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# AFFINE QUANTUM SCHUR ALGEBRAS AND AFFINE HECKE ALGEBRAS

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**Let $\mathsf{F}$ be the Schur functor from the category of finite-dimensional $\mathcal{H}_\vartriangle(r)_\mathbb{C}$-modules to that of finite-dimensional $\mathcal{S}_\vartriangle(n,r)_\mathbb{C}$-modules, where $\mathcal{H}_\vartriangle(r)_\mathbb{C}$ is the extended affine Hecke algebra of type $A$ over $\mathbb{C}$ and $\mathcal{S}_\vartriangle(n,r)_\mathbb{C}$ is the affine quantum Schur algebras over $\mathbb{C}$. The Drinfeld polynomials associated with $\mathsf{F}(V)$, where $V$ is an irreducible $\mathcal{H}_\vartriangle(r)_\mathbb{C}$-module, have been previously determined when $n > r$. Here we generalize these results to the case $n \leqslant r$. As an application, we recover the classification of finite-dimensional irreducible $\mathcal{S}_\vartriangle(n,r)_\mathbb{C}$-modules proved by Deng, Du and Fu using a different method. As another application, we generalize a result of Green to the affine case.**

## 1. Introduction

Finite-dimensional irreducible modules for quantum affine algebras were classified by Chari and Pressley [1991; 1994; 1995; 1997] in terms of Drinfeld polynomials. Finite-dimensional irreducible modules for $\mathcal{H}_\vartriangle(r)_\mathbb{C}$ were classified in [Zelevinsky 1980; Rogawski 1985], where $\mathcal{H}_\vartriangle(r)_\mathbb{C}$ is the extended affine Hecke algebra of type $A$ over the complex field $\mathbb{C}$ with a non-root of unity. The category of finite-dimensional $\mathcal{H}_\vartriangle(r)_\mathbb{C}$-modules and the category of finite-dimensional $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$-modules which are of level $r$ are related by a functor $\mathcal{F}$ defined in [Chari and Pressley 1996, §4.2]. Here $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$ is quantum affine $\mathfrak{sl}_n$ over $\mathbb{C}$. Chari and Pressley [loc. cit.] proved that $\mathcal{F}$ is an equivalence of categories if $n > r$. Furthermore the Drinfeld polynomials associated with $\mathcal{F}(V)$ were determined in [loc. cit., §7.6] in the case of $n > r$, where $V$ is an irreducible $\mathcal{H}_\vartriangle(r)_\mathbb{C}$-module.

Let $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ be quantum affine $\mathfrak{gl}_n$ over $\mathbb{C}$. In [Frenkel and Mukhin 2002], finite-dimensional irreducible polynomial representations of $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ were classified. It was proved in [Deng, Du and Fu 2012, Theorem 3.8.1] that the natural algebra homomorphism $\zeta_r$ from $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ to the affine quantum Schur algebra $\mathcal{S}_\vartriangle(n,r)_\mathbb{C}$ is

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surjective. Every $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-module can be regarded as a $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$-module via $\zeta_r$. Let $\mathsf{F}$ be the Schur functor from the category of finite-dimensional $\mathcal{H}_\triangle(r)_\mathbb{C}$-modules to the category of finite-dimensional $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules. It was proved in [Deng, Du and Fu 2012, Theorem 4.1.3 and Proposition 4.2.1] that $\mathsf{F}$ is an equivalence of categories in the case of $n \geqslant r$ and that $\mathsf{F}(V)|_{\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)}$ is isomorphic to $\mathscr{F}(V)$ for any $\mathcal{H}_\triangle(r)_\mathbb{C}$-module $V$. Furthermore, using [Chari and Pressley 1996, §7.6], the Drinfeld polynomials associated with $\mathsf{F}(V)$ were determined in [Deng, Du and Fu 2012, Theorem 4.4.2] in the case of $n > r$, where $V$ is an irreducible $\mathcal{H}_\triangle(r)_\mathbb{C}$-module. We will generalize these results to the case of $n \leqslant r$ in Theorem 4.9. Using this result, we will prove in Corollary 4.10 the classification theorem of finite-dimensional irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules, which was established in [Deng, Du and Fu 2012, Theorem 4.6.8]. Finally, we will relate the parametrization of irreducible $\mathcal{S}_\triangle(N,r)_\mathbb{C}$-modules, via the functor $\mathsf{G}$ defined in (4.10.1), to the parametrization of irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules in Theorem 4.11. This result is the affine version of [Green 2007, (6.5f)].

## 2. Quantum affine $\mathfrak{gl}_n$

Let $v \in \mathbb{C}^*$ be a complex number which is not a root of unity, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Let $(c_{i,j})$ be the Cartan matrix of affine type $A_{n-1}$. We recall the Drinfeld's new realization of quantum affine $\mathfrak{gl}_n$ as follows.

**Definition 2.1.** The *quantum loop algebra* $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ (or *quantum affine* $\mathfrak{gl}_n$) is the $\mathbb{C}$-algebra generated by $\mathrm{x}^\pm_{i,s}$ $(1 \leqslant i < n,\ s \in \mathbb{Z})$, $\mathrm{k}_i^{\pm 1}$, and $\mathrm{g}_{i,t}$ $(1 \leqslant i \leqslant n,\ t \in \mathbb{Z} \setminus \{0\})$ with the following relations:

(QLA1) $\mathrm{k}_i\mathrm{k}_i^{-1} = 1 = \mathrm{k}_i^{-1}\mathrm{k}_i,\ [\mathrm{k}_i, \mathrm{k}_j] = 0,$

(QLA2) $\mathrm{k}_i\mathrm{x}^\pm_{j,s} = v^{\pm(\delta_{i,j}-\delta_{i,j+1})}\mathrm{x}^\pm_{j,s}\mathrm{k}_i,\ [\mathrm{k}_i, \mathrm{g}_{j,s}] = 0,$

(QLA3) $[\mathrm{g}_{i,s}, \mathrm{x}^\pm_{j,t}] = \begin{cases} 0 & \text{if } i \neq j,\ j+1, \\ \pm v^{-js}([s]/s)\mathrm{x}^\pm_{j,s+t} & \text{if } i = j, \\ \mp v^{-js}([s]/s)\mathrm{x}^\pm_{j,s+t} & \text{if } i = j+1, \end{cases}$

(QLA4) $[\mathrm{g}_{i,s}, \mathrm{g}_{j,t}] = 0,$

(QLA5) $[\mathrm{x}^+_{i,s}, \mathrm{x}^-_{j,t}] = \delta_{i,j}(\phi^+_{i,s+t} - \phi^-_{i,s+t})/(v - v^{-1}),$

(QLA6) $\mathrm{x}^\pm_{i,s}\mathrm{x}^\pm_{j,t} = \mathrm{x}^\pm_{j,t}\mathrm{x}^\pm_{i,s}$ for $|i-j| > 1$, and $[\mathrm{x}^\pm_{i,s+1}, \mathrm{x}^\pm_{j,t}]_{v^{\pm c_{ij}}} = -[\mathrm{x}^\pm_{j,t+1}, \mathrm{x}^\pm_{i,s}]_{v^{\pm c_{ij}}},$

(QLA7) $[\mathrm{x}^\pm_{i,s}, [\mathrm{x}^\pm_{j,t}, \mathrm{x}^\pm_{i,p}]_v]_v = -[\mathrm{x}^\pm_{i,p}, [\mathrm{x}^\pm_{j,t}, \mathrm{x}^\pm_{i,s}]_v]_v$ for $|i-j| = 1,$

where $[x, y]_a = xy - ayx$, $[s] = (v^s - v^{-s})/(v - v^{-1})$, and the $\phi^\pm_{i,s}$ are defined via generating functions in the indeterminate $u$ by

$$\Phi^\pm_i(u) := \widetilde{\mathrm{k}}_i^{\pm 1} \exp\Big(\pm(v - v^{-1}) \sum_{m \geqslant 1} \mathrm{h}_{i,\pm m} u^{\pm m}\Big) = \sum_{s \geqslant 0} \phi^\pm_{i,\pm s} u^{\pm s}$$

with $\widetilde{\mathtt{k}}_i = \mathtt{k}_i/\mathtt{k}_{i+1}$ ($\mathtt{k}_{n+1} = \mathtt{k}_1$) and $\mathtt{h}_{i,\pm m} = v^{\pm(i-1)m}\mathtt{g}_{i,\pm m} - v^{\pm(i+1)m}\mathtt{g}_{i+1,\pm m}$ ($1 \leqslant i < n$).

The algebra $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ has another presentation which we now describe. Let $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ be the double Ringel–Hall algebra of the cyclic quiver $\triangle(n)$. By [Deng, Du and Fu 2012, Theorem 2.3.1], the algebra $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ has the following presentation.

**Lemma 2.2.** *The double Ringel–Hall algebra $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ of the cyclic quiver $\triangle(n)$ is the $\mathbb{C}$-algebra generated by $E_i$, $F_i$, $K_i$, $K_i^{-1}$, $\mathsf{z}_s^+$, $\mathsf{z}_s^-$, for $1 \leqslant i \leqslant n$, $s \in \mathbb{Z}^+$, and relations*:

(QGL1) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1,$

(QGL2) $K_iE_j = v^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i,\ K_iF_j = v^{-\delta_{i,j}+\delta_{i,j+1}}F_jK_i,$

(QGL3) $E_iF_j - F_jE_i = \delta_{i,j}(\widetilde{K}_i - \widetilde{K}_i^{-1})/(v - v^{-1}),$ *where* $\widetilde{K}_i = K_iK_{i+1}^{-1},$

(QGL4) $\displaystyle\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1-c_{i,j} \\ a \end{bmatrix} E_i^aE_jE_i^b = 0$ *for* $i \neq j,$

(QGL5) $\displaystyle\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1-c_{i,j} \\ a \end{bmatrix} F_i^aF_jF_i^b = 0$ *for* $i \neq j,$

(QGL6) $\mathsf{z}_s^+\mathsf{z}_t^+ = \mathsf{z}_t^+\mathsf{z}_s^+,\ \mathsf{z}_s^-\mathsf{z}_t^- = \mathsf{z}_t^-\mathsf{z}_s^-,\ \mathsf{z}_s^+\mathsf{z}_t^- = \mathsf{z}_t^-\mathsf{z}_s^+,$

(QGL7) $K_i\mathsf{z}_s^+ = \mathsf{z}_s^+K_i,\ K_i\mathsf{z}_s^- = \mathsf{z}_s^-K_i,$

(QGL8) $E_i\mathsf{z}_s^+ = \mathsf{z}_s^+E_i,\ E_i\mathsf{z}_s^- = \mathsf{z}_s^-E_i,\ F_i\mathsf{z}_s^- = \mathsf{z}_s^-F_i,$ *and* $\mathsf{z}_s^+F_i = F_i\mathsf{z}_s^+,$

*where* $1 \leqslant i, j \leqslant n,\ s, t \in \mathbb{Z}^+$, *and*

$$\begin{bmatrix} c \\ a \end{bmatrix} = \prod_{s=1}^{a} \frac{v^{c-s+1} - v^{-c+s-1}}{v^s - v^{-s}} \quad \textit{for } c \in \mathbb{Z}.$$

*It is a Hopf algebra with comultiplication $\Delta$, counit $\varepsilon$, and antipode $\sigma$ defined by*

$$\Delta(E_i) = E_i \otimes \widetilde{K}_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + \widetilde{K}_i^{-1} \otimes F_i,$$
$$\Delta(K_i^{\pm1}) = K_i^{\pm1} \otimes K_i^{\pm1}, \quad \Delta(\mathsf{z}_s^{\pm}) = \mathsf{z}_s^{\pm} \otimes 1 + 1 \otimes \mathsf{z}_s^{\pm},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0 = \varepsilon(\mathsf{z}_s^{\pm}), \quad \varepsilon(K_i) = 1,$$
$$\sigma(E_i) = -E_i\widetilde{K}_i^{-1}, \quad \sigma(F_i) = -\widetilde{K}_iF_i, \quad \sigma(K_i^{\pm1}) = K_i^{\mp1}, \quad \sigma(\mathsf{z}_s^{\pm}) = -\mathsf{z}_s^{\pm},$$

*where* $1 \leqslant i \leqslant n$ *and* $s \in \mathbb{Z}^+$.

Let $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$ be the subalgebra of $\mathfrak{D}_{\triangle,\mathbb{C}}(n)$ generated by $E_i$, $F_i$, $\widetilde{K}_i$, $\widetilde{K}_i^{-1}$ for $i \in [1, n]$. Beck [1994] proved that $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$ is isomorphic to the subalgebra of $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ generated by all $\mathtt{x}_{i,s}^{\pm}$, $\widetilde{\mathtt{k}}_i^{\pm1}$, and $\mathtt{h}_{i,t}$. The following result extends Beck's isomorphism.

**Lemma 2.3** [Deng, Du and Fu 2012, Proposition 4.4.1]. *There is a Hopf algebra isomorphism*

$$f : \mathfrak{D}_{\triangle,\mathbb{C}}(n) \to \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$$

*such that*

$$K_i^{\pm 1} \mapsto \mathtt{k}_i^{\pm 1}, \quad E_j \mapsto \mathtt{x}_{j,0}^{+}, \quad F_j \mapsto \mathtt{x}_{j,0}^{-} \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j < n),$$
$$E_n \mapsto v\mathscr{X}\widetilde{\mathtt{k}}_n, \quad F_n \mapsto v^{-1}\widetilde{\mathtt{k}}_n^{-1}\mathscr{Y}, \quad \mathtt{z}_s^{\pm} \mapsto \mp s v^{\pm s}\theta_{\pm s}\ (s \geqslant 1),$$

*where*

$$\theta_{\pm s} = \mp\frac{1}{[s]}(\mathtt{g}_{1,\pm s} + \cdots + \mathtt{g}_{n,\pm s}),$$
$$\mathscr{X} = [\mathtt{x}_{n-1,0}^{-}, [\mathtt{x}_{n-2,0}^{-}, \ldots, [\mathtt{x}_{2,0}^{-}, \mathtt{x}_{1,1}^{-}]_{v^{-1}} \cdots ]_{v^{-1}}]_{v^{-1}},$$
$$\mathscr{Y} = [\cdots [[\mathtt{x}_{1,-1}^{+}, \mathtt{x}_{2,0}^{+}]_v, \mathtt{x}_{3,0}^{+}]_v, \ldots, \mathtt{x}_{n-1,0}^{+}]_v.$$

We now review the classification theorem of finite-dimensional irreducible polynomial $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-modules. We first need to introduce the elements $\mathscr{Q}_{i,s} \in \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$, which will be used to define pseudo-highest weight modules. For $1 \leqslant i \leqslant n$ and $s \in \mathbb{Z}$, define the elements $\mathscr{Q}_{i,s} \in \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ through the generating functions

$$\mathscr{Q}_i^{\pm}(u) := \exp\left(-\sum_{t\geqslant 1}\frac{1}{[t]}g_{i,\pm t}(vu)^{\pm t}\right) = \sum_{s\geqslant 0}\mathscr{Q}_{i,\pm s}u^{\pm s} \in \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)[[u, u^{-1}]].$$

For a representation $V$ of $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$, a nonzero vector $w \in V$ is called a *pseudo-highest weight vector* if there exists some $Q_{i,s} \in \mathbb{C}$ such that

$$\text{(2.3.1)} \qquad \mathtt{x}_{j,s}^{+}w = 0, \quad \mathscr{Q}_{i,s}w = Q_{i,s}w, \quad \mathtt{k}_i w = v^{\lambda_i}w$$

for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$ and $s \in \mathbb{Z}$. The module $V$ is called a *pseudo-highest weight module* if $V = \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)w$ for some pseudo-highest weight vector $w$. We also write the short form $\mathscr{Q}_i^{\pm}(u)w = Q_i^{\pm}(u)w$ for the relations $\mathscr{Q}_{i,s}w = Q_{i,s}w$ $(s \in \mathbb{Z})$, where

$$Q_i^{\pm}(u) = \sum_{s\geqslant 0} Q_{i,\pm s}u^{\pm s}.$$

Let $V$ be a finite-dimensional polynomial representation of $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ of type 1. Then $V = \oplus_{\lambda\in\mathbb{N}^n} V_\lambda$, where

$$V_\lambda = \{x \in V \mid \mathtt{k}_j x = v^{\lambda_j}x,\ 1 \leqslant j \leqslant n\},$$

and, since all $\mathscr{Q}_{i,s}$ commute with the $\mathtt{k}_j$, each $V_\lambda$ is a direct sum of generalized eigenspaces of the form

$$\text{(2.3.2)} \quad V_{\lambda,\gamma} = \{x \in V_\lambda \mid (\mathscr{Q}_{i,s} - \gamma_{i,s})^p x = 0 \quad \text{for some } p\ (1 \leqslant i \leqslant n, s \in \mathbb{Z})\},$$

where $\gamma = (\gamma_{i,s})$ with $\gamma_{i,s} \in \mathbb{C}$. Let $\Gamma_i^{\pm}(u) = \sum_{s\geqslant 0}\gamma_{i,\pm s}u^{\pm s}$.

A finite-dimensional $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module $V$ is called a *polynomial representation* if the restriction of $V$ to $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$ is a polynomial representation of type 1 and, for every weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ of $V$, the formal power series $\Gamma_i^{\pm}(u)$ associated to the eigenvalues $(\gamma_{i,s})_{s\in\mathbb{Z}}$ defining the generalized eigenspaces $V_{\lambda,\gamma}$ as given in (2.3.2), are polynomials in $u^{\pm}$ of degree $\lambda_i$ so that the zeroes of the functions $\Gamma_i^{+}(u)$ and $\Gamma_i^{-}(u)$ are the same.

Following [Frenkel and Mukhin 2002], an $n$-tuple of polynomials

$$\boldsymbol{Q} = (Q_1(u), \ldots, Q_n(u))$$

with constant terms 1 is called *dominant* if, for each $1 \leqslant i \leqslant n-1$, the ratio $Q_i(v^{i-1}u)/Q_{i+1}(v^{i+1}u)$ is a polynomial. Let $\mathcal{Q}(n)$ be the set of dominant $n$-tuples of polynomials.

For $g(u) = \prod_{1\leqslant i\leqslant m} (1 - a_i u) \in \mathbb{C}[u]$ with constant term 1 and $a_i \in \mathbb{C}^*$, define

$$g^{\pm}(u) = \prod_{1\leqslant i\leqslant m} (1 - a_i^{\pm 1} u^{\pm 1}). \tag{2.3.3}$$

For $\boldsymbol{Q} = (Q_1(u), \ldots, Q_n(u)) \in \mathcal{Q}(n)$, define $Q_{i,s} \in \mathbb{C}$, for $1 \leqslant i \leqslant n$ and $s \in \mathbb{Z}$, by the formula

$$Q_i^{\pm}(u) = \sum_{s\geqslant 0} Q_{i,\pm s} u^{\pm s},$$

where $Q_i^{\pm}(u)$ is defined using (2.3.3). Let $I(\boldsymbol{Q})$ be the left ideal of $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ generated by $\mathtt{x}^{+}_{j,s}$, $\mathcal{Q}_{i,s} - Q_{i,s}$, and $\mathtt{k}_i - v^{\lambda_i}$, for $1 \leqslant j \leqslant n-1$, $1 \leqslant i \leqslant n$, and $s \in \mathbb{Z}$, where $\lambda_i = \deg Q_i(u)$, and define

$$M(\boldsymbol{Q}) = \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)/I(\boldsymbol{Q}).$$

Then $M(\boldsymbol{Q})$ has a unique irreducible quotient, denoted by $L(\boldsymbol{Q})$. The polynomials $Q_i(u)$ are called *Drinfeld polynomials* associated with $L(\boldsymbol{Q})$.

**Theorem 2.4** [Frenkel and Mukhin 2002]. *The* $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$*-modules* $L(\boldsymbol{Q})$ *with* $\boldsymbol{Q} \in \mathcal{Q}(n)$ *are all nonisomorphic finite-dimensional irreducible polynomial representations of* $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$.

If $\boldsymbol{Q}, \boldsymbol{Q}' \in \mathcal{Q}(n)$ satisfies $Q_j(v^{j-1}u)/Q_{j+1}(v^{j+1}u) = Q'_j(v^{j-1}u)/Q'_{j+1}(v^{j+1}u)$ and $\deg Q_j(u) - \deg Q_{j+1}(u) = \deg Q'_j(u) - \deg Q'_{j+1}(u)$ for $1 \leqslant j \leqslant n-1$, then $L(\boldsymbol{Q})|_{\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)} \cong L(\boldsymbol{Q}')|_{\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)}$, by [Deng, Du and Fu 2012, Lemma 4.7.1, Corollary 4.7.2]. Thus we can denote $L(\boldsymbol{Q})|_{\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)}$ by $\bar{L}(\boldsymbol{P})$, where $\boldsymbol{P} = (P_1(u), \ldots, P_{n-1}(u))$ with $P_j(u) = Q_j(v^{j-1}u)/Q_{j+1}(v^{j+1}u)$.

Let $\mathcal{P}(n)$ be the set of $(n-1)$-tuples of polynomials with constant term 1. The following result is due to Chari and Pressley [1991; 1994; 1995].

**Theorem 2.5.** *The modules* $\bar{L}(\boldsymbol{P})$ *with* $\boldsymbol{P} \in \mathcal{P}(n)$ *are all nonisomorphic finite-dimensional irreducible* $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$*-modules of type* 1.

## 3. Affine quantum Schur algebras

In this section we collect some facts about extended affine Hecke algebras and affine quantum Schur algebras, which will be used in Section 4. The extended affine Hecke algebra $\mathcal{H}_\triangle(r)_\mathbb{C}$ is defined to be the algebra generated by

$$T_i, \quad X_j^{\pm 1} \quad (1 \leqslant i \leqslant r-1,\ 1 \leqslant j \leqslant r),$$

and relations

$$\begin{gathered}(T_i+1)(T_i-v^2)=0,\\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, \quad T_iT_j=T_jT_i\ (|i-j|>1),\\ X_iX_i^{-1}=1=X_i^{-1}X_i, \quad X_iX_j=X_jX_i,\\ T_iX_iT_i=v^2X_{i+1}, \quad X_jT_i=T_iX_j\ (j\neq i,i+1).\end{gathered}$$

Let $\mathfrak{S}_r$ be the symmetric group with generators $s_i:=(i,i+1)$ for $1\leqslant i\leqslant r-1$. Let $I(n,r)=\{(i_1,\dots,i_r)\in\mathbb{Z}^r \mid 1\leqslant i_k\leqslant n,\ \forall k\}$. The symmetric group $\mathfrak{S}_r$ acts on the set $I(n,r)$ by place permutation:

$$\boldsymbol{i}w=(i_{w(1)},\dots,i_{w(r)}), \quad \text{for } \boldsymbol{i}\in I(n,r) \text{ and } w\in\mathfrak{S}_r.$$

Let $\Omega_\mathbb{C}$ be a vector space over $\mathbb{C}$ with basis $\{\omega_i \mid i\in\mathbb{Z}\}$. For $\boldsymbol{i}=(i_1,\dots,i_r)\in\mathbb{Z}^r$, write

$$\omega_{\boldsymbol{i}}=\omega_{i_1}\otimes\omega_{i_2}\otimes\cdots\otimes\omega_{i_r}=\omega_{i_1}\omega_{i_2}\cdots\omega_{i_r}\in\Omega_\mathbb{C}^{\otimes r}.$$

The tensor space $\Omega_\mathbb{C}^{\otimes r}$ admits a right $\mathcal{H}_\triangle(r)_\mathbb{C}$-module structure defined by

$$\begin{cases}\omega_{\boldsymbol{i}}\cdot X_t^{-1}=\omega_{i_1}\cdots\omega_{i_{t-1}}\omega_{i_t+n}\omega_{i_{t+1}}\cdots\omega_{i_r} & \text{for all } \boldsymbol{i}\in\mathbb{Z}^r,\\ \omega_{\boldsymbol{i}}\cdot T_k=\begin{cases}v^2\omega_{\boldsymbol{i}} & \text{if } i_k=i_{k+1},\\ v\omega_{\boldsymbol{i}s_k} & \text{if } i_k<i_{k+1},\\ v\omega_{\boldsymbol{i}s_k}+(v^2-1)\omega_{\boldsymbol{i}} & \text{if } i_{k+1}<i_k,\end{cases} & \text{for all } \boldsymbol{i}\in I(n,r),\end{cases}$$

where $1\leqslant k\leqslant r-1$ and $1\leqslant t\leqslant r$.

The algebra

$$\mathcal{S}_\triangle(n,r)_\mathbb{C}:=\mathrm{End}_{\mathcal{H}_\triangle(r)_\mathbb{C}}(\mathcal{T}_\triangle(n,r))$$

is called an affine $q$-Schur algebra, where $\mathcal{T}_\triangle(n,r)=\Omega_\mathbb{C}^{\otimes r}$. Let $\Omega_{n,\mathbb{C}}$ be the subspace of $\Omega_\mathbb{C}$ spanned by $\omega_i$ with $1\leqslant i\leqslant n$ and $\mathcal{H}(r)_\mathbb{C}$ be the subalgebra of $\mathcal{H}_\triangle(r)_\mathbb{C}$ generated by $T_k$ for $1\leqslant k\leqslant r-1$. Then the algebra $\mathcal{S}(n,r)_\mathbb{C}:=\mathrm{End}_{\mathcal{H}(r)_\mathbb{C}}(\mathcal{T}(n,r))$ is called a $q$-Schur algebra, where $\mathcal{T}(n,r)=\Omega_{n,\mathbb{C}}^{\otimes r}$.

The algebras $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ and $\mathcal{S}_\triangle(n,r)_\mathbb{C}$ are related by an algebra homomorphism $\zeta_r$, which we now describe. For $i\in\mathbb{Z}$, let $\bar{i}$ denotes the corresponding integer modulo $n$.

The complex vector space $\Omega_{\mathbb{C}}$ is a natural $\mathfrak{D}_{\vartriangle,\mathbb{C}}(n)$-module with the action

$$E_i \cdot \omega_s = \delta_{\overline{i+1},\bar{s}}\omega_{s-1}, \quad F_i \cdot \omega_s = \delta_{\bar{i},\bar{s}}\omega_{s+1}, \quad K_i^{\pm 1} \cdot \omega_s = v^{\pm\delta_{\bar{i},\bar{s}}}\omega_s, \\ \mathsf{z}_t^+ \cdot \omega_s = \omega_{s-tn}, \quad \mathsf{z}_t^- \cdot \omega_s = \omega_{s+tn}. \tag{3.0.1}$$

The Hopf algebra structure induces a $\mathfrak{D}_{\vartriangle,\mathbb{C}}(n)$-module $\Omega_{\mathbb{C}}^{\otimes r}$. By [Deng, Du and Fu 2012, Proposition 3.5.5], the actions of $\mathfrak{D}_{\vartriangle,\mathbb{C}}(n)$ and $\mathscr{H}_{\vartriangle}(r)_{\mathbb{C}}$ on $\Omega_{\mathbb{C}}^{\otimes r}$ are commute. We will identify $\mathfrak{D}_{\vartriangle,\mathbb{C}}(n)$ and $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ via the algebra isomorphism $f$ defined in Lemma 2.3. Consequently, there is an algebra homomorphism

$$\zeta_r : \mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n) = \mathfrak{D}_{\vartriangle,\mathbb{C}}(n) \to \mathscr{S}_{\vartriangle}(n,r)_{\mathbb{C}}.$$

It is proved in [Deng, Du and Fu 2012, Theorem 3.8.1] that $\zeta_r$ is surjective. Let $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$ be the subalgebra of $\mathfrak{D}_{\vartriangle,\mathbb{C}}(n)$ generated by $E_i, F_i, K_j, K_j^{-1}$ for $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n$. The restriction of $\zeta_r$ to $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$ induces a surjective algebra homomorphism $\zeta_r : \mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n) \to \mathscr{S}(n,r)_{\mathbb{C}}$ (see [Jimbo 1986]). Every $\mathscr{S}_{\vartriangle}(n,r)_{\mathbb{C}}$-module (resp., $\mathscr{S}(n,r)_{\mathbb{C}}$-module) will be inflated into a $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module (resp., $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$-module) via $\zeta_r$.

The following easy lemma relates $\Omega_{\mathbb{C}}^{\otimes r}$ with $\Omega_{n,\mathbb{C}}^{\otimes r}$.

**Lemma 3.1** [Deng, Du and Fu 2012, Lemma 4.1.1]. *There is a* $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$-$\mathscr{H}_{\vartriangle}(r)_{\mathbb{C}}$-*bimodule isomorphism*

$$\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} \mathscr{H}_{\vartriangle}(r)_{\mathbb{C}} \xrightarrow{\sim} \Omega_{\mathbb{C}}^{\otimes r}, \quad x \otimes h \mapsto xh.$$

The irreducible $\mathscr{H}_{\vartriangle}(r)_{\mathbb{C}}$-modules were classified in [Zelevinsky 1980; Rogawski 1985], which we now describe. For $\boldsymbol{a} = (a_1, \dots, a_r) \in (\mathbb{C}^*)^r$, let $M_{\boldsymbol{a}} = \mathscr{H}_{\vartriangle}(r)_{\mathbb{C}}/J_{\boldsymbol{a}}$, where $J_{\boldsymbol{a}}$ is the left ideal of $\mathscr{H}_{\vartriangle}(r)_{\mathbb{C}}$ generated by $X_j - a_j$ for $1 \leqslant j \leqslant r$.

A *segment* $\mathsf{s}$ with center $a \in \mathbb{C}^*$ is by definition an ordered sequence

$$\mathsf{s} = (av^{-k+1}, av^{-k+3}, \dots, av^{k-1}) \in (\mathbb{C}^*)^k.$$

Here $k$ is called the length of the segment, denoted by $|\mathsf{s}|$. If $\boldsymbol{s} = \{\mathsf{s}_1, \dots, \mathsf{s}_p\}$ is an unordered collection of segments, define $\wp(\boldsymbol{s})$ to be the partition associated with the sequence $(|\mathsf{s}_1|, \dots, |\mathsf{s}_p|)$. That is, $\wp(\boldsymbol{s}) = (|\mathsf{s}_{i_1}|, \dots, |\mathsf{s}_{i_p}|)$ with $|\mathsf{s}_{i_1}| \geqslant \cdots \geqslant |\mathsf{s}_{i_p}|$, where $|\mathsf{s}_{i_1}|, \dots, |\mathsf{s}_{i_p}|$ is a permutation of $|\mathsf{s}_1|, \dots, |\mathsf{s}_p|$. We also call $|\boldsymbol{s}| := |\wp(\boldsymbol{s})|$ the length of $\boldsymbol{s}$.

Let $\mathscr{S}_r$ be the set of unordered collections of segments $\boldsymbol{s}$ with $|\boldsymbol{s}| = r$. Then $\mathscr{S}_r = \bigcup_{\mu\in\Lambda^+(r)} \mathscr{S}_{r,\mu}$, where $\mathscr{S}_{r,\mu} = \{\boldsymbol{s} \in \mathscr{S}_r \mid \wp(\boldsymbol{s}) = \mu\}$ and $\Lambda^+(r)$ is the set of partitions of $r$.

If $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ is reduced let $T_w = T_{i_1}T_{i_2}\cdots T_{i_m}$. For $p \geqslant 1$ let

$$\Lambda(p,r) = \Big\{\mu \in \mathbb{N}^p \Bigm| \sum_{1\leqslant i\leqslant p} \mu_i = r\Big\} \tag{3.1.1}$$

For $\mu \in \Lambda(p, r)$ let $\mathfrak{S}_\mu$ be the corresponding standard Young subgroup of the symmetric group $\mathfrak{S}_r$, and let $\mathcal{D}_\mu = \{d \in \mathfrak{S}_r \mid \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_\mu\}$. For $\mu \in \Lambda(p, r)$ let

$$\mathcal{I}_\mu = \mathcal{H}(r)_{\mathbb{C}} y_\mu, \tag{3.1.2}$$

where

$$y_\mu = \sum_{w \in \mathfrak{S}_\mu} (-v^2)^{-\ell(w)} T_w \in \mathcal{H}(r)_{\mathbb{C}}.$$

For $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\} \in \mathcal{S}_{r,\mu}$, let $\boldsymbol{a}(\boldsymbol{s}) = (\mathsf{s}_1, \ldots, \mathsf{s}_p) \in (\mathbb{C}^*)^r$ be the $r$-tuple obtained by juxtaposing the segments in $\boldsymbol{s}$. Let $\iota : \mathcal{H}(r)_{\mathbb{C}} \to M_{\boldsymbol{a}(\boldsymbol{s})}$ be the natural $\mathcal{H}(r)_{\mathbb{C}}$-module isomorphism defined by sending $h$ to $\bar{h}$. Let

$$\bar{\mathcal{I}}_\mu = \iota(\mathcal{I}_\mu) = \mathcal{H}(r)_{\mathbb{C}} \bar{y}_\mu = \mathcal{H}_\triangle(r)_{\mathbb{C}} \bar{y}_\mu.$$

Then,

$$\mathcal{H}(r)_{\mathbb{C}} y_\mu \cong E_\mu \oplus \Bigg( \bigoplus_{\substack{\nu \vdash r \\ \nu \rhd \lambda}} m_{\nu,\mu} E_\nu \Bigg), \tag{3.1.3}$$

where $E_\nu$ is the left cell module defined by the Kazhdan–Lusztig's C-basis [1979] associated with the left cell containing $w_{0,\nu}$.

Let $V_{\boldsymbol{s}}$ be the unique composition factor of the $\mathcal{H}_\triangle(r)_{\mathbb{C}}$-module $\mathcal{H}_\triangle(r)_{\mathbb{C}} \bar{y}_\mu$ such that the multiplicity of $E_\mu$ in $V_{\boldsymbol{s}}$ as an $\mathcal{H}(r)_{\mathbb{C}}$-module is nonzero.

The following classification theorem is due to [Zelevinsky 1980; Rogawski 1985].

**Theorem 3.2.** *The modules $V_{\boldsymbol{s}}$ with $\boldsymbol{s} \in \mathcal{S}_r$ are all nonisomorphic finite-dimensional irreducible $\mathcal{H}_\triangle(r)_{\mathbb{C}}$-modules.*

Let $\mathcal{S}_\triangle(n, r)_{\mathbb{C}}$-mod (resp., $\mathcal{H}_\triangle(r)_{\mathbb{C}}$-mod) be the category of finite-dimensional $\mathcal{S}_\triangle(n, r)_{\mathbb{C}}$-modules (resp., $\mathcal{H}_\triangle(r)_{\mathbb{C}}$-modules). The categories $\mathcal{S}_\triangle(n, r)_{\mathbb{C}}$-mod and $\mathcal{H}_\triangle(r)_{\mathbb{C}}$-mod are related by the Schur functor $\mathsf{F}$, which we now define. Using the $\mathcal{S}_\triangle(n, r)_{\mathbb{C}}$-$\mathcal{H}_\triangle(r)_{\mathbb{C}}$-bimodule $\Omega_{\mathbb{C}}^{\otimes r}$, we define a functor

$$\mathsf{F} = \mathsf{F}_{n,r} : \mathcal{H}_\triangle(r)_{\mathbb{C}}\text{-mod} \to \mathcal{S}_\triangle(n, r)_{\mathbb{C}}\text{-mod}, \quad V \mapsto \Omega_{\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}_\triangle(r)_{\mathbb{C}}} V. \tag{3.2.1}$$

Let

$$\mathcal{S}_r^{(n)} = \{\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\} \in \mathcal{S}_r,\ p \geqslant 1,\ |\mathsf{s}_i| \leqslant n,\ \forall i\}.$$

The following classification theorem is given in [Deng, Du and Fu 2012, Theorems 4.3.4 and 4.5.3].

**Lemma 3.3.** *For $\boldsymbol{s} \in \mathcal{S}_r$ we have $\mathsf{F}(V_{\boldsymbol{s}}) \neq 0$ if and only if $\boldsymbol{s} \in \mathcal{S}_r^{(n)}$. Furthermore, the set*

$$\{\mathsf{F}(V_{\boldsymbol{s}}) \mid \boldsymbol{s} \in \mathcal{S}_r^{(n)}\}$$

*is a complete set of nonisomorphic finite-dimensional irreducible $\mathcal{S}_\vartriangle(n,r)_\mathbb{C}$-modules.*

The following result, which will be used in Theorem 4.9, is taken from [Chari and Pressley 1996, §7.6; Deng, Du and Fu 2012, Theorem 4.4.2 and Lemma 4.6.5].

**Lemma 3.4.** *Assume $n \geqslant r$. Let $\boldsymbol{s} = (av^{-r+1}, av^{-r+3}, \ldots, av^{r-1})$ be a single segment and $\mu = \wp(\boldsymbol{s}) = (r)$. Then $V_{\boldsymbol{s}} = \bar{\mathcal{I}}_\mu$ and $\mathsf{F}(V_{\boldsymbol{s}}) \cong L(\boldsymbol{Q})$, where $\boldsymbol{Q} = (Q_1(u), \ldots, Q_n(u))$ with*

$$Q_n(u) = (1 - av^{-n+1}u)^{\delta_{n,r}},$$
$$\frac{Q_i(uv^{i-1})}{Q_{i+1}(uv^{i+1})} = (1-au)^{\delta_{i,r}} \quad \textit{for } 1 \leqslant i \leqslant n-1.$$

## 4. Identification of irreducible $\mathcal{S}_\vartriangle(n,r)_\mathbb{C}$-modules

In this section we will prove that $\mathsf{F}(\bar{\mathcal{I}}_{\wp(\boldsymbol{s})})$ is isomorphic to the tensor product of irreducible $\mathcal{S}_\vartriangle(n,r)_\mathbb{C}$-modules for $\boldsymbol{s} \in \mathcal{S}_r^{(n)}$ and $\mathsf{F}(\bar{\mathcal{I}}_{\wp(\boldsymbol{s})}) = 0$ for $\boldsymbol{s} \notin \mathcal{S}_r^{(n)}$ in Proposition 4.6. Using this result, we will relate the parametrization of irreducible $\mathcal{H}_\vartriangle(r)_\mathbb{C}$-modules, via the functor $\mathsf{F}$ defined in (3.2.1), to the parametrization of finite-dimensional irreducible polynomial representations of $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ in Theorem 4.9. As applications, we will classify finite-dimensional irreducible $\mathcal{S}_\vartriangle(n,r)_\mathbb{C}$-modules in Corollary 4.10, and generalize [Green 2007, (6.5f)] to the affine case.

To compute $\mathsf{F}(\bar{\mathcal{I}}_{\wp(\boldsymbol{s})})$, we need Proposition 4.3 of [Rogawski 1985], which we now describe. For $1 \leqslant j \leqslant p$, let $\mathcal{H}_{\mu,j}$ be the subalgebra of $\mathcal{H}(r)_\mathbb{C}$ generated by $T_i$ with $s_i \in \mathfrak{S}_{\mu^{(j)}}$, where

$$\mu^{(j)} = (1^{\mu_{[1,j-1]}}, \mu_j, 1^{r-\mu_{[1,j]}}),$$

and $\mu_{[1,j]} = \mu_1 + \mu_2 + \cdots + \mu_j$. Since $\mathcal{H}_{\mu,j} \cong \mathcal{H}(\mu_j)_\mathbb{C}$ for $1 \leqslant j \leqslant p$ and $\Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ is a right $\mathcal{H}(\mu_j)_\mathbb{C}$-module, $\Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ can be also regarded as a right $\mathcal{H}_{\mu,j}$-module.

Recall the notation $\mathcal{I}_\mu$ defined in (3.1.2). For $\mu \in \Lambda(p,r)$ and $1 \leqslant j \leqslant p$ let

$$\mathcal{J}_\mu = \bigcap_{\substack{s_i \in \mathfrak{S}_\mu \\ 1 \leqslant i \leqslant r-1}} \mathcal{H}(r)_\mathbb{C} C_i, \quad \mathcal{J}_{\mu,j} = \bigcap_{\substack{s_i \in \mathfrak{S}_{\mu^{(j)}} \\ 1 \leqslant i \leqslant r-1}} \mathcal{H}_{\mu,j} C_i, \quad \text{and} \quad \mathcal{I}_{\mu,j} = \mathcal{H}_{\mu,j} y_{\mu^{(j)}}.$$

where $C_i = v^{-1}T_i - v$ and $y_{\mu^{(j)}} = \sum_{w \in \mathfrak{S}_{\mu^{(j)}}} (-v^2)^{-\ell(w)} T_w$. By Proposition 4.3 of [Rogawski 1985] we have:

**Lemma 4.1.** *We have $\mathcal{I}_\mu = \mathcal{J}_\mu$, $\mathcal{I}_{\mu,j} = \mathcal{J}_{\mu,j}$ for $\mu \in \Lambda(p,r)$ and $1 \leqslant j \leqslant p$.*

**Lemma 4.2.** *Assume $I$ is a left ideal of $\mathcal{H}(r)_\mathbb{C}$. Then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} I \cong \Omega_{n,\mathbb{C}}^{\otimes r} I$.*

*Proof.* Since $\mathcal{H}(r)_{\mathbb{C}}$ is semisimple, there exist a left ideal $J$ of $\mathcal{H}(r)_{\mathbb{C}}$ such that $\mathcal{H}(r)_{\mathbb{C}} = I \oplus J$. Then $\Omega_{n,\mathbb{C}}^{\otimes r} \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} \mathcal{H}(r)_{\mathbb{C}} \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \oplus \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} J$. Thus the natural linear map $f : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \to \Omega_{n,\mathbb{C}}^{\otimes r}$ defined by sending $w \otimes h$ to $wh$ is injective. Consequently, $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} I \cong \operatorname{Im}(f) = \Omega_{n,\mathbb{C}}^{\otimes r} I$. □

By Lemmas 3.1, 4.1, and 4.2 we conclude that $\mathsf{F}(\bar{\mathcal{I}}_\mu) \cong \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_{\mathbb{C}}} \bar{\mathcal{I}}_\mu \cong \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu$, where $\mu = \wp(\boldsymbol{s})$ for some $\boldsymbol{s} \in \mathcal{S}_r$. We now compute $\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu$.

**Lemma 4.3.** *For $\mu \in \Lambda(p, r)$, we have*

$$\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu = \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{I}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{I}_{\mu,p}.$$

*Proof.* Since $\mathcal{I}_\mu = \bigcap_{1 \leqslant j \leqslant p} \mathcal{I}_{\mu^{(j)}}$ we have $\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu \subseteq \bigcap_{1 \leqslant j \leqslant p} \big(\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_{\mu^{(j)}}\big)$. Furthermore by Lemma 4.1 we have $\mathcal{I}_{\mu^{(j)}} = \mathcal{I}_{\mu^{(j)}} = \mathcal{X}_{\mu,j} \mathcal{I}_{\mu,j} = \mathcal{X}_{\mu,j} \mathcal{I}_{\mu,j}$ where $\mathcal{X}_{\mu,j} = \operatorname{span}\{T_w \mid w \in \mathcal{D}_{\mu^{(j)}}^{-1}\}$. This implies that

$$\Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_{\mu^{(j)}} = \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_{\mu,j} = \Omega_{n,\mathbb{C}}^{\mu_1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\mu_{j-1}} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_j} \mathcal{I}_{\mu_j} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_{j+1}} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p}$$

for $1 \leqslant j \leqslant p$. Thus,

$$\begin{aligned} \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu &\subseteq \bigcap_{1 \leqslant j \leqslant p} \big(\Omega_{n,\mathbb{C}}^{\mu_1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\mu_{j-1}} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_j} \mathcal{I}_{\mu_j} \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_{j+1}} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p}\big) \\ &= \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{I}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{I}_{\mu,p}. \end{aligned}$$

On the other hand, we assume $w_1 h_1 \otimes \cdots \otimes w_p h_p \in \Omega_{n,\mathbb{C}}^{\otimes \mu_1} \mathcal{I}_{\mu,1} \otimes \cdots \otimes \Omega_{n,\mathbb{C}}^{\otimes \mu_p} \mathcal{I}_{\mu,p}$, where $w_j \in \Omega_{n,\mathbb{C}}^{\otimes \mu_j}$ and $h_j \in \mathcal{I}_{\mu,j}$. Since $h_k h_l = h_l h_k$ for any $k, l$ and $h_j \in \mathcal{I}_{\mu,j}$, we have $h_1 h_2 \cdots h_p = (h_1 \cdots h_{j-1} h_{j+1} \cdots h_p) h_j \in \mathcal{H}(r)_{\mathbb{C}} \mathcal{I}_{\mu,j} \subseteq \mathcal{H}(r)_{\mathbb{C}} C_i$ for $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant p$ with $s_i \in \mathfrak{S}_{\mu^{(j)}}$. This implies that $h_1 h_2 \cdots h_p \in \mathcal{I}_\mu$. It follows that $w_1 h_1 \otimes \cdots \otimes w_p h_p = (w_1 \otimes \cdots \otimes w_p) h_1 \cdots h_p \in \Omega_{n,\mathbb{C}}^{\otimes r} \mathcal{I}_\mu$. The assertion follows. □

For $\mu \in \Lambda(p, r)$ and $1 \leqslant j \leqslant p$, let $\widetilde{\mathcal{H}}_{\mu,j}$ be the subalgebra of $\mathcal{H}_\vartriangle(r)_{\mathbb{C}}$ generated by $T_i$ and $X_{\mu_{[1,j-1]}+1}, \ldots, X_{\mu_{[1,j]}}$ with $s_i \in \mathfrak{S}_{\mu^{(j)}}$. Since $\widetilde{\mathcal{H}}_{\mu,j} \cong \mathcal{H}_\vartriangle(\mu_j)_{\mathbb{C}}$ and $\Omega_{\mathbb{C}}^{\otimes \mu_j}$ is a right $\mathcal{H}_\vartriangle(\mu_j)_{\mathbb{C}}$-module, one can regard $\Omega_{\mathbb{C}}^{\otimes \mu_j}$ as a right $\widetilde{\mathcal{H}}_{\mu,j}$-module.

For $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\} \in \mathcal{S}_{r,\mu}$, let $\boldsymbol{a} = (\mathsf{s}_1, \ldots, \mathsf{s}_p) \in (\mathbb{C}^*)^r$ be the $r$-tuple obtained by juxtaposing the segments in $\boldsymbol{s}$. For $1 \leqslant j \leqslant p$ let $\mathfrak{I}_{\mu,j}$ be the left ideal of $\widetilde{\mathcal{H}}_{\mu,j}$ generated by $X_k - a_k$ for $\mu_{[1,j-1]} + 1 \leqslant k \leqslant \mu_{[1,j]}$. Let $\iota_j : \mathcal{H}_{\mu,j} \to \widetilde{\mathcal{H}}_{\mu,j}/\mathfrak{I}_{\mu,j}$ be the natural $\mathcal{H}_{\mu,j}$-module isomorphism defined by sending $h$ to $\bar{h}$. Let

$$\bar{\mathcal{I}}_{\mu,j} = \iota_j(\mathcal{I}_{\mu,j}) = \mathcal{H}_{\mu,j} \bar{y}_{\mu^{(j)}} = \widetilde{\mathcal{H}}_{\mu,j} \bar{y}_{\mu^{(j)}}.$$

By Lemma 4.3 we have the following corollary.

**Corollary 4.4.** *Maintain the notation above. There is a $\mathrm{U}_{\mathbb{C}}(\mathfrak{gl}_n)$-module isomorphism*

$$\varphi : (\Omega_{\mathbb{C}}^{\otimes \mu_1} \otimes_{\widetilde{\mathcal{H}}_{\mu,1}} \bar{\mathcal{I}}_{\mu,1}) \otimes \cdots \otimes (\Omega_{\mathbb{C}}^{\otimes \mu_p} \otimes_{\widetilde{\mathcal{H}}_{\mu,p}} \bar{\mathcal{I}}_{\mu,p}) \to \mathsf{F}(\bar{\mathcal{I}}_\mu)$$

*such that* $\varphi(w_1\otimes\bar h_1\otimes\cdots\otimes w_p\otimes\bar h_p)=w_1\otimes\cdots\otimes w_p\otimes\overline{h_1\cdots h_p}$ *for* $w_j\in\Omega_{n,\mathbb{C}}^{\otimes\mu_j}$ *and* $h_j\in\mathscr{I}_{\mu,j}$ *with* $1\leqslant j\leqslant p$.

*Proof.* Combining Lemmas 3.1, 4.1 with 4.2 yields $\mathsf{F}(\bar{\mathscr{I}}_\mu)\cong\Omega_{n,\mathbb{C}}^{\otimes r}\otimes_{\mathscr{H}(r)_{\mathbb{C}}}\bar{\mathscr{I}}_\mu\cong\Omega_{n,\mathbb{C}}^{\otimes r}\mathscr{I}_\mu$ and $\Omega_{\mathbb{C}}^{\otimes\mu_j}\otimes_{\widetilde{\mathscr{H}}_{\mu,j}}\bar{\mathscr{I}}_{\mu,j}\cong\Omega_{n,\mathbb{C}}^{\otimes^{\mu_j}}\otimes_{\mathscr{H}_{\mu,j}}\bar{\mathscr{I}}_{\mu,j}\cong\Omega_{n,\mathbb{C}}^{\otimes^{\mu_j}}\mathscr{I}_{\mu,j}$ for $1\leqslant j\leqslant p$. This, together with Lemma 4.3, implies the assertion. □

We now prove that $\varphi$ is in fact a $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module isomorphism.

**Lemma 4.5.** *The map* $\varphi$ *is a* $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$*-module homomorphism.*

*Proof.* Let $u\in\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$ and $w=w_1\otimes\bar h_1\otimes\cdots\otimes w_p\otimes\bar h_p\in(\Omega_{\mathbb{C}}^{\otimes\mu_1}\otimes_{\widetilde{\mathscr{H}}_{\mu,1}}\bar{\mathscr{I}}_{\mu,1})\otimes\cdots\otimes(\Omega_{\mathbb{C}}^{\otimes\mu_p}\otimes_{\widetilde{\mathscr{H}}_{\mu,p}}\bar{\mathscr{I}}_{\mu,p})$, where $w_i\in\Omega_{n,\mathbb{C}}^{\otimes\mu_i}$ and $h_i\in\mathscr{I}_{\mu,i}$ for $1\leqslant i\leqslant p$. Assume $\Delta^{(p-1)}(u)=\sum_{(u)}u_1\otimes\cdots\otimes u_p$, $u_iw_i=\sum_{k_i}w_{i,k_i}g_{i,k_i}$ and $g_{i,k_i}h_i=\sum_{j_i}g_{i,k_i,j_i}X_{j_i}$, where $w_{i,k_i}\in\Omega_{n,\mathbb{C}}^{\otimes\mu_i}$, $g_{i,k_i}\in\widetilde{\mathscr{H}}_{\mu,i}$, and $g_{i,k_i,j_i}\in\mathscr{H}_{\mu,i}$, $X_{j_i}\in\widetilde{\mathscr{H}}_{\mu,i}$. Then

$$g_{i,k_i}(\iota_i(h_i))=g_{i,k_i}\overline{h_i}=\sum_{j_i}a_{j_i}\overline{g_{i,k_i,j_i}}.$$

Hence,

$$\begin{aligned}uw&=\sum_{(u)}u_1w_1\otimes\bar h_1\otimes\cdots\otimes u_pw_p\otimes\bar h_p\\&=\sum_{(u)}\sum_{k_1,\dots,k_p}w_{1,k_1}\otimes g_{1,k_1}\bar h_1\otimes\cdots\otimes w_{p,k_p}\otimes g_{p,k_p}\bar h_p\\&=\sum_{(u)}\sum_{\substack{k_1,\dots,k_p\\j_1,\dots,j_p}}a_{j_1}\cdots a_{j_p}w_{1,k_1}\otimes\overline{g_{1,k_1,j_1}}\otimes\cdots\otimes w_{p,k_p}\otimes\overline{g_{p,k_p,j_p}}.\end{aligned}$$

Since

$$g_{1,k_1}\cdots g_{p,k_p}\overline{h_1\cdots h_p}=\overline{g_{1,k_1}h_1\cdots g_{p,k_p}h_p}=\sum_{j_1,\dots,j_p}a_{j_1}\cdots a_{j_p}\overline{g_{1,k_1,j_1}\cdots g_{p,k_p,j_p}},$$

we conclude that

$$\begin{aligned}\varphi(uw)&=\sum_{(u)}\sum_{\substack{k_1,\dots,k_p\\j_1,\dots,j_p}}a_{j_1}\cdots a_{j_p}w_{1,k_1}\otimes\cdots\otimes w_{p,k_p}\otimes\overline{g_{1,k_1,j_1}\cdots g_{p,k_p,j_p}}\\&=\sum_{(u)}\sum_{k_1,\dots,k_p}w_{1,k_1}\otimes\cdots\otimes w_{p,k_p}\otimes g_{1,k_1}\cdots g_{p,k_p}\overline{h_1\cdots h_p}\\&=\sum_{(u)}u_1w_1\otimes\cdots\otimes u_pw_p\otimes\overline{h_1\cdots h_p}\\&=u(w_1\otimes\cdots\otimes w_p\otimes\overline{h_1\cdots h_p})\\&=u\varphi(w).\end{aligned}$$

□

We can now describe $\mathsf{F}(\bar{\mathcal{I}}_{\wp(s)})$ as follows.

**Proposition 4.6.** *Let $s = \{\mathsf{s}_1, \dots, \mathsf{s}_p\} \in \mathcal{S}_{r,\mu}$ with $\mathsf{s}_i = (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \dots, a_i v^{\mu_i-1})$. Then $\mathsf{F}(\bar{\mathcal{I}}_\mu) = 0$ for $s \notin \mathcal{S}_r^{(n)}$ and $\mathsf{F}(\bar{\mathcal{I}}_\mu) \cong L(\boldsymbol{Q}_1) \otimes \cdots \otimes L(\boldsymbol{Q}_p)$ for $s \in \mathcal{S}_r^{(n)}$, where $\boldsymbol{Q}_i = (Q_{i,1}(u), \dots, Q_{i,n}(u))$ with $Q_{i,n}(u) = (1 - a_i v^{-n+1} u)^{\delta_{\mu_i,n}}$ and $Q_{i,j}(uv^{j-1})/Q_{i,j+1}(uv^{j+1}) = (1 - a_i u)^{\delta_{j,\mu_i}}$ for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant n-1$.*

*Proof.* Since $\bar{\mathcal{I}}_{\mu_i} \cong V_{\mathsf{s}_i}$ for $1 \leqslant i \leqslant p$, by Corollary 4.4 and Lemma 4.5 we conclude that $\mathsf{F}(\bar{\mathcal{I}}_\mu) = \mathsf{F}_{n,r}(\bar{\mathcal{I}}_\mu) \cong \mathsf{F}_{n,\mu_1}(V_{\mathsf{s}_1}) \otimes \cdots \otimes \mathsf{F}_{n,\mu_p}(V_{\mathsf{s}_p})$. If $s \notin \mathcal{S}_r^{(n)}$, then there exists $1 \leqslant k \leqslant p$ such that $|\mathsf{s}_k| = \mu_k > n$. By Lemma 3.3 we have $\mathsf{F}_{n,\mu_k}(V_{\mathsf{s}_k}) = 0$ and hence $\mathsf{F}(\bar{\mathcal{I}}_\mu) = 0$. If $s \in \mathcal{S}_r^{(n)}$, then by Lemma 3.4 we have $\mathsf{F}_{n,\mu_i}(V_{\mathsf{s}_i}) \cong L(\boldsymbol{Q}_i)$ for $1 \leqslant i \leqslant p$. Consequently, $\mathsf{F}(\bar{\mathcal{I}}_\mu) \cong L(\boldsymbol{Q}_1) \otimes \cdots \otimes L(\boldsymbol{Q}_p)$. □

We now turn to studying $\mathsf{F}(V_s)$ for $s \in \mathcal{S}_r^{(n)}$. To compute $\mathsf{F}(V_s)$, we need to generalize [Chari and Pressley 1996, §7.2] to the case of $n \leqslant r$. Recall the notation $\Lambda(n, r)$ defined in (3.1.1). Let $\Lambda^+(n, r) = \Lambda(n, r) \cap \Lambda^+(r)$. For $\lambda \in \mathbb{N}^n$ let $L(\lambda)$ be the irreducible $\mathrm{U}_\mathbb{C}(\mathfrak{gl}_n)$-module with highest weight $\lambda$. For $1 \leqslant i \leqslant n$, let $\mathfrak{k}_i = \zeta_r(K_i)$ and

$$\begin{bmatrix} \mathfrak{k}_i; 0 \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{\mathfrak{k}_i v^{-s+1} - \mathfrak{k}_i^{-1} v^{s-1}}{v^s - v^{-s}}.$$

For $\mu \in \mathbb{N}^n$ let $\mathfrak{k}_\mu = \begin{bmatrix} \mathfrak{k}_1;0 \\ \mu_1 \end{bmatrix} \cdots \begin{bmatrix} \mathfrak{k}_n;0 \\ \mu_n \end{bmatrix}$. The following result is the generalization of [Chari and Pressley 1996, §7.2].

**Lemma 4.7.** *Let $\mu \in \Lambda^+(r)$. Then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} E_\mu \neq 0$ if and only if $\mu' \in \Lambda(n, r)$, where $\mu'$ is the dual partition of $\mu$. Furthermore if $\mu' \in \Lambda^+(n, r)$, then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} E_\mu \cong L(\mu')$.*

*Proof.* We choose $N$ such that $N > \max\{n, r\}$. Let $e = \sum_{\mu \in \Lambda(n,r)} \mathfrak{k}_\mu \in \mathcal{S}(N, r)_\mathbb{C}$. It is well known that for $\mu \in \Lambda^+(N, r)$, $eL(\mu) \neq 0$ if and only if $\mu \in \Lambda(n, r)$ (see [Green 2007, (6.5f)]). Furthermore by [Chari and Pressley 1996, §7.2; Deng, Du and Fu 2012, Lemma 4.3.3] we have $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} E_\mu \cong e\big(\Omega_{N,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} E_\mu\big) \cong e(L(\mu'))$. Thus $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} E_\mu \neq 0$ if and only if $\mu' \in \Lambda(n, r)$. If $\mu' \in \Lambda^+(n, r)$, then $\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathcal{H}(r)_\mathbb{C}} E_\mu \cong e(L(\mu')) \cong L(\mu')$. □

In the case of $n > r$, the Drinfeld polynomials associated with $\mathsf{F}(V_s)$ were calculated for $s \in \mathcal{S}_r^{(n)}$ in [Chari and Pressley 1996, §7.6; Deng, Du and Fu 2012, Theorem 4.4.2]. We are now prepared to use Proposition 4.6 and Lemma 4.7 to generalize these results to the case of $n \leqslant r$ in Theorem 4.9.

Let $\mathfrak{Q}(n)_r = \{\boldsymbol{Q} \in \mathfrak{Q}(n) \mid \sum_{1 \leqslant i \leqslant n} \deg Q_i(u) = r\}$. For $s = \{\mathsf{s}_1, \dots, \mathsf{s}_p\} \in \mathcal{S}_r^{(n)}$ with

$$\mathsf{s}_i = (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \dots, a_i v^{\mu_i-1}) \in (\mathbb{C}^*)^{\mu_i},$$

define $\boldsymbol{Q}_s = (Q_1(u), \ldots, Q_n(u))$ by setting $Q_n(u) = \prod_{\substack{1\leqslant i\leqslant p\\ \mu_i=n}} (1-a_i u v^{-n+1})$ and

$$Q_i(u) = P_i(uv^{-i+1})P_{i+1}(uv^{-i+2})\cdots P_{n-1}(uv^{n-2i})Q_n(uv^{2(n-i)})$$

for $1 \leqslant i \leqslant n-1$, where

$$P_i(u) = \prod_{\substack{1\leqslant j\leqslant p\\ \mu_j=i}} (1-a_j u).$$

Then

$$\sum_{1\leqslant i\leqslant n} \deg Q_i(u) = n \deg Q_n(u) + \sum_{1\leqslant i\leqslant n-1} i \deg P_i(u) = \sum_{1\leqslant i\leqslant p} \mu_i = r.$$

So $\boldsymbol{Q}_s \in \mathcal{Q}(n)_r$. Consequently, we obtain a map $\partial_{n,r} : \mathcal{S}_r^{(n)} \to \mathcal{Q}(n)_r$ defined by sending $\boldsymbol{s}$ to $\boldsymbol{Q}_s$.

**Lemma 4.8.** *The map $\partial_{n,r} : \mathcal{S}_r^{(n)} \to \mathcal{Q}(n)_r$ is bijective.*

*Proof.* It is clear that $\partial_{n,r}$ is injective. Let $\boldsymbol{Q} = (Q_1(u), \ldots, Q_n(u)) \in \mathcal{Q}(n)_r$ and let $\lambda \in \Lambda(n,r)$, with $\lambda_i = \deg Q_i(u)$. For $1 \leqslant j \leqslant n-1$ let

$$P_j(u) = \frac{Q_j(uv^{j-1})}{Q_{j+1}(uv^{j+1})}$$

and $\nu_j = \deg P_j(u) = \lambda_j - \lambda_{j+1}$. We write, for $1 \leqslant i \leqslant n-1$,

$$P_i(u) = (1-a_{\nu_1+\cdots+\nu_{i-1}+1}u)(1-a_{\nu_1+\cdots+\nu_{i-1}+2}u)\cdots(1-a_{\nu_1+\cdots+\nu_{i-1}+\nu_i}u),$$

and $Q_n(u) = (1-b_1u)\cdots(1-b_{\lambda_n}u)$. Let $p' = \sum_{1\leqslant i\leqslant n-1} \nu_i$ and $p = p' + \lambda_n$. Let $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\}$, where

$$\mathsf{s}_i = \begin{cases} (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \ldots, a_i v^{\mu_i-1}) & \text{for } 1 \leqslant i \leqslant p', \\ (b_{i-p'}, b_{i-p'}v^2, \ldots, b_{i-p'}v^{2(n-1)}) & \text{for } p'+1 \leqslant i \leqslant p, \end{cases}$$

and $(\mu_1, \ldots, \mu_{p'}) = (1^{\nu_1}, \ldots, (n-1)^{\nu_{n-1}})$. Since

$$\sum_{1\leqslant i\leqslant p} |\mathsf{s}_i| = \sum_{1\leqslant j\leqslant p'} \mu_j + n\lambda_n = \sum_{1\leqslant i\leqslant n-1} i\nu_i + n\lambda_n = \sum_{1\leqslant i\leqslant n} \lambda_i = r,$$

we have $\boldsymbol{s} \in \mathcal{S}_r^{(n)}$. It is easy to see that $\partial_{n,r}(\boldsymbol{s}) = \boldsymbol{Q}$. Thus $\partial_{n,r}$ is surjective. □

**Theorem 4.9.** *For $\boldsymbol{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_p\} \in \mathcal{S}_r^{(n)}$ with $\mathsf{s}_i = (a_i v^{-\mu_i+1}, a_i v^{-\mu_i+3}, \ldots, a_i v^{\mu_i-1})$, we have $\mathsf{F}(V_s) \cong L(\boldsymbol{Q}_s)$, where $\boldsymbol{Q}_s = \partial_{n,r}(\boldsymbol{s})$. In particular, we have $\mathsf{F}(V_s)|_{\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)} \cong \bar{L}(\boldsymbol{P})$, where*

$$P_i(u) = \prod_{\substack{1\leqslant j\leqslant p\\ \mu_j=i}} (1-a_j u) \quad \textit{for } 1 \leqslant i \leqslant n-1.$$

*Proof.* Let $W = \mathsf{F}(\bar{\mathscr{I}}_\mu)$. By Proposition 4.6 we have $W \cong L(\boldsymbol{Q}_1) \otimes \cdots \otimes L(\boldsymbol{Q}_p)$, where $\boldsymbol{Q}_i = (Q_{i,1}(u), \ldots, Q_{i,n}(u))$ with $Q_{i,n}(u) = (1 - a_i v^{-n+1} u)^{\delta_{\mu_i,n}}$ and

$$P_{i,j}(u) := \frac{Q_{i,j}(uv^{j-1})}{Q_{i,j+1}(uv^{j+1})} = (1 - a_i u)^{\delta_{j,\mu_i}}$$

for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant n-1$. We will identify $W$ with $L(\boldsymbol{Q}_1) \otimes \cdots \otimes L(\boldsymbol{Q}_p)$. Let $w = w_1 \otimes \cdots \otimes w_p \in W$, where $w_i$ is the pseudo-highest weight vector in $L(\boldsymbol{Q}_i)$. Then by [Chari and Pressley 1996, §6.3; Frenkel and Mukhin 2002, Lemma 4.1] we conclude that $w$ is the pseudo-highest weight vector in $W$ such that $\mathtt{k}_i w = v^{\lambda_i} w$ and $\mathscr{Q}_i^{\pm}(u) w = Q_i^{\pm}(u) w$ for $1 \leqslant i \leqslant n$, where $\lambda_i = \deg Q_i^+(u)$,

$$\begin{aligned} Q_n^{\pm}(u) &= \prod_{1 \leqslant i \leqslant p} Q_{i,n}^{\pm}(u) = \prod_{1 \leqslant i \leqslant p} (1 - (a_i u)^{\pm 1} v^{\pm(-n+1)})^{\delta_{\mu_i,n}} \\ &= \prod_{\substack{1 \leqslant i \leqslant p \\ \mu_i = n}} (1 - (a_i u)^{\pm 1} v^{\pm(-n+1)}) \end{aligned}$$

and

$$\begin{aligned} P_j^{\pm}(u) &:= \frac{Q_j^{\pm}(v^{j-1}u)}{Q_{j+1}^{\pm}(v^{j+1}u)} = \prod_{1 \leqslant i \leqslant p} P_{i,j}^{\pm}(u) \\ &= \prod_{1 \leqslant i \leqslant p} (1 - (a_i u)^{\pm 1})^{\delta_{j,\mu_i}} = \prod_{\substack{1 \leqslant i \leqslant p \\ \mu_i = j}} (1 - (a_i u)^{\pm 1}) \end{aligned}$$

for $1 \leqslant j \leqslant n-1$. By definition we have $\boldsymbol{Q}_s = (Q_1^+(u), \ldots, Q_n^+(u))$. Since

$$\lambda_j = \deg Q_j^+(u) = \lambda_n + \sum_{j \leqslant s \leqslant n-1} \deg P_s^+(u) = \big|\{1 \leqslant i \leqslant p \mid \mu_i \geqslant j\}\big|$$

for $1 \leqslant j \leqslant n$, we have $\lambda = (\lambda_1, \ldots, \lambda_n) = \mu'$.

Let $L = \mathsf{F}(V_s)$. Since $V_s$ is a semisimple $\mathscr{H}(r)_{\mathbb{C}}$-module, by Lemmas 3.1 and 4.7 we have $[L : L(\lambda)] = [L : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu] = [\Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} V_s : \Omega_{n,\mathbb{C}}^{\otimes r} \otimes_{\mathscr{H}(r)_{\mathbb{C}}} E_\mu] = [V_s : E_\mu] = 1$. Thus

(4.9.1) $$\dim L_\lambda = 1.$$

Since $V_s$ is the irreducible subquotient of $\bar{\mathscr{I}}_\mu$, there is a surjective $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module homomorphism $f : M \to L$, where $M$ is a certain submodule of $W$. Since $1 = \dim L_\lambda \leqslant \dim M_\lambda \leqslant \dim W_\lambda = 1$, we conclude that $\dim M_\lambda = \dim W_\lambda = 1$. Hence $M_\lambda = W_\lambda = \mathrm{span}\{w\}$ and $L_\lambda = \mathrm{span}\{f(w)\}$. By (4.9.1) we have $f(w) \neq 0$. Since $f$ is a $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-module homomorphism, $f(w)$ is the pseudo-highest weight vector in $L$ such that $\mathtt{k}_i f(w) = f(\mathtt{k}_i w) = v^{\lambda_i} f(w)$ and $\mathscr{Q}_i^{\pm}(u) f(w) = f(\mathscr{Q}_i^{\pm}(u) w) = Q_i^{\pm}(u) f(w)$ for $1 \leqslant i \leqslant n$. This implies that $L$ is the irreducible quotient module of $M(\boldsymbol{Q}_s)$ and hence $L \cong L(\boldsymbol{Q}_s)$. □

Combining Lemmas 3.3, 4.8 with 4.9 yields the following classification theorem of irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules, which was proved as Theorem 4.6.8 in [Deng, Du and Fu 2012] using a different approach.

**Corollary 4.10.** *The set* $\{L(\boldsymbol{Q}) \mid \boldsymbol{Q} \in \mathfrak{Q}(n)_r\}$ *is a complete set of nonisomorphic finite-dimensional irreducible* $\mathcal{S}_\triangle(n,r)_\mathbb{C}$*-modules.*

Finally we will use Theorem 4.9 to generalize [Green 2007, (6.5f)] to the affine case in Theorem 4.11. Assume $N \geqslant n$. Let $e = \sum_{\lambda\in\Lambda(n,r)} \mathfrak{k}_\lambda \in \mathcal{S}_\triangle(N,r)_\mathbb{C}$. Then $e\mathcal{S}_\triangle(N,r)_\mathbb{C}e \cong \mathcal{S}_\triangle(n,r)_\mathbb{C}$. Consequently, the categories $e\mathcal{S}_\triangle(N,r)_\mathbb{C}e$-mod and $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-mod may be identified. With this identification, we define a functor

$$\mathsf{G} = \mathsf{G}_{N,n,r} \;:\; \mathcal{S}_\triangle(N,r)_\mathbb{C}\text{-mod} \to \mathcal{S}_\triangle(n,r)_\mathbb{C}\text{-mod}, \qquad V \mapsto eV. \tag{4.10.1}$$

Then by definition we have $\mathsf{G}_{N,n,r} \circ \mathsf{F}_{N,r} = \mathsf{F}_{n,r}$. For $\boldsymbol{Q} = (Q_1(u), \ldots, Q_n(u)) \in \mathfrak{Q}(n)_r$ let $\widetilde{\boldsymbol{Q}} = (Q_1(u), \ldots, Q_n(u), 1, \ldots, 1) \in \mathfrak{Q}(N)_r$. Let $\widetilde{\mathfrak{Q}}(n)_r = \{\widetilde{\boldsymbol{Q}} \mid \boldsymbol{Q} \in \mathfrak{Q}(n)_r\} \subseteq \mathfrak{Q}(N)_r$. Clearly, by definition, we have

$$\partial_{N,r}(\boldsymbol{s}) = \widetilde{\partial_{n,r}(\boldsymbol{s})} \quad \text{for } \boldsymbol{s} \in \mathcal{S}_r^{(n)}. \tag{4.10.2}$$

**Theorem 4.11.** *Assume* $N \geqslant n$. *Then* $\mathsf{G}(L(\widetilde{\boldsymbol{Q}})) \cong L(\boldsymbol{Q})$ *for* $\boldsymbol{Q} \in \mathfrak{Q}(n)_r$. *In particular we have* $\dim L(\widetilde{\boldsymbol{Q}})_\alpha = \dim L(\boldsymbol{Q})_\alpha$ *for* $\alpha \in \Lambda(n,r)$. *Furthermore, for* $\boldsymbol{Q}' \in \mathfrak{Q}(N)_r$, $\mathsf{G}(L(\boldsymbol{Q}')) \neq 0$ *if and only if* $\boldsymbol{Q}' \in \widetilde{\mathfrak{Q}}(n)_r$.

*Proof.* If $\boldsymbol{Q} \in \mathfrak{Q}(n)_r$ then by Lemma 4.8 we conclude that there exist $\boldsymbol{s} \in \mathcal{S}_r^{(n)}$ such that $\boldsymbol{Q} = \partial_{n,r}(\boldsymbol{s})$. By Theorem 4.9 and (4.10.2) we have $L(\widetilde{\boldsymbol{Q}}) \cong \mathcal{T}_\triangle(N,r) \otimes_{\mathcal{H}_\triangle(r)_\mathbb{C}} V_{\boldsymbol{s}}$. So by [Deng, Du and Fu 2012, Lemma 4.3.3] and Theorem 4.9 we have

$$\mathsf{G}(L(\widetilde{\boldsymbol{Q}})) \cong (e\mathcal{T}_\triangle(N,r)) \otimes_{\mathcal{H}_\triangle(r)_\mathbb{C}} V_{\boldsymbol{s}} \cong \mathcal{T}_\triangle(n,r) \otimes_{\mathcal{H}_\triangle(r)_\mathbb{C}} V_{\boldsymbol{s}} \cong L(\boldsymbol{Q}).$$

By [Green 2007, (6.2g)], the set $\{\mathsf{G}(L(\boldsymbol{Q}')) \neq 0 \mid \boldsymbol{Q}' \in \mathfrak{Q}(N)_r\}$ forms a complete set of non-isomorphic irreducible $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules. This together with Corollary 4.10 implies that $\{\mathsf{G}(L(\boldsymbol{Q}')) \neq 0 \mid \boldsymbol{Q}' \in \mathfrak{Q}(N)_r\} = \{\mathsf{G}(L(\widetilde{\boldsymbol{Q}})) \mid \boldsymbol{Q} \in \mathfrak{Q}(n)_r\}$. Consequently, $\mathsf{G}(L(\boldsymbol{Q}')) \neq 0$ if and only if $\boldsymbol{Q}' \in \widetilde{\mathfrak{Q}}(n)_r$. □

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