$$\bar{v}_{K,N}(r) = \int_0^r \mathrm{sn}^{N-1}_{K/(N-1)}(t)\,dt.$$

A metric measure space (M, m) satisfies the *Bishop–Gromov volume growth estimate* $\mathrm{BG}(K, N)$ if for any $x \in \mathrm{supp}(m)$, the function

$$v_x(r)/\bar{v}_{K,N}(r)$$

is nonincreasing in $r \in (0, \infty)$ (with $r \le \pi\sqrt{(N-1)/K}$ if $K > 0$).

Since $\bar{v}_{K,N}(r)$ is continuous in K, $\mathrm{BG}(K-, N)$ implies $\mathrm{BG}(K, N)$. The Bishop–Gromov volume growth estimate is implied by several lower Ricci curvature bounds, for instance the measure contraction property.

Theorem 6.4 [Ohta 2007, Theorem 5.1; Sturm 2006b, Remark 5.3]. *If (M, m) satisfies $\mathrm{MCP}(K, N)$, then it satisfies $\mathrm{BG}(K, N)$.*

Summarizing the above facts, we can state the following implications: Let $K, N \in \mathbb{R}$ with $N \geq 1$. For a nonbranching metric measure space (M, m),

$$\begin{aligned} \text{(6-1)} \qquad \mathrm{CD}^*_{\mathrm{loc}}(K, N) &\Longrightarrow \mathrm{CD}^*_{\mathrm{loc}}(K-, N) \Longleftrightarrow \mathrm{CD}_{\mathrm{loc}}(K-, N) \\ &\Longrightarrow \mathrm{MCP}(K-, N) \Longrightarrow \mathrm{BG}(K-, N) \Longrightarrow \mathrm{BG}(K, N). \end{aligned}$$

Universal covering space with lifted measure. Let X be a semilocally simply connected space. Then, there is a universal covering $\pi : Y \to X$. In addition, if X is a length space, then Y can also be considered as a length space. The map π becomes a local isometry.

In addition, we assume that (X, m) is a proper metric measure space. Let $\mathcal{V}$ be the family of all open sheets of the universal covering $\pi : Y \to X$. We define a set function $m_Y : \mathcal{V} \to [0, \infty]$ by

$$m_Y(V) = m(\pi(V)).$$

One can naturally extend m_Y to a Borel measure on Y. We also write this measure as m_Y, and call it the *lift* of m. Since m is locally finite, so is m_Y.

In general, for a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(M)$, if $\Gamma(0)$ and $\Gamma(1)$ are supported on $U(x, r)$ for some $x \in X$ and $r > 0$, then $\Gamma(t)$ is supported on $U(x, 2r)$ for every $t \in (0, 1)$ [Sturm 2006a, Lemma 2.11]. Therefore, we obtain:

Proposition 6.5 [Bacher and Sturm 2010, Theorem 7.10]. *The local* (*reduced*) *curvature-dimension condition is inherited by the lift. Namely*, *let* $K, N \in \mathbb{R}$ *with* $N \geq 1$ *and let* (X, m) *and* (Y, m_Y) *be as above. If* (X, m) *satisfies* $\mathrm{CD}_{\mathrm{loc}}(K, N)$ (*resp.* $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$), *then* (Y, m_Y) *also satisfies* $\mathrm{CD}_{\mathrm{loc}}(K, N)$ (*resp.* $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$).

Proof of Theorem 1.8. Let X be an n-dimensional compact orientable Alexandrov space without boundary. Let m be a locally finite Borel measure on X with full support. We assume that (X, m) satisfies $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$ for $K < 0$ and $N \geq 1$. By Proposition 6.5, the universal covering Y of X with lift m_Y of m also satisfies $\mathrm{CD}^*_{\mathrm{loc}}(K, N)$. And, Y is an n-dimensional Alexandrov space. Since m has full support, so does m_Y. By the implication (6-1), (Y, m_Y) satisfies $\mathrm{BG}(K, N)$. Therefore, as mentioned in the preface of this section, the original proof of Gromov's theorem relying on the Bishop–Gromov volume comparison works in our setting (see [Gromov 1982, §2; Yamaguchi 1997, Appendix]). Hence, we can prove Theorem 1.8 with a similar such an argument. For undefined terms appearing and for facts used in the following argument, we refer to [Gromov 1982; Yamaguchi 1997].

Let $\mathcal{M}$ (resp. $\mathcal{M}_+$) be the Banach space (resp. the set) of all finite signed (resp. positive) Borel measures on Y, where $\mathcal{M}$ is equipped with the norm $\|\mu\| = \int_Y d|\mu| \in [0, \infty)$. Due to the general theory established in [Gromov 1982, §2] and [Yamaguchi 1997, Appendix], if a differentiable averaging operator $S : Y \to \mathcal{M}_+$ exists, then for any $\alpha \in H_n(X)$,

$$\|\alpha\|_1 \leq n!\,(\mathscr{L}[S])^n \operatorname{mass}(\alpha) \tag{6-2}$$

holds. Here, the value $\mathscr{L}[S]$ is defined as follows: For $y \in Y$,

$$\mathscr{L}S_y = \limsup_{z\to y} \frac{\|S(z)-S(y)\|}{d(z,y)} \quad \text{and} \quad \mathscr{L}[S] = \sup_{y\in Y} \frac{\mathscr{L}S_y}{\|S(y)\|}.$$

We recall a concrete construction of a differentiable averaging operator. For $R > 0$ and $y \in Y$, we set $S_R(y) \in \mathcal{M}_+$ to be

$$S_R(y) = 1_{B(y,R)} \cdot m_Y.$$

Here, 1_A is the characteristic function of $A \subset Y$. For $\epsilon > 0$, we define $S_{R,\epsilon} : Y \to \mathcal{M}_+$ by

$$S_{R,\epsilon}(y) = \frac{1}{\epsilon}\int_{R-\epsilon}^{R} S_{R'}(y)\,dR'.$$

Its norm is $\|S_{R,\epsilon}(y)\| = (1/\epsilon)\int_{R-\epsilon}^{R} v_y(R')\,dR'$ and is not less than $v_y(R-\epsilon)$. Here, $v_z(r) = m_Y(B(z,r))$ for $z \in Y$ and $r > 0$. Given the Lipschitz function $\psi = \psi_{R,\epsilon} : [0,\infty) \to [0,1]$ defined by

$$\psi(t) = \begin{cases} 1 & \text{if } t \leq R-\epsilon, \\ (R-t)/\epsilon & \text{if } t \in [R-\epsilon, R], \\ 0 & \text{if } t \geq R, \end{cases}$$

we can write $S_{R,\epsilon}(y) = \psi(d(y,\cdot))\,m_Y$ for any $y \in Y$.

We can check $S_{R,\epsilon}$ is a differentiable averaging operator as follows: Since m_Y is $\pi_1(X)$-invariant, the maps S_R and $S_{R,\epsilon}$ are $\pi_1(X)$-equivariant. Since m is absolutely continuous in $\mathscr{H}^n_X$, so is m_Y in $\mathscr{H}^n_Y$. One can check that $S_{R,\epsilon}$ is differentiable m_Y-almost everywhere with respect to the differentiable structure of Y, where the differentiable structure on Alexandrov spaces are defined by Otsu and Shioya [1994]. Indeed, the differential $D_yS_{R,\epsilon}(\gamma^+(0))$ of $S_{R,\epsilon}$ at y along a geodesic γ starting from $y = \gamma(0)$ is calculated by

$$\big(D_yS_{R,\epsilon}(\gamma^+(0))\big)(A) = \frac{1}{\epsilon}\int_{A\cap A(y;R-\epsilon,R)} \cos\angle(z'_y, \gamma^+(0))\,dm_Y(z)$$

for any Borel set $A \subset Y$, where $A(z; r, r')$ is the annulus around $z \in Y$ of inner radius r and outer radius r', for $r \leq r'$.

To estimate $\mathscr{L}[S_{R,\epsilon}]$, we use the Bishop–Gromov volume growth estimate as follows. We obtain

$$\mathscr{L}(S_{R,\epsilon})_y = \sup_{\xi\in\Sigma_y} \|D_yS_{R,\epsilon}(\xi)\| \leq \frac{m_Y(A(y;\,R-\epsilon,\,R))}{\epsilon}.$$

It follows from $\mathrm{BG}(K, N)$ that

$$\frac{\mathcal{L}(S_{R,\epsilon})_y}{\|S_{R,\epsilon}(y)\|} \leq \frac{v_y(R) - v_y(R-\epsilon)}{\epsilon \cdot v_y(R-\epsilon)} \leq C_{K,N}(R,\epsilon).$$

Here, setting

$$\bar{v}(R') = \bar{v}_{K,N}(R') = \int_0^{R'} \mathrm{sn}_{K/(N-1)}^{N-1}(t)\, dt,$$

we have

$$C_{K,N}(R,\epsilon) := \frac{\bar{v}(R) - \bar{v}(R-\epsilon)}{\epsilon \cdot \bar{v}(R-\epsilon)}.$$

Since $\mathrm{mass}([X]) = \mathcal{H}^n(X)$ [Yamaguchi 1997, Theorem 0.1], by using (6-2) and by letting $\epsilon \to 0$ and $R \to \infty$, we obtain

$$\|X\| \leq n! \sqrt{-K(N-1)}^{\,n} \mathcal{H}^n(X).$$

This completes the proof of Theorem 1.8. □

Remark 6.6. By [Petrunin 2011] and [Zhang and Zhu 2010], it is known that for an n-dimensional Alexandrov space X of curvature $\geq \kappa$, the metric measure space $(X, \mathcal{H}^n)$ satisfies the curvature-dimension condition $\mathrm{CD}((n-1)\kappa, n)$. Therefore, Corollary 1.7 is implied by Theorem 1.8.

If there exists a compact orientable n-dimensional Alexandrov space X, without boundary, of curvature $\geq \kappa$, with $\kappa < 0$, which has nonnegative Ricci curvature with respect to some reference measure m such that $m \ll \mathcal{H}^n$ and $\mathrm{supp}(m) = X$, then Theorem 1.8 yields $\|X\| = 0$.

Acknowledgements

Mitsuishi is supported by Research Fellowship for Young Scientists of the Japan Society for the Promotion of Science.

References

[Bacher and Sturm 2010] K. Bacher and K.-T. Sturm, “Localization and tensorization properties of the curvature-dimension condition for metric measure spaces”, *J. Funct. Anal.* **259**:1 (2010), 28–56. MR 2011i:53050 Zbl 1196.53027

[Burago et al. 1992] Y. Burago, M. Gromov, and G. Perelman, “Пространства А. Д. Александрова с органиченными снизу кривизнами”, *Uspekhi Mat. Nauk* **47**:2(284) (1992), 3–51. Translated as “A. D. Alexandrov spaces with curvature bounded below” in *Russian Math. Surveys* **47**:2 (1992), 1–58. MR 93m:53035 Zbl 0802.53018

[Burago et al. 2001] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics **33**, American Mathematical Society, Providence, RI, 2001. MR 2002e:53053 Zbl 0981.51016

[Cavalletti and Sturm 2012] F. Cavalletti and K.-T. Sturm, "Local curvature-dimension condition implies measure-contraction property", *J. Funct. Anal.* **262**:12 (2012), 5110–5127. MR 2916062 Zbl 1244.53050

[Gromov 1982] M. Gromov, "Volume and bounded cohomology", *Inst. Hautes Études Sci. Publ. Math.* **56** (1982), 5–99. MR 84h:53053 Zbl 0516.53046

[Kapovitch 2005] V. Kapovitch, "Restrictions on collapsing with a lower sectional curvature bound", *Math. Z.* **249**:3 (2005), 519–539. MR 2005k:53048 Zbl 1068.53024

[Kapovitch 2007] V. Kapovitch, "Perelman's stability theorem", pp. 103–136 in *Surveys in differential geometry, XI*, edited by J. Cheeger, Surv. Differ. Geom. **11**, International Press, 2007. MR 2009g:53057 Zbl 1151.53038

[Kapovitch et al. 2010] V. Kapovitch, A. Petrunin, and W. Tuschmann, "Nilpotency, almost nonnegative curvature, and the gradient flow on Alexandrov spaces", *Ann. of Math.* (2) **171**:1 (2010), 343–373. MR 2011d:53063 Zbl 1192.53040

[Korevaar and Schoen 1993] N. J. Korevaar and R. M. Schoen, "Sobolev spaces and harmonic maps for metric space targets", *Comm. Anal. Geom.* **1**:3-4 (1993), 561–659. MR 95b:58043 Zbl 0862.58004

[Mese and Zulkowski 2010] C. Mese and P. R. Zulkowski, "The Plateau problem in Alexandrov spaces", *J. Differential Geom.* **85**:2 (2010), 315–356. MR 2011k:53051 Zbl 1250.53066

[Ohta 2007] S.-i. Ohta, "On the measure contraction property of metric measure spaces", *Comment. Math. Helv.* **82**:4 (2007), 805–828. MR 2008j:53075 Zbl 1176.28016

[Otsu and Shioya 1994] Y. Otsu and T. Shioya, "The Riemannian structure of Alexandrov spaces", *J. Differential Geom.* **39**:3 (1994), 629–658. MR 95e:53062 Zbl 0808.53061

[Perelman 1991] G. Perelman, "A. D. Alexandrov spaces with curvatures bounded below II", preprint, Steklov Institute, St. Petersburg, 1991, Available at http://www.math.psu.edu/petrunin/papers/alexandrov/perelmanASWCBFB2+.pdf.

[Perelman 1993] G. Y. Perelman, "Начала теории Морса на пространствах Александрова", *Algebra i Analiz* **5**:1 (1993), 232–241. Translated as "Elements of Morse theory on Alexandrov spaces" in *St. Petersburg Math. J.* **5**:1 (1994), 205–213. MR 94h:53054

[Perelman and Petrunin 1994] G. Perelman and A. Petrunin, "Quasigeodesics and gradient curves in Alexandrov spaces", preprint, Pennsylvania State University, University Park, PA, 1994, Available at http://www.math.psu.edu/petrunin/papers/alexandrov/perelman-petrunin-QG.pdf.

[Petrunin 1995] A. Petrunin, *Quasigeodesics in multidimensional Alexandrov spaces*, thesis, University of Illinois, Urbana-Champaign, IL, 1995, Available at http://search.proquest.com/docview/304191433. MR 2693118

[Petrunin 2007] A. Petrunin, "Semiconcave functions in Alexandrov's geometry", pp. 137–201 in *Metric and comparison geometry*, edited by J. Cheeger and K. Grove, Surveys in differential geometry **11** (2006), International Press, Somerville, MA, 2007. MR 2010a:53052 Zbl 1166.53001

[Petrunin 2011] A. Petrunin, "Alexandrov meets Lott–Villani–Sturm", *Münster J. Math.* **4** (2011), 53–64. MR 2012m:53087 Zbl 1247.53038

[Plaut 1996] C. Plaut, "Spaces of Wald curvature bounded below", *J. Geom. Anal.* **6**:1 (1996), 113–134. MR 97j:53043 Zbl 0859.53023

[Sturm 2006a] K.-T. Sturm, "On the geometry of metric measure spaces, I", *Acta Math.* **196**:1 (2006), 65–131. MR 2007k:53051a Zbl 1105.53035

[Sturm 2006b] K.-T. Sturm, "On the geometry of metric measure spaces, II", *Acta Math.* **196**:1 (2006), 133–177. MR 2007k:53051b Zbl 1106.53032

[Yamaguchi 1997] T. Yamaguchi, "Simplicial volumes of Alexandrov spaces", *Kyushu J. Math.* **51**:2 (1997), 273–296. MR 98e:53071 Zbl 0914.53028

[Zhang and Zhu 2010] H.-C. Zhang and X.-P. Zhu, "Ricci curvature on Alexandrov spaces and rigidity theorems", *Comm. Anal. Geom.* **18**:3 (2010), 503–553. MR 2012d:53128 Zbl 1230.53064

Received February 17, 2013. Revised August 21, 2013.

AYATO MITSUISHI
MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
6-3, AOBA, ARAMAKI, AOBA-KU
SENDAI 980-8578
JAPAN
mitsuishi@math.tohoku.ac.jp

TAKAO YAMAGUCHI
DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY
KITASHIRAKAWA
KYOTO 606-8502
JAPAN
takao@math.kyoto-u.ac.jp

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.423

SEQUENCES OF OPEN RIEMANNIAN MANIFOLDS WITH BOUNDARY

RAQUEL PERALES AND CHRISTINA SORMANI

We consider sequences of open Riemannian manifolds with boundary that have no regularity conditions on the boundary. To define a reasonable notion of a limit of such a sequence, we examine δ*-inner regions*, that avoid the boundary by a distance δ. We prove Gromov–Hausdorff compactness theorems for sequences of these δ-inner regions. We then build "glued limit spaces" out of the Gromov–Hausdorff limits of δ-inner regions and study the properties of these glued limit spaces. Our main applications assume the sequence is noncollapsing and has nonnegative Ricci curvature. We include open questions.

1. Introduction

Recall that Gromov's Ricci compactness theorem states that a sequence of compact Riemannian manifolds with nonnegative Ricci curvature and a uniform upper bound on diameter has a subsequence that converges in the Gromov–Hausdorff sense to a metric space [8]. When the sequence of manifolds is noncollapsing, Gromov–Hausdorff limit spaces have a variety of properties, particularly restrictions on their metrics, their Hausdorff measures, and their topologies. These properties were proven by Cheeger, Colding, Naber, Wei and the second author [3; 4; 13; 5].

Here we consider an open Riemannian manifold (M^m, g) endowed with the length metric d_M, as in (3). We define the boundary to be

$$\partial M = \overline{M} \setminus M, \tag{1}$$

where $\overline{M}$ is the metric completion of M. For example, (M^m, g) may be a smooth manifold with boundary. However, we do not require any smoothness conditions on this boundary.

First observe that Gromov's Ricci compactness theorem does not hold for precompact open manifolds with boundary that have a uniform upper bound on diameter, even if they are flat and two-dimensional:

Perales is a doctoral student at Stony Brook. Sormani's research is partially supported by NSF DMS 10060059.
MSC2010: 53C23.
Keywords: Gromov–Hausdorff, manifold, boundary.

Figure 1. Models of Example 1.1: M_2, M_3, $M_4, \ldots$.

Example 1.1. The j-fold covering spaces M_j of the annuli $\mathrm{Ann}_0(1/j,1) \subset \mathbb{E}^2$, depicted in Figure 1, are flat surfaces such that

$$\mathrm{Diam}(M_j) \le 2+\pi \quad \text{and} \quad \mathrm{Vol}(M_j) = j(\pi - \pi(1/j)^2). \tag{2}$$

See Remark 5.5 for the proof that there is no subsequence of these spaces with a Gromov–Hausdorff limit.

Assuming both a uniform upper bound on volume and diameter, we still do not have Gromov–Hausdorff compactness:

Example 1.2. The smooth regions $M_j \subset \mathbb{E}^2$ with many spikes, depicted in Figure 2, have no subsequence with a Gromov–Hausdorff limit. See Example 2.13 for details.

Compactness theorems for sequences of Riemannian manifolds with boundary, assuming curvature controls on the boundary, have been proven by Kodani [11], Anderson, Katsuda, Kurylev, Lassas, and Taylor [1], Wong [14] and Knox [10]. A survey of these results has been written by the first author [12]. Since we do not wish to assume the boundary is smooth, we prove compactness theorems for regions which avoid the boundary (Theorem 1.4). We then glue together the limits of these regions (Theorem 6.3) and prove that these glued limit spaces have nice properties (Theorem 8.8).

Figure 2. Models of Example 1.2: M_4, M_6, M_8, $M_{12}, \ldots$.

Definition 1.3. Given an open Riemannian manifold (M, g_M) and $\delta > 0$, we define the δ-*inner region* as

$$M^\delta = \{x \in M : d_M(x, \partial M) > \delta\},$$

where ∂M is defined as in (1),

$$d_M(x, y) := \inf\{L_g(C) \,:\, C : [0, 1] \to M,\ C(0) = x,\ C(1) = y\}, \tag{3}$$

and

$$L_g(C) = \int_0^1 g(C'(t), C'(t))\, dt.$$

There are two metrics on the δ-inner region M^δ: the restricted metric d_M and the induced length metric

$$d_{M^\delta}(x, y) := \inf\{L_g(C) \,:\, C : [0, 1] \to M^\delta,\ C(0) = x,\ C(1) = y\}. \tag{4}$$

Note that d_{M^δ} is only defined between points in the same connected component of M^δ. The intrinsic diameter

$$\mathrm{Diam}(M^\delta, d_{M^\delta}) = \sup\{d_{M^\delta}(x, y) : x, y \in M^\delta\}$$

will be infinite if M^δ is not connected by rectifiable paths.

Theorem 1.4. *Given $m \in \mathbb{N}$, $\delta > 0$, $D > 0$, $V > 0$, and $\theta > 0$, let $\mathcal{M}_\theta^{m,\delta,D,V}$ be the class of open m-dimensional Riemannian manifolds M with boundary with nonnegative Ricci curvature*, $\mathrm{Vol}(M) \le V$, *and*

$$\mathrm{Diam}(M^\delta, d_{M^\delta}) \le D \tag{5}$$

that are noncollapsing at a point, in the sense that

$$\mathrm{Vol}(B_q(\delta)) \ge \theta \delta^m \quad \textit{for some } q \in M^\delta. \tag{6}$$

If $(M_j, g_j) \subset \mathcal{M}_\theta^{m,\delta,D,V}$, there is a subsequence $(M_{j_k}^\delta, d_{M_{j_k}})$ such that the metric completions with the restricted metric d_{M_j} converge in the Gromov–Hausdorff sense to a metric space (Y^δ, d).

Example 1.2 satisfies the conditions of this theorem, demonstrating why we can only obtain Gromov–Hausdorff convergence of the M_j^δ instead of the M_j themselves. The M_j^δ of Example 1.1 do not have Gromov–Hausdorff convergent subsequences (see Remark 5.5), demonstrating the necessity of the hypothesis requiring an upper bound on the volume. In Theorem 5.2, we remove the intrinsic diameter condition (5) and the noncollapsing condition (6), and assume conditions on closed geodesics and constant sectional curvature instead.

Theorem 1.4 and Theorem 5.2 are proved in Section 5. We start by reviewing Gromov–Hausdorff convergence in Section 2. In Sections 3 and 4 we study the limits of inner regions in sequences of manifolds that have Gromov–Hausdorff limits. See in particular Theorem 4.1. These sections contain many examples.

In Section 6 we define glued limit spaces for any sequence of open Riemannian manifolds (M_j, g_j), assuming that for all $\delta > 0$, the (M_j^δ, d_j) converge in the Gromov–Hausdorff sense to a metric space (Y^δ, d_δ). We build a "glued limit space" (Y, d_Y) from these Y^δ in Theorem 6.1 and Theorem 6.3. The metric completion of a glued limit space is called a "completed glued limit space."

Note that this glued limit space may exist even when (M_j, d_j) has no Gromov–Hausdorff limit, as in Example 2.13 (see Remark 6.10). The glued limit may not be precompact even when one has a sequence of flat Riemannian manifolds with boundary (Examples 6.11 and 6.12).

In general the completed glued limit space of a sequence of M_j need not be unique (Example 6.16). However, if the (M_j, d_{M_j}) have a Gromov–Hausdorff limit (X, d_X), then the completed glued limit space is unique and is embedded isometrically into X (Theorem 6.6). The completed glued limit space need not be isometric to the Gromov–Hausdorff limit (Example 4.10) even when the (M_j, g_j) are regions in the Euclidean plane satisfying all the hypothesis of Theorem 1.4 (Remark 6.7). Intuitively, regions which collapse relative to the boundary disappear, while regions which collapse that lie far from the boundary need not disappear.

In Section 7 we apply Theorems 5.2 and 1.4 to construct glued limit spaces for sequences of manifolds with curvature bounds (Theorems 7.1 and 7.4). In Section 8 we explore the properties of these glued limit spaces. First we present an example where the curvature bounds in the sequence of manifolds is lost in the Gromov–Hausdorff limit (Example 8.1). Then we prove Proposition 8.4 concerning glued limits of manifolds with constant sectional curvature. We close with Theorem 8.8, proving that glued limits constructed under the conditions of Theorem 1.4 have Hausdorff dimension m, Hausdorff measure at most V, and positive density everywhere. This final theorem is proved using Theorem 8.3, which proves certain balls in glued limit spaces are the Gromov–Hausdorff limits of nice balls in the open manifolds, combined with the Bishop–Gromov volume comparison theorem [8] and Colding's volume convergence theorem [4].

Throughout the paper we state open questions at 6.14, 8.6, 8.7, 8.10, and 8.9. The first author is in the process of proving Open question 8.10 as part of her doctoral dissertation. Please contact us if you would like to work on one of the other open questions or if you are interested in extending our theorems to the setting where the sequence has a negative uniform lower Ricci curvature bound or is allowed to collapse.

2. Background

Here we review Gromov–Hausdorff convergence and Gromov's compactness theorem [8]. A good resource for this material is [2].

2A. *Hausdorff convergence.* In [8], Gromov defined the Gromov–Hausdorff distance between pairs of compact metric spaces. We review this definition here.

Definition 2.1 (Hausdorff). The Hausdorff distance between two compact subsets A_1, A_2 of a metric space Z with metric d_Z is defined as

$$d_H^Z(A_1, A_2) = \inf\{r : A_1 \subset T_r(A_2),\ A_2 \subset T_r(A_1)\},$$

where the tubular neighborhood $T_r(A)$ is the set $T_r(A) = \{x \in Z : d_Z(x, A) < r\}$.

Observe that if one has a sequence of compact subsets $A_j \subset Z$ such that $d_H(A_j, A_\infty) \to 0$, then for all $a \in A_\infty$ there exists $a_j \in A_j$ such that $\lim_{j\to\infty} a_j = a$.

Lemma 2.2. *Suppose $A_j \subset Z$ are compact, $d_H^Z(A_j, A_\infty) = h_j \to 0$, $a_j \in A_j$ and $a_\infty \in A_\infty$ such that $d_Z(a_j, a_\infty) = \delta_j \to 0$. Then for all $r > 0$ there exist $r_j = r + \delta_j + h_j \to r$ such that the closed balls converge*:

$$d_H^Z\big(\bar{B}_{a_j}(r_j) \cap A_j, \bar{B}_{a_\infty}(r) \cap A_\infty\big) \to 0.$$

Here we are not assuming that A_∞ or A_j are length spaces. For completeness of exposition we include the proof of this well-known lemma:

Proof. Suppose $x \in \bar{B}_{a_\infty}(r) \cap A_\infty$; then $d_Z(x, a_\infty) \le r$ and $x \in A_\infty \subset T_{h_j}(A_j)$. So there exists $y_j \in A_j$ such that $d_Z(x, y_j) < h_j$. By the triangle inequality,

$$d(y_j, a_j) \le d(y_j, x) + d(x, a_\infty) + d(a_\infty, a_j) \le h_j + r + \delta_j = r_j.$$

Thus

$$\bar{B}_{a_\infty}(r) \cap A_\infty \subset T_{h_j}(\bar{B}_{a_j}(r_j) \cap A_j).$$

Now we need only show that for all $\varepsilon > 0$ the following inclusion holds for all sufficiently large j:

$$\bar{B}_{a_j}(r_j) \cap A_j \subset T_\varepsilon(\bar{B}_{a_\infty}(r) \cap A_\infty).$$

Suppose not. Then there exist $\varepsilon_0 > 0$, a subsequence $j \to \infty$ and elements

$$x_j \in (\bar{B}_{a_j}(r_j) \cap A_j) \setminus T_{\varepsilon_0}(\bar{B}_{a_\infty}(r) \cap A_\infty). \tag{7}$$

Since Z is compact and $T_{\varepsilon_0}(\bar{B}_{a_\infty}(r) \cap A_\infty)$ is open, a subsequence of the x_j converges to some

$$x_\infty \notin T_{\varepsilon_0}(\bar{B}_{a_\infty}(r) \cap A_\infty).$$

Since $d(x_j, a_j) \le r_j$, we have $d(x_\infty, a_\infty) \le r$. Since $x_j \in A_j$, there exists $y_j \in A_\infty$

such that $d(x_j, y_j) < h_j$. By the triangle inequality,

$$y_j \in B_{a_\infty}(r + h_j) \cap A_\infty.$$

Observe that, for our subsequence, $y_j \to x_\infty$; thus

$$x_\infty \in \bar{B}_{a_\infty}(r) \cap A_\infty \subset T_{\varepsilon_0}(\bar{B}_{a_\infty}(r) \cap A_\infty),$$

which is a contradiction. □

2B. *Gromov–Hausdorff convergence.*

Definition 2.3. An isometric embedding $\varphi : (X, d_X) \to (Z, d_Z)$ between metric spaces is a mapping which preserves distances:

$$d_Z(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2).$$

Definition 2.4 (Gromov). The Gromov–Hausdorff distance between a pair of compact metric spaces, (X_1, d_{X_1}) and (X_2, d_{X_2}), is defined as

$$d_{\mathrm{GH}}\big((X_1, d_{X_1}), (X_2, d_{X_2})\big) = \inf\{d_Z(\varphi_1(X_1), \varphi_2(X_2)) : \varphi_i : X_i \to Z\} \tag{8}$$

where the infimum is taken over all isometric embeddings $\varphi_i : X_i \to Z$ and all metric spaces Z.

Gromov proved that the Gromov–Hausdorff distance is a distance on the space of compact metric spaces. When studying metric spaces X_i which are only precompact, one takes the metric completions $\bar{X}_i$ before comparing such spaces using the Gromov–Hausdorff distance:

Definition 2.5. Given a precompact metric space (X, d_X), the metric completion $(\bar{X}, d_X)$ consists of equivalence classes of Cauchy sequences $\{x_1, x_2, x_3, \dots\}$ in X, where

$$d_X(\{x_j\}, \{y_j\}) = \lim_{j\to\infty} d_X(x_j, y_j),$$

and two Cauchy sequences are equivalent if the distance between them is 0. There is an isometric embedding

$$\varphi : X \to \bar{X} \quad \text{given by} \quad \varphi(x) = \{x, x, x, \dots\}.$$

In this paper we define the boundary of an open metric space to be

$$\partial X = \bar{X} \setminus X.$$

When M is a smooth Riemannian manifold with boundary, then this notion of boundary agrees with the standard notion of boundary. However, if M is a smooth Riemannian manifold with a singular point removed, then the boundary in our setting is just the missing singular point.

2C. *Lattices and Gromov–Hausdorff convergence.* One technique that can be applied to produce amazingly complicated Gromov–Hausdorff limits from surfaces is the construction of lattices. The basic, well-known lemma is as follows:

Lemma 2.6. *Let $X = [a_1, b_1] \times \cdots \times [a_k, b_k]$ with the taxi product metric*

$$d_X\big((x_1, \ldots, x_k), (y_1, \ldots, y_k)\big) = \sum_{i=1}^{k} |x_i - y_i|.$$

Then for any $\varepsilon > 0$ there exists a 2-dimensional manifold M_ε such that

$$d_{\mathrm{GH}}(M_\varepsilon, X) < \varepsilon.$$

The classic application of this lemma is to construct a Gromov–Hausdorff limit of Riemannian surfaces which is infinite-dimensional:

Example 2.7. Let $X_j = [0,1] \times \big[0, \tfrac{1}{2}\big] \times \cdots \times \big[0, \big(\tfrac{1}{2}\big)^j\big]$ with the taxi metric, and let

$$X = [0,1] \times \big[0, \tfrac{1}{2}\big] \times \cdots \times \big[0, \big(\tfrac{1}{2}\big)^j\big] \times \cdots$$

be the infinite-dimensional space also with the taxi metric

$$d_X\big((x_1, x_2, \ldots), (y_1, y_2, \ldots)\big) = \sum_{i=1}^{\infty} |x_i - y_i|.$$

Then

$$d_{\mathrm{GH}}(X_k, X) \le \sum_{j=k+1}^{\infty} \big(\tfrac{1}{2}\big)^j = \big(\tfrac{1}{2}\big)^k \to 0. \tag{9}$$

Thus, by Lemma 2.6, we have a sequence of surfaces M_k converging to X as well.

Since we are interested in manifolds with boundary, we will prove a stronger version of Lemma 2.6 that can be applied to produce examples later in the paper.

Proposition 2.8. *Suppose that $X = [a_1, b_1] \times \cdots \times [a_k, b_k]$ with the taxi product metric, and let $A \subset \partial X$ (possibly empty). Then for any $\varepsilon > 0$, there exists an open Riemannian surface M with boundary ∂M (possibly empty) such that*

$$d_{\mathrm{GH}}(M, X) < \varepsilon \quad \textit{and} \quad d_{\mathrm{GH}}(\partial M, A) < \varepsilon.$$

Suppose we have a collection of X_k and $A_k \subset \partial X_k$ as above, with subsets $B_k \subset X_k$ and isometric embeddings $\psi_k : B_{k+1} \to B_k$, and we glue $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_k$ via these isometric embeddings, and set $A = \bigcup A_k \subset X$. Then for any $\varepsilon > 0$ we have an open Riemannian surface M with boundary ∂M (possible empty) such that

$$d_{\mathrm{GH}}(M, X) < \varepsilon \quad \textit{and} \quad d_{\mathrm{GH}}(\partial M, A) < \varepsilon.$$

In fact, for any $\delta > 0$, using the restricted distances, we have

$$d_{\mathrm{GH}}\big((M \setminus T_\delta(\partial M), d_M), (X \setminus T_\delta(A), d_X)\big) < \varepsilon.$$

Proof. For the first part, we take a lattice $Y'_\varepsilon \subset Y_\varepsilon \subset X$ such that $X \subset T_{\varepsilon/2}(Y_\varepsilon)$. Here we use Y'_ε to denote the points and Y_ε to include 1-dimensional edges between the points in the lattice. Observe that $d_{Y_\varepsilon}(y_1, y_2) = d_X(y_1, y_2)$ because we are using the taxi norm. Let $A_\varepsilon \subset Y'_\varepsilon$ be chosen such that $A_\varepsilon \subset T_{\varepsilon/2}(A)$, so

$$d_{\mathrm{GH}}(Y_\varepsilon, X) < \frac{\varepsilon}{2} \quad \text{and} \quad d_{\mathrm{GH}}(A_\varepsilon, A) < \frac{\varepsilon}{2}. \tag{10}$$

Note that we may now view Y_ε as a graph. For example, if $X = [0,5] \times [0,6]$, $A = [0,5] \times \{6\}$, and $\varepsilon = 1$, then the left side of Figure 3 is the graph Y_ε, with A_ε depicted in red.

Next we construct a smooth surface M by replacing the lattice points in $A_\varepsilon \subset Y'_\varepsilon$ by small hemispheres of diameter $\ll \varepsilon$ and lattice points in $Y'_\varepsilon \setminus A_\varepsilon$ by small spheres of diameter $\ll \varepsilon$. We replace the line segments in Y_ε by arbitrarily thin cylinders of the same length, small enough that we can glue them to their corresponding spheres, smoothly replacing disjoint balls in those spheres or hemispheres. This creates a smooth manifold M such that ∂M is a union of the boundaries of the hemispheres, and such that

$$d_{\mathrm{GH}}(Y_\varepsilon, M) < \frac{\varepsilon}{2} \quad \text{and} \quad d_{\mathrm{GH}}(A_\varepsilon, \partial M) < \frac{\varepsilon}{2}.$$

See the right side of Figure 3, where M^2 is depicted in gray and ∂M^2 is in red. This completes the first claim in the proposition.

To complete the rest, we take M_k consisting of tubes joined at spheres and hemispheres close to X_k, as above, such that

$$d_{\mathrm{GH}}(X_k, M_k) < \frac{\varepsilon}{k} \quad \text{and} \quad d_{\mathrm{GH}}(A_k, \partial M_k) < \frac{\varepsilon}{k}.$$

Note that in the construction above we could have created $B'_k \subset Y'_k$ corresponding to B_k. We have $\varepsilon/(2k)$ almost distance-preserving maps $\psi'_k : B'_{k+1} \to B'_k$. So now we glue together the M_k to form M as follows. If $b \in B'_k$ maps to $\psi'_k(b) \in B'_k$,

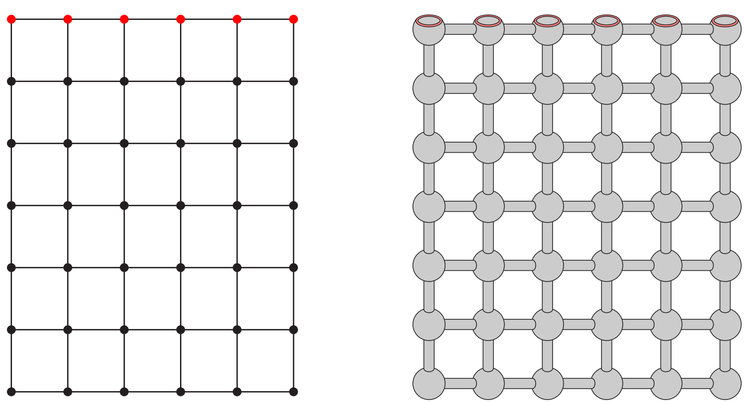

Figure 3. $A_\varepsilon \subset Y_\varepsilon$ and $\partial M \subset M$ as in the proof of Proposition 2.8.

we connect the sphere or hemisphere corresponding to b in M_k to a sphere or hemisphere corresponding to $\psi_k(b)$ in M_{k+1} by a very short, very thin tube. □

2D. ***Review of Gromov's compactness theorem.*** In [8], Gromov proved a compactness theorem for sequences of compact metric spaces. We review this theorem and related propositions here.

Theorem 2.9 (Gromov). *Given $D > 0$ and a function $N : (0, D] \to \mathbb{N}$, we define the collection $\mathcal{M}^{D,N}$ of compact metric spaces (X, d_X) with diameter $\leq D$ that can be covered by $N(\epsilon)$ balls of radius $\epsilon > 0$:*

$$X \subset \bigcup_{i=1}^{N(\epsilon)} B_{x_i}(\epsilon). \tag{11}$$

This collection $\mathcal{M}^{D,N}$ is compact with respect to the Gromov–Hausdorff distance.

It is standard to determine whether a metric space lies in such a compact collection by examining maximal collections of disjoint balls:

Proposition 2.10. *Given a metric space (X, d_X), let N be the maximum number of pairwise disjoint balls of radius $\epsilon/2$ that can lie in X. Then the minimum number of balls of radius ϵ required to cover X is at most N.*

Proof. Let $\{B_{x_i}(\epsilon/2) : i = 1, \dots, N\}$ be a maximal collection of pairwise disjoint balls of radius $\epsilon/2$. Let $x \in X$. Then $B_{x_i}(\epsilon/2) \cap B_x(\epsilon/2)$ is nonempty for some $i \in \{1, \dots, N\}$. Thus $d_X(x, x_i) < \epsilon$, and

$$X \subset \bigcup_{i=1}^{N} B_{x_i}(\epsilon). \qquad \square$$

In a Riemannian manifold or metric measure space, the volumes of balls may thus be applied to determine the function N.

Proposition 2.11. *If there exists $\Theta > 0$ such that*

$$\mathrm{Vol}(B_p(\epsilon)) / \mathrm{Vol}(M) \geq \Theta,$$

then the maximum number of disjoint balls of radius ϵ is at most $1/\Theta$.

Proof. We have

$$\mathrm{Vol}(M) \geq \sum_{i=1}^{N} \mathrm{Vol}(B_{x_i}(\epsilon)) \geq \sum_{i=1}^{N} \Theta \,\mathrm{Vol}(M) = N\Theta \,\mathrm{Vol}(M). \qquad \square$$

Gromov applies his compactness theorem in conjunction with these propositions to study the compactness of sequences of compact Riemannian manifolds for which one is able to control the volumes of balls. We will apply the same idea to study sequences of metric completions of open manifolds.

One of the beauties of Gromov's compactness theorem is that the converse holds as well:

Theorem 2.12 (Gromov). *Suppose (X_j, d_j) are compact metric spaces. Suppose that there exists $\epsilon_0 > 0$ such that X_j contains at least j disjoint balls of radius ϵ_0. Then no subsequence of the X_j has a Gromov–Hausdorff limit.*

In particular, if $(X_j, d_{X_j}) \xrightarrow{\text{GH}} (X, d_X)$, then they have a uniform upper bound on diameter. Nor can they have many spikes, as in the following example:

Example 2.13. Let

$$M_j = \{(\theta, r) : \theta \in S^1,\ r \in (1, 3 + \cos(j\theta))\}$$

with metric $g_j = dr^2 + r^2 d\theta^2$. Then $\mathrm{Vol}(M_j) \le \pi 4^2$, $\mathrm{Diam}(M_j) \le 3 + \pi + 3$, and M_j has 0 sectional curvature.

Observe that in M_j the balls of radius 1 about $(2\pi k/j, 3)$ are disjoint because paths between these points in M_j must reach within $r \le 2$ between the spikes and so have length at least $2(3-2)$. Thus, there are j disjoint balls of radius 1 in M_j, and no subsequence of the metric completions of the M_j converges in the Gromov–Hausdorff sense.

Example 2.14. Let

$$X_j = \big([0,1] \times [0,1]\big) \sqcup \big([0,1] \times \big[0, \tfrac{1}{2}\big]\big) \sqcup \cdots \sqcup \big([0,1] \times \big[0, \big(\tfrac{1}{2}\big)^j\big]\big) \tag{12}$$

be a disjoint union of spaces with taxicab metrics glued via the map $\psi(0, y) = (0, y)$. Then X_j has no Gromov–Hausdorff convergent subsequence, because it has j disjoint balls of radius 1 about the points $(1, 0)$. If we take the surfaces M_j as constructed in Proposition 2.8, such that

$$d_{\mathrm{GH}}(M_j, X_j) \to 0,$$

they also have no Gromov–Hausdorff convergent subsequence.

In a later paper, Gromov proved the following useful theorem [7, page 65] by defining an appropriate compact metric space and applying Theorem 2.16.

Theorem 2.15 (Gromov). *If one has a sequence of compact metric spaces (X_j, d_{X_j}) such that $(X_j, d_{X_j}) \xrightarrow{\text{GH}} (X_\infty, d_{X_\infty})$, then there exists a common compact metric space Z and isometric embeddings $\varphi_j : (X_j, d_{X_j}) \to (Z, d_Z)$ such that $d_H(\varphi_j(X_j), \varphi_\infty(X_\infty)) \to 0$.*

Theorem 2.16 (Blaschke). *If Z is a compact metric space then every sequence of closed subsets of Z has a subsequence that converges in the Hausdorff sense to a closed subset.*

Theorem 2.15 implies the Gromov–Hausdorff Arzelà–Ascoli theorem:

Theorem 2.17 (Gromov). *Assume $X_j \xrightarrow{\text{GH}} X$, $Y_j \xrightarrow{\text{GH}} Y$, and let $f_j : X_j \to Y_j$ be an equicontinuous sequence; i.e., for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that*

$$d_{X_j}(p,q) < \delta_\epsilon \implies d_{Y_j}(f_j(p), f_j(q)) < \epsilon.$$

Then there is a subsequence with a continuous limit function $f : X \to Y$. If the f_j are isometric embeddings, then so is f.

In particular, if the X_j are geodesic spaces, then so is the limit space [8].

2E. ***Gromov's Ricci compactness theorem.*** In this section we review Gromov's Ricci compactness theorem, which is based on the Bishop–Gromov volume comparison theorem [8]:

Theorem 2.18 (Bishop–Gromov). *If M is an m-dimensional Riemannian manifold with boundary having nonnegative Ricci curvature, and $B_p(R) \subset M^m$ does not reach the boundary, then for all $r \in (0, R)$ we have*

$$\frac{\mathrm{Vol}(B_p(r))}{\mathrm{Vol}(B_p(R))} \geq \left(\frac{r}{R}\right)^m. \tag{13}$$

Gromov's Ricci compactness theorem was originally stated for compact manifolds without boundary:

Theorem 2.19 (Gromov). *Let $m \in \mathbb{N}$, $D > 0$ and let $\mathcal{M}^{m,D}$ be the class of compact m-dimensional Riemannian manifolds M with nonnegative Ricci curvature and* $\mathrm{Diam}(M) \leq D$. *Here the manifolds do not have boundary. Then $\mathcal{M}^{m,D}$ is precompact with respect to the Gromov–Hausdorff distance.*

In fact, Gromov's compactness theorem has a commonly used version applied to balls, which we state as follows:

Theorem 2.20 (Gromov). *Let $m \in \mathbb{N}$, $D > 0$ and let $\mathcal{M}^m$ be the class of compact m-dimensional Riemannian manifolds M with nonnegative Ricci curvature. If $M_j \in \mathcal{M}^m$ and $p_j \in M_j$ such that $d(p_j, \partial M_j) > D$, there exists a subsequence such that $(B_{p_j}(D/3), d_{M_j})$ converges in the Gromov–Hausdorff distance.*

For completeness of exposition we show how Gromov's original proof implies Theorem 2.20.

Proof. Let $q \in B_{p_j}(D/3)$. Then $B_{p_j}(D/3) \subset B_q(2D/3) \subset B_{p_j}(D)$ does not reach the boundary of M_j, so we may apply the Bishop–Gromov volume comparison theorem to see that

$$\begin{aligned}\frac{\mathrm{Vol}(B_q(r))}{\mathrm{Vol}(B_{p_j}(D/3))} &\geq \frac{r^m}{(2D/3)^m}\frac{\mathrm{Vol}(B_q(2D/3))}{\mathrm{Vol}(B_{p_j}(D/3))} \\ &\geq \frac{r^m}{(2D/3)^m}\frac{\mathrm{Vol}(B_{p_j}(D/3))}{\mathrm{Vol}(B_{p_j}(D/3))} = \frac{(3r)^m}{(2D)^m}.\end{aligned}$$

So now we may apply Proposition 2.11 to complete the proof. □

2F. ***Volume convergence theorems.*** In [4], Colding proved the following volume convergence theorem:

Theorem 2.21 (Colding). *Let M_j^m be complete Riemannian manifolds with nonnegative Ricci curvature and $p_j \in M_j$ such that*

$$B_{p_j}(1) \xrightarrow{\mathrm{GH}} B_0(1) \subset \mathbb{E}^m,$$

where $\mathbb{E}^m$ is Euclidean space of dimension m. Then

$$\lim_{j\to\infty} \mathrm{Vol}(B_{p_j}(1)) = \mathrm{Vol}(B_0(1)).$$

Remark 2.22. The proof of this theorem does not require global nonnegative Ricci curvature on a complete manifold. In fact, M_j^m could be an open manifold as long as $B_{p_j}(2) \subset \overline{M}_j^m$ does not hit the boundary. One may not even need a radius of 2.

Colding applied this theorem to prove a number of theorems, including one in which the Gromov–Hausdorff limit is an arbitrary compact Riemannian manifold of the same dimension (also [4]):

Theorem 2.23 (Colding). *Let M_j^m and M_∞^m be compact Riemannian manifolds with nonnegative Ricci curvature for $j = 1, 2, 3, \dots$ such that*

$$M_j^m \xrightarrow{\mathrm{GH}} M_\infty^m.$$

Then for all $r > 0$ and for all $p_j \in M_j$ such that $p_j \to p_\infty$, we have

$$\lim_{j\to\infty} \mathrm{Vol}(B_{p_j}(r)) = \mathrm{Vol}(B_{p_\infty}(r)). \tag{14}$$

Remark 2.24. Again, Colding's proof does not really require M_j to be complete. These M_j could be open Riemannian manifolds as long as $B_{p_j}(r) \subset \overline{M}_j$ does not hit the boundary. Here we do not need to worry about twice the radius because the proof involves estimating countable collections of small balls $B_{q_{j,i}}(\epsilon_{j,i})$ in $B_{p_j}(r)$ and applying Theorem 2.21 to those small balls, and one can always ensure the $B_{q_{j,i}}(2\epsilon_{j,i})$ avoid the boundary as in Remark 2.22.

Cheeger and Colding then conducted a study of the properties of Gromov–Hausdorff limits of manifolds of nonnegative Ricci curvature in [3]. They improve upon Theorem 2.23, allowing M_∞ to be an arbitrary limit space as long as the sequence is noncollapsing:

Theorem 2.25 (Cheeger and Colding). *Let $V_0 > 0$ and let M_j^m be compact Riemannian manifolds with nonnegative Ricci curvature for $j = 1, 2, 3, \dots$, such that*

$$M_j^m \xrightarrow{\mathrm{GH}} M_\infty^m \quad \textit{and} \quad \mathrm{Vol}(M_j^m) \geq V_0.$$

Then for all $r > 0$ and for all $p_j \in M_j$ such that $p_j \to p_\infty \in M_\infty$, we have

$$\lim_{j\to\infty} \mathrm{Vol}(B_{p_j}(r)) = \mathcal{H}^m(B_{p_\infty}(r)), \tag{15}$$

where $\mathcal{H}^m$ is the Hausdorff measure of dimension m.

Remark 2.26. Again this theorem is proved locally, so as in Remark 2.24 this theorem holds when M_j^m are open Riemannian manifolds as long as $B_{p_j}(r) \subset \bar{M}_j^m$ do not touch the boundary.

Of course, Cheeger and Colding studied more than just manifolds with nonnegative Ricci curvature and more than just noncollapsing sequences in their work, but these theorems are the only ones needed in this paper. See also work of the second author with Wei for an adaption of their volume convergence theorem which deals with Hausdorff measures defined using restricted versus intrinsic distances [13].

3. Properties of inner regions

We defined in Definition 1.3 the inner regions M^δ of an open Riemannian manifold M. These spaces are open Riemannian manifolds; however, we will study them using the restricted distance d_M rather than the intrinsic length metric d_{M^δ} defined in (4). There are natural isometric embeddings of (M^δ, d_M) and its metric completion $(\bar{M}^\delta, d_M)$ into (M, d_M). Thus the metric completion is, in fact, compact when M is precompact. This occurs, for example, when M has finite diameter.

Example 3.1. In Figure 4, we depict a single flat manifold M^2, which is a flat disk with a spike attached. For a sequence $\delta_1 < \delta_2 < \delta_3 < \delta_4$, the gray inner regions depict M^{δ_i}. For δ sufficiently large, M^δ is an empty set.

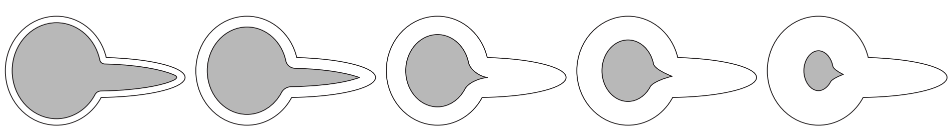

Figure 4. Example 3.1: Single M, varying δ.

Lemma 3.2. *For any sequence $\delta_i \to 0$, we have*

$$M = \bigcup_{i=1}^{\infty} M^{\delta_i}, \quad \textit{and in fact} \quad M = \bigcup_{\delta>0} M^{\delta}. \tag{16}$$

Proof. Let $x \in M$. Since M is open, $\varepsilon = d_M(x, \partial M) > 0$. Then $x \in M^{\varepsilon/2}$. □

Lemma 3.3. *Let $\delta > \delta' > 0$. If $y \in M^{\delta}$, then for any $\varepsilon < \delta - \delta'$ we have*

$$B(y, \varepsilon) = \{x \in M : d_M(x, y) < \varepsilon\} \subset M^{\delta'}.$$

Proof. Let $x \in B(y, \varepsilon)$, so $d_M(x, y) < \delta - \delta'$. Since $y \in M^{\delta}$, for all $z \in \partial M$ we have $d_M(y, z) > \delta$. By the triangle inequality,

$$d_M(x, z) \geq d_M(y, z) - d_M(x, y) > \delta - (\delta - \delta') = \delta'. \qquad \square$$

Inner regions M^{δ} with restricted metrics d_M are not necessarily length spaces:

Example 3.4. In the flat open manifold

$$M = \{(x, y) : x^2 + y^2 \in (1, 25)\} \subset \mathbb{E}^2,$$

the distance between $(3, 1)$ and $(-3, 1)$ is

$$d_M\big((3, 1), (-3, 1)\big) = 6$$

because they are joined by curves of length arbitrarily close to 6. However, for $\delta = 1$ we have

$$M^{\delta} = \{(x, y) : x^2 + y^2 \in (4, 16)\} \subset \mathbb{E}^2.$$

The length of any curve in M^{δ} between $(3, 1)$ and $(-3, 1)$ must go around $(0, 2)$ and thus has length at least $2\sqrt{9+1} > 6$.

In fact, inner regions of path connected manifolds need not be connected:

Example 3.5. Let M be the connected union of balls in the Euclidean plane:

$$M = B_{(4,0)}(5) \cup B_{(-4,0)}(5) \subset \mathbb{E}^2.$$

Then

$$\partial M = A_+ \cup A_-,$$

where

$$A_+ = \partial B_{(4,0)}(5) \cap \{(x, y) : x \geq 0\},$$

$$A_- = \partial B_{(-4,0)}(5) \cap \{(x, y) : -x \geq 0\}.$$

Note that

$$(0, 3), (0, -3) \in \partial M.$$

Thus, for $\delta > 3$,

$$M^{\delta} \cap \{(0, y) : y \in \mathbb{R}\} = \varnothing.$$

However, for $\delta < 5$, we have

$$(4,0), (-4,0) \in M^\delta.$$

Thus M^δ is not connected for $\delta \in (3,5)$.

4. Manifolds with Gromov–Hausdorff limits have converging inner regions

In this section we will prove:

Theorem 4.1. *Let M_j be precompact open metric spaces, (X, d_X) a compact metric space, and assume $(\overline{M}_j, d_{M_j}) \xrightarrow{\mathrm{GH}} (X, d_X)$. For each $\delta > 0$, there exists a sequence of indices $\{j_k\} \to \infty$ and a compact metric space $Y^\delta_{\{j_k\}} \subset X$ such that the subsequence of δ-inner regions $\overline{M}^\delta_{j_k}$ converges to $Y^\delta_{\{j_k\}}$:*

$$\big(\overline{M}^\delta_{j_k}, d_{M_{j_k}}\big) \xrightarrow{\mathrm{GH}} \big(Y^\delta_{\{j_k\}}, d_\delta\big). \tag{17}$$

If (17) *holds for $\delta = \delta_1$ and $\delta = \delta_2$, with $0 < \delta_2 < \delta_1$, then*

$$Y^{\delta_1}_{\{j_k\}} \subset Y^{\delta_2}_{\{j_k\}}.$$

Given a sequence of decreasing positive numbers $\delta_i \to 0$, one can choose the sequence $\{j_k\} \to \infty$ so that (17) *holds for all $\delta = \delta_i$; moreover, the union*

$$U_{\{\delta_i\},\{j_k\}} = \bigcup_i Y^{\delta_i}_{\{j_k\}}.$$

is an open subset of X.

Given two sequences $\{\delta_i\}, \{\beta_i\}$ such that (17) *holds for all $\delta \in \{\delta_i\} \cup \{\beta_i\}$, then*

$$U_{\{\delta_i\},\{j_k\}} = U_{\{\beta_i\},\{j_k\}}. \tag{18}$$

Note that M^δ_j can be an empty space; see Example 4.8. Consider the Gromov–Hausdorff limit of an empty metric space to be an empty metric space.

Remark 4.2. In Example 4.9 we will see that a subsequence j_k may be necessary to obtain GH convergence of the δ-inner regions, and that $U_{\{\delta_i\},\{j_k\}}$ depends on the choice of the subsequence. Even the closure of $U_{\{\delta_i\},\{j_k\}}$ may depend on the choice of subsequence j_k; see Example 4.11. The $U_{\{\delta_i\},\{j_k\}}$ may be disjoint and not isometric; see Example 4.12.

4A. ***Hausdorff convergence of δ-inner regions.*** We begin with a basic theorem:

Theorem 4.3. *Let (Z, d_Z) be a compact metric space. Suppose $M_j \subset Z$ are open metric spaces with the induced metric and $X \subset Z$ is closed and such that $\overline{M}_j \xrightarrow{\mathrm{H}} X$. Then, for each $\delta > 0$, there exist a sequence of indices $\{j_k\} \to \infty$ and a compact set $W^\delta_{\{j_k\}} \subset X$ such that*

$$\overline{M}^\delta_{j_k} \xrightarrow{\mathrm{H}} W^\delta_{\{j_k\}}. \tag{19}$$

If (19) *holds for* $\delta = \delta_1, \delta_2$, *with* $0 < \delta_2 < \delta_1$, *then*

$$W^{\delta_1}_{\{j_k\}} \subset W^{\delta_2}_{\{j_k\}}.$$

Given a sequence of positive numbers $\delta_i \to 0$, *there exists a sequence of indices* $\{j_k\} \to \infty$ *such that* (19) *holds for all* $\delta = \delta_i$; *moreover the union*

$$U'_{\{\delta_i\},\{j_k\}} = \bigcup_i W^{\delta_i}_{\{j_k\}}$$

is an open subset of X.

Given two sequences $\{\delta_i\}, \{\beta_i\}$ *such that* (19) *holds for all* $\delta \in \{\delta_i\} \cup \{\beta_i\}$, *we have*

$$U'_{\{\delta_i\},\{j_k\}} = U'_{\{\beta_i\},\{j_k\}}. \tag{20}$$

Again here, M^δ_j can be an empty space. We consider the Hausdorff limit of an empty metric space to be an empty metric space.

Before we prove this theorem, we provide an example demonstrating that even if

$$\overline{M}^{\delta_1}_{j_k} \xrightarrow{\text{H}} W^{\delta_1}_{\{j_k\}} \quad \text{and} \quad \overline{M}^{\delta_2}_{j_k} \xrightarrow{\text{H}} W^{\delta_2}_{\{j_k\}}$$

for some $\delta_1 > \delta_2 > 0$, there may be $\delta \in (\delta_2, \delta_1)$ for which $\overline{M}^{\delta}_{j_k}$ does not converge:

Example 4.4. Fix $\varepsilon < \frac{1}{3}$. In 2-dimensional Euclidean space $\mathbb{E}^2$, consider the sequence M_j, where M_{2j} is a ball of radius 1 with a spike of width 4ε attached to it, as depicted in Figure 4, and M_{2j+1} is a ball of radius 1 with a spike whose width decreases from 6ε to 4ε as $j \to \infty$. Then $\overline{M}^{\varepsilon}_j$ converges to ball of radius $1-\varepsilon$ with a spike of width 2ε, and $\overline{M}^{3\varepsilon}_j$ converges to a ball of radius $1-3\varepsilon$ with no spike attached. But $\overline{M}^{2\varepsilon}_{2j}$ converges to a ball of radius $1-2\varepsilon$, while $\overline{M}^{2\varepsilon}_{2j+1}$ converges to a ball of radius $1-2\varepsilon$ with a line segment attached to it. Thus $M^{2\varepsilon}_j$ does not converge in the Hausdorff sense.

In the proof of Theorem 4.3 we will apply the following fact:

Remark 4.5. Recall that if $\{A_j\}$ is a sequence of closed subsets of a metric space A such that $A_j \xrightarrow{\text{H}} A_\infty$, then

$$A_\infty = \bigl\{a \in A : \text{for all } j \in \mathbb{N}, \text{ there exist } a_j \in A_j \text{ such that } \lim_{j\to\infty} a_j = a\bigr\}.$$

Any subsequence $\{A_{j_k}\}$ of $\{A_j\}$ also converges in the Hausdorff sense to A_∞. Then

$$A_\infty = \bigl\{a \in A : \text{for all } k \in \mathbb{N}, \text{ there exist } a_{j_k} \in A_{j_k} \text{ such that } \lim_{k\to\infty} a_{j_k} = a\bigr\}.$$

Proof of Theorem 4.3. Apply Theorem 2.16 to the sequence $\{\overline{M}^\delta_j\}_{j=1}^\infty$ to get a subsequence $\{\overline{M}^\delta_{j_k}\}_{k=1}^\infty$ and a compact set $W^\delta_{\{j_k\}}$ such that (19) is satisfied. Since $\overline{M}^\delta_{j_k} \subset \overline{M}_{j_k}$, we have $W^\delta_{\{j_k\}} \subset X$. Similarly, $W^{\delta_1}_{\{j_k\}} \subset W^{\delta_2}_{\{j_k\}}$ when (19) holds for $0 < \delta_2 < \delta_1$.

Given $\delta_i \to 0$, start with δ_1. By Theorem 2.16 there exists a sequence of integers $\{\iota_k(\delta_1)\} \to \infty$ and a compact set $W^{\delta_1}_{\{\iota_k(\delta_1)\}}$ such that

$$\overline{M}^{\delta_1}_{\iota_k(\delta_1)} \xrightarrow{\text{H}} W^{\delta_1}_{\{\iota_k(\delta_1)\}}.$$

For $n > 1$, there exists a subsequence $\{\iota_k(\delta_n)\}_k$ of $\{\iota_k(\delta_{n-1})\}_k$ and a compact set $W^{\delta_n}_{\{\iota_k(\delta_n)\}}$ such that

$$\overline{M}^{\delta_n}_{\iota_k(\delta_n)} \xrightarrow{\text{H}} W^{\delta_n}_{\{\iota_k(\delta_n)\}}.$$

Define $j_k = \iota_k(\delta_k)$. Then $\{j_k\}_{k=n}^{\infty}$ is a subsequence of $\{\iota_k(\delta_n)\}_{k=1}^{\infty}$, and thus (19) holds for all n.

Let y be an element of $U'_{\{\delta_i\},\{j_k\}}$. There exists $N \in \mathbb{N}$ such that $y \in W^{\delta_i}_{\{j_k\}}$ for $i \geq N$. Suppose that $x \in X$ and $d_Z(x,y) < \delta_N/6$. Since $y \in W^{\delta_i}_{\{j_k\}}$, choose $y_{j_k} \in \overline{M}^{\delta_N}_{j_k}$ such that $y = \lim_{j\to\infty} y_{j_k}$ and $d_Z(y, y_{j_k}) < \delta_N/6$. Analogously, take $x_j \in \overline{M}_j$ such that $x = \lim_{j\to\infty} x_j$ and $d_Z(x, x_j) < \delta_N/6$. Then

$$d_Z(x_{j_k}, y_{j_k}) < d_Z(x_{j_k}, x) + d_Z(x,y) + d_Z(y, y_{j_k}) < \frac{\delta_N}{2}.$$

This implies that $d_Z(x_{j_k}, \partial(M_{j_k})) > \delta_N/2$. Then $x \in W^{\delta_i}_{\{j_k\}} \subset U'_{\{\delta_i\},\{j_k\}}$ for some $i > N$.

Given another sequence $\beta_i \to 0$ such that (19) holds for all $\delta = \beta_i$, select for each i some $l(i)$ such that $\delta_{l(i)} < \beta_i$. Then

$$W^{\beta_i}_{\{j_k\}} \subset W^{\delta_{l(i)}}_{\{j_k\}},$$

and so $U'_{\{\beta_i\},\{j_k\}} \subset U'_{\{\delta_{l(i)}\},\{j_k\}} \subset U'_{\{\delta_i\},\{j_k\}}$. Conversely, $U'_{\{\delta_i\},\{j_k\}} \subset U'_{\{\beta_i\},\{j_k\}}$. □

Definition 4.6. With the hypotheses of Theorem 4.3, define

$$U'_{\{j_k\}} = \bigcup W^{\delta}_{\{j_k\}}, \tag{21}$$

where the union is taken over all δ for which $\overline{M}^{\delta}_{j_k}$ is a sequence that converges in the Hausdorff sense to a metric space $W^{\delta}_{\{j_k\}}$, and define

$$U' = \bigcup_{\delta>0} W^{\delta}, \tag{22}$$

where W^{δ} is the Hausdorff limit space of some convergent subsequence of $\overline{M}^{\delta}_j$.

4B. *Finding limits of inner regions in the Gromov–Hausdorff limits.*

Proof of Theorem 4.1. By Theorem 2.15 there exists a common metric space Z and isometric embeddings $\varphi_j : (\overline{M}_j, d_{M_j}) \to (Z, d_Z)$, $\varphi : (X, d_X) \to (Z, d_Z)$ such that

$$d^Z_H(\varphi_j(\overline{M}_j), \varphi(X)) \to 0.$$

Now we can apply Theorem 4.3. For each $\delta > 0$, there exist a subsequence $\varphi_{j_k}(\overline{M}^\delta_{j_k})$ and a compact set $W^\delta_{\{j_k\}} \subset \varphi(X)$ such that

$$\varphi_{j_k}(\overline{M}^\delta_{j_k}) \xrightarrow{\mathrm{H}} W^\delta_{\{j_k\}}.$$

Let $Y^\delta_{\{j_k\}} = \varphi^{-1}(W^\delta_{\{j_k\}})$. Clearly, (17) holds and $Y^{\delta_1}_{\{j_k\}} \subset Y^{\delta_2}_{\{j_k\}}$ when (17) holds for $0 < \delta_2 < \delta_1$. Given a sequence of positive numbers $\delta_i \to 0$, there exists a subsequence $\{j_k\} \subset \mathbb{N}$ such that

$$\varphi_{j_k}(\overline{M}^{\delta_i}_{j_k}) \xrightarrow{\mathrm{H}} W^{\delta_i}_{\{j_k\}}$$

for all i. Then (17) holds for all i, and $U_{\{\delta_i\},\{j_k\}} = \varphi^{-1}(U'_{\{\delta_i\},\{j_k\}})$ is an open subset of X that does not depend on the sequence δ_i. □

4C. *Unions of limits of inner regions in Gromov–Hausdorff limits.* The following notion of an "inner union" has some interesting properties.

Definition 4.7. In the situation of Theorem 4.1, let $D_{\{j_k\}}$ denote the set of $\delta > 0$ such that (17) holds for a given sequence $\{j_k\}$. We put

$$U_{\{j_k\}} = \bigcup_{\delta \in D_{\{j_k\}}} Y^\delta_{\{j_k\}} \tag{23}$$

and call this set an *inner union of limits* for the sequence $\{M_j\}$. Observe that, by (18), we have

$$U_{\{j_k\}} = U_{\{\delta_i\},\{j_k\}} \tag{24}$$

for any sequence $\{\delta_i\} \to 0$ of elements of $D_{\{j_k\}}$.

In Theorem 6.6 we will prove that $U_{\{j_k\}}$ is a special case of the glued limits we will construct in Theorem 6.3. Since it is easy to understand the properties of these $U_{\{j_k\}}$, we present a few examples of them here so that we may refer to them later as examples of glued limit spaces.

Example 4.8. Let M_j be a Euclidean disk of radius $1/j$. Then $\overline{M}_j \xrightarrow{\mathrm{GH}} X$, where X is a single point. For any $\delta > 0$, taking $j > 1/\delta$, we have $M^\delta_j = \varnothing$. Thus the inner union of limits is empty for any choice of subsequence.

In the following example we see that $U_{\{j_k\}}$ depends on the subsequence $\{j_k\}$, and in the next we see that X is not necessarily contained in the closure of $U_{\{j_k\}}$, even if the closure is nonempty.

Example 4.9. Let M_{2j} be the Euclidean disk of radius 1 and M_{2j+1} the Euclidean disk with the center point removed. Then $\overline{M}_j$ is a closed Euclidean disk as is the limit space X. Given $\delta \in (0, 1)$, M^δ_{2j} is the Euclidean disk of radius $1-\delta$. Their metric completions converge to the closed disk of radius $1-\delta$. $U_{\{2j\}}$ is the open Euclidean disk of radius 1. However, M^δ_{2j+1} is a Euclidean annulus $\mathrm{Ann}_0(\delta, 1-\delta)$, and the metric completions converge to the closure of this annulus. $U_{\{2j+1\}}$ is the open Euclidean disk of radius 1 with the center point removed. In this example $U = U_{\{2j\}}$.

Example 4.10. In 2-dimensional Euclidean space, consider the sequence of balls with attached spikes depicted in Figure 4. The Gromov–Hausdorff limit of the sequence is a ball with an interval attached, while the closure of U is just the closed ball.

In Example 4.9, we saw that $U_{\{2j\}} \neq U_{\{2j+1\}}$, yet their closures are the same. This is not always the case; $U_{\{j_k\}}$ could even be an empty set.

Example 4.11. For $j \in \mathbb{N}$, let M_{2j} be a flat torus (so it has no boundary), and let the M_{2j+1} be flat tori with increasingly dense small holes cut out, the holes getting smaller and smaller so the M_{2j+1} still converge to the flat torus X. Then $U_{\{2j\}} = X$, but for any $\delta > 0$, M^{δ}_{2j+1} becomes an empty set. So $U_{\{2j+1\}}$ is the empty set.

Example 4.12. For $j \in \mathbb{N}$, let M_{2j} be a flat torus $S^1 \times S^1$, with increasingly many dense small holes in $W \times S^1$, where $W = (0, \pi/4) \subset S^1$, and let M_{2j+1} be a flat torus $S^1 \times S^1$, with increasingly many dense small holes in $(S^1 \setminus W) \times S^1$. Then

$$U_{\{2j\}} = (S^1 \setminus W) \times S^1 \quad \text{and} \quad U_{\{2j+1\}} = W \times S^1 \tag{25}$$

with the restricted distance from $S^1 \times S^1$, which are disjoint and not isometric to each other.

Example 4.13. It is possible for a sequence of open Riemannian manifolds M_j to have δ-inner regions M^{δ}_j which converge in the Gromov–Hausdorff sense to some Y^{δ} for all $\delta > 0$, and yet the limit has two distinct inner unions $U_{\{2j\}} \neq U_{\{2j+1\}}$. This can be seen, for example, with the following F-shaped regions:

$$M_j = (0, 1/j) \times (-1, 0] \;\cup\; (0,1) \times (0,3) \;\cup\; [1,3) \times (0,1) \;\cup\; [1,3) \times (2,3) \setminus A_j$$

in the Euclidean plane, where A_{2j} is an increasingly dense collection of increasingly tiny balls in $(1,3) \times (0,1)$, and A_{2j+1} is an increasingly dense collection of increasingly tiny balls in $(1,3) \times (2,3)$. Then

$$M_j \xrightarrow{\mathrm{GH}} X = \;(0,1) \times (0,3) \;\cup\; [1,3) \times (0,1) \;\cup\; [1,3) \times (2,3).$$

For $\delta > 0$ fixed, taking j large enough that $1/j < 2\delta$, we see that

$$M^{\delta}_{2j} \xrightarrow{\mathrm{GH}} Y^{\delta} = (\delta, 1-\delta) \times (\delta, 3-\delta) \;\cup\; [1-\delta, 3-\delta) \times (2+\delta, 3-\delta),$$

which is isometric to

$$M^{\delta}_{2j+1} \xrightarrow{\mathrm{GH}} Y^{\delta} = (\delta, 1-\delta) \times (\delta, 3-\delta) \;\cup\; [1-\delta, 3-\delta) \times (\delta, 1-\delta).$$

Thus, the M_j^δ have a GH limit without taking a subsequence. On the other hand, the inner unions of limits are not equal, only isometric:

$$U_{\{2j\}} = (0,1)\times(0,3) \;\cup\; [1,3)\times(2,3) \subset X,$$
$$U_{\{2j+1\}} = (0,1)\times(0,3) \;\cup\; [1,3)\times(0,1) \subset X.$$

We will prove in Theorem 6.6 that when M_j^δ have GH limits for all δ, all closures of inner unions of limits are isometric.

5. Converging inner regions of sequences with curvature bounds

In this section, we prove that δ-inner regions converge under certain geometric hypotheses on the manifolds even when the manifolds themselves have no Gromov–Hausdorff limits.

5A. *Constant sectional curvature.* Here we prove that the inner regions of a sequence of manifolds in the following class have a subsequence which converges in the Gromov–Hausdorff sense.

Definition 5.1. Given $m \in \mathbb{N}$, $H \in \mathbb{R}$, $V > 0$, and $l > 0$, we define $\mathcal{M}_H^{m,V,l}$ to be the class of connected open Riemannian manifolds M of dimension at most m with constant sectional curvature $\mathrm{Sect}_M = H$, $\mathrm{Vol}(M) \le V$, and

$$L_{\min}(M) = \inf\{L_g(C) : C \text{ is a closed geodesic in M }\} > l,$$

where a closed geodesic is any geodesic which starts and ends at the same point.

Recall that complete simply connected manifolds with constant sectional curvature $H \le 0$ have no closed geodesics, by Hadamard's theorem, while those with $H > 0$ have $L(M) = 2\pi/\sqrt{H}$. (See [6].) Here we are requiring that the closed geodesic lies in an open manifold M, and we do not have completeness.

Theorem 5.2. *Given any $\delta > 0$, if $(M_j, g_j) \subset \mathcal{M}_H^{m,V,l}$, then there is a subsequence $(M_{j_k}^\delta, d_{M_{j_k}})$ such that the metric completion with the restricted metric converges in the Gromov–Hausdorff sense to a metric space (Y^δ, d). In particular, the extrinsic diameters measured using the restricted metric are bounded uniformly*:

$$\mathrm{Diam}(M_{j_k}^\delta, d_{M_{j_k}}) \le \epsilon_0 \frac{V}{V_H^m(\epsilon_0)}, \quad \mathrm{Diam}(Y^\delta, d) \le \epsilon_0 \frac{V}{V_H^m(\epsilon_0)},$$

where

$$\epsilon_0 = \begin{cases} \frac{1}{2}\min\{\delta, \frac{l}{2}, \frac{\pi}{\sqrt{H}}\} & \text{if } H > 0, \\ \frac{1}{2}\min\{\delta, \frac{l}{2}\} & \text{otherwise}, \end{cases} \tag{26}$$

and $V_H^m(\epsilon_0)$ is the volume of a ball of radius ϵ_0 in the complete simply connected space with constant sectional curvature H.

Remark 5.3. There are no closed geodesics in the M_j of Examples 1.1 and 1.2, so $L(M_j) = \infty$. These examples have $H = 0$ and $m = 2$. Since Example 1.2 also has a uniform upper bound on volume, it demonstrates why we can only obtain Gromov–Hausdorff convergence of the M_j^δ instead of the M_j themselves. The M_j^δ of Example 1.1 do not have GH-convergent subsequences (see Remark 5.5), demonstrating the necessity of an upper bound on volume.

Proof of Theorem 5.2. Let $M \in \mathcal{M}_H^{m,V,l}$ and $p \in M^\delta$. In view of (26), we see that if $0 < \epsilon < \epsilon_0$, then $B_p(\epsilon)$ does not reach the boundary of M and does not contain any conjugate point to p, since one does not reach a conjugate point before one would in the comparison space.

We claim that there are also no cut points to p in $B_p(\epsilon)$. If there was a cut point q, then proceeding in a similar way to Klingenberg [9], we see that there exists a closed geodesic starting at p of length at most $2d(p,q) < 2\epsilon_0$. By hypothesis, the length of this closed geodesic is greater than l, which is a contradiction.

Thus there is a Riemannian isometric diffeomorphism

$$\psi : B_p(\epsilon_0) \to B_x(\epsilon_0) \subset M_H^m, \tag{27}$$

where M_H^m is the simply connected space of constant sectional curvature H. In particular, $\mathrm{Vol}(B_p(\epsilon))$ is greater than or equal to the volume of a ball of the same radius in a simply connected space form of constant curvature H. By combining Proposition 2.11 with Proposition 2.10 and then Gromov's compactness theorem, there is a subsequence $(M_{j_k}^\delta, d_{M_{j_k}})$ such that the metric completion with the restricted metric converges in the Gromov–Hausdorff sense to a metric space (Y^δ, d). Notice that by Proposition 2.11, the maximum number of disjoint balls of radius $\epsilon_0/2$ that lie in M is at most $(V/V_H^m)(\epsilon_0/2)$. Thus, by Proposition 2.10, the minimum number of balls of radius ϵ_0 needed to cover M is at most $(V/V_H^m)(\epsilon_0/2)$. From this it follows that

$$\mathrm{Diam}(M_{j_k}^\delta, d_{M_{j_k}}) \le \epsilon_0 \frac{V}{V_H^m(\epsilon_0/2)}.$$

Since

$$\mathrm{Diam}(M_{j_k}^\delta, d_{M_{j_k}}) \to \mathrm{Diam}(Y^\delta, d),$$

we conclude that

$$\mathrm{Diam}(Y^\delta, d) \le \epsilon_0 \frac{V}{V_H^m(\epsilon_0/2)}. \qquad \square$$

Remark 5.4. If the injectivity radius for each $p \in M_j^\delta$ is bounded above by a positive constant, then the condition on the length of closed geodesics in Theorem 5.2 is satisfied.

5B. ***Examples with constant sectional curvature.*** The volume condition in Theorem 5.2 may not be replaced by a condition on diameter:

Remark 5.5. Let (M_j, g_j) be the j-th covering space of $\mathrm{Ann}_{(0,0)}(1/j, 1) \subset \mathbb{E}^2$.

Since every point in M_j has distance less than 1 from the inner boundary, and the inner boundary has length $j2\pi(1/j) = 2\pi$, we know

$$\mathrm{Diam}(M_j, d_{M_j}) \leq 2\pi + 2. \tag{28}$$

Yet the number of disjoint balls of radius $\delta < \frac{1}{4}$ centered on the cover of $\partial B_{(0,0)}\left(\frac{1}{2}\right)$ is greater than $2j$. So there is no subsequence of M_j^δ which converges in the Gromov–Hausdorff sense.

This sequence fails to satisfy the volume condition of Theorem 5.2:

$$\mathrm{Vol}(M_j) = j(\pi 1^2 - \pi/j^2) = \pi \frac{j^2-1}{j}.$$

It is worth observing that the intrinsic diameters

$$\mathrm{Diam}(M_j^\delta, M_j^\delta) \geq j2\pi\left(\delta + \frac{1}{j}\right)$$

also diverge to infinity.

Remark 5.6. The flat manifolds of Example 1.2, described more explicitly in Example 2.13, satisfy the hypothesis of Theorem 5.2. See Figure 5. In fact, for fixed $\delta > 0$, once $(2\pi/j)4 < \delta$, every point with $r \geq 2$ lies within a distance δ from the boundary because the spike is less than δ wide. So all the M_j^δ eventually lie within $r < 2$, where the metric is just the standard Euclidean metric, and there is a uniform bound on the number of disjoint balls. So the Gromov–Hausdorff limit also lies within the Euclidean ball of radius 2. On the other hand, every point within the ball of radius $1+\delta < r < 2-\delta$ lies in M_j^δ, so the Gromov–Hausdorff limit Y^δ contains $\mathrm{Ann}_{(0,0)}(1+\delta, 2-\delta)$. In fact, Y^δ is the metric completion of this annulus with the flat Euclidean metric.

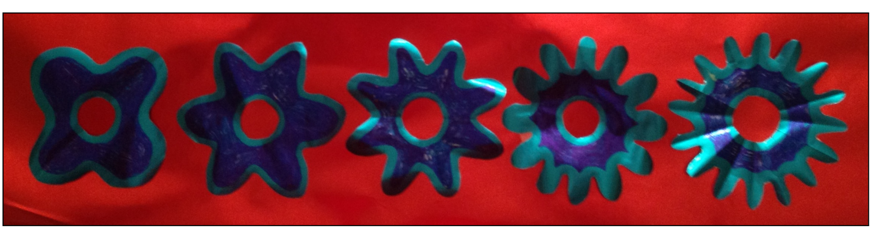

Figure 5. Models of Example 1.2: M_4^δ, M_6^δ, M_8^δ, M_{12}^δ, $M_{16}^\delta, \ldots$

5C. ***Manifolds with nonnegative Ricci curvature.*** Here we prove Theorem 1.4 by applying Gromov's compactness theorem (Theorem 2.9) combined with the following proposition:

Proposition 5.7. *If (M, g_M) is a compact Riemannian manifold with boundary having nonnegative Ricci curvature, then for any $\delta > 0$ and any $\epsilon \in (0, \delta/2)$, the δ-inner region M^δ contains a finite collection of points $\{p_1, p_2, \ldots, p_N\}$ such that*

$$M^\delta \subset \bigcup_{i=1}^{N} B_{p_i}(\epsilon),$$

where

$$N \le N(\delta, \epsilon, D_\delta, V, \theta) = \frac{V}{\theta}\left(\frac{2^{2D_\delta/\epsilon}}{\epsilon}\right)^m,$$

$m = \dim(M)$, $\mathrm{Vol}(M) \le V$, $\mathrm{Diam}(M^\delta, d_{M^\delta}) \le D_\delta$, *and*

$$\sup\{\mathrm{Vol}(B_q(\delta)) : q \in M^\delta\} \ge \theta\delta^m. \tag{29}$$

Remark 5.8. In this proposition we can use the volume of any ball centered in M^δ to estimate θ in (29). This allows us to study sequences like those in Example 3.1. One does not need a Ricci curvature condition if one has a uniform lower bound on the volumes of all balls centered in M^δ, as can be seen in Proposition 2.11 in the review of Gromov–Hausdorff convergence.

Proof of Proposition 5.7. By Propositions 2.10 and 2.11 in the review of Gromov–Hausdorff convergence, we need only to find a uniform lower bound on the volume of an arbitrary ball $B_p(\epsilon)$ centered at $p \in M^\delta$.

Fix q as in (6). Then by the fact that $B_q(\delta)$ does not hit ∂M and M has nonnegative Ricci curvature, we may apply the Bishop–Gromov volume comparison theorem to see that

$$\theta \le \frac{\mathrm{Vol}(B_q(\delta))}{\delta^m} \le \frac{\mathrm{Vol}(B_q(\epsilon))}{\epsilon^m} \tag{30}$$

because $\delta > \delta/2 > \epsilon$.

Let $C : [0, 1] \to M^\delta$ be the shortest curve from p to q. Then

$$L = L(C) \le \mathrm{Diam}(M^\delta, d_{M^\delta}) \le D_\delta.$$

Let $n > L/\epsilon$ and $x_j = C(t_j)$, where $t_j = jL/n$, so that

$$x_j \in M^\delta \quad \text{and} \quad d_M(x_{j-1}, x_j) = L/n < \epsilon.$$

In particular, $B_{x_j}(2\epsilon)$ lies within the interior of M and has nonnegative Ricci curvature. Thus, by the Bishop–Gromov volume comparison theorem,

$$\mathrm{Vol}(B_{x_j}(\epsilon)) \ge \frac{1}{2^m}\mathrm{Vol}(B_{x_j}(2\epsilon)) \ge \frac{1}{2^m}\mathrm{Vol}(B_{x_{j+1}}(\epsilon)).$$

Applying this repeatedly for $j = 1, \dots, n$, and finally applying (30), we have

$$\mathrm{Vol}(B_p(\epsilon)) \geq \frac{1}{2^{mn}} \mathrm{Vol}(B_q(\epsilon)) \geq \frac{1}{2^{mD_\delta/\epsilon}} \mathrm{Vol}(B_q(\epsilon)) \geq \frac{1}{2^{mD_\delta/\epsilon}} \theta \epsilon^m.$$

The estimate on $N(\delta, \epsilon, D_\delta, V, \theta)$ then follows immediately from Propositions 2.10 and 2.11. □

6. Glued limit spaces

In this section we define *glued limit spaces* and *completed glued limit spaces* and study their properties without making any curvature assumptions. We begin by constructing isometric embeddings

$$\varphi_{\delta_{i+1},\delta_i} : Y^{\delta_{i+1}} \to Y^{\delta_i}$$

between the Gromov-Hausdorff limits Y^{δ_i} of inner regions $M_j^{\delta_i}$ (Theorem 6.1). We then apply these isometric embeddings to glue together the Y^{δ_i} and construct a glued limit space $Y = Y(\{\delta_i\}, \{\varphi_{\delta_{i+1},\delta_i}\})$ (Theorem 6.3).

We next study sequences of M_j which converge in the Gromov–Hausdorff sense. We prove that if the sequence has a completed glued limit space, then it is unique (Theorem 6.6). However, the glued limit is not the Gromov–Hausdorff limit (Remark 6.7), it might even be empty (Remark 6.8), and it need not exist (Remark 6.9).

Finally, we construct some important examples of glued limit spaces for sequences which do not have Gromov–Hausdorff limits. In Remark 6.10 we describe how the sequence from Example 2.13 has a bounded and precompact glued limit space. We provide another example with a bounded glued limit space which is not precompact (Example 6.12). We provide an example where the glued limit space is not a length space (Example 6.13). We close this section with Example 6.16 demonstrating that these glued limit spaces and their completions depend on the isometric embeddings used to define them and need not be unique.

6A. ***Gluing inner regions together.*** Here we prove the existence of isometric embeddings which we will later apply as glue to connect inner regions together.

Theorem 6.1. *Let $\delta_i \to 0$ be a decreasing sequence and M_j a sequence of open manifolds such that*

$$(\overline{M}_j^{\delta_i}, d_{M_j}) \xrightarrow{\mathrm{GH}} (Y^{\delta_i}, d_{Y^{\delta_i}}) \tag{31}$$

for all i, where possibly some of these sequences and their limits are eventually empty sets. Then there exist subsequential limit isometric embeddings

$$\varphi_{\delta_{i+1},\delta_i} : Y^{\delta_i} \to Y^{\delta_{i+1}}, \tag{32}$$

which are just the identity when $\delta_i = \delta_{i+1}$. *If* $\delta \in (0, \delta_0]$, *there exists a compact metric space* $Y^{\delta_i} \subset Y^\delta \subset Y^{\delta_{i+1}}$ *with the restricted metric* $d_{Y^\delta} = d_{Y^{\delta_{i+1}}}$ *and a convergent subsequence*

$$(\overline{M}^\delta_{j_k}, d_{M_{j_k}}) \xrightarrow{\mathrm{GH}} (Y^\delta, d_{Y^\delta}), \tag{33}$$

and when $\delta \in (\delta_{i+1}, \delta_i)$, *for any such* Y^δ, *the restriction map* $\varphi_{\delta,\delta_i} : Y^{\delta_i} \to Y^\delta$ *and the inclusion map* $\varphi_{\delta_{i+1},\delta} : Y^\delta \to Y^{\delta_{i+1}}$ *are isometric embeddings.*

Proof. By Theorem 2.15, for each i there exists a compact metric space Z_i and isometric embeddings

$$\varphi_j : \overline{M}^{\delta_{i+1}}_j \to Z_i \quad \text{and} \quad \varphi_\infty : Y^{\delta_{i+1}} \to Z_i$$

such that

$$\varphi_j(\overline{M}^{\delta_{i+1}}_j) \xrightarrow{\mathrm{H}} \varphi_\infty(Y^{\delta_{i+1}}).$$

By Theorem 2.16, we can choose a subsequence $\{j_k\}_{k=1}^\infty$ such that the $\varphi_{j_k}(\overline{M}^{\delta_i}_{j_k})$ converge in the Hausdorff sense to a compact subspace $X^{\delta_i} \subset \varphi_\infty(Y^{\delta_{i+1}})$. By (31),

$$\overline{M}^{\delta_i}_{j_k} \xrightarrow{\mathrm{GH}} Y^{\delta_i}.$$

Then, by uniqueness, up to an isometry of the Gromov–Hausdorff limit space there exists an isometric embedding

$$\varphi_{\delta_{i+1},\delta_i} : Y^{\delta_i} \to Y^{\delta_{i+1}} \quad \text{such that} \quad \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i}) = \varphi_\infty^{-1}(X^{\delta_i}).$$

By Theorem 2.12 there is a uniform upper bound $D_i > 0$ of the diameters of $(\overline{M}^{\delta_i}_j, d_{M_j})$ and a function $N_i : (0, D_i] \to \mathbb{N}$ such that $N_i(\epsilon)$ is an upper bound for the number of ϵ-balls needed to cover $\overline{M}^{\delta_i}_j$ for all $\epsilon \in (0, D_i]$ and for all $j \in \mathbb{N}$. If $\delta \in (\delta_{i+1}, \delta_i)$, define $N : (0, D_{i+1}/2] \to \mathbb{N}$ by $N(\epsilon) = N_{i+1}(2\epsilon)$. Then

$$\mathrm{Diam}(\overline{M}^\delta_j, d_{M_j}) \le \mathrm{Diam}(\overline{M}^{\delta_{i+1}}_j, d_{M_j}) \le D_{i+1}.$$

Apply Theorem 2.9 to get a subsequence $\{l_k\}_{k=1}^\infty$ of $\{j_k\}_{k=1}^\infty$ such that the $\varphi_{l_k}(\overline{M}^{\delta_i}_{l_k})$ converge in the Hausdorff sense to a closed subset $X^\delta \subset \varphi_\infty(Y^{\delta_{i+1}})$.

We define

$$Y^\delta = \varphi_\infty^{-1}(X_\delta) \subset Y^{\delta_{i+1}}.$$

The choice of the subsequence $\{l_k\}$ implies that $X^{\delta_i} \subset Y^\delta$, so $Y^{\delta_i} \subset Y^\delta$. The rest of the theorem immediately follows. $\square$

Remark 6.2. The choice of isometric embeddings $\varphi_{\delta_{i+1},\delta_i}$ is not unique. In Example 6.16 we provide two distinct isometric embeddings $\varphi_{\delta_{i+1},\delta_i} \neq \varphi'_{\delta_{i+1},\delta_i}$.

6B. ***Glued limit spaces are defined.*** We now define a glued limit space for a sequence of Riemannian manifolds satisfying the hypothesis of Theorem 6.1. We prove that this glued limit space is a metric space unless it is the empty set. We prove that it contains isometric images of all Gromov–Hausdorff limits of convergent subsequences of inner regions (which may be empty). An example of a sequence of open Riemannian manifolds which has an empty glued limit space will be given in Remark 6.8. Our definitions of a glued limit space and a completed glued limit space are stated along with their construction in the following theorem:

Theorem 6.3. *Given a sequence of open Riemannian manifolds M_j with a sequence $\delta_i \to 0$ satisfying the hypothesis of Theorem 6.1, one can define a* ***glued limit space*** *Y using the subsequential limit isometric embeddings of* (32) *as follows*:

$$Y = Y(\{\delta_i\}, \{\varphi_{\delta_{i+1},\delta_i}\}) = Y^{\delta_0} \sqcup \bigsqcup_{i=1}^{\infty} \big(Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i})\big) \tag{34}$$

with the metric

$$d_Y(x,y) = \begin{cases} d_{Y^{\delta_0}}(x,y) & \text{if } x, y \in Y^{\delta_0}, \\ d_{Y^{\delta_{i+1}}}(x,y) & \text{if } x, y \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i}), \\ d_{Y^{\delta_{i+1}}}(x, \varphi_{\delta_{i+1},\delta_0}(y)) & \text{if } x \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i}) \text{ for some } i \in \mathbb{N} \\ & \text{and } y \in Y^{\delta_0}, \\ d_{Y^{\delta_{i+j+1}}}(x, \varphi_{\delta_{i+j+1},\delta_{i+1}}(y)) & \text{if } x \in Y^{\delta_{i+j+1}} \setminus \varphi_{\delta_{i+j+1},\delta_{i+j}}(Y^{\delta_{i+j}}) \text{ and} \\ & y \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i}) \text{ for some } i, j \in \mathbb{N}, \end{cases}$$

where we have set

$$\varphi_{\delta_{i+j},\delta_i} = \varphi_{\delta_{i+j},\delta_{i+j-1}} \circ \cdots \circ \varphi_{\delta_{i+1},\delta_i}.$$

This glued limit is not defined using an arbitrary collection of isometric embeddings, but rather only those achieved as in Theorem 6.1.

Furthermore, for all $\delta \in (0, \delta_0]$ there exists a subsequence $M^{\delta}_{j_k}$ which converges in the Gromov–Hausdorff sense to a compact metric space Y^{δ}, and for any such Y^{δ} there exists an isometric embedding

$$F_\delta = F_{\delta,\{\delta_i\}} : Y^\delta \to Y$$

such that for the δ_i in our sequence we have

$$F_{\delta_i}(Y^{\delta_i}) \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}).$$

If β_j is any sequence decreasing to 0, *then*

$$Y = \bigcup_{j=1}^{\infty} F_{\beta_j}(Y^{\beta_j}).$$

*We say that a sequence of open Riemannian manifolds M_j has a glued limit space Y if there exists a sequence $\delta_i \to 0$ satisfying the hypothesis of this theorem. A **completed glued limit** is defined to be the metric completion $\overline{Y}$ of a glued limit space Y, and the **boundary** of a glued limit space is defined to be the set $\overline{Y} \setminus Y$.*

Remark 6.4. In Example 4.10, for sufficiently large δ_0, each limit Y^{δ} is a ball in Euclidean 2-dimensional space. According to Theorem 6.3, the glued limit space of this sequence is constructed by taking the disjoint union of the ball Y^{δ_0} with the concentric annulus $Y^{\delta_0/i+1} \setminus Y^{\delta_0/i}$.

Remark 6.5. The definition of the glued limit space depends on the choice of δ_i and the isometric embeddings in Theorem 6.1. Even if one fixes the sequence $\delta_i \to 0$, the glued limit need not be unique; see Example 6.16.

Proof of Theorem 6.3. We first prove that d_Y is positive definite. For the first and second cases of the definition of d_Y, we immediately see that $d_Y(x, y) = 0$ if and only if $x = y$. For the third and fourth cases, notice that

$$\varphi_{\delta_{i+j+1},\delta_{i+1}}(y) = (\varphi_{\delta_{i+j+1},\delta_{i+j}} \circ \varphi_{\delta_{i+j},\delta_{i+1}})(y),$$

so

$$\varphi_{\delta_{i+j+1},\delta_{i+1}}(y) \in \varphi_{\delta_{i+j+1},\delta_{i+j}}(Y^{\delta_{i+j}}). \tag{35}$$

Thus $x \neq \varphi_{\delta_{i+j+1},\delta_{i+1}}(y)$ and

$$d_Y(x, y) = d_{Y^{\delta_{i+j+1}}}(x, \varphi_{\delta_{i+j+1},\delta_{i+1}}(y)) \neq 0.$$

Define $F_{\delta_i} : Y^{\delta_i} \to Y$ by

$$F_{\delta_i}(y) = \begin{cases} y & \text{if } i = 1, \\ y & \text{if } i > 1 \text{ and } y \in Y^{\delta_i} \setminus \varphi_{\delta_i,\delta_{i-1}}(Y^{\delta_{i-1}}) \\ \varphi^{-1}_{\delta_i,\delta_0}(y) & \text{if } i > 1 \text{ and } \varphi^{-1}_{\delta_i,\delta_0}(y) \in Y^{\delta_0}, \\ \varphi^{-1}_{\delta_i,\delta_j}(y) & \text{if } i > 1 \text{ and } \varphi^{-1}_{\delta_i,\delta_j}(y) \in Y^{\delta_j} \setminus \varphi_{\delta_j,\delta_{j-1}}(Y^{\delta_{j-1}}) \text{ for some } j > 1. \end{cases}$$

What we are doing in the third and fourth part of the definition of F_{δ_i} is the following. Suppose that $y \in Y^{\delta_0/i}$. Then either

$$y \in Y^{\delta_i} \setminus \varphi_{\delta_i,\delta_{i-1}}(Y^{\delta_{i-1}})$$

or $y \in \varphi_{\delta_i,\delta_{i-1}}(Y^{\delta_{i-1}})$. In the latter case, there exists $y_{i-1} \in Y^{\delta_{i-1}}$ such that $y = \varphi_{\delta_i,\delta_{i-1}}(y_{i-1})$. If $i - 1 > 1$, either

$$y_{i-1} \in Y^{\delta_{i-1}} \setminus \varphi_{\delta_{i-1},\delta_{i-2}}(Y^{\delta_{i-2}}) \quad \text{or} \quad y_{i-1} \in \varphi_{\delta_{i-1},\delta_{i-2}}(Y^{\delta_{i-2}}).$$

Proceeding in the same way, if necessary, we find j such that there exists $y_j \in Y^{\delta_j}$ satisfying the condition $y_j \notin \varphi_{\delta_j,\delta_{j-1}}(Y^{\delta_{j-1}})$ if $j > 1$, and also $y = \varphi_{\delta_i,\delta_j}(y_j)$.

It is easy to see that

$$F_{\delta_i}(Y^{\delta_i}) = Y^{\delta_0} \cup \bigsqcup_{j=1}^{i-1} \big(Y^{\delta_{j+1}} \setminus \varphi_{\delta_{j+1},\delta_j}(Y^{\delta_j})\big), \tag{36}$$

and for $j < i$,

$$F_{\delta_j} = F_{\delta_i} \circ \varphi_{\delta_i,\delta_j}.$$

For arbitrary δ, by Theorem 6.1 there exists a subsequence $M^{\delta}_{j_k}$ which converges in the Gromov–Hausdorff sense to a limit Y^{δ}. Define $F_\delta : Y^\delta \to Y$ by

$$F_\delta = F_{\{\delta,\{\delta_i\}\}} = \begin{cases} F_{\delta_0} \circ \varphi_{\delta_0,\delta} & \text{if } \delta_0 < \delta, \\ F_{\delta_{i+1}} \circ \varphi_{\delta_{i+1},\delta} & \text{if } \delta_{i+1} \le \delta < \delta_i, \end{cases}$$

where $\varphi_{\delta_0,\delta}, \varphi_{\delta_{i+1},\delta}$ are given in Theorem 6.1.

Observe that in the latter case of the definition of F_δ, $F_{\delta_i} = F_\delta \circ \varphi_{\delta,\delta_i}$. This, together with the definition of F_δ, gives

$$F_{\delta_i}(Y^{\delta_i}) \subset F_\delta(Y^\delta) \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}). \tag{37}$$

Now that we have β_j decreasing to 0, there exists N sufficiently large that $\beta_j \le \delta_0$, and for all $j \ge N$, there exists i such that $\beta_j \in [\delta_{i+1}, \delta_i)$. From (36) and (37), taking $\delta = \beta_i$, we conclude that

$$Y = \bigcup_{j=N}^{\infty} F_{\beta_j}(Y^{\beta_j}) = \bigcup_{j=1}^{\infty} F_{\beta_j}(Y^{\beta_j})$$

because $F_{\beta_0}(Y^{\beta_0}) \subset F_{\beta_N}(Y^{\beta_N})$.

To prove that F_δ is an isometric embedding, it is enough to prove that each F_{δ_i} is an isometric embedding. F_{δ_0} is an isometric embedding by definition of Y. For $F_{\delta_{i+1}}$ we must check three cases. Let $x, y \in Y^{\delta_{i+1}}$.

Case (i): If $x, y \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i})$, then $F_{\delta_{i+1}}(x) = x$, and $F_{\delta_{i+1}}(y) = y$.

Case (ii): If $x \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i})$ and $y \in \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i})$, then

$$F_{\delta_{i+1}}(y) = \varphi^{-1}_{\delta_{i+1},\delta_{i+1-j}}(y) \in Y^{\delta_{i+1-j}} \setminus \varphi_{\delta_{i+1-j},\delta_{i-j}}(Y^{\delta_{i-j}})$$

for some j, so

$$\begin{aligned} d_Y\big(F_{\delta_{i+1}}(x), F_{\delta_{i+1}}(y)\big) &= d_{Y^{\delta_{i+1-j}}}\big(F_{\delta_{i+1}}(x), \varphi_{\delta_{i+1},\delta_{i+1-j}}(F_{\delta_{i+1}}(y))\big) \\ &= d_{Y^{\delta_{i+1}}}(x, y). \end{aligned}$$

Case (iii): If $F_{\delta_{i+1}}(x) = \varphi^{-1}_{\delta_{i+1},\delta_{i+1-k}}(x)$ and $F_{\delta_{i+1}}(y) = \varphi^{-1}_{\delta_{i+1},\delta_{i+1-j}}(y)$, then take $k \le j$. Recall that $\varphi_{\delta_{i+1},\delta_{i+1-k}} \circ \varphi_{\delta_{i+1-k},\delta_{i+1-j}} = \varphi_{\delta_{i+1},\delta_{i+1-j}}$, so we have

$$\begin{aligned}
&d_Y(F_{\delta_{i+1}}(x), F_{\delta_{i+1}}(y)) \\
&\qquad = d_{Y^{\delta_{i+1-k}}}(F_{\delta_{i+1}}(x), \varphi_{\delta_{i+1-k},\delta_{i+1-j}}(F_{\delta_{i+1}}(y))) \\
&\qquad = d_{Y^{\delta_{i+1}}}(\varphi_{\delta_{i+1},\delta_{i+1-k}}(F_{\delta_{i+1}}(x)), \varphi_{\delta_{i+1},\delta_{i+1-j}}(F_{\delta_{i+1}}(y))) \\
&\qquad = d_{Y^{\delta_{i+1}}}(x, y).
\end{aligned}$$

The triangle inequality follows from the above paragraphs. For $x, y, z \in Y$, find δ such that $x, y, z \in F_\delta(Y^\delta)$. The triangle inequality holds for the preimages of x, y, z, and since F_δ is an isometric embedding, it also holds for x, y, z. $\square$

6C. ***Glued limits within Gromov–Hausdorff limits.*** Recall that in Theorem 4.3 we proved that if a sequence of open Riemannian manifolds M_j has a Gromov–Hausdorff limit X, then subsequences of the inner regions M_j^δ have Gromov–Hausdorff limits. Here we assume that the M_j also have a (possibly empty) completed glued limit space as in Theorem 6.3. We prove that this completed glued limit space is unique and provide a precise description as to how to find this completed glued limit space as a subset of the Gromov–Hausdorff limit (Theorem 6.6).

Note that the completed glued limit need not agree with the Gromov–Hausdorff limit (Remark 6.7). In fact, we provide an example where the completed glued limit space is empty (Remark 6.8).

It should be emphasized that we must assume the M_j have a completed glued limit to obtain uniqueness. It is possible that a sequence M_j has a Gromov–Hausdorff limit and that one needs a subsequence to obtain a glued limit, and that different subsequences provide different completed glued limits (see Remark 6.9).

Theorem 6.6. *Let $\{M_j\}$ be a sequence of open manifolds that converges in the Gromov–Hausdorff sense to a compact metric space (X, d_X). Suppose Y is a glued limit space of the $\{M_j\}$ defined as in Theorem 6.3. Then the completed glued limit $\overline{Y}$ is isometric to the closure $\overline{U}_{\{j_k\}} \subset X$ of any limit's inner union $U_{\{j_k\}} \subset X$ defined as in Definition 4.7 for any subsequence j_k. In particular, any completed glued limit and the closure of any of the limit's inner regions are isometric.*

We do not claim all the limit's inner regions are the same subset of X, and in fact this is not true, even after taking a closure. They are only isometric to one another. See Example 4.13.

Proof. Let Y be a glued limit space defined using Theorems 6.3 and 6.1 via a sequence of isometric embeddings φ_j of $M_j^{\delta_i} \subset M_j^{\delta_{i+1}}$ into a sequence of compact metric spaces Z_i rather than a single compact metric space Z.

Since we have assumed the original sequence of Riemannian manifolds has a glued limit space Y without requiring a subsequence, the following spaces are isometric:

$$Y_{\{\delta_i\},\{j_k\}} \cong Y_{\{\delta_i\}} \cong Y_{\{\delta_i\},\{j'_k\}}$$

for any pair of subsequences $\{j_k\}$ and $\{j'_k\}$.

Recall that Theorem 6.3 provides, for each $\delta > 0$, an isometric embedding $F_\delta : Y^\delta \to Y$, with

$$Y = \bigcup_{i=1}^{\infty} F_{\delta_i}(Y^{\delta_i}) \quad \text{and} \quad F_{\delta_i}(Y^{\delta_i}) \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}). \tag{38}$$

Since Y^δ is the Gromov–Hausdorff limit of the inner regions M_j^δ, it is isometric to the limit of the inner regions $Y^\delta_{\{j_k\}} \subset U_{\{j_k\}} \subset X$ of Theorem 4.1. Note that we need a subsequence for each δ to produce the limit of the inner regions. We can produce a diagonal subsequence (also denoted $\{j_k\}$) such that

$$Y^\delta_{\{j_k\}} \subset U_{\{j_k\}} \subset X \quad \text{ is defined for all } \delta \in \{\delta_i\},$$

so we have isometric embeddings

$$\psi_{\delta_i} : F_{\delta_i}(Y^{\delta_i}) \subset Y \to Y^{\delta}_{\{j_k\}} \subset X.$$

Since $F_{\delta_i}(Y^{\delta_i}) \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}})$, for each i and any h we may study the restriction

$$\psi_{\delta_{i+h}} : F_{\delta_i}(Y^{\delta_i}) \subset Y \to Y^{\delta_{i+h}}_{\{j_k\}} \subset U_{\{j_k\}} \subset X.$$

Since $F_{\delta_i}(Y^{\delta_i})$ and X are compact, we can find a subsequence h_k depending on i which converges to a limit isometric embedding:

$$\psi_{i,\infty} : F_{\delta_i}(Y^{\delta_i}) \subset Y \to \bar{U}_{\{j_k\}} \subset X.$$

We may do this for each i and diagonalize the subsequences if we wish. Since $\psi_{\delta_{i+h}}$ is a restriction of $\psi_{\delta_{i+1+h}}$, we see that $\psi_{i,\infty}$ is a restriction of $\psi_{i+1,\infty}$. Thus we may define an isometric embedding

$$\psi_\infty : Y \to \bar{U}_{\{j_k\}} \subset X.$$

Extending this, we have an isometric embedding

$$\bar{\psi}_\infty : \bar{Y} \to \bar{U}_{\{j_k\}}.$$

Since X is compact, $\bar{U}_{\{j_k\}}$ is compact and thus so is $\bar{Y}$.

We need only construct an isometric embedding from $\bar{U}_{\{j_k\}}$ to $\bar{Y}$ to prove that these spaces are isometric, because they are compact metric spaces. We repeat the same trick as above but now use the fact that we have isometries

$$F'_{\delta_i} : Y^{\delta_i}_{\{j_k\}} \to F_{\delta_i}(Y^{\delta_i}) \subset Y$$

and

$$Y = \bigcup_{i=1}^{\infty} F'_{\delta_i}(Y^{\delta_i}_{\{j_k\}}),$$

with

$$F'_{\delta_i}(Y^{\delta_i}_{\{j_k\}}) \subset F'_{\delta_{i+1}}(Y^{\delta_{i+1}}_{\{j_k\}}).$$

Since $Y^{\delta_i}_{\{j_k\}} \subset Y^{\delta_{i+1}}(j_k)$, we may study for each i and any h the restriction

$$F'_{i+h} : Y^{\delta_i}_{\{j_k\}} \to Y \subset \overline{Y}.$$

Since we have shown $\overline{Y}$ is compact, a subsequence converges for each i (and we can diagonalize these subsequences), so that we obtain isometric embeddings

$$F'_{i,\infty} : Y^{\delta_i}_{\{j_k\}} \to \overline{Y}.$$

Since $F'_{i,\infty}$ is a restriction of $F'_{i+1,\infty}$, we can define an isometric embedding

$$F'_{\infty} : U_{\{j_k\}} \to \overline{Y}. \tag{39}$$

This extends to an isometric embedding from $\overline{U}_{\{j_k\}}$ to $\overline{Y}$. Since we have a pair of isometric embeddings between a pair of compact metric spaces, these metric spaces are isometric. □

Remark 6.7. It is possible that the completed glued limit is not the same as the Gromov–Hausdorff limit. Example 4.10 has a glued limit which is an open disk in Euclidean space; its completed glued limit is the closed disk, while its Gromov–Hausdorff limit is a disk with a line segment attached.

Remark 6.8. The glued limit of a sequence of open Riemannian manifolds may exist but be the empty set. See, for example, the sequence M_{2j+1} in Example 4.11. This sequence converges in the Gromov–Hausdorff sense but U is an empty set. It only satisfies the conditions of Theorem 6.1 in a trivial way: for each $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that $M_j^\delta = \varnothing$ for all $j \geq N_\delta$.

Remark 6.9. A sequence of M_j which converges in the Gromov–Hausdorff sense may not have a glued limit space. In fact, one may need to take a subsequence to obtain a glued limit, and different subsequences might produce different glued limit spaces. In Examples 4.9–4.12, the subsequence M_{2j} has a completed glued limit space which is isometric to $\overline{U}_{\{2j\}}$ and the subsequence M_{2j+1} has a completed glued limit space which is isometric to $\overline{U}_{\{2j+1\}}$, but the sequence M_j itself does not have a glued limit space. We thus see that the different glued limits obtained using different subsequences are quite different. In particular, in Example 4.11 the completed glued limit of the M_{2j} agrees with the Gromov–Hausdorff limit of the M_j, while the completed glued limit of the M_{2j+1} is empty.

6D. ***Glued limit spaces when there are no Gromov–Hausdorff limits.*** In the setting of Theorem 6.1, the subsequence of manifolds M_j such that $M_j^\delta \xrightarrow{\text{GH}} Y^\delta$ need not have any Gromov–Hausdorff limit. Here we discuss an old example and present two new examples.

Remark 6.10. The manifolds M_j described in Example 2.13 have increasingly many spikes, and the sequence does not have a Gromov–Hausdorff limit. However, the sequence M_j^δ converges to the metric completion of the annulus $\text{Ann}_{(0,0)}(1+\delta, 2-\delta)$ with the flat metric; see Remark 5.6. Start with $\delta_0 < \frac{1}{2}$; then

$$Y^{\delta_0} = \text{Ann}_{(0,0)}(1+\delta_0, 2-\delta_0)$$

and

$$Y^{\delta_0/(i+1)} \setminus \varphi_{\delta_0/(i+1),\delta_0/i}(Y^{\delta_0/i}) = A_1 \cup A_2,$$

where

$$A_1 = \text{Ann}_{(0,0)}\Big(1+\frac{\delta_0}{i+1}, 1+\frac{\delta_0}{i}\Big) \quad \text{and} \quad A_2 = \text{Ann}_{(0,0)}\Big(2-\frac{\delta_0}{i}, 2-\frac{\delta_0}{i+1}\Big).$$

Thus $Y = \text{Ann}_{(0,0)}(1,2)$ with the flat length metric. This glued limit space Y is precompact.

A similar example, also constructed using flat manifolds $M_j \subset \mathbb{E}^2$ with no Gromov–Hausdorff limit, has convergent M_j^δ and a glued limit space which is a flat open manifold that is bounded but not precompact:

Example 6.11. We define a flat open manifold with j spikes of decreasing width:

$$M_j = U_j \cup V_j,$$

where

$$U_j = \Big\{(r\cos\theta, r\sin\theta) : r < 4 + \sin\frac{4\pi^2}{\theta},\ \theta \in (2\pi/j, 2\pi]\Big\}, \tag{40}$$

$$V_j = \big\{(r\cos\theta, r\sin\theta) : r < 4,\ \theta \in (0, 2\pi/j]\big\}. \tag{41}$$

As in Example 2.13, the (M_j, d_{M_j}) have no Gromov–Hausdorff limit because they have increasingly many spikes. Unlike Example 2.13, for any number N, there exists δ_N sufficiently small such that $M_j^{\delta_N}$ has N spikes. In fact,

$$(M_j^\delta, d_{M_j}) \xrightarrow{\text{GH}} (Y^\delta, d_Y),$$

where Y^δ is the δ-inner region of the flat open manifold

$$Y = \Big\{(r\cos\theta, r\sin\theta) : r < 4 + \sin\frac{4\pi^2}{\theta},\ \theta \in (0, 2\pi]\Big\}.$$

Taking the identity maps for the isometric embeddings, we see that Y is also a glued limit space for the M_j, even though it is bounded but not precompact.

Recall Example 2.14 of a sequence of surfaces having no Gromov–Hausdorff limit. We modify it to obtain a sequence of manifolds with boundary that has no Gromov–Hausdorff limit, but whose δ-inner regions have Gromov–Hausdorff limits, and we construct the glued limit space and see that it is also bounded and not precompact. This glued limit space is not a manifold.

Example 6.12. Let

$$X_j = \big([0,1]\times[0,1]\big) \sqcup \big([0,1]\times\big[0,\tfrac{1}{2}\big]\big) \sqcup\cdots\sqcup \big([0,1]\times\big[0,\big(\tfrac{1}{2}\big)^j\big]\big)$$

be a disjoint union of spaces with taxicab metrics, glued via the map $\psi(0,y)=(0,y)$. One may think of X_j as a book with j pages of decreasing height glued along a spike on the left. Within X_j, choose A_j to be the union of the top edges of each of the pages. If we take surfaces M_j as constructed in Proposition 2.8, they now have boundary, and

$$d_{\mathrm{GH}}(M_j, X_j)\to 0 \quad\text{and}\quad d_{\mathrm{GH}}(M_j^\delta, X_j\setminus T_\delta(A_j))\to 0.$$

As in Example 2.14, the M_j have no GH-convergent subsequence because the X_j have no GH-convergent subsequence.

Observe that there exists k_δ such that, for all $j > k_\delta$,

$$X_j\setminus T_\delta(A_j) = \big([0,1]\times[0,\delta)\big) \sqcup \big([0,1]\times[0,\tfrac{1}{2}-\delta]\big) \sqcup\cdots\sqcup \big([0,1]\times\big[0,\big(\tfrac{1}{2}\big)^{k_\delta}-\delta\big]\big).$$

Since this sequence does not depend on j, it clearly converges in the Gromov–Hausdorff sense. Thus, the M_j^δ converge to the same Gromov–Hausdorff limit space. In fact, they converge to $X_\infty\setminus T_\delta(A_\infty)$, where

$$X_\infty = \big([0,1]\times[0,1]\big) \sqcup \big([0,1]\times\big[0,\tfrac{1}{2}\big]\big) \sqcup\cdots\sqcup \big([0,1]\times\big[0,\big(\tfrac{1}{2}\big)^j\big]\big)\sqcup\cdots,$$

and A_∞ is the union of the tops of all of these pages. In fact, X_∞ is the glued limit space.

6E. *A glued limit space which is not geodesic.* Here we present an example whose glued limit space is not geodesic or even a length space (and neither is its metric completion):

Example 6.13. In Euclidean space $\mathbb{E}^2$, define

$$M_j = \big((-1,1)\times(-1,1)\big)\setminus\big(\big[-\tfrac{1}{2},\tfrac{1}{2}\big]\times\big[0,1-\tfrac{1}{j}\big]\big). \tag{42}$$

Then, for $\delta<\frac{1}{4}$, there is $J=J(\delta)$ such that

$$M_j^\delta = \big((-1+\delta,1-\delta)\times(-1+\delta,-\delta)\big) \sqcup \big(\big(-1+\delta,-\tfrac{1}{2}-\delta\big)\times(-\delta,1-\delta)\big) \\ \sqcup \big(\big(\tfrac{1}{2}-\delta,1-\delta\big)\times(-\delta,1-\delta)\big)$$

for $j\geq J$.

Thus $\overline{M}_j^\delta$ is a constant sequence for $j \geq J$, and $\overline{M}_j^\delta \xrightarrow{\text{GH}} Y^\delta$, where

$$Y^\delta = [-1+\delta, 1-\delta] \times [-1+\delta, 0] \cup \left[-1+\delta, -\tfrac{1}{2}+\delta\right] \times [0, 1-\delta] \\ \cup \left[\tfrac{1}{2}-\delta, 1-\delta\right] \times [0, 1-\delta].$$

The completed glued limit is not a length space:

$$\overline{Y} = [-1, 1] \times [-1, 0] \cup \left[-1, -\tfrac{1}{2}\right] \times [0, 1] \cup \left[\tfrac{1}{2}, 1\right] \times [0, 1] \subset \mathbb{E}^2.$$

Note that $\overline{M}_j \xrightarrow{\text{GH}} X = Y \cup \left(\{1\} \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]\right)$.

Open question 6.14. Is a glued limit space locally geodesic: for all $y \in Y$, does there exist $\epsilon_y > 0$ such that $B(y, \epsilon_y)$ is geodesic? If there is a counterexample, what conditions can be imposed on the space to guarantee that it is locally geodesic?

6F. ***Balls in glued limit spaces.*** Recall from Lemma 3.3 that for any $p \in M^{\delta_i}$, if $x \in B_p(\delta_i - \delta_{i+1}) \subset M$, then $x \in M^{\delta_{i+1}}$. This is not true for glued limit spaces. That is, it is possible for $p \in F_{\delta_i}(Y^{\delta_i})$ to have an $x \in B_p(\delta_i - \delta_{i+1}) \subset Y$ such that $x \notin F_{\delta_{i+1}}(Y^{\delta_{i+1}})$. In fact, we can take the ball of arbitrarily small radius and still have $x \notin F_{\delta_{i+1}}(Y^{\delta_{i+1}})$:

Example 6.15. In Example 6.12 we constructed a sequence M_j having no Gromov–Hausdorff limit, but such that the M_j^δ converge in the Gromov–Hausdorff sense to $Y^\delta = X_\infty \setminus T_\delta(A_\infty)$, where

$$X_\infty = \left([0,1] \times [0,1]\right) \sqcup \left([0,1] \times \left[0, \tfrac{1}{2}\right]\right) \sqcup \cdots \sqcup \left([0,1] \times \left[0, \left(\tfrac{1}{2}\right)^j\right]\right) \sqcup \cdots,$$

where each piece is connected along $(0, y) \sim (0, y)$ and A_∞ is the union of the tops of all of these pages. This X_∞ is a glued limit space for this example.

Then $F_\delta(Y^\delta) = X_\infty \setminus T_\delta(A_\infty)$. Take any ball about the common point $(0,0)$ in X_∞. For any radius $r > 0$, $B_{(0,0)}(r)$ contains infinitely many points

$$y_j = \left(\frac{r}{2}, 0\right) \in [0,1] \times \left[0, \left(\tfrac{1}{2}\right)^j\right]. \tag{43}$$

However, $y_j \notin F_\delta(Y^\delta)$ for j sufficiently large that $\left(\tfrac{1}{2}\right)^j < \delta$.

6G. ***Nonuniqueness of the glued limit space.*** We now see that glued limit spaces and completed glued limit spaces are not necessarily unique. Recall that in Theorem 6.6 we explained that if the M_j have a Gromov–Hausdorff limit, then the completed glued limit space is unique. So we need to construct a sequence of manifolds M_j having no Gromov–Hausdorff limit. In fact, we will imitate Example 6.12, applying Proposition 2.8 to construct the following example:

Example 6.16. There are a sequence $\{M_j\}$ of Riemannian surfaces with boundary, a sequence $\delta_i \to 0$, and metric spaces Y^{δ_i} such that

$$d_{\text{GH}}(M_j^{\delta_i}, Y^{\delta_i}) \to 0,$$

with two different glued limit spaces

$$Y_1 = Y(\delta_{2i}, \varphi_{\delta_{2i},\delta_{2i+2}}) \quad \text{and} \quad Y_2 = Y(\delta_{2i}, \varphi'_{\delta_{2i},\delta_{2i+2}}),$$

constructed as in Theorem 6.3 and Theorem 6.1, whose metric completions are not isometric.

Proof. Let

$$P_j = [0,1] \times \left[-\frac{1}{2j}, \frac{1}{2j}\right] \quad \text{for } j = 1, 2, \dots, \tag{44}$$

and let

$$X_j = P_1 \sqcup (P_2 \sqcup P_2) \sqcup \cdots \sqcup (P_j \sqcup \cdots \sqcup P_j) \quad (2^{i-1} \text{ copies of } P_i,\ 1 \le i \le j)$$

be a disjoint union of $N_j = 1 + 2 + 4 + \cdots + 2^{j-1}$ spaces endowed with taxicab metrics, glued via with the map $\psi(0, y) = (0, y)$. One may think of X_j as a book with N_j pages of different heights glued along a spike on the left.

Let $H_j \subset P_j$ be defined by

$$H_j = [0,1] \times \left\{-\frac{1}{2j}\right\} \ \cup\ \{1\} \times \left[-\frac{1}{2j}, \frac{1}{2j}\right] \ \cup\ [0,1] \times \left\{\frac{1}{2j}\right\} \subset P_j,$$

and let $A_j \subset X_j$ be defined by

$$A_j = H_1 \sqcup (H_2 \sqcup H_2) \sqcup \cdots \sqcup (H_j \sqcup \cdots \sqcup H_j) \ \ (2^{i-1} \text{ copies of } H_i,\ 1 \le i \le j).$$

If we take surfaces M_j as constructed in Proposition 2.8, they now have a boundary, and we have

$$d_{\mathrm{GH}}(M_j, X_j) \to 0 \quad \text{and} \quad d_{\mathrm{GH}}(M_j^{\delta}, X_j \setminus T_\delta(A_j)) \to 0.$$

As in Example 2.14, the M_j have no GH-convergent subsequence because the X_j have no GH-convergent subsequence.

Now

$$\begin{aligned} X_j \setminus T_\delta(A_j) = {} & \big(P_1 \setminus T_\delta(H_1)\big) \sqcup \big(P_2 \setminus T_\delta(H_2)\big) \sqcup \big(P_2 \setminus T_\delta(H_2)\big) \\ & \sqcup \big(P_3 \setminus T_\delta(H_3)\big) \sqcup \cdots \sqcup \big(P_3 \setminus T_\delta(H_3)\big) \\ & \vdots \\ & \sqcup \big(P_j \setminus T_\delta(H_j)\big) \sqcup \cdots \sqcup \big(P_j \setminus T_\delta(H_j)\big). \end{aligned}$$

Observe that

$$P_j \setminus T_\delta(H_j) = [0, 1-\delta] \times \left[-\frac{1}{2j} + \delta, \frac{1}{2j} - \delta\right].$$

Taking $\delta = \delta_{2i} = 1/(2i)$ and $j > i$, we have

$$P_i \setminus T_\delta(H_i) = \left[0, 1 - \frac{1}{2i}\right] \times \{0\}$$

and

$$P_j \setminus T_\delta(H_j) = \varnothing.$$

Thus

$$\begin{aligned} X_j \setminus T_\delta(A_j) = &\big(P_1 \setminus T_\delta(H_1)\big) \sqcup \big(P_2 \setminus T_\delta(H_2)\big) \sqcup \big(P_2 \setminus T_\delta(H_2)\big) \\ &\sqcup \big(P_3 \setminus T_\delta(H_3)\big) \sqcup \cdots \sqcup \big(P_3 \setminus T_\delta(H_3)\big) \\ &\vdots \\ &\sqcup \big(P_{i-1} \setminus T_\delta(H_{i-1})\big) \sqcup \cdots \sqcup \big(P_{i-1} \setminus T_\delta(H_{i-1})\big) \\ &\sqcup \left[0, 1-\frac{1}{2i}\right] \times \{0\} \sqcup \cdots \sqcup \left[0, 1-\frac{1}{2i}\right] \times \{0\}, \end{aligned}$$

endowed with taxicab metrics and glued together with the map $\psi(0, y) = (0, y)$. There are $1+2+4+\cdots+2^{(i-1)-1}$ rectangular pages and 2^{i-1} pages that are just intervals of length $1-1/(2i)$. Taking $j \to \infty$, we get

$$d_{\mathrm{GH}}(X_j \setminus T_\delta(A_j), Y^\delta) \to 0, \tag{45}$$

where

$$Y^{\delta_{2i}} = Y^{1/(2i)} = X_j \setminus T_{\delta_{2i}}(A_j) \quad \text{for all } j > i.$$

So

$$\begin{aligned} Y^{\delta_{2i}} = &\big(P_1 \setminus T_{1/(2i)}(H_1)\big) \sqcup \big(P_2 \setminus T_{1/(2i)}(H_2)\big) \sqcup \big(P_2 \setminus T_{1/(2i)}(H_2)\big) \\ &\sqcup \big(P_3 \setminus T_{1/(2i)}(H_3)\big) \sqcup \cdots \sqcup \big(P_3 \setminus T_{1/(2i)}(H_3)\big) \\ &\vdots \\ &\sqcup \big(P_{2i-1} \setminus T_{1/(2i)}(H_{2i-1})\big) \sqcup \cdots \sqcup \big(P_{2i-1} \setminus T_{1/(2i)}(H_{2i-1})\big) \\ &\sqcup \left[0, 1-\frac{1}{2i}\right] \times \{0\} \sqcup \cdots \sqcup \left[0, 1-\frac{1}{4i}\right] \times \{0\}, \end{aligned}$$

endowed with taxicab metrics and glued together with the map $\psi(0, y) = (0, y)$; there are $1+2+4+\cdots+2^{(i-1)-1}$ rectangular pages and 2^{i-1} pages that are just intervals of length $1-1/(2i)$.

If we define $\varphi_{\delta_{2i},\delta_{2i+2}} : Y^{\delta_{2i}} \to Y^{\delta_{2i+2}}$ to be the inclusion map and then construct the glued limit space as in Theorem 6.1, we obtain

$$\begin{aligned} Y_1 = Y(\delta_{2i}, \varphi_{\delta_{2i},\delta_{2i+2}}) &= Y \\ &= (P_1 \setminus H_1) \sqcup (P_2 \setminus H_2) \sqcup (P_2 \setminus H_2) \\ &\quad \sqcup (P_3 \setminus H_3) \sqcup \cdots \sqcup (P_3 \setminus H_3) \\ &\quad \vdots \\ &\quad \sqcup (P_j \setminus H_j) \sqcup \cdots \sqcup (P_j \setminus H_j) \cdots \end{aligned}$$

endowed with taxicab metrics glued with a gluing map $\psi(0, y) = (0, y)$. This has infinitely many pages, all shaped like rectangles.

Now we define $\varphi'_{\delta_{2i},\delta_{2i+2}} : Y^{\delta_{2i}} \to Y^{\delta_{2i+2}}$ to be an isometric embedding which maps a point

$$(x, y) \in P_k \setminus T_{\delta_{2i}}(H_k) \subset Y^{\delta_{2i}}$$

for $k < i$ to

$$(x, y) \in P_k \setminus T_{\delta_{2i+2}}(H_k) \subset Y^{\delta_{2i+2}}$$

via the inclusion map, and which maps

$$(x, y) \in P_i \setminus T_{\delta_{2i}}(H_i) = \left[0, 1 - \frac{1}{2i}\right] \times \{0\} \subset Y^{\delta_{2i}}$$

to

$$(x, y - \delta_{2i} + \delta_{2i+2}) \in P_{i+1} \setminus T_{\delta_{2i+2}}(H_{i+1}) = \left[0, 1 - \frac{1}{2i+2}\right] \times \{0\} \subset Y^{\delta_{2i+2}}.$$

This is possible because we have enough copies of $P_{i+1} \setminus T_{\delta_{2i+2}}(H_{i+1})$ in $Y^{\delta_{2i+2}}$.

In particular, $\varphi'_{\delta_{2i},\delta_{2i+2}}$ maps the interval pages into interval pages. If we then construct the glued limit space as in Theorem 6.1, we obtain

$$Y_2 = Y(\delta_{2i}, \varphi'_{\delta_{2i},\delta_{2i+2}}) = Y \sqcup [0,1]\times\{0\} \sqcup [0,1]\times\{0\} \sqcup [0,1]\times\{0\} \sqcup \cdots,$$

which has infinitely many pages that are intervals in addition to all the pages shaped like rectangles. So we have two distinct glued limit spaces for the sequence $\delta_{2i} = 1/(2i)$, and their metric completions are not isometric. □

7. Glued limits under curvature bounds

In this section we prove the existence of glued limits of sequences of manifolds with certain natural geometric conditions (Theorems 7.1 and 7.4). We do not require the sequences of manifolds themselves to have Gromov–Hausdorff limits.

7A. ***Constructing glued limits of manifolds with constant sectional curvature.*** In this section we prove that if $M_j \in \mathcal{M}_H^{m,V,l}$ for all j (see Definition 5.1) then the sequence has a glued limit space (Theorem 7.1). The sequence need not have a Gromov–Hausdorff limit (see Remark 7.2).

Theorem 7.1. *Given any $\delta_0 > 0$, if $(M_j, g_j) \subset \mathcal{M}_H^{m,V,l}$, then there is a Gromov–Hausdorff convergent subsequence $\{M_{j_k}^{\delta_0}\}$ and a glued limit space Y such that for all $\delta \in (0, \delta_0]$ there exists a further subsequence $\{j'_k\}$ of $\{j_k\}$ for which $M_{j'_k}^{\delta}$ converges in the Gromov–Hausdorff sense to a compact metric space Y^δ, and for any such Y^δ there exists an isometric embedding*

$$F_\delta : Y^\delta \to Y. \tag{46}$$

Remark 7.2. The sequences of flat surfaces $M_j \subset \mathbb{E}^2$ defined in Example 2.13 and Example 6.11 have a common finite upper volume bound, but there is no common finite upper bound for the number of disjoint balls in M_j of radius less than 1. Thus, these two sequences do not have a Gromov–Hausdorff limit. Nonetheless, since

$$L_{\min}(M_j) = \inf\{L_g(C) : C \text{ is a closed geodesic in } M_j\ \} > l,$$

Theorem 7.1 demonstrates that we can construct glued limits for these spaces.

Remark 7.3. The choice of a further subsequence $\{j'_k\}$ of $\{j_k\}$ in Theorem 7.1 is necessary. Let $(M_j, g_j) \subset \mathcal{M}_0^{2,V,l}$ be the sequence defined in Example 4.4. Take $\delta_0 = 3\varepsilon$. Then $\{M_j^{\delta_0}\}$ is a Gromov–Hausdorff convergent sequence. Choosing $2\varepsilon \in (0, \delta_0]$, we see that $\overline{M}_{2j}^{2\varepsilon}$ converges in the Gromov–Hausdorff sense but $\overline{M}_j^{2\varepsilon}$ does not.

Proof. Consider the sequence δ_0, $\delta_i = \delta_0/i$, $i \in \mathbb{N}$. Start with δ_0. By Theorem 5.2, there exist a sequence $\{\iota_k(\delta_0)\}$ of integers and a compact metric space Y^{δ_0} such that

$$(\overline{M}_{\iota_k(\delta_0)}, d_{M_{\iota_k(\delta_0)}}) \xrightarrow{\text{GH}} (Y^{\delta_0}, d_{Y^{\delta_0}}). \tag{47}$$

Proceeding as before, for each $n \in \mathbb{N}$ there is a subsequence $\{\iota_k(\delta_n)\}_k$ of $\{\iota_k(\delta_{n-1})\}_k$ and a compact metric space $Y^{\delta_n}_{\{\iota_k(\delta_n)\}}$ such that

$$(\overline{M}^{\delta_n}_{\iota_k(\delta_n)}, d_{M_{\iota_k(\delta_n)}}) \xrightarrow{\text{GH}} Y^{\delta_n}.$$

Define $j_k = \iota_k(\delta_k)$. We have

$$(\overline{M}^{\delta_n}_{j_k}, d_{M_{j_k}}) \xrightarrow{\text{GH}} Y^{\delta_n}$$

for $n = 0, 1, 2, \dots$ since $\{j_k\}_{k=n}^{\infty}$ is a subsequence of $\{\iota_k(\delta_n)\}_{k=1}^{\infty}$. We may now apply Theorem 6.3 to complete the proof. $\square$

7B. *Constructing glued limits with Ricci curvature bounds.* Here we prove that glued limits exist for noncollapsing sequences of manifolds with nonnegative Ricci curvature and bounded volume which have control on the intrinsic diameters of their inner regions (defined in (4)):

Theorem 7.4. *Given $m \in \mathbb{N}$, a decreasing sequence $\delta_i \to 0$, $(i \geq 0)$, $V > 0$, $\theta > 0$, and $D_i > 0$, let (M_j, g_j) be a sequence of m-dimensional open Riemannian manifolds with nonnegative Ricci curvature such that* $\mathrm{Vol}(M_j) \leq V$,

$$\sup\{\mathrm{Diam}(M_j^{\delta_i}, d_{M_j^{\delta_i}}) : j \in \mathbb{N}\} < D_i \quad \textit{for all } i \in \mathbb{N},$$

and such that

$$\textit{for all } j \in \mathbb{N}, \textit{ there exists } q_j \in M_j^{\delta_0} \textit{ such that } \mathrm{Vol}(B_{q_j}(\delta_0)) \geq \theta\delta_0^m.$$

Then there exists a sequence $\{j_k\} \to \infty$ *such that for all* δ_i *the sequence* $\{M_{j_k}^{\delta_i}\}$ *converges in the Gromov–Hausdorff sense to a compact metric space* Y^{δ_i}. *Thus, the* M_{j_k} *have a glued limit space* Y *such that for all* $\delta \in (0, \delta_0]$ *there is a further subsequence* $\{j_k'\}$ *of* $\{j_k\}$ *for which* $M_{j_k'}^{\delta}$ *converges in the Gromov–Hausdorff sense to a compact metric space* Y^{δ} *isometrically embedded in* Y:

$$F_\delta : Y^\delta \to Y. \tag{48}$$

Remark 7.5. If there is $D > 0$ such that

$$\sup_{\delta \in (0,\delta_0]} \{\mathrm{Diam}(M_j^\delta, d_{M_j^\delta})\} \le D,$$

then we could take $D_i = D$ for all i. But this requirement is unnecessarily strong.

Remark 7.6. The choice of a further subsequence $\{j_k'\}$ of $\{j_k\}$ in Theorem 7.1 is necessary. For the sequence (M_j, g_j) defined in Example 4.4, consider a decreasing sequence $\delta_i \to 0$ such that $\delta_0 = 3\varepsilon$ and $\delta_1 = \varepsilon$. Then the hypotheses of the theorem are satisfied. For all δ_i, $\{M_j^{\delta_i}\}$ converges in the Gromov–Hausdorff sense. However, for $2\varepsilon \in (0, \delta_0]$, $\{M_j^{2\varepsilon}\}$ does not have a Gromov–Hausdorff limit.

Proof of Theorem 7.4. Take $\delta \in (0, \delta_0]$; by hypothesis and the Bishop–Gromov volume comparison (Theorem 2.19),

$$\mathrm{Vol}(B_{q_j}(\delta)) \ge \mathrm{Vol}(B_{q_j}(\delta_0))\Big(\frac{\delta}{\delta_0}\Big)^m \ge \theta\delta^m.$$

This and the hypotheses of the theorem imply that, for each i,

$$\{(M_j, g_j)\} \subset \mathcal{M}_\theta^{m,\delta_i,D_i,V}.$$

Start with δ_0. By Theorem 1.4 there exists a sequence $\{\iota_k(\delta_0)\}$ of integers such that

$$\big(\overline{M}^{\delta_0}_{\iota_k(\delta_0)}, d_{M_{\iota_k(\delta_0)}}\big) \xrightarrow{\mathrm{GH}} \big(Y^{\delta_0}, d_{Y^{\delta_0}}\big).$$

Proceeding as before, for each $n \in \mathbb{N}$ there exists a subsequence $\{\iota_k(\delta_n)\}_k$ of $\{\iota_k(\delta_{n-1})\}_k$ and a compact metric space $Y^{\delta_n}_{\{\iota_k(\delta_n)\}}$ such that

$$\big(\overline{M}^{\delta_n}_{\iota_k(\delta_n)}, d_{M_{\iota_k(\delta_n)}}\big) \xrightarrow{\mathrm{GH}} \big(Y^{\delta_n}, d_{Y^{\delta_n}}\big).$$

Define $j_k = \iota_k(\delta_k)$. We have

$$\big(\overline{M}^{\delta_n}_{j_k}, d_{M_{j_k}}\big) \xrightarrow{\mathrm{GH}} \big(Y^{\delta_n}, d_{Y^{\delta_n}}\big)$$

since $\{j_k\}_{k=n}^\infty$ is a subsequence of $\{\iota_k(\delta_n)\}_{k=1}^\infty$. Finally, apply Theorem 6.3. $\square$

8. Properties of glued limit spaces under curvature bounds

In this final section of the paper we consider the local properties of the glued limits of sequences of manifolds with constant sectional curvature as in Theorem 7.1 and manifolds with nonnegative Ricci curvature as in Theorem 7.4. We begin with an example indicating how even when the sequences of manifolds have a Gromov–Hausdorff limit, one need not retain curvature conditions on the Gromov–Hausdorff limit space (Example 8.1). This is in sharp contrast with the setting where the Riemannian manifolds are compact without boundary. In this example, the glued limit space is empty. Then we have a subsection about balls in glued limit spaces without any assumption on curvature (Theorem 8.3). We apply this control on the balls to prove that local curvature properties do persist on glued limit spaces. In particular, we prove (Proposition 8.4) that the glued limits of manifolds with constant sectional curvature bounds (and other conditions) are unions of manifolds with constant sectional curvature. We close with Theorem 8.8, concerning the metric measure properties of glued limits of manifolds with nonnegative Ricci curvature.

8A. ***An example with no curvature control.*** We now construct a sequence of flat open manifolds whose Gromov–Hausdorff limit is not flat:

Example 8.1. Let $B_p(1) \subset \mathbb{H}^2$ be a unit ball in hyperbolic space and $B_0(1) \subset \mathbb{E}^2$ be the unit ball in Euclidean space. Then $\exp_p : B_0(1) \to B_p(1)$. Let

$$S_j = \left\{ \left(\frac{i}{j}, \frac{k}{j} \right) : i, k \in \mathbb{Z} \right\} \cap B_0(1) \subset \mathbb{E}^2 \tag{49}$$

and $S'_j = \exp_p(S_j)$. Form a graph A_j whose vertices are in S_j and whose edges form a triangulation, by connecting $(i/j, k/j)$ to $((i+1)/j, k/j)$, $(i/j, (k+1)/j)$ and $((i+1)/j, (k+1)/j)$. We let $A'_j = \exp_p(A_j)$, and set the lengths of the edges in A'_j to be the distances between the vertices viewed as points in $\mathbb{H}^2$. Then A'_j converges to $B_p(1) \subset \mathbb{H}^2$ in the Gromov–Hausdorff sense.

Now define A''_j to be the simplicial complex formed by filling in the triangles in A'_j with flat Euclidean triangles. Observe that $\{A''_j\}$ converges to $B_p(1) \subset \mathbb{H}^2$ in the Gromov–Hausdorff sense as well. Finally, for each j we remove tiny balls of radius $\ll 1/j$ around the vertices in A''_j, to create a flat open manifold M_j. These M_j converge in the Gromov–Hausdorff sense to $B_p(1) \subset \mathbb{H}^2$.

Remark 8.2. Example 8.1 has an empty glued limit space. In the next subsections we will see that the glued limit spaces do retain some of the curvature properties of the initial sequence of manifolds. Thus the glued limit space is a more natural object of study than the Gromov–Hausdorff limit, even when the Gromov–Hausdorff limit exists.

8B. ***Balls to glued limit spaces.*** Generally when one wishes to study the properties of a complete noncompact limit space, one studies balls in the limit space as Gromov–Hausdorff limits of balls in the sequence. Here we cannot control balls in the limit space, but we can control balls of radius $\epsilon < \delta_i - \delta_{i+1}$ centered in $F_{\delta_i}(Y^{\delta_i})$ intersected with $F_{\delta_{i+1}}(Y^{\delta_{i+1}})$. This will suffice to study the geometric properties of the glued limit spaces.

Theorem 8.3. *Let Y be a glued limit of a sequence $\{M_j\}$ of Riemannian manifolds, as in Theorem 6.3. If $y \in Y^{\delta_i}$ and $\epsilon < \delta_i - \delta_{i+1}$, then there exist a subsequence $\{M_{j_k}^{\delta_i}\}$ containing points y_{j_k} and a sequence $\epsilon_{j_k} \to \epsilon$ such that*

$$B(y_{j_k}, \epsilon_{j_k}) = \{x \in M_{j_k} : d_M(x, y_{j_k}) < \epsilon_{j_k}\} \subset M_{j_k}^{\delta_{i+1}} \tag{50}$$

and

$$d_{\mathrm{GH}}\Big(\big(\bar{B}(y_{j_k}, \epsilon_{j_k}), d_{M_{j_k}}\big), \big(\bar{B}(F_{\delta_i}(y), \epsilon) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}), d_Y\big)\Big) \to 0. \tag{51}$$

Note that in Example 6.15 we saw that $B(F_{\delta_i}(y), \epsilon) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}})$ need not be isometric to $B(F_{\delta_i}(y), \epsilon) \subset Y$, even when ϵ is taken arbitrarily small.

Proof. Recall that in Theorem 6.1 we found $\varphi_{\delta_{i+1},\delta_i}$ defined in the following way. We picked isometric embeddings

$$\varphi_j : M_j^{\delta_{i+1}} \to Z \quad \text{and} \quad \varphi_\infty : Y^{\delta_{i+1}} \to Z$$

such that

$$d_H^Z\big(\varphi_j(M_j^{\delta_{i+1}}), \varphi_\infty(Y_j^{\delta_{i+1}})\big) \to 0.$$

Then we found a subsequence such that

$$d_H^Z\big(\varphi_{j_k}(M_{j_k}^{\delta_i}), X^{\delta_i}\big) \to 0$$

and chose $\varphi_{\delta_{i+1},\delta_i}$ to be an isometry such that

$$\varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i}) = \varphi_\infty^{-1}(X^{\delta_i}).$$

Then there exist

$$y_{j_k} \in M_{j_k}^{\delta_i} \subset M_{j_k}^{\delta_{i+1}} \subset M_{j_k}$$

such that

$$d_Z\big(\varphi_{j_k}(y_{j_k}), \varphi_\infty(\varphi_{\delta_{i+1},\delta_i}(y))\big) \to 0.$$

Let $\epsilon' \in (0, \delta_i - \delta_{i+1})$. Then by Lemma 3.3 we have

$$B(y_{j_k}, \epsilon') = \{x \in M_{j_k} : d_M(x, y_{j_k}) < \epsilon'\} \subset M_{j_k}^{\delta_{i+1}}.$$

From this, and since $\varphi_{j_k} : M_{j_k}^{\delta_{i+1}} \to Z$ is an isometry into its image, we see that

$$\big(B(y_{j_k}, \epsilon'), d_{M_{j_k}^{\delta_{i+1}}}\big) \quad \text{is isometric to} \quad \big(B(\varphi_{j_k}(y_{j_k}), \epsilon') \cap \varphi_{j_k}(M_{j_k}^{\delta_{i+1}}), d_Z\big).$$

By Lemma 2.2, for any $\epsilon \in (0, \delta_i - \delta_{i+1})$, there exists $\epsilon_{j_k} \to \epsilon$ eventually in $(0, \delta_i - \delta_{i+1})$, such that

$$\bar{B}\big(\varphi_{j_k}(y_{j_k}), \epsilon_{j_k}\big) \cap \varphi_{j_k}\big(M_{j_k}^{\delta_{i+1}}\big) \xrightarrow{\text{H}} \bar{B}\big(\varphi_\infty \varphi_{\delta_{i+1},\delta_i}(y), \epsilon\big) \cap \varphi_\infty\big(Y^{\delta_{i+1}}\big).$$

Now,

$$\big(\bar{B}(\varphi_\infty \varphi_{\delta_{i+1},\delta_i}(y), \epsilon) \cap \varphi_\infty(Y^{\delta_{i+1}}), d_Z\big)$$

is isometric to

$$\big(\bar{B}(\varphi_{\delta_{i+1},\delta_i}(y), \epsilon), d_{Y^{\delta_{i+1}}}\big),$$

which is isometric to

$$\big(F_{\delta_{i+1}} \bar{B}(\varphi_{\delta_{i+1},\delta_i}(y), \epsilon), d_{F_{\delta_{i+1}}(Y^{\delta_{i+1}})}\big),$$

which is isometric to

$$\big(\bar{B}(F_{\delta_{i+1}} \varphi_{\delta_{i+1},\delta_i}(y), \epsilon) \cap F_{\delta_{i+1}} Y^{\delta_{i+1}}, d_Y\big).$$

Hence

$$d_{\text{GH}}\Big(\big(\bar{B}(y_{j_k}, \epsilon_{j_k}), d_{M_{j_k}}\big), \big(\bar{B}(F_{\delta_{i+1}}(\varphi_{\delta_{i+1},\delta_i}(y)), \epsilon) \cap F_{\delta_{i+1}} Y^{\delta_{i+1}}, d_Y\big)\Big) \to 0. \quad \square$$

8C. ***Properties of glued limits of manifolds with constant sectional curvature.*** Here we prove a proposition, present a key example and state two open questions concerning the glued limits of manifolds with constant sectional curvature.

Proposition 8.4. *Let Y be a glued limit space obtained as in Theorem 7.1 from a sequence $M_j \in \mathcal{M}_H^{m,V,l}$. Then there exists a countable collection of sets $W_i \subset Y$, each of which is isometric to an m-dimensional smooth open manifold of constant sectional curvature H, such that*

$$Y \subset \bigcup_{i=1}^{\infty} W_i. \tag{52}$$

In fact,

$$F_{\delta_i}(Y^{\delta_i}) \subset W_i \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \subset Y.$$

See Example 8.5, in which the glued limit space is a countable collection of flat tori which are not connected to one another but have a metric restricted from a larger compact metric space of finite volume.

Proof. Recall that any glued limit space Y defined as in Theorem 6.3 depends on a sequence $\delta_i \to 0$ and gluings $\varphi_{\delta_{i+1},\delta_i} : Y^{\delta_{i+1}} \to Y^{\delta_i}$ via the subsequential limit isometric embeddings of (32). There are isometric embeddings $F_{\delta_i} : Y^{\delta_i} \to Y$ such that

$$Y \subset \bigcup_{i=1}^{\infty} F_{\delta_i}(Y^{\delta_i}) \tag{53}$$

and

$$F_{\delta_i}(Y^{\delta_i}) \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}). \tag{54}$$

Let

$$\epsilon_i = \tfrac{1}{2}\min\left\{\delta_i - \delta_{i+1}, \frac{l}{2}, \frac{\pi\sqrt{h}}{2}\right\},$$

where $h = H$ when $H > 0$ and $h = (l/\pi)^2$ otherwise.

Let

$$W_i = T_{\epsilon_i}\big(F_{\delta_i}(Y^{\delta_i})\big) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \subset Y.$$

First observe that by (54) we have

$$F_{\delta_i}(Y^{\delta_i}) \subset W_i.$$

So combined with (53), we have (52). So we need only show W_i is a smooth m-dimensional open manifold of constant sectional curvature H.

For all $w \in W_i$, there exists $y_\infty \in F_{\delta_i}(Y^{\delta_i})$ such that $w \in B_{y_\infty}(\epsilon_i) \subset Y$. Since

$$B_{y_\infty}(\epsilon_i) \subset T_{\epsilon_i}\big(F_{\delta_i}(Y^{\delta_i})\big)$$

we have

$$U = B_{y_\infty}(\epsilon_i) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}) = B_{y_\infty}(\epsilon_i) \cap W_i. \tag{55}$$

We need only show that U is isometric to a ball of radius ϵ_i in M_H^m, the m-dimensional simply connected manifold with constant sectional curvature H.

There exists $y \in Y^{\delta_i}$ such that $y_\infty = F_{\delta_i}(y)$. By Theorem 8.3, and the fact that $\epsilon_i < \delta_i - \delta_{i+1}$, there exists a subsequence $M_{j_k}^{\delta_i}$ containing points y_{j_k} and $\epsilon_{j_k} \to \epsilon_i$ such that (50) and (51) are satisfied.

Since $\epsilon_i < l/2$, we have $\epsilon_{j_k} < l/2$ for k sufficiently large, and so by (50) the M_j satisfy the conditions of Theorem 5.2, and by (27) we know there is a Riemannian isometric diffeomorphism from $B(y_{j_k}, \epsilon_{j_k})$ to a ball in M_H^m, the m-dimensional simply connected manifold with constant sectional curvature H. Since $\epsilon_i < \sqrt{H}\,\pi/2$ when $H > 0$, we have a convex ball, so that, as metric spaces,

$$\big(B(y_{j_k}, \epsilon_{j_k}), d_M\big) \quad \text{is isometric to} \quad \big(B(p, \epsilon_{j_k}), d_{M_H^m}\big).$$

Taking $k \to \infty$, the closures of these latter balls converge in the Gromov–Hausdorff sense to $(\bar{B}(p, \epsilon_i), d_{M_H^m})$. Thus, by (51) and the uniqueness of GH limits,

$$\big(\bar{B}(y_\infty, \epsilon_i) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}), d_Y\big) \quad \text{is isometric to} \quad \big(\bar{B}(p, \epsilon_i), d_{M_H^m}\big).$$

Thus we have (55), and we are done. □

Example 8.5. In this example we construct a glued limit space Y for a sequence of manifolds M_j^m satisfying the conditions of Theorem 5.2. In addition, the M_j^m converge in the Gromov–Hausdorff sense to a metric space X, so that the glued limit space is unique. The glued limit Y is a countable union of connected flat manifolds with the restricted metric from X.

Proof. Let M_1 be two flat square annuli connected by a slanted strip of width 1 and length $\sqrt{2}$:

$$M_1 = C_{0,1} \cup C_{1,1} \cup S_{0,1} \subset \mathbb{R}^3,$$

where

$$\begin{aligned}
C_{0,1} &= \Big(\big((-1,1)\times(-1,1)\big) \setminus \big((-\tfrac{1}{2},\tfrac{1}{2})\times[-\tfrac{1}{2},\tfrac{1}{2}]\big)\Big)\times\{0\},\\
C_{1,1} &= \Big(\big((-1,1)\times(-1,1)\big) \setminus \big((-\tfrac{1}{2},\tfrac{1}{2})\times[-\tfrac{1}{2},\tfrac{1}{2}]\big)\Big)\times\{1\},\\
S_{0,1} &= \big\{(x,y,z) : (x,y)\in(-\tfrac{1}{2},\tfrac{1}{2})\times[-\tfrac{1}{2},\tfrac{1}{2}],\ z = x+\tfrac{1}{2}\big\}.
\end{aligned}$$

Endowed with the length metric, this is isometric to an open manifold with constant sectional curvature 0. Note that, for $\delta > \frac{1}{4}$,

$$M_1^{\delta} \subset C_{0,1} \cup C_{1,1}.$$

Let M_2 be three flat square annuli of total area at most $4+4+4\big(\frac{1}{4}\big)$ connected by two slanted strips of width $\frac{1}{2}$:

$$M_2 = C_{0,2} \cup C_{1,2} \cup C_{2,2} \cup S_{1,2} \cup S_{2,2} \subset \mathbb{R}^3,$$

where

$$\begin{aligned}
C_{0,2} &= \Big(\big((-1,1)\times(-1,1)\big) \setminus \big((-\tfrac{1}{4},\tfrac{1}{4})\times[-\tfrac{1}{4},\tfrac{1}{4}]\big)\Big)\times\{0\},\\
C_{1,2} &= \Big(\big((-\tfrac{1}{2},\tfrac{1}{2})\times(-\tfrac{1}{2},\tfrac{1}{2})\big) \setminus \big((-\tfrac{1}{4},\tfrac{1}{4})\times[-\tfrac{1}{4},\tfrac{1}{4}]\big)\Big)\times\{\tfrac{1}{2}\},\\
C_{2,2} &= \Big(\big((-1,1)\times(-1,1)\big) \setminus \big((-\tfrac{1}{4},\tfrac{1}{4})\times[-\tfrac{1}{4},\tfrac{1}{4}]\big)\Big)\times\{\tfrac{2}{2}\},\\
S_{0,2} &= \big\{(x,y,z) : (x,y)\in(-\tfrac{1}{4},\tfrac{1}{4})\times[-\tfrac{1}{4},\tfrac{1}{4}],\ z = x+\tfrac{1}{4}\big\},\\
S_{1,2} &= \big\{(x,y,z) : (x,y)\in(-\tfrac{1}{4},\tfrac{1}{4})\times[-\tfrac{1}{4},\tfrac{1}{4}],\ z = x+\tfrac{3}{4}\big\}.
\end{aligned}$$

Endowed with the length metric, this is isometric to an open manifold with constant sectional curvature 0. Note that, for $\delta > \frac{1}{8}$,

$$M_1^{\delta} \subset C_{0,2} \cup C_{1,2} \cup C_{2,2} \setminus \big(B((0,0),\delta)\times[0,1]\big).$$

Let M_j be $(j+1)$ flat square annuli of total area at most $4+4\sum_{i=0}^{j}\big(\frac{1}{2}\big)^{j}$, connected by j slanted strips of width $\big(\frac{1}{2}\big)^{j}$:

$$M_j = \bigcup_{i=0}^{j} C_{i,j} \cup \bigcup_{i=0}^{j-1} S_{i,j} \subset \mathbb{R}^3, \tag{56}$$

where, with the notation

$$I_k = \left(-\left(\tfrac{1}{2}\right)^k, \left(\tfrac{1}{2}\right)^k\right), \quad \bar{I}_k = \left[-\left(\tfrac{1}{2}\right)^k, \left(\tfrac{1}{2}\right)^k\right], \quad \text{and} \quad m_j = \frac{2^{i+1-j} - 2^{i-j}}{\left(\tfrac{1}{2}\right)^{j+1} - \left(-\tfrac{1}{2}\right)^{j+1}},$$

we define

$$\begin{aligned}
C_{0,j} &= \big((I_0 \times I_0) \setminus (I_{j+1} \times \bar{I}_{j+1})\big) \times \{0\},\\
C_{i,j} &= \big((I_{j-i} \times I_{j-i}) \setminus (I_{j+1} \times \bar{I}_{j+1})\big) \times \{2^{i-j}\},\\
C_{j,j} &= \big((I_0 \times I_0) \setminus (I_{j+1} \times \bar{I}_{j+1})\big) \times \{2^{j-j}\},\\
S_{0,j} &= \big\{(x,y,z) : (x,y) \in I_{j+1} \times \bar{I}_{j+1},\ z = x + \left(\tfrac{1}{2}\right)^{j+1}\big\},\\
S_{i,j} &= \big\{(x,y,z) : (x,y) \in I_{j+1} \times \bar{I}_{j+1},\ z = m_j\big(x + \left(\tfrac{1}{2}\right)^{j+1}\big) + 2^{i-j}\big\}.
\end{aligned}$$

Endowed with the length metric, this is isometric to an open manifold with constant sectional curvature 0. Note that, for $\delta > \left(\tfrac{1}{2}\right)^{j+2}$,

$$M_j^\delta \subset C_{0,j} \cup \cdots \cup C_{j,j} \setminus \big(B((0,0),\delta) \times [0,1]\big).$$

The Gromov–Hausdorff limit of the M_j exists and can be see to be

$$X = \bigcup_{j=0}^{\infty} C_j \cup S_0 \subset \mathbb{R}^3, \tag{57}$$

where

$$\begin{aligned}
C_0 &= I_0 \times I_0 \times \{0\},\\
C_i &= I_{j-i} \times I_{j-i} \times \{2^{i-j}\},\\
S_0 &= \{0\} \times \{0\} \times [0,1],
\end{aligned}$$

endowed with the length metric. The Gromov–Hausdorff limit Y^δ of the M_j^δ exists, and

$$Y^\delta \subset X \setminus \big(B((0,0),\delta) \times [0,1]\big).$$

In fact, $Y = X \setminus S_0$. □

Open question 8.6. Are the glued limits of sequences of manifolds with constant sectional curvature open manifolds with constant sectional curvature? We know they need not be connected by Example 8.5.

Open question 8.7. Are the glued limits of sequences of manifolds with constant sectional curvature unique? Perhaps an adaptation of Example 8.5 could be applied to show that they are not.

8D. ***Properties of glued limits of manifolds with nonnegative Ricci curvature.*** We now prove the final theorem of our paper and state the last two open questions:

Theorem 8.8. *Suppose that we have a sequence of m-dimensional open Riemannian manifolds M_j with nonnegative Ricci curvature and* $\mathrm{Vol}(M_j) \le V_0$, *and there* *exists a sequence* $\delta_i \to 0$ *such that the inner regions* $M_j^{\delta_i}$ *converge in the Gromov–Hausdorff sense as* $j \to \infty$ *to* Y^{δ_i} *without collapsing. Suppose that* Y *is a glued limit constructed as in Theorem 6.3. Then* Y *has Hausdorff dimension* m, $\mathcal{H}^m(Y) \le V_0$, *and its Hausdorff measure has positive lower density everywhere.*

Note that this theorem may be applied to study the glued limits of sequences of manifolds satisfying the conditions of Theorem 7.4.

To prove this theorem we will apply Cheeger and Colding's volume convergence theorem [3; 4], which was reviewed in Section 2F. See Theorem 2.25 and Remark 2.26 for the precise statement we will use here.

Proof. First we prove that

$$W_i = T_{(\delta_i - \delta_{i+1})/2}\big(F_{\delta_i}(Y^{\delta_i})\big) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \subset Y$$

have Hausdorff dimension m and have doubling Hausdorff measures. For any $w \in W_i$, let

$$U_w = B\Big(w, \frac{\delta_i - \delta_{i+1}}{2}\Big) \cap W_i.$$

We can find $y \in Y^{\delta_i}$ such that $d_Y(y, w) < (\delta_i - \delta_{i+1})/2$. Then we have

$$U_w = B\Big(w, \frac{\delta_i - \delta_{i+1}}{2}\Big) \cap B(F_{\delta_i}(y), \delta_i - \delta_{i+1}) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}). \tag{58}$$

By Theorem 8.3, we have a sequence $\{j_k\}$, points $y_{j_k} \in M_{j_k}^{\delta_i}$, and a sequence $\{\epsilon_{j_k}\} \to \epsilon = (\delta_i - \delta_{i+1})/2$ satisfying (50) and (51):

$$d_{\mathrm{GH}}\Big(\big(\bar{B}(y_{j_k}, \epsilon_{j_k}), d_{M_{j_k}}\big), \big(\bar{B}(F_{\delta_i}(y), \epsilon) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}), d_Y\big)\Big) \to 0.$$

Combining this with the fact that

$$w \in B\Big(y, \frac{\delta_i - \delta_{i+1}}{2}\Big) \subset \bar{B}(F_{\delta_i}(y), \epsilon) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \subset Y,$$

there exist

$$z_{j,k} \in \bar{B}\Big(y_{j_k}, \frac{\delta_i - \delta_{i+1}}{2}\Big) \subset \bar{B}(y_{j_k}, \epsilon_{j_k}) \subset M_{j_k}$$

such that

$$d_{\mathrm{GH}}\Big(\Big(\bar{B}\Big(z_{j_k}, \frac{\delta_i - \delta_{i+1}}{2}\Big), d_{M_{j_k}}\Big), (\bar{U}_w, d_Y)\Big) \to 0.$$

Since we assumed this is noncollapsing, then by the Cheeger–Colding volume convergence theorem mentioned above we have

$$\mathcal{H}_m(B_w(r) \cap U_w) = \lim_{k \to \infty} \mathcal{H}_m(B_{z_{j_k}}(r))$$

for all $r \le r_i = (\delta_i - \delta_{i+1})/2$. By (58) and Bishop's volume comparison theorem, we see that

$$\mathcal{H}_m(B_w(r) \cap W_i) = \mathcal{H}_m(B_w(r) \cap U_w) \le \omega_m r^m \quad \text{for all } r \le r_i$$

is positive and finite for any $w \in W_i$. By the Bishop–Gromov volume comparison theorem,

$$\frac{\mathcal{H}_m(B_w(r_1) \cap W_i)}{\mathcal{H}_m(B_w(r_2) \cap W_i)} \geq \frac{r_1^m}{r_2^m} \quad \text{for all } w \in W_i,\ r_1 < r_2 \leq r_i. \tag{59}$$

Since W_i is a subset of the compact $F_{\delta_{i+1}}(Y^{\delta_{i+1}})$, it is precompact. Choose a maximal collection $\{w_1, \dots, w_N\} \subset W_i$ such that the $B(w_i, r_i/2)$ are disjoint. Then

$$W_i \subset \bigcup_{n=1}^{N} B(w_n, r_i)$$

and

$$\mathcal{H}^m(W_i) \leq \sum_{n=1}^{N} \mathcal{H}^m(B(w_n, r_i) \leq \left(\tfrac{1}{4}\right)^m \sum_{n=1}^{N} \mathcal{H}^m(B(w_n, r_i/4)).$$

But it is not hard to see, examining (50), that $B(w_n, r_i/2)$ are the limits of disjoint balls in M_j, so

$$\sum_{n=1}^{N} \mathcal{H}^m(B(w_n, r_i/4)) \leq \limsup_{j\to\infty} \mathcal{H}_m(M_j^{\delta_i}) \leq V_0.$$

So W_i has Hausdorff dimension m and

$$\mathcal{H}_m(W_i) \leq V_0.$$

Now

$$Y = \bigcup_{i=1}^{\infty} W_i,$$

so it has Hausdorff dimension m and

$$\mathcal{H}_m(Y) \leq V_0.$$

Now to see that Y has positive density everywhere, we must show

$$\Theta_*(y, \mathcal{H}^m) = \liminf_{r\to 0} \frac{\mathcal{H}_m(B(y,r))}{r^m} > 0.$$

For fixed $i \geq I_y$, we have

$$\mathcal{H}_m(B(y,r)) \geq \mathcal{H}_m(B(y,r) \cap W_i).$$

Combining this with (59), we have

$$\begin{aligned}\Theta_*(y, \mathcal{H}^m) &= \liminf_{r\to 0} \frac{\mathcal{H}_m(B(y,r)\cap W_i)}{r^m} \\ &\geq \liminf_{r\to 0} \frac{\mathcal{H}_m(B(y,r_i)\cap W_i)}{r_i^m} \\ &\geq \frac{\mathcal{H}_m(B(y,r_i)\cap W_i)}{r_i^m} > 0. \qquad \square\end{aligned}$$

Open question 8.9. Are glued limit spaces of sequences as in Theorem 8.8 unique?

Open question 8.10. Are glued limit spaces of sequences as in Theorem 8.8 countably $\mathcal{H}^m$-rectifiable?

Acknowledgements

We would like to thank Stephanie Alexander (UIUC) for informing us about the work of Wong and Kodani when we first began to explore the question. We'd like to thank Frank Morgan (Williams) and David Johnson (Lehigh) for their interest and encouragement. We'd like to thank Pedro Solórzano (UC Riverside) for looking over some of the proofs. Thanks to Tabitha (IS 25), Penelope (IS 25) and Kendall (PS 32) for building the models of Examples 1.1–1.2 depicted in Figures 1, 2 and 5 and for computing the areas of the manifolds in Example 1.1 as part of a K–12 outreach. Finally we would like to thank Jorge Basilio (CUNY), Christine Briener (MIT), Maria Hempel (ETH Zurich), Sajjad Lakzian (CUNY), Christopher Lonke (Carnegie Mellon), Mike Munn (University of Missouri at Columbia), Jacobus Portegies (Courant, NYU), and Timothy Susse (CUNY) for actively participating with us in the CUNY Metric Geometry Reading Seminar in the summer of 2012 and Kenneth Knox (Stony Brook) for joining us in the spring of 2013.

References

[1] Michael Anderson, Atsushi Katsuda, Yaroslav Kurylev, Matti Lassas, and Michael Taylor. Boundary regularity for the Ricci equation, geometric convergence, and Gel'fand's inverse boundary problem. *Invent. Math.*, 158(2):261–321, 2004.

[2] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*. Number 33 in Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001.

[3] Jeff Cheeger and Tobias Holck Colding. On the structure of spaces with Ricci curvature bounded below, I. *J. Differential Geom.*, 46(3):406–480, 1997.

[4] Tobias Holck Colding. Ricci curvature and volume convergence. *Ann. of Math.* (2), 145(3):477–501, 1997.

[5] Tobias Holck Colding and Aaron Naber. Characterization of tangent cones of noncollapsed limits with lower Ricci bounds and applications. *Geom. Funct. Anal.*, 23(1):134–148, 2013.

[6] Manfredo Perdigão do Carmo. *Geometria Riemanniana*. Number 10 in Projeto Euclides. Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 2nd edition, 1988. Translated in *Riemannian Geometry*, Birkhäuser, Boston, 1992.

[7] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, 53:53–73, 1981.

[8] Mikhael Gromov. *Structures métriques pour les variétés Riemanniennes*. CEDIC, Paris, 1981. Translated in *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser, Boston, 2007.

[9] W. Klingenberg. Contributions to Riemannian geometry in the large. *Ann. of Math.* (2), 69:654–666, 1959.

[10] Kenneth S. Knox. A compactness theorem for Riemannian manifolds with boundary and applications. preprint, 2013.

[11] Shigeru Kodani. Convergence theorem for Riemannian manifolds with boundary. *Compositio Math.*, 75(2):171–192, 1990.

[12] Raquel Perales. A survey on the convergence of manifolds with boundary. preprint, 2013.

[13] Christina Sormani and Guofang Wei. Hausdorff convergence and universal covers. *Trans. Amer. Math. Soc.*, 353(9):3585–3602, 2001.

[14] Jeremy Wong. An extension procedure for manifolds with boundary. *Pacific J. Math.*, 235(1):173–199, 2008.

Received February 8, 2013. Revised June 18, 2013.

RAQUEL PERALES
DEPARTMENT OF MATHEMATICS
STONY BROOK UNIVERSITY
100 NICOLLS RD
STONY BROOK, NY 11794
UNITED STATES
praquel@math.sunysb.edu

CHRISTINA SORMANI
DEPARTMENT OF MATHEMATICS
CUNY GRADUATE CENTER AND LEHMAN COLLEGE
365 FIFTH AVENUE
NEW YORK, NY 10016
UNITED STATES
sormanic@member.ams.org

PACIFIC JOURNAL OF MATHEMATICS
Vol. 270, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.270.473

INVARIANT DIFFERENTIAL OPERATORS ON A CLASS OF MULTIPLICITY-FREE SPACES

HUBERT RUBENTHALER

If (G, V) is a multiplicity-free space with a one-dimensional quotient, we give generators and relations for the noncommutative algebra $D(V)^{G'}$ of invariant differential operators under the semisimple part G' of the reductive group G. More precisely we show that $D(V)^{G'}$ is the quotient of a Smith algebra by a completely described two-sided ideal.

1. Introduction

Let H be a reductive algebraic group over $\mathbb{C}$ and let X be a smooth irreducible H-variety. Let $\mathbb{C}[X]$ be the algebra of regular functions on X and let $D(X)$ be the algebra of differential operators on X. Then the H-action on X extends naturally to $\mathbb{C}[X]$ and $D(X)$. Let $\mathbb{C}[X]^H$ (resp. $D(X)^H$) be the subalgebras of H-invariants in $\mathbb{C}[X]$ (resp. $D(X)$). The ring $\mathbb{C}[X]^H$ is the ring of regular functions on the categorical quotient $X /\!/ H$. The problem of determining the structure of $D(X)^H$ was investigated by several authors [Levasseur and Stafford 1989; Van den Bergh 1996; Schwarz 2002]. On the other hand under the above mentioned hypothesis there exists an H-equivariant restriction map

$$\delta : D(X)^H \to D(X /\!/ H),$$

obtained by applying elements in $D(X)^H$ to functions in $\mathbb{C}[X]^H$. It is expected that $D(X)^H$, as well as its image under δ (the so-called algebra of radial components), should share many properties of enveloping algebras [Schwarz 2002; Levasseur 2009]. In this paper we obtain the precise structure of $D(V)^{G'}$ in the case where (G, V) is a so-called *multiplicity-free space with a one-dimensional quotient* (here G is reductive and $G' = [G, G]$ is the derived group). These spaces are defined to be the multiplicity-free spaces (G, V) for which the quotient $V /\!/ G'$ is one-dimensional. To be more precise we show that the (noncommutative) algebra $D(V)^{G'}$ is a quotient of a generalized *Smith algebra*. Over $\mathbb{C}$ this kind of algebra was introduced by S. P. Smith [1990] as a natural generalization of the enveloping algebra of $\mathfrak{sl}_2$. As a corollary we describe by generators and relations the algebras of radial components

MSC2010: primary 22E46, 16S32; secondary 11S90.
Keywords: multiplicity-free space, invariant differential operator, Smith algebra.

attached to the G'-isotypic components in the polynomial algebra $\mathbb{C}[V]$ (the image under δ above corresponds to the trivial representation of G').

According to the classification obtained in [Rubenthaler 2013], the class of multiplicity-free spaces with a one-dimensional quotient is a rather large class inside the multiplicity-free spaces. It contains both irreducible and nonirreducible representations.

The representations $(\mathrm{Str}(V), V)$, where V is a simple Jordan algebra over $\mathbb{C}$ and where $\mathrm{Str}(V)$ is the structure group of V, are examples of irreducible multiplicity-free spaces with a one-dimensional quotient (see Remark 2.2.7 and Example 2.3.3 below). Among these there is the natural representation of $\mathrm{GL}(n, \mathbb{C})$ on the space $\mathrm{Sym}_n(\mathbb{C})$ of $n \times n$ symmetric matrices and also the irreducible 27-dimensional representation of $E_6 \times \mathbb{C}^*$.

The spin representation of $\mathrm{Spin}(7) \times \mathbb{C}^*$ and the irreducible 7-dimensional representation of $G_2 \times \mathbb{C}^*$ are other irreducible examples.

The representation $(\mathrm{SL}(n, \mathbb{C}) \times (\mathbb{C}^*)^2, \Lambda_1 \oplus \Lambda^2(\Lambda_1))$ (n odd and $n \geq 5$), where Λ_1 is the natural representation of $\mathrm{SL}(n, \mathbb{C})$ and $\Lambda^2(\Lambda_1)$ is its second exterior power, provides a nonirreducible example.

Let us now give a more precise description of our paper.

In Section 2 we give basic definitions and properties of, and notation for, multiplicity-free spaces, including multiplicity-free spaces with a one-dimensional quotient. If (G, V) is a multiplicity-free space then G has an open orbit on V (i.e., (G, V) is a prehomogeneous vector space). We also prove that in the so-called regular case the G-invariant differential operators on the open orbit of a multiplicity-free space always have polynomial coefficients (in fact a slightly more general result is proved; see Theorem 2.2.6).

In Section 3 we introduce the various algebras of differential operators we are interested in. We define their natural gradings and we define the so-called Bernstein–Sato polynomial of a homogeneous operator of any degree, not only for degree-zero operators as usual. We obtain there the first results concerning these algebras. Using the Harish-Chandra isomorphism for multiplicity-free spaces [Knop 1998], we prove a key lemma on invariant polynomials under the so-called little Weyl group which enables us to prove that $D(V)^G$ is a polynomial algebra over the center $\mathfrak{Z}(\mathcal{T})$ of $D(V)^{G'}$, with the Euler operator as generator (Theorem 3.2.6). We also give generators of the center $\mathfrak{Z}(\mathcal{T})$ (Theorem 3.2.10) and obtain some specific results in the case of prehomogeneous vector spaces of commutative parabolic type (Theorem 3.3.1).

Section 4, which is the main section, is devoted to the structure of $D(V)^{G'}$. We first briefly define and study Smith algebras over a commutative ring $\boldsymbol{A}$ with unit and no zero divisors (the original definition by Smith was over $\mathbb{C}$). These algebras are defined by generators and relations (involving a polynomial in $\boldsymbol{A}[t]$), and their center is a polynomial algebra $\boldsymbol{A}[\Omega_1]$, where Ω_1 is a generalized Casimir element. Our main result asserts that $D(V)^{G'}$ is isomorphic to the quotient of a Smith algebra over its center $\mathscr{Z}(\mathscr{T})$ by the two-sided ideal generated by the element Ω_1. Concretely, we give generators and relations for $D(V)^{G'}$ (see Theorem 4.2.2).

Section 5 is devoted to the study of the algebras of radial components. By radial component of a differential operator in $D(V)^{G'}$ we mean the restriction of D to a G'-isotypic component of $\mathbb{C}[V]$. As a corollary of the preceding results we prove that these algebras are quotients of “classical” Smith algebras, that is, Smith algebras over $\mathbb{C}$ (see Theorem 5.2.3). Of course the defining relations depend on the G'-isotypic component. We also give generators of the kernel of the radial component map. In the case of the trivial representation of G', the structure of the algebra of radial components was first obtained by Levasseur [2009], by other methods.

2. Multiplicity-free spaces with a one-dimensional quotient

2.1. ***Prehomogeneous vector spaces, basic definitions and properties.*** Let G be a connected algebraic group over $\mathbb{C}$, and let (G, ρ, V) be a rational representation of G on the (finite-dimensional) vector space V. Then the triplet (G, ρ, V) is called a *prehomogeneous vector space* (abbreviated to PV) if the action of G on V has a Zariski open orbit $\Omega \subset V$. For the general theory of PVs, we refer the reader to the book of Kimura [2003] or to [Sato and Kimura 1977]. The elements in Ω are called *generic*. The PV is said to be *irreducible* if the corresponding representation is irreducible. The *singular set* S of (G, ρ, V) is defined by $S = V \setminus \Omega$. Elements in S are called *singular*. If no confusion can arise we often simply denote the PV by (G, V). We will also write $g.x$ instead of $\rho(g)x$, for $g \in G$ and $x \in V$. It is easy to see that the condition for a rational representation (G, ρ, V) to be a PV is in fact an infinitesimal condition. More precisely let $\mathfrak{g}$ be the Lie algebra of G and let $d\rho$ be the derived representation of ρ. Then (G, ρ, V) is a PV if and only if there exists $v \in V$ such that the map

$$\begin{aligned} \mathfrak{g} &\to V, \\ X &\mapsto d\rho(X)v, \end{aligned}$$

is surjective (we will often write $X.v$ instead of $d\rho(X)v$). Therefore we will call $(\mathfrak{g}, V)$ a PV if the preceding condition is satisfied.

Let (G, V) be a PV. A rational function f on V is called a *relative invariant* of (G, V) if there exists a rational character χ of G such that $f(g.x) = \chi(g)P(x)$ for $g \in G$ and $x \in V$. From the existence of an open orbit it is easy to see that a character χ which is trivial on the isotropy subgroup of an element $x \in \Omega$ determines a unique relative invariant P. Let $S_1, S_2, \dots, S_k$ denote the irreducible components of codimension one of the singular set S. Then there exist irreducible polynomials

$P_1, P_2, \dots, P_k$ such that $S_i = \{x \in V \mid P_i(x) = 0\}$. The polynomials P_i are unique up to nonzero constants; they are relative invariants of (G, V) and any nonzero relative invariant f can be written in a unique way as $f = cP_1^{n_1} P_2^{n_2} \cdots P_k^{n_k}$, where $n_i \in \mathbb{Z}$ and $c \in \mathbb{C}^*$. The polynomials $P_1, P_2, \dots, P_k$ are called the *fundamental relative invariants* of (G, V). Moreover if the representation (G, V) is irreducible then there exists at most one irreducible polynomial which is relatively invariant.

The prehomogeneous vector space (G, V) is called *regular* if there exists a relative invariant polynomial P whose Hessian $H_P(x)$ is nonzero on Ω. If G is reductive, then (G, V) is regular if and only if the singular set S is a hypersurface, or if and only if the isotropy subgroup of a generic point is reductive. If the PV (G, V) is regular, or if G is reductive, then the contragredient representation (G, V^*) is again a PV.

2.2. ***Multiplicity-free spaces.*** For the results concerning multiplicity-free spaces we refer the reader to the survey by Benson and Ratcliff [2004] or to [Knop 1998]. Let (G, V) be a finite-dimensional rational representation of a connected reductive algebraic group G. Let $\mathbb{C}[V]$ be the algebra of polynomials on V. Then G acts on $\mathbb{C}[V]$ by

$$g.\varphi(x) = \varphi(g^{-1}x) \quad (g \in G,\ \varphi \in \mathbb{C}[V]).$$

As the space $\mathbb{C}[V]^n$ of homogeneous polynomials of degree n is stable under this action, the representation $(G, \mathbb{C}[V])$ is completely reducible. Let $D(V)$ be the algebra of differential operators on V with polynomial coefficients. The group G acts also on $D(V)$ by

$$(g.D)(\varphi) = g.(D(g^{-1}.\varphi)) \quad (g \in G,\ D \in D(V),\ \varphi \in \mathbb{C}[V]).$$

Recall the G-equivariant identifications between $\mathbb{C}[V]$ and the symmetric algebra $S(V^*)$ of the dual space V^* and between $\mathbb{C}[V^*]$ and the symmetric algebra $S(V)$ of V. The embedding

$$\begin{aligned} V &\to D(V), \\ v &\mapsto D_v, \end{aligned}$$

where $D_v P(x) = \lim_{t\to 0}(P(x + tv) - P(x))/t$, extends uniquely to an embedding $S(V) \to D(V)$ whose image is the ring of differential operators with constant coefficients. If $f \in S(V) \simeq \mathbb{C}[V^*]$ we denote by $f(\partial)$ the corresponding differential operator. Another way to construct $f(\partial)$ for $f \in \mathbb{C}[V^*]$ is to say that $f(\partial)$ is the unique differential operator on V satisfying

$$f(\partial_x)e^{\langle x,y\rangle} = f(y)e^{\langle x,y\rangle} \quad (x \in V,\ y \in V^*). \tag{2-2-1}$$

Recall also that the $\mathbb{C}[V]$-module $D(V)$ can be identified with $\mathbb{C}[V]\otimes S(V)$ through the multiplication map

$$\begin{aligned} m:\mathbb{C}[V]\otimes S(V) &\xrightarrow{\ \simeq\ } D(V),\\ \varphi\otimes f &\longmapsto \varphi f(\partial). \end{aligned}$$

The preceding map is in fact G-equivariant and therefore the G-module $D(V)$ is isomorphic to the G-module $\mathbb{C}[V]\otimes S(V)$. The duality pairing $V\otimes V^*\to\mathbb{C}$ extends uniquely to the nondegenerate G-equivariant pairing

$$\begin{aligned} S(V)\otimes S(V^*)\simeq \mathbb{C}[V^*]\otimes\mathbb{C}[V] &\to \mathbb{C},\\ f\otimes\varphi &\mapsto \langle f,\varphi\rangle = f(\partial)\varphi(0), \end{aligned} \tag{2-2-2}$$

which gives rise to an embedding $\mathbb{C}[V^*]\hookrightarrow\mathbb{C}[V]^*$. It is easy to see that if $i\neq j$, $\langle\mathbb{C}^i[V^*],\mathbb{C}^j[V]\rangle=\{0\}$.

Definition 2.2.1. Let G be a connected reductive algebraic group, and let V be the space of a finite-dimensional (complex) rational representation of G. The representation (G, V) is said to be multiplicity-free (abbreviated to *MF*) if each irreducible representation of G occurs at most once in the representation $(G,\mathbb{C}[V])$.

Remark 2.2.2. Historically the classification of MF spaces goes as follows. Kac [1980] determined all the MF spaces where the representation (G, V) is irreducible. Brion [1985] did the case where $G'=[G,G]$ is (almost) simple. Finally, Benson and Ratcliff [1996; 2004] and independently Leahy [1998] (see also [Knop 1998]) classified all indecomposable saturated MF spaces up to geometric equivalence.

The following theorem summarizes some basic results concerning MF spaces (see [Howe and Umeda 1991; Knop 1998; Benson and Ratcliff 2004]):

Theorem 2.2.3. 1) *A finite-dimensional representation (G, V) is MF if and only if (B, V) is a prehomogeneous vector space for any Borel subgroup B of G (and hence each MF space (G, V) is a PV).*

2) *A finite-dimensional representation (G, V) is MF if and only if the algebra $D(V)^G$ of invariant differential operators with polynomial coefficients is commutative.*

3) *If (G, V) is a MF space, then the dual space (G, V^*) is also MF.*

Proof. The first assertion is due to [Vinberg and Kimelfeld 1978]; another proof can be found in [Knop 1998]. The second assertion is due to [Howe and Umeda 1991, Theorem 7.1]. For the third assertion note that as $\langle\mathbb{C}^i[V^*],\mathbb{C}^j[V]\rangle=\{0\}$ for $i\neq j$, we obtain that $f\mapsto\langle f,\ \ \rangle$ is a G-equivariant isomorphism between $\mathbb{C}^i[V^*]$ and $\mathbb{C}^i[V]^*$, and hence (G, V^*) is multiplicity-free. $\square$

Let us be more precise about the decomposition of the polynomials under the action of the group G or a Borel subgroup. Therefore we need more notation. We can write $G = G'C$, where $G' = [G, G]$ is the subgroup of commutators and $C = Z(G)^\circ \simeq (\mathbb{C}^*)^p$ is the connected component of the center of G. Let T' be a maximal torus in G', and let $B' = T'U$ be a Borel subgroup of G', where U is the nilradical of B'. The group $T = T'C$ is a maximal torus in G and $B = TU$ is a Borel subgroup of G. We will denote by $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{t}$, $\mathfrak{t}'$, $\mathfrak{c}$, $\mathfrak{b}$, $\mathfrak{b}'$, $\mathfrak{u}$ the corresponding Lie algebras. Let R be the set of roots of $(\mathfrak{g}', \mathfrak{t}')$, let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ be the basis of simple roots corresponding to $\mathfrak{b}'$ and let R^+ be the corresponding set of positive roots.

Denote by Λ' the lattice of weights of $(\mathfrak{g}', \mathfrak{t}')$. Then $\Lambda' = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \cdots \oplus \mathbb{Z}\omega_\ell$, where the ω_i are the fundamental weights. Let $\Lambda'^+ = \mathbb{N}\omega_1 \oplus \mathbb{N}\omega_2 \oplus \cdots \oplus \mathbb{N}\omega_\ell$ be the set of dominant weights. Denote by $X(C)$ the group of algebraic characters of C, which we will sometimes consider as linear forms on $\mathfrak{c}$. Set

$$\Lambda = \Lambda' \oplus X(C), \quad \Lambda^+ = \Lambda'^+ \oplus X(C).$$

For $\lambda \in \Lambda^+$ (resp. $\lambda' \in \Lambda'^+$) let us denote by $V_{-\lambda}$ (resp. $V_{-\lambda'}$) an irreducible $\mathfrak{g}$-module (resp. $\mathfrak{g}'$-module) with the highest weight λ (resp. λ'). We use this unusual notation because we want to index the modules occurring in $\mathbb{C}[V]$ by the character of their highest weight polynomial, rather than by the highest weight.

For a multiplicity-free space (G, V) we have the decomposition

$$\mathbb{C}[V] = \bigoplus_{\lambda \in \Lambda^+} V_{-\lambda}^{m(\lambda)},$$

where $m(\lambda) = 0$ or 1. If $m(\lambda) = 1$, then there exists a uniquely defined positive integer $d(\lambda)$ such that $V_{-\lambda} \in \mathbb{C}[V]^{d(\lambda)}$. The integer $d(\lambda)$ is called the *degree* of λ. Let us denote by $\Delta_0, \Delta_1, \dots, \Delta_k, \dots, \Delta_r$ the fundamental relative invariants of the PV (B, V), indexed in such a way that $\Delta_0, \Delta_1, \dots, \Delta_k$ are the fundamental relative invariants of the PV (G, V) and such that the other invariants are ordered by decreasing degree. We denote by d_i the degree of Δ_i $(i = 0, \dots, r)$. It is worthwhile noticing that at least Δ_r is of degree one as the highest weight vectors of the irreducible components of V^* must occur. Then any relative invariant of (B, V) is of the form $c\Delta^{\boldsymbol{a}}$, where $\boldsymbol{a} = (a_0, a_1, \dots, a_r) \in \mathbb{Z}^{r+1}$ and $\Delta^{\boldsymbol{a}} = \Delta_0^{a_0} \cdots \Delta_r^{a_r}$. The nonnegative integer $r + 1$ is called the *rank* of the MF space (G, V). The algebra of U-invariants is the subalgebra generated by the Δ_i; i.e., $\mathbb{C}[V]^U$ is given by $\mathbb{C}[\Delta_0, \dots, \Delta_r]$. As the polynomials Δ_i are algebraically independent, this latter algebra is a polynomial algebra. Let λ_i be the character of Δ_i (we use the same notation λ_i for the character of the group and for its derivative, which is an element of Λ^+). Hence the (infinitesimal) character of $\Delta^{\boldsymbol{a}}$ is $\lambda_{\boldsymbol{a}} = a_0\lambda_0 + \cdots + a_r\lambda_r$. Of course by definition the elements $\Delta^{\boldsymbol{a}}$ $(a_i \geq 0,\ i = 0, \dots, r)$ are the highest weights vectors in $\mathbb{C}[V]$. Due to the fact that the group action on $\Delta^{\boldsymbol{a}}$ is given by $g.\Delta^{\boldsymbol{a}}(x) = \Delta^{\boldsymbol{a}}(g^{-1}x)$, the infinitesimal highest weight of $\Delta^{\boldsymbol{a}}$ is $-\lambda_{\boldsymbol{a}} = -a_0\lambda_0 - \cdots - a_r\lambda_r$.

If we set $V_{\boldsymbol{a}} = V_{-\lambda_{\boldsymbol{a}}}$, we therefore can write

$$\mathbb{C}[V] = \bigoplus_{a_0\geq 0,\dots,a_r\geq 0} V_{\boldsymbol{a}}. \tag{2-2-3}$$

Sometimes, if $\lambda = a_0\lambda_0 + \cdots + a_r\lambda_r$, we simply write V_λ instead of $V_{\boldsymbol{a}}$. If we denote by d_i the degree of Δ_i, one can notice that all elements in $V_{\boldsymbol{a}}$ are of degree $d(\boldsymbol{a}) = a_0d_0 + a_1d_1 + \cdots + a_rd_r$. It is also worthwhile noticing that we have

$$V_{\boldsymbol{a}} = \Delta_0^{a_0}\Delta_1^{a_1}\cdots\Delta_k^{a_k} V_{0,\dots,0,a_{k+1},\dots,a_r}. \tag{2-2-4}$$

The proof of the following lemma is straightforward.

Lemma 2.2.4. *Define* $\mathbb{O} = \{x \in V \mid \Delta_i(x) \neq 0,\ i = 0, \dots, k\}$. *Let* $\mathbb{C}[\mathbb{O}]$ *be the ring of regular functions on* $\mathbb{O}$ (*elements of* $\mathbb{C}[\mathbb{O}]$ *are just rational functions whose denominators are of the form* $\Delta_0^{a_0}\cdots\Delta_k^{a_k}$, *with* $a_0, \dots, a_k \geq 0$). *As the polynomials* $\Delta_0, \dots, \Delta_k$ *are relative invariants under* G, *the open set* $\mathbb{O}$ *is* G*-stable*, *and therefore* G *acts on* $\mathbb{C}[\mathbb{O}]$. *Then* $\mathbb{C}[\mathbb{O}]$ *decomposes without multiplicities under the action of* G. *More precisely the decomposition into irreducibles is given by*

$$\mathbb{C}[\mathbb{O}] = \bigoplus_{\substack{(a_0,\dots,a_k)\in\mathbb{Z}^{k+1}\\ (a_{k+1},\dots,a_r)\in\mathbb{N}^{r-k}}} V_{\boldsymbol{a}},$$

where $V_{\boldsymbol{a}} = \Delta_0^{a_0}\Delta_1^{a_1}\cdots\Delta_k^{a_k} V_{0,\dots,0,a_{k+1},\dots,a_r}$ *is the irreducible subspace of* $\mathbb{C}[\mathbb{O}]$ *generated by the highest weight vector* $\Delta^{\boldsymbol{a}} = \Delta_0^{a_0}\Delta_1^{a_1}\cdots\Delta_r^{a_r}$.

Remark 2.2.5. We want to draw the attention of the reader to the fact that if (G, V) is not a regular PV, then the open set $\mathbb{O}$ may be distinct from the open G-orbit Ω.

The preceding lemma has the following consequence.

Theorem 2.2.6. *Let* (G, V) *be a multiplicity-free space. As before set*

$$\mathbb{O} = \{x \in V \mid \Delta_i(x) \neq 0,\ i = 0, \dots, k\}.$$

Then $D(V)^G = D(\mathbb{O})^G$. *In other words any* G*-invariant differential operator with coefficients in* $\mathbb{C}[\mathbb{O}]$ *has in fact polynomial coefficients.*

Proof. Let $D \in D(\mathbb{O})^G$. As we know from the preceding lemma that $\mathbb{C}[\mathbb{O}]$ decomposes without multiplicities under G, we obtain that D defines a G-equivariant endomorphism on each $V_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathbb{Z}^{k+1}\times\mathbb{N}^{r-k}$. Thus D stabilizes $\mathbb{C}[V] = \bigoplus_{a_0\geq 0,\dots,a_r\geq 0} V_{\boldsymbol{a}}$. It is easy to see that a differential operator with rational coefficients and which stabilizes the polynomials must have polynomial coefficients. $\square$

Remark 2.2.7. Let V be a simple Jordan algebra over $\mathbb{C}$ or $\mathbb{R}$. Let Ω be the set of invertible elements in V and let G be the structure group of V. It is known that (G, V) is a multiplicity-free space with Ω as open G-orbit. Then the preceding theorem implies that $D(V)^G = D(\Omega)^G$. This result was already known in this context and is usually obtained by computing an explicit set of generators of $D(\Omega)^G$ (see [Nomura 1989; Faraut and Korányi 1994; Yan 2000]). Through the so-called Kantor–Koecher–Tits construction there is a one-to-one correspondence between these spaces and the PVs of commutative parabolic type (see Example 2.3.3 below).

Proposition 2.2.8. *Let (G, V) be a MF space. For $\tilde{\boldsymbol{a}} = (a_{k+1}, \dots, a_r) \in \mathbb{N}^{r-k}$ we define $V_{\tilde{\boldsymbol{a}}} = V_{(0,\dots,0,a_{k+1},\dots,a_r)}$. Then for $\boldsymbol{a} = (a_0, \dots, a_k, a_{k+1}, \dots, a_r)$ the spaces $V_{\boldsymbol{a}} = \Delta_0^{a_0} \cdots \Delta_k^{a_k} V_{\tilde{\boldsymbol{a}}}$ are G'-equivalent if $\tilde{\boldsymbol{a}}$ is fixed and if $(a_0, \dots, a_k) \in \mathbb{Z}^{k+1}$. If we define*

$$U_{\tilde{\boldsymbol{a}}} = \bigoplus_{(a_0,\dots,a_k)\in\mathbb{N}^{k+1}} \Delta_0^{a_0} \cdots \Delta_k^{a_k} V_{\tilde{\boldsymbol{a}}}, \quad W_{\tilde{\boldsymbol{a}}} = \bigoplus_{(a_0,\dots,a_k)\in\mathbb{Z}^{k+1}} \Delta_0^{a_0} \cdots \Delta_k^{a_k} V_{\tilde{\boldsymbol{a}}},$$

the decompositions of $\mathbb{C}[V]$ and $\mathbb{C}[\mathbb{O}]$ into G'-isotypic components are given by

$$\mathbb{C}[V] = \bigoplus_{\tilde{\boldsymbol{a}}} U_{\tilde{\boldsymbol{a}}}, \quad \mathbb{C}[\mathbb{O}] = \bigoplus_{\tilde{\boldsymbol{a}}} W_{\tilde{\boldsymbol{a}}}.$$

Proof. The map $P \mapsto \Delta_0^{a_0} \cdots \Delta_k^{a_k} P$ is a G'-equivariant isomorphism between $V_{\tilde{\boldsymbol{a}}}$ and $\Delta_0^{a_0} \cdots \Delta_k^{a_k} V_{\tilde{\boldsymbol{a}}}$; hence all these spaces are G'-equivalent. To prove the second assertion it is enough to prove that if $\tilde{\boldsymbol{a}} \neq \tilde{\boldsymbol{b}}$, then the spaces $V_{\tilde{\boldsymbol{a}}}$ and $V_{\tilde{\boldsymbol{b}}}$ are not G'-equivalent. Suppose that this would be the case and let $\Delta^{\tilde{\boldsymbol{a}}}$ and $\Delta^{\tilde{\boldsymbol{b}}}$ be the corresponding highest weight vectors with characters $\lambda_{\tilde{\boldsymbol{a}}}$ and $\lambda_{\tilde{\boldsymbol{b}}}$ respectively. From the G'-equivalence we know that $\lambda_{\tilde{\boldsymbol{a}}|\mathfrak{t}'} = \lambda_{\tilde{\boldsymbol{b}}|\mathfrak{t}'}$ and hence $P = \Delta^{\tilde{\boldsymbol{a}}}/\Delta^{\tilde{\boldsymbol{b}}}$ is a relative invariant under B whose character is trivial on $\mathfrak{t}'$. Therefore it generates a one-dimensional representation; hence P is a relative invariant under G. Finally we obtain that $\Delta^{\tilde{\boldsymbol{a}}} = \Delta_0^{a_0} \cdots \Delta_k^{a_k} \Delta^{\tilde{\boldsymbol{b}}}$, and this is not possible if $\tilde{\boldsymbol{a}} \neq \tilde{\boldsymbol{b}}$. ☐

As (G, V^*) is multiplicity-free (Theorem 2.2.3) and $\mathbb{C}^i[V^*] \simeq \mathbb{C}^i[V]^*$, we have

$$\mathbb{C}[V^*] = \bigoplus_{a_0\geq 0,\dots,a_r\geq 0} V_{\boldsymbol{a}}^*, \tag{2-2-5}$$

where $V_{\boldsymbol{a}}^*$ is the irreducible G-submodule of $\mathbb{C}[V^*]$ generated by a lowest weight vector $\Delta^{*\boldsymbol{a}} \in \mathbb{C}[V^*]$, defined up to a multiplicative constant, whose character with respect to the opposite Borel subgroup B^- is equal to $-\lambda_{\boldsymbol{a}} = -a_0\lambda_0 - \cdots - a_r\lambda_r$. Let us fix a lowest weight vector Δ_i^* $(i = 0, \dots, r)$ with character $-\lambda_i$ (with respect to B^-). Then we can choose $\Delta^{*\boldsymbol{a}} = \Delta_0^{*a_0} \Delta_1^{*a_1} \cdots \Delta_r^{*a_r}$. Of course the module $V_{\boldsymbol{a}}^*$ is the dual module of $V_{\boldsymbol{a}}$ through $f \mapsto \langle f, \quad \rangle$ (see (2-2-2)).

As $V_{\boldsymbol{a}}$ is a G-irreducible module, it is well known that the tensor G-module $V_{\boldsymbol{a}} \otimes V_{\boldsymbol{a}}^*$ contains, up to a constant, a unique G-invariant vector $R_{\boldsymbol{a}}$ and that $V_{\boldsymbol{a}} \otimes V_{\boldsymbol{b}}^*$ does not contain any nontrivial G-invariant vector if $\boldsymbol{a} \neq \boldsymbol{b}$ (see for example [Howe and Umeda 1991]). To be more precise we define $R_{\boldsymbol{a}}$ to be the operator corresponding to the "unit matrix" in $V_{\boldsymbol{a}} \otimes V_{\boldsymbol{a}}^* \simeq \mathrm{Hom}(V_{\boldsymbol{a}}, V_{\boldsymbol{a}})$. Moreover as $\mathbb{C}[V] \otimes \mathbb{C}[V^*]$ is G-isomorphic to $D(V)$, the element $R_{\boldsymbol{a}}$ can be viewed as a G-invariant differential operator with polynomial coefficients. The operators $R_{\boldsymbol{a}}$ are sometimes called *Capelli operators*. They are also called *unnormalized canonical invariants* in [Benson and Ratcliff 2004]. Moreover the family of elements $R_{\boldsymbol{a}}$ ($\boldsymbol{a} \in \mathbb{N}^{r+1}$) is a vector basis of the vector space $D(V)^G = D(\mathbb{O})^G$.

The Capelli operators R_i corresponding to the space V_{λ_i} $(i = 0, \ldots, r)$ will be of particular importance because of the result below.

Theorem 2.2.9 (Howe and Umeda). *Let (G, V) be a MF space. The Capelli operators R_i $(i = 0, \ldots, r)$ are algebraically independent and $D(V)^G = \mathbb{C}[R_0, \ldots, R_r]$.*

Proof. See [Howe and Umeda 1991, Theorem 9.1; Benson and Ratcliff 2004, Corollary 7.4.4]. ☐

Remark 2.2.10. a) Recall that for $i = 0, 1, \ldots, k$ the polynomials $\Delta_0, \Delta_1, \ldots, \Delta_k$ are the fundamental relative invariants under the action of the full group G. Once these polynomials are fixed, let us define the polynomial $\Delta_i^* \in \mathbb{C}[V^*]$ as the unique fundamental relative invariant of (G, V^*) with character λ_i^{-1}, such that $\Delta_i^*(\partial)\Delta_i(0) = 1$, for $i = 0, \ldots, k$. Then the Capelli operators R_i $(i = 0, \ldots, k)$ are given by $R_i = \Delta_i(x)\Delta_i^*(\partial)$, and the Capelli operator corresponding to the irreducible component $V_{a_0\lambda_0+\cdots+a_k\lambda_k}$ is a scalar multiple of $\Delta_0^{a_0}(x)\cdots\Delta_k^{a_k}(x)\Delta_0^*(\partial)^{a_0}\cdots\Delta_k^*(\partial)^{a_k}$. More generally the Capelli operator $R_{\boldsymbol{a}}$ corresponding to $V_{\boldsymbol{a}}$, where

$$\boldsymbol{a} = a_0\lambda_0 + \cdots + a_k\lambda_k + a_{k+1}\lambda_{k+1} + \cdots + a_r\lambda_r,$$

is a scalar multiple of $\Delta_0^{a_0}(x) \cdots \Delta_k^{a_k}(x) R_{a_{k+1}\lambda_{k+1}+\cdots+a_r\lambda_r} \Delta_0^*(\partial)^{a_0} \cdots \Delta_k^*(\partial)^{a_k}$.

b) Moreover, in the case where (G, V) is irreducible, as Δ_r is the highest weight vector in V^*, the operator R_r is nothing but the Euler operator E.

c) More generally, if $V = V_1 \oplus \cdots \oplus V_\ell$, where the representations (G, V_i) are irreducible, the various Euler operators E_i on V_i are the Capelli operators associated to the irreducible subspaces $V_i^* \in \mathbb{C}[V]$. Of course the global Euler operator E on V is given by $E = E_1 + \cdots + E_\ell$. As the highest weight vectors of the spaces (G, V_i^*) occur as the last ℓ elements of $\Delta_0, \ldots, \Delta_r$, we have $R_{r-\ell+1} = E_1, \ldots, R_r = E_\ell$.

d) According to b) and c), one can always take $\{R_0, R_1, \ldots, R_{r-1}, E\}$ as a set of algebraically independent generators of $D(V)^G$.

2.3. ***Multiplicity-free spaces with a one-dimensional quotient.*** Let us now define the main objects this paper deals with, namely the MF spaces with a one-dimensional quotient, which were introduced by T. Levasseur.

Definition 2.3.1 [Levasseur 2009, Sections 3.2 and 4.2]. 1) A prehomogeneous vector space (G, V) is said to be of rank one* if there exists a homogeneous polynomial Δ_0 on V such that $\Delta_0 \notin \mathbb{C}[V]^G$ and $\mathbb{C}[V]^{G'} = \mathbb{C}[\Delta_0]$.

2) A multiplicity-free space (G, V) is said to have a one-dimensional quotient if it is a PV of rank one.

Remark 2.3.2. a) The classification of multiplicity-free spaces with a one-dimensional quotient has been obtained in [Rubenthaler 2013].

b) It can be shown that if (G, V) is a PV of rank one, then the polynomial Δ_0 is the unique fundamental relative invariant of (G, V). More precisely a PV (G, V) is of rank one if and only if it has a unique fundamental relative invariant [ibid.]. Hence in the notation of Section 2.2 we have $k = 0$, in other words Δ_0 is the unique fundamental G relative invariant among the B relative invariants $\Delta_0, \Delta_1, \dots, \Delta_r$.

We give now some examples of MF spaces with a one-dimensional quotient.

Example 2.3.3. PVs of commutative parabolic type (for details we refer to [Muller et al. 1986]; [Rubenthaler and Schiffmann 1987] is also relevant).

Let $\tilde{\mathfrak{g}}$ be a simple complex Lie algebra. Assume we are given a 3-grading of $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+.$$

Then $\mathfrak{g}$ is a reductive Lie subalgebra and it is well known that the representation $(\mathfrak{g}, V^+)$ is prehomogeneous (here $\mathfrak{g}$ acts on V^+ via the bracket). Let $\widetilde{G}$ be the adjoint group of $\tilde{\mathfrak{g}}$ and let G be the connected subgroup of $\widetilde{G}$ whose Lie algebra is $\mathfrak{g}$. Then the space (G, V^+) is multiplicity-free. Moreover such a space has a one-dimensional quotient if and only if it is a regular PV. Up to local isomorphism one obtains the following list:

1) $(\mathrm{SL}(n, \mathbb{C}) \times \mathrm{SL}(n, \mathbb{C}) \times \mathbb{C}^*, M_n(\mathbb{C}))$ acting via $(g_1, g_2, t).x = tg_1xg_2^{-1}$, where $g_1, g_2 \in \mathrm{SL}(n, \mathbb{C})$, $t \in \mathbb{C}^*$, $x \in M_n(\mathbb{C})$; here $\Delta_0(x) = \det(x)$.
2) $(O(n, \mathbb{C}) \times \mathbb{C}^*, \mathbb{C}^n)$ with the natural action. Here $\Delta_0(x) = Q(x) = \sum_{i=1}^{i=n} x_i^2$.
3) $(\mathrm{GL}(n, \mathbb{C}), \mathrm{Sym}_n(\mathbb{C}))$, where $\mathrm{Sym}_n(\mathbb{C})$ denotes the $n \times n$ symmetric matrices, with the action $g.x = gx^tg$. Then $\Delta_0(x) = \det(x)$.
4) $(\mathrm{GL}(n, \mathbb{C}), \mathrm{Skew}_n(\mathbb{C}))$, n even, with the action $g.x = gx^tg$, where $\mathrm{Skew}_n(\mathbb{C})$ denotes the $n \times n$ skew-symmetric matrices. Then $\Delta_0(x) = Pf(x)$, where $Pf(x)$ denotes the pfaffian of the even skew-symmetric matrix x.

*If (G, V) is also multiplicity-free, its rank as a PV is not the same as its rank as an MF space.

5) $(E_6 \times \mathbb{C}^*, \mathbb{C}^{27})$ (the irreducible 27-dimensional representation of E_6). The fundamental relative invariant is of degree 3; it is known as the Freudenthal cubic.

Example 2.3.4. We consider $(\mathrm{GL}(2) \times \mathrm{Sp}(n), \mathbb{C}^2 \otimes \mathbb{C}^{2n})$ (tensor product of the natural representations). Here the action is given by

$$(g_1, g_2).X = g_2 X({}^t g_1), \quad g_1 \in \mathrm{SL}(2), g_2 \in \mathrm{Sp}(n), X \in M_{2n,2}$$

The relative invariant Δ_0 is given by $Pf({}^t X J X)$, where

$$J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix},$$

and where $Pf(\cdot)$ is the pfaffian of a 2×2 skew symmetric matrix. The rank is equal to 3 and it is a regular PV. For details see [Howe and Umeda 1991, case 11.6; Rubenthaler 2013, case 4.1.7].

Example 2.3.5. $(\mathrm{GL}(n) \times \mathrm{GL}(n-1), M_{n,1} \oplus M_{n,n-1})$. The action is given by

$$(g_1, g_2)(v, x) = (g_1 v, g_1 x g_2^{-1}), \; g_1 \in \mathrm{GL}(n), g_2 \in \mathrm{GL}(n-1), v \in M_{n,1}, x \in M_{n,n-1}.$$

The relative invariant Δ_0 is given by $\Delta_0(x) = \det(v; x)$, where $(v; x)$ is the $n \times n$ matrix obtained by putting the column vector v left to the $n \times (n-1)$ matrix x. The rank is equal to $2n-1$ and it is a regular PV. For details see [Benson and Ratcliff 2000, case 4.2.4; Rubenthaler 2013, case 4.2.5].

3. Algebras of differential operators

From now on we suppose that (G, V) is an MF space with a one-dimensional quotient.

3.1. ***Gradings and Bernstein–Sato polynomials.*** Recall that $\Delta_0, \dots, \Delta_r$ denote the fundamental relative invariants under a fixed Borel subgroup B of G. As the space has a one-dimensional quotient, Δ_0 is the unique polynomial among them which is relatively invariant under G (this means that $k = 0$ in the notation of Section 2.2). We also set $\mathbb{O} = \{x \in V \mid \Delta_0(x) \neq 0\}$.

Of course the Euler operator E on V, defined for $P \in \mathbb{C}[V]$ by

$$EP(x) = \frac{\partial}{\partial t} P(tx)_{t=1} = P'(x)x,$$

is invariant by any element in $\mathrm{GL}(E)$.

Once and for all we also define the following two elements in $D(V)$:

$$X = \Delta_0 \text{ (multiplication by } \Delta_0), \quad Y = \Delta_0^*(\partial).$$

The operator
$$X^{-1} \text{ (multiplication by } \Delta_0^{-1}),$$
which belongs to $D(\mathbb{O})$, will also play an important role. From the definition of the G action on $\mathbb{C}[V]$ and on $D(V)$ we have
$$g.X = \lambda_0(g^{-1})X, \quad g.X^{-1} = \lambda_0(g)X^{-1}, \quad g.Y = \lambda_0(g)Y, \tag{3-1-1}$$
and hence $X, Y \in D(V)^{G'}$ and $X^{-1} \in D(\mathbb{O})^{G'}$.

We now introduce some notation used in the rest of the paper:
$$\mathcal{T} = D(\mathbb{O})^{G'}, \quad \mathcal{T}_0 = D(V)^G = D(\mathbb{O})^G$$
(the last equality comes from Theorem 2.2.6). Remember that $\mathcal{T}_0$ is a polynomial algebra in $r+1$ variables (Theorem 2.2.9). We have the inclusions
$$\mathcal{T}_0 = D(V)^G = D(\mathbb{O})^G \subset D(V)^{G'} \subset \mathcal{T} = D(\mathbb{O})^{G'}.$$
An element D in $\mathcal{T}$ is said to be of degree m if $[E, D] = mD$. As differential operators in $\mathcal{T}$ have coefficients which are fractions whose denominators are homogeneous (powers of Δ_0), it is clear that $\mathcal{T}$ is graded by its homogeneous components. But on the other hand any homogeneous element D in $\mathcal{T}$ preserves the G'-isotypic components $W_{\tilde{\boldsymbol{a}}} = \bigoplus_{n\in\mathbb{N}} \Delta_0^n V_{\tilde{\boldsymbol{a}}}$ (see Proposition 2.2.8). Therefore a homogeneous element D maps $\Delta_0^n V_{\tilde{\boldsymbol{a}}}$ on $\Delta_0^{n+j} V_{\tilde{\boldsymbol{a}}}$ for some j and hence only multiples of d_0 (the degree of Δ_0) occur as homogeneous degrees in $\mathcal{T}$. If we define, for $p \in \mathbb{Z}$, $\mathcal{T}_p = \{D \in \mathcal{T} \mid [E, D] = pd_0D\}$, then
$$\mathcal{T} = \bigoplus_{p\in\mathbb{Z}} \mathcal{T}_p \tag{3-1-2}$$
(At this point it is not completely evident that the two definitions of $\mathcal{T}_0$ coincide, that is, $D(V)^G = \{D \in \mathcal{T} \mid [E, D] = 0\}$. This will be a consequence of the proof of Proposition 3.1.6 below.)

Similarly if we define
$$D(V)_p^{G'} = \left\{D \in D(V)^{G'} \;\middle|\; [E, D] = pd_0D\right\},$$
we have $D(V)^{G'} = \bigoplus_{p\in\mathbb{Z}} D(V)_p^{G'}$.

Definition 3.1.1. For $\boldsymbol{a} = (a_0, a_1, \ldots, a_r)$ and $p \in \mathbb{N}$, we define
$$\boldsymbol{a} + p = (a_0 + p, a_1, \ldots, a_r).$$
Then if $D \in \mathcal{T}_p$, the Schur Lemma ensures that if $P \in V_{\boldsymbol{a}}$ we have $DP = b_D(\boldsymbol{a})X^pP$, where $b_D(\boldsymbol{a}) \in \mathbb{C}$. It is easy to see that b_D is a polynomial in the variables $(a_0, a_1, \ldots, a_r)$ (see for example [Knop 1998, proof of Corollary 4.4]). This polynomial is called the Bernstein–Sato polynomial of D.

Example 3.1.2. Relations (3-1-1) imply that $X \in \mathcal{T}_1$, $X^{-1} \in \mathcal{T}_{-1}$ and $Y \in \mathcal{T}_{-1}$. And of course $E \in \mathcal{T}_0$. Obviously, from the definition, we have $b_X(\boldsymbol{a}) = b_{X^{-1}}(\boldsymbol{a}) = 1$, $b_E(\boldsymbol{a}) = d_0a_0 + d_1a_1 + \cdots + d_ra_r =$ the degree of $V_{\boldsymbol{a}}$ (recall that d_i is the degree of Δ_i). The computation of b_Y is more difficult. However it is known in the case of PVs of commutative parabolic type (see Example 2.3.3). In this case, for $\boldsymbol{X} = (X_0, X_1, \ldots, X_r)$ it is given by

$$b_Y(\boldsymbol{X}) = c\prod_{j=0}^{r}\left(X_0 + \cdots + X_j + j\frac{d}{2}\right), \tag{3-1-3}$$

where the constant c can be made explicit (see [Bopp and Rubenthaler 1993, Théorème 3.19]) and where $d/2 = (\dim(V) - d_0)/((d_0 - 1)d_0)$. This explicit computation of the polynomial b_Y in the particular case of PVs of commutative parabolic type has been obtained by several authors, using distinct methods (see [Kostant and Sahi 1991; Wallach 1992; Bopp and Rubenthaler 1993; Faraut and Korányi 1994]). The constant d is the same as the constant d which is familiar to specialists of Jordan algebras.

The following lemma is obvious, but useful.

Lemma 3.1.3. *Let $D_1, D_2 \in \mathcal{T}_p$. Then $D_1 = D_2$ if and only if $b_{D_1} = b_{D_2}$.*

Definition 3.1.4. The automorphism τ of $\mathcal{T} = D(\mathbb{O})^{G'}$ is defined by

$$\tau(D) = XDX^{-1} \quad \text{for all } D \in \mathcal{T}.$$

Proposition 3.1.5. *The algebra $\mathcal{T}_0$ is stable under τ and for any $D \in \mathcal{T}_0$ we have*

$$XD = \tau(D)X, \tag{3-1-4}$$
$$DY = Y\tau(D). \tag{3-1-5}$$

Proof. By definition, $\mathcal{T}_0 = D(V)^G$. From relations (3-1-1) we see that if D is G-invariant so is $\tau(D)$. Obviously $\tau(D) \in D(\mathbb{O})^G$. But $D(\mathbb{O})^G = D(V)^G$ by Theorem 2.2.6; hence $\mathcal{T}_0$ is τ-stable. Relation (3-1-4) is just the definition of τ. We will now prove that (3-1-5) holds on each subspace $V_{\boldsymbol{a}}$. Let b_D be the Bernstein–Sato polynomial of D. Then an easy calculation shows that the left and right sides of (3-1-5) act on $V_{\boldsymbol{a}}$ by $b_D(\boldsymbol{a} - 1)b_Y(\boldsymbol{a})X^{-1}$. Then Lemma 3.1.3 implies (3-1-5). □

Let us denote by $\mathcal{T}_0[X, Y]$ the subalgebra of $\mathcal{T}$ generated by $\mathcal{T}_0$, X and Y. From the preceding proposition and from the fact that XY and YX belong to $\mathcal{T}_0$ we know that any element $D \in \mathcal{T}_0[X, Y]$ can be written as a finite sum $D = \sum_{p,q\in\mathbb{N}} a_{p,q}X^pY^q$ with $a_{p,q} \in \mathcal{T}_0$. Similarly, let $\mathcal{T}_0[X, X^{-1}]$ denote the subalgebra of $\mathcal{T}$ generated by $\mathcal{T}_0$, X and X^{-1}. Also any element D in $\mathcal{T}_0[X, X^{-1}]$ can we written as a finite sum $D = \sum_{p\in\mathbb{Z}} a_pX^p$. The following proposition shows that $D(V)^{G'} = \mathcal{T}_0[X, Y]$ and that $\mathcal{T} = D(\mathbb{O})^{G'} = \mathcal{T}_0[X, X^{-1}]$ and makes the gradings more precise.

Proposition 3.1.6. 1) *We have*

$$D(V)^{G'} = \mathcal{T}_0[X,Y] = \Big(\bigoplus_{p\in\mathbb{N}^*} \mathcal{T}_0 Y^p\Big) \oplus \mathcal{T}_0 \oplus \Big(\bigoplus_{p\in\mathbb{N}^*} \mathcal{T}_0 X^p\Big)$$

(*in particular* $D(V)^{G'}_p = \mathcal{T}_0 X^p$ *if* $p \geq 0$, *and* $D(V)^{G'}_p = \mathcal{T}_0 Y^{-p}$ *if* $p < 0$). *Equivalently*,

$$D(V)^{G'} = \mathcal{T}_0[X,Y] = \Big(\bigoplus_{p\in\mathbb{N}^*} Y^p \mathcal{T}_0\Big) \oplus \mathcal{T}_0 \oplus \Big(\bigoplus_{p\in\mathbb{N}} X^p \mathcal{T}_0\Big).$$

2) *We have* $\mathcal{T} = D(\mathbb{O})^{G'} = \mathcal{T}_0[X, X^{-1}] = \bigoplus_{p\in\mathbb{Z}} \mathcal{T}_0 X^p = \bigoplus_{p\in\mathbb{Z}} X^p \mathcal{T}_0$.

3) *Any element* D *in* $\mathcal{T}_0[X,Y]$ *can be written uniquely in the form*

$$D = \sum_{i>0} u_i Y^i + \sum_{i\geq 0} v_i X^i \quad \textit{or} \quad D = \sum_{i>0} Y^i u_i + \sum_{i\geq 0} X^i v_i \quad (\textit{finite sums})$$

with $u_i, v_i \in \mathcal{T}_0$.

Any element $D \in \mathcal{T}$ *can be written uniquely in the form*

$$D = \sum_{i\in\mathbb{Z}} u_i X^i \quad \textit{or} \quad D = \sum_{i\in\mathbb{Z}} X^i u_i \quad (\textit{finite sums})$$

with $u_i \in \mathcal{T}_0$.

Proof. 1) For the moment we define $\mathcal{T}_0$ by $\mathcal{T}_0 = D(V)^G$. From Proposition 2.2.8 we know that the decomposition of $\mathbb{C}[V]$ into G'-isotypic components is given by

$$\mathbb{C}[V] = \bigoplus_{\tilde{\boldsymbol{a}}\in\mathbb{N}^r} U_{\tilde{\boldsymbol{a}}}, \quad \text{where } U_{\tilde{\boldsymbol{a}}} = \bigoplus_{a_0\in\mathbb{N}} \Delta_0^{a_0} V_{\tilde{\boldsymbol{a}}} \text{ and } \tilde{\boldsymbol{a}} = (0, a_1, \dots, a_r).$$

We will now use the technique of [Howe and Umeda 1991] which we have already mentioned before Theorem 2.2.9. As $\mathbb{C}[V]\otimes\mathbb{C}[V^*]$ is G'-isomorphic to $D(V)$, each subspace $\Delta^{a_0} V_{\tilde{\boldsymbol{a}}} \otimes (\Delta^{b_0} V_{\tilde{\boldsymbol{a}}})^*$ will give rise to a unique G'-invariant differential operator $R_{a_0,b_0,\tilde{\boldsymbol{a}}}$. Then by the same arguments as in Remark 2.2.10, it is easy to see that $R_{a_0,b_0,\tilde{\boldsymbol{a}}} = \Delta_0(x)^{a_0} R_{0,0,\tilde{\boldsymbol{a}}} \Delta_0^*(\partial)^{b_0} = X^{a_0} R_{0,0,\tilde{\boldsymbol{a}}} Y^{b_0}$. The elements $X^{a_0} R_{0,0,\tilde{\boldsymbol{a}}} Y^{b_0}$ ($a_0, b_0 \in \mathbb{N}$, $\tilde{\boldsymbol{a}} \in \mathbb{N}^r$) form a vector basis of $D(V)^{G'}$. Remark now that $R_{0,0,\tilde{\boldsymbol{a}}}$ is in $D(V)^G = \mathcal{T}_0$. Then from Proposition 3.1.5, we get $X^{a_0} R_{0,0,\tilde{\boldsymbol{a}}} Y^{b_0} = \tau^{a_0}(R_{0,0,\tilde{\boldsymbol{a}}}) X^{a_0} Y^{b_0}$ and $\tau^{a_0}(R_{0,0,\tilde{\boldsymbol{a}}}) \in \mathcal{T}_0$. If now $a_0 \leq b_0$, then $X^{a_0} R_{0,0,\tilde{\boldsymbol{a}}} Y^{b_0} = R Y^{b_0-a_0}$, where $R = \tau^{a_0}(R_{0,0,\tilde{\boldsymbol{a}}}) X^{a_0} Y^{a_0} \in \mathcal{T}_0$. If $a_0 > b_0$, then $X^{a_0} R_{0,0,\tilde{\boldsymbol{a}}} Y^{b_0} = R X^{a_0-b_0}$, where $R = \tau^{a_0}(R_{0,0,\tilde{\boldsymbol{a}}}) \tau^{a_0-b_0}(X^{b_0} Y^{b_0}) \in \mathcal{T}_0$. The first decomposition in assertion 1) is proved. The second decomposition is a consequence of relations (3-1-4) and (3-1-5).

2) A slight extension of (2-2-2) shows that $\mathbb{C}[\mathbb{O}]\otimes\mathbb{C}[V^*]$ is G-isomorphic to $D(\mathbb{O})$ through the map $\varphi\otimes f \mapsto \varphi f(\partial)$. Then the same proof as in 1) above shows that the

elements $X^{a_0}R_{0,0,\tilde{\boldsymbol{a}}}Y^{b_0}$ ($a_0 \in \mathbb{Z}$, $b_0 \in \mathbb{N}$, $\tilde{\boldsymbol{a}} \in \mathbb{N}^r$) form a vector basis of $D(\mathbb{O})^{G'} = \mathcal{T}$. Consider now an element $D \in \mathcal{T}$ such that $[E, D] = 0$. Then necessarily D is a linear combination of elements of the form $X^{a_0}R_{0,0,\tilde{\boldsymbol{a}}}Y^{a_0}$ with $a_0 \in \mathbb{N}$. Then, as announced previously, the two definitions of $\mathcal{T}_0$ coincide ($\mathcal{T}_0 = D(V)^G$ and $\mathcal{T}_0 = \{D \in \mathcal{T} \mid [E, D] = 0\}$). Now if $D \in \mathcal{T}_p$, then $D = DX^{-p}X^p$ and $DX^{-p} \in \mathcal{T}_0$. Hence $\mathcal{T}_p = \mathcal{T}_0 X^p = X^p \mathcal{T}_0$.

Assertion 3) is then obvious. □

Remark 3.1.7. The inclusion $D(V)^{G'} \subset D(\mathbb{O})^{G'}$ is obviously strict (note that $X^{-1} \in D(\mathbb{O})^{G'} \setminus D(V)^{G'}$), but the preceding results show that these two graded algebras have the same "positive part" $\big(\bigoplus_{p\in\mathbb{N}} \mathcal{T}_0 X^p\big)$.

The following proposition, whose proof is straightforward, shows that all the Bernstein–Sato polynomials are known if one knows the Bernstein–Sato polynomials of Y and of the elements of $\mathcal{T}_0$.

Proposition 3.1.8. *Let $D = D_0X^n$ ($n \in \mathbb{Z}$) and $D' = D_0Y^n$ ($n \in \mathbb{N}^*$), $D_0 \in \mathcal{T}_0$, be generic homogeneous elements in $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$ and $\mathcal{T}_0[X, Y]$. Then $b_D(\boldsymbol{a}) = b_{D_0}(\boldsymbol{a} + n)$ and $b_{D'}(\boldsymbol{a}) = b_{D_0}(\boldsymbol{a} - n)b_Y(\boldsymbol{a})b_Y(\boldsymbol{a} - 1)\cdots b_Y(\boldsymbol{a} - n + 1)$.*

3.2. ***The Harish-Chandra isomorphism and the center of*** $\mathcal{T}$**.** The aim of this subsection is to describe $\mathcal{T}_0 = D(V)^G$ as a module over the center of $\mathcal{T}$. For this we will use the Harish-Chandra isomorphism for MF spaces due to F. Knop.

Let (G, V) be an MF space with a one-dimensional quotient. Let B be a fixed Borel subgroup of G. Remember that (B, V) is a PV. Recall also that we denote by $\Delta_0, \Delta_1, \ldots, \Delta_r$ the set of fundamental relative invariants of (B, V) and that Δ_0 is the unique fundamental relative invariant under G. We denote by d_i (resp. λ_i) the degree (resp. the infinitesimal character) of Δ_i. Let $\mathfrak{b}$ be the Lie algebra of B, let $\mathfrak{t} \subset \mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ and let Σ be the set of roots of the pair $(\mathfrak{g}, \mathfrak{t})$. Denote by W the Weyl group of Σ. Denote by Σ^+ the set of positive roots such that $\mathfrak{b} = \mathfrak{t} + \sum_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} \alpha$. We define

$$\mathfrak{a}^* = \bigoplus_{i=0}^{r} \mathbb{C}\lambda_i \subset \mathfrak{t}^* \quad \text{and} \quad A = \mathfrak{a}^* + \rho \subset \mathfrak{t}^*.$$

Let $\mathcal{Z}(\mathfrak{g})$ be the center of the enveloping algebra of $\mathfrak{g}$. Denote by $\mathbb{C}[\mathfrak{t}^*]^W$ the W-invariant polynomials on $\mathfrak{t}^*$. One knows that the classical Harish-Chandra isomorphism is an isomorphism $H : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{t}^*]^W$ which can be computed the following way. For any $\lambda \in \mathfrak{t}^*$, let V_λ be the irreducible highest weight module with highest weight λ. It is well known that $\mathcal{Z}(\mathfrak{g})$ acts by scalar multiplication on V_λ. The scalar by which an element $z \in \mathcal{Z}(\mathfrak{g})$ acts on V_λ is precisely $H(z)(\lambda + \rho)$.

The natural representation of G on $\mathbb{C}[V]$ extends to a representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ on the same space $\mathbb{C}[V]$. Hence $z \in \mathcal{Z}(\mathfrak{g})$ acts on $V_{\boldsymbol{a}}$ by the scalar $H(z)(-\lambda_{\boldsymbol{a}}+\rho)$,[†] where $\lambda_{\boldsymbol{a}}=\sum_{i=0}^{r}a_i\lambda_i$ (remember that $\boldsymbol{a}=(a_0,\dots,a_r)$). Conversely if $\lambda=a_0\lambda_0+\cdots+a_r\lambda_r$ we define $\boldsymbol{a}_\lambda=(a_0,\dots,a_r)\in\mathbb{C}^{r+1}$. By abuse of notation if b_D is the Bernstein–Sato polynomial of $D\in\mathcal{T}_0$, we set $b_D(\lambda)=b_D(\boldsymbol{a}_\lambda)$.

On the other hand any $D\in D(V)^G=\mathcal{T}_0$ acts on each $V_{\boldsymbol{a}}$ by the scalar $b_D(\boldsymbol{a})$, where $b_D(\boldsymbol{a})$ is the Bernstein–Sato polynomial of D. This allows us to define the map

$$\begin{aligned} h: D(V)^G &\to \mathbb{C}[A],\\ D &\mapsto h(D): -\lambda+\rho\mapsto h(D)(-\lambda+\rho)=b_D(\lambda), \end{aligned}$$

where $\mathbb{C}[A]$ denotes the algebra of polynomials on the affine space $A=\mathfrak{a}^*+\rho\subset\mathfrak{t}^*$.

Let $\pi(z)$ be the operator in $D(V)^G$ which represents the action of z on $\mathbb{C}[V]$ and let $r:\mathbb{C}[\mathfrak{t}^*]^W\to\mathbb{C}[A]$ be the restriction homomorphism. It is clear from the definitions that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{Z}(\mathfrak{g}) \arrow[r, "H"] \arrow[d, "\pi"'] & \mathbb{C}[\mathfrak{t}^*]^W \arrow[d, "r"] \\ D(V)^G \arrow[r, "h"] & \mathbb{C}[A] \end{tikzcd}$$

Theorem 3.2.1 [Knop 1998, Theorem 4.8 and Corollary 4.9; Benson and Ratcliff 2004, Theorem 9.2.1]. *The homomorphism h is injective and there exists a finite group W_0 (sometimes called the little Weyl group) which is a subgroup of the stabilizer of A in W, such that the image of h is $\mathbb{C}[A]^{W_0}$. Hence h is an isomorphism between $D(V)^G$ and $\mathbb{C}[A]^{W_0}$. The isomorphism h is called the Harish-Chandra isomorphism for the MF space (G,V). Moreover W_0 acts as a reflection group on $\mathfrak{a}^*$.*

Let us see what is the automorphism of $\mathbb{C}[A]^{W_0}$ which corresponds to the action of τ on $D(V)^G$ through the Harish-Chandra isomorphism h. Let $D\in D(V)^G$. Then $h(\tau(D))(-\lambda+\rho)=h(XDX^{-1})(-\lambda+\rho)=b_{XDX^{-1}}(\lambda)=b_D(\lambda-\lambda_0)$. This calculation proves of course that $\mathbb{C}[A]^{W_0}$ is stable under $P(\lambda+\rho)\mapsto P((\lambda-\lambda_0)+\rho)$. Therefore we make the following definition.

Definition 3.2.2. By abuse of notation τ will also denote the automorphism of $\mathbb{C}[A]^{W_0}$ which is defined by $\tau(P)(\lambda+\rho)=P((\lambda-\lambda_0)+\rho)$ $(P\in\mathbb{C}[A]^{W_0})$. Let $\mathbb{C}[A]^{W_0,\tau}$ denote the set of elements in $\mathbb{C}[A]^{W_0}$ that are invariant under τ.

Proposition 3.2.3. *Let $\mathfrak{Z}(\mathcal{T})$ be the center of $\mathcal{T}=D(\mathbb{O})^{G'}$. Then $\mathfrak{Z}(\mathcal{T})$ is also the center of $\mathcal{T}_0[X,Y]=D(V)^{G'}$. Moreover the following assertions are equivalent*:

i) $D\in\mathfrak{Z}(\mathcal{T})$.

ii) $D\in\mathcal{T}_0$ *and* $\tau(D)=D$ *(i.e., D commutes with X).*

[†] The change of sign is due to the fact that we consider here characters of relative invariants instead of highest weights.

iii) *$D \in \mathcal{T}_0$ and the Bernstein–Sato polynomial $b_D(a_0, a_1, \dots, a_r)$ does not depend on a_0.*

iv) *$D \in \mathcal{T}_0$ and D commutes with Y.*

v) *$D \in \mathcal{T}_0$ and $h(D) \in \mathbb{C}[A]^{W_0,\tau}$.*

Proof. i) $\Rightarrow$ ii): Let $D \in \mathfrak{Z}(\mathcal{T})$. Then $[E, D] = 0$, hence $D \in \mathcal{T}_0$, and $[D, X] = 0$.

ii) $\Rightarrow$ iii): Let $D \in \mathcal{T}_0$. If $XD = DX$ then, from the definitions we have

$$\begin{aligned} b_{XD}(a_0, a_1, \dots, a_r) &= b_D(a_0, a_1, \dots, a_r) = b_{DX}(a_0, a_1, \dots, a_r) \\ &= b_D(a_0 + 1, a_1, \dots, a_r); \end{aligned}$$

hence $b_D(a_0, a_1, \dots, a_r)$ does not depend on a_0.

iii) $\Rightarrow$ i): Suppose that for $D \in \mathcal{T}_0$, the Bernstein–Sato polynomial does not depend on a_0. Then the elements XD and DX in $\mathcal{T}_1$ have the same Bernstein–Sato polynomial. Hence $XD = DX$ (Lemma 3.1.3). Then from Proposition 3.1.6(2) we see that $D \in \mathfrak{Z}(\mathcal{T})$.

iii) $\Rightarrow$ iv): Let $D \in \mathcal{T}_0$ such that b_D does not depend on a_0. Then

$$\begin{aligned} b_{DY}(a_0, a_1, \dots, a_r) &= b_D(a_0 - 1, a_1, \dots, a_r) b_Y(a_0, a_1, \dots, a_r) \\ &= b_D(a_0, a_1, \dots, a_r) b_Y(a_0, a_1, \dots, a_r) \\ &= b_{YD}(a_0, a_1, \dots, a_r). \end{aligned}$$

Hence $DY = YD$.

iv) $\Rightarrow$ iii): If $DY = YD$, then

$$\begin{aligned} b_{DY}(a_0, a_1, \dots, a_r) &= b_D(a_0 - 1, a_1, \dots, a_r) b_Y(a_0, a_1, \dots, a_r) \\ &= b_{YD}(a_0, a_1, \dots, a_r) \\ &= b_Y(a_0, a_1, \dots, a_r) b_D(a_0, a_1, \dots, a_r). \end{aligned}$$

Hence b_D does not depend on a_0.

The equivalence of iii) and v) is obvious since $h(D)(-\lambda + \rho) = b_D(\lambda)$.

From ii) we obtain that $\mathfrak{Z}(\mathcal{T})$ is also the center of $\mathcal{T}_0[X, Y]$. □

Remark 3.2.4. As a consequence of the preceding proposition it is worthwhile noticing that if $D \in D(V)^{G'}$ (or $D \in \mathcal{T}$) commutes with two operators among (X, E, Y), then D commutes with the third one. This is a well known property if (X, E, Y) is an $\mathfrak{sl}_2$-triple. But we know from [Igusa 1981] that except if Δ_0 is quadratic or linear the Lie algebra generated by (X, E, Y) is infinite-dimensional.

We will see in Theorem 4.2.2 that the associative algebra generated by (X, E, Y) over $\mathfrak{X}(\mathcal{T})$ is "similar" to $\mathcal{U}(\mathfrak{sl}_2(\mathfrak{X}(\mathcal{T}))$.

Define a linear form μ on $\mathfrak{a}^*$ by

$$\mu(a_0\lambda_0+\cdots+a_r\lambda_r)=\sum_{i=0}^{r}a_id_i=b_E(\boldsymbol{a})\quad(\boldsymbol{a}=(a_0,\ldots,a_r)\in\mathbb{C}^{r+1})$$

(μ is the *degree form*, as its value on $\boldsymbol{a}=(a_0,\ldots,a_r)\in\mathbb{N}^{r+1}$ is equal to the degree of the polynomials in $V_{\boldsymbol{a}}$). Define also

$$\mathcal{M}=\{\lambda\in\mathfrak{a}^*\mid\mu(\lambda)=0\}\quad\text{and}\quad M=\mathcal{M}+\rho\subset A.$$

Note that $M=\{\lambda+\rho\in A\mid h(E)(-\lambda+\rho)=0\}$. As $h(E)$ is W_0-invariant, so is the set M. Set

$$I(M)=\left\{P\in\mathbb{C}[A]^{W_0}\mid P_{|M}=0\right\}.$$

The key lemma is the following.

Lemma 3.2.5. *We have $I(M)=\mathbb{C}[A]^{W_0}h(E)$ and*

$$\mathbb{C}[A]^{W_0}=\mathbb{C}[A]^{W_0,\tau}\oplus I(M).$$

Proof. Let $P\in I(M)$. Then P is a polynomial on the affine subspace $A\subset\mathfrak{t}^*$ vanishing on M, the set of zeros of the irreducible polynomial $h(E)$. Therefore $P=h(E)Q$. As P and $h(E)$ are W_0-invariant, so is also the polynomial Q. Hence $I(M)\subset\mathbb{C}[A]^{W_0}h(E)$. The reverse inclusion is obvious.

Let $F=\mathbb{C}\lambda_0\subset\mathfrak{a}^*$. As obviously $\mathfrak{a}^*=\mathcal{M}\oplus F$, we have $A=M\oplus F$. Remember that $\mathfrak{t}=\mathfrak{c}\oplus\mathfrak{t}'$, where $\mathfrak{c}$ is the center of $\mathfrak{g}$. The infinitesimal character λ_0 is a character of $\mathfrak{g}$, and is therefore trivial on $\mathfrak{t}'\subset\mathfrak{g}'$. As any $w_0\in W_0$ fixes pointwise the center $\mathfrak{c}$ of $\mathfrak{g}$, we see that F is pointwise fixed by W_0.

Let $Q\in\mathbb{C}[M]^{W_0}$. Define

$$\widetilde{Q}(m+f)=Q(m),\quad\text{for all } m\in M,\ f\in F.$$

From the preceding discussion we obtain that $\widetilde{Q}$ is W_0-invariant; in other words $\widetilde{Q}\in\mathbb{C}[A]^{W_0}$. But in fact $\widetilde{Q}$ is also τ-invariant:

$$\tau(\widetilde{Q})(m+f)=\widetilde{Q}(m+f-\lambda_0)=Q(m)=\widetilde{Q}(m+f).$$

Hence $\widetilde{Q}\in\mathbb{C}[A]^{W_0,\tau}$; in other words any W_0-invariant polynomial on M can be extended to a (W_0,τ)-invariant polynomial on A. This extension is in fact unique: for any τ-invariant extension $\widetilde{\widetilde{Q}}$ of Q we have $\widetilde{\widetilde{Q}}(m+x\lambda_0)=\widetilde{\widetilde{Q}}(m+(x+1)\lambda_0)$

and hence $\tilde{\tilde{Q}} = \tilde{Q}$. Hence we have proved that the restriction map

$$\begin{aligned} \mathbb{C}[A]^{W_0,\tau} &\to \mathbb{C}[M]^{W_0}, \\ P &\mapsto P_{|M}, \end{aligned}$$

is bijective (and therefore $\mathbb{C}[A]^{W_0,\tau} \cap I(M) = \{0\}$) and the inverse map is $Q \mapsto \tilde{Q}$. Now for $P \in \mathbb{C}[A]^{W_0}$ we can write

$$P = \widetilde{P_{|M}} + \big(P - \widetilde{P_{|M}}\big).$$

From the discussion above we have $\widetilde{P_{|M}} \in \mathbb{C}[A]^{W_0,\tau}$, and $\big(P - \widetilde{P_{|M}}\big) \in I(M)$. □

Theorem 3.2.6. 1) $\mathscr{T}_0 = D(V)^G = \mathfrak{X}(\mathscr{T}) \oplus E\mathscr{T}_0$.

2) *Any element $H \in D(V)^G$ can be uniquely written in the form*

$$H = H_0 + EH_1 + E^2H_2 + \cdots + E^kH_k,$$

where $H_i \in \mathfrak{X}(\mathscr{T})$, $i = 1, 2, \ldots, k \in \mathbb{N}$.

Proof. Through the Harish-Chandra isomorphism h, the algebra $D(V)^G = \mathscr{T}_0$ corresponds to $\mathbb{C}[A]^{W_0}$, the algebra $\mathfrak{X}(\mathscr{T})$ corresponds to $\mathbb{C}[A]^{W_0,\tau}$ and the ideal $E\mathscr{T}_0$ corresponds to $I(M)$. Therefore the first assertion is just the pullback by h of the decomposition obtained in Lemma 3.2.5.

An element $H \in D(V)^G$ can therefore be uniquely written $H = H_0 + EH^1$, with $H_0 \in \mathfrak{X}(\mathscr{T})$, and $H^1 \in \mathscr{T}_0$. By induction we obtain a decomposition

$$H = H_0 + EH_1 + E^2H_2 + \cdots + E^{k-1}H_{k-1} + E^kH^k,$$

where $H_0, \ldots, H_{k-1} \in \mathfrak{X}(\mathscr{T})$, and $H^k \in \mathscr{T}_0$. The process stops because if k is greater than the degree in a_0 of b_H, then necessarily $H^k = 0$ (see Proposition 3.2.3). □

From this theorem and Proposition 3.1.6 we obtain immediately this consequence:

Corollary 3.2.7. 1) *Let $D \in \mathscr{T}$. Then D can be written uniquely in the form*

$$D = \sum_{\substack{k\in\mathbb{Z}\\ \ell\in\mathbb{N}}} H_{k,\ell}E^\ell X^k \quad \textit{or} \quad D = \sum_{\substack{k\in\mathbb{Z}\\ \ell\in\mathbb{N}}} H_{k,\ell}X^k E^\ell \quad (\textit{finite sums}),$$

where $H_{k,\ell} \in \mathfrak{X}(\mathscr{T})$.

2) *Let $D \in \mathscr{T}_0[X, Y]$. Then D can be written uniquely in the form*

$$D = \sum_{\substack{k\in\mathbb{N}^*\\ \ell\in\mathbb{N}}} H_{k,\ell}E^\ell Y^k + \sum_{\substack{r\in\mathbb{N}\\ s\in\mathbb{N}}} H'_{r,s}E^s X^r \quad (\textit{finite sums}) \quad \textit{or}$$

$$D = \sum_{\substack{k\in\mathbb{N}^*\\ \ell\in\mathbb{N}}} H_{k,\ell}Y^k E^\ell + \sum_{\substack{r\in\mathbb{N}\\ s\in\mathbb{N}}} H'_{r,s}X^r E^s \quad (\textit{finite sums}),$$

where $H_{k,\ell}$, $H'_{r,s} \in \mathfrak{X}(\mathscr{T})$.

Corollary 3.2.8. *Let $P \in \mathbb{C}[A]^{W_0}$. Then P can be uniquely written in the form*

$$P(-\lambda+\rho) = \sum_{i=0}^{p} \alpha_i(-\lambda+\rho)(a_0d_0 + a_1d_1 + \cdots + a_rd_r)^i,$$

where $\alpha_i \in \mathbb{C}[A]^{W_0,\tau}$ and $\lambda = a_0\lambda_0 + a_1\lambda_1 + \cdots + a_r\lambda_r \in \mathfrak{a}^*$.

Proof. As $h(E)(-\lambda+\rho) = a_0d_0 + a_1d_1 + \cdots + a_rd_r$, the preceding decomposition is just the image through the Harish-Chandra isomorphism of the decomposition in Theorem 3.2.6(2). □

Remark 3.2.9. It is easy to see that as W_0 stabilizes the affine space $A = \mathfrak{a}^* + \rho$ it also stabilizes $\mathfrak{a}^*$ (this is implicit in Theorem 3.2.1). Moreover if we denote by 0_ρ the barycenter of the W_0-orbit of ρ, then 0_ρ is a fixed point of the W_0-action on A which is in M. As $\mathbb{C}[A]^{W_0} = \mathbb{C}[\mathfrak{a}^* + \rho]^{W_0} = \mathbb{C}[\mathfrak{a}^* + 0_\rho]^{W_0} \simeq \mathbb{C}[\mathfrak{a}^*]^{W_0}$, and as $\mathcal{T}_0 = D(V)^G \simeq \mathbb{C}[A]^{W_0}$ is a polynomial algebra in $r+1$ variables by Theorem 2.2.9, the group W_0 acts as a reflection group on $\mathfrak{a}^*$ by the Shephard–Todd–Chevalley theorem (this is a part of Knop's argument for Theorem 3.2.1). Hence by the theorem of Chevalley, the $r+1$ algebraically independent generators of the algebra $\mathbb{C}[A]^{W_0} \simeq \mathbb{C}[\mathfrak{a}^*]^{W_0}$ can be chosen to be homogeneous, either as functions on the vector space $\mathfrak{a}^*$, or as functions on A, for the vector space structure on A defined by taking 0_ρ as origin.

We will now describe more precisely the algebra $\mathfrak{Z}(\mathcal{T})$.

Theorem 3.2.10. 1) *$\mathfrak{Z}(\mathcal{T})$ is a polynomial algebra in r variables. For $D \in \mathcal{T}_0$, let us denote by $\overline{D}$ the projection of D on $\mathfrak{Z}(\mathcal{T})$ according to the decomposition $\mathcal{T}_0 = \mathfrak{Z}(\mathcal{T}) \oplus E\mathcal{T}_0$. Remember from Theorem 2.2.9 that the set $R_0, \ldots, R_{r-1}, R_r$ of Capelli operators associated to the invariants $\Delta_0, \Delta_1, \ldots, \Delta_r$ ordered by decreasing degree is a set algebraically independent generators of $\mathcal{T}_0$. Then $\{\overline{R_0}, \ldots, \overline{R_{r-1}}\}$ is a set of algebraically independent generators of $\mathfrak{Z}(\mathcal{T})$.*

2) *Let D be an element of $\mathcal{T}_0$ and let b_D be its Bernstein–Sato polynomial. Then the Bernstein–Sato polynomial of $\overline{D}$ is given by*

$$b_{\overline{D}}(a_0, a_1, \ldots, a_r) = b_D\biggl(-\frac{a_1d_1 + \cdots + a_rd_r}{d_0}, a_1, \ldots, a_r\biggr).$$

Proof. 1) Let us remark first that $\mathfrak{Z}(\mathcal{T})$ is already known to be a polynomial algebra from a result of Knop [1994]. He has proved that for a regular action of a reductive group on a smooth affine variety the center of the ring of invariant differential operators is always a polynomial algebra. We give here a direct proof and obtain some extra information. We know from Proposition 3.2.3 that $\mathfrak{Z}(\mathcal{T})$ is isomorphic, through the Harish-Chandra isomorphism h, to $\mathbb{C}[A]^{W_0,\tau}$. From the proof of Lemma 3.2.5 we know that W_0 stabilizes M and that $\mathbb{C}[A]^{W_0,\tau} \simeq \mathbb{C}[M]^{W_0} = (\mathbb{C}[A]^{W_0})_{|M}$. As

W_0 is a reflection group on A (this means that it is generated by the reflections it contains), so is $W_{0|M}$. Therefore $\mathbb{C}[M]^{W_0}$ (and hence $\mathscr{Z}(\mathscr{T})$) is a polynomial algebra in $r = \dim M$ variables by Chevalley's Theorem. We know from Remark 2.2.10(d) that $\{R_0, \ldots, R_{r-1}, E\}$ is also a set algebraically independent generators of $\mathscr{T}_0$; hence $\{h(R_0), \ldots, h(R_{r-1}), h(E)\}$ is a set of algebraically independent generators of $\mathbb{C}[A]^{W_0}$. We obtain that $\mathbb{C}[M]^{W_0} = \mathbb{C}[h(R_0)_{|M}, \ldots, h(R_{r-1})_{|M}]$ as $h(E)_{|M} = 0$. As the transcendence degree of $\mathrm{Frac}(\mathbb{C}[M]^{W_0})$ over $\mathbb{C}$ is r, the generators $h(R_0)_{|M}, \ldots, h(R_{r-1})_{|M}$ are algebraically independent. Taking their inverse image under h gives the first assertion of the theorem.

2) As we have seen the decomposition $\mathscr{T}_0 = \mathscr{Z}(\mathscr{T}) \oplus E\mathscr{T}_0$ is nothing else but the inverse image under h of the decomposition $\mathbb{C}[A]^{W_0} = \mathbb{C}[A]^{W_0,\tau} \oplus I(M)$. Let $D \in \mathscr{T}_0$. From the proof of Lemma 3.2.5 we have $h(\overline{D}) = \widetilde{h(D)_{|M}}$, where $\widetilde{h(D)_{|M}}$ is the unique (W_0, τ)-invariant extension to A of $h(D)_{|M}$. For $\lambda = a_0\lambda_0 + \cdots + a_r\lambda_r \in \mathfrak{a}^*$, we have $h(E)(\lambda+\rho) = b_E(-\lambda) = -(a_0d_0 + \cdots + a_rd_r) = -\mu(\lambda)$ (the degree form). Remember also that $\mathfrak{a}^* = \mathscr{M} \oplus F$, where $\mathscr{M} = \ker(\mu)$ and $F = \mathbb{C}\lambda_0$. Let us write $\lambda = m_\lambda + \alpha\lambda_0$, according to this decomposition. Then $b_E(\lambda) = \alpha b_E(\lambda_0) = \alpha d_0$. Hence $\alpha = \mu(\lambda)/d_0$ and $m_\lambda = \lambda - (\mu(\lambda)/d_0)\lambda_0$. Then we obtain

$$\begin{aligned}
b_{\overline{D}}(\lambda) &= h(\overline{D})(-\lambda+\rho) = \widetilde{h(D)_{|M}}(-\lambda+\rho)\\
&= \widetilde{h(D)_{|M}}\left(-\lambda + \frac{\mu(\lambda)}{d_0}\lambda_0 - \frac{\mu(\lambda)}{d_0}\lambda_0 + \rho\right) = \widetilde{h(D)_{|M}}\left(-\lambda + \frac{\mu(\lambda)}{d_0}\lambda_0 + \rho\right)\\
&= h(D)_{|M}\left(-\lambda + \frac{\mu(\lambda)}{d_0}\lambda_0 + \rho\right) = h(D)\left(-\lambda + \frac{\mu(\lambda)}{d_0}\lambda_0 + \rho\right)\\
&= b_D\left(\lambda - \frac{\mu(\lambda)}{d_0}\lambda_0\right).
\end{aligned}$$

If we translate this into the $(a_0, \ldots, a_r)$-variables we obtain the second assertion. □

Corollary 3.2.11. *Let b_Y be the Bernstein–Sato operator of Y. For any $\ell \in \mathbb{N}$ the element of* $\mathrm{End}(\mathbb{C}[V])$ *that acts on each space $V_{\boldsymbol{a}}$ as scalar multiplication by $b_Y(-(a_1d_1 + \cdots + a_rd_r)/d_0 + \ell, a_1, \ldots, a_r)$ is the differential operator*

$$\overline{X^{1-\ell}YX^\ell} \in \mathscr{Z}(\mathscr{T}).$$

Moreover, if (G, V^+) is a PV of commutative parabolic type, the differential operators $\overline{X^{1-\ell}YX^\ell}$ $(\ell = 0, 1, \ldots, r)$ are generators of $\mathscr{Z}(\mathscr{T})$.

Proof. As $b_{X^{1-\ell}YX^\ell}(a_0, \ldots, a_r) = b_Y(a_0 + \ell, a_1, \ldots, a_r)$, the first assertion follows immediately from Theorem 3.2.10. If (G, V^+) is a PV of commutative parabolic type, we know from Theorem 3.3.1 below that the operators $X^{1-\ell}YX^\ell$ $(\ell = 0, \ldots, r)$ are (algebraically independent) generators of $\mathscr{T}_0$. □

3.3. ***The case of regular PV's of commutative parabolic type.*** In the case where (G, V^+) is a regular PV of commutative parabolic type (see Example 2.3.3), we obtain some specific results.

Theorem 3.3.1. *Let (G, V^+) be a regular PV of commutative parabolic type.*

1) *The degree of Δ_0 is equal to $r+1$ which is the rank of (G, V^+) as a MF space. More generally the degree of Δ_i is equal to $r+1-i$.*
2) *For $\ell \in \mathbb{Z}$ set $D_\ell = X^{1-\ell} Y X^\ell$. Then $D_0, D_1, \dots, D_r$ are algebraically independent generators of $\mathcal{T}_0 = D(V^+)^G$ (i.e., $\mathcal{T}_0 = \mathbb{C}[D_0, D_1, \dots, D_r]$).*
3) *We have $\mathcal{T} = D(\Omega^+)^{G'} = \mathbb{C}[X, X^{-1}, Y]$, where $\mathbb{C}[X, X^{-1}, Y]$ is the associative subalgebra of $D(\Omega^+)$ generated by X, X^{-1}, Y.*
4) *We have $\mathcal{T}_0[X, Y] = D(V)^{G'} = \mathbb{C}[X, Y, R_1, \dots, R_r]$, where the R_i are the Capelli operators introduced before Theorem 2.2.9 and $\mathbb{C}[X, Y, R_1, \dots, R_r]$ is the associative subalgebra of $D(V^+)$ generated by $X, Y, R_1, \dots, R_r$.*

Proof. 1) This first assertion is proved in [Muller et al. 1986, Proposition 2.16 and Lemme 3.7].

2) We need now to use some technical results from the structure theory of commutative PVs of parabolic type. For details see [Muller et al. 1986; Rubenthaler and Schiffmann 1987]. We need also results concerning the symmetric space structure of the open G orbit Ω^+ in V^+; they can be found in [Bopp and Rubenthaler 1993]. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$; then $\mathfrak{t}$ is also a Cartan subalgebra of $\tilde{\mathfrak{g}}$ (see the notation in Example 2.3.3). Let $\widetilde{\Sigma}$ and Σ be the root systems of $(\tilde{\mathfrak{g}}, \mathfrak{t})$ and $(\mathfrak{g}, \mathfrak{t})$, respectively. We choose an order on $\widetilde{\Sigma}$ such that the roots occurring in V^+ are positive. We know from Proposition 2.9. in [Bopp and Rubenthaler 1993] that the open G-orbit $\Omega^+ = \{x \in V^+ \mid \Delta_0(x) \neq 0\}$ is a symmetric space G/H, where H is the isotropy subgroup of a point $I^+ \in \Omega^+$. The choice of I^+ can be made the following way. It is known that any maximal set of strongly orthogonal long roots occurring in V^+ has $r+1 = \operatorname{rank}(G, V^+)$ elements. There is a canonical way to construct such a maximal set, called the "descent"; see [Muller et al. 1986, Theorem 2.7, p. 101]. If $\{\alpha_0, \alpha_1, \dots, \alpha_r\}$ is such a maximal set of strongly orthogonal long roots, then the element $I^+ = X_{\alpha_0} + X_{\alpha_1} + \cdots + X_{\alpha_r}$ is generic (here as usual the X_{α_i} are nonzero root vectors). Let $\mathfrak{h} = Z_{\mathfrak{g}}(I^+)$ be the Lie algebra of H, and let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form of $\tilde{\mathfrak{g}}$. Let $H_{\alpha_i} \in \mathfrak{t}$ be the coroot of α_i. Set $\mathfrak{a} = \sum_{i=0}^r \mathbb{C} H_{\alpha_i}$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ [Bopp and Rubenthaler 1993, Proposition 5.4] and the dual space $\mathfrak{a}^*$ can be identified with the space of restrictions of the fundamental characters $\lambda_0, \lambda_1, \dots, \lambda_r$ [ibid., Lemme 2.5]. Hence this definition of $\mathfrak{a}^*$ is coherent with the direct definition of $\mathfrak{a}^*$ given in Section 3.2 in the general case $\big(\mathfrak{a}^* = \sum_{i=0}^r \mathbb{C}\lambda_i\big)$.

For $\lambda \in \mathfrak{t}^*$, we will denote by $\overline{\lambda}$ the restriction of λ to $\mathfrak{a}$. Through the "classical" Harish-Chandra isomorphism γ for symmetric spaces [Heckman and Schlichtkrull 1994, Part II, Theorem 4.3] the algebra $\mathcal{T}_0$ is isomorphic to $S(\mathfrak{a})^{W_R} = \mathbb{C}[\mathfrak{a}^*]^{W_R}$, where W_R is the Weyl group of the root system R of $(\mathfrak{g}, \mathfrak{a})$. This root system is known to be of type A_r (the proof is the same as for Theorem 3.11 in [Bopp and Rubenthaler 2005]). Hence W_R is the symmetric group of $r+1$ variables and it acts by permutations on the $\overline{\alpha}_i$. We will choose an order on R such that $\overline{\Sigma^+} \subset R^+$. As in [Muller et al. 1986; Rubenthaler and Schiffmann 1987] we consider here relative invariants $\Delta_0, \Delta_1, \dots, \Delta_r$ with respect to the Borel subgroup defined by Σ^-. Define $\rho = \frac{1}{2}\sum_{\beta \in R^-} \beta$. It is well known that for $D \in \mathcal{T}_0$ and $\lambda = \sum_{i=0}^r a_i \lambda_i \in \mathfrak{a}^*$, $\gamma(D)(-\overline{\lambda} + \rho)$ is equal to the eigenvalue of D acting on $\Delta_0^{a_0} \cdots \Delta_r^{a_r}$. In other words we have

$$\gamma(D)(-\overline{\lambda} + \rho) = b_D(\lambda).$$

From [Rubenthaler and Schiffmann 1990, Lemme 3.9, p. 155], we know that

$$\rho = \frac{d}{4}\sum_{i<j}(\overline{\alpha}_i - \overline{\alpha}_j) = \frac{d}{4}\sum_{i=0}^{r}(r-2i)\overline{\alpha}_i$$

and from [ibid., Lemme 3.8, p. 155], we also have

$$\overline{\lambda} = a_0\overline{\alpha}_0 + (a_0 + a_1)\overline{\alpha}_1 + \cdots + (a_0 + \cdots + a_r)\overline{\alpha}_r.^{\ddagger}$$

Let us now make the following change of variables:

$$s_i = a_0 + \cdots + a_i, \text{ for } i = 0, \dots, r.$$

As $b_{D_\ell}(\lambda) = b_Y(s_0 + \ell, \dots, s_r + \ell) = c\prod_{i=0}^r (s_i + \ell + id/2)$ (see Example 3.1.2) we obtain

$$\gamma(D_\ell)(\overline{\lambda}) = b_{D_\ell}(-\lambda + \rho) = b_{D_\ell}\Bigl(\sum_{i=0}^{r} -s_i\overline{\alpha}_i + \frac{d}{4}\sum_{i=0}^{r}(r-2i)\overline{\alpha}_i\Bigr)$$
$$= c\prod_{i=0}^{r}\Bigl(-s_i + \frac{d}{4}r + \ell\Bigr).$$

As expected the polynomials $\gamma(D_\ell)$ are symmetric in the s_i variables (i.e., invariant under W_R). Moreover it is easy to prove that these polynomials, for $\ell = 0, \dots, r$, are algebraically independent generators of the algebra of symmetric polynomials. This proves 2).

‡The change of sign with respect to Lemme 3.8 in [Rubenthaler and Schiffmann 1990] is again due to the fact that we consider here characters of relative invariants instead of the highest weights.

3) As $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$ (see Proposition 3.1.6), and as, from 2), the elements of $\mathcal{T}_0$ are polynomials in X, X^{-1}, Y we obtain that $\mathcal{T} \subset \mathbb{C}[X, X^{-1}, Y]$. The inverse inclusion is obvious.

4) The inclusion $\mathbb{C}[X, Y, R_1, \dots, R_r] \subset D(V^+)^{G'} = \mathcal{T}_0[X, Y]$ is obvious. Conversely, from Theorem 2.2.9 we have $\mathcal{T}_0[X, Y] = \mathbb{C}[R_0, R_1, \dots, R_r][X, Y]$. As $R_0 = XY$ (see Remark 2.2.10), we have $\mathcal{T}_0[X, Y] \subset \mathbb{C}[X, Y, R_1, \dots, R_r]$. $\square$

Remark 3.3.2. According to [Terras 1988, p. 208], the operators D_ℓ were first considered by Selberg on positive definite symmetric matrices. They appear also in [Maaß 1971], in the same context of positive definite symmetric matrices. In the setting of symmetric cones, the analogue of assertion 2) of the preceding theorem can be found in [Faraut and Korányi 1994, Corollary XIV.1.6].

Remark 3.3.3. Note that for PVs of commutative parabolic type we have $R_r = E$. In the special case where $G \simeq \mathrm{SO}(k) \times \mathbb{C}^*$ and $V^+ \simeq \mathbb{C}^k$, we have always $r = 1$, and assertion 4) of the preceding theorem yields

$$D(\mathbb{C}^k)^{\mathrm{SO}(k)} = \mathbb{C}[Q(x), Q(\partial), E],$$

where $Q(x) = X = \sum_{i=1}^k x_i^2$, $Q(\partial) = Y = \sum_{i=1}^k \partial^2/\partial x_i^2$.

This was proved by S. Rallis and G. Schiffmann [1980, Lemma 5.2, p. 112].

4. The structure of $D(V)^{G'}$

4.1. *Smith algebras over rings.* As usual if a, b are elements of an associative algebra we define $[a, b] = ab - ba$.

Definition 4.1.1. Let $\boldsymbol{A}$ be a commutative associative algebra over $\mathbb{C}$, with unit element 1 and without zero divisors. Let $f, u \in \boldsymbol{A}[t]$ be two polynomials in one variable with coefficients in $\boldsymbol{A}$. Let $n \in \mathbb{N}^*$.

1) The Smith algebra $S(\boldsymbol{A}, f, n)$ is the associative algebra over $\boldsymbol{A}$ with generators (x, y, e) subject to the relations $[e, x] = nx$, $[e, y] = -ny$, $[y, x] = f(e)$.

2) The algebra $U(\boldsymbol{A}, u, n)$ is the associative algebra over $\boldsymbol{A}$ with generators $(\tilde{x}, \tilde{y}, \tilde{e})$ subject to the relations $[\tilde{e}, \tilde{x}] = n\tilde{x}$, $[\tilde{e}, \tilde{y}] = -n\tilde{y}$, $\tilde{x}\tilde{y} = u(\tilde{e})$, $\tilde{y}\tilde{x} = u(\tilde{e} + n)$.

Remark 4.1.2. 1) The algebras $S(\mathbb{C}, f, n)$ were introduced and intensively studied by Smith [1990], who called them "algebras similar to $\mathcal{U}(\mathfrak{sl}_2)$", where $\mathcal{U}(\mathfrak{sl}_2)$ is the enveloping algebra of $\mathfrak{sl}_2$. In fact they share many interesting properties with $\mathcal{U}(\mathfrak{sl}_2)$, in particular they have a very rich representation theory.

2) One can prove, as in [Smith 1990], that if the degree of f is one and $n \neq 0$, and if the leading coefficient is invertible in $\boldsymbol{A}$, then $S(\boldsymbol{A}, f, n)$ is isomorphic to the enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\boldsymbol{A}))$.

Let $\mathcal{R}$ be a ring and let $\sigma \in \mathrm{Aut}(\mathcal{R})$. Let us recall that a σ-derivation of $\mathcal{R}$ is an additive map $\delta : \mathcal{R} \to \mathcal{R}$ such that $\delta(su) = s\delta(u) + \delta(s)\sigma(u)$. Given a σ-derivation δ, the skew polynomial ring over $\mathcal{R}$ determined by σ and δ is the ring $\mathcal{R}[t, \sigma, \delta] := \langle \mathcal{R}, t\rangle / \{st - t\sigma(s) - \delta(s) \mid s \in \mathcal{R}\}$, where $\langle \mathcal{R}, t\rangle$ stands for the ring freely generated by $\mathcal{R}$ and an element t with the relations given by the ring structure on $\mathcal{R}$ (for details see [McConnell and Robson 1987, Section 1.2, p. 15; Goodearl and Warfield 2004, p. 34]).

Proposition 4.1.3. *Let $\mathfrak{b}$ the 2-dimensional Lie algebra over $\boldsymbol{A}$, with basis $\{\varepsilon, \alpha\}$ and relation $[\varepsilon, \alpha] = n\alpha$. Let $\mathcal{U}(\mathfrak{b})$ be the enveloping algebra of $\mathfrak{b}$. Define an automorphism σ of $\mathcal{U}(\mathfrak{b})$ by $\sigma(\alpha) = \alpha$ and $\sigma(\varepsilon) = \varepsilon - n$ and define also a σ-derivation δ of $\mathcal{U}(\mathfrak{b})$ by $\delta(\alpha) = f(\varepsilon)$ and $\delta(\varepsilon) = 0$. Then $S(\boldsymbol{A}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$.*

Proof. The proof is almost the same as the one given by Smith [1990, Proposition 1.2]. The isomorphism $S(\boldsymbol{A}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$ is given by $e \mapsto \varepsilon$, $x \mapsto \alpha$ and $y \mapsto t$. □

Corollary 4.1.4. *$S(\boldsymbol{A}, f, n)$ is a noetherian domain with $\boldsymbol{A}$-basis*

$$\{y^i x^j e^k \mid i, j, k \in \mathbb{N}\}$$

(*or any similar family of ordered monomials obtained by permutation of the elements (y, x, e)*).

Proof. (compare with [Smith 1990, proof of Corollary 1.3, p. 288]). We know from [McConnell and Robson 1987, Theorem 1.2.9], that as $\mathcal{U}(\mathfrak{b})$ is a noetherian domain, so is $S(\boldsymbol{A}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$. Since

$$\begin{aligned} \mathcal{U}(\mathfrak{b})[t, \sigma, \delta] &= \mathcal{U}(\mathfrak{b}) \oplus \mathcal{U}(\mathfrak{b})t \oplus \mathcal{U}(\mathfrak{b})t^2 \oplus \mathcal{U}(\mathfrak{b})t^3 \oplus \cdots \oplus \mathcal{U}(\mathfrak{b})t^\ell \oplus \cdots \\ &= \mathcal{U}(\mathfrak{b}) \oplus t\mathcal{U}(\mathfrak{b}) \oplus t^2\mathcal{U}(\mathfrak{b}) \oplus t^3\mathcal{U}(\mathfrak{b}) \oplus \cdots \oplus t^\ell\mathcal{U}(\mathfrak{b}) \oplus \cdots \end{aligned}$$

(direct sums of $\boldsymbol{A}$-modules) and since the Poincaré–Birkhoff–Witt theorem is still true for enveloping algebras of Lie algebras which are free over rings (see [Bourbaki 1971]), the ordered monomials in (y, x, e) beginning or ending with y form a basis of the algebra $S(\boldsymbol{A}, f, n)$. To obtain the basis $\{e^i y^j x^k\}$ or $\{x^k y^j e^i\}$ it suffices to replace the algebra $\mathfrak{b}$ by the algebra $\mathfrak{b}_-$ which is generated by e and y. □

Remark 4.1.5. The adjoint action of e ($u \mapsto [e, u]$) on $S(\boldsymbol{A}, f, n)$ is semisimple and gives a decomposition of $S(\boldsymbol{A}, f, n)$ into weight spaces:

$$S(\boldsymbol{A}, f, n) = \bigoplus_{\nu \in \mathbb{Z}} S(\boldsymbol{A}, f, n)^\nu,$$

where $S(\boldsymbol{A}, f, n)^\nu = \{u \in S(\boldsymbol{A}, f, n) \mid [e, u] = \nu n u\}$. As $[e, x^j y^i e^k] = n(j-i) y^i x^j e^k$, we obtain, using Corollary 4.1.4, that the ordered monomials of the form $x^i y^i e^k$ form an $\boldsymbol{A}$-basis for $S(\boldsymbol{A}, f, n)^0$. Moreover as $yx = xy + f(e)$, it is easy to see that $S(\boldsymbol{A}, f, n)^0 = \boldsymbol{A}[xy, e] = \boldsymbol{A}[yx, e]$, where $\boldsymbol{A}[xy, e]$ (resp. $\boldsymbol{A}[yx, e]$) denotes the $\boldsymbol{A}$-subalgebra generated by xy (resp. yx) and e.

The proof of the following lemma is straightforward.

Lemma 4.1.6. *Let $n \in \mathbb{N}^*$ and let $f \in \boldsymbol{A}[t]$. There exists an element $u \in \boldsymbol{A}[t]$, which is unique up to addition of an element of $\boldsymbol{A}$, such that*

$$f(t) = u(t+n) - u(t) \tag{4-1-1}$$

Proposition 4.1.7 (compare with [Smith 1990, Proposition 1.5]). *Let u be as in the preceding lemma. Define*

$$\Omega_1 = xy - u(e).$$

Then the center of $S(\boldsymbol{A}, f, n)$ is $\boldsymbol{A}[\Omega_1]$ which is isomorphic to the polynomial algebra $\boldsymbol{A}[t]$.

Proof. Let us now prove that Ω_1 is central. Obviously Ω_1 commutes with e.

From the defining relations of $S(\boldsymbol{A}, f, n)$ we have $ex = x(e+n)$ and therefore, for any $k \in \mathbb{N}$, $e^k x = x(e+n)^k$.

This implies of course that for any polynomial $P \in \boldsymbol{A}[t]$ we have

$$P(e)x = xP(e+n) \quad \text{or} \quad P(e-n)x = xP(e). \tag{4-1-2}$$

Similarly one proves that

$$P(e)y = yP(e-n) \quad \text{or} \quad P(e+n)y = yP(e). \tag{4-1-3}$$

Let us show that Ω_1 commutes with x. Using Lemma 4.1.6 and (4-1-2) we obtain

$$\begin{aligned} x\Omega_1 &= x(xy - u(e)) = x^2y - xu(e) = x(yx - f(e)) - xu(e) \\ &= x(yx - u(e+n) + u(e)) - xu(e) = xyx - xu(e+n) = xyx - u(e)x \\ &= \Omega_1 x. \end{aligned}$$

A similar calculation using (4-1-3) shows that Ω_1 commutes also with y. Hence Ω_1 belongs to the center of $S(\boldsymbol{A}, f, n)$.

Let now z be a central element of $S(\boldsymbol{A}, f, n)$. Then $z \in S(\boldsymbol{A}, f, n)^0$. We have $S(\boldsymbol{A}, f, n)^0 = \boldsymbol{A}[xy, e] = \boldsymbol{A}[\Omega_1, e]$, and hence z can be written as follows:

$$z = \sum c_i(e)\Omega_1^i \quad \text{(finite sum)},$$

where $c_i(e) \in \boldsymbol{A}[e]$.

We have

$$\begin{aligned} 0 = [z, x] &= \Big[\sum c_i(e)\Omega_1^i, x\Big] = \sum [c_i(e), x]\,\Omega_1^i \\ &= \sum (c_i(e)x - xc_i(e))\Omega_1^i \\ &= \sum x(c_i(e+n) - c_i(e))\Omega_1^i \quad \text{(using (4-1-2))} \\ &= x\sum (c_i(e+n) - c_i(e))\Omega_1^i. \end{aligned}$$

As the algebra $S(\boldsymbol{A}, f, n)$ has no zero divisors we get

$$\sum (c_i(e+n) - c_i(e))\Omega_1^i = 0.$$

As $\Omega_1 = xy - u(e)$, we have $\Omega_1^i = x^i y^i$ modulo monomials of the form $e^k x^p y^p$ with $p < i$. Then from Corollary 4.1.4 above we obtain $c_i(e+n) - c_i(e) = 0$ for all i. As the elements e^k are free over $\boldsymbol{A}$ (Corollary 4.1.4) we obtain from Lemma 4.1.6 that $c_i \in \boldsymbol{A}$, for all i. □

Remark 4.1.8. Conversely, let us start with $u \in \boldsymbol{A}[t]$. Define $f \in \boldsymbol{A}[t]$ by $f(t) = u(t+n) - u(t)$. From the definitions we have

$$U(\boldsymbol{A}, u, n) = S(\boldsymbol{A}, f, n)/(xy - u(e)) = S(\boldsymbol{A}, f, n)/(\Omega_1),$$

where $(xy - u(e)) = (\Omega_1)$ is the ideal generated by $xy - u(e) = \Omega_1$. Again, as for $S(\boldsymbol{A}, f, n)$, the adjoint action of $\tilde{e}$ gives a decomposition of $U(\boldsymbol{A}, u, n)$ into weight spaces:

$$U(\boldsymbol{A}, u, n) = \bigoplus_{\nu \in \mathbb{Z}} U(\boldsymbol{A}, u, n)^\nu, \tag{4-1-4}$$

where $U(\boldsymbol{A}, u, n)^\nu = \{\tilde{v} \in U(\boldsymbol{A}, u, n) \mid [\tilde{e}, \tilde{v}] = \nu n \tilde{v}\}$.

Proposition 4.1.9. *Let $u \in \boldsymbol{A}[t]$ and $s \in \mathbb{N}$. The $\boldsymbol{A}$-linear mappings*

$$\varphi, \psi : \boldsymbol{A}[t] \to U(\boldsymbol{A}, u, n)$$

given by

$$\varphi(P) = \tilde{x}^s P(\tilde{e}), \quad \psi(P) = \tilde{y}^s P(\tilde{e})$$

are injective (*in particular the subalgebra $\boldsymbol{A}[\tilde{e}] \subset U(\boldsymbol{A}, u, n)$ generated by $\tilde{e}$ is a polynomial algebra*).

Proof. Define $f(t) = u(t+n) - u(t)$. Every element of $S(\boldsymbol{A}, f, n)$ can be written uniquely in the form

$$\sum a_{k,\ell,m} e^k x^\ell y^m \quad (a_{k,\ell,m} \in \boldsymbol{A})$$

(Corollary 4.1.4). Therefore, from Remark 4.1.8, every element in $U(\boldsymbol{A}, u, n)$ can be written in the form

$$\sum a_{k,\ell,m}\tilde{e}^k\tilde{x}^\ell\tilde{y}^m \quad (a_{k,\ell,m} \in \boldsymbol{A}).$$

Let $P(t) = \sum_{i=0}^{p} a_i t^i$ $(a_i \in \boldsymbol{A})$ be a polynomial such that $\tilde{x}^s P(\tilde{e}) = 0$ (i.e., $P \in \ker\varphi$). As $U(\boldsymbol{A}, u, n) = S(\boldsymbol{A}, f, n)/(\Omega_1)$, we see that there exists $\alpha \in S(\boldsymbol{A}, f, n)$ such that

$$x^s \sum_{i=0}^{p} a_i e^i = \alpha\Omega_1 = \alpha(xy - u(e)).$$

If $\alpha = \sum a_{k,\ell,m}e^k x^\ell y^m$, using the fact that $\Omega_1 = xy - u(e)$ is central and relation (4-1-2) we get

$$\begin{aligned} x^s \sum_{i=0}^{p} a_i e^i &= \Big(\sum_{k,\ell,m} a_{k,\ell,m}e^k x^\ell y^m\Big)(xy - u(e)) = \sum_{k,\ell,m} a_{k,\ell,m}e^k x^\ell (xy - u(e))y^m \\ &= \sum_{k,\ell,m} a_{k,\ell,m}e^k x^{\ell+1} y^{m+1} - \sum_{k,\ell,m} a_{k,\ell,m}e^k x^\ell u(e) y^m \\ &= \sum_{k,\ell,m} a_{k,\ell,m}e^k x^{\ell+1} y^{m+1} - \sum_{k,\ell,m} a_{k,\ell,m}e^k u(e - \ell n) x^\ell y^m. \end{aligned} \tag{4-1-5}$$

Suppose now that $\alpha \neq 0$; then one can define

$$\ell_0 = \max\{\ell \in \mathbb{N} \mid \exists k, m, a_{k,\ell,m} \neq 0\}.$$

Let k_0, m_0 be such that $a_{k_0,\ell_0,m_0} \neq 0$. From (4-1-5) we get

$$x^s \sum_{i=0}^{p} a_i e^i + \sum_{k,\ell,m} a_{k,\ell,m}e^k u(e - \ell n) x^\ell y^m = \sum_{k,\ell,m} a_{k,\ell,m}e^k x^{\ell+1} y^{m+1}.$$

Using again (4-1-2) we obtain

$$\sum_{i=0}^{p} a_i (e - ns)^i x^s + \sum_{k,\ell,m} a_{k,\ell,m}e^k u(e - \ell n) x^\ell y^m = \sum_{k,\ell,m} a_{k,\ell,m}e^k x^{\ell+1} y^{m+1}.$$

The left side of this equality does not contain the monomial $e^{k_0}x^{\ell_0+1}y^{m_0+1}$, but the right side does. As the elements $e^k x^\ell y^m$ are a basis over $\boldsymbol{A}$ (Corollary 4.1.4), we obtain a contradiction. Therefore $\alpha = 0$, and hence $x^s \sum_{i=0}^{p} a_i e^i$ vanishes. Again from Corollary 4.1.4, we obtain that $a_i = 0$ for all i. This proves that $\ker\varphi = \{0\}$. The proof for ψ is similar. $\square$

Corollary 4.1.10. *Every element $\tilde{u}$ in $U(\boldsymbol{A}, u, n)$ can be written uniquely in the form*

$$\tilde{u} = \sum_{\ell>0,\, k\geq 0} \alpha_{k,\ell}\tilde{y}^{\ell}\tilde{e}^{k} + \sum_{m\geq 0,\, r\geq 0} \beta_{m,r}\tilde{x}^{m}\tilde{e}^{r}$$

with $\alpha_{k,\ell}, \beta_{m,r} \in \boldsymbol{A}$.

Proof. We have already noticed that any element in $U(\boldsymbol{A}, u, n)$ can be written (in a non unique way) as a linear combination, with coefficients in $\boldsymbol{A}$, of the elements $\tilde{x}^{i}\tilde{y}^{j}\tilde{e}^{k}$.

Suppose that $i \geq j$. Then we have $\tilde{x}^{i}\tilde{y}^{j}\tilde{e}^{k} = \tilde{x}^{i-j}\tilde{x}^{j}\tilde{y}^{j}\tilde{e}^{k}$. As $\tilde{x}\tilde{y} = u(\tilde{e})$, we see that $\tilde{x}^{j}\tilde{y}^{j} = Q_{j}(\tilde{e})$, where Q_j is a polynomial with coefficients in $\boldsymbol{A}$. Therefore $\tilde{x}^{i}\tilde{y}^{j}\tilde{e}^{k} = \sum_{\ell} \gamma_{\ell}\tilde{x}^{i-j}\tilde{e}^{\ell}$, with $\gamma_{\ell} \in \boldsymbol{A}$. Similarly one can prove that if $i < j$, we have $\tilde{x}^{i}\tilde{y}^{j}\tilde{e}^{k} = \sum_{\ell} \delta_{\ell}\tilde{y}^{j-i}\tilde{e}^{\ell}$, with $\delta_{\ell} \in \boldsymbol{A}$. This shows that any element $\tilde{u}$ in $U(\boldsymbol{A}, u, n)$ can be written in the expected form.

Suppose now that

$$\sum_{\ell>0,\, k\geq 0} \alpha_{k,\ell}\tilde{y}^{\ell}\tilde{e}^{k} + \sum_{m\geq 0,\, r\geq 0} \beta_{m,r}\tilde{x}^{m}\tilde{e}^{r} = 0.$$

Then, as $\tilde{y}^{\ell}\tilde{e}^{k} \in U(\boldsymbol{A}, u, n)^{-\ell}$ and $\tilde{x}^{m}\tilde{e}^{r} \in U(\boldsymbol{A}, u, n)^{m}$, we deduce from (4-1-4) that

$$\sum_{k} \alpha_{k,\ell}\tilde{y}^{\ell}\tilde{e}^{k} = 0 \quad \text{for all } \ell > 0, \qquad \sum_{r} \beta_{m,r}\tilde{x}^{m}\tilde{e}^{r} = 0 \quad \text{for all } m \geq 0.$$

Then from Proposition 4.1.9, we deduce that $\alpha_{k,\ell} = 0$ and $\beta_{m,r} = 0$. □

4.2. ***Generators and relations for $\boldsymbol{D(V)^{G'}}$.*** Let $\mathfrak{X}(\mathcal{T})[t]$ be the polynomials in one variable with coefficients in $\mathfrak{X}(\mathcal{T})$. From the commutation rules $[E, X] = d_0X$ and $[E, Y] = -d_0Y$, we easily deduce that for $P \in \mathfrak{X}(\mathcal{T})[t]$ we have

$$YP(E) = P(E + d_0)Y, \quad XP(E) = P(E - d_0)X. \tag{4-2-1}$$

From Theorem 3.2.6 above, we know that any element in $D(V)^{G}$ can be written uniquely as a polynomial in E with coefficients in $\mathfrak{X}(\mathcal{T})$. As XY and YX belong to $D(V)^{G}$, there exist therefore two uniquely determined polynomials u_{XY} and u_{YX} in $\mathfrak{X}(\mathcal{T})[t]$ such that $XY = u_{XY}(E)$ and $YX = u_{YX}(E)$. From (4-2-1) we obtain that

$$YXY = u_{YX}(E)Y = Yu_{XY}(E) = u_{XY}(E + d_0)Y$$

and therefore

$$u_{YX}(E) = u_{XY}(E + d_0). \tag{4-2-2}$$

As the polynomial u_{XY} will play an important role in Theorem 4.2.2 below, let us emphasize the connection between u_{XY} and the Bernstein–Sato polynomial b_Y.

Remark first that $b_Y = b_{XY}$. We know from Corollary 3.2.8 that

$$\begin{aligned} h(XY)(-\lambda+\rho) &= b_{XY}(\lambda) = b_Y(\lambda) \\ &= \sum_{i=0}^{p} \alpha_i(-\lambda+\rho)(a_0 d_0 + a_1 d_1 + \cdots + a_r d_r)^i \\ &= \sum_{i=0}^{p} \alpha_i(-\lambda+\rho)(h(E)(-\lambda+\rho))^i \end{aligned}$$

with uniquely defined polynomials $\alpha_i \in \mathbb{C}[A]^{W_0,\tau}$. Therefore we obtain

Proposition 4.2.1. *Keeping the notation above, we have*

$$u_{XY}(t) = \sum_{i=0}^{p} h^{-1}(\alpha_i) t^i.$$

Theorem 4.2.2. *Let $f_{XY}(t) = u_{XY}(t+d_0) - u_{XY}(t)$. The mapping*

$$\tilde{x} \mapsto X, \quad \tilde{y} \mapsto Y, \quad \tilde{e} \mapsto E$$

extends uniquely to an isomorphism of $\mathfrak{Z}(\mathcal{T})$-algebras between $U(\mathfrak{Z}(\mathcal{T}), u_{XY}, d_0)$ (which is isomorphic to $S(\mathfrak{Z}(\mathcal{T}), f_{XY}, d_0)/(\Omega_1)$) and $D(V)^{G'} = \mathcal{T}_0[X, Y]$.

Proof. As $[E, X] = d_0 X$, $[E, Y] = -d_0 Y$, $XY = u_{XY}(E)$ and $YX = u_{XY}(E+d_0)$ (see (4-2-2)), and as from Theorem 3.2.6 the algebra $D(V)^{G'} = \mathcal{T}_0[X, Y]$ is generated over $\mathfrak{Z}(\mathcal{T})$ by X, Y, E, we know (universal property) that the mapping

$$\tilde{x} \mapsto X, \quad \tilde{y} \mapsto Y, \quad \tilde{e} \mapsto E$$

extends uniquely to a surjective morphism of $\mathfrak{Z}(\mathcal{T})$-algebras:

$$\varphi : U(\mathfrak{Z}(\mathcal{T}), u_{XY}, d_0) \to D(V)^{G'}.$$

From Corollary 4.1.10 any element $\tilde{u}$ in $U(\mathfrak{Z}(\mathcal{T}), u_{XY}, d_0)$ can be written uniquely in the form

$$\tilde{u} = \sum_{\ell>0,\, k\geq 0} \alpha_{k,\ell} \tilde{y}^\ell \tilde{e}^k + \sum_{m\geq 0,\, r\geq 0} \beta_{m,r} \tilde{x}^m \tilde{e}^r$$

with $\alpha_{k,\ell}, \beta_{m,r} \in \mathfrak{Z}(\mathcal{T})$. Suppose now that $\tilde{u} \in \ker(\varphi)$, then

$$\varphi(\tilde{u}) = \sum_{\ell>0,\, k\geq 0} \alpha_{k,\ell} Y^\ell E^k + \sum_{m\geq 0,\, r\geq 0} \beta_{m,r} X^m E^r = 0,$$

with $\alpha_{k,\ell}, \beta_{m,r} \in \mathfrak{Z}(\mathcal{T})$. Then Corollary 3.2.7 implies that $\alpha_{k,\ell} = \beta_{m,r} = 0$. Hence φ is an isomorphism. □

5. Radial components

5.1. ***Radial components and Bernstein–Sato polynomials.*** Remember that for $\tilde{\boldsymbol{a}} = (a_1, a_2, \dots, a_r) \in \mathbb{N}^r$ we have defined $V_{\tilde{\boldsymbol{a}}} = V_{(0,a_1,\dots,a_r)}$. Remember also that for $\boldsymbol{a} = (a_0, a_1, \dots, a_r)$ we have $V_{\boldsymbol{a}} = \Delta^{a_0} V_{\tilde{\boldsymbol{a}}}$. We know from Proposition 2.2.8 that the spaces $U_{\tilde{\boldsymbol{a}}} = \bigoplus_{a_0 \in \mathbb{N}} \Delta_0^{a_0} V_{\tilde{\boldsymbol{a}}}$ are the G'-isotypic components of $\mathbb{C}[V]$ and that the spaces $W_{\tilde{\boldsymbol{a}}} = \bigoplus_{a_0 \in \mathbb{Z}} \Delta_0^{a_0} V_{\tilde{\boldsymbol{a}}}$ are the G'-isotypic components of $\mathbb{C}[\mathbb{O}]$. Therefore the algebra $D(V)^{G'} = \mathcal{T}_0[X, Y]$ stabilizes each space $U_{\tilde{\boldsymbol{a}}}$ and the algebra $D(\mathbb{O})^{G'} = \mathcal{T}_0[X, X^{-1}] = \mathcal{T}$ stabilizes each space $W_{\tilde{\boldsymbol{a}}}$.

Let us consider the restriction map

$$\begin{aligned} D(\mathbb{O})^{G'} &\to \operatorname{End}(W_{\tilde{\boldsymbol{a}}}), \\ D &\mapsto r_{\tilde{\boldsymbol{a}}}(D) = D_{|W_{\tilde{\boldsymbol{a}}}}. \end{aligned}$$

Definition 5.1.1. Let $D \in D(\mathbb{O})^{G'} = \mathcal{T}_0[X, X^{-1}] = \mathcal{T}$. The operator $r_{\tilde{\boldsymbol{a}}}(D) = D_{|W_{\tilde{\boldsymbol{a}}}}$ is called the radial component of D with respect to $\tilde{\boldsymbol{a}}$.

Example 5.1.2. Consider the case where $\tilde{\boldsymbol{a}} = 0$. Then $W_{\tilde{\boldsymbol{a}}} = \mathbb{C}[\Delta_0, \Delta_0^{-1}]$, and $r_0(D)$ is the endomorphism of $\mathbb{C}[t, t^{-1}]$ defined by $D(\varphi \circ \Delta_0) = r_0(D)(\varphi) \circ \Delta_0$. The operator $r_0(D)$ is the usual radial component of D (we will see below that it is a differential operator).

Notice now that the space $W_{\tilde{\boldsymbol{a}}} = \bigoplus_{a_0 \in \mathbb{Z}} \Delta_0^{a_0} V_{\tilde{\boldsymbol{a}}}$ can be viewed as the space of Laurent polynomials in Δ_0, with coefficients in $V_{\tilde{\boldsymbol{a}}}$, in other words any $P \in W_{\tilde{\boldsymbol{a}}}$ can be written uniquely under the form

$$P = \sum \Delta_0^p \gamma_p$$

with $\gamma_p \in V_{\tilde{\boldsymbol{a}}}$. This can also be written as $P = \varphi \circ (\Delta_0)$, with $\varphi(t) = \sum t^p \gamma_p$ in $V_{\tilde{\boldsymbol{a}}}[t, t^{-1}]$ (this being precisely the set of linear combinations $\sum t^p \gamma_p$, with $\gamma_p \in V_{\tilde{\boldsymbol{a}}}$).

There is a natural action of $D(\mathbb{C}^*) = \mathbb{C}[t, t^{-1}, t\, d/dt]$ on $V_{\tilde{\boldsymbol{a}}}[t, t^{-1}]$ given by $(d/dt) t^p \gamma_p = p t^{p-1} \gamma_p$.

Proposition 5.1.3. *Let $D \in \mathcal{T}_n$ a homogeneous element of degree n. Let b_D be its Bernstein–Sato polynomial. Let $\varphi \in V_{\tilde{\boldsymbol{a}}}[t, t^{-1}]$. Then*

$$D(\varphi \circ \Delta_0) = \big(t^n b_D(t\, d/dt, a_1, \dots, a_r) \varphi\big) \circ \Delta_0;$$

in other words, $r_{\tilde{\boldsymbol{a}}}(D) = t^n b_D(t\, d/dt, a_1, \dots, a_r)$.

Proof. It is enough to show that the two operators coincide on elements of the form $\Delta_0^p \gamma_p$, with $\gamma_p \in V_{\tilde{\boldsymbol{a}}}$. Then $\varphi = t^p \gamma_p$. Let us write

$$b_D(\boldsymbol{a}) = \sum_k c_k(a_1, \dots, a_r) a_0^k.$$

We have

$$\begin{aligned}\Big(t^n b_D\Big(t\frac{d}{dt}, a_1, \dots, a_r\Big)\varphi\Big) \circ \Delta_0 &= t^n\Big(\sum_k c_k(a_1, \dots, a_r)\Big(t\frac{d}{dt}\Big)^k \varphi\Big) \circ \Delta_0 \\ &= t^n\Big(\sum_k c_k(a_1, \dots, a_r) p^k t^p \gamma_p\Big) \circ \Delta_0 \\ &= (t^n b_D(p, a_1, \dots, a_r) t^p \gamma_p) \circ \Delta_0 \\ &= b_D(p, a_1, \dots, a_r)\Delta_0^{p+n}\gamma_p \\ &= D(\Delta_0^p \gamma_p). \qquad \square\end{aligned}$$

Corollary 5.1.4. *If (G, V) is a PV of commutative parabolic type of rank $r+1$, then the radial component of Y is given by*

$$r_{\tilde{\boldsymbol{a}}}(Y) = ct^{-1}\prod_{j=0}^{r}\Big(t\frac{d}{dt} + a_1 + \cdots + a_j + j\frac{d}{2}\Big).$$

Proof. This is just a consequence of the formula for b_Y given in Example 3.1.2. $\square$

Example 5.1.5. Consider case 1) in Example 2.3.3: then $G = (\mathrm{SL}(n)\times\mathrm{SL}(n))\times\mathbb{C}^*$ acting on $x \in V = M_n(\mathbb{C})$ by $(g_1, g_2, t).x = tg_1xg_2^{-1}$. Then $\Delta_0 = X = \det x$ and

$$Y = \Delta_0^*(\partial) = \det\Big(\frac{\partial}{\partial x_{ij}}\Big),$$

where x_{ij} are the coefficients of the matrix X. As in this case $d/2 = 1$ (see [Muller et al. 1986, Table 2, p. 122]), we have

$$b_Y(a_0, a_1, \dots, a_{n-1}) = \prod_{j=0}^{n-1}(a_0 + a_1 + \cdots + a_j + j).$$

Hence the radial component $r_0(Y)$ defined by $\det\Big(\dfrac{\partial}{\partial x_{ij}}\Big)(\varphi\circ\det) = (r_0(Y)\varphi)\circ\det$ is given by

$$r_0(Y) = t^{-1}\prod_{j=0}^{n-1}\Big(t\frac{d}{dt} + j\Big).$$

This radial component has already been calculated by Raïs [1972, p. 22], by other methods. He obtained that $r_0(Y) = \big[\prod_{j=2}^{n}(t\,d/dt + j)\big]\,d/dt$. A simple calculation shows that the two operators are the same.

5.2. *Algebras of radial components.*

Definition 5.2.1. The radial component algebra $R_{\tilde{\boldsymbol{a}}}$ is the image of $D(V)^{G'} = \mathcal{T}_0[X, Y]$ under the map $D \mapsto r_{\tilde{\boldsymbol{a}}}(D)$.

Remember from Proposition 3.2.3 that the elements D in $\mathscr{X}(\mathscr{T})$ are characterized by the fact that the corresponding Bernstein–Sato polynomial b_D does not depend on the a_0 variable. Therefore such a D acts by the scalar $b_D(0, \tilde{\boldsymbol{a}})$ on $W_{\tilde{\boldsymbol{a}}}$; that is, $r_{\tilde{\boldsymbol{a}}}(D) = b_D(0, \tilde{\boldsymbol{a}})Id_{W_{\tilde{\boldsymbol{a}}}}$.

Let us consider the polynomial $u_{XY} \in \mathscr{X}(\mathscr{T})[t]$ introduced in Section 4.2. If $u_{XY} = \sum_i c_i t^i$, with $c_i \in \mathscr{X}(\mathscr{T})$, we define

$$r_{\tilde{\boldsymbol{a}}}(u_{XY}) = \sum_i r_{\tilde{\boldsymbol{a}}}(c_i)t^i \in \mathbb{C}[t].$$

Lemma 5.2.2. *Let $\boldsymbol{a} = (a_0, a_1, \dots, a_r) \in \mathbb{N}^{r+1}$. Suppose that $a_0 > 0$. Then the map $P \mapsto YP$ from $V_{\boldsymbol{a}}$ to $V_{\boldsymbol{a}-1}$ is a G'-equivariant isomorphism.*

Sketch of proof. It is enough to prove that this map is not 0. As $\Delta_0^{*a_0} \cdots \Delta_r^{*a_r}$ is the lowest weight vector of $V_{\boldsymbol{a}}^* \subset \mathbb{C}[V^*]$ we have $\Delta_0^*(\partial)^{a_0} \cdots \Delta_r^*(\partial)^{a_r} \Delta_0^{a_0} \cdots \Delta_r^{a_r}(0) \neq 0$. Hence $\Delta_0^*(\partial)\Delta_0^{a_0} \cdots \Delta_r^{a_r} \neq 0$. □

Theorem 5.2.3. *The radial component algebra $R_{\tilde{\boldsymbol{a}}}$ is isomorphic, as an associative algebra over $\mathbb{C}$, to the algebra $U(\mathbb{C}, r_{\tilde{\boldsymbol{a}}}(u_{XY}), d_0)$ introduced in Definition 4.1.1.*

Proof. The algebra $R_{\tilde{\boldsymbol{a}}}$ is generated over $\mathbb{C}$ by the elements $r_{\tilde{\boldsymbol{a}}}(E)$, $r_{\tilde{\boldsymbol{a}}}(X)$, $r_{\tilde{\boldsymbol{a}}}(Y)$. The defining relations of $U(\mathbb{C}, r_{\tilde{\boldsymbol{a}}}(u_{XY}), d_0)$ are satisfied:

$$\begin{aligned}
[r_{\tilde{\boldsymbol{a}}}(E), r_{\tilde{\boldsymbol{a}}}(X)] &= r_{\tilde{\boldsymbol{a}}}([E, X]) = d_0 r_{\tilde{\boldsymbol{a}}}(X),\\
[r_{\tilde{\boldsymbol{a}}}(E), r_{\tilde{\boldsymbol{a}}}(Y)] &= r_{\tilde{\boldsymbol{a}}}([E, Y]) = -d_0 r_{\tilde{\boldsymbol{a}}}(Y),\\
r_{\tilde{\boldsymbol{a}}}(X) r_{\tilde{\boldsymbol{a}}}(Y) &= r_{\tilde{\boldsymbol{a}}}(XY) = r_{\tilde{\boldsymbol{a}}}(u_{XY})(r_{\tilde{\boldsymbol{a}}}(E)),\\
r_{\tilde{\boldsymbol{a}}}(Y) r_{\tilde{\boldsymbol{a}}}(X) &= r_{\tilde{\boldsymbol{a}}}(YX) = r_{\tilde{\boldsymbol{a}}}(u_{XY})(r_{\tilde{\boldsymbol{a}}}(E) + d_0).
\end{aligned}$$

Therefore the mapping

$$\tilde{x} \mapsto r_{\tilde{\boldsymbol{a}}}(X), \quad \tilde{y} \mapsto r_{\tilde{\boldsymbol{a}}}(Y), \quad \tilde{e} \mapsto r_{\tilde{\boldsymbol{a}}}(E)$$

extends uniquely to a surjective morphism of $\mathbb{C}$-algebras

$$\varphi_{\tilde{\boldsymbol{a}}} : U(\mathbb{C}, r_{\tilde{\boldsymbol{a}}}(u_{XY}), d_0) \to R_{\tilde{\boldsymbol{a}}}.$$

From Corollary 4.1.10 any element $\tilde{u}$ in $U(\mathbb{C}, r_{\tilde{\boldsymbol{a}}}(u_{XY}), d_0)$ can be written uniquely in the form

$$\tilde{u} = \sum_{\ell>0,\, k\geq 0} \alpha_{k,\ell} \tilde{y}^{\ell} \tilde{e}^{k} + \sum_{m\geq 0,\, s\geq 0} \beta_{m,s} \tilde{x}^{m} \tilde{e}^{s}$$

with $\alpha_{k,\ell}, \beta_{m,s} \in \mathbb{C}$. Suppose now that $\tilde{u} \in \ker(\varphi_{\tilde{\boldsymbol{a}}})$, then

$$\varphi_{\tilde{\boldsymbol{a}}}(\tilde{u}) = \sum_{\ell>0,\, k\geq 0} \alpha_{k,\ell} r_{\tilde{\boldsymbol{a}}}(Y)^{\ell} r_{\tilde{\boldsymbol{a}}}(E)^{k} + \sum_{m\geq 0,\, s\geq 0} \beta_{m,s} r_{\tilde{\boldsymbol{a}}}(X)^{m} r_{\tilde{\boldsymbol{a}}}(E)^{s} = 0.$$

Applying this operator to a function of the form $\Delta^{a_0}P$, with $P \in V_{\tilde{\boldsymbol{a}}}$, we obtain

$$\sum_{\ell>0} Y^\ell\Big(\sum_{k\geq 0}\alpha_{k,\ell}E^k\Delta^{a_0}P\Big)+\sum_{m\geq 0}X^m\Big(\sum_{s\geq 0}\beta_{m,s}E^s\Delta^{a_0}P\Big)=0.$$

As the operators X and Y have degree d_0 and $-d_0$, respectively, this implies that

$$Y^\ell\Big(\sum_{k\geq 0}\alpha_{k,\ell}E^k\Delta^{a_0}P\Big)=0 \quad \text{for all } \ell,$$
$$X^m\Big(\sum_{s\geq 0}\beta_{m,s}E^s\Delta^{a_0}P\Big)=0 \quad \text{for all } m.$$

Therefore, by Lemma 5.2.2 we obtain $\sum_{k\geq 0}\alpha_{k,\ell}E^k\Delta^{a_0}P=0$ for all ℓ and all $a_0>\ell$, and $\sum_{s\geq 0}\beta_{m,s}E^s\Delta^{a_0}P=0$ for all m, a_0. As $E\Delta^{a_0}P=(a_0d_0+d(\tilde{\boldsymbol{a}}))\Delta^{a_0}P$, where $d(\tilde{\boldsymbol{a}})=a_1d_1+\cdots+a_rd_r$, we have $\sum_{k\geq 0}\alpha_{k,\ell}(a_0d_0+d(\tilde{\boldsymbol{a}}))^k\Delta^{a_0}P=0$ for all ℓ and all $a_0>\ell$, and $\sum_{s\geq 0}\beta_{m,s}(a_0d_0+d(\tilde{\boldsymbol{a}}))^s\Delta^{a_0}P=0$ for all m, a_0. Hence $\sum_{k\geq 0}\alpha_{k,\ell}(a_0d_0+d(\tilde{\boldsymbol{a}}))^k=0$ for all ℓ and $a_0>\ell$, and $\sum_{s\geq 0}\beta_{m,s}(a_0d_0+d(\tilde{\boldsymbol{a}}))^s=0$ for all m, a_0. This implies that $\alpha_{k,\ell}=0$ and $\beta_{m,s}=0$ for all ℓ, k, m, s. Hence $\tilde{u}=0$ and $\varphi_{\tilde{\boldsymbol{a}}}$ is injective. □

Remark 5.2.4. For $\tilde{\boldsymbol{a}}=0$, the preceding result was first obtained by Levasseur [2009], by other methods.

Now define $J_{\tilde{\boldsymbol{a}}}=\ker(r_{\tilde{\boldsymbol{a}}}|_{D(V)^{G'}})$. This is a two-sided ideal of $D(V)^{G'}=\mathcal{T}_0[X,Y]$. Remember from Proposition 3.1.6 that any $D\in D(V)^{G'}$ can be written uniquely in the form

$$D=\sum_{k\in\mathbb{N}^*}u_kY^k+\sum_{n\in\mathbb{N}}v_nX^n \quad \text{(finite sums)},$$

where $u_k, v_n\in\mathcal{T}_0=D(V)^G$.

Lemma 5.2.5. $$J_{\tilde{\boldsymbol{a}}}=\Big\{D=\sum_{k\in\mathbb{N}^*}u_kY^k+\sum_{n\in\mathbb{N}}v_nX^n \;\Big|\; u_k, v_n\in J_{\tilde{\boldsymbol{a}}}\cap\mathcal{T}_0\Big\}.$$

Proof. From Theorem 5.2.3 the algebra $R_{\tilde{\boldsymbol{a}}}$ is isomorphic to $U(\mathbb{C}, r_{\tilde{\boldsymbol{a}}}(u_{XY}), d_0)$. If $r_{\tilde{\boldsymbol{a}}}(D)=\sum_{k\in\mathbb{N}^*}r_{\tilde{\boldsymbol{a}}}(u_k)r_{\tilde{\boldsymbol{a}}}(Y)^k+\sum_{n\in\mathbb{N}}r_{\tilde{\boldsymbol{a}}}(v_n)r_{\tilde{\boldsymbol{a}}}(X)^n=0$, then from Corollary 4.1.10 we obtain that $r_{\tilde{\boldsymbol{a}}}(u_k)=0$ and $r_{\tilde{\boldsymbol{a}}}(v_n)=0$ for all k and all n. □

Let us now give a set of generators for the ideal $\ker(r_{\tilde{\boldsymbol{a}}})$ in $D(V)^{G'}=\mathcal{T}_0[X,Y]$. From Proposition 5.1.3 we obtain that $r_{\tilde{\boldsymbol{a}}}(E)=d_0(t\,d/dt)+d(\tilde{\boldsymbol{a}})$. Therefore $r_{\tilde{\boldsymbol{a}}}((E-d(\tilde{\boldsymbol{a}}))/d_0)=t\,d/dt$. Define $G_i^{\tilde{\boldsymbol{a}}}=R_i-b_{R_i}((E-d(\tilde{\boldsymbol{a}}))/d_0,\tilde{\boldsymbol{a}})$, where the R_i are the Capelli operators introduced in Section 2.2. Obviously $G_i^{\tilde{\boldsymbol{a}}}\in D(V)^G=\mathcal{T}_0$.

Using Proposition 5.1.3 again we obtain

$$\begin{aligned} r_{\tilde{\boldsymbol{a}}}(G_i^{\tilde{\boldsymbol{a}}}) &= r_{\tilde{\boldsymbol{a}}}\left(R_i - b_{R_i}\left(\frac{E-d(\tilde{\boldsymbol{a}})}{d_0}, \tilde{\boldsymbol{a}}\right)\right) \\ &= r_{\tilde{\boldsymbol{a}}}(R_i) - b_{R_i}\left(r_{\tilde{\boldsymbol{a}}}\left(\frac{E-d(\tilde{\boldsymbol{a}})}{d_0}\right), \tilde{\boldsymbol{a}}\right) \\ &= b_{R_i}\left(t\frac{d}{dt}, \tilde{\boldsymbol{a}}\right) - b_{R_i}\left(t\frac{d}{dt}, \tilde{\boldsymbol{a}}\right) = 0. \end{aligned}$$

Hence the elements $G_i^{\tilde{\boldsymbol{a}}}$ belong to $J_{\tilde{\boldsymbol{a}}} \cap \mathcal{T}_0$.

Theorem 5.2.6. *The elements $G_i^{\tilde{\boldsymbol{a}}}$ are generators of $J_{\tilde{\boldsymbol{a}}}$:*

$$J_{\tilde{\boldsymbol{a}}} = \ker\big(r_{\tilde{\boldsymbol{a}}}|_{D(V)^{G'}}\big) = \sum_{i=0}^{r} D(V)^{G'} G_i^{\tilde{\boldsymbol{a}}} = \sum_{i=0}^{r} G_i^{\tilde{\boldsymbol{a}}} D(V)^{G'}.$$

Proof. From Lemma 5.2.5, it is now enough to prove that

$$J_{\tilde{\boldsymbol{a}}} \cap \mathcal{T}_0 \subset \sum_{i=0}^{r} D(V)^{G} G_i^{\tilde{\boldsymbol{a}}} = \sum_{i=0}^{r} \mathcal{T}_0 G_i^{\tilde{\boldsymbol{a}}}.$$

Let $D \in J_{\tilde{\boldsymbol{a}}} \cap \mathcal{T}_0$. As $\mathcal{T}_0 = \mathbb{C}[R_0, \dots, R_r]$ (Theorem 2.2.9), we have also $\mathcal{T}_0 = \mathbb{C}[G_0^{\tilde{\boldsymbol{a}}}, \dots, G_r^{\tilde{\boldsymbol{a}}}, E]$. Therefore $D = \sum Q_i E^i$, where $Q_i \in \mathbb{C}[G_0^{\tilde{\boldsymbol{a}}}, \dots, G_r^{\tilde{\boldsymbol{a}}}]$. Hence $Q_i \in Q_i(0) + \sum_{i=0}^{r} D(V)^G G_i^{\tilde{\boldsymbol{a}}}$. Then

$$0 = r_{\tilde{\boldsymbol{a}}}(D) = \sum_i Q_i(0) r_{\tilde{\boldsymbol{a}}}(E^i) = \sum_i Q_i(0)\Big(d_0\Big(t\frac{d}{dt}\Big) + d(\tilde{\boldsymbol{a}})\Big)^i.$$

Therefore $Q_i(0) = 0$ $(i = 0, \dots, r)$. Hence $Q_i \in \sum_{i=0}^{r} D(V)^G G_i^{\tilde{\boldsymbol{a}}}$, which yields $D \in \sum_{i=0}^{r} D(V)^G G_i^{\tilde{\boldsymbol{a}}}$. □

Remark 5.2.7. For $\tilde{\boldsymbol{a}} = 0$, the result of the preceding theorem is due to [Levasseur 2009, Theorem 4.11(v)].

5.3. *Rational radial component algebras.*

Definition 5.3.1. The rational radial component algebra $R_{\tilde{\boldsymbol{a}}}^r$ is the image of

$$D(\mathbb{O})^{G'} = \mathcal{T}_0[X, X^{-1}] = \mathcal{T}$$

under the map $D \mapsto r_{\tilde{\boldsymbol{a}}}(D)$.

In fact as shown in the following proposition the structure of the algebras $R_{\tilde{\boldsymbol{a}}}^r$ is simpler than the structure of $R_{\tilde{\boldsymbol{a}}}$, and the ideal $I_{\tilde{\boldsymbol{a}}} = \ker(r_{\tilde{\boldsymbol{a}}}) \subset \mathcal{T}$ has the same generators as $J_{\tilde{\boldsymbol{a}}}$.

Proposition 5.3.2. 1) *For all $\tilde{\boldsymbol{a}}$, the rational radial component algebra $R_{\tilde{\boldsymbol{a}}}^r$ is isomorphic to $\mathbb{C}[t, t^{-1}, t\,d/dt]$.*

2) $I_{\tilde{\boldsymbol{a}}} = \ker(r_{\tilde{\boldsymbol{a}}}) = \sum_{i=0}^{r} \mathcal{T} G_i^{\tilde{\boldsymbol{a}}} = \sum_{i=0}^{r} G_i^{\tilde{\boldsymbol{a}}} \mathcal{T}$.

Proof. 1) We have $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$. And $\mathcal{T}_0 = \mathfrak{X}(\mathcal{T})[E]$, from Theorem 3.2.6. Therefore $\mathcal{T} = \mathfrak{X}(\mathcal{T})[X, X^{-1}, E]$. On the other hand we have $r_{\tilde{\boldsymbol{a}}}(\mathfrak{X}(\mathcal{T})) = \mathbb{C}$, $r_{\tilde{\boldsymbol{a}}}(X) = t$, $r_{\tilde{\boldsymbol{a}}}(X^{-1}) = t^{-1}$ and $r_{\tilde{\boldsymbol{a}}}(E) = d_0(t\,d/dt) + d(\tilde{\boldsymbol{a}})$. Hence

$$R^r_{\tilde{\boldsymbol{a}}} = r_{\tilde{\boldsymbol{a}}}(\mathcal{T}) = \mathbb{C}\Big[t, t^{-1}, d_0\Big(t\frac{d}{dt}\Big) + d(\tilde{\boldsymbol{a}})\Big] = \mathbb{C}\Big[t, t^{-1}, t\frac{d}{dt}\Big].$$

2) Obviously $\sum_{i=0}^{r} \mathcal{T} G_i^{\tilde{\boldsymbol{a}}} \subset I_{\tilde{\boldsymbol{a}}}$. As $I_{\tilde{\boldsymbol{a}}}$ is a two-sided ideal of $\mathcal{T}$, it is easily seen to be graded. If $D \in I_{\tilde{\boldsymbol{a}}} \cap \mathcal{T}_p$, then $X^{-p}D \in \mathcal{T}_0 \cap I_{\tilde{\boldsymbol{a}}} = \mathcal{T}_0 \cap J_{\tilde{\boldsymbol{a}}} = \sum_{i=0}^{r} \mathcal{T}_0 G_i^{\tilde{\boldsymbol{a}}}$. Therefore $D \in \sum_{i=0}^{r} \mathcal{T} G_i^{\tilde{\boldsymbol{a}}}$. □

Acknowledgements

I would like to thank Thierry Levasseur for providing me with the manuscript of [Levasseur 2009]. I would also like to thank Sylvain Rubenthaler who provided me with a first proof of Proposition 4.1.9 which was important for my understanding.

References

[Benson and Ratcliff 1996] C. Benson and G. Ratcliff, "A classification of multiplicity free actions", *J. Algebra* **181**:1 (1996), 152–186. MR 97c:14046 Zbl 0869.14021

[Benson and Ratcliff 2000] C. Benson and G. Ratcliff, "Rationality of the generalized binomial coefficients for a multiplicity free action", *J. Austral. Math. Soc. Ser. A* **68**:3 (2000), 387–410. MR 2001a:20071 Zbl 0964.20021

[Benson and Ratcliff 2004] C. Benson and G. Ratcliff, "On multiplicity free actions", pp. 221–304 in *Representations of real and p-adic groups* (Singapore, 2002–2003), edited by E.-C. Tan and C.-B. Zhu, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. **2**, World Scientific, River Edge, NJ, 2004. MR 2005k:20107 Zbl 1061.22017

[Van den Bergh 1996] M. Van den Bergh, "Some rings of differential operators for Sl_2-invariants are simple", *J. Pure Appl. Algebra* **107**:2-3 (1996), 309–335. MR 97c:16032 Zbl 0871.16014

[Bopp and Rubenthaler 1993] N. Bopp and H. Rubenthaler, "Fonction zêta associée à la série principale sphérique de certains espaces symétriques", *Ann. Sci. École Norm. Sup.* (4) **26**:6 (1993), 701–745. MR 95f:11095 Zbl 0806.14039

[Bopp and Rubenthaler 2005] N. Bopp and H. Rubenthaler, "Local zeta functions attached to the minimal spherical series for a class of symmetric spaces", *Mem. Amer. Math. Soc.* **174**:821 (2005). MR 2006c:11140 Zbl 1076.11059

[Bourbaki 1971] N. Bourbaki, *Groupes et algèbres de Lie, chapitre I: Algèbres de Lie*, 2nd ed., Actualités Scientifiques et Industrielles **1285**, Hermann, Paris, 1971. MR 42 #6159 Zbl 0213.04103

[Brion 1985] M. Brion, "Représentations exceptionnelles des groupes semi-simples", *Ann. Sci. École Norm. Sup.* (4) **18**:2 (1985), 345–387. MR 87e:14043 Zbl 0588.22010

[Faraut and Korányi 1994] J. Faraut and A. Korányi, *Analysis on symmetric cones*, Clarendon/Oxford University Press, New York, 1994. MR 98g:17031 Zbl 0841.43002

[Goodearl and Warfield 2004] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, 2nd ed., London Mathematical Society Student Texts **61**, Cambridge University Press, 2004. MR 2005b:16001 Zbl 1101.16001

[Heckman and Schlichtkrull 1994] G. Heckman and H. Schlichtkrull, *Harmonic analysis and special functions on symmetric spaces*, Perspectives in Mathematics **16**, Academic Press, San Diego, CA, 1994. MR 96j:22019 Zbl 0836.43001

[Howe and Umeda 1991] R. Howe and T. Umeda, "The Capelli identity, the double commutant theorem, and multiplicity-free actions", *Math. Ann.* **290**:3 (1991), 565–619. MR 92j:17004 Zbl 0733.20019

[Igusa 1981] J.-I. Igusa, "On Lie algebras generated by two differential operators", pp. 187–195 in *Manifolds and Lie groups* (Notre Dame, IN, 1980), edited by J. Hano et al., Progr. Math. **14**, Birkhäuser, Boston, 1981. MR 83e:17013 Zbl 0479.17009

[Kac 1980] V. G. Kac, "Some remarks on nilpotent orbits", *J. Algebra* **64**:1 (1980), 190–213. MR 81i:17005 Zbl 0431.17007

[Kimura 2003] T. Kimura, *Introduction to prehomogeneous vector spaces*, Translations of Mathematical Monographs **215**, American Mathematical Society, Providence, RI, 2003. MR 2003k:11180 Zbl 1035.11060

[Knop 1994] F. Knop, "A Harish-Chandra homomorphism for reductive group actions", *Ann. of Math.* (2) **140**:2 (1994), 253–288. MR 95h:14045 Zbl 0828.22017

[Knop 1998] F. Knop, "Some remarks on multiplicity free spaces", pp. 301–317 in *Representation theories and algebraic geometry* (Montreal, PQ, 1997), edited by A. Broer and G. Sabidussi, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **514**, Kluwer, Dordrecht, 1998. MR 99i:20056 Zbl 0915.20021

[Kostant and Sahi 1991] B. Kostant and S. Sahi, "The Capelli identity, tube domains, and the generalized Laplace transform", *Adv. Math.* **87**:1 (1991), 71–92. MR 92h:22033 Zbl 0748.22008

[Leahy 1998] A. S. Leahy, "A classification of multiplicity free representations", *J. Lie Theory* **8**:2 (1998), 367–391. MR 2000g:22024 Zbl 0910.22015

[Levasseur 2009] T. Levasseur, "Radial components, prehomogeneous vector spaces, and rational Cherednik algebras", *Int. Math. Res. Not.* **2009**:3 (2009), 462–511. MR 2010g:22028 Zbl 1167.22006

[Levasseur and Stafford 1989] T. Levasseur and J. T. Stafford, "Rings of differential operators on classical rings of invariants", *Mem. Amer. Math. Soc.* **81**:412 (1989). MR 90i:17018 Zbl 0691.16019

[Maaß 1971] H. Maaß, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Mathematics **216**, Springer, Berlin, 1971. MR 49 #8938 Zbl 0224.10028

[McConnell and Robson 1987] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Wiley, Chichester, 1987. MR 89j:16023 Zbl 0644.16008

[Muller et al. 1986] I. Muller, H. Rubenthaler, and G. Schiffmann, "Structure des espaces préhomogènes associés à certaines algèbres de Lie graduées", *Math. Ann.* **274**:1 (1986), 95–123. MR 88e:17025 Zbl 0568.17007

[Nomura 1989] T. Nomura, "Algebraically independent generators of invariant differential operators on a symmetric cone", *J. Reine Angew. Math.* **400** (1989), 122–133. MR 91h:22021 Zbl 0667.43007

[Raïs 1972] M. Raïs, "Distributions homogènes sur des espaces de matrices", *Mém. Soc. Math. France* **30** (1972), 3–109. MR 58 #22412 Zbl 0245.46045

[Rallis and Schiffmann 1980] S. Rallis and G. Schiffmann, "Weil representation, I: Intertwining distributions and discrete spectrum", *Mem. Amer. Math. Soc.* **25**:231 (1980). MR 81j:22007 Zbl 0442.22006

[Rubenthaler 2013] H. Rubenthaler, "Multiplicity free spaces with a one-dimensional quotient", *J. Lie Theory* **23**:2 (2013), 433–458. MR 3113517 Zbl 1273.14095

[Rubenthaler and Schiffmann 1987] H. Rubenthaler and G. Schiffmann, "Opérateurs différentiels de Shimura et espaces préhomogènes", *Invent. Math.* **90**:2 (1987), 409–442. MR 89e:11030 Zbl 0638.10024

[Rubenthaler and Schiffmann 1990] H. Rubenthaler and G. Schiffmann, "SL_2-triplet associé à un polynôme homogène", *J. Reine Angew. Math.* **408** (1990), 136–158. MR 91k:22033 Zbl 0694.17004

[Sato and Kimura 1977] M. Sato and T. Kimura, "A classification of irreducible prehomogeneous vector spaces and their relative invariants", *Nagoya Math. J.* **65** (1977), 1–155. MR 55 #3341 Zbl 0321.14030

[Schwarz 2002] G. W. Schwarz, "Finite-dimensional representations of invariant differential operators", *J. Algebra* **258**:1 (2002), 160–204. MR 2004c:16042a Zbl 1020.22005

[Smith 1990] S. P. Smith, "A class of algebras similar to the enveloping algebra of sl(2)", *Trans. Amer. Math. Soc.* **322**:1 (1990), 285–314. MR 91b:17013 Zbl 0732.16019

[Terras 1988] A. Terras, *Harmonic analysis on symmetric spaces and applications, II*, Springer, Berlin, 1988. MR 89k:22017 Zbl 0668.10033

[Vinberg and Kimelfeld 1978] È. B. Vinberg and B. N. Kimelfeld, "Odnorodnye oblasti na flagovykh mnogoobraziyakh i sfericheskie podgruppy poluprostykh grupp Li", *Funktsional. Anal. i Prilozhen.* **12**:3 (1978), 12–19. Translated as "Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups", *Funct. Anal. Appl.* **12** (1979), 168–174. MR 82e:32042 Zbl 0439.53055

[Wallach 1992] N. R. Wallach, "Polynomial differential operators associated with Hermitian symmetric spaces", pp. 76–94 in *Representation theory of Lie groups and Lie algebras* (Fuji-Kawaguchiko, 1990), edited by T. Kawazoe et al., World Scientific, River Edge, NJ, 1992. MR 94a:22031 Zbl 1226.22018

[Yan 2000] Z. Yan, "Invariant differential operators and holomorphic function spaces", *J. Lie Theory* **10**:1 (2000), 1–31. MR 2001e:22011 Zbl 0948.43004

Received March 26, 2013. Revised January 8, 2014.

HUBERT RUBENTHALER
INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE
UNIVERSITÉ DE STRASBOURG ET CNRS
7 RUE RENÉ DESCARTES
67084 STRASBOURG CEDEX
FRANCE

rubenth@math.unistra.fr

CONTENTS

Volume 270, no. 1 and no. 2

Guidelines for Authors

Authors may submit manuscripts at msp.berkeley.edu/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 270 No. 2 August 2014

0030-8730(201408)270:2;1-2