Pacific Journal of Mathematics

EIGENVALUE ESTIMATE AND COMPACTNESS FOR CLOSED f-MINIMAL SURFACES

XU CHENG, TITO MEJIA AND DETANG ZHOU

Volume 271 No. 2 October 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 271, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.271.347

EIGENVALUE ESTIMATE AND COMPACTNESS FOR CLOSED f-MINIMAL SURFACES

XU CHENG, TITO MEJIA AND DETANG ZHOU

Let Ω be a bounded domain with convex boundary in a complete noncompact Riemannian manifold with Bakry–Émery Ricci curvature bounded below by a positive constant. We prove a lower bound on the first eigenvalue of the weighted Laplacian for closed embedded f-minimal hypersurfaces contained in Ω. Using this estimate, we prove a compactness theorem for the space of closed embedded f-minimal surfaces with uniform upper bounds on genus and diameter in a complete 3-manifold with Bakry–Émery Ricci curvature bounded below by a positive constant and admitting an exhaustion by bounded domains with convex boundary.

1. Introduction

A hypersurface Σ immersed in a Riemannian manifold $(M, \bar{g})$ is said to be f-minimal if its mean curvature H satisfies, for any $p \in \Sigma$,

$$H = \langle \overline{\nabla} f, \nu \rangle,$$

where ν is the unit normal at $p \in \Sigma$, f is a smooth function defined on M, and $\overline{\nabla} f$ denotes the gradient of f on M. When f is a constant function, an f-minimal hypersurface is just a minimal hypersurface. One nontrivial class of f-minimal hypersurfaces is that of self-shrinkers. Recall that a self-shrinker (for the mean curvature flow in the Euclidean space $(\mathbb{R}^{n+1}, g_{\text{can}})$) is a hypersurface immersed in $(\mathbb{R}^{n+1}, g_{\text{can}})$ satisfying

$$H = \tfrac{1}{2} \langle x, \nu \rangle,$$

where x is the position vector in $\mathbb{R}^{n+1}$. Hence a self-shrinker is an f-minimal hypersurface Σ with $f = |x|^2/4$ (see more information on self-shrinkers in [Colding and Minicozzi 2012a] and references therein).

In the study of f-minimal hypersurfaces, it is convenient to consider the ambient space as a smooth metric measure space $(M, \bar{g}, e^{-f} d\mu)$, where $d\mu$ is the volume

Cheng and Zhou were partially supported by CNPq and FAPERJ of Brazil. Mejia was supported by CNPq of Brazil.
MSC2010: primary 58J50; secondary 58E30.
Keywords: Riemannian manifold, eigenvalue, drifted Laplacian, minimal surface.

form of $\bar{g}$. For $(M, \bar{g}, e^{-f} d\mu)$, an important and natural tensor is the Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f := \overline{\mathrm{Ric}} + \overline{\nabla}^2 f$. There are many interesting examples of smooth metric measure spaces $(M, \bar{g}, e^{-f} d\mu)$ with $\overline{\mathrm{Ric}}_f \geq k$, for a positive constant k. A nontrivial class of examples is the shrinking gradient Ricci solitons. It is known that, after a normalization, a shrinking gradient Ricci soliton $(M, \bar{g}, f)$ satisfies the equation $\overline{\mathrm{Ric}} + \overline{\nabla}^2 f = \bar{g}/2$ or, equivalently, $\overline{\mathrm{Ric}}_f = \frac{1}{2}$. We refer to [Cao 2010], a survey of this topic where some compact and noncompact examples are explained. Even though the asymptotic growth of the potential function f of a noncompact shrinking gradient Ricci soliton is the same as that of a Gaussian shrinking soliton [Cao and Zhou 2010], both the geometry and topology can be quite different from known examples. We may consider f-minimal hypersurfaces in a shrinking gradient Ricci soliton. For instance, a self-shrinker in $\mathbb{R}^{n+1}$ can be viewed as an f-minimal hypersurface in the Gaussian shrinking soliton $(\mathbb{R}^{n+1}, g_{\mathrm{can}}, |x|^2/4)$.

There are other examples of f-minimal hypersurfaces. Let M be the hyperbolic space $\mathbb{H}^{n+1}(-1)$. Let r denote the distance function from a fixed point $p \in M$ and $f(x) = nar^2(x)$, where $a > 0$ is a constant. Then $\overline{\mathrm{Ric}}_f \geq n(2a-1)$, and the geodesic sphere of radius r centered at p in $\mathbb{H}^{n+1}(-1)$ is an f-minimal hypersurface if it satisfies $2ar = \coth r$.

An f-minimal hypersurface Σ has two aspects to view. One is that Σ is f-minimal if and only if Σ is a critical point of the weighted volume functional $e^{-f} d\sigma$, where $d\sigma$ is the volume element of Σ. Another one is that Σ is f-minimal if and only if Σ is minimal in the new conformal metric $\tilde{g} = e^{-2f/n}\bar{g}$ (see Section 2). f-minimal hypersurfaces, even more general stationary hypersurfaces for parametric elliptic functionals, have been studied before. See, for instance, the work of White [1987] and Colding and Minicozzi [2002].

In this paper, we will first estimate the lower bound on the first eigenvalue of the weighted Laplacian $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ for closed (i.e., compact and without boundary) embedded f-minimal hypersurfaces in a complete metric measure space $(M, \bar{g}, e^{-f} d\mu)$. Subsequently using the eigenvalue estimate, we study compactness for the space of closed embedded f-minimal surfaces in a complete noncompact 3-manifold. To explain our result, we give some background.

Choi and Wang [1983] estimated the lower bound for the first eigenvalue of closed minimal hypersurfaces in a complete Riemannian manifold with Ricci curvature bounded below by a positive constant and proved the following:

Theorem 1. *If M is a simply connected complete Riemannian manifold with Ricci curvature bounded below by a constant $k > 0$ and Σ is a closed embedded minimal hypersurface, then the first eigenvalue of the Laplacian Δ on Σ is at least $k/2$.*

Later, using a covering argument, Choi and Schoen [1985] proved that the assumption that M is simply connected is not needed. Recently Ma and Du [2010] extended Theorem 1 to the first eigenvalue of the weighted Laplacian Δ_f on a closed embedded f-minimal hypersurface in a simply connected compact manifold with positive Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f$. Very recently Li and Wei [2012] also used the covering argument to delete the assumption that the ambient space is simply connected in the result of Ma and Du.

The Bonnet–Myers theorem says that a complete manifold with Ricci curvature bounded below by a positive constant must be compact. But the corresponding result is not true for complete manifolds with Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f$ bounded below by a positive constant. One example is the Gaussian shrinking soliton $(\mathbb{R}^{n+1}, g_{\mathrm{can}}, e^{-|x|^2/4}d\mu)$, with $\overline{\mathrm{Ric}}_f = \frac{1}{2}$. Hence the theorems of Ma and Du and Li and Wei cannot be applied to self-shrinkers.

For self-shrinkers, Ding and Xin [2013] recently obtained a lower bound on the first eigenvalue $\lambda_1(\mathscr{L})$ of the weighted Laplacian $\mathscr{L} = \Delta - \langle x, \nabla\cdot\rangle/2$ (i.e., Δ_f) on a closed n-dimensional embedded self-shrinker in the Euclidean space $\mathbb{R}^{n+1}$, that is, $\lambda_1(\mathscr{L}) \geq \frac{1}{4}$.

We will discuss a lower bound for the first eigenvalue of Δ_f of a closed embedded f-minimal hypersurface in the case that the ambient space is complete and noncompact. Precisely, we prove the following:

Theorem 2. *Let $(M^{n+1}, \bar{g}, e^{-f}d\mu)$ be a complete noncompact smooth metric measure space with Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Let Σ be a closed embedded f-minimal hypersurface in M. If there is a bounded domain D in M with convex boundary ∂D so that Σ is contained in D, then the first eigenvalue $\lambda_1(\Delta_f)$ of the weighted Laplacian Δ_f on Σ satisfies*

$$\lambda_1(\Delta_f) \geq \frac{k}{2}. \tag{1}$$

Here and below the boundary ∂D is called convex if, for any $p \in \partial D$, the second fundamental form A of ∂D at p is nonnegative with respect to the outer unit normal of ∂D.

A closed self-shrinker Σ^n in $\mathbb{R}^{n+1}$ satisfies the assumption of Theorem 2 since there always exists a ball D of $\mathbb{R}^{n+1}$containing Σ. Therefore Theorem 2 implies the result of Ding and Xin for self-shrinkers mentioned before. Also we give a different and hence alternative proof of their result.

Remark. If M is a Cartan–Hadamard manifold, all geodesic balls are convex. If M is a complete noncompact Riemannian manifold with nonnegative sectional curvature, the work of Cheeger and Gromoll [1972] asserts that M admits an exhaustion by convex domains.

Choi and Wang [1983] used the lower bound estimate of the first eigenvalue in Theorem 1 to obtain an upper bound on the area of a simply connected closed embedded minimal surface Σ in a 3-manifold, depending on the genus g of Σ and the positive lower bound k of Ricci curvature of M. Further the lower bound on the first eigenvalue and the upper bound on the area were used in [Choi and Schoen 1985] to prove a smooth compactness theorem for the space of closed embedded minimal surfaces of genus g in a closed 3-manifold M^3 with positive Ricci curvature. Very recently Li and Wei [2012] proved a compactness theorem for closed embedded f-minimal surfaces in a compact 3-manifold with Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f \geq k$, for a constant $k > 0$.

On the other hand, Ding and Xin [2013] recently applied the lower bound estimate of the first eigenvalue of the weighted Laplacian on a self-shrinker to prove a compactness theorem for closed self-shrinkers in $\mathbb{R}^3$ with uniform bounds on genus and diameter. As was mentioned before, a self-shrinker in $\mathbb{R}^3$ is an f-minimal surface in a complete noncompact $\mathbb{R}^3$ with $\overline{\mathrm{Ric}}_f \geq \frac{1}{2}$. Motivated by this example, we consider compactness for f-minimal surfaces in a complete noncompact manifold. We prove:

Theorem 3. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Assume that M admits an exhaustion by bounded domains with convex boundary. Then the space, denoted by $S_{D,g}$, of closed embedded f-minimal surfaces in M with genus at most g and diameter at most D is compact in the C^m topology, for any $m \geq 2$. Namely any sequence in $S_{D,g}$ has a subsequence that converges in the C^m topology on compact subsets to a surface in $S_{D,g}$, for any $m \geq 2$.*

Theorem 3 implies especially the compactness theorem of Ding and Xin for self-shrinkers. We also prove the following compactness theorem, which implies Theorem 3.

Theorem 4. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Given a bounded domain $\Omega \subset M$, let S be the space of closed embedded f-minimal surfaces in M with genus at most g and contained in the closure $\overline{\Omega}$. If there is a bounded domain $U \subset M$ with convex boundary so that $\overline{\Omega} \subset U$, then S is compact in the C^m topology, for any $m \geq 2$. Namely any sequence in S has a subsequence that converges in the C^m topology on compact subsets to a surface in S, for any $m \geq 2$.*

If M admits an exhaustion by bounded domains with convex boundary, such U as in Theorem 4 always exists. Also the assumption that f-minimal surfaces are contained in the closure of a bounded domain Ω in Theorem 4 is equivalent to there being a uniform upper bound on the extrinsic diameter of f-minimal surfaces (see remark on page 361).

We mention that, for self-shrinkers in $\mathbb{R}^3$, Colding and Minicozzi [2012b] proved a smooth compactness theorem for complete embedded self-shrinkers with uniform upper bound on genus and uniform scale-invariant area growth. In [Cheng et al. 2012], we generalized their result to the complete embedded f-minimal surfaces in a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, for a constant $k > 0$.

Theorems 3 and 4 have some immediate corollaries. First they imply the corresponding compactness theorems for embedded closed f-minimal surfaces of fixed topological type and bounded diameter; see Theorems 7 and 8. Second, by using an argument as in [Choi and Schoen 1985], we have the following uniform curvature estimates:

Corollary of Theorem 3. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Assume that M admits an exhaustion by bounded domains with convex boundary. Then, for any integer g and a positive constant D, there exists a constant C depending only on M, g and D such that if Σ is a closed embedded f-minimal surface of genus g and diameter at most D in M, the norm $|A|$ of the second fundamental form of Σ satisfies*

$$\max_{x \in \Sigma} |A| \leq C.$$

Corollary of Theorem 4. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Let Ω be a bounded domain whose closure is contained in a bounded domain U with convex boundary. Then, for any integer g, there exists a constant C depending only on U, g such that if Σ is a closed embedded f-minimal surface of genus g contained in $\overline{\Omega}$, the norm $|A|$ of the second fundamental form of Σ satisfies*

$$\max_{x \in \Sigma} |A| \leq C.$$

An argument similar to the proof of Theorem 2 also works for the case where the ambient space is a compact manifold with convex boundary. Hence we have the following estimate:

Theorem 5. *Let $(M^{n+1}, \bar{g})$ be a simply connected compact manifold with convex boundary ∂M and f a nonconstant smooth function on M. Assume that $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. If Σ is a closed f-minimal hypersurface embedded in M and does not intersect the boundary ∂M, then the first eigenvalue of the weighted Laplacian on Σ satisfies*

$$\lambda_1(\Delta_f) \geq \frac{k}{2}. \tag{2}$$

Here we give a remark: the assumption in Theorem 5 that f is a nonconstant smooth function on M is necessary. The reason is that under the assumption

$\overline{\mathrm{Ric}} \geq k > 0$, any closed minimal hypersurface Σ must intersect the convex boundary ∂M by a standard argument similar to the one in Frankel's intersection theorem.

The rest of this paper is organized as follows: In Section 2, some definitions and notation are given. In Section 3, we give some facts which will be used later. In Section 4, we prove Theorems 2 and 5. In Section 6, we prove Theorems 3 and 4. For completeness, we give in an appendix the proof of the known Reilly formula for a weighted metric measure space.

2. Definitions and notation

In general, a smooth metric measure space, denoted by $(N, g, e^{-w} d\mathrm{vol})$, is a Riemannian manifold (N, g) together with a weighted volume form $e^{-w} d\mathrm{vol}$ on N, where w is a smooth function on N and $d\mathrm{vol}$ the volume element induced by the Riemannian metric g. The associated weighted Laplacian Δ_w is defined by

$$\Delta_w u := \Delta u - \langle \nabla w, \nabla u \rangle,$$

where Δ and ∇ are the Laplacian and gradient on (N, g), respectively.

The second-order operator Δ_w is a self-adjoint operator on the space of square integrable functions on N with respect to the measure $e^{-w} d\mathrm{vol}$. For a closed manifold N, the first eigenvalue of Δ_w, denoted by $\lambda_1(\Delta_w)$, is the lowest nonzero real number λ_1 satisfying

$$\Delta_w u = -\lambda_1 u, \quad \text{on } N.$$

It is well known that the definition of $\lambda_1(\Delta_w)$ is equivalent to

$$\lambda_1(\Delta_w) = \inf_{\substack{\int_N u e^{-w} d\mathrm{vol} = 0 \\ u \not\equiv 0}} \frac{\int_N |\nabla u|^2 e^{-w} \, d\mathrm{vol}}{\int_N u^2 e^{-w} \, d\mathrm{vol}}.$$

The ∞-Bakry–Émery Ricci curvature tensor Ric_w (for simplicity, Bakry–Émery Ricci curvature) on $(N, g, e^{-w} d\mathrm{vol})$ is defined by

$$\mathrm{Ric}_w := \mathrm{Ric} + \nabla^2 w,$$

where Ric denotes the Ricci curvature of (N, g) and $\nabla^2 w$ is the Hessian of w on N. If w is constant, Δ_w and Ric_w are the Laplacian Δ and Ricci curvature Ric on N, respectively.

Now let $(M^{n+1}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold. Assume that f is a smooth function on M so that $(M^{n+1}, \bar{g}, e^{-f} d\mu)$ is a smooth metric measure space, where $d\mu$ is the volume element induced by $\bar{g}$.

Let $i : \Sigma^n \to M^{n+1}$ be an n-dimensional smooth immersion. Then

$$i : (\Sigma^n, i^* \bar{g}) \to (M^{n+1}, \bar{g})$$

is an isometric immersion with the induced metric $i^*\bar{g}$. For simplicity, we still denote $i^*\bar{g}$ by $\bar{g}$ whenever there is no confusion. Let $d\sigma$ denote the volume element of $(\Sigma, \bar{g})$. Then the function f induces a weighted measure $e^{-f} d\sigma$ on Σ. Thus we have an induced smooth metric measure space $(\Sigma^n, \bar{g}, e^{-f} d\sigma)$.

In this paper, unless otherwise specified, we denote by a bar all quantities on $(M, \bar{g})$, for instance by $\overline{\nabla}$ and $\overline{\mathrm{Ric}}$, the Levi-Civita connection and the Ricci curvature tensor of $(M, \bar{g})$, respectively. Also we denote, for example, by ∇, Ric, Δ and Δ_f, the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the weighted Laplacian on $(\Sigma, \bar{g})$, respectively. Let $p \in \Sigma$ and ν a unit normal at p. The second fundamental form A, the mean curvature H, and the mean curvature vector $\boldsymbol{H}$ of hypersurface $(\Sigma, \bar{g})$ are defined, respectively, by

$$A : T_p\Sigma \to T_p\Sigma, \quad A(X) := \overline{\nabla}_X \nu, \quad X \in T_p\Sigma,$$
$$H := \operatorname{tr} A = -\sum_{i=1}^{n} \langle \overline{\nabla}_{e_i} e_i, \nu \rangle, \quad \boldsymbol{H} := -H\nu.$$

Define the weighted mean curvature vector $\boldsymbol{H}_f$ and the weighted mean curvature H_f of $(\Sigma, \bar{g})$ by

$$\boldsymbol{H}_f := \boldsymbol{H} - (\overline{\nabla} f)^{\perp} \quad \text{and} \quad \boldsymbol{H}_f = -H_f \nu,$$

where $\perp$ denotes the projection to the normal bundle of Σ. It follows that

$$H_f = H - \langle \overline{\nabla} f, \nu \rangle.$$

Definition. A hypersurface Σ immersed in $(M^{n+1}, \bar{g}, e^{-f} d\mu)$ with the induced metric $\bar{g}$ is called f-minimal if its weighted mean curvature H_f vanishes identically or, equivalently, if it satisfies

$$H = \langle \overline{\nabla} f, \nu \rangle. \tag{3}$$

Definition. The weighted volume of $(\Sigma, \bar{g})$ is defined by

$$V_f(\Sigma) := \int_\Sigma e^{-f} \, d\sigma. \tag{4}$$

It is well known that Σ is f-minimal if and only if Σ is a critical point of the weighted volume functional. Namely it holds that

Proposition 1. *If T is a compactly supported normal variational vector field on Σ (i.e., $T = T^{\perp}$), then the first variation formula of the weighted volume of $(\Sigma, \bar{g})$ is given by*

$$\frac{d}{dt} V_f(\Sigma_t)\Big|_{t=0} = -\int_\Sigma \langle T, \boldsymbol{H}_f \rangle_{\bar{g}} e^{-f} \, d\sigma. \tag{5}$$

On the other hand, an f-minimal hypersurface can be viewed as a minimal hypersurface under a conformal metric. More precisely, define the new metric $\tilde{g} = e^{-2f/n}\bar{g}$ on M, which is conformal to $\bar{g}$. Then the immersion $i : \Sigma \to M$ induces a metric $i^*\tilde{g}$ on Σ from $(M, \tilde{g})$. In the following, $i^*\tilde{g}$ is still denoted by $\tilde{g}$ for simplicity of notation. The volume of $(\Sigma, \tilde{g})$ is

$$\widetilde{V}(\Sigma) := \int_\Sigma d\tilde{\sigma} = \int_\Sigma e^{-f}\, d\sigma = V_f(\Sigma). \tag{6}$$

Hence Proposition 1 and (6) imply that

$$\int_\Sigma \langle T, \widetilde{\boldsymbol{H}}\rangle_{\tilde{g}}\, d\tilde{\sigma} = \int_\Sigma \langle T, \boldsymbol{H}_f\rangle_{\bar{g}} e^{-f}\, d\sigma, \tag{7}$$

where $d\tilde{\sigma} = e^{-f} d\sigma$ and $\widetilde{\boldsymbol{H}}$ denote the volume element and the mean curvature vector of Σ with respect to the conformal metric $\tilde{g}$, respectively.

Equation (7) implies that $\widetilde{\boldsymbol{H}} = e^{2f/n}\boldsymbol{H}_f$. Therefore $(\Sigma, \bar{g})$ is f-minimal in $(M, \bar{g})$ if and only if $(\Sigma, \tilde{g})$ is minimal in $(M, \tilde{g})$.

In this paper, for a closed hypersurface, we choose ν to be the outer unit normal.

3. Some facts on the weighted Laplacian and f-minimal hypersurfaces

In this section, we give some known results which will be used later in this paper. Recall that Reilly [1977] proved an integral version of the Bochner formula for compact domains of a Riemannian manifold, which is called the Reilly formula. Ma and Du [2010] obtained a Reilly formula for metric measure spaces, which is the following proposition. We include its proof in an appendix for the sake of completeness.

Proposition 2. *Let Ω be a compact Riemannian manifold with boundary $\partial\Omega$ and $(\Omega, \bar{g}, e^{-f} d\mu)$ a smooth metric measure space. Then*

$$\begin{aligned}\int_\Omega (\overline{\Delta}_f u)^2 e^{-f} &= \int_\Omega |\overline{\nabla}^2 u|^2 e^{-f} + \int_\Omega \overline{\mathrm{Ric}}_f(\overline{\nabla} u, \overline{\nabla} u) e^{-f} \\ &\quad + 2\int_{\partial\Omega} u_\nu (\Delta_f u) e^{-f} + \int_{\partial\Omega} A(\nabla u, \nabla u) e^{-f} + \int_{\partial\Omega} u_\nu^2 H_f e^{-f},\end{aligned} \tag{8}$$

where ν is the outward pointing unit normal to $\partial\Omega$ and A is the second fundamental form of $\partial\Omega$ with respect to the normal ν, the quantities with bars denote the ones on $(\Omega, \bar{g})$ (for instance, $\overline{\mathrm{Ric}}_f$ denotes the Bakry–Émery Ricci curvature on $(\Omega, \bar{g})$), and Δ_f and H_f denote the weighted Laplacian on $\partial\Omega$ and the weighted mean curvature of $\partial\Omega$, respectively.

A Riemannian manifold with Bakry–Émery Ricci curvature bounded below by a positive constant has some properties similar to a Riemannian manifold with Ricci curvature bounded below by a positive constant. We refer to [Wei and Wylie 2009; Munteanu and Wang 2014; 2012] and the references therein.

Proposition 3 [Morgan 2005] (see also [Wei and Wylie 2009, Corollary 5.1]). *If a complete smooth metric measure space $(N, g, e^{-\omega} d\mu)$ has $\mathrm{Ric}_w \geq k$, where k is a positive constant, then N has finite weighted volume and finite fundamental group.*

For f-minimal hypersurfaces, the following intersection theorem holds.

Proposition 4 [Wei and Wylie 2009, Theorem 7.4]. *Any two closed f-minimal hypersurfaces immersed in a complete smooth metric measure space $(M, \bar{g}, e^{-f} d\mu)$ with $\overline{\mathrm{Ric}}_f > 0$ must intersect. Thus a closed f-minimal hypersurface in M must be connected.*

In [Cheng and Zhou 2013] it was proved that the weighted volume of a self-shrinker Σ^n immersed in $\mathbb{R}^m$ being finite implies it is properly immersed. This result extends to f-minimal submanifolds:

Proposition 5 [Cheng et al. 2012]. *Let Σ^n be an n-dimensional complete f-minimal submanifold immersed in an m-dimensional Riemannian manifold M^m, $n < m$. If Σ has finite weighted volume, then Σ is properly immersed in M.*

An f-minimal hypersurface is an f-minimal submanifold with codimension 1. See more properties of f-minimal submanifolds in [Cheng et al. 2012].

4. Lower bound for $\lambda_1(\Delta_f)$

In this section, we apply the Reilly formula for metric measure spaces to prove Theorems 2 and 5.

Proof of Theorem 2. Since $\overline{\mathrm{Ric}}_f \geq k$, where $k > 0$ is constant, Proposition 3 implies that M has finite fundamental group. We first assume that M is simply connected. Since Σ is connected (Proposition 4) and embedded in M, Σ is orientable and divides M into two components (see its proof in [Choi and Schoen 1985]). Thus Σ divides D into two bounded components Ω_1 and Ω_2. That is $D \backslash \Sigma = \Omega_1 \cup \Omega_2$ with $\partial\Omega_1 = \Sigma$ and $\partial\Omega_2 = \partial D \cup \Sigma$.

For simplicity, we denote by λ_1 the first eigenvalue $\lambda_1(\Delta_f)$ of the weighted Laplacian Δ_f on Σ. Let h be a corresponding eigenfunction so that on Σ

$$\Delta_f h + \lambda_1 h = 0 \quad \text{with} \int_\Sigma h^2 e^{-f} = 1.$$

Consider the solution of the Dirichlet problem on Ω_1 so that

$$\begin{cases} \bar{\Delta}_f u = 0 & \text{in } \Omega_1, \\ u = h & \text{on } \partial\Omega_1 = \Sigma. \end{cases} \tag{9}$$

Substitute Ω_1 for Ω and put the solution u of (9) in Proposition 2. Then the assumption on $\overline{\mathrm{Ric}}_f$ implies that

$$0 \geq k \int_{\Omega_1} |\overline{\nabla} u|^2 e^{-f} - 2\lambda_1 \int_{\Sigma} u_\nu h e^{-f} + \int_{\Sigma} A(\nabla h, \nabla h) e^{-f},$$

where ν is the outer unit normal of Σ with respect to Ω_1. By Stokes' theorem and (9),

$$\int_{\Sigma} u_\nu h e^{-f} = \int_{\Omega_1} (|\overline{\nabla} u|^2 + u\overline{\Delta}_f u) e^{-f} = \int_{\Omega_1} |\overline{\nabla} u|^2 e^{-f}.$$

Thus

$$0 \geq (k - 2\lambda_1) \int_{\Omega_1} |\overline{\nabla} u|^2 e^{-f} + \int_{\Sigma} A(\nabla h, \nabla h) e^{-f}.$$

If $\int_\Sigma A(\nabla h, \nabla h) e^{-f} \geq 0$, by $u \not\equiv C$, we have

$$\lambda_1 \geq \frac{k}{2}.$$

If $\int_\Sigma A(\nabla h, \nabla h) e^{-f} < 0$, we consider the compact domain Ω_2 with the boundary $\partial\Omega_2 = \Sigma \cup \partial D$. Let u be the solution of the mixed problem

$$\begin{cases} \overline{\Delta}_f u = 0 & \text{in } \Omega_2, \\ u = h & \text{on } \Sigma, \\ u_{\tilde{\nu}} = 0 & \text{on } \partial D, \end{cases} \tag{10}$$

where $\tilde{\nu}$ denotes the outer unit normal of ∂D with respect to Ω_2.

Substituting Ω_2 for Ω and putting the solution u of (10) in Proposition 2, we have

$$0 \geq \int_{\Omega_2} |\overline{\nabla}^2 u|^2 e^{-f} + k \int_{\Omega_2} |\overline{\nabla} u|^2 e^{-f} - 2\lambda_1 \int_{\Sigma} h u_{\tilde{\nu}} e^{-f} + \int_{\Sigma} \widetilde{A}(\nabla h, \nabla h) e^{-f} + \int_{\partial D} \widetilde{A}(\nabla u, \nabla u) e^{-f},$$

where $\tilde{\nu}$ denotes the outer unit normal of Σ with respect to Ω_2 and $\widetilde{A}$ denotes the second fundamental form of Σ with respect to the normal $\tilde{\nu}$.

On the other hand, Stokes' theorem and (10) imply

$$\int_{\Omega_2} |\overline{\nabla} u|^2 e^{-f} = \int_{\partial\Omega_2} u u_{\tilde{\nu}} e^{-f} = \int_{\Sigma} h u_{\tilde{\nu}} e^{-f}.$$

Thus we have

$$0 \geq (k - 2\lambda_1) \int_{\Omega_2} |\overline{\nabla} u|^2 e^{-f} + \int_{\Sigma} \widetilde{A}(\nabla h, \nabla h) e^{-f} + \int_{\partial D} \widetilde{A}(\nabla u, \nabla u) e^{-f}. \tag{11}$$

Since ∂D is assumed convex, the last term on the right side of (11) is nonnegative. Observe that the orientations of Σ are opposite for Ω_1 and Ω_2. Namely $\tilde{\nu} = -\nu$.

Then $\widetilde{A}(\nabla u, \nabla u) = -A(\nabla u, \nabla u)$ on Σ. This implies that the second term on the right side of (11) is nonnegative. Thus

$$0 \geq (k - 2\lambda_1) \int_{\Omega_2} |\overline{\nabla} u|^2 e^{-f}.$$

Since u is not a constant function, we conclude that $k - 2\lambda_1 \leq 0$. Again we have

$$\lambda_1 \geq \frac{k}{2}.$$

Therefore we obtain that $\lambda_1(\Delta_f) \geq k/2$ if M is simply connected.

Second, if M is not simply connected, we consider its universal covering $\widehat{M}$, which is a finite $|\pi_1|$-fold covering. $\widehat{M}$ is simply connected and the covering map $\pi : \widehat{M} \to M$ is a locally isometry.

Take $\hat{f} = f \circ \pi$. Obviously $\widehat{M}$ has $\widehat{\mathrm{Ric}}_{\hat{f}} \geq k$, and the lift $\widehat{\Sigma}$ of Σ is also $\hat{f}$-minimal, embedded and closed. By Proposition 4, $\widehat{\Sigma}$ must be connected. Since $\widehat{M}$ is simply connected, the closed embedded connected $\widehat{\Sigma}$ must be orientable and thus divides $\widehat{M}$ into two components. Moreover the connectedness of $\widehat{\Sigma}$ implies that the lift $\widehat{D}$ of D is also a connected domain. Also $\partial\widehat{D} = \widehat{\partial D}$ is smooth and convex. Hence the assertion obtained for the simply connected ambient space can be applied here. Thus the first eigenvalue of the weighted Laplacian $\widehat{\Delta}_{\hat{f}}$ on $\widehat{\Sigma}$ satisfies $\lambda_1(\widehat{\Delta}_{\hat{f}}) \geq k/2$.

Observing the lift of the first eigenfunction of Σ is also an eigenfunction of $\widehat{M}$, we have

$$\lambda_1(\Delta_f) \geq \lambda_1(\widehat{\Delta}_{\hat{f}}) \geq \frac{k}{2}. \qquad \square$$

Remark. In Theorem 2, the boundary ∂D is not necessarily smooth. ∂D can be assumed to be C^1, which is sufficient for the existence of the solution of the mixed problem (10).

Theorem 5 holds by the same argument as that of Theorem 2.

5. Upper bound on area and total curvature of f-minimal surfaces

In this section, we study surfaces in a 3-manifold. First we estimate the corresponding upper bounds on the area and weighted area of an embedded closed f-minimal surface by applying the first eigenvalue estimate in Section 4. Next we discuss the upper bound on the total curvature. We begin with a result of Yang and Yau [1980]:

Proposition 6. *Let Σ^2 be a closed orientable Riemannian surface with genus g. Then the first eigenvalue $\lambda_1(\Delta)$ of the Laplacian Δ on Σ satisfies*

$$\lambda_1(\Delta)\,\mathrm{Area}(\Sigma) \leq 8\pi(1 + g).$$

Using Theorem 2 and Proposition 6, we obtain the following area estimates for closed embedded f-minimal surfaces if the ambient space is simply connected.

Proposition 7. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a simply connected complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Let $\Sigma^2 \subset M$ be a closed embedded f-minimal surface with genus g. If Σ is contained in a bounded domain D with convex boundary ∂D, then its area and weighted area satisfy*

$$\mathrm{Area}(\Sigma) \leq \frac{16\pi(1+g)}{k} e^{\mathrm{osc}_\Sigma f} \tag{12}$$

and

$$\mathrm{Area}_f(\Sigma) \leq \frac{16\pi(1+g)}{k} e^{-\inf_\Sigma f}, \tag{13}$$

where $\mathrm{osc}_\Sigma f = \sup_\Sigma f - \inf_\Sigma f$.

Proof. Consider the conformal metric $\tilde{g} = e^{-f} \bar{g}$ on M. Let $\lambda_1(\widetilde{\Delta})$ be the first eigenvalue of the Laplacian $\widetilde{\Delta}$ on $(\Sigma, \tilde{g})$, which satisfies

$$\lambda_1(\widetilde{\Delta}) = \inf_{\substack{\int_\Sigma u\, d\tilde{\sigma}=0 \\ u\neq 0}} \frac{\int_\Sigma |\widetilde{\nabla} u|^2_{\tilde{g}}\, d\tilde{\sigma}}{\int_\Sigma u^2\, d\tilde{\sigma}},$$

where $\widetilde{\Delta}$, $\widetilde{\nabla}$ and $d\tilde{\sigma}$ are the Laplacian, gradient and area element of Σ with respect to the metric $\tilde{g}$, respectively.

On the other hand, the first eigenvalue of the weighted Laplacian $\lambda_1(\Delta_f)$ on $(\Sigma, \bar{g})$ satisfies

$$\lambda_1(\Delta_f) = \inf_{\substack{\int_\Sigma u e^{-f}\, d\sigma=0 \\ u\neq 0}} \frac{\int_\Sigma |\nabla u|^2_{\bar{g}} e^{-f}\, d\sigma}{\int_\Sigma u^2 e^{-f}\, d\sigma}.$$

Since $\widetilde{\nabla} u = e^f \nabla u$, $d\tilde{\sigma} = e^{-f} d\sigma$ and $\tilde{g} = e^{-f} \bar{g}$,

$$\begin{aligned} \lambda_1(\widetilde{\Delta}) &= \inf_{\substack{\int_\Sigma u e^{-f}\, d\sigma=0 \\ u\neq 0}} \frac{\int_\Sigma |\nabla u|^2_{\bar{g}}\, d\sigma}{\int_\Sigma u^2 e^{-f}\, d\sigma} \\ &\geq \inf_{\substack{\int_\Sigma u e^{-f}\, d\sigma=0 \\ u\neq 0}} \frac{\int_\Sigma |\nabla u|^2_{\bar{g}} e^{-f+\inf_\Sigma(f)}\, d\sigma}{\int_\Sigma u^2 e^{-f}\, d\sigma} \\ &= e^{\inf_\Sigma f} \lambda_1(\Delta_f). \end{aligned}$$

By this inequality, Theorem 2 and Proposition 6, we have the estimate

$$\mathrm{Area}(\Sigma, \tilde{g}) \leq \frac{16\pi(1+g)}{k} e^{-\inf_\Sigma(f)}. \tag{14}$$

Since $\mathrm{Area}_f(\Sigma) = \int_\Sigma e^{-f}\, d\sigma = \mathrm{Area}(\Sigma, \tilde{g})$,

$$\mathrm{Area}_f(\Sigma) \le \frac{16\pi(1+g)}{k} e^{-\inf_\Sigma(f)},$$

which is (13). Thus

$$\mathrm{Area}(\Sigma) \le \frac{16\pi(1+g)}{k} e^{\sup_\Sigma(f) - \inf_\Sigma(f)} = \frac{16\pi(1+g)}{k} e^{\mathrm{osc}_\Sigma(f)}.$$

That is, (12) holds. □

Now, suppose that M is not simply connected. We use a covering argument as in [Choi and Schoen 1985].

Proposition 8. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \ge k$, where k is a positive constant. Let Σ^2 be a closed embedded f-minimal surface. If Σ is contained in a bounded domain D of M with convex boundary ∂D, then*

$$\mathrm{Area}_f(\Sigma) \le \frac{16\pi}{k}\left(\frac{2}{|\pi_1|} - \frac{1}{2}\chi(\Sigma)\right) e^{-\inf_\Sigma f} \tag{15}$$

and

$$\mathrm{Area}(\Sigma) \le \frac{16\pi}{k}\left(\frac{2}{|\pi_1|} - \frac{1}{2}\chi(\Sigma)\right) e^{\mathrm{osc}_\Sigma f}, \tag{16}$$

where $|\pi_1|$ is the order of the fundamental group of M, and $\chi(\Sigma)$ is the Euler characteristic of Σ.

Proof. Let $\widehat{M}$ be the universal covering manifold of M. By Proposition 3, the covering is a finite $|\pi_1|$-fold covering. Let $\widehat{\Sigma}$ be the lifting of Σ. In the proof of Theorem 2, we have shown that $\widehat{\Sigma}$ is orientable and satisfies the assumption of Theorem 2. Hence Theorem 2 implies that the first eigenvalue of the weighted Laplacian of $\widehat{\Sigma}$ satisfies $\lambda_1(\widehat{\Delta}_{\hat{f}}) \ge k/2$, where $\hat{f}$ is the lift of f. By Proposition 7, we conclude that

$$\mathrm{Area}(\widehat{\Sigma}) \le \frac{16\pi}{k}\big(2 - \tfrac{1}{2}\chi(\widehat{\Sigma})\big) e^{\mathrm{osc}_{\widehat{\Sigma}}(\tilde{f})}$$

and

$$\mathrm{Area}_{\hat{f}}(\widehat{\Sigma}) = \int_{\widehat{\Sigma}} e^{-\hat{f}}\, d\sigma \le \frac{16\pi}{k}\big(2 - \tfrac{1}{2}\chi(\widehat{\Sigma})\big) e^{-\inf_{\widehat{\Sigma}}(\hat{f})}.$$

Thus (15) and (16) follow from the equalities

$$\chi(\widehat{\Sigma}) = |\pi_1| \cdot \chi(\Sigma), \quad \inf_{\widehat{\Sigma}}(\hat{f}) = \inf_\Sigma(f), \quad \mathrm{osc}_{\widehat{\Sigma}}(\hat{f}) = \mathrm{osc}_\Sigma(f),$$
$$\mathrm{Area}(\widehat{\Sigma}) = |\pi_1| \cdot \mathrm{Area}(\Sigma) \quad \text{and} \quad \mathrm{Area}_{\hat{f}}(\widehat{\Sigma}) = |\pi_1| \cdot \mathrm{Area}_f(\Sigma).$$ □

In the following, we will give the upper bound for the total curvature of f-minimal surfaces. Here the term the *total curvature* of Σ means $\int_\Sigma |A|^2\, d\sigma$ not $\int_\Sigma K\, d\sigma$.

Proposition 9. *Let $(M^3, \bar{g}, e^{-f}\, d\mu)$ be a complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Let $\Sigma^2 \subset M$ be a closed embedded f-minimal surface with genus g. If Σ is contained in a bounded domain D of M with convex boundary ∂D, then Σ satisfies*

$$\int_\Sigma |A|^2\, d\sigma \leq C, \tag{17}$$

where A is the second fundamental form of $(\Sigma, \bar{g})$ and C is a constant depending on the genus g of Σ, the order $|\pi_1|$ of the fundamental group of M, the maximum $\sup_\Sigma \bar{K}$ of the sectional curvature of M on Σ, the lower bound k of the Bakry–Émery Ricci curvature of M, the oscillation $\operatorname{osc}_\Sigma(f)$ and the maximum $\sup_\Sigma |\bar{\nabla} f|$ on Σ.

Proof. By the Gauss equation and Gauss–Bonnet formula,

$$\begin{aligned}\int_\Sigma |A|^2\, d\sigma = \int_\Sigma H^2 - 2\int_\Sigma (K - \bar{K}) &= \int_\Sigma \langle \bar{\nabla} f, \boldsymbol{n}\rangle^2 - 4\pi\chi(\Sigma) + 2\int_\Sigma \bar{K} \\ &\leq (\sup_\Sigma |\bar{\nabla} f|)^2 \operatorname{Area}(\Sigma) + 8\pi(g-1) + 2(\sup_\Sigma \bar{K}) \operatorname{Area}(\Sigma).\end{aligned}$$

Using (16), we have the conclusion of the theorem. □

To prove the compactness theorem in Section 6, we need the following total curvature estimate for $(\Sigma, \tilde{g})$, which is a minimal surface in $(M, \tilde{g})$.

Proposition 10. *Let $(M^3, \bar{g}, e^{-f}\, d\mu)$ be a complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Let $\Sigma^2 \subset M$ be a closed embedded f-minimal surface with genus g. If Σ is contained in a bounded domain D of M with convex boundary $\partial\Omega$, then Σ satisfies*

$$\int_\Sigma |\widetilde{A}|^2_{\tilde{g}}\, d\tilde{\sigma} \leq C, \tag{18}$$

where $\widetilde{A}$ is the second fundamental form of $(\Sigma, \tilde{g})$ with respect to the conformal metric $\tilde{g} = e^{-f}\bar{g}$ of M and C is a constant depending on the genus g of Σ, the order $|\pi_1|$ of the fundamental group of M, the maximum $\sup_\Sigma \widetilde{K}$ of the sectional curvature of $(M, \tilde{g})$ on Σ, the lower bound k of the Bakry–Émery Ricci curvature of M and the oscillation $\operatorname{osc}_\Sigma(f)$ on Σ.

Proof. By the Gauss equation and the Gauss–Bonnet formula, we have

$$\begin{aligned}\int_\Sigma |\widetilde{A}|^2_{\tilde{g}}\, d\tilde{\sigma} = \int_\Sigma \widetilde{H}^2 - 2\int_\Sigma (\widetilde{K}^\Sigma - \widetilde{K}^M)\, d\tilde{\sigma} &= -4\pi\chi(\Sigma) + 2\int_\Sigma \widetilde{K}\, d\tilde{\sigma} \\ &\leq 8\pi(g-1) + 2(\sup_\Sigma \widetilde{K}) \operatorname{Area}((\Sigma, \tilde{g})) \\ &= 8\pi(g-1) + 2(\sup_\Sigma \widetilde{K}) \operatorname{Area}_f(\Sigma).\end{aligned}$$

We have used $\widetilde{H} = e^f H_f = 0$ and $\mathrm{Area}((\Sigma, \tilde{g})) = \mathrm{Area}_f(\Sigma)$. Now (18) follows from (15). □

6. Compactness of compact f-minimal surfaces

We will prove some compactness theorems for closed embedded f-minimal surfaces in a 3-manifold. We have two ways to prove Theorem 4.

The first proof roughly follows the one in [Colding and Minicozzi 2011] (cf. [Choi and Schoen 1985]) with some modifications. The modifications can be made because we have the assumptions that f-minimal surfaces are contained in the closure of a bounded domain Ω of M and $\overline{\Omega}$ is contained in a bounded domain U with convex boundary. The second proof will need a compactness theorem of complete embedded f-minimal surfaces that was proved in [Cheng et al. 2012].

We prefer to give two proofs here since the first one is independent of the compactness theorem of complete embedded f-minimal surfaces. But the compactness theorem of complete embedded f-minimal surfaces needs a theorem about nonexistence of L_f-stable minimal surfaces (see [Cheng et al. 2012, Theorem 3]).

First proof. We first prove a singular compactness theorem, which is a variation of a result from [Choi and Schoen 1985] (compare [Colding and Minicozzi 2011, Proposition 7.14; Anderson 1985; White 1987]):

Proposition 11. *Let $(M^3, \bar{g})$ be a 3-manifold. Assume that Ω is a bounded domain in M. Let Σ_i be a sequence of closed embedded minimal surfaces contained in $\overline{\Omega}$, with genus g, and satisfying*

$$\mathrm{Area}(\Sigma_i) \le C_1 \tag{19}$$

and

$$\int_{\Sigma_i} |A_{\Sigma_i}|^2 \le C_2. \tag{20}$$

Then there exists a finite set of points $\mathcal{S} \subset \overline{\Omega}$ and a subsequence, still denoted by Σ_i, that converges uniformly in the C^m topology ($m \ge 2$) on compact subsets of $M \setminus \mathcal{S}$ to a complete minimal surface $\Sigma \subset \overline{\Omega}$ (possibly with multiplicity).

The subsequence also converges to Σ in extrinsic Hausdorff distance. Σ is smooth, embedded in M, has genus at most g and satisfies (19) *and* (20).

Proof. We may use the same argument as that of [Colding and Minicozzi 2011, Proposition 7.14]. Moreover $\Sigma_i \subset \overline{\Omega}$ implies that the singular set $S \subset \overline{\Omega}$ and the smooth surface $\Sigma \subset \overline{\Omega}$. Here we omit the details of proof. □

We can apply Proposition 11 to the f-minimal surfaces which are minimal in the conformal metric.

Lemma. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a smooth metric measure space. Assume that Ω is a bounded domain in M. Let $\Sigma_i \subset \overline{\Omega}$ be a sequence of closed embedded f-minimal surfaces of genus g. Suppose that $\tilde{g} = e^{-f}\bar{g}$ on M and $(\Sigma_i, \tilde{g})$ satisfy*

$$\text{Area}((\Sigma_i, \tilde{g})) = \text{Area}_f(\Sigma_i) \le C_1 \tag{21}$$

and

$$\int_{\Sigma_i} |\widetilde{A}_{\Sigma_i}|^2_{\tilde{g}}\, d\tilde{\sigma} \le C_2, \tag{22}$$

where $\widetilde{A}_{\Sigma_i}$ and $d\tilde{\sigma}$ denote the second fundamental form and the volume element of $(\Sigma_i, \tilde{g})$, respectively. Then there exists a finite set of points $\mathcal{S} \subset \overline{\Omega}$ and a subsequence, still denoted by Σ_i, that converges uniformly in the C^m topology ($m \ge 2$) on compact subsets of $M \setminus \mathcal{S}$ to a complete f-minimal surface $\Sigma \subset \overline{\Omega}$ (possibly with multiplicity).

The subsequence also converges to Σ in extrinsic Hausdorff distance. Σ is smooth, embedded in M, has genus at most g, and satisfies (21) *and* (22).

Proof. Since an f-minimal surface in the original metric $\bar{g}$ is equivalent to it being minimal in the conformal metric $\tilde{g}$, we can apply Proposition 11 to get the conclusion of the lemma. □

Proof of Theorem 4. First assume M is simply connected. Since $\Sigma_i \subset \overline{\Omega} \subset U$, we see from Proposition 7 and Proposition 10 that

$$\text{Area}((\Sigma_i, \tilde{g})) = \text{Area}_f(\Sigma_i) \le C_1$$

and

$$\int_{\Sigma_i} |\widetilde{A}_{\Sigma_i}|^2_{\tilde{g}}\, d\sigma_{\tilde{g}} \le C_2,$$

where C_1 and C_2 depend on g, $\sup_{\Omega_j} f$, $\sup_{\Omega_j} \widetilde{K}$ and k.

By the lemma, there exists a finite set of points $\mathcal{S} \subset \widetilde{\Omega}$ and a subsequence $\Sigma_{i'}$ that converges uniformly in the C^m topology (any $m \ge 2$) on compact subsets of $M \setminus \mathcal{S}$ to a complete f-minimal surface $\Sigma \subset \overline{\Omega}$ without boundary (possibly with multiplicity). Σ is smooth, embedded in M and has genus at most g. Equivalently, with respect to the conformal metric $\tilde{g}$, a subsequence $\Sigma_{i'}$ of minimal surfaces converges uniformly in the C^m topology on compact subsets of $M \setminus \mathcal{S}$ to a complete minimal surface Σ, where $\Sigma \subset \overline{\Omega}$.

Since complete embedded $\Sigma \subset \overline{\Omega}$ satisfies (21), it must be properly embedded (Proposition 5), thus closed and orientable.

We need to prove that the convergence is smooth across the points $\mathcal{S}$. By Allard's regularity theorem, it suffices to prove that the convergence has multiplicity one. If the multiplicity is not one, by a proof similar to that of [Choi and Schoen 1985] (see also [Colding and Minicozzi 2011, p. 249]), we can show that there is an i big enough and a Σ_i in the convergent subsequence, so that the first eigenvalue of the Laplacian $\widetilde{\Delta}^{\Sigma_i}$ on Σ_i with the conformal metric $\tilde{g}$ satisfies $\lambda_1(\widetilde{\Delta}^{\Sigma_i}) < ke^{\inf_\Omega f}/2$. We have

$$\begin{aligned}\lambda_1(\widetilde{\Delta}^{\Sigma_i}) &= \inf\left\{\frac{\int_{\Sigma_i}|\widetilde{\nabla}\phi|^2_{\tilde{g}}\,d\tilde{\sigma}}{\int_{\Sigma_i}\phi^2\,d\tilde{\sigma}}, \int_{\Sigma_i}\phi\,d\tilde{\sigma}=0\right\}\\ &= \inf\left\{\frac{\int_{\Sigma_i}|\nabla\phi|^2 d\sigma}{\int_{\Sigma_i}\phi^2 e^{-f}\,d\sigma}, \int_{\Sigma_i}\phi e^{-f}\,d\sigma=0\right\}\\ &\geq \lambda_1(\Delta_f^{\Sigma_i})e^{\inf_\Omega f}.\end{aligned}$$

By Theorem 2, $\Sigma_i \subset \overline{\Omega} \subset U$ implies $\lambda_1(\Delta_f^{\Sigma_i}) \geq k/2$. Thus we have a contradiction.

When M is not simply connected, we use a covering argument. The assumption of $\overline{\mathrm{Ric}}_f \geq k$, where $k > 0$ is constant, implies that M has finite fundamental group π_1 (Proposition 3). We consider the finite-fold universal covering $\widehat{M}$. By the proof of Theorem 2, we know that the corresponding lifts of Σ_i, $\overline{\Omega}$ and U satisfy $\widehat{\Sigma}_i \subset \widehat{\overline{\Omega}} \subset \widehat{U}$. Then Propositions 8 and 10 give uniform bounds on the area and total curvature in the conformal metric $\hat{\tilde{g}}$ on $\widehat{M}$. By the assertion on the simply connected ambient manifold before, we have the smooth convergence of a subsequence of $\widehat{\Sigma}_i$. This implies the smooth convergence of a subsequence of Σ_i. □

Second Proof. In [Cheng et al. 2012], we proved the following:

Theorem 6. *Let $(M^3, \bar{g}, e^{-f}d\mu)$ be a complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Given an integer $g \geq 0$ and a constant $V > 0$, the space $S_{g,V}$ of smooth complete embedded f-minimal surfaces $\Sigma \subset M$ with*

- *genus at most g,*
- *$\partial\Sigma = \varnothing$, and*
- *$\int_\Sigma e^{-f}\,d\sigma \leq V$*

is compact in the C^m topology, for any $m \geq 2$. Namely any sequence of $S_{g,V}$ has a subsequence that converges in the C^m topology on compact subsets to a surface in $S_{D,g}$, for any $m \geq 2$.

Proof of Theorem 4. Since a surface in S is contained in $\bar{\Omega} \subset U$, by Proposition 8, we have the uniform bound V of the weighted volume of closed embedded f-minimal surfaces in S. Hence Theorem 6 can be applied. Moreover $\Sigma_i \subset \overline{\Omega}$ implies that the smooth limit surface $\Sigma \subset \overline{\Omega}$. Otherwise, since the subsequence $\{\Sigma_i\}$ converges uniformly in the C^m topology ($m \geq 2$) on any compact subset of M to Σ, there is a surface Σ_i (with index i big enough) in the subsequence that would not satisfy $\Sigma_i \subset \overline{\Omega}$.

By Proposition 5, Σ must be properly embedded. Thus Σ must be closed. $\square$

To prove Theorem 3 we require a lemma.

Lemma. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k > 0$. If Σ is any closed f-minimal surface in M with genus at most g and diameter at most D, then $\Sigma \subset \overline{B}_r(p)$ for some $r > 0$ (independent of Σ), where $B_r(p)$ is a ball in M with radius r centered at $p \in M$.*

Proof. Fix a closed f-minimal surface Σ_0. Obviously $\Sigma_0 \subset B_{r_0}(p)$ for some $r_0 > 0$. Proposition 4 says that Σ and Σ_0 must intersect. Then, for $x \in \Sigma$,

$$d(p, x) \leq d(p, x_0) + d(x_0, x) \leq r_0 + D, x_0 \in \Sigma_0.$$

Taking $r = r_0 + D$, we have $\Sigma \subset \overline{B}_{r_0+D}$. $\square$

Remark. In the lemma and hence in Theorem 3, D is a bound on the intrinsic diameter of closed f-minimal surfaces or a bound on the extrinsic diameter of closed f-minimal surfaces. Also, by Proposition 4, the assumption that f-minimal surfaces are contained in the closure of a bounded domain Ω in Theorem 4 is equivalent to that of a uniform upper bound on the extrinsic diameter of f-minimal surfaces.

Proof of Theorem 3. By the lemma immediately above, we may apply Theorem 4 to the space $S_{D,g}$. Next the closed embedded limit Σ must have diameter at most D. Otherwise, since the subsequence $\{\Sigma_i\}$ converges uniformly in the C^m topology ($m \geq 2$) on any compact subset of M to Σ, there is a surface Σ_i (with the index i big enough) in the subsequence that would have diameter greater than D. So Σ must be in $S_{D,g}$. $\square$

As a corollary, Theorem 3 implies:

Theorem 7. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Assume that M admits an exhaustion by bounded domains with convex boundary. Then the space of closed embedded f-minimal surface in M of fixed topological type and diameter at most D is compact in the C^m topology, for any $m \geq 2$.*

Proof of Theorem 7. By Theorem 3, it suffices to prove that the limit f-minimal surface of a convergent subsequence in the given space has the same topological type, which holds by the Gauss–Bonnet formula and smooth convergence. $\square$

Similar to the proof of Theorem 7, Theorem 4 implies:

Theorem 8. *Let $(M^3, \bar{g}, e^{-f}d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Assume that Ω is a bounded domain and U is a bounded domain with convex boundary so that $\overline{\Omega} \subset U$. Then the space of closed embedded f-minimal surface in M of fixed topological type and contained in the closure $\overline{\Omega}$ is compact in the C^m topology, for any $m \geq 2$.*

Appendix: Proof of Proposition 2

The Bochner formula implies that

$$\tfrac{1}{2}\overline{\Delta}_f|\overline{\nabla}u|^2 - \langle\overline{\nabla}u, \overline{\nabla}(\overline{\Delta}_f u)\rangle = |\overline{\nabla}^2 u|^2 + \overline{\mathrm{Ric}}_f(\overline{\nabla}u, \overline{\nabla}u).$$

Integrating this equation on Ω with respect to the weighted measure $e^{-f}d\mu$, we obtain

$$\int_\Omega \big(\tfrac{1}{2}\overline{\Delta}_f|\overline{\nabla}u|^2 - \langle\overline{\nabla}u, \overline{\nabla}(\overline{\Delta}_f u)\rangle\big)e^{-f} = \int_\Omega |\overline{\nabla}^2 u|^2 e^{-f} + \int_\Omega \overline{\mathrm{Ric}}_f(\overline{\nabla}u, \overline{\nabla}u)e^{-f}.$$

On the other hand, by the divergence formula, we have

$$\begin{aligned}\tfrac{1}{2}\overline{\Delta}_f|\overline{\nabla}u|^2 &- \langle\overline{\nabla}u, \overline{\nabla}(\overline{\Delta}_f u)\rangle \\ &= \tfrac{1}{2}\overline{\mathrm{div}}\big(e^{-f}\overline{\nabla}|\overline{\nabla}u|^2\big)e^f - \overline{\mathrm{div}}\big(e^{-f}\overline{\Delta}_f(u)\overline{\nabla}u\big)e^f + (\overline{\Delta}_f u)^2.\end{aligned}$$

Integrating and applying Stokes' theorem, we have

$$\begin{aligned}(23)\quad \int_\Omega \big(\tfrac{1}{2}\overline{\Delta}_f|\overline{\nabla}u|^2 &- \langle\overline{\nabla}u, \overline{\nabla}(\overline{\Delta}_f u)\rangle\big)e^{-f} \\ &= \int_{\partial\Omega}\big(\tfrac{1}{2}|\overline{\nabla}u|^2_\nu - (\overline{\Delta}_f u)u_\nu\big)e^{-f} + \int_\Omega (\overline{\Delta}_f u)^2 e^{-f}.\end{aligned}$$

Then

$$\begin{aligned}(24)\quad \tfrac{1}{2}|\overline{\nabla}u|^2_\nu &- (\overline{\Delta}_f u)u_\nu \\ &= \langle\overline{\nabla}_\nu\overline{\nabla}u, \overline{\nabla}u\rangle - (\overline{\Delta}_f u)u_\nu = \langle\overline{\nabla}_{\overline{\nabla}u}\overline{\nabla}u, \nu\rangle - (\overline{\Delta}_f u)u_\nu \\ &= \langle\overline{\nabla}_\nu\overline{\nabla}u, \nu\rangle u_\nu + \langle\overline{\nabla}_{\nabla u}\overline{\nabla}u, \nu\rangle - (\overline{\Delta}_f u)u_\nu \\ &= \big(\langle\overline{\nabla}_\nu\overline{\nabla}u, \nu\rangle - \overline{\Delta}u + \langle\overline{\nabla}f, \overline{\nabla}u\rangle\big)u_\nu + \langle\nabla u, \nabla u_\nu\rangle - \langle\overline{\nabla}u, \overline{\nabla}_{\nabla u}\nu\rangle \\ &= \big(-\Delta u - Hu_\nu + \langle\nabla f, \nabla u\rangle + \langle\overline{\nabla}f, \nu\rangle u_\nu\big)u_\nu + \langle\nabla u, \nabla u_\nu\rangle - \langle\nabla u, \overline{\nabla}_{\nabla u}\nu\rangle \\ &= -(\Delta_f u + H_f u_\nu)u_\nu + \langle\nabla u, \nabla u_\nu\rangle - A(\nabla u, \nabla u),\end{aligned}$$

where $H_f = H - \langle \overline{\nabla} f, \nu \rangle$. By substituting (24) into (23), we obtain

$$\begin{aligned}
&\int_\Omega \big(\tfrac{1}{2}\bar{\Delta}_f |\overline{\nabla} u|^2 - \langle \overline{\nabla} u, \overline{\nabla}(\bar{\Delta}_f u)\rangle\big)e^{-f} \\
&\quad = -\int_{\partial\Omega} (\Delta_f u) u_\nu e^{-f} - \int_{\partial\Omega} H_f u_\nu^2 e^{-f} + \int_{\partial\Omega} \big(\langle \nabla u, \nabla u_\nu\rangle - A(\nabla u, \nabla u)\big)e^{-f} \\
&\qquad\qquad + \int_\Omega (\bar{\Delta}_f u)^2 e^{-f} \\
&\quad = -2\int_{\partial\Omega} (\Delta_f u) u_\nu e^{-f} - \int_{\partial\Omega} H_f u_\nu^2 e^{-f} - \int_{\partial\Omega} A(\nabla u, \nabla u) e^{-f} + \int_\Omega (\bar{\Delta}_f u)^2 e^{-f}.
\end{aligned}$$

This immediately implies (8).

References

[Anderson 1985] M. T. Anderson, "Curvature estimates for minimal surfaces in 3-manifolds", *Ann. Sci. École Norm. Sup.* (4) **18**:1 (1985), 89–105. MR 87e:53098 Zbl 0578.49027

[Cao 2010] H.-D. Cao, "Recent progress on Ricci solitons", pp. 1–38 in *Recent advances in geometric analysis* (Taipei, 2007), edited by Y.-I. Lee et al., Adv. Lect. Math. **11**, International Press, Somerville, MA, 2010. MR 2011d:53061 Zbl 1201.53046 arXiv 0908.2006

[Cao and Zhou 2010] H.-D. Cao and D. Zhou, "On complete gradient shrinking Ricci solitons", *J. Differential Geom.* **85**:2 (2010), 175–185. MR 2011k:53040 Zbl 1246.53051

[Cheeger and Gromoll 1972] J. Cheeger and D. Gromoll, "On the structure of complete manifolds of nonnegative curvature", *Ann. of Math.* (2) **96** (1972), 413–443. MR 46 #8121 Zbl 0246.53049

[Cheng and Zhou 2013] X. Cheng and D. Zhou, "Volume estimate about shrinkers", *Proc. Amer. Math. Soc.* **141**:2 (2013), 687–696. MR 2996973 Zbl 1262.53030

[Cheng et al. 2012] X. Cheng, T. Mejia, and D. Zhou, "Stability and compactness for complete f-minimal surfaces", preprint, 2012. To appear in Trans. Amer. Math. Soc. arXiv 1210.8076

[Choi and Schoen 1985] H. I. Choi and R. Schoen, "The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature", *Invent. Math.* **81**:3 (1985), 387–394. MR 87a:58040 Zbl 0577.53044

[Choi and Wang 1983] H. I. Choi and A. N. Wang, "A first eigenvalue estimate for minimal hypersurfaces", *J. Differential Geom.* **18**:3 (1983), 559–562. MR 85d:53028 Zbl 0523.53055

[Colding and Minicozzi 2002] T. H. Colding and W. P. Minicozzi, II, "Estimates for parametric elliptic integrands", *Int. Math. Res. Not.* **2002**:6 (2002), 291–297. MR 2002k:53060 Zbl 1002.53035

[Colding and Minicozzi 2011] T. H. Colding and W. P. Minicozzi, II, *A course in minimal surfaces*, Graduate Studies in Mathematics **121**, American Mathematical Society, Providence, RI, 2011. MR 2780140 Zbl 1242.53007

[Colding and Minicozzi 2012a] T. H. Colding and W. P. Minicozzi, II, "Generic mean curvature flow, I: Generic singularities", *Ann. of Math.* (2) **175**:2 (2012), 755–833. MR 2993752 Zbl 1239.53084

[Colding and Minicozzi 2012b] T. H. Colding and W. P. Minicozzi, II, "Smooth compactness of self-shrinkers", *Comment. Math. Helv.* **87**:2 (2012), 463–475. MR 2914856 Zbl 1258.53069

[Ding and Xin 2013] Q. Ding and Y. L. Xin, "Volume growth, eigenvalue and compactness for self-shrinkers", *Asian J. Math.* **17**:3 (2013), 443–456. MR 3119795 Zbl 1283.53062

[Li and Wei 2012] H. Li and Y. Wei, "f-minimal surface and manifold with positive m-Bakry–Émery Ricci curvature", preprint, 2012. To appear in J. Geom. Anal. arXiv 1209.0895v1

[Ma and Du 2010] L. Ma and S.-H. Du, "Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians", *C. R. Math. Acad. Sci. Paris* **348**:21-22 (2010), 1203–1206. MR 2011m:58051 Zbl 1208.58028

[Morgan 2005] F. Morgan, "Manifolds with density", *Notices Amer. Math. Soc.* **52**:8 (2005), 853–858. MR 2006g:53044 Zbl 1118.53022

[Munteanu and Wang 2012] O. Munteanu and J. Wang, "Analysis of weighted Laplacian and applications to Ricci solitons", *Comm. Anal. Geom.* **20**:1 (2012), 55–94. MR 2903101 Zbl 1245.53039

[Munteanu and Wang 2014] O. Munteanu and J. Wang, "Geometry of manifolds with densities", *Adv. Math.* **259** (2014), 269–305. MR 3197658 Zbl 1290.53048

[Reilly 1977] R. C. Reilly, "Applications of the Hessian operator in a Riemannian manifold", *Indiana Univ. Math. J.* **26**:3 (1977), 459–472. MR 57 #13799 Zbl 0391.53019

[Wei and Wylie 2009] G. Wei and W. Wylie, "Comparison geometry for the Bakry–Émery Ricci tensor", *J. Differential Geom.* **83**:2 (2009), 377–405. MR 2011a:53064 Zbl 1189.53036

[White 1987] B. White, "Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals", *Invent. Math.* **88**:2 (1987), 243–256. MR 88g:58037 Zbl 0615.53044

[Yang and Yau 1980] P. C. Yang and S. T. Yau, "Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **7**:1 (1980), 55–63. MR 81m:58084 Zbl 0446.58017

Received July 4, 2013. Revised May 14, 2014.

XU CHENG
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE FEDERAL FLUMINENSE - UFF
24020-140 CENTRO, NITERÓI-RJ
BRAZIL
xcheng@impa.br

TITO MEJIA
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE FEDERAL FLUMINENSE - UFF
24020-140 CENTRO, NITERÓI-RJ
BRAZIL
tmejia.uff@gmail.com

DETANG ZHOU
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE FEDERAL FLUMINENSE - UFF
24020-140 CENTRO, NITERÓI-RJ
BRAZIL
zhou@impa.br

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US $410/year for the electronic version, and $535/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 271 No. 2 October 2014

0030-8730(201410)271:2;1-6