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DISJOINTIFICATION INEQUALITIES IN SYMMETRIC QUASI-BANACH SPACES AND THEIR APPLICATIONS

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We demonstrate the relevance of the Prokhorov inequality to the study of Khintchine-type inequalities in symmetric function spaces. Our main result shows that the latter inequalities hold for a pair of quasi-Banach symmetric function spaces X and Y if and only if the Kruglov operator K acts from X to Y. We also obtain an extension of von Bahr–Esseen and Esseen–Janson L_p-estimates for sums of independent mean zero random variables to the class of quasi-Banach symmetric spaces. In particular, in contrast to the well-known Esseen–Janson theorem, we do not assume that the summands are equidistributed.

1. Introduction

The classical Khintchine inequality [1923] describes the span of independent centered $\{\pm 1\}$-valued Bernoulli random variables in quasi-Banach L_p-spaces. A particular case of the latter sequence is given by the Rademacher functions $r_n(t) := \operatorname{sgn} \sin(2^n \pi t)$, $t \in [0, 1)$, $n \geq 1$. In this case, for all $p \in (0, \infty)$ the sequence $\{r_n\}_{n=1}^{\infty}$ in the L_p-spaces on the interval $(0, 1)$ (equipped with Lebesgue measure m) is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of l_2. A famous extension of this inequality to a more general case of random variables was given later by Marcinkiewicz and Zygmund (see [1937, Theorem 13, p. 87] and [1938, Theorem 5, p. 109]): for every $1 \leq p < \infty$ there are constants $A_p > 0$ and $B_p > 0$ such that for any $n \in \mathbb{N}$ and for an arbitrary sequence of independent mean zero random variables $(f_k)_{k \in \mathbb{N}}$ from $L_p(0, 1)$ we have

$$(1) \qquad A_p \left\| \left(\sum_{k=1}^{n} f_k^2 \right)^{1/2} \right\|_p \leq \left\| \sum_{k=1}^{n} f_k \right\|_p \leq B_p \left\| \left(\sum_{k=1}^{n} f_k^2 \right)^{1/2} \right\|_p .$$

In the special setting of Banach symmetric function spaces Johnson and Schechtman [1988] proved a far reaching generalization of the Marcinkiewicz–Zygmund inequality (1). More precisely, they established that if such a space X is either separable or has the Fatou property (for the relevant definitions see the following

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section) and the lower Boyd index of X is strictly positive, then (1) holds (even for a more general case of martingale differences). Later on, Astashkin [2008] showed that inequality (1) holds in a Banach symmetric space X if and only if X satisfies the so-called Kruglov property. The latter property, introduced by Braverman [1994], has its origin in a remarkable result due to Rosenthal [1970] that for sequences $\{f_n\}_{n=1}^{\infty}$ of independent mean zero random variables in $L_p(0,1)$, $p \geq 2$, the mapping $f_n \to f_n(t-n+1)\chi_{[n-1,n)}(t)$, $t \in \mathbb{R}$, extends to an isomorphism between the closed linear span $[f_n]_{n=1}^{\infty}$ (taken in $L_p(0,1)$) and the closed linear span $[f_n(t-n+1)\chi_{[n-1,n)}]_{n=1}^{\infty}$ (taken in $L_p(0,\infty) \cap L_2(0,\infty)$). The main focus of the present paper is to establish optimal conditions on a *quasinormed* symmetric function space in which inequalities of the type (1) hold. Our techniques are centered around the so-called Kruglov operator, a natural generalization of the Kruglov property, which was introduced in [Astashkin and Sukochev 2005] (see also [Astashkin and Sukochev 2010]). The usage of this operator allows us to make a straightforward connection between sums of independent random variables and their disjoint translates. Another major ingredient of our approach consists in utilizing Prokhorov's famous inequality [1959] (see also Theorem 17 below) which allows us to treat the problem in the full generality.

Using our present method, we also provide a far-reaching extension of the well-known von Bahr–Esseen and Esseen–Janson L_p-estimates for sums of independent mean zero random variables (see [von Bahr and Esseen 1965] and [Esseen and Janson 1985]). We extend inequalities of such type to the class of quasi-Banach symmetric spaces, and, at the same time, we do not assume that the summands are equally distributed (which is in strong contrast with Esseen and Janson's approach [1985, Theorem 4]). Note that earlier, Braverman [1994, § II.2] generalized the von Bahr–Esseen inequality to (Banach) symmetric spaces with the Kruglov property.

2. Preliminaries

2.1. ***Quasi-Banach spaces.*** Let X be a linear space over the field of real numbers $\mathbb{R}$. A function $\|\cdot\|_X : X \to \mathbb{R}$ is called a *quasinorm* if the following conditions hold:

(a) $\|x+y\|_X \leq C(\|x\|_X + \|y\|_X)$ for every $x, y \in X$ and some constant $C > 0$.

(b) $\|cx\|_X = |c| \cdot \|x\|_X$ for every $x \in X$ and $c \in \mathbb{R}$.

(c) $\|x\|_X \geq 0$. Moreover, $\|x\|_X = 0$ if and only if $x = 0$.

The least of all constants C satisfying condition (a) above is called *the modulus of concavity* of the quasinorm $\|\cdot\|_X$ and is denoted by $C(X)$.

If X is a linear space over $\mathbb{R}$ and if $\|\cdot\|_X : X \to \mathbb{R}$ is a quasinorm, then $X = (X, \|\cdot\|_X)$ is called a *quasinormed space*. If every Cauchy sequence in a quasinormed space X converges, then X is called a *quasi-Banach* space.

For example, $L_p(0,1)$ and $L_p(0,\infty)$, $0<p<1$, are quasi-Banach spaces with modulus of concavity $C(p)=C(L_p)=2^{1/p-1}$.

Recall that a quasinorm $\|\cdot\|_X$ in X is said to be a *p-norm*, $0<p<1$, if for any $x_1,x_2\in X$ we have

$$\|x_1+x_2\|_X^p \le \|x_1\|_X^p+\|x_2\|_X^p.$$

By the Aoki–Rolewicz theorem [Kalton et al. 1984], for any quasinorm $\|\cdot\|_X$ there exists $0<p<1$ such that $\|\cdot\|_X$ is a p-norm.

2.2. ***Symmetric function spaces.*** We are interested in those quasi-Banach spaces which consist of Lebesgue-measurable functions either on $(0,1)$ or on $(0,\infty)$.

For a Lebesgue-measurable, a.e. finite function x on $(0,1)$ (or $(0,\infty)$) we define its *distribution function* by

$$d_x(s):=m(\{t:\ x(t)>s\}),\quad s\in\mathbb{R},$$

where m stands for Lebesgue measure. Let $S(0,1)$ (respectively, $S(0,\infty)$) denote the space of all Lebesgue-measurable functions x on $(0,1)$ (respectively, on $(0,\infty)$ with $d_{|x|}(s)<\infty$ for sufficiently large s).

Two measurable functions x and y are called *equimeasurable* (written $x\sim y$) if their distribution functions d_x and d_y coincide. In particular, for every measurable function x, the function $|x|$ is equimeasurable with its *decreasing rearrangement* x^*, defined by the formula

$$x^*(t):=\inf\{\tau\ge 0: d_{|x|}(\tau)<t\},\quad t>0.$$

If $x,y\ge 0$, then $x^*=y^*$ if and only if x and y are equimeasurable. We recall that a function x is said to be symmetrically distributed if x and $-x$ are equimeasurable.

As it is traditional in probability theory, we denote by ϕ_x the characteristic function of an element $x\in S(0,1)$; that is, $\phi_x(t)=\int_0^1 e^{itx(s)}\,ds$. Recall that functions $x,y\in S(0,1)$ are equimeasurable if and only if their characteristic functions ϕ_x and ϕ_y coincide.

Definition 1. Let $X\subset S(0,1)$ (or $X\subset S(0,\infty)$) be a quasi-Banach space.

(a) X is said to be a quasi-Banach function space if, from $x\in X$, $y\in S(0,1)$ (or $y\in S(0,\infty)$) and $|y|\le|x|$, it follows that $y\in X$ and $\|y\|_X\le\|x\|_X$.

(b) A quasi-Banach function space X is said to be symmetric if, for every $x\in X$ and any measurable function y, the assumption $y^*=x^*$ implies that $y\in X$ and $\|y\|_X=\|x\|_X$.

Without loss of generality, in what follows we assume that $\|\chi_{(0,1)}\|_X=1$, where χ_E denotes the indicator function of a Lebesgue measurable set E.

The following assertion is well known in the Banach-space setting (see, for instance, [Lindenstrauss and Tzafriri 1979, Proposition 1.d.2]). For the reader's convenience, we provide a short proof.

Lemma 2. *Let X be a quasi-Banach function space. If $0 \le x$ and $y \in X$, then* $\|(xy)^{1/2}\|_X \le C(X)\|x\|_X^{1/2}\|y\|_X^{1/2}$.

Proof. It is easy to see that

$$(xy)^{1/2} \le \tfrac{1}{2}(\theta x + \theta^{-1}y), \quad \theta > 0,$$

and, therefore,

$$\|(xy)^{1/2}\|_X \le \frac{C(X)}{2}(\theta\|x\|_X + \theta^{-1}\|y\|_X).$$

Taking the infimum over all $\theta > 0$, we infer

$$\|(xy)^{1/2}\|_X \le C(X)\|x\|_X^{1/2}\|y\|_X^{1/2}. \qquad \square$$

Let X be a quasi-Banach symmetric function space and let $x_n \in X$, $n \in \mathbb{N}$, be such that $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$ and $x_n \to x$ almost everywhere. If, for every such sequence, we have $x \in X$ and $\|x\|_X \le \liminf_{n\to\infty} \|x_n\|_X$, then X is said to satisfy the *Fatou property*.

Suppose that X is a separable quasi-Banach symmetric space on $(0, 1)$. Denote by $\overline{X}$ the set of all $x \in S(0, 1)$ such that $\lim_{a\to+\infty} \| [|x|]_a\|_X < \infty$, where $[|x|]_a := |x|$ if $|x| < a$ and $[|x|]_a := 0$ if $|x| \ge a$. The set $\overline{X}$, equipped with the norm $\|x\|_{\overline{X}} := \lim_{a\to+\infty} \| [|x|]_a\|_X$, becomes a quasi-Banach symmetric space with the Fatou property. Moreover, X embeds isometrically into $\overline{X}$. It can be easily checked that for every quasi-Banach symmetric space X on $(0, 1)$ the continuous embedding $X \supset L_\infty(0, 1)$ holds. Then, the closure of $L_\infty(0, 1)$ in X, denoted by X_0, is a separable quasi-Banach symmetric space with the norm $\|\cdot\|_X$ whenever $X \ne L_\infty(0, 1)$.

If $\tau > 0$, the dilation operator σ_τ is defined by setting $\sigma_\tau x(s) = x(s/\tau)$, $s > 0$, in the case of the semiaxis. In the case of the interval $(0, 1)$, the operator σ_τ is defined by

$$\sigma_\tau x(s) := \begin{cases} x(s/\tau) & \text{if } s \le \min\{1, \tau\}, \\ 0 & \text{if } \tau < s \le 1. \end{cases}$$

Below we shall often consider the probability product space

$$(\Omega, \mathbb{P}) := \prod_{k=0}^{\infty} ((0, 1), m_k),$$

(m_k is the Lebesgue measure on $(0, 1)$, $k \ge 0$). Observe that in an arbitrary symmetric space the norms of any two elements with identical distribution coincide.

Hence, using a one-to-one measure-preserving transformation between measure spaces $(\Omega, \mathbb{P})$ and $((0,1), m)$, we will identify an arbitrary measurable function $x(\omega) = x(\omega_0, \omega_1, \ldots, \omega_n, \ldots)$ on $(\Omega, \mathbb{P})$ with the corresponding element from $S(0,1)$. Since a particular form of the measure-preserving transformation used in such identification is not important, we completely suppress it from the notations. Thus, we will view the set Ω as $(0,1)$ and any measurable function on $(\Omega, \mathbb{P})$ as a function from $S(0,1)$ and vice versa. A reader interested in more details of such identification is referred to [Astashkin and Sukochev 2010].

Let x_k, $k \geq 0$, be elements from $S(0,1)$ and let $y_k \in S(0,\infty)$, $k \geq 0$, be their disjoint copies; that is, $x_k \sim y_k$ for all $k \geq 0$, and $y_l y_m = 0$ if $l \neq m$. For the function $\sum_{k\geq 0} y_k$, which is frequently called the *disjoint sum* of x_k, $k \geq 0$, we shall use the suggestive notation $\bigoplus_{k\geq 0} x_k$. It is important to observe that the distribution function of a disjoint sum $\bigoplus_{k\geq 0} x_k$ does not depend on the particular choice of elements y_k, $k \geq 0$. In the special case when $\sum_{k=1}^{n} m(\operatorname{supp}(x_k)) \leq 1$, $n \in \mathbb{N}$, it is convenient to view the sum $\bigoplus_{k\geq 0} x_k$ as a measurable function on $(0,1)$.

The following useful construction was introduced in [Johnson et al. 1979] (see also [Lindenstrauss and Tzafriri 1979, 2.f]). If X is a quasi-Banach symmetric function space on $(0,1)$ and $0 < p \leq \infty$, then the set Z_X^p consists of all $f \in S(0,\infty)$ such that

$$\|f\|_{Z_X^p} := \|f^* \chi_{(0,1)}\|_X + \|\min\{f^*, f^*(1)\}\|_p < \infty.$$

It can be easily checked that the functional $\|\cdot\|_{Z_X^p}$ is a quasinorm on Z_X^p.

2.3. ***Kruglov operator and Kruglov property.*** The Kruglov property was introduced by Braverman [1994] when he compared sums of independent functions with sums of their disjoint copies in (Banach) symmetric spaces. Such terminology stems from related probabilistic constructions, due to Kruglov [1970], used in the study of infinitely divisible distributions (e.g., in analysis of the classical Levy–Khintchine formula).

Let $x \in S(0,1)$. By $\pi(x)$ we denote the random variable $\sum_{i=1}^{N} x_i$, where x_i, $i = 1, \ldots, N$, are independent copies of x and N is a random variable having Poisson distribution with parameter 1 and independent with respect to the sequence $\{x_i\}$.

Definition 3. A quasi-Banach symmetric space X on $(0,1)$ is said to have the *Kruglov property* ($X \in \mathbb{K}$) if from $x \in X$ it follows that $\pi(x) \in X$.

Simplifying the situation, the Kruglov property holds for spaces sufficiently "remote" from the space $L_\infty(0,1)$. For example, if a symmetric Banach function space X contains $L_p(0,1)$ for some $p < \infty$, then X possesses the Kruglov property (see, e.g., [Braverman 1994, Theorem 1.2] or [Astashkin and Sukochev 2010]). For a more precise characterization of various classes of (Banach) symmetric function spaces possessing the Kruglov property, we refer the reader to [Astashkin and Sukochev 2005; 2007; 2010; Braverman 1994].

Now, we recall the definition of the Kruglov operator, which can be viewed as a natural generalization of the notion of the Kruglov property. Let $\{B_n\}_{n=0}^{\infty}$ be a fixed sequence of mutually disjoint measurable subsets of $(0,1)$ such that $m(B_n) = 1/(en!)$. Define the operator $K : S(0,1) \to S(0,1)$ by setting

$$Kx(\omega) := \sum_{n=1}^{\infty} \sum_{k=1}^{n} x(\omega_k) \chi_{B_n}(\omega_0).$$

It is not difficult to see that

$$\phi_{Kx}(t) = \phi_{\pi(x)}(t) = \exp(\phi_x(t) - 1), \quad t \in \mathbb{R}. \tag{2}$$

Therefore, by the definition of the Kruglov property, a quasi-Banach symmetric function space X has the Kruglov property if and only if the operator K acts boundedly in X. Though the following crucial theorem originated in [Astashkin and Sukochev 2005], the first explicit statement (with a proof) appeared in [Astashkin et al. 2011].

Theorem 4. *If a sequence $\{x_k\}_{k=1}^n \subset S(0,1)$, $n \in \mathbb{N}$, consists of disjointly supported functions, then the sequence $\{Kx_k\}_{k=1}^n$ consists of independent functions.*

We will need also the following assertion, which is an immediate consequence of [Astashkin and Sukochev 2010, Theorem 15].

Theorem 5. *If X is a separable quasi-Banach symmetric space on $(0,1)$ such that $K : \overline{X} \to \overline{X}$, then $K : X \to X$ and $\|K\|_{X\to X} = \|K\|_{\overline{X}\to\overline{X}}$.*

3. Disjointification inequalities for positive functions

We will use the following approximation to the function Kx, where x is an arbitrary measurable function on the interval $(0,1)$. For every $n \in \mathbb{N}$ define the operator $H_n : S(0,1) \to S(0,1)$ by the formula

$$H_n x(\omega) := \sum_{k=1}^{n} (\sigma_{1/n} x)(\omega_k). \tag{3}$$

The following result is well known (see the proof of Lemma 1.6 in [Braverman 1994] or of Theorem 22 in [Astashkin and Sukochev 2010]). However, we present its proof for the reader's convenience.

Lemma 6. *The sequence of functions $\{H_n x\}_{n=1}^{\infty}$ converges to the function Kx in distribution.*

Proof. It is not difficult to see that $\phi_{H_n x} = \phi^n_{\sigma_{1/n}x}$. On the other hand,

$$\phi_{\sigma_{1/n}x}(t) = \int_0^1 e^{it\sigma_{1/n}x(s)}\,ds = \left(1 - \frac{1}{n}\right) + \frac{1}{n}\phi_x(t).$$

Therefore, by (2), we obtain

$$\phi_{H_n x} = \left(1 + \frac{\phi_x - 1}{n}\right)^n \to \exp(\phi_x - 1) = \phi_{Kx}.$$

Since the convergence of distributions follows from the convergence of characteristic functions [Borovkov 1998, Theorem 6.2.1], the result follows. ☐

Theorem 7. *Let X and Y be quasi-Banach symmetric spaces on* $(0,1)$ *and let Y have the Fatou property. Suppose that there exists a positive constant $C > 0$ such that for every sequence of nonnegative independent functions $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, with $\sum_{k=1}^n m(\mathrm{supp}(x_k)) \le 1$, we have*

$$\left\| \sum_{k=1}^n x_k \right\|_Y \le C \cdot \left\| \bigoplus_{k=1}^n x_k \right\|_X. \tag{4}$$

Then the operator K maps X into Y and $\|K\|_{X\to Y} \le C$.

The assertion remains valid under the assumption that the inequality (4) *holds for $X = Y$, where X is a separable quasi-Banach symmetric space.*

Proof. For every $x \in X$, let us define $x_k(\omega) = (\sigma_{1/n}x)(\omega_k)$, $\omega \in \Omega$. It follows from the definition of disjoint sum that

$$\bigoplus_{k=1}^n x_k \sim x \quad \text{for every } n \in \mathbb{N}.$$

Therefore, applying (3) and (4), we obtain $\|H_n x\|_F \le C\|x\|_E$. Furthermore, by Lemma 6, the sequence $\{H_n x\}_{n\ge 1}$ converges to the function Kx in distribution when $n \to \infty$ and hence $(H_n x)^* \to (Kx)^*$ almost everywhere on $(0,1)$. Since Y has the Fatou property, it follows that $Kx \in Y$ and $\|Kx\|_Y \le C\|x\|_X$.

Suppose now that X is a separable quasi-Banach symmetric space such that (4) holds for every sequence of nonnegative independent functions $\{x_k\}_{k=1}^n \subset X$ such that $\sum_{k=1}^n m(\mathrm{supp}(x_k)) \le 1$, $n \in \mathbb{N}$. From the definition of the space $\overline{X}$ (see Section 2), it follows that a similar inequality with the same constant C holds also for every sequence of nonnegative independent functions $\{x_k\}_{k=1}^n \subset \overline{X}$ with $\sum_{k=1}^n m(\mathrm{supp}(x_k)) \le 1$, $n \in \mathbb{N}$. Therefore, since $\overline{X}$ has the Fatou property, by the first part of theorem, we conclude that $K : \overline{X} \to \overline{X}$ and $\|K\|_{\overline{X}\to\overline{X}} \le C$. An application of Theorem 5 completes the proof. ☐

Our next purpose is to establish the main result of this section (Theorem 16), which is in a sense converse to the assertion of the preceding theorem. The first step in its proof is Proposition 9 below. We also need some preparatory results.

Lemma 8. *For every positive $x \in S(0,1)$, we have $\sigma_{1/2}x^* \le (Kx)^*$.*

Proof. Let B_n, $n \ge 1$, be the sets from the definition of the Kruglov operator K. Since the B_n are pairwise disjoint and

$$\sum_{n=1}^{\infty} m(B_n) = \frac{e-1}{e} > \frac{1}{2},$$

we may select a measurable set $B \subset \bigcup_{n\ge1} B_n$ such that $m(B) = 1/2$. It is clear that $(Kx)(\omega) \ge x(\omega_1)\chi_B(\omega_0)$ for every $\omega \in \Omega$. Since the function $x(\omega_1)\chi_B(\omega_0)$ is equimeasurable with the function $\sigma_{1/2}x^*$, the assertion follows immediately. $\square$

Proposition 9. *Suppose that the operator K maps boundedly X into Y, where X and Y are quasi-Banach symmetric spaces on $(0,1)$. If $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, is a sequence of independent functions from X and if $\sum_{k=1}^n m(\mathrm{supp}(x_k)) \le 1$, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \le 2C(Y)\|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^n x_k\Big\|_X.$$

Proof. Without loss of generality, it may be assumed that $x_k \ge 0$, $1 \le k \le n$. Let $y_k \in S(0,1)$ be pairwise disjoint copies of x_k, $1 \le k \le n$. By Theorem 4, the sequence $\{Ky_k\}_{k=1}^n$ consists of independent functions. Observing that $K(\bigoplus_{k=1}^n x_k)$ is equimeasurable with $\sum_{k=1}^n Ky_k$, and the latter is equimeasurable with the function $\sum_{k=1}^n (Kx_k)^*(\omega_k)$, we arrive at

$$\Big\|\sum_{k=1}^n (Kx_k)^*(\omega_k)\Big\|_Y = \Big\|\sum_{k=1}^n Ky_k\Big\|_Y \le \|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^n x_k\Big\|_X.$$

By Lemma 8, we have

$$\sum_{k=1}^n (\sigma_{1/2}x_k^*)(\omega_k) \le \sum_{k=1}^n (Kx_k)^*(\omega_k),$$

and, therefore,

$$\Big\|\sum_{k=1}^n (\sigma_{1/2}x_k^*)(\omega_k)\Big\|_Y \le \|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^n x_k\Big\|_X. \tag{5}$$

For an arbitrary $k \in \mathbb{N}$, let $x_k^{(1)}$ and $x_k^{(2)}$ be disjointly supported elements of $S(0,1)$ equimeasurable with the function $\sigma_{1/2}x_k^*$. A moment's reflection shows that the sum $x_k^{(1)} + x_k^{(2)}$ is equimeasurable with the function x_k^*, $k \in \mathbb{N}$. Hence, the function $\sum_{k=1}^{n} x_k$ is equimeasurable with the sum $y_0 + y_1$, where

$$y_i(\omega) := \sum_{k=1}^{n} x_k^{(i)}(\omega_k), \quad i = 0, 1,$$

which immediately implies

$$\left\| \sum_{k=1}^{n} x_k \right\|_Y = \|y_0 + y_1\|_Y \le C(Y)(\|y_0\|_Y + \|y_1\|_Y) \le 2C(Y) \left\| \sum_{k=1}^{n} \sigma_{1/2} x_k^*(\omega_k) \right\|_Y.$$

The assertion follows now from inequality (5). □

Our next objective is to omit the assumption $\sum_{k=1}^{n} m(\operatorname{supp}(x_k)) \le 1$. The main step is a disjointification inequality for bounded functions obtained below in Proposition 14. Let us start with some technical lemmas.

Lemma 10. *Let*

$$s_k := \sum_{n=k}^{\infty} \frac{1}{e \cdot n!}, \quad k \in \mathbb{N}.$$

Then $4ks_{k+1} \ge s_k$ *for every* $k \in \mathbb{N}$.

Proof. Clearly,

$$4ks_{k+1} \ge \frac{(k+1)^2}{k} s_{k+1} \ge \frac{(k+1)^2}{k} \cdot \frac{1}{e \cdot (k+1)!} = \frac{k+1}{k} \cdot \frac{1}{e \cdot k!}.$$

On the other hand, since $k! \cdot (k+1)^n \le (k+n)!$, we have that

$$\begin{aligned} \frac{k+1}{k} \cdot \frac{1}{e \cdot k!} &= \frac{1}{e \cdot k!} \cdot \frac{1}{1 - 1/(k+1)} \\ &= \frac{1}{e \cdot k!} \left(1 + \frac{1}{k+1} + \frac{1}{(k+1)^2} + \cdots \right) \ge \sum_{n=k}^{\infty} \frac{1}{e \cdot n!}. \end{aligned}$$ □

By the definition of the Kruglov operator, the function $K\chi_{[0,1]}$ has the Poisson distribution with parameter 1. Let

$$\psi_0(t) := \int_0^t (K\chi_{[0,1]})^*(s)\, ds.$$

It is clear that $K : L_\infty(0,1) \to M_{\psi_0}$ and $\|K\|_{L_\infty \to M_{\psi_0}} = 1$. Here M_{ψ_0} is the Marcinkiewicz space consisting of all elements $x \in S(0,1)$ such that

$$\|x\|_{M_{\psi_0}} := \sup_{0 < t \le 1} \frac{\int_0^t x^*(s)\, ds}{\psi_0(t)} < \infty.$$

Lemma 11. *The following inequality holds*:

$$\inf_{0 < t < 1 - 1/e} \frac{t\psi_0'(t)}{\psi_0(t)} \ge \frac{1}{4}.$$

Proof. Let s_k be as in Lemma 10. Since $\psi_0' = (K\chi_{[0,1]})^*$ is a Poisson random variable with parameter 1, it follows that

$$\psi_0'(t) = k \quad \text{for all } t \in (s_{k+1}, s_k),\ k \in \mathbb{N}.$$

Therefore,

$$\psi_0(s_{k+1}) = \int_0^{s_{k+1}} \psi_0'(t)\,dt = \sum_{n=k+1}^{\infty} \frac{n}{e \cdot n!} = \sum_{n=k}^{\infty} \frac{1}{e \cdot n!} = s_k, \quad k \in \mathbb{N}.$$

Now, let $0 < t < 1 - 1/e$. Then $t \in [s_{k+1}, s_k)$ for some $k \geq 1$, and so $\psi_0'(t) = k$. Since ψ_0 is concave, the function $t/\psi_0(t)$ increases. Therefore, by Lemma 10,

$$\frac{t\psi_0'(t)}{\psi_0(t)} = \frac{kt}{\psi_0(t)} \geq \frac{ks_{k+1}}{\psi_0(s_{k+1})} = \frac{ks_{k+1}}{s_k} \geq \frac{1}{4}. \qquad \square$$

Lemma 12. *If Y is a quasi-Banach symmetric space on* $(0,1)$ *such that the operator K maps $L_\infty(0,1)$ into Y, then $Y \supset M_{\psi_0}$ and*

$$\|x\|_Y \leq 8\,C(Y)\,\|x\|_{M_{\psi_0}} \cdot \|K\|_{L_\infty \to Y}, \quad x \in M_{\psi_0}.$$

Proof. It follows from Lemma 11 that

$$\begin{aligned}\|x\|_{M_{\psi_0}} &= \sup_{0<t\leq 1}\left(\frac{1}{\psi_0(t)}\int_0^t x^*(s)\,ds\right) \geq \sup_{0<t<1/2}\left(\frac{tx^*(t)}{\psi_0(t)}\right) \\ &\geq \inf_{0<t<1/2}\left(\frac{t\psi_0'(t)}{\psi_0(t)}\right) \cdot \sup_{0<t<1/2}\left(\frac{x^*(t)}{\psi_0'(t)}\right) \geq \frac{1}{4}\sup_{0<t<1/2}\left(\frac{x^*(t)}{\psi_0'(t)}\right).\end{aligned}$$

Therefore,

$$x^*(t) \leq 4\|x\|_{M_{\psi_0}}\psi_0'(t), \quad 0 < t \leq \tfrac{1}{2},$$

whence

$$x^*(t) \leq \sigma_2 x^*(t) \leq 4\|x\|_{M_{\psi_0}}\sigma_2\psi_0'(t), \quad 0 < t \leq 1.$$

Combining the last inequality with the obvious equalities

$$\|K\|_{L_\infty \to Y} = \|K\chi_{[0,1]}\|_Y = \|\psi_0'\|_Y,$$

we obtain

$$\|x\|_Y \leq \|\sigma_2 x^*\|_Y \leq 4\|x\|_{M_{\psi_0}}\|\sigma_2\psi_0'\|_Y \leq 8\,C(Y)\|x\|_{M_{\psi_0}}\|K\|_{L_\infty \to Y}. \quad \square$$

In the following lemma, we use the classical notion of majorization. Let $0 \leq x, y \in L_1(0,1)$. We write $y \prec x$ if $\int_0^t y^*(s)\,ds \leq \int_0^t x^*(s)\,ds$ for all $t \in (0,1)$ and $\int_0^1 y^*(s)\,ds = \int_0^1 x^*(s)\,ds$.

Lemma 13. *Let* $\{x_k\}_{k=1}^n$ *and* $\{y_k\}_{k=1}^n$, $n \in \mathbb{N}$, *be sequences of positive and independent functions from* $L_1(0,1)$. *If* $y_k \prec x_k$ *for each* k, *then*

$$\sum_{k=1}^n y_k \prec \sum_{k=1}^n x_k.$$

Proof. Define the functions $x, y \in L_1(0,1)$ by setting

$$x(\omega) := \sum_{k=1}^n x_k(\omega_k), \quad y(\omega) := \sum_{k=1}^n y_k(\omega_k).$$

It follows from the assumption that for every $1 \le k \le n$ there exists a bistochastic operator A_k (on $L_1(0,1)$) such that $A_k x_k = y_k$ [Bennett and Sharpley 1988, Proposition 3.2.9]. A moment's reflection shows that the operator $A := \bigotimes_{k=1}^n A_k$ is a bistochastic operator on $L_1(\Omega, \mathbb{P})$ (which we identify with $L_1(0,1)$) and that $Ax = \sum_{k=1}^n A_k x_k(\omega_k)$. Applying Proposition 3.2.4 of the same reference, we arrive at

$$y = \sum_{k=1}^n A_k x_k(\omega_k) = Ax \prec x.$$

Since $\sum_{k=1}^n x_k$ (respectively, $\sum_{k=1}^n y_k$) is equimeasurable with x (respectively, y), the assertion follows. ☐

Proposition 14. *If* $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, *is a sequence of bounded independent functions, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_{M_{\psi_0}} \le 2\Big\|\bigoplus_{k=1}^n x_k\Big\|_{L_1 \cap L_\infty(0,\infty)}.$$

Proof. Without loss of generality, we can assume that $x_k \ge 0$ for $1 \le k \le n$. Suppose that

$$\Big\|\bigoplus_{k=1}^n x_k\Big\|_\infty = 1 \quad \text{and} \quad \|x_k\|_1 = \alpha_k.$$

If $\alpha = \sum_{k=1}^n \alpha_k > 1$, then $x_k \prec \alpha \chi_{[0,\alpha^{-1}\alpha_k]}$ for $1 \le k \le n$. Applying Lemma 13, we obtain

$$\sum_{k=1}^n x_k \prec \alpha \sum_{k=1}^n \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k).$$

From the definition of the norm of a Marcinkiewicz space, Proposition 9 and the equalities $\|K\|_{L_\infty \to M_{\psi_0}} = 1$ and $C(M_{\psi_0}) = 1$, we obtain

$$\Big\|\sum_{k=1}^n x_k\Big\|_{M_{\psi_0}} \le \alpha\Big\|\sum_{k=1}^n \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k)\Big\|_{M_{\psi_0}}$$

$$\le 2\alpha\Big\|\bigoplus_{k=1}^n \chi_{[0,\alpha^{-1}\alpha_k]}\Big\|_\infty = 2\Big\|\bigoplus_{k=1}^n x_k\Big\|_{L_1(0,\infty)}. \tag{6}$$

If $\alpha=\sum_{k=1}^n\alpha_k<1$, then $x_k\prec\chi_{[0,\alpha_k]}$ for $1\le k\le n$. It follows from Lemma 13 that

$$\sum_{k=1}^n x_k \prec \sum_{k=1}^n \chi_{[0,\alpha_k]}(\omega_k).$$

Therefore, by Proposition 9, we have

$$\Big\|\sum_{k=1}^n x_k\Big\|_{M_{\psi_0}} \le \Big\|\sum_{k=1}^n \chi_{[0,\alpha_k]}(\omega_k)\Big\|_{M_{\psi_0}} \le 2\Big\|\bigoplus_{k=1}^n \chi_{[0,\alpha_k]}\Big\|_\infty = 2.$$

Combining this estimate with inequality (6), we are done. □

The following statement is an immediate consequence of Proposition 14 and Lemma 12.

Corollary 15. *Let Y be a quasi-Banach symmetric space on* $(0,1)$ *such that the operator K maps $L_\infty(0,1)$ into Y. If $\{x_k\}_{k=1}^n$, $n\in\mathbb{N}$, is a sequence of bounded and independent functions, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \le 16\,C(Y)\,\|K\|_{L_\infty\to Y}\Big\|\bigoplus_{k=1}^n x_k\Big\|_{L_1\cap L_\infty(0,\infty)}.$$

Now, we are ready to prove the main result of this section related to the comparison of sums of independent functions and their disjoint copies in quasi-Banach symmetric function spaces.

Theorem 16. *Let X and Y be quasi-Banach symmetric spaces on* $(0,1)$ *such that the operator K acts boundedly from X into Y. If $\{x_k\}_{k=1}^n\subset X$, $n\in\mathbb{N}$ is a sequence of independent functions, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \le 16\,C^2(Y)\,\|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^n x_k\Big\|_{Z_X^1}. \tag{7}$$

Proof. Let us write x for $\bigoplus_{k=1}^n x_k$. Define the functions

$$x_{k,1}:=x_k\chi_{\{|x_k|>x^*(1)\}},\quad x_{k,2}:=x_k-x_{k,1},\quad 1\le k\le n.$$

The functions $x_{k,1}$, $1\le k\le n$, are independent, as are the functions $x_{k,2}$, $1\le k\le n$.

Moreover, it is easy to see that

$$\bigoplus_{k=1}^{n} |x_{k,1}| \sim x^* \chi_{(0,1)} \quad \text{and} \quad \bigoplus_{k=1}^{n} |x_{k,2}| \sim x^* \chi_{(1,\infty)}.$$

Since $L_\infty(0,1) \subset X$ and $\|x\|_X \le \|x\|_\infty$, $x \in L_\infty(0,1)$, it follows from the assumption of the theorem that $K : L_\infty(0,1) \to Y$ and $\|K\|_{L_\infty \to Y} \le \|K\|_{X \to Y}$. Therefore, applying Proposition 9 and Corollary 15, we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{n} x_k\Big\|_Y &\le C(Y)\bigg(\Big\|\sum_{k=1}^{n} x_{k,1}\Big\|_Y + \Big\|\sum_{k=1}^{n} x_{k,2}\Big\|_Y\bigg)\\ &\le 16\, C^2(Y)\|K\|_{X\to Y}\bigg(\Big\|\bigoplus_{k=1}^{n} x_{k,1}\Big\|_X + \Big\|\bigoplus_{k=1}^{n} x_{k,2}\Big\|_{L_1\cap L_\infty(0,\infty)}\bigg)\\ &\le 16\, C^2(Y)\|K\|_{X\to Y}(\|x^*\chi_{(0,1)}\|_X + \|\min\{x^*, x^*(1)\}\|_{L_1\cap L_\infty(0,\infty)}).\end{aligned}$$

□

4. Disjointification inequalities for symmetrically distributed (mean zero) functions

If we assume that the independent functions x_k, $1 \le k \le n$, in the statement of Theorem 16 are symmetrically distributed, then the disjointification inequality (7) can be significantly improved. In particular, we are able to extend estimates from [Astashkin and Sukochev 2007] for symmetric Banach function spaces to the quasi-Banach setting. Our main tool is the following remarkable inequality due to Prokhorov [1959], which we restate here using the direct sum notation.

Theorem 17. *If* $\{x_k\}_{k=1}^n$ $(n \in \mathbb{N})$ *is a sequence of bounded independent symmetrically distributed random variables on* $(0,1)$, *then for all* $t > 0$

$$m\bigg(\bigg\{\sum_{k=1}^{n} x_k > t\bigg\}\bigg) \le \exp\bigg(-\frac{t}{2\,\|\bigoplus_{k=1}^{n} x_k\|_\infty} \operatorname{arcsinh} \frac{t\,\|\bigoplus_{k=1}^{n} x_k\|_\infty}{2\,\|\bigoplus_{k=1}^{n} x_k\|_2^2}\bigg). \tag{8}$$

Let the function ψ_0 be as in the previous section.

Proposition 18. *If* $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, *is a sequence of bounded independent symmetrically distributed functions on* $(0,1)$, *then*

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_{M_{\psi_0}} \le C_{\mathrm{abs}} \Big\|\bigoplus_{k=1}^{n} x_k\Big\|_{L_2\cap L_\infty(0,\infty)},$$

for some absolute constant C_{abs}.

Proof. For every $m \geq 1$, we define a linear operator $A_m : L_2 \cap L_\infty(0,\infty) \to M_{\psi_0}$ by setting for $x \in L_2 \cap L_\infty(0,\infty)$

$$A_m x(\omega) := \sum_{k=1}^{m} x(k-1+\omega_{2k-1}) r(\omega_{2k}),$$

where $r(t) = 1$ if $0 \leq t \leq \frac{1}{2}$ and $r(t) = -1$ if $\frac{1}{2} < t \leq 1$. It is clear that

$$\|A_m\|_{L_2 \cap L_\infty \to M_{\psi_0}} \leq m, \quad m \in \mathbb{N}.$$

Our objective is to show that for every fixed $x \in L_2 \cap L_\infty(0,\infty)$ the orbit $\{A_m x\}_{m=1}^{\infty}$ is uniformly bounded in M_{ψ_0}. Provided we have done so, the uniform boundedness principle guarantees that the sequence $\{\|A_m\|_{L_2 \cap L_\infty \to M_\psi}\}_{m=1}^{\infty}$ is uniformly bounded, and the assertion of the theorem would follow from this fact since the sum $\sum_{k=1}^{n} x_k$ for a given sequence $\{x_k\}_{k=1}^{n}$ of bounded independent symmetrically distributed functions on $(0,1)$ is equidistributed with the function $A_n z$, where

$$z := \bigoplus_{k=1}^{n} x_k.$$

Fix $x \in L_2 \cap L_\infty(0,\infty)$, and set

$$\alpha(x) := \|x\|_\infty + \sup_n \frac{\|x\chi_{[0,n]}\|_2^2}{\|x\chi_{[0,n]}\|_\infty}$$

(here, $0/0$ is set to be 0). Clearly, $\alpha(x) < \infty$ and our objective would be achieved if we show that

$$\|A_m x\|_{M_{\psi_0}} \leq 4e \cdot \alpha(x) \quad \text{for all } m \in \mathbb{N}. \tag{9}$$

Fix $m \in \mathbb{N}$. Since

$$\Big(\bigoplus_{k=1}^{m} x(k-1+\omega_{2k-1}) r(\omega_{2k})\Big)^* = (x\chi_{[0,m]})^*,$$

it follows from (8) that for every $t > 0$, we have

$$m(\{|A_m x| > t\alpha(x)\}) \leq \exp\Big(-\frac{t\alpha(x)}{2\|x\chi_{[0,m]}\|_\infty} \operatorname{arcsinh} \frac{t\alpha(x)\|x\chi_{[0,m]}\|_\infty}{2\|x\chi_{[0,m]}\|_2^2}\Big).$$

Combining this estimate with the obvious inequalities

$$\frac{t\alpha(x)}{2\|x\chi_{[0,m]}\|_\infty} \geq \frac{t}{2}, \quad \operatorname{arcsinh} \frac{t\alpha(x)\|x\chi_{[0,m]}\|_\infty}{2\|x\chi_{[0,m]}\|_2^2} \geq \operatorname{arcsinh} \frac{t}{2},$$

we arrive at

$$m(\{|A_m x| > t\alpha(x)\}) \leq \exp\Big(-\frac{t}{2} \operatorname{arcsinh} \frac{t}{2}\Big). \tag{10}$$

The right-hand side of the preceding inequality is in fact directly related to the distribution function of the function ψ_0'. Indeed, in the proof of Lemma 11 we have already pointed out that $\psi_0' := (K\chi_{[0,1]})^*$ is a Poisson random variable with parameter 1. A direct calculation yields the estimate

$$m(\{\psi_0' > t\}) \geq \exp(-1 - 2t \cdot \operatorname{arcsinh}(2t)), \quad t > 0,$$

which, in turn, implies

$$m(\{4\psi_0' > t\}) \geq \exp\Big(-1 - \frac{t}{2} \operatorname{arcsinh} \frac{t}{2}\Big), \quad t > 0.$$

Combining this with (10), we infer

$$m(\{|A_m x| > t\alpha(x)\}) \leq e \cdot m(\{4\psi_0' > t\}).$$

Since $\|\psi_0'\|_{M(\psi_0)} = 1$, from the preceding estimate and [Braverman 1994, Proposition 1.2], inequality (9) follows. □

The corollary below follows from Proposition 18 and Lemma 12.

Corollary 19. *Let Y be a quasi-Banach symmetric space on* $(0, 1)$ *such that the operator K maps $L_\infty(0, 1)$ into Y. If $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, is a sequence of bounded independent symmetrically distributed functions, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \leq 8\, C_{\mathrm{abs}}\, C(Y)\, \|K\|_{L_\infty \to Y} \Big\|\bigoplus_{k=1}^n x_k\Big\|_{L_2 \cap L_\infty(0,\infty)}.$$

We need the following assertion proved by Braverman [1994, Proposition 1.11] in the Banach setting. The proof in the quasi-Banach setting is identical.

Lemma 20. *If a quasi-Banach symmetric space X on* $(0, 1)$ *embeds into* $L_1(0, 1)$, *then there exists a constant $C_0(X)$ such that*

$$\|x\|_X \leq C_0(X)\, \|x(\omega_1) - x(\omega_2)\|_X$$

for every mean zero function $x \in X$.

We are now ready to present the main result of this section.

Theorem 21. *Let X and Y be quasi-Banach symmetric spaces on* $(0, 1)$ *such that* $K : X \to Y$.

(a) *If $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, is a sequence of independent symmetrically distributed functions, then*

$$\Big\|\sum_{k=1}^n x_k\Big\|_Y \leq 8\, C_{\mathrm{abs}}\, C^2(Y)\, \|K\|_{X \to Y} \Big\|\bigoplus_{k=1}^n x_k\Big\|_{Z_X^2}. \tag{11}$$

(b) *If* $X \subset L_1(0,1)$, *then the inequality*

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_Y \le 16\, C_{\mathrm{abs}}\, C_0(Y)\, C^2(Y)\, C(X)\, \|K\|_{X\to Y} \Big\|\bigoplus_{k=1}^{n} x_k\Big\|_{Z_X^2} \tag{12}$$

holds for every sequence $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, *of independent mean zero functions from* X.

Proof. The proof of the first assertion is similar to the proof of Theorem 16, with the only difference being that the reference to Corollary 15 should be replaced with a reference to Corollary 19.

In the proof of the second assertion we use the standard symmetrization trick. Define the functions $y_k \in X$, $1 \le k \le n$, by setting

$$y_k(\omega) := x_k(\omega_{2k-1}) - x_k(\omega_{2k}).$$

By Lemma 20,

$$\Big\|\sum_{k=1}^{n} x_k\Big\|_Y \le C_0(Y)\Big\|\sum_{k=1}^{n} x_k(\omega_{2k-1}) - \sum_{k=1}^{n} x_k(\omega_{2k})\Big\|_Y = C_0(Y)\Big\|\sum_{k=1}^{n} y_k\Big\|_Y.$$

Evidently, y_k, $1 \le k \le n$, are independent and symmetrically distributed. Therefore, by (a), we obtain

$$\Big\|\sum_{k=1}^{n} y_k\Big\|_Y \le 8\, C_{\mathrm{abs}}\, C^2(Y)\, \|K\|_{X\to Y}\Big\|\bigoplus_{k=1}^{n} y_k\Big\|_{Z_X^2}.$$

Observing that for every $t > 0$, we have

$$m\Big(\Big\{\Big|\bigoplus_{k=1}^{n} y_k\Big| > t\Big\}\Big) \le 2m\Big(\Big\{s > 0 : \Big|\bigoplus_{k=1}^{n} x_k\Big| > t\Big\}\Big),$$

and appealing to the fact that Z_X^2 is a quasi-Banach symmetric space with modulus of concavity $C(X)$, we infer

$$\Big\|\bigoplus_{k=1}^{n} y_k\Big\|_{Z_X^2} \le 2\, C(X)\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_{Z_X^2}.$$

Combining these inequalities, we conclude the proof. □

5. Khintchine inequality in quasi-Banach spaces

In this section, we provide an extension of the classical Khintchine inequality to general quasi-Banach symmetric function spaces. We begin with the formulation of the main results of this section.

Theorem 22. *Let X and Y be quasi-Banach symmetric spaces on the interval* $(0, 1)$ *such that the operator K is bounded from X into Y. If $\{x_k\}_{k=1}^n$, $n \in \mathbb{N}$, is a sequence of independent symmetrically distributed random variables from X, then*

$$\Big\| \sum_{k=1}^n x_k \Big\|_Y \le 512\, C_{\text{abs}}\, C^6(X)\, C^2(Y)\, \|K\|_{X\to Y} \Big\| \Big(\sum_{k=1}^n x_k^2 \Big)^{1/2} \Big\|_X . \tag{13}$$

The next theorem shows that in the case when $X = Y$ the boundedness of the Kruglov operator is a necessary and sufficient condition for the inequalities of the type (13). In the Banach setting, an analogous result was earlier proved in [Astashkin 2008].

Theorem 23. *Let X be a quasi-Banach symmetric function space on* $(0, 1)$ *which is separable or has the Fatou property. The following conditions are equivalent*:

(a) *There is a constant $C > 0$ such that the inequality*

$$\Big\| \sum_{k=1}^n x_k \Big\|_X \le C \Big\| \Big(\sum_{k=1}^n x_k^2 \Big)^{1/2} \Big\|_X$$

holds for every sequence $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, of independent symmetrically distributed functions.

(b) $K : X \to X$.

For the proof we will need a series of lemmas. The first two of them are well known; however, we present their short proofs for the reader's convenience.

Lemma 24. *Let X be a quasi-Banach symmetric space on* $(0, 1)$. *If we set $p := \frac{1}{2}\log_2^{-1}(2\,C(X))$, then $X \subset L_p(0, 1)$ and*

$$\|x\|_p \le 8\, C^3(X)\, \|x\|_X, \quad x \in X.$$

Proof. Define an increasing function ψ on $(0, 1)$ by the formula $\psi(u) := \|\chi_{[0,u]}\|_X$, $0 < u < 1$. It follows from the definition of a quasinorm that

$$\psi(2u) \le 2\, C(X)\psi(u), \quad 0 < u \le 1,$$

whence

$$\psi(2^{-n}) \ge (2\, C(X))^{-n}, \quad n \ge 0.$$

If $u \in (0, 1]$ is arbitrary, then $u \in [2^{-n-1}, 2^{-n}]$ for some $n \ge 0$. Hence,

$$\psi(u) \ge \psi(2^{-n-1}) \ge 2^{-(n+1)\log_2(2C(X))} \ge \frac{1}{2\, C(X)}\, u^{\log_2(2C(X))}.$$

If $x \in X$, then for every $0 < t \le 1$ we have

$$\|x\|_X \ge \|x^*(t)\chi_{[0,t]}\|_X \ge x^*(t)\frac{1}{2\, C(X)}\, t^{\log_2(2C(X))}.$$

Hence,

$$x^*(t) \leq 2\,\|x\|_X\,C(X)\,t^{-\log_2(2C(X))}, \quad 0 < t \leq 1.$$

The assertion follows immediately. □

Lemma 25. *If* $0 < p < 1$ *and* $x, y \in L_1(0,1)$ *are positive*, *then from* $y \prec x$ *it follows that* $\|y\|_p \geq \|x\|_p$.

Proof. Fix $\varepsilon > 0$. Passing to step-function approximation, we easily infer that there exist $n \in \mathbb{N}$ and a function

$$z := \sum_{k=1}^{n} \lambda_k x_k \qquad \text{with} \quad x_k \geq 0,\ x_k^* = x^* \ \text{and} \ \sum_{k=1}^{n} \lambda_k = 1,\ \lambda_k \geq 0,$$

such that $\|y - z\|_1 \leq \varepsilon$. It follows now from the Minkowski inequality that

$$\|z\|_p = \Big\|\sum_{k=1}^{n} \lambda_k x_k\Big\|_p \geq \sum_{k=1}^{n} \lambda_k \|x_k\|_p = \|x\|_p.$$

Since $\varepsilon > 0$ is arbitrarily small and the quasinorm in $L_p(0,1)$, $0 < p < 1$, is continuous with respect to L_1-convergence, the proof is complete. □

Lemma 26. *Let* $0 < p < 1$ *and let* $\{y_k\}_{k=1}^n$, $n \in \mathbb{N}$, *be a sequence of positive bounded independent functions on* $(0,1)$. *We have*

$$\Big\|\bigoplus_{k=1}^{n} y_k\Big\|_1 \leq 2^{1/p} \max\Big\{\sup_{1\leq k\leq n} \|y_k\|_\infty, \Big\|\sum_{k=1}^{n} y_k\Big\|_p\Big\}. \tag{14}$$

Proof. Without loss of generality, we can assume that

$$\sup_{1\leq k\leq n} \|y_k\|_\infty = 1, \quad \|y_k\|_1 = \alpha_k, \quad 1 \leq k \leq n.$$

Let $\alpha = \sum_{k=1}^{n} \alpha_k$. If $\alpha \leq 1$, then the assertion is evident. If $\alpha \geq 1$, then

$$y_k \prec \alpha \chi_{[0,\alpha^{-1}\alpha_k]}, \quad 1 \leq k \leq n.$$

From Lemma 13 it follows that

$$\sum_{k=1}^{n} y_k \prec \alpha \sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k),$$

whence, according to Lemma 25, we have

$$\Big\|\sum_{k=1}^{n} y_k\Big\|_p \geq \alpha \Big\|\sum_{k=1}^{n} \chi_{[0,\alpha^{-1}\alpha_k]}(\omega_k)\Big\|_p.$$

Combining this inequality with [Johnson and Schechtman 1989, Lemma 3], we infer

$$2\,C(p)\Big\|\sum_{k=1}^{n} y_k\Big\|_p \geq \alpha\Big\|\bigoplus_{k=1}^{n}\chi_{[0,\alpha^{-1}\alpha_k]}\Big\|_p = \alpha.$$

Since $C(p) = 2^{1/p-1}$ for $0 < p < 1$, the assertion follows. □

Lemma 27. *Let X be a quasi-Banach symmetric space on* $(0,1)$. *If* $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, *is a sequence of bounded independent functions, then*

$$\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_2 \leq 32\,C^5(X)\max\Big\{\sup_{1\leq k\leq n}\|x_k\|_\infty,\ \Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X\Big\}.$$

Proof. If $p = \frac{1}{2}\log_2^{-1}(2\,C(X))$, then by Lemma 24 we have

$$8\,C^3(X)\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X \geq \Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_p = \Big\|\sum_{k=1}^{n} x_k^2\Big\|_{p/2}^{1/2}.$$

Clearly,

$$\Big\|\bigoplus_{k=1}^{n} x_k\Big\|_2 = \Big\|\bigoplus_{k=1}^{n} x_k^2\Big\|_1^{1/2} \quad\text{and}\quad \|x_k\|_\infty = \|x_k^2\|_\infty^{1/2}.$$

Now, applying Lemma 26 to the functions $y_k = x_k^2$, $1 \leq k \leq n$, we obtain the result. □

Lemma 28. *Let X be a quasi-Banach symmetric space on* $(0,1)$. *If* $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, *is a sequence of independent functions and if* $x := \bigoplus_{k=1}^n x_k$, *then*

$$2C(X)\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X \geq x^*(1).$$

Proof. A simple argument shows that it is sufficient to consider the case when

$$\sum_{k=1}^{n} m(\operatorname{supp}(x_k)) = 1. \tag{15}$$

Since $|x_k| \geq x^*(1)\chi_{\operatorname{supp}(x_k)}$, $1 \leq k \leq n$, we have

$$\sum_{k=1}^{n} x_k^2 \geq (x^*(1))^2\sum_{k=1}^{n}\chi_{\operatorname{supp}(x_k)}.$$

Since the functions x_k, $1 \le k \le n$, are independent, the support of the function at the right-hand side of the inequality above has Lebesgue measure equal to

$$1 - \prod_{k=1}^{n} \bigl(1 - m(\mathrm{supp}(x_k))\bigr),$$

which is bigger than $\frac{1}{2}$ (thanks to the condition (15) and to the arithmetic-geometric mean inequality). Therefore, since $\|\chi_{(0,1)}\|_X = 1$, we obtain

$$\Bigl\|\Bigl(\sum_{k=1}^{n} x_k^2\Bigr)^{1/2}\Bigr\|_X \ge x^*(1)\,\|\chi_{[0,1/2]}\|_X \ge \frac{x^*(1)}{2\,C(X)},$$

and the proof is complete. □

Let $1 \le p < \infty$ and let X be a quasi-Banach symmetric function space on $(0,1)$ or $(0,\infty)$. The *p-concavification* of X, $X^{1/p}$, is defined by

$$X^{1/p} := \{x \in S(0,1) \text{ (or } S(0,\infty)) : |x|^{1/p} \in X\}, \quad \|x\|_{X^{1/p}} := \|\,|x|^{1/p}\,\|_X^p.$$

Note that the space $X^{1/p}$, equipped with the quasinorm $\|\cdot\|_{X^{1/p}}$, is also a quasi-Banach symmetric function space (see, for instance, [Lindenstrauss and Tzafriri 1979]).

We are now ready to prove the main result of this section.

Proof of Theorem 22. Setting $x := \bigoplus_{k=1}^{n} x_k$, by Theorem 21, we have

$$\Bigl\|\sum_{k=1}^{n} x_k\Bigr\|_Y \le 8\,C_{\mathrm{abs}}\,C^2(Y)\,\|K\|_{X\to Y}\bigl(\|x^*\chi_{(0,1)}\|_X + \|x^*\chi_{(1,\infty)}\|_2\bigr). \tag{16}$$

Arguing in the same way as in the proof of Theorem 16, we can define two sequences of independent functions $\{x_{k,1}\}$ and $\{x_{k,2}\}$ such that $x_{1k} + x_{2k} = x_k$, $|x_{k,1}| \le |x_k|$, $|x_{k,2}| \le |x_k|$, for $1 \le k \le n$, and the disjoint sums $\bigoplus_{k=1}^{n} |x_{k,1}|$ and $\bigoplus_{k=1}^{n} |x_{k,2}|$ are equimeasurable with the functions $x^*\chi_{(0,1)}$ and $x^*\chi_{(1,\infty)}$, respectively. Applying Lemma 27 to the sequence $\{x_{k,2}\}_{k=1}^{n}$, we obtain

$$\begin{aligned}\|x^*\chi_{(1,\infty)}\|_2 &= \Bigl\|\bigoplus_{k=1}^{n} x_{k,2}\Bigr\|_2 \\ &\le 32\,C^5(X)\max\Bigl\{\sup_{1\le k\le n}\|x_{k,2}\|_\infty,\ \Bigl\|\Bigl(\sum_{k=1}^{n} x_{k,2}^2\Bigr)^{1/2}\Bigr\|_X\Bigr\}.\end{aligned}$$

Note that $\|x_{k,2}\|_\infty \le x^*(1)$ for $1 \le k \le n$. Using Lemma 28, we obtain

$$\|x^*\chi_{(1,\infty)}\|_2 \le 64\,C^6(X)\,\Bigl\|\Bigl(\sum_{k=1}^{n} x_k^2\Bigr)^{1/2}\Bigr\|_X. \tag{17}$$

On the other hand,

$$\|x^*\chi_{(0,1)}\|_X = \Big\|\bigoplus_{k=1}^{n} x_{k,1}\Big\|_X = \Big\|\bigoplus_{k=1}^{n} x_{k,1}^2\Big\|_{X^{1/2}}^{1/2},$$

and

$$\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X \geq \Big\|\Big(\sum_{k=1}^{n} x_{k,1}^2\Big)^{1/2}\Big\|_X = \Big\|\sum_{k=1}^{n} x_{k,1}^2\Big\|_{X^{1/2}}^{1/2}.$$

Applying [Johnson and Schechtman 1989, Lemma 3] to the space $X^{1/2}$ and the functions $x_{k,1}^2$, we obtain

$$\|x^*\chi_{(0,1)}\|_X \leq (2\,C(X^{1/2}))^{1/2}\Big\|\Big(\sum_{k=1}^{n} x_k^2\Big)^{1/2}\Big\|_X.$$

Since $C(X^{1/2}) \leq 4C^2(X)$, the assertion follows now from the last inequality and inequalities (16) and (17). □

Lemma 29. *Let $x \in S(0,1)$, $x \geq 0$, and let $n \in \mathbb{N}$. If x_k, $k = 1, 2, \ldots, 2n$, are independent copies of the function $\sigma_{1/n}x$, then for all sufficiently large $n \in \mathbb{N}$ we have*

$$\Big(\sum_{k=1}^{n} x_{2k}\Big)^* \leq \sigma_3\Big(\sum_{k=1}^{2n} (-1)^k x_k\Big)^*.$$

Proof. It is clear that the functions $x_{2k-1} - x_{2k}$, $1 \leq k \leq n$, are independent. Therefore,

$$\begin{aligned} m\Big(\Big\{\sum_{k=1}^{n} x_{2k} - x_{2k-1} > t\Big\}\Big) &\geq m\Big(\Big\{\sum_{k=1}^{n} x_{2k} > t,\ \sum_{k=1}^{n} x_{2k-1} = 0\Big\}\Big) \\ &= m\Big(\Big\{\sum_{k=1}^{n} x_{2k} > t\Big\}\Big) \cdot m\Big(\Big\{\sum_{k=1}^{n} x_{2k-1} = 0\Big\}\Big) \\ &= \Big(1 - \frac{1}{n}\Big)^n m\Big(\Big\{\sum_{k=1}^{n} x_{2k} > t\Big\}\Big). \end{aligned}$$

Hence, for all sufficiently large $n \in \mathbb{N}$,

$$m\Big(\Big\{\Big|\sum_{k=1}^{n} x_{2k-1} - x_{2k}\Big| > t\Big\}\Big) \geq \frac{1}{3}\, m\Big(\Big\{\sum_{k=1}^{n} x_{2k} > t\Big\}\Big). \qquad □$$

Proof of Theorem 23. We have to prove only the implication (a) $\Longrightarrow$ (b).

Let $x \in X$, $x \geq 0$, and $n \in \mathbb{N}$. Taking for x_k, $k = 1, 2, \dots, 2n$, independent copies of the function $\sigma_{1/n}x$, by Lemma 29 we have

$$\Big\|\sum_{k=1}^{n} x_{2k}\Big\|_X \leq \Big\|\sigma_3\Big(\sum_{k=1}^{2n}(-1)^k x_k\Big)\Big\|_X \leq 3\,C(X)^2\Big\|\sum_{k=1}^{2n}(-1)^k x_k\Big\|_X.$$

On the other hand, the functions $x_{2k-1} - x_{2k}$, $1 \leq k \leq n$, are independent and symmetrically distributed. Therefore, by the assumption, we have

$$\begin{aligned}\Big\|\sum_{k=1}^{2n}(-1)^k x_k\Big\|_X &\leq C\Big\|\Big(\sum_{k=1}^{n}(x_{2k-1}-x_{2k})^2\Big)^{1/2}\Big\|_X \\ &\leq C\Big\|\Big(\sum_{k=1}^{n}x_{2k-1}^2\Big)^{1/2} + \Big(\sum_{k=1}^{n}x_{2k}^2\Big)^{1/2}\Big\|_X \\ &\leq 2\,C\cdot C(X)\Big\|\Big(\sum_{k=1}^{n}x_{2k}^2\Big)^{1/2}\Big\|_X.\end{aligned}$$

Combining these inequalities, we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{n} x_{2k}\Big\| &\leq 6\,C\cdot C(X)^3\Big\|\Big(\sum_{k=1}^{n}x_{2k}^2\Big)^{1/2}\Big\|_X \\ &\leq 6\,C\cdot C(X)^3\Big\|\Big(\max_{1\leq k\leq n} x_{2k} \cdot \sum_{k=1}^{n}x_{2k}\Big)^{1/2}\Big\|_X.\end{aligned}$$

It follows now from Lemma 2 that

$$\Big\|\sum_{k=1}^{n} x_{2k}\Big\|_X \leq 6\,C\cdot C(X)^4\,\Big\|\max_{1\leq k\leq n} x_{2k}\Big\|_X^{1/2}\cdot\Big\|\sum_{k=1}^{n}x_{2k}\Big\|_X^{1/2}.$$

Hence,

$$\Big\|\sum_{k=1}^{n} x_{2k}\Big\|_X \leq 36\,C^2\,C(X)^8\Big\|\max_{1\leq k\leq n} x_{2k}\Big\|_X \leq 36\,C^2\,C(X)^8\Big\|\bigoplus_{k=1}^{n}x_{2k}\Big\|_X.$$

Appealing to the definition of x_k, $1 \leq k \leq 2n$, we obtain

$$\Big(\bigoplus_{k=1}^{n}x_{2k}\Big)^* = x^* \quad \text{and} \quad \Big(\sum_{k=1}^{n}x_{2k}\Big)^* = (H_n x)^*,$$

where the operator H_n is defined by (3).

Recall that, by Lemma 6, $(H_n x)^* \to (Kx)^*$ almost everywhere on $(0, 1)$. Therefore, if X has the Fatou property, it follows that $\|Kx\|_X \le 36C^2 C(X)^8 \|x\|_X$, and the proof in this case is complete. If X is separable, we can repeat almost verbatim the arguments used in the second part of the proof of Theorem 7. □

6. Von Bahr–Esseen type inequalities

We have the following remarkable theorem.

Theorem 30 [von Bahr and Esseen 1965, Theorem 2]. *If* $1 \le p \le 2$ *and* $\{f_k\}_{k=1}^n \subset L_p(0, 1)$, $n \in \mathbb{N}$, *is a sequence of independent mean zero functions*, *then*

$$\Big\| \sum_{k=1}^n f_k \Big\|_p \le \Big(2 \sum_{k=1}^n \|f_k\|_p^p \Big)^{1/p}. \tag{18}$$

In [Braverman 1994, § II,2], Theorem 30 is extended to Banach symmetric function spaces with the Kruglov property. Versions of disjointification inequalities obtained in Sections 3 and 4 for quasi-Banach symmetric spaces allow us to extend Braverman's result to the quasi-Banach setting. Moreover, we shall consider different quasinorms at the left- and right-hand sides of (18). Our proofs appear to be more straightforward (and simpler) than the proofs for the special case considered in [Braverman 1994].

Definition 31. Quasi-Banach symmetric function spaces X and Y (in this order) satisfy the von Bahr–Esseen r-estimate (written $(X, Y) \in (BE)_r$) if there exists a constant $B > 0$ such that

$$\Big\| \sum_{k=1}^n f_k \Big\|_Y \le B \Big(\sum_{k=1}^n \|f_k\|_X^r \Big)^{1/r} \tag{19}$$

for every sequence of independent symmetrically distributed functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$. If, in addition, $X = Y$, then we say that X satisfies the von Bahr–Esseen r-estimate (written $X \in (BE)_r$).

In view of this definition, we may restate Theorem 30 as $L_p(0, 1) \in (BE)_p$.

Remark 32. If $Y \subset L_1(0, 1)$, then an application of Lemma 20 yields the estimate (19) for all mean zero independent functions.

Clearly, $(X, Y) \in (BE)_r$ implies that $X \subset Y$. Taking Rademacher functions (see Section 1) as the f_k, it is easy to see that we always have $0 < r \le 2$. Finally, if X is p-normed, then $p \le r \le 2$.

Recall that a quasi-Banach lattice X satisfies an *upper r-estimate*, $r > 0$, if there is a constant $C > 0$ such that

$$\Big\| \sum_{k=1}^{n} x_k \Big\|_X \le C \Big(\sum_{k=1}^{n} \|x_k\|_X^r \Big)^{1/r}$$

for every sequence of mutually disjoint elements $\{x_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$.

Recall also that a quasi-Banach symmetric space $L_{r,\infty}$, $r > 0$, consists of all $x \in S(0,1)$ such that

$$\|x\|_{r,\infty} := \sup_{0<t\le 1} x^*(t)\, t^{1/r} < \infty.$$

Theorem 33. *Let $0 < r < 2$. For all quasi-Banach symmetric function spaces X and Y the following statements hold*:

(a) *If $K : X \to Y$ and X satisfies an upper r-estimate, then $(X, Y) \in (BE)_r$.*

(b) *If $K : Y \to Y$ and, for some $C > 0$ and for every sequence of mutually disjoint functions $\{f_k\}_{k=1}^n \subset X$ $(n \in \mathbb{N})$, we have*

$$\Big\| \sum_{k=1}^{n} f_k \Big\|_Y \le C \Big(\sum_{k=1}^{n} \|f_k\|_X^r \Big)^{1/r}, \tag{20}$$

then $(X, Y) \in (BE)_r$.

(c) *If $(X, Y) \in (BE)_r$, then* (20) *holds for every sequence of mutually disjoint functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$.*

The main part of the proof of Theorem 33 is given below in Lemma 36.

Let $0 < p < r < 2$ and let $r > 1$. Recall that $L_{r,\infty}$ satisfies an upper r-estimate (see, for example, [Braverman 1994, Theorem 1.12]) and that $K : L_{r,\infty} \to L_{r,\infty}$ by Theorem 1.3 of the same reference. Setting $X = L_{r,\infty}$ and $Y = L_p(0,1)$ and taking into account Remark 32, we obtain the well-known Esseen–Janson theorem (see [Esseen and Janson 1985, Theorem 4]). It is worth noting that, in contrast to the previous reference, we do not require that the functions f_k are equidistributed.

Lemma 34. *Let $r > 0$ and let X and Y be quasi-Banach symmetric function spaces. Suppose that there is a constant $C > 0$ such that for every sequence of mutually disjoint functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, inequality* (20) *holds. Then $X \subset L_{r,\infty}$.*

Proof. Fix $t \in (0,1]$ and let $n \in \mathbb{N}$ be such that $1/2 < nt \le 1$. Since $\chi_{(0,tn)} = \sum_{k=1}^n \chi_{(t(k-1),tk)}$, the functions $\varphi_X(t) := \|\chi_{(0,t)}\|_X$ and $\varphi_Y(t) := \|\chi_{(0,t)}\|_Y$ satisfy the estimate

$$\varphi_Y(tn) \le C \Big(\sum_{k=1}^{n} \|\chi_{(t(k-1),tk)}\|_X^r \Big)^{1/r} = C\, \varphi_X(t)\, n^{1/r},$$

by (20). Hence, we obtain that

$$\varphi_X(t) \geq C^{-1} \varphi_Y(tn)\, n^{-1/r} \geq C^{-1} \varphi_Y(1/2)\, t^{1/r} = C_1^{-1} t^{1/r},$$

whence for every $x \in X$

$$\|x\|_X \geq x^*(t)\, \|\chi_{(0,t)}\|_X = x^*(t)\varphi_X(t) \geq C_1^{-1} x^*(t)\, t^{1/r}, \quad 0 < t \leq 1.$$

Therefore, $\|x\|_{r,\infty} \leq C_1 \|x\|_X$ for all $x \in X$ and the proof is completed. □

Lemma 35. *Let X be a quasi-Banach symmetric function space on* $(0, 1)$ *satisfying an upper r-estimate*, $0 < r < 2$. *There exists $C_X > 0$ such that for every sequence $\{x_k\}_{k=1}^{\infty} \subset X$ we have*

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z^2_{L_{r,\infty}}} \leq C_X \Big(\sum_{k=1}^{\infty} \|x_k\|_X^r \Big)^{1/r}.$$

Proof. By Lemma 34, we have $X \subset L_{r,\infty}$. Therefore, $x_k^* \leq \|x_k\|_{r,\infty}\, \xi_r$, where $\xi_r(t) = t^{-1/r}$, $0 < t \leq 1$, whence

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z^2_{L_{r,\infty}}} \leq C \Big\| \bigoplus_{k=1}^{\infty} \|x_k\|_{r,\infty}\, \xi_r \Big\|_{Z^2_{L_{r,\infty}}}.$$

Note that for any $a_k \geq 0$ we have

$$\bigoplus_{k=1}^{\infty} a_k \xi_r \sim \Big(\sum_{k=1}^{\infty} a_k^r \Big)^{1/r} \xi_r.$$

Hence,

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z^2_{L_{r,\infty}}} \leq C \Big(\sum_{k=1}^{\infty} \|x_k\|_{r,\infty}^r \Big)^{1/r} \|\xi_r\|_{Z^2_{L_{r,\infty}}} \leq C' \|\xi_r\|_{Z^2_{L_{r,\infty}}} \Big(\sum_{k=1}^{\infty} \|x_k\|_X^r \Big)^{1/r},$$

and the result follows. □

Lemma 36. *Let X be a quasi-Banach symmetric function space on* $(0, 1)$ *satisfying an upper r-estimate*, $0 < r < 2$. *There exists a constant $B_X > 0$ such that for every sequence $\{x_k\}_{k=1}^{\infty} \subset X$ we have*

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z^2_X} \leq B_X \Big(\sum_{k=1}^{\infty} \|x_k\|_X^r \Big)^{1/r}.$$

Proof. By the definition of the quasinorm in Z^2_X, we have that

$$\|z\|_{Z^2_X} \leq \|z^* \chi_{(0,1)}\|_X + \|z\|_{Z^2_{L_{r,\infty}}}, \quad z \in Z^2_X. \tag{21}$$

Denote $\bigoplus_{k=1}^{\infty} |x_k|$ by x, for brevity. Without loss of generality, we can assume that x^* does not have any interval of constancy. Setting $y_k = x_k \chi_{\{|x_k|>x^*(1)\}}$, we have

$$\bigoplus_{k=1}^{\infty} |y_k| \sim x^* \chi_{(0,1)}.$$

Therefore, since X satisfies an upper r-estimate, we obtain

$$\|x^* \chi_{(0,1)}\|_X = \Big\| \bigoplus_{k=1}^{\infty} y_k \Big\|_X \le C \Big(\sum_{k=1}^{\infty} \|y_k\|_X^r \Big)^{1/r} \le C \Big(\sum_{k=1}^{\infty} \|x_k\|_X^r \Big)^{1/r}.$$

The assertion follows now from inequality (21) and the preceding lemma. □

Proof of Theorem 33. The first assertion follows from Theorem 21 and Lemma 36. The proof of the second assertion is identical.

Now, we prove the third assertion. Suppose that $(X, Y) \in (BE)_r$. Let the functions $f_k \in X$, $1 \le k \le n$, be pairwise disjoint and let g_k, $1 \le k \le n$, be their independent copies. Without loss of generality, we can assume that the f_k (and therefore the g_k as well) are symmetrically distributed. By [Johnson and Schechtman 1989, Theorem 1], we have

$$\Big\| \sum_{k=1}^{n} f_k \Big\|_Y = \Big\| \sum_{k=1}^{n} f_k \Big\|_{Z_Y^2} \le C' \Big\| \sum_{k=1}^{n} g_k \Big\|_Y \le C' B \Big(\sum_{k=1}^{n} \|f_k\|_X^r \Big)^{1/r},$$

which is (20) with $C = C'B$. □

If $X = Y$, then estimate (20) means that X satisfies an upper r-estimate and we obtain the following corollary.

Corollary 37. *Let $0 < r < 2$ and let X be a quasi-Banach symmetric function space such that $K : X \to X$. Then $X \in (BE)_r$ if and only if X satisfies an upper r-estimate.*

In the Banach-space setting this result may be found in [Braverman 1994, Theorem 2.3].

For $r = 2$, we have the following result.

Theorem 38. *Let X and Y be quasi-Banach symmetric function spaces.*

(a) *Suppose that $X \supset L_2(0,1)$. If $K : X \to Y$ and X satisfies an upper 2-estimate, or if $K : Y \to Y$ and for some $C > 0$ and for every sequence of mutually disjoint functions $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, we have*

$$\Big\| \sum_{k=1}^{n} f_k \Big\|_Y \le C \Big(\sum_{k=1}^{n} \|f_k\|_X^2 \Big)^{1/2}, \tag{22}$$

then $(X, Y) \in (BE)_2$.

(b) *If* $(X, Y) \in (BE)_2$, *then* $X \supset L_2(0,1)$ *and inequality* (22) *holds for some* $C > 0$ *and for every sequence of mutually disjoint functions* $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$.

Proof. (a) The proof is identical to that of the preceding theorem, substituting the reference to Lemma 36 with the reference to the following assertion.

Lemma 39. *Let a quasi-Banach symmetric space* X *satisfy an upper* 2*-estimate and let* $X \supset L_2(0,1)$. *There exists a constant* $B_X > 0$ *such that for every sequence* $\{x_k\}_{k=1}^\infty \subset X$ *we have*

$$\Big\| \bigoplus_{k=1}^{\infty} x_k \Big\|_{Z_X^2} \le B_X \bigg(\sum_{k=1}^{\infty} \|x_k\|_X^2 \bigg)^{1/2}.$$

(b) Inequality (22) can be proved in exactly the same way as in Theorem 33. Therefore, it remains to show that $X \subset L_2(0,1)$.

Let $f \in X$ be symmetrically distributed and let $\{f_k\}_{k=1}^\infty$ be a sequence of its independent copies. By assumption, $(X, Y) \in (BE)_2$ and, therefore,

$$\Big\| n^{-1/2} \sum_{k=1}^{n} f_k \Big\|_Y \le C \bigg(n^{-1} \sum_{k=1}^{n} \|f_k\|_X^2 \bigg)^{1/2} = C\|f\|_X, \quad n = 1, 2, \ldots.$$

By Lemma 24, there exists $p > 0$ such that $Y \subset L_p(0,1)$. Hence, by the previous inequality, we have

$$\sup_{n\ge 1} \int_0^1 \Big| n^{-1/2} \sum_{k=1}^{n} f_k(t) \Big|^p \, dt < \infty.$$

Applying [Esseen and Janson 1985, Theorem 2], we obtain that $f \in L_2(0,1)$. Since both X and $L_2(0,1)$ are symmetric, the assertion follows. □

Corollary 40. *Let* X *be a quasi-Banach symmetric space such that* $K : X \to X$. *Then* $X \in (BE)_2$ *if and only if* X *satisfies an upper* 2*-estimate and* $X \subset L_2(0,1)$.

This assertion was proved by Braverman [1994, Theorem 2.4] in the Banach setting.

Remark 41. Though the condition $K : X \to X$ is essential in both Theorem 33 and Theorem 38, it is not necessary. For example, $\operatorname{Exp} L_2 \in (BE)_2$ [Braverman 1994, Theorem 2.9], but $K : \operatorname{Exp} L_2 \not\to \operatorname{Exp} L_2$.

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