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Let Ω be a bounded domain with convex boundary in a complete noncompact Riemannian manifold with Bakry–Émery Ricci curvature bounded below by a positive constant. We prove a lower bound on the first eigenvalue of the weighted Laplacian for closed embedded f-minimal hypersurfaces contained in Ω. Using this estimate, we prove a compactness theorem for the space of closed embedded f-minimal surfaces with uniform upper bounds on genus and diameter in a complete 3-manifold with Bakry–Émery Ricci curvature bounded below by a positive constant and admitting an exhaustion by bounded domains with convex boundary.

1. Introduction

A hypersurface Σ immersed in a Riemannian manifold $(M, \bar{g})$ is said to be f-minimal if its mean curvature H satisfies, for any $p \in \Sigma$,

$$H = \langle \overline{\nabla} f, \nu \rangle,$$

where ν is the unit normal at $p \in \Sigma$, f is a smooth function defined on M, and $\overline{\nabla} f$ denotes the gradient of f on M. When f is a constant function, an f-minimal hypersurface is just a minimal hypersurface. One nontrivial class of f-minimal hypersurfaces is that of self-shrinkers. Recall that a self-shrinker (for the mean curvature flow in the Euclidean space $(\mathbb{R}^{n+1}, g_{\text{can}})$) is a hypersurface immersed in $(\mathbb{R}^{n+1}, g_{\text{can}})$ satisfying

$$H = \tfrac{1}{2}\langle x, \nu \rangle,$$

where x is the position vector in $\mathbb{R}^{n+1}$. Hence a self-shrinker is an f-minimal hypersurface Σ with $f = |x|^2/4$ (see more information on self-shrinkers in [Colding and Minicozzi 2012a] and references therein).

In the study of f-minimal hypersurfaces, it is convenient to consider the ambient space as a smooth metric measure space $(M, \bar{g}, e^{-f} d\mu)$, where $d\mu$ is the volume

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form of $\bar{g}$. For $(M, \bar{g}, e^{-f}\, d\mu)$, an important and natural tensor is the Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f := \overline{\mathrm{Ric}} + \overline{\nabla}^2 f$. There are many interesting examples of smooth metric measure spaces $(M, \bar{g}, e^{-f}\, d\mu)$ with $\overline{\mathrm{Ric}}_f \geq k$, for a positive constant k. A nontrivial class of examples is the shrinking gradient Ricci solitons. It is known that, after a normalization, a shrinking gradient Ricci soliton $(M, \bar{g}, f)$ satisfies the equation $\overline{\mathrm{Ric}} + \overline{\nabla}^2 f = \bar{g}/2$ or, equivalently, $\overline{\mathrm{Ric}}_f = \frac{1}{2}$. We refer to [Cao 2010], a survey of this topic where some compact and noncompact examples are explained. Even though the asymptotic growth of the potential function f of a noncompact shrinking gradient Ricci soliton is the same as that of a Gaussian shrinking soliton [Cao and Zhou 2010], both the geometry and topology can be quite different from known examples. We may consider f-minimal hypersurfaces in a shrinking gradient Ricci soliton. For instance, a self-shrinker in $\mathbb{R}^{n+1}$ can be viewed as an f-minimal hypersurface in the Gaussian shrinking soliton $(\mathbb{R}^{n+1}, g_{\mathrm{can}}, |x|^2/4)$.

There are other examples of f-minimal hypersurfaces. Let M be the hyperbolic space $\mathbb{H}^{n+1}(-1)$. Let r denote the distance function from a fixed point $p \in M$ and $f(x) = nar^2(x)$, where $a > 0$ is a constant. Then $\overline{\mathrm{Ric}}_f \geq n(2a-1)$, and the geodesic sphere of radius r centered at p in $\mathbb{H}^{n+1}(-1)$ is an f-minimal hypersurface if it satisfies $2ar = \coth r$.

An f-minimal hypersurface Σ has two aspects to view. One is that Σ is f-minimal if and only if Σ is a critical point of the weighted volume functional $e^{-f}\, d\sigma$, where $d\sigma$ is the volume element of Σ. Another one is that Σ is f-minimal if and only if Σ is minimal in the new conformal metric $\tilde{g} = e^{-2f/n}\bar{g}$ (see Section 2). f-minimal hypersurfaces, even more general stationary hypersurfaces for parametric elliptic functionals, have been studied before. See, for instance, the work of White [1987] and Colding and Minicozzi [2002].

In this paper, we will first estimate the lower bound on the first eigenvalue of the weighted Laplacian $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ for closed (i.e., compact and without boundary) embedded f-minimal hypersurfaces in a complete metric measure space $(M, \bar{g}, e^{-f}\, d\mu)$. Subsequently using the eigenvalue estimate, we study compactness for the space of closed embedded f-minimal surfaces in a complete noncompact 3-manifold. To explain our result, we give some background.

Choi and Wang [1983] estimated the lower bound for the first eigenvalue of closed minimal hypersurfaces in a complete Riemannian manifold with Ricci curvature bounded below by a positive constant and proved the following:

Theorem 1. *If M is a simply connected complete Riemannian manifold with Ricci curvature bounded below by a constant $k > 0$ and Σ is a closed embedded minimal hypersurface, then the first eigenvalue of the Laplacian Δ on Σ is at least $k/2$.*

Later, using a covering argument, Choi and Schoen [1985] proved that the assumption that M is simply connected is not needed. Recently Ma and Du [2010] extended Theorem 1 to the first eigenvalue of the weighted Laplacian Δ_f on a closed embedded f-minimal hypersurface in a simply connected compact manifold with positive Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f$. Very recently Li and Wei [2012] also used the covering argument to delete the assumption that the ambient space is simply connected in the result of Ma and Du.

The Bonnet–Myers theorem says that a complete manifold with Ricci curvature bounded below by a positive constant must be compact. But the corresponding result is not true for complete manifolds with Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f$ bounded below by a positive constant. One example is the Gaussian shrinking soliton $(\mathbb{R}^{n+1}, g_{\mathrm{can}}, e^{-|x|^2/4}d\mu)$, with $\overline{\mathrm{Ric}}_f = \frac{1}{2}$. Hence the theorems of Ma and Du and Li and Wei cannot be applied to self-shrinkers.

For self-shrinkers, Ding and Xin [2013] recently obtained a lower bound on the first eigenvalue $\lambda_1(\mathscr{L})$ of the weighted Laplacian $\mathscr{L} = \Delta - \langle x, \nabla \cdot \rangle/2$ (i.e., Δ_f) on a closed n-dimensional embedded self-shrinker in the Euclidean space $\mathbb{R}^{n+1}$, that is, $\lambda_1(\mathscr{L}) \geq \frac{1}{4}$.

We will discuss a lower bound for the first eigenvalue of Δ_f of a closed embedded f-minimal hypersurface in the case that the ambient space is complete and noncompact. Precisely, we prove the following:

Theorem 2. *Let $(M^{n+1}, \bar{g}, e^{-f}d\mu)$ be a complete noncompact smooth metric measure space with Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Let Σ be a closed embedded f-minimal hypersurface in M. If there is a bounded domain D in M with convex boundary ∂D so that Σ is contained in D, then the first eigenvalue $\lambda_1(\Delta_f)$ of the weighted Laplacian Δ_f on Σ satisfies*

$$\lambda_1(\Delta_f) \geq \frac{k}{2}. \tag{1}$$

Here and below the boundary ∂D is called convex if, for any $p \in \partial D$, the second fundamental form A of ∂D at p is nonnegative with respect to the outer unit normal of ∂D.

A closed self-shrinker Σ^n in $\mathbb{R}^{n+1}$ satisfies the assumption of Theorem 2 since there always exists a ball D of $\mathbb{R}^{n+1}$containing Σ. Therefore Theorem 2 implies the result of Ding and Xin for self-shrinkers mentioned before. Also we give a different and hence alternative proof of their result.

Remark. If M is a Cartan–Hadamard manifold, all geodesic balls are convex. If M is a complete noncompact Riemannian manifold with nonnegative sectional curvature, the work of Cheeger and Gromoll [1972] asserts that M admits an exhaustion by convex domains.

Choi and Wang [1983] used the lower bound estimate of the first eigenvalue in Theorem 1 to obtain an upper bound on the area of a simply connected closed embedded minimal surface Σ in a 3-manifold, depending on the genus g of Σ and the positive lower bound k of Ricci curvature of M. Further the lower bound on the first eigenvalue and the upper bound on the area were used in [Choi and Schoen 1985] to prove a smooth compactness theorem for the space of closed embedded minimal surfaces of genus g in a closed 3-manifold M^3 with positive Ricci curvature. Very recently Li and Wei [2012] proved a compactness theorem for closed embedded f-minimal surfaces in a compact 3-manifold with Bakry–Émery Ricci curvature $\overline{\mathrm{Ric}}_f \geq k$, for a constant $k > 0$.

On the other hand, Ding and Xin [2013] recently applied the lower bound estimate of the first eigenvalue of the weighted Laplacian on a self-shrinker to prove a compactness theorem for closed self-shrinkers in $\mathbb{R}^3$ with uniform bounds on genus and diameter. As was mentioned before, a self-shrinker in $\mathbb{R}^3$ is an f-minimal surface in a complete noncompact $\mathbb{R}^3$ with $\overline{\mathrm{Ric}}_f \geq \frac{1}{2}$. Motivated by this example, we consider compactness for f-minimal surfaces in a complete noncompact manifold. We prove:

Theorem 3. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Assume that M admits an exhaustion by bounded domains with convex boundary. Then the space, denoted by $S_{D,g}$, of closed embedded f-minimal surfaces in M with genus at most g and diameter at most D is compact in the C^m topology, for any $m \geq 2$. Namely any sequence in $S_{D,g}$ has a subsequence that converges in the C^m topology on compact subsets to a surface in $S_{D,g}$, for any $m \geq 2$.*

Theorem 3 implies especially the compactness theorem of Ding and Xin for self-shrinkers. We also prove the following compactness theorem, which implies Theorem 3.

Theorem 4. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Given a bounded domain $\Omega \subset M$, let S be the space of closed embedded f-minimal surfaces in M with genus at most g and contained in the closure $\overline{\Omega}$. If there is a bounded domain $U \subset M$ with convex boundary so that $\overline{\Omega} \subset U$, then S is compact in the C^m topology, for any $m \geq 2$. Namely any sequence in S has a subsequence that converges in the C^m topology on compact subsets to a surface in S, for any $m \geq 2$.*

If M admits an exhaustion by bounded domains with convex boundary, such U as in Theorem 4 always exists. Also the assumption that f-minimal surfaces are contained in the closure of a bounded domain Ω in Theorem 4 is equivalent to there being a uniform upper bound on the extrinsic diameter of f-minimal surfaces (see remark on page 361).

We mention that, for self-shrinkers in $\mathbb{R}^3$, Colding and Minicozzi [2012b] proved a smooth compactness theorem for complete embedded self-shrinkers with uniform upper bound on genus and uniform scale-invariant area growth. In [Cheng et al. 2012], we generalized their result to the complete embedded f-minimal surfaces in a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, for a constant $k > 0$.

Theorems 3 and 4 have some immediate corollaries. First they imply the corresponding compactness theorems for embedded closed f-minimal surfaces of fixed topological type and bounded diameter; see Theorems 7 and 8. Second, by using an argument as in [Choi and Schoen 1985], we have the following uniform curvature estimates:

Corollary of Theorem 3. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Assume that M admits an exhaustion by bounded domains with convex boundary. Then, for any integer g and a positive constant D, there exists a constant C depending only on M, g and D such that if Σ is a closed embedded f-minimal surface of genus g and diameter at most D in M, the norm $|A|$ of the second fundamental form of Σ satisfies*

$$\max_{x \in \Sigma} |A| \leq C.$$

Corollary of Theorem 4. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Let Ω be a bounded domain whose closure is contained in a bounded domain U with convex boundary. Then, for any integer g, there exists a constant C depending only on U, g such that if Σ is a closed embedded f-minimal surface of genus g contained in $\overline{\Omega}$, the norm $|A|$ of the second fundamental form of Σ satisfies*

$$\max_{x \in \Sigma} |A| \leq C.$$

An argument similar to the proof of Theorem 2 also works for the case where the ambient space is a compact manifold with convex boundary. Hence we have the following estimate:

Theorem 5. *Let $(M^{n+1}, \bar{g})$ be a simply connected compact manifold with convex boundary ∂M and f a nonconstant smooth function on M. Assume that $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. If Σ is a closed f-minimal hypersurface embedded in M and does not intersect the boundary ∂M, then the first eigenvalue of the weighted Laplacian on Σ satisfies*

$$\lambda_1(\Delta_f) \geq \frac{k}{2}. \tag{2}$$

Here we give a remark: the assumption in Theorem 5 that f is a nonconstant smooth function on M is necessary. The reason is that under the assumption

$\overline{\mathrm{Ric}} \geq k > 0$, any closed minimal hypersurface Σ must intersect the convex boundary ∂M by a standard argument similar to the one in Frankel's intersection theorem.

The rest of this paper is organized as follows: In Section 2, some definitions and notation are given. In Section 3, we give some facts which will be used later. In Section 4, we prove Theorems 2 and 5. In Section 6, we prove Theorems 3 and 4. For completeness, we give in an appendix the proof of the known Reilly formula for a weighted metric measure space.

2. Definitions and notation

In general, a smooth metric measure space, denoted by $(N, g, e^{-w} d\mathrm{vol})$, is a Riemannian manifold (N, g) together with a weighted volume form $e^{-w} d\mathrm{vol}$ on N, where w is a smooth function on N and $d\mathrm{vol}$ the volume element induced by the Riemannian metric g. The associated weighted Laplacian Δ_w is defined by

$$\Delta_w u := \Delta u - \langle \nabla w, \nabla u \rangle,$$

where Δ and ∇ are the Laplacian and gradient on (N, g), respectively.

The second-order operator Δ_w is a self-adjoint operator on the space of square integrable functions on N with respect to the measure $e^{-w} d\mathrm{vol}$. For a closed manifold N, the first eigenvalue of Δ_w, denoted by $\lambda_1(\Delta_w)$, is the lowest nonzero real number λ_1 satisfying

$$\Delta_w u = -\lambda_1 u, \quad \text{on } N.$$

It is well known that the definition of $\lambda_1(\Delta_w)$ is equivalent to

$$\lambda_1(\Delta_w) = \inf_{\substack{\int_N u e^{-w} d\mathrm{vol}=0 \\ u \not\equiv 0}} \frac{\int_N |\nabla u|^2 e^{-w} \, d\mathrm{vol}}{\int_N u^2 e^{-w} \, d\mathrm{vol}}.$$

The ∞-Bakry–Émery Ricci curvature tensor Ric_w (for simplicity, Bakry–Émery Ricci curvature) on $(N, g, e^{-w} d\mathrm{vol})$ is defined by

$$\mathrm{Ric}_w := \mathrm{Ric} + \nabla^2 w,$$

where Ric denotes the Ricci curvature of (N, g) and $\nabla^2 w$ is the Hessian of w on N. If w is constant, Δ_w and Ric_w are the Laplacian Δ and Ricci curvature Ric on N, respectively.

Now let $(M^{n+1}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold. Assume that f is a smooth function on M so that $(M^{n+1}, \bar{g}, e^{-f} d\mu)$ is a smooth metric measure space, where $d\mu$ is the volume element induced by $\bar{g}$.

Let $i : \Sigma^n \to M^{n+1}$ be an n-dimensional smooth immersion. Then

$$i : (\Sigma^n, i^* \bar{g}) \to (M^{n+1}, \bar{g})$$

is an isometric immersion with the induced metric $i^*\bar{g}$. For simplicity, we still denote $i^*\bar{g}$ by $\bar{g}$ whenever there is no confusion. Let $d\sigma$ denote the volume element of $(\Sigma, \bar{g})$. Then the function f induces a weighted measure $e^{-f}\,d\sigma$ on Σ. Thus we have an induced smooth metric measure space $(\Sigma^n, \bar{g}, e^{-f}\,d\sigma)$.

In this paper, unless otherwise specified, we denote by a bar all quantities on $(M, \bar{g})$, for instance by $\overline{\nabla}$ and $\overline{\mathrm{Ric}}$, the Levi-Civita connection and the Ricci curvature tensor of $(M, \bar{g})$, respectively. Also we denote, for example, by ∇, Ric, Δ and Δ_f, the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the weighted Laplacian on $(\Sigma, \bar{g})$, respectively. Let $p \in \Sigma$ and ν a unit normal at p. The second fundamental form A, the mean curvature H, and the mean curvature vector $\boldsymbol{H}$ of hypersurface $(\Sigma, \bar{g})$ are defined, respectively, by

$$A : T_p\Sigma \to T_p\Sigma, \quad A(X) := \overline{\nabla}_X \nu, \quad X \in T_p\Sigma,$$
$$H := \operatorname{tr} A = -\sum_{i=1}^{n} \langle \overline{\nabla}_{e_i} e_i, \nu \rangle, \quad \boldsymbol{H} := -H\nu.$$

Define the weighted mean curvature vector $\boldsymbol{H}_f$ and the weighted mean curvature H_f of $(\Sigma, \bar{g})$ by

$$\boldsymbol{H}_f := \boldsymbol{H} - (\overline{\nabla} f)^{\perp} \quad \text{and} \quad \boldsymbol{H}_f = -H_f \nu,$$

where $\perp$ denotes the projection to the normal bundle of Σ. It follows that

$$H_f = H - \langle \overline{\nabla} f, \nu \rangle.$$

Definition. A hypersurface Σ immersed in $(M^{n+1}, \bar{g}, e^{-f}\,d\mu)$ with the induced metric $\bar{g}$ is called f-minimal if its weighted mean curvature H_f vanishes identically or, equivalently, if it satisfies

$$H = \langle \overline{\nabla} f, \nu \rangle. \tag{3}$$

Definition. The weighted volume of $(\Sigma, \bar{g})$ is defined by

$$V_f(\Sigma) := \int_\Sigma e^{-f}\,d\sigma. \tag{4}$$

It is well known that Σ is f-minimal if and only if Σ is a critical point of the weighted volume functional. Namely it holds that

Proposition 1. *If T is a compactly supported normal variational vector field on Σ (i.e., $T = T^{\perp}$), then the first variation formula of the weighted volume of $(\Sigma, \bar{g})$ is given by*

$$\frac{d}{dt} V_f(\Sigma_t)\Big|_{t=0} = -\int_\Sigma \langle T, \boldsymbol{H}_f \rangle_{\bar{g}} e^{-f}\,d\sigma. \tag{5}$$

On the other hand, an f-minimal hypersurface can be viewed as a minimal hypersurface under a conformal metric. More precisely, define the new metric $\tilde{g} = e^{-2f/n}\bar{g}$ on M, which is conformal to $\bar{g}$. Then the immersion $i : \Sigma \to M$ induces a metric $i^*\tilde{g}$ on Σ from $(M, \tilde{g})$. In the following, $i^*\tilde{g}$ is still denoted by $\tilde{g}$ for simplicity of notation. The volume of $(\Sigma, \tilde{g})$ is

$$\widetilde{V}(\Sigma) := \int_\Sigma d\tilde{\sigma} = \int_\Sigma e^{-f}\, d\sigma = V_f(\Sigma). \tag{6}$$

Hence Proposition 1 and (6) imply that

$$\int_\Sigma \langle T, \widetilde{\boldsymbol{H}}\rangle_{\tilde{g}}\, d\tilde{\sigma} = \int_\Sigma \langle T, \boldsymbol{H}_f\rangle_{\bar{g}} e^{-f}\, d\sigma, \tag{7}$$

where $d\tilde{\sigma} = e^{-f}\, d\sigma$ and $\widetilde{\boldsymbol{H}}$ denote the volume element and the mean curvature vector of Σ with respect to the conformal metric $\tilde{g}$, respectively.

Equation (7) implies that $\widetilde{\boldsymbol{H}} = e^{2f/n}\boldsymbol{H}_f$. Therefore $(\Sigma, \bar{g})$ is f-minimal in $(M, \bar{g})$ if and only if $(\Sigma, \tilde{g})$ is minimal in $(M, \tilde{g})$.

In this paper, for a closed hypersurface, we choose ν to be the outer unit normal.

3. Some facts on the weighted Laplacian and f-minimal hypersurfaces

In this section, we give some known results which will be used later in this paper. Recall that Reilly [1977] proved an integral version of the Bochner formula for compact domains of a Riemannian manifold, which is called the Reilly formula. Ma and Du [2010] obtained a Reilly formula for metric measure spaces, which is the following proposition. We include its proof in an appendix for the sake of completeness.

Proposition 2. *Let Ω be a compact Riemannian manifold with boundary $\partial\Omega$ and $(\Omega, \bar{g}, e^{-f}\, d\mu)$ a smooth metric measure space. Then*

$$\begin{aligned} \int_\Omega (\overline{\Delta}_f u)^2 e^{-f} = \int_\Omega |\overline{\nabla}^2 u|^2 e^{-f} + \int_\Omega \overline{\mathrm{Ric}}_f(\overline{\nabla} u, \overline{\nabla} u) e^{-f} \\ + 2\int_{\partial\Omega} u_\nu (\Delta_f u) e^{-f} + \int_{\partial\Omega} A(\nabla u, \nabla u) e^{-f} + \int_{\partial\Omega} u_\nu^2 H_f e^{-f}, \end{aligned} \tag{8}$$

where ν is the outward pointing unit normal to $\partial\Omega$ and A is the second fundamental form of $\partial\Omega$ with respect to the normal ν, the quantities with bars denote the ones on $(\Omega, \bar{g})$ (for instance, $\overline{\mathrm{Ric}}_f$ denotes the Bakry–Émery Ricci curvature on $(\Omega, \bar{g})$), and Δ_f and H_f denote the weighted Laplacian on $\partial\Omega$ and the weighted mean curvature of $\partial\Omega$, respectively.

A Riemannian manifold with Bakry–Émery Ricci curvature bounded below by a positive constant has some properties similar to a Riemannian manifold with Ricci curvature bounded below by a positive constant. We refer to [Wei and Wylie 2009; Munteanu and Wang 2014; 2012] and the references therein.

Proposition 3 [Morgan 2005] (see also [Wei and Wylie 2009, Corollary 5.1]). *If a complete smooth metric measure space $(N, g, e^{-\omega} d\mu)$ has $\mathrm{Ric}_w \geq k$, where k is a positive constant, then N has finite weighted volume and finite fundamental group.*

For f-minimal hypersurfaces, the following intersection theorem holds.

Proposition 4 [Wei and Wylie 2009, Theorem 7.4]. *Any two closed f-minimal hypersurfaces immersed in a complete smooth metric measure space $(M, \bar{g}, e^{-f} d\mu)$ with $\overline{\mathrm{Ric}}_f > 0$ must intersect. Thus a closed f-minimal hypersurface in M must be connected.*

In [Cheng and Zhou 2013] it was proved that the weighted volume of a self-shrinker Σ^n immersed in $\mathbb{R}^m$ being finite implies it is properly immersed. This result extends to f-minimal submanifolds:

Proposition 5 [Cheng et al. 2012]. *Let Σ^n be an n-dimensional complete f-minimal submanifold immersed in an m-dimensional Riemannian manifold M^m, $n < m$. If Σ has finite weighted volume, then Σ is properly immersed in M.*

An f-minimal hypersurface is an f-minimal submanifold with codimension 1. See more properties of f-minimal submanifolds in [Cheng et al. 2012].

4. Lower bound for $\boldsymbol{\lambda_1(\Delta_f)}$

In this section, we apply the Reilly formula for metric measure spaces to prove Theorems 2 and 5.

Proof of Theorem 2. Since $\overline{\mathrm{Ric}}_f \geq k$, where $k > 0$ is constant, Proposition 3 implies that M has finite fundamental group. We first assume that M is simply connected. Since Σ is connected (Proposition 4) and embedded in M, Σ is orientable and divides M into two components (see its proof in [Choi and Schoen 1985]). Thus Σ divides D into two bounded components Ω_1 and Ω_2. That is $D \backslash \Sigma = \Omega_1 \cup \Omega_2$ with $\partial\Omega_1 = \Sigma$ and $\partial\Omega_2 = \partial D \cup \Sigma$.

For simplicity, we denote by λ_1 the first eigenvalue $\lambda_1(\Delta_f)$ of the weighted Laplacian Δ_f on Σ. Let h be a corresponding eigenfunction so that on Σ

$$\Delta_f h + \lambda_1 h = 0 \quad \text{with} \int_\Sigma h^2 e^{-f} = 1.$$

Consider the solution of the Dirichlet problem on Ω_1 so that

$$\begin{cases} \bar{\Delta}_f u = 0 & \text{in } \Omega_1, \\ u = h & \text{on } \partial\Omega_1 = \Sigma. \end{cases} \tag{9}$$

Substitute Ω_1 for Ω and put the solution u of (9) in Proposition 2. Then the assumption on $\overline{\mathrm{Ric}}_f$ implies that

$$0 \geq k \int_{\Omega_1} |\overline{\nabla} u|^2 e^{-f} - 2\lambda_1 \int_\Sigma u_\nu h e^{-f} + \int_\Sigma A(\nabla h, \nabla h) e^{-f},$$

where ν is the outer unit normal of Σ with respect to Ω_1. By Stokes' theorem and (9),

$$\int_\Sigma u_\nu h e^{-f} = \int_{\Omega_1} (|\overline{\nabla} u|^2 + u \overline{\Delta}_f u) e^{-f} = \int_{\Omega_1} |\overline{\nabla} u|^2 e^{-f}.$$

Thus

$$0 \geq (k - 2\lambda_1) \int_{\Omega_1} |\overline{\nabla} u|^2 e^{-f} + \int_\Sigma A(\nabla h, \nabla h) e^{-f}.$$

If $\int_\Sigma A(\nabla h, \nabla h) e^{-f} \geq 0$, by $u \not\equiv C$, we have

$$\lambda_1 \geq \frac{k}{2}.$$

If $\int_\Sigma A(\nabla h, \nabla h) e^{-f} < 0$, we consider the compact domain Ω_2 with the boundary $\partial \Omega_2 = \Sigma \cup \partial D$. Let u be the solution of the mixed problem

$$\begin{cases} \overline{\Delta}_f u = 0 & \text{in } \Omega_2, \\ u = h & \text{on } \Sigma, \\ u_{\tilde{\nu}} = 0 & \text{on } \partial D, \end{cases} \tag{10}$$

where $\tilde{\nu}$ denotes the outer unit normal of ∂D with respect to Ω_2.

Substituting Ω_2 for Ω and putting the solution u of (10) in Proposition 2, we have

$$0 \geq \int_{\Omega_2} |\overline{\nabla}^2 u|^2 e^{-f} + k \int_{\Omega_2} |\overline{\nabla} u|^2 e^{-f} - 2\lambda_1 \int_\Sigma h u_{\tilde{\nu}} e^{-f} + \int_\Sigma \widetilde{A}(\nabla h, \nabla h) e^{-f} + \int_{\partial D} \widetilde{A}(\nabla u, \nabla u) e^{-f},$$

where $\tilde{\nu}$ denotes the outer unit normal of Σ with respect to Ω_2 and $\widetilde{A}$ denotes the second fundamental form of Σ with respect to the normal $\tilde{\nu}$.

On the other hand, Stokes' theorem and (10) imply

$$\int_{\Omega_2} |\overline{\nabla} u|^2 e^{-f} = \int_{\partial \Omega_2} u u_{\tilde{\nu}} e^{-f} = \int_\Sigma h u_{\tilde{\nu}} e^{-f}.$$

Thus we have

$$0 \geq (k - 2\lambda_1) \int_{\Omega_2} |\overline{\nabla} u|^2 e^{-f} + \int_\Sigma \widetilde{A}(\nabla h, \nabla h) e^{-f} + \int_{\partial D} \widetilde{A}(\nabla u, \nabla u) e^{-f}. \tag{11}$$

Since ∂D is assumed convex, the last term on the right side of (11) is nonnegative. Observe that the orientations of Σ are opposite for Ω_1 and Ω_2. Namely $\tilde{\nu} = -\nu$.

Then $\widetilde{A}(\nabla u, \nabla u) = -A(\nabla u, \nabla u)$ on Σ. This implies that the second term on the right side of (11) is nonnegative. Thus

$$0 \geq (k - 2\lambda_1) \int_{\Omega_2} |\overline{\nabla} u|^2 e^{-f}.$$

Since u is not a constant function, we conclude that $k - 2\lambda_1 \leq 0$. Again we have

$$\lambda_1 \geq \frac{k}{2}.$$

Therefore we obtain that $\lambda_1(\Delta_f) \geq k/2$ if M is simply connected.

Second, if M is not simply connected, we consider its universal covering $\widehat{M}$, which is a finite $|\pi_1|$-fold covering. $\widehat{M}$ is simply connected and the covering map $\pi : \widehat{M} \to M$ is a locally isometry.

Take $\hat{f} = f \circ \pi$. Obviously $\widehat{M}$ has $\widehat{\mathrm{Ric}}_{\hat{f}} \geq k$, and the lift $\widehat{\Sigma}$ of Σ is also $\hat{f}$-minimal, embedded and closed. By Proposition 4, $\widehat{\Sigma}$ must be connected. Since $\widehat{M}$ is simply connected, the closed embedded connected $\widehat{\Sigma}$ must be orientable and thus divides $\widehat{M}$ into two components. Moreover the connectedness of $\widehat{\Sigma}$ implies that the lift $\widehat{D}$ of D is also a connected domain. Also $\partial\widehat{D} = \widehat{\partial D}$ is smooth and convex. Hence the assertion obtained for the simply connected ambient space can be applied here. Thus the first eigenvalue of the weighted Laplacian $\widehat{\Delta}_{\hat{f}}$ on $\widehat{\Sigma}$ satisfies $\lambda_1(\widehat{\Delta}_{\hat{f}}) \geq k/2$.

Observing the lift of the first eigenfunction of Σ is also an eigenfunction of $\widehat{M}$, we have

$$\lambda_1(\Delta_f) \geq \lambda_1(\widehat{\Delta}_{\hat{f}}) \geq \frac{k}{2}. \qquad \square$$

Remark. In Theorem 2, the boundary ∂D is not necessarily smooth. ∂D can be assumed to be C^1, which is sufficient for the existence of the solution of the mixed problem (10).

Theorem 5 holds by the same argument as that of Theorem 2.

5. Upper bound on area and total curvature of f-minimal surfaces

In this section, we study surfaces in a 3-manifold. First we estimate the corresponding upper bounds on the area and weighted area of an embedded closed f-minimal surface by applying the first eigenvalue estimate in Section 4. Next we discuss the upper bound on the total curvature. We begin with a result of Yang and Yau [1980]:

Proposition 6. *Let Σ^2 be a closed orientable Riemannian surface with genus g. Then the first eigenvalue $\lambda_1(\Delta)$ of the Laplacian Δ on Σ satisfies*

$$\lambda_1(\Delta)\,\mathrm{Area}(\Sigma) \leq 8\pi(1+g).$$

Using Theorem 2 and Proposition 6, we obtain the following area estimates for closed embedded f-minimal surfaces if the ambient space is simply connected.

Proposition 7. *Let $(M^3, \bar{g}, e^{-f}\,d\mu)$ be a simply connected complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Let $\Sigma^2 \subset M$ be a closed embedded f-minimal surface with genus g. If Σ is contained in a bounded domain D with convex boundary ∂D, then its area and weighted area satisfy*

$$\mathrm{Area}(\Sigma) \leq \frac{16\pi(1+g)}{k} e^{\operatorname{osc}_\Sigma f} \tag{12}$$

and

$$\mathrm{Area}_f(\Sigma) \leq \frac{16\pi(1+g)}{k} e^{-\inf_\Sigma f}, \tag{13}$$

where $\operatorname{osc}_\Sigma f = \sup_\Sigma f - \inf_\Sigma f$.

Proof. Consider the conformal metric $\tilde{g} = e^{-f}\bar{g}$ on M. Let $\lambda_1(\widetilde{\Delta})$ be the first eigenvalue of the Laplacian $\widetilde{\Delta}$ on $(\Sigma, \tilde{g})$, which satisfies

$$\lambda_1(\widetilde{\Delta}) = \inf_{\substack{\int_\Sigma u\,d\tilde{\sigma}=0\\ u\neq 0}} \frac{\int_\Sigma |\widetilde{\nabla} u|^2_{\tilde{g}}\,d\tilde{\sigma}}{\int_\Sigma u^2\,d\tilde{\sigma}},$$

where $\widetilde{\Delta}$, $\widetilde{\nabla}$ and $d\tilde{\sigma}$ are the Laplacian, gradient and area element of Σ with respect to the metric $\tilde{g}$, respectively.

On the other hand, the first eigenvalue of the weighted Laplacian $\lambda_1(\Delta_f)$ on $(\Sigma, \bar{g})$ satisfies

$$\lambda_1(\Delta_f) = \inf_{\substack{\int_\Sigma u e^{-f}\,d\sigma=0\\ u\neq 0}} \frac{\int_\Sigma |\nabla u|^2_{\bar{g}} e^{-f}\,d\sigma}{\int_\Sigma u^2 e^{-f}\,d\sigma}.$$

Since $\widetilde{\nabla} u = e^f \nabla u$, $d\tilde{\sigma} = e^{-f}\,d\sigma$ and $\tilde{g} = e^{-f}\bar{g}$,

$$\begin{aligned}
\lambda_1(\widetilde{\Delta}) &= \inf_{\substack{\int_\Sigma u e^{-f}\,d\sigma=0\\ u\neq 0}} \frac{\int_\Sigma |\nabla u|^2_{\bar{g}}\,d\sigma}{\int_\Sigma u^2 e^{-f}\,d\sigma} \\
&\geq \inf_{\substack{\int_\Sigma u e^{-f}\,d\sigma=0\\ u\neq 0}} \frac{\int_\Sigma |\nabla u|^2_{\bar{g}} e^{-f+\inf_\Sigma(f)}\,d\sigma}{\int_\Sigma u^2 e^{-f}\,d\sigma} \\
&= e^{\inf_\Sigma f}\lambda_1(\Delta_f).
\end{aligned}$$

By this inequality, Theorem 2 and Proposition 6, we have the estimate

$$\mathrm{Area}(\Sigma, \tilde{g}) \leq \frac{16\pi(1+g)}{k} e^{-\inf_\Sigma(f)}. \tag{14}$$

Since $\mathrm{Area}_f(\Sigma) = \int_\Sigma e^{-f}\, d\sigma = \mathrm{Area}(\Sigma, \tilde{g})$,

$$\mathrm{Area}_f(\Sigma) \le \frac{16\pi(1+g)}{k} e^{-\inf_\Sigma(f)},$$

which is (13). Thus

$$\mathrm{Area}(\Sigma) \le \frac{16\pi(1+g)}{k} e^{\sup_\Sigma(f) - \inf_\Sigma(f)} = \frac{16\pi(1+g)}{k} e^{\mathrm{osc}_\Sigma(f)}.$$

That is, (12) holds. □

Now, suppose that M is not simply connected. We use a covering argument as in [Choi and Schoen 1985].

Proposition 8. *Let $(M^3, \bar{g}, e^{-f}\, d\mu)$ be a complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \ge k$, where k is a positive constant. Let Σ^2 be a closed embedded f-minimal surface. If Σ is contained in a bounded domain D of M with convex boundary ∂D, then*

$$\mathrm{Area}_f(\Sigma) \le \frac{16\pi}{k}\left(\frac{2}{|\pi_1|} - \frac{1}{2}\chi(\Sigma)\right) e^{-\inf_\Sigma f} \tag{15}$$

and

$$\mathrm{Area}(\Sigma) \le \frac{16\pi}{k}\left(\frac{2}{|\pi_1|} - \frac{1}{2}\chi(\Sigma)\right) e^{\mathrm{osc}_\Sigma f}, \tag{16}$$

where $|\pi_1|$ is the order of the fundamental group of M, and $\chi(\Sigma)$ is the Euler characteristic of Σ.

Proof. Let $\widehat{M}$ be the universal covering manifold of M. By Proposition 3, the covering is a finite $|\pi_1|$-fold covering. Let $\widehat{\Sigma}$ be the lifting of Σ. In the proof of Theorem 2, we have shown that $\widehat{\Sigma}$ is orientable and satisfies the assumption of Theorem 2. Hence Theorem 2 implies that the first eigenvalue of the weighted Laplacian of $\widehat{\Sigma}$ satisfies $\lambda_1(\widehat{\Delta}_{\hat{f}}) \ge k/2$, where $\hat{f}$ is the lift of f. By Proposition 7, we conclude that

$$\mathrm{Area}(\widehat{\Sigma}) \le \frac{16\pi}{k}\left(2 - \tfrac{1}{2}\chi(\widehat{\Sigma})\right) e^{\mathrm{osc}_{\widehat{\Sigma}}(\tilde{f})}$$

and

$$\mathrm{Area}_{\hat{f}}(\widehat{\Sigma}) = \int_{\widehat{\Sigma}} e^{-\hat{f}}\, d\sigma \le \frac{16\pi}{k}\left(2 - \tfrac{1}{2}\chi(\widehat{\Sigma})\right) e^{-\inf_{\widehat{\Sigma}}(\hat{f})}.$$

Thus (15) and (16) follow from the equalities

$$\chi(\widehat{\Sigma}) = |\pi_1| \cdot \chi(\Sigma), \quad \inf_{\widehat{\Sigma}}(\hat{f}) = \inf_\Sigma(f), \quad \mathrm{osc}_{\widehat{\Sigma}}(\hat{f}) = \mathrm{osc}_\Sigma(f),$$
$$\mathrm{Area}(\widehat{\Sigma}) = |\pi_1| \cdot \mathrm{Area}(\Sigma) \quad \text{and} \quad \mathrm{Area}_{\hat{f}}(\widehat{\Sigma}) = |\pi_1| \cdot \mathrm{Area}_f(\Sigma).$$ □

In the following, we will give the upper bound for the total curvature of f-minimal surfaces. Here the term the *total curvature* of Σ means $\int_\Sigma |A|^2\, d\sigma$ not $\int_\Sigma K\, d\sigma$.

Proposition 9. *Let* $(M^3, \bar{g}, e^{-f} d\mu)$ *be a complete smooth metric measure space with* $\overline{\mathrm{Ric}}_f \geq k$, *where* k *is a positive constant. Let* $\Sigma^2 \subset M$ *be a closed embedded* f*-minimal surface with genus* g. *If* Σ *is contained in a bounded domain* D *of* M *with convex boundary* ∂D, *then* Σ *satisfies*

$$\int_\Sigma |A|^2 \, d\sigma \leq C, \tag{17}$$

where A *is the second fundamental form of* $(\Sigma, \bar{g})$ *and* C *is a constant depending on the genus* g *of* Σ, *the order* $|\pi_1|$ *of the fundamental group of* M, *the maximum* $\sup_\Sigma \bar{K}$ *of the sectional curvature of* M *on* Σ, *the lower bound* k *of the Bakry–Émery Ricci curvature of* M, *the oscillation* $\mathrm{osc}_\Sigma(f)$ *and the maximum* $\sup_\Sigma |\bar{\nabla} f|$ *on* Σ.

Proof. By the Gauss equation and Gauss–Bonnet formula,

$$\begin{aligned}\int_\Sigma |A|^2 \, d\sigma = \int_\Sigma H^2 - 2\int_\Sigma (K - \bar{K}) &= \int_\Sigma \langle \bar{\nabla} f, \boldsymbol{n}\rangle^2 - 4\pi\chi(\Sigma) + 2\int_\Sigma \bar{K} \\ &\leq (\sup_\Sigma |\bar{\nabla} f|)^2 \,\mathrm{Area}(\Sigma) + 8\pi(g-1) + 2(\sup_\Sigma \bar{K})\,\mathrm{Area}(\Sigma).\end{aligned}$$

Using (16), we have the conclusion of the theorem. □

To prove the compactness theorem in Section 6, we need the following total curvature estimate for $(\Sigma, \tilde{g})$, which is a minimal surface in $(M, \tilde{g})$.

Proposition 10. *Let* $(M^3, \bar{g}, e^{-f} d\mu)$ *be a complete smooth metric measure space with* $\overline{\mathrm{Ric}}_f \geq k$, *where* k *is a positive constant. Let* $\Sigma^2 \subset M$ *be a closed embedded* f*-minimal surface with genus* g. *If* Σ *is contained in a bounded domain* D *of* M *with convex boundary* $\partial\Omega$, *then* Σ *satisfies*

$$\int_\Sigma |\widetilde{A}|^2_{\tilde{g}} \, d\tilde{\sigma} \leq C, \tag{18}$$

where $\widetilde{A}$ *is the second fundamental form of* $(\Sigma, \tilde{g})$ *with respect to the conformal metric* $\tilde{g} = e^{-f}\bar{g}$ *of* M *and* C *is a constant depending on the genus* g *of* Σ, *the order* $|\pi_1|$ *of the fundamental group of* M, *the maximum* $\sup_\Sigma \widetilde{K}$ *of the sectional curvature of* $(M, \tilde{g})$ *on* Σ, *the lower bound* k *of the Bakry–Émery Ricci curvature of* M *and the oscillation* $\mathrm{osc}_\Sigma(f)$ *on* Σ.

Proof. By the Gauss equation and the Gauss–Bonnet formula, we have

$$\begin{aligned}\int_\Sigma |\widetilde{A}|^2_{\tilde{g}} \, d\tilde{\sigma} = \int_\Sigma \widetilde{H}^2 - 2\int_\Sigma (\widetilde{K}^\Sigma - \widetilde{K}^M)\, d\tilde{\sigma} &= -4\pi\chi(\Sigma) + 2\int_\Sigma \widetilde{K}\, d\tilde{\sigma} \\ &\leq 8\pi(g-1) + 2(\sup_\Sigma \widetilde{K})\,\mathrm{Area}((\Sigma, \tilde{g})) \\ &= 8\pi(g-1) + 2(\sup_\Sigma \widetilde{K})\,\mathrm{Area}_f(\Sigma).\end{aligned}$$

We have used $\widetilde{H} = e^f H_f = 0$ and $\mathrm{Area}((\Sigma, \tilde{g})) = \mathrm{Area}_f(\Sigma)$. Now (18) follows from (15). □

6. Compactness of compact f-minimal surfaces

We will prove some compactness theorems for closed embedded f-minimal surfaces in a 3-manifold. We have two ways to prove Theorem 4.

The first proof roughly follows the one in [Colding and Minicozzi 2011] (cf. [Choi and Schoen 1985]) with some modifications. The modifications can be made because we have the assumptions that f-minimal surfaces are contained in the closure of a bounded domain Ω of M and $\overline{\Omega}$ is contained in a bounded domain U with convex boundary. The second proof will need a compactness theorem of complete embedded f-minimal surfaces that was proved in [Cheng et al. 2012].

We prefer to give two proofs here since the first one is independent of the compactness theorem of complete embedded f-minimal surfaces. But the compactness theorem of complete embedded f-minimal surfaces needs a theorem about nonexistence of L_f-stable minimal surfaces (see [Cheng et al. 2012, Theorem 3]).

First proof. We first prove a singular compactness theorem, which is a variation of a result from [Choi and Schoen 1985] (compare [Colding and Minicozzi 2011, Proposition 7.14; Anderson 1985; White 1987]):

Proposition 11. *Let $(M^3, \bar{g})$ be a 3-manifold. Assume that Ω is a bounded domain in M. Let Σ_i be a sequence of closed embedded minimal surfaces contained in $\overline{\Omega}$, with genus g, and satisfying*

$$\mathrm{Area}(\Sigma_i) \le C_1 \tag{19}$$

and

$$\int_{\Sigma_i} |A_{\Sigma_i}|^2 \le C_2. \tag{20}$$

Then there exists a finite set of points $\mathcal{S} \subset \overline{\Omega}$ and a subsequence, still denoted by Σ_i, that converges uniformly in the C^m topology $(m \ge 2)$ on compact subsets of $M \setminus \mathcal{S}$ to a complete minimal surface $\Sigma \subset \overline{\Omega}$ (possibly with multiplicity).

The subsequence also converges to Σ in extrinsic Hausdorff distance. Σ is smooth, embedded in M, has genus at most g and satisfies (19) and (20).

Proof. We may use the same argument as that of [Colding and Minicozzi 2011, Proposition 7.14]. Moreover $\Sigma_i \subset \overline{\Omega}$ implies that the singular set $S \subset \overline{\Omega}$ and the smooth surface $\Sigma \subset \overline{\Omega}$. Here we omit the details of proof. □

We can apply Proposition 11 to the f-minimal surfaces which are minimal in the conformal metric.

Lemma. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a smooth metric measure space. Assume that Ω is a bounded domain in M. Let $\Sigma_i \subset \overline{\Omega}$ be a sequence of closed embedded f-minimal surfaces of genus g. Suppose that $\tilde{g} = e^{-f}\bar{g}$ on M and $(\Sigma_i, \tilde{g})$ satisfy*

$$\operatorname{Area}((\Sigma_i, \tilde{g})) = \operatorname{Area}_f(\Sigma_i) \leq C_1 \tag{21}$$

and

$$\int_{\Sigma_i} |\widetilde{A}_{\Sigma_i}|^2_{\tilde{g}} \, d\tilde{\sigma} \leq C_2, \tag{22}$$

where $\widetilde{A}_{\Sigma_i}$ and $d\tilde{\sigma}$ denote the second fundamental form and the volume element of $(\Sigma_i, \tilde{g})$, respectively. Then there exists a finite set of points $\mathcal{S} \subset \overline{\Omega}$ and a subsequence, still denoted by Σ_i, that converges uniformly in the C^m topology $(m \geq 2)$ on compact subsets of $M \backslash \mathcal{S}$ to a complete f-minimal surface $\Sigma \subset \overline{\Omega}$ (possibly with multiplicity).

The subsequence also converges to Σ in extrinsic Hausdorff distance. Σ is smooth, embedded in M, has genus at most g, and satisfies (21) and (22).

Proof. Since an f-minimal surface in the original metric $\bar{g}$ is equivalent to it being minimal in the conformal metric $\tilde{g}$, we can apply Proposition 11 to get the conclusion of the lemma. □

Proof of Theorem 4. First assume M is simply connected. Since $\Sigma_i \subset \overline{\Omega} \subset U$, we see from Proposition 7 and Proposition 10 that

$$\operatorname{Area}((\Sigma_i, \tilde{g})) = \operatorname{Area}_f(\Sigma_i) \leq C_1$$

and

$$\int_{\Sigma_i} |\widetilde{A}_{\Sigma_i}|^2_{\tilde{g}} \, d\sigma_{\tilde{g}} \leq C_2,$$

where C_1 and C_2 depend on g, $\sup_{\Omega_j} f$, $\sup_{\Omega_j} \widetilde{K}$ and k.

By the lemma, there exists a finite set of points $\mathcal{S} \subset \widetilde{\Omega}$ and a subsequence $\Sigma_{i'}$ that converges uniformly in the C^m topology (any $m \geq 2$) on compact subsets of $M \backslash \mathcal{S}$ to a complete f-minimal surface $\Sigma \subset \overline{\Omega}$ without boundary (possibly with multiplicity). Σ is smooth, embedded in M and has genus at most g. Equivalently, with respect to the conformal metric $\tilde{g}$, a subsequence $\Sigma_{i'}$ of minimal surfaces converges uniformly in the C^m topology on compact subsets of $M \backslash \mathcal{S}$ to a complete minimal surface Σ, where $\Sigma \subset \overline{\Omega}$.

Since complete embedded $\Sigma \subset \overline{\Omega}$ satisfies (21), it must be properly embedded (Proposition 5), thus closed and orientable.

We need to prove that the convergence is smooth across the points $\mathcal{S}$. By Allard's regularity theorem, it suffices to prove that the convergence has multiplicity one. If the multiplicity is not one, by a proof similar to that of [Choi and Schoen 1985] (see also [Colding and Minicozzi 2011, p. 249]), we can show that there is an i big enough and a Σ_i in the convergent subsequence, so that the first eigenvalue of the Laplacian $\widetilde{\Delta}^{\Sigma_i}$ on Σ_i with the conformal metric $\tilde{g}$ satisfies $\lambda_1(\widetilde{\Delta}^{\Sigma_i}) < ke^{\inf_\Omega f}/2$. We have

$$\begin{aligned}\lambda_1(\widetilde{\Delta}^{\Sigma_i}) &= \inf\left\{\frac{\int_{\Sigma_i}|\widetilde{\nabla}\phi|^2_{\tilde{g}}\,d\tilde{\sigma}}{\int_{\Sigma_i}\phi^2\,d\tilde{\sigma}}, \int_{\Sigma_i}\phi\,d\tilde{\sigma}=0\right\}\\ &= \inf\left\{\frac{\int_{\Sigma_i}|\nabla\phi|^2 d\sigma}{\int_{\Sigma_i}\phi^2 e^{-f}\,d\sigma}, \int_{\Sigma_i}\phi e^{-f}\,d\sigma=0\right\}\\ &\geq \lambda_1(\Delta_f^{\Sigma_i})e^{\inf_\Omega f}.\end{aligned}$$

By Theorem 2, $\Sigma_i \subset \overline{\Omega} \subset U$ implies $\lambda_1(\Delta_f^{\Sigma_i}) \geq k/2$. Thus we have a contradiction.

When M is not simply connected, we use a covering argument. The assumption of $\overline{\mathrm{Ric}}_f \geq k$, where $k > 0$ is constant, implies that M has finite fundamental group π_1 (Proposition 3). We consider the finite-fold universal covering $\widehat{M}$. By the proof of Theorem 2, we know that the corresponding lifts of Σ_i, $\overline{\Omega}$ and U satisfy $\widehat{\Sigma}_i \subset \widehat{\overline{\Omega}} \subset \widehat{U}$. Then Propositions 8 and 10 give uniform bounds on the area and total curvature in the conformal metric $\hat{\tilde{g}}$ on $\widehat{M}$. By the assertion on the simply connected ambient manifold before, we have the smooth convergence of a subsequence of $\widehat{\Sigma}_i$. This implies the smooth convergence of a subsequence of Σ_i. □

Second Proof. In [Cheng et al. 2012], we proved the following:

Theorem 6. *Let $(M^3, \bar{g}, e^{-f}d\mu)$ be a complete smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Given an integer $g \geq 0$ and a constant $V > 0$, the space $S_{g,V}$ of smooth complete embedded f-minimal surfaces $\Sigma \subset M$ with*

- *genus at most g,*
- *$\partial\Sigma = \varnothing$, and*
- *$\int_\Sigma e^{-f}\,d\sigma \leq V$*

is compact in the C^m topology, for any $m \geq 2$. Namely any sequence of $S_{g,V}$ has a subsequence that converges in the C^m topology on compact subsets to a surface in $S_{D,g}$, for any $m \geq 2$.

Proof of Theorem 4. Since a surface in S is contained in $\bar{\Omega} \subset U$, by Proposition 8, we have the uniform bound V of the weighted volume of closed embedded f-minimal surfaces in S. Hence Theorem 6 can be applied. Moreover $\Sigma_i \subset \overline{\Omega}$ implies that the smooth limit surface $\Sigma \subset \overline{\Omega}$. Otherwise, since the subsequence $\{\Sigma_i\}$ converges uniformly in the C^m topology ($m \geq 2$) on any compact subset of M to Σ, there is a surface Σ_i (with index i big enough) in the subsequence that would not satisfy $\Sigma_i \subset \overline{\Omega}$.

By Proposition 5, Σ must be properly embedded. Thus Σ must be closed. □

To prove Theorem 3 we require a lemma.

Lemma. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k > 0$. If Σ is any closed f-minimal surface in M with genus at most g and diameter at most D, then $\Sigma \subset \overline{B}_r(p)$ for some $r > 0$ (independent of Σ), where $B_r(p)$ is a ball in M with radius r centered at $p \in M$.*

Proof. Fix a closed f-minimal surface Σ_0. Obviously $\Sigma_0 \subset B_{r_0}(p)$ for some $r_0 > 0$. Proposition 4 says that Σ and Σ_0 must intersect. Then, for $x \in \Sigma$,

$$d(p,x) \leq d(p,x_0) + d(x_0,x) \leq r_0 + D, x_0 \in \Sigma_0.$$

Taking $r = r_0 + D$, we have $\Sigma \subset \overline{B}_{r_0+D}$. □

Remark. In the lemma and hence in Theorem 3, D is a bound on the intrinsic diameter of closed f-minimal surfaces or a bound on the extrinsic diameter of closed f-minimal surfaces. Also, by Proposition 4, the assumption that f-minimal surfaces are contained in the closure of a bounded domain Ω in Theorem 4 is equivalent to that of a uniform upper bound on the extrinsic diameter of f-minimal surfaces.

Proof of Theorem 3. By the lemma immediately above, we may apply Theorem 4 to the space $S_{D,g}$. Next the closed embedded limit Σ must have diameter at most D. Otherwise, since the subsequence $\{\Sigma_i\}$ converges uniformly in the C^m topology ($m \geq 2$) on any compact subset of M to Σ, there is a surface Σ_i (with the index i big enough) in the subsequence that would have diameter greater than D. So Σ must be in $S_{D,g}$. □

As a corollary, Theorem 3 implies:

Theorem 7. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Assume that M admits an exhaustion by bounded domains with convex boundary. Then the space of closed embedded f-minimal surface in M of fixed topological type and diameter at most D is compact in the C^m topology, for any $m \geq 2$.*

Proof of Theorem 7. By Theorem 3, it suffices to prove that the limit f-minimal surface of a convergent subsequence in the given space has the same topological type, which holds by the Gauss–Bonnet formula and smooth convergence. □

Similar to the proof of Theorem 7, Theorem 4 implies:

Theorem 8. *Let $(M^3, \bar{g}, e^{-f} d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Assume that Ω is a bounded domain and U is a bounded domain with convex boundary so that $\overline{\Omega} \subset U$. Then the space of closed embedded f-minimal surface in M of fixed topological type and contained in the closure $\overline{\Omega}$ is compact in the C^m topology, for any $m \geq 2$.*

Appendix: Proof of Proposition 2

The Bochner formula implies that

$$\tfrac{1}{2}\bar{\Delta}_f|\overline{\nabla}u|^2 - \langle\overline{\nabla}u, \overline{\nabla}(\bar{\Delta}_f u)\rangle = |\overline{\nabla}^2 u|^2 + \overline{\mathrm{Ric}}_f(\overline{\nabla}u, \overline{\nabla}u).$$

Integrating this equation on Ω with respect to the weighted measure $e^{-f} d\mu$, we obtain

$$\int_\Omega \big(\tfrac{1}{2}\bar{\Delta}_f|\overline{\nabla}u|^2 - \langle\overline{\nabla}u, \overline{\nabla}(\bar{\Delta}_f u)\rangle\big)e^{-f} = \int_\Omega |\overline{\nabla}^2 u|^2 e^{-f} + \int_\Omega \overline{\mathrm{Ric}}_f(\overline{\nabla}u, \overline{\nabla}u)e^{-f}.$$

On the other hand, by the divergence formula, we have

$$\begin{aligned}\tfrac{1}{2}\bar{\Delta}_f|\overline{\nabla}u|^2 &- \langle\overline{\nabla}u, \overline{\nabla}(\bar{\Delta}_f u)\rangle\\ &= \tfrac{1}{2}\overline{\mathrm{div}}\big(e^{-f}\overline{\nabla}|\overline{\nabla}u|^2\big)e^f - \overline{\mathrm{div}}\big(e^{-f}\bar{\Delta}_f(u)\overline{\nabla}u\big)e^f + (\bar{\Delta}_f u)^2.\end{aligned}$$

Integrating and applying Stokes' theorem, we have

$$\begin{aligned}\int_\Omega \big(\tfrac{1}{2}\bar{\Delta}_f|\overline{\nabla}u|^2 &- \langle\overline{\nabla}u, \overline{\nabla}(\bar{\Delta}_f u)\rangle\big)e^{-f}\\ &= \int_{\partial\Omega}\big(\tfrac{1}{2}|\overline{\nabla}u|^2_\nu - (\bar{\Delta}_f u)u_\nu\big)e^{-f} + \int_\Omega (\bar{\Delta}_f u)^2 e^{-f}.\end{aligned}\tag{23}$$

Then

$$\begin{aligned}\tfrac{1}{2}|\overline{\nabla}u|^2_\nu &- (\bar{\Delta}_f u)u_\nu\\ &= \langle\overline{\nabla}_\nu\overline{\nabla}u, \overline{\nabla}u\rangle - (\bar{\Delta}_f u)u_\nu = \langle\overline{\nabla}_{\overline{\nabla}u}\overline{\nabla}u, \nu\rangle - (\bar{\Delta}_f u)u_\nu\\ &= \langle\overline{\nabla}_\nu\overline{\nabla}u, \nu\rangle u_\nu + \langle\overline{\nabla}_{\nabla u}\overline{\nabla}u, \nu\rangle - (\bar{\Delta}_f u)u_\nu\\ &= \big(\langle\overline{\nabla}_\nu\overline{\nabla}u, \nu\rangle - \bar{\Delta}u + \langle\overline{\nabla}f, \overline{\nabla}u\rangle\big)u_\nu + \langle\nabla u, \nabla u_\nu\rangle - \langle\overline{\nabla}u, \overline{\nabla}_{\nabla u}\nu\rangle\\ &= \big(-\Delta u - Hu_\nu + \langle\nabla f, \nabla u\rangle + \langle\overline{\nabla}f, \nu\rangle u_\nu\big)u_\nu + \langle\nabla u, \nabla u_\nu\rangle - \langle\nabla u, \overline{\nabla}_{\nabla u}\nu\rangle\\ &= -(\Delta_f u + H_f u_\nu)u_\nu + \langle\nabla u, \nabla u_\nu\rangle - A(\nabla u, \nabla u),\end{aligned}\tag{24}$$

where $H_f = H - \langle \overline{\nabla} f, \nu \rangle$. By substituting (24) into (23), we obtain

$$\begin{aligned}
&\int_\Omega \left(\tfrac{1}{2}\overline{\Delta}_f |\overline{\nabla} u|^2 - \langle \overline{\nabla} u, \overline{\nabla}(\overline{\Delta}_f u)\rangle\right) e^{-f} \\
&= -\int_{\partial\Omega} (\Delta_f u) u_\nu e^{-f} - \int_{\partial\Omega} H_f u_\nu^2 e^{-f} + \int_{\partial\Omega} \left(\langle \nabla u, \nabla u_\nu\rangle - A(\nabla u, \nabla u)\right) e^{-f} \\
&\qquad + \int_\Omega (\overline{\Delta}_f u)^2 e^{-f} \\
&= -2\int_{\partial\Omega} (\Delta_f u) u_\nu e^{-f} - \int_{\partial\Omega} H_f u_\nu^2 e^{-f} - \int_{\partial\Omega} A(\nabla u, \nabla u) e^{-f} + \int_\Omega (\overline{\Delta}_f u)^2 e^{-f}.
\end{aligned}$$

This immediately implies (8).

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