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**UNITARY REPRESENTATIONS OF $GL(n, K)$ DISTINGUISHED
BY A GALOIS INVOLUTION FOR A p -ADIC FIELD K**

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Let F be a p -adic field and K a quadratic extension of F . Using Tadić's classification of the unitary dual of $GL(n, K)$, we give the list of irreducible unitary representations of this group distinguished by $GL(n, F)$ in terms of distinguished discrete series. It is known that a generalised Steinberg representation $St(\rho, k)$ is distinguished if and only if the cuspidal representation ρ is η^{k-1} -distinguished for η , the character of F^* with kernel consisting of the norms of K^* . This actually gives a classification of distinguished unitary representations in terms of distinguished cuspidal representations.

Introduction

In the present work, for F a p -adic field and K a quadratic extension of F , smooth and complex unitary (which will be synonymous with unitarisable for us), we study representations of $GL(n, K)$ which admit on their space a nonzero invariant linear form under $GL(n, F)$. These unitary representations are called $GL(n, F)$ -distinguished (or simply distinguished) and are conjectured to be the unitary part of the image of a functorial lift, in the Langlands' program, from $U(n, K/F)$ to $GL(n, K)$.

Distinguished generic representations of $GL(n, K)$ have been classified in [Matringe 2011b], in terms of distinguished quasi-discrete series, using Zelevinsky's classification of generic representations. Here we do the same for distinguished irreducible unitary representations using Tadić's classification of irreducible unitary representations. Our main result (Theorem 2.13) is similar to the main result of [Matringe 2011b]. However, to extend the result from generic unitary to irreducible unitary representations, we use different techniques. Our main tools are the Bernstein–Zelevinsky derivative functors, and we apply ideas from [Bernstein 1984]. For instance, the building blocks for unitary representations (the so-called Speh representations) are not parabolically induced; hence one needs new methods to deal with these representations. That is what we do in the second part of Section 2 to obtain a definitive statement in Corollary 2.9, which we state here as a theorem.

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Theorem. *Let k and m be two positive integers, and let n be equal to km . If Δ is a discrete series of $\mathrm{GL}(m, K)$ and $u(\Delta, k)$ is the corresponding Speh representation of $\mathrm{GL}(n, K)$ (see Definition 1.3), then $u(\Delta, k)$ is distinguished if and only if Δ is.*

One direction is given by the fact that if π is a distinguished irreducible unitary representation, then it is also the case of its highest shifted derivative (see Proposition 2.4). The other direction is a nontrivial generalisation of the following simple observation: if ρ is a distinguished cuspidal representation of $\mathrm{GL}(n, K)$, then it is known that the parabolically induced representation $\nu^{1/2}\rho \times \nu^{-1/2}\rho$ is distinguished, and it is also known that its irreducible submodule $St(\rho, 2)$ is not distinguished. Hence its quotient $u(\rho, 2)$, which is a Speh representation of $\mathrm{GL}(2n, K)$, is distinguished. The case of general irreducible unitary representations of $\mathrm{GL}(n, K)$ distinguished by $\mathrm{GL}(n, F)$ is treated in the third part of Section 2. We obtain the main result of the paper in Theorem 2.13. Denoting by σ the nontrivial element of the Galois group of K over F , by π^\vee the smooth contragredient of a representation π of $\mathrm{GL}(n, K)$, and by π^σ the representation $\pi \circ \sigma$, its statement is as follows.

Theorem. *Let n be a positive integer and π an irreducible unitary representation of $\mathrm{GL}(n, K)$. By Tadić's classification (see Theorem 1.9), the representation π is a commutative product (in the sense of normalised parabolic induction) of representations of the form $u(\Delta, k)$ for $k > 0$ and Δ a discrete series, and representations of the form $\pi(u(\Delta, k), \alpha)$ (see Definition 1.8) for Δ and k as before and α an element of $(0, 1/2)$. Then the representation π is distinguished if and only if π^\vee is isomorphic to π^σ and the Speh representations $u(\Delta, k)$ occurring in the product π with odd multiplicity are distinguished.*

1. Preliminaries

Basic facts and notations. First, in the following, we fix a nonarchimedean local field F of characteristic 0 and an algebraic closure \bar{F} of F . We denote by K a quadratic extension of F in \bar{F} . We denote by \mathfrak{O}_F and \mathfrak{P}_F the ring of integers of F and the unique maximal ideal of F respectively. We similarly define \mathfrak{O}_K and \mathfrak{P}_K . We denote by $|\cdot|_F$ and $|\cdot|_K$ the normalised absolute values, which satisfy $|x|_K = |x|_F^2$ for x in F . We fix a nontrivial smooth character θ of K which is trivial on F . We denote by σ the nontrivial element of the Galois group $\mathrm{Gal}_F(K)$ of K over F and by η the quadratic character of F^* , whose kernel is the set of norms of K^* . For n and $m \geq 1$, we denote by $\mathcal{M}_{n,m}$ the space of matrices $\mathcal{M}(n, m, K)$, by \mathcal{M}_n the algebra $\mathcal{M}_{n,n}$, and by G_n the group of invertible elements in \mathcal{M}_n . We will denote by G_0 the trivial group. If m belongs to \mathcal{M}_n , we denote by m^σ the matrix obtained from m by applying σ to each entry. If S is a subset of \mathcal{M}_n , we denote by S^σ the subset of S consisting of elements fixed by σ . For

$m \in \mathcal{M}_n$, we denote by $|m|_K$ or $\nu_K(m)$ the real number $|\det m|_K$, and we define similarly $|m|_F$ or $\nu_F(m)$ for m in \mathcal{M}_n^σ .

When G is a closed subgroup of G_n , we denote by $\text{Alg}(G)$ the category of smooth complex G -modules. If (π, V) belongs to $\text{Alg}(G)$, H is a closed subgroup of G , and χ is a character of H , we denote by $V(H, \chi)$ the subspace of V generated by vectors of the form $\pi(h)v - \chi(h)v$ for h in H and v in V . This space is stable under the action of the subgroup $N_G(\chi)$ of the normalizer $N_G(H)$ of H in G , which fixes χ . We denote by δ_H the positive character of $N_G(H)$ such that if μ is a right Haar measure on H and int is the action of $N_G(H)$ on smooth functions f with compact support in H , given by $(\text{int}(n)f)(h) = f(n^{-1}hn)$, then $\mu \circ \text{int}(n) = \delta_H(n)\mu$ for n in $N_G(H)$. The space $V(H, \chi)$ is $N_G(\chi)$ -stable. Thus, if L is a closed subgroup of $N_G(\chi)$ and δ' is a (smooth) character of L (which will be a normalising character dual to that of normalised induction later), the quotient $V_{H,\chi} = V/V(H, \chi)$ (which we simply denote by V_H when χ is trivial) becomes a smooth L -module for the (normalised) action $l.(v + V(H, \chi)) = \delta'(l)\pi(l)v + V(H, \chi)$ of L on $V_{H,\chi}$. If (ρ, W) belongs to $\text{Alg}(H)$, we define the objects

$$(\text{ind}_H^G(\rho), V_c = \text{ind}_H^G(W)) \quad \text{and} \quad (\text{Ind}_H^G(\rho), V = \text{Ind}_H^G(W))$$

of $\text{Alg}(G)$ as follows. The space V is the space $\mathcal{C}^\infty(H \backslash G, \rho)$ of smooth functions from G to W fixed under right translation by the elements of a compact open subgroup U_f of G , and satisfying $f(hg) = \rho(h)f(g)$ for all h in H and g in G . The space V_c is the subspace $\mathcal{C}_c^\infty(H \backslash G, \rho)$ of V consisting of functions with support compact mod H . In both cases, the action of G is by right translation on the functions. By definition, the real part $\text{Re}(\chi)$ of a character χ of F^* is the real number r such that $|\chi(t)|_{\mathbb{C}} = |t|^r$, where $|z|_{\mathbb{C}} = \sqrt{z\bar{z}}$ for z in \mathbb{C} .

Irreducible representations of $GL(n)$. We will only consider smooth representations of G_n and its closed subgroups. We denote by A_n the maximal torus of diagonal matrices in G_n . It will sometimes be useful to parametrise A_n with simple roots, that is, to write an element $t = \text{diag}(t_1, \dots, t_n)$ of A_n as $t = z_1 \cdots z_n$, where $z_n = t_n I_n$ and $z_i = \text{diag}((t_i/t_{i+1})I_i, I_{n-i})$ belongs to the centre of G_i embedded in G_n , which we denote Z_i . For $z_i = \text{diag}(t_i I_i, I_{n-i})$ in Z_i , we denote t_i by $t(z_i)$. If $n \geq 1$, let $\bar{n} = (n_1, \dots, n_t)$ be a partition of n of length t (i.e., an ordered set of t positive integers whose sum is n). We denote by $M_{\bar{n}}$ the Levi subgroup of G_n of matrices of the form $\text{diag}(g_1, \dots, g_t)$ with each g_i in G_{n_i} , by $N_{\bar{n}}$ the unipotent subgroup of matrices of the form

$$\begin{pmatrix} I_{n_1} & \star & \star \\ & \ddots & \star \\ & & I_{n_t} \end{pmatrix},$$

and by $P_{\bar{n}}$ the standard parabolic subgroup $M_{\bar{n}}N_{\bar{n}}$ (where $M_{\bar{n}}$ normalises $N_{\bar{n}}$). Note that $M_{(1,\dots,1)}$ is equal to A_n , and we set $N_{(1,\dots,1)} = N_n$. For each i , let π_i be a smooth representation of G_{n_i} . Then the tensor product $\pi_1 \otimes \dots \otimes \pi_t$ is a representation of $M_{\bar{n}}$, which can be considered as a representation of $P_{\bar{n}}$ that is trivial on $N_{\bar{n}}$. We will use the product notation

$$\pi_1 \times \dots \times \pi_t = \text{Ind}_{P_{\bar{n}}}^{G_n} (\delta_{P_{\bar{n}}}^{1/2} \pi_1 \otimes \dots \otimes \pi_t)$$

for the normalised parabolic induction.

We say that an irreducible representation (ρ, V) of G_n is cuspidal if the Jacquet module $V_{N_{\bar{n}}}$ is zero when \bar{n} is a proper partition of n . Suppose that $\bar{n} = (m, \dots, m)$ is a partition of n of length l and that ρ is a cuspidal representation of G_m . Let a and b be two integers with $a \leq b$ such that $b - a + 1 = l$. Then [Zelevinsky 1980, Theorem 9.3] implies that the G_n -module $v_K^a \rho \times \dots \times v_K^b \rho$ has a unique irreducible quotient, which we denote by $\Delta(\rho, b, a)$. We call it a segment or a quasi-discrete series of G_n . If, in addition, a quasi-discrete series is unitary (which amounts to saying that its central character is unitary), we will call it a discrete series or a unitary segment. We will sometimes write $St(\rho, l) = \Delta(\rho, (l - 1)/2, -(l - 1)/2)$.

We end this section with a word about induced representations of Langlands' type and their quotients.

Definition 1.1. Let $\Delta_1, \dots, \Delta_t$ be segments of G_{n_1}, \dots, G_{n_t} respectively, and suppose that the central characters satisfy the relation $\text{Re}(c_{\Delta_i}) \geq \text{Re}(c_{\Delta_{i+1}})$. Let $n = n_1 + \dots + n_t$. Then the representation $\Delta_1 \times \dots \times \Delta_t$ of G_n is said to be induced of Langlands' type.

The following result is well known and can be found in [Rodier 1982].

Proposition 1.2. *Let $\pi = \Delta_1 \times \dots \times \Delta_t$ be an induced representation of Langlands' type as above. Then π has a unique irreducible quotient, which we denote by $L(\Delta_1, \dots, \Delta_t)$. If $\Delta'_1, \dots, \Delta'_s$ are other segments with $\text{Re}(c_{\Delta'_j}) \geq \text{Re}(c_{\Delta'_{j+1}})$ such that $L(\Delta_1, \dots, \Delta_t) = L(\Delta'_1, \dots, \Delta'_s)$, then we have the equality of nonordered sets $\{\Delta_1, \dots, \Delta_t\} = \{\Delta'_1, \dots, \Delta'_s\}$.*

A particular class of Langlands' quotients is the class of Speh representations, which are the building blocks of the unitary dual of G_n in Tadić's classification.

Definition 1.3. Let k and m be two positive integers, and set $n = km$. If Δ is a segment of G_m , we denote by $u(\Delta, k)$ the representation $L(v_K^{(k-1)/2} \Delta, \dots, v_K^{(1-k)/2} \Delta)$ of G_n .

We now recall some basic facts about Bernstein–Zelevinsky derivatives.

Derivatives. We define a character of N_n , denoted again by θ , by the formula $\theta(m) = \theta(\sum_{i=1}^{n-1} m_{i,i+1})$. For $n \geq 2$, we denote by U_n the group of matrices of the form $\begin{pmatrix} I_{n-1} & v \\ & 1 \end{pmatrix}$. For $n > k \geq 1$, the group G_k embeds naturally in G_n and is given by matrices of the form $\text{diag}(g, I_{n-k})$. We denote by P_n the mirabolic subgroup $G_{n-1}U_n$ of G_n for $n \geq 2$, and we set $P_1 = \{1_{G_1}\}$. If one sees P_{n-1} as a subgroup of G_{n-1} itself embedded in G_n , then P_{n-1} is the normaliser of $\theta|_{U_n}$ in G_{n-1} (i.e., if $g \in G_{n-1}$, then $\theta(g^{-1}ug)$ for all $u \in U_n$ if and only if $g \in P_{n-1}$). We define the following functors:

- The functor Φ^+ from $\text{Alg}(P_{k-1})$ to $\text{Alg}(P_k)$ such that for π in $\text{Alg}(P_{k-1})$, one has $\Phi^+ \pi = \text{ind}_{P_{k-1}U_k}^{P_k} (\delta_{U_k}^{1/2} \pi \otimes \theta)$.
- The functor $\hat{\Phi}^+$ from $\text{Alg}(P_{k-1})$ to $\text{Alg}(P_k)$ such that for π in $\text{Alg}(P_{k-1})$, one has $\hat{\Phi}^+ \pi = \text{Ind}_{P_{k-1}U_k}^{P_k} (\delta_{U_k}^{1/2} \pi \otimes \theta)$.
- The functor Φ^- from $\text{Alg}(P_k)$ to $\text{Alg}(P_{k-1})$ such that if (π, V) is a smooth P_k -module, $\Phi^- V = V_{U_k, \theta}$, and P_{k-1} acts on $\Phi^-(V)$ by $\Phi^- \pi(p)(v + V(U_k, \theta)) = \delta_{U_k}(p)^{-1/2} \pi(p)(v + V(U_k, \theta))$.
- The functor Ψ^- from $\text{Alg}(P_k)$ to $\text{Alg}(G_{k-1})$ such that if (π, V) is a smooth P_k -module, $\Psi^- V = V_{U_k, 1}$, and G_{k-1} acts on $\Psi^-(V)$ by $\Psi^- \pi(g)(v + V(U_k, 1)) = \delta_{U_k}(g)^{-1/2} \pi(g)(v + V(U_k, 1))$.
- The functor Ψ^+ from $\text{Alg}(G_{k-1})$ to $\text{Alg}(P_k)$ such that for π in $\text{Alg}(G_{k-1})$, one has $\Psi^+ \pi = \text{ind}_{G_{k-1}U_k}^{P_k} (\delta_{U_k}^{1/2} \pi \otimes 1) = \delta_{U_k}^{1/2} \pi \otimes 1$.

If τ is a representation of P_n (or a representation of G_n , which we consider as a P_n -module by restriction), the representation $\tau^{(k)}$ of G_{n-k} will be defined as $\Psi^-(\Phi^-)^{k-1} \tau$ and will be called the k -th derivative of τ . It is shown in [Bernstein and Zelevinsky 1977, Section 3.5] that these representations give a natural filtration of any P_n -module.

Lemma 1.4. *If τ is an object of $\text{Alg}(P_n)$, then τ has a natural filtration of P_n -modules $0 \subset \tau_n \subset \dots \subset \tau_1 = \tau$, where $\tau_k = \Phi^{+k-1} \Phi^{-k-1} \tau$. Moreover, the quotient τ_k / τ_{k+1} is isomorphic to $(\Phi^+)^{k-1} \Psi^+ \tau^{(k)}$ as a P_n -module.*

It is shown in [Zelevinsky 1980, Section 8] that if π is an irreducible representation of G_n , then its highest derivative π^- , which is the derivative $\pi^{(k)}$ for $k \leq n$ that is maximal for the condition $\pi^{(k)} \neq 0$, is an irreducible representation of G_{n-k} . The following lemma is an immediate consequence of [Bernstein and Zelevinsky 1977, Lemma 4.5].

Lemma 1.5. *Let π_i be an irreducible representation of G_{n_i} for positive integers n_1, \dots, n_t . Then the highest derivative of $\pi_1 \times \dots \times \pi_t$ is the representation $\pi_1^- \times \dots \times \pi_t^-$.*

As we study unitary representations, we will need some further properties of these derivatives, which are extracted from [Bernstein 1984]. First, as in this reference, we introduce the following definition.

Definition 1.6. Let τ be a P_n -module. We denote by $\tau^{[k]}$ the representation $v_K^{1/2} \tau^{(k)}$ of G_{n-k} and call it the k -th shifted derivative of τ . We denote by $\tau^{[-]}$ the highest shifted derivative of τ .

We then recall the following consequence of the unitarisability criterion given in [ibid., Section 7.3].

Proposition 1.7. *If π is an irreducible unitary representation of G_n with highest derivative $\pi^{(h)}$, then $\pi^{[h]}$ is unitary and the central characters of the irreducible subquotients of $\pi^{[k]}$ all have positive real parts for $0 < k < h$.*

Unitary representations of $GL(n)$. We now recall results from [Tadić 1986] about the classification of irreducible unitary representations of G_n .

Definition 1.8. For $\alpha \in \mathbb{R}$, $m > 0$, $k > 0$, and Δ a segment of G_m , we denote by $\pi(u(\Delta, k), \alpha)$ the representation $v_K^\alpha u(\Delta, k) \times v_K^{-\alpha} u(\Delta, k)$ of G_n for $n = 2mk$.

Theorem 1.9 [Tadić 1986, Theorem D]. *Let π be an irreducible unitary representation of G_n . Then there is a partition (n_1, \dots, n_t) of n and representations π_i of G_{n_i} , each of which is either of the form $\pi(u(\Delta_i, k_i), \alpha_i)$ for Δ_i a unitary segment, $k_i \geq 1$, and $0 < \alpha_i < 1/2$ or of the form $u(\Delta_i, k_i)$ for Δ_i a unitary segment and $k_i \geq 1$, such that $\pi = \pi_1 \times \dots \times \pi_t$. Moreover, the representation π is equal to $\pi'_1 \times \dots \times \pi'_s$ for representations π'_j of the same type as the representations π_i if and only if $\{\pi_1, \dots, \pi_t\} = \{\pi'_1, \dots, \pi'_s\}$ as nonordered sets.*

If all the representations π_i in the above theorem are such that $k_i = 1$, we say that π is a *generic* unitary representation of G_n .

We will also need the description of the composition series of the so-called end of complementary series, which is proved in [Tadić 1987a] (see [Badulescu 2011, Theorem 2] for a quick proof). If Δ is the segment $St(\rho, l)$ for $l \geq 1$, we write $\Delta_+ = St(\rho, l + 1)$ and $\Delta_- = St(\rho, l - 1)$, where $St(\rho, 0)$ is $\mathbf{1}_{G_0}$ by convention.

Theorem 1.10. *Let m be a positive integer, Δ a segment of G_m , $k \geq 2$ an integer, and $n = 2mk$. The representation $\pi(u(\Delta, k), 1/2)$ of G_n is of length 2, and its irreducible subquotients are $u(\Delta_-, k) \times u(\Delta_+, k)$ and $u(\Delta, k - 1) \times u(\Delta, k + 1)$.*

Finally, we recall the formula which gives the highest shifted derivative of a Sp h representation, from [Tadić 1987b, Section 6.1] (see [Offen and Sayag 2008, (3.3)] for the proof).

Proposition 1.11. *Let $m > 0$ and $k > 1$ be two integers, and let Δ be a segment of G_m . The highest shifted derivative of the representation $u(\Delta, k)$ is equal to $u(\Delta, k)^{[m]} = u(\Delta, k - 1)$. The highest (shifted) derivative of Δ is equal to $\mathbf{1}_{G_0}$.*

Distinguished representations of $GL(n)$. In this paragraph, we recall results from [Matringe 2011b]. First, we introduce some notations and definitions.

Definition 1.12. Let G be a closed subgroup of G_n , H a closed subgroup of G , and χ a character of H . We say that a representation π in $\text{Alg}(G)$ is (H, χ) -distinguished if the space $\text{Hom}_H(\pi, \chi)$ is nonzero. If H is clear, we say χ -distinguished instead of (H, χ) -distinguished, and if χ is trivial, we say H -distinguished (or distinguished if H is clear). If $G = G_n$ and $H = G_n^\sigma$, we will sometimes say (σ, χ) -distinguished instead of (H, χ) -distinguished, and if χ is trivial, we will simply say σ -distinguished.

We recall the following general facts from [Flicker 1991] about σ -distinguished representations of G_n . We denote by π^σ the representation $g \mapsto \pi(g^\sigma)$ for π a representation of G_n .

Proposition 1.13. *Let $n \geq 1$ be an integer and π be an irreducible representation of G_n . If π is σ -distinguished, then $\pi^\vee = \pi^\sigma$ and $\text{Hom}_{G_n^\sigma}(\pi, \mathbf{1})$ is of dimension 1.*

We now introduce the class of σ -induced irreducible unitary representations of G_n . They will turn out to be the σ -distinguished irreducible unitary representations of G_n .

Definition 1.14. For $n \geq 1$, let π be an irreducible unitary representation

$$\pi = u(\Delta_1, k_1) \times \cdots \times u(\Delta_s, k_s) \times \pi(u(\Delta_{s+1}, k_{s+1}), \alpha_{s+1}) \times \cdots \times \pi(u(\Delta_t, k_t), \alpha_t)$$

of G_n with unitary segments Δ_i , positive integers k_i , and $\alpha_i \in (0, 1/2)$. The representation π is said to be σ -induced if it satisfies $\pi^\vee = \pi^\sigma$ and if for every $i \leq s$ such that $u(\Delta_i, k_i)$ occurs with odd multiplicity in the product π , the segment Δ_i is σ -distinguished.

Remark 1.15. Maybe the preceding definition is not completely transparent to the reader. Let us try to explain what σ -induced irreducible unitary representations look like. Let

$$\pi = u(\Delta_1, k_1) \times \cdots \times u(\Delta_t, k_s) \times \pi(u(\Delta_{s+1}, k_{s+1}), \alpha_{k_{s+1}}) \times \cdots \times \pi(u(\Delta_t, k_t), \alpha_{k_t})$$

be an irreducible unitary representation of G_n . First, if one has $\pi^\vee = \pi^\sigma$ (call this relation σ -self-duality), then it means the two following things:

- (a) For i between 1 and s , either $u(\Delta_i, k_i)$ is σ -self-dual or, if this relation is not satisfied, there exists $j \neq i$ between 1 and s such that $u(\Delta_j, k_j)^\vee = u(\Delta_i, k_i)^\sigma$.
- (b) For i between $s + 1$ and t , either $\pi(u(\Delta_i, k_i), \alpha_i)$ is σ -self-dual or, if this relation is not satisfied, there exists $j \neq i$ between $s + 1$ and t such that $\pi(u(\Delta_j, k_j), \alpha_j)^\vee = \pi(u(\Delta_i, k_i), \alpha_i)^\sigma$.

In (a) above, if you have $u(\Delta_i, k_i)^\vee = u(\Delta_i, k_i)^\sigma$ which occurs with multiplicity at least 2, that is, if there is $j \neq i$ between 1 and s such that $u(\Delta_j, k_j) = u(\Delta_i, k_i)$, then one has $u(\Delta_j, k_j)^\vee = u(\Delta_i, k_i)^\sigma$. Hence (a) can also be stated as:

(a') $u(\Delta_1, k_1) \times \cdots \times u(\Delta_s, k_s)$ is a product of representations of the form $u(\Delta_i, k_i) \times (u(\Delta_i, k_i)^\vee)^\sigma$ and of σ -self dual representations $u(\Delta_j, k_j)$ which occur with odd multiplicity.

Now in (b), if $\pi(u(\Delta_i, k_i), \alpha_i)$ is σ -self dual, it is equal to

$$v_K^{\alpha_i} u(\Delta_i, k_i) \times ((v_K^{\alpha_i} u(\Delta_i, k_i))^\vee)^\sigma$$

(because Δ_i^\vee must be equal to Δ_i^σ). All in all, π is σ -self dual if and only if it is a product of representations of the form

$$v_K^\alpha u(\Delta, k) \times ((v_K^\alpha u(\Delta, k))^\vee)^\sigma$$

for $0 \leq \alpha < 1/2$, Δ a discrete series, and k a positive integer (we allow here α to be equal to zero in order to take in account representations $u(\Delta_i, k_i) \times (u(\Delta_i, k_i)^\vee)^\sigma$ occurring in (a')), of representations of the form

$$\pi(u(\Delta, k), \alpha) \times (\pi(u(\Delta, k), \alpha)^\vee)^\sigma$$

for α in $(0, 1/2)$ and Δ and k as above, and of representations of the form $u(\Delta', k')$ (Δ' unitary and $k' > 0$) occurring with odd multiplicity and which are σ -self dual. In this situation, π is σ -induced if and only if these representations $u(\Delta', k')$ are such that Δ' is σ -distinguished.

Theorem 5.2 of [Matringe 2011b] then classifies distinguished generic representations.

Theorem 1.16. *For $n \geq 1$, a generic unitary representation of G_n is σ -distinguished if and only if it is σ -induced.*

We also recall [Matringe 2009, Corollary 3.1] about distinction of discrete series.

Proposition 1.17. *Let ρ be a cuspidal representation of G_r for $r \geq 1$ and $\Delta = St(\rho, l)$ for $l \geq 1$. The segment Δ of G_{lr} is σ -distinguished if and only if ρ is (σ, η^{l-1}) -distinguished.*

Finally, [Anandavardhanan et al. 2004, Corollary 1.6] says that the segment Δ above cannot be σ -distinguished and (σ, η) -distinguished at the same time. This has the following immediate corollary.

Corollary 1.18. *Let Δ be a segment of G_n for $n \geq 2$. Then Δ is σ -distinguished if and only if Δ_+ is (σ, η) -distinguished. In particular, if Δ is distinguished, then Δ_+ is not.*

2. Distinguished unitary representations

We will first prove the convergence of integrals defining invariant linear forms.

Asymptotics in the degenerate Kirillov model. We denote by $N_{n,h}$ the group of matrices $h(a, n) = \begin{pmatrix} a & x \\ 0 & n \end{pmatrix}$ with a in G_{n-h} , n in N_h , and x in $\mathcal{M}_{n-h,h}$. It is proved in [Zelevinsky 1980, Section 5] that any irreducible representation π of G_n has a “degenerate Kirillov model” (which is just the standard Kirillov model in the nondegenerate case). This means that the restriction of π to P_n embeds as a unique P_n -submodule $K(\pi, \theta)$ of $(\hat{\Phi}^+)^{h-1}\Psi^+(\pi^{(h)})$, where $\pi^{(h)} = \pi^-$. The space $K(\pi, \theta)$ consists of smooth functions W from P_n to $V_{\pi^{(h)}}$ which are fixed under right translation by an open subgroup U_W and satisfy the relation

$$W(h(a, n)p) = |a|_K^{h/2} \theta(n) \pi^{(h)}(a) W(p)$$

for $h(a, n)$ in $N_{n,h}$ and p in P_n . It can be handy to identify such a function with a map from P_n to $V_{\pi^{[h]}}$ which satisfies the relation

$$(1) \quad W(h(a, n)p) = |a|_K^{(h-1)/2} \theta(n) \pi^{[h]}(a) W(p)$$

for $h(a, n)$ in $N_{n,h}$ and p in P_n .

We now give an asymptotic expansion of the elements of $K(\pi, \theta)$ in terms of the exponents of π . The proof, which is omitted, is an easy adaptation of the proof of [Matringe 2011a, Theorem 2.1]. We write $\mathcal{C}_c^\infty(F, V)$ for the space of smooth functions with compact support from F to a complex vector space V .

Theorem 2.1. *Let π be an irreducible representation of G_n for $n \geq 2$. Let $\pi^{(h)}$ be the highest derivative of π , and let W belong to $K(\pi, \theta)$. We suppose that we have $h \geq 2$, and we denote by $(c_{k,i_k})_{1 \leq k \leq r_k}$ the family of central characters of the irreducible subquotients of $\pi^{(k)}$. In this situation, the restriction $W(z_{n-h+1} \cdots z_{n-1})$ of W to the torus $Z_{n-h+1} \cdots Z_{n-1}$ is a linear combination of functions of the form*

$$\prod_{k=n-h+1}^{n-1} [c_{i_k,k} \delta_{U_{k+1}}^{1/2} \cdots \delta_{U_n}^{1/2}](z_k) v_F(z_k)^{m_k} \phi_k(t(z_k))$$

for i_k between 1 and r_k , nonnegative integers m_k , and functions ϕ_k in $\mathcal{C}_c^\infty(F, V_{\pi^{(h)}})$.

From this, we deduce the convergence of the following integrals, which we will need later.

Proposition 2.2. *Let π be an irreducible unitary representation of G_n for $n \geq 1$. Let $\pi^{(h)}$ be the highest derivative of π , and let W belong to $K(\pi, \theta)$. We suppose that there is a nonzero G_{n-h}^σ -invariant linear form L on the space of $\pi^{[h]}$, and for every element W of $K(\pi, \theta)$, we define the map $f_{L,W} = L \circ W$. Then for all W in $K(\pi, \theta)$, the integral*

$$\Lambda(W) = \int_{N_{n,h}^\sigma \backslash P_n^\sigma} f_{L,W}(p) dp$$

is absolutely convergent and Λ defines a nonzero P_n^σ -invariant linear form on V_π .

Proof. If h equals 1, then $\Lambda(W)$ is equal to $L(W(I_n))$ up to normalisation, and the result is obvious. For $h \geq 2$, first, thanks to Relation (1), the restriction of the map $f_{L,W}$ to P_n^σ satisfies the relation

$$f_{L,W}(h(a, n)p) = |a|_F^{h-1} f_{L,W}(p)$$

for p in P_n^σ and $h(a, n)$ in $N_{n,h}^\sigma$. We notice that $|a|_F^{h-1}$ is indeed equal to

$$\frac{\delta_{N_{n,h}^\sigma}(h(a, n))}{\delta_{P_n^\sigma}} = \frac{|a|_F^h}{|a|_F}.$$

Actually, the integral $\Lambda(W)$ is equal to

$$\int_{N_{n-1,h}^\sigma \backslash G_{n-1}^\sigma} f_{L,W}(p) dp.$$

Hence, thanks to the Iwasawa decomposition, the integral Λ will converge absolutely for any W in $K(\pi, \theta)$ if and only if the following integral does as well:

$$\int_{Z_{n-h+1} \dots Z_{n-1}} f_{L,W}(z_{n-h+1} \dots z_n) \delta_{N_{n-1,h}^\sigma}^{-1}(z_{n-h+1} \dots z_{n-1}) d^* z_{n-h+1} \dots d^* z_{n-1}$$

for any W in $K(\pi, \theta)$. As $\delta_{N_{n-1,h}^\sigma}(z_{n-h+1} \dots z_{n-1})$ is equal to the product

$$\prod_{k=n-h+1}^{n-1} \delta_{U_{k+1}^\sigma} \dots \delta_{U_{n-1}^\sigma}(z_k) = \prod_{k=n-h+1}^{n-1} \delta_{U_{k+1}}^{1/2} \dots \delta_{U_{n-1}}^{1/2}(z_k)$$

for the z_i in Z_i^σ , we obtain that the integral

$$\int_{Z_{n-h+1} \dots Z_{n-1}} |f_{L,W}(z_{n-h+1} \dots z_n)| \delta_{N_{n-1,h}^\sigma}^{-1}(z_{n-h+1} \dots z_{n-1}) d^* z_{n-h+1} \dots d^* z_{n-1}$$

is majorized by a sum of integrals of the form

$$\begin{aligned} & \prod_{k=n-h+1}^{n-1} \int_{Z_k} c_{i_k,k} \delta_{U_n}^{1/2}(z_k) v_F(z_k)^{m_k} f_k(t(z_k)) d^* z_k \\ &= \prod_{k=n-h+1}^{n-1} \int_{Z_k} c_{i_k,k} \delta_{U_{k+1}}^{1/2}(z_k) v_F(z_k)^{m_k} f_k(t(z_k)) d^* z_k \end{aligned}$$

for functions $f_k = L \circ \phi_k$ in $\mathcal{C}_c^\infty(F)$, thanks to Theorem 2.1. These last integrals are convergent, as, according to Proposition 1.7, the real part $\text{Re}(c_{i_k,k} \delta_{U_{k+1}}^{1/2})$ is positive. This concludes the proof of the convergence. To show that Λ is nonzero, we just need to remember that π contains as a P_n -submodule the space $(\Phi^+)^{h-1}(\Psi^+(\pi^{(h)}))$ and the restriction to P_n^σ of elements of $(\Phi^+)^{h-1}(\Psi^+(\pi^{(h)}))$ is surjective on the space

$$\mathcal{C}_c^\infty \left(N_{n,h}^\sigma \backslash P_n^\sigma, \frac{\delta_{N_{n,h}^\sigma}}{\delta_{P_n^\sigma}} \pi^{[k]} \otimes \mathbf{1} \right). \quad \square$$

The case of Speh representations. The aim of this section is to prove that a representation $u(\Delta, k)$ is σ -distinguished if and only if Δ is, independently of k . Oddly enough, the trickiest part is to prove that when Δ is σ -distinguished, so is $u(\Delta, k)$. We first recall, as a lemma, [Kable 2004, Proposition 1], which is the key ingredient of the proof of the functional equation of the local Asai L -function.

Lemma 2.3. *Let τ be a representation of P_n for $n \geq 1$. Then the space $\text{Hom}_{P_n^\sigma}(\tau, \mathbf{1})$ is isomorphic to $\text{Hom}_{P_{n+1}^\sigma}(\Phi^+(\tau), \mathbf{1})$.*

This implies the following generalisation of [Anandavardhanan et al. 2004, Theorem 1.1]:

Proposition 2.4. *Let π be an irreducible unitary representation of G_n for $n \geq 1$. The representation π is P_n^σ -distinguished if and only if its highest shifted derivative $\pi^{[-]}$ is σ -distinguished.*

Proof. One implication follows from Proposition 2.2. For the other one, we first notice that by the definition of Ψ^+ , if π' is a representation of G_k for $k \geq 0$, then the space $\text{Hom}_{P_{k+1}^\sigma}(\Psi^+(\pi'), \mathbf{1})$ is isomorphic to $\text{Hom}_{G_k^\sigma}(v^{1/2}\pi', \mathbf{1})$. Hence, thanks to Lemma 2.3, the space $\text{Hom}_{P_{k+l}^\sigma}((\Phi^+)^{l-1}\Psi^+(\tau), \mathbf{1})$ is isomorphic to $\text{Hom}_{G_k^\sigma}(v^{1/2}\pi', \mathbf{1})$. Now, if τ is an irreducible unitary representation of G_n , let h be the integer such that $\pi^- = \pi^{(h)}$. The restriction of π to P_n has a filtration with factors $(\Phi^+)^{k-1}\Psi^+(\pi^{(k)})$ for k between 1 and h , according to Lemma 1.4. If L is a nonzero P_n^σ -invariant linear form on π , it must induce a nonzero element of $\text{Hom}_{P_n^\sigma}((\Phi^+)^{k-1}\Psi^+(\pi^{(k)}), \mathbf{1}) \simeq \text{Hom}_{G_{n-k}^\sigma}(\pi^{[k]}, \mathbf{1})$ for some k in $\{1, \dots, h\}$. But if the space $\text{Hom}_{G_{n-k}^\sigma}(\pi^{[k]}, \mathbf{1})$ is nonzero, it implies that the central character of one of the irreducible subquotients of $\pi^{[k]}$ has real part equal to zero because F^* must act trivially on at least one irreducible subquotient of $\pi^{[k]}$. Hence, according to Proposition 1.7, this means that the space $\text{Hom}_{P_n^\sigma}((\Phi^+)^{k-1}\Psi^+(\pi^{(k)}), \mathbf{1})$ is reduced to zero for k between 1 and $h-1$ and that the space

$$\text{Hom}_{P_n^\sigma}((\Phi^+)^{h-1}\Psi^+(\pi^{(h)}), \mathbf{1}) \simeq \text{Hom}_{G_{n-h}^\sigma}(\pi^{[h]}, \mathbf{1})$$

is nonzero. The result is thus proved. □

The proof of the preceding proposition implicitly contains the following statement.

Proposition 2.5. *Let π be an irreducible unitary representation of G_n which is P_n^σ -distinguished. Then its highest shifted derivative $\pi^{[-]}$ is σ -distinguished, and the space $\text{Hom}_{P_n^\sigma}(\pi, \mathbf{1})$ is of dimension 1 with basis equal to a certain linear form L . Moreover, the restriction of L to $\tau_0 = (\Phi^+)^{h-1}\Psi^+(\pi^-)$ is nonzero, and if τ is any P_n -submodule of π which is P_n^σ -distinguished, then τ contains τ_0 and the space $\text{Hom}_{P_n^\sigma}(\tau, \mathbf{1})$ is spanned by the restriction $L|_\tau$.*

From this, we deduce a statement which will be used twice in a crucial way.

Proposition 2.6. *Let n_1 and n_2 be two positive integers and π_1 and π_2 be two irreducible unitary representations of G_{n_1} and G_{n_2} respectively. Suppose that π_1 is $G_{n_1}^\sigma$ -distinguished and that π_2 is $P_{n_2}^\sigma$ -distinguished. In this situation, if $\pi = \pi_1 \times \pi_2$ is G_n^σ -distinguished, then π_2 is $G_{n_2}^\sigma$ -distinguished.*

Proof. We write $\pi_1 \times \pi_2$ as induced from the lower parabolic subgroup $P^- = P_{(n_1, n_2)}^-$ obtained by transposing $P_{(n_1, n_2)}$. It is thus the space $\mathcal{C}_c^\infty(P^- \backslash G_n, \delta_{P^-}^{1/2} \pi_1 \otimes \pi_2)$. The double class $P^- P_n$, being open in G_n , contains

$$\tau = \mathcal{C}_c^\infty(P^- \backslash P^- P_n, \delta_{P^-}^{1/2} \pi_1 \otimes \pi_2),$$

which is a P_n -submodule of π . Let L_1 be a basis of $\text{Hom}_{G_{n_1}}(\pi_1, \mathbf{1})$, L_2 be a basis of $\text{Hom}_{P_{n_2}^\sigma}(\pi_2, \mathbf{1})$, and denote by λ the linear form $L_1 \otimes L_2$ on $\pi_1 \otimes \pi_2$. We now introduce the following linear form on τ :

$$L : f \mapsto \int_{P^- \cap P_n^\sigma \backslash P_n^\sigma} \lambda(f(p)) dp.$$

It is well-defined because the restriction of f to P_n^σ has compact support modulo $P^- \cap P_n^\sigma$ because it satisfies $f(hp) = |a|_F^{-n_2} |b|_F^{n_1} f(p)$ for

$$h = \begin{pmatrix} a & 0 & 0 \\ x & b & y \\ 0 & 0 & 1 \end{pmatrix} \in P^- \cap P_n^\sigma$$

written in blocks according to the partition $(n_1, n_2 - 1, 1)$ of n and because of the relation

$$\frac{\delta_{P^- \cap P_n^\sigma}}{\delta_{P_n^\sigma}}(h) = \frac{|a|_F^{1-n_2} |b|_F^{1+n_1}}{|a|_F |b|_F} = |a|_F^{-n_2} |b|_F^{n_1}.$$

Let's now show that L is nonzero. For v_1 in V_{π_1} and v_2 in V_{π_2} , let U be a congruence subgroup of G_n such that $U \cap G_{n_1}$ fixes v_1 and $U \cap G_{n_2}$ fixes v_2 . As U has an Iwahori decomposition with respect to P^- , the map defined by $f_{U, v_1, v_2}(p^- u) = \delta_{P^-}^{1/2} \pi_1 \otimes \pi_2(p^-)(v_1 \otimes v_2)$ for u in U , p^- in P^- and by zero outside $P^- U$ belongs to V_π . Moreover, $L(f_{U, v_1, v_2})$ is a positive multiple of $L_1(v_1)L_2(v_2)$. In particular, L is nonzero. This implies that L belongs to $\text{Hom}_{P_n^\sigma}(\tau, \mathbf{1}) - \{0\}$. It remains to prove that π_2 is $G_{n_2}^\sigma$ -distinguished. We are going to prove that L_2 is actually $G_{n_2}^\sigma$ -invariant. By Proposition 2.5, as π is irreducible, unitary, and σ -distinguished, we know that $\text{Hom}_{P_n^\sigma}(\pi, \mathbf{1})$ is one-dimensional, spanned by a linear form L' . Moreover, by the same proposition, up to multiplying L' by a scalar, the restriction of L' to τ is equal to L . Hence we denote L' by L . The fact that $\text{Hom}_{P_n^\sigma}(\pi, \mathbf{1})$ is one-dimensional also implies that L is in fact G_n^σ -invariant. Now take h of the form $\text{diag}(I_{n_1}, b)$ with b in $G_{n_2}(\mathfrak{O}_K)$. We have $\rho(h)f_{U, v_1, v_2} = f_{U, v_1, \rho(b)v_2}$. Moreover, if b belongs to $G_{n_2}(\mathfrak{O}_K)^\sigma$, the relation $L(\rho(h)f_{U, v_1, v_2}) = L(f_{U, v_1, v_2})$ implies the equality $L_1(v_1)L_2(\rho(b)v_2) = L_1(v_1)L_2(v_2)$. This implies that L_2 is $G_{n_2}(\mathfrak{O}_K)^\sigma$ -invariant.

In particular, it is w_{n_2} -invariant, where w_{n_2} is the antidiagonal matrix with ones on the second diagonal. As L_2 is $P_{n_2}^\sigma$ -invariant by hypothesis, it is $G_{n_2}^\sigma$ -invariant because w_{n_2} and $P_{n_2}^\sigma$ span the group $G_{n_2}^\sigma$, and this concludes the proof. \square

For Speh representations, we first obtain the following criterion of P_n^σ -distinction.

Proposition 2.7. *Let r be a positive integer, k be an integer ≥ 2 , and $n = kr$. Let Δ be a discrete series of G_r . Then the representation $u(\Delta, k)$ is P_n^σ -distinguished if and only if $u(\Delta, k - 1)$ is σ -distinguished.*

Proof. We recall from Proposition 1.11 that $u(\Delta, k)^{[-]}$ is equal to $u(\Delta, k - 1)$. We then apply Proposition 2.4. \square

Proposition 2.4 also has the following corollary.

Corollary 2.8. *Let n_1, \dots, n_t and k be positive integers and Δ_i be a unitary segment of G_{n_i} for each i . If the product $u(\Delta_1, k) \times \dots \times u(\Delta_t, k)$ is σ -distinguished, then the product $\Delta_1 \times \dots \times \Delta_t$ is σ -distinguished as well.*

Proof. First, according to Theorem 1.9, the product $u(\Delta_1, k) \times \dots \times u(\Delta_t, k)$ is unitary. According to Lemma 1.5 and Proposition 1.11, the highest shifted derivative of this product is $u(\Delta_1, k - 1) \times \dots \times u(\Delta_t, k - 1)$. It is σ -distinguished according to Proposition 2.4. Hence, by induction, the product $\Delta_1 \times \dots \times \Delta_t$ is σ -distinguished as well. \square

In particular, if $u(\Delta, k)$ is σ -distinguished, then Δ is σ -distinguished. We are now able to prove the main result of this section.

Corollary 2.9. *Let k and m be two positive integers and Δ be a discrete series of G_m . The representation $u(\Delta, k)$ is σ -distinguished if and only if Δ is σ -distinguished.*

Proof. If $u(\Delta, k)$ is σ -distinguished, we already noticed that Δ is σ -distinguished as a consequence of Corollary 2.8. For the converse, we do an induction on k .

The case $k = 1$ is clear, so let's suppose that $u(\Delta, l)$ is σ -distinguished for $l \leq k$ with $k \geq 1$. We recall from Theorem 1.10 that $v^{1/2}u(\Delta, k) \times v^{-1/2}u(\Delta, k)$ is of length two and has $u(\Delta_-, k) \times u(\Delta_+, k)$ and $u(\Delta, k - 1) \times u(\Delta, k + 1)$ as irreducible subquotients. Now, as $u(\Delta, k)^\vee = u(\Delta, k)^\sigma$, according to the main theorem of [Blanc and Delorme 2008], the representation $v^{1/2}u(\Delta, k) \times v^{-1/2}u(\Delta, k)$ is σ -distinguished. But $u(\Delta_-, k) \times u(\Delta_+, k)$ can't be distinguished, otherwise $\Delta_- \times \Delta_+$ would be distinguished thanks to Corollary 2.8, and this would in turn imply that both Δ_- and Δ_+ are also distinguished according to Theorem 1.16, which contradicts Corollary 1.18. Hence, the representation $u(\Delta, k - 1) \times u(\Delta, k + 1)$ must be σ -distinguished. We recall that the representation $u(\Delta, k - 1)$ is σ -distinguished by the induction hypothesis. As $u(\Delta, k)$ is σ -distinguished by hypothesis as well, the representation $u(\Delta, k + 1)$ is $P_{(k+1)m}^\sigma$ -distinguished by Proposition 2.7. Then,

the representation $u(\Delta, k + 1)$ is σ -distinguished according to Proposition 2.6, and this provides the induction step. \square

As a corollary, we obtain the following result.

Corollary 2.10. *Let k and m be positive integers. If Δ is a segment of G_m and $u(\Delta, k)^\vee$ is isomorphic to $u(\Delta, k)^\sigma$, then $u(\Delta, k)$ is either σ -distinguished or (σ, η) -distinguished and not both at the same time.*

Proof. The representation $u(\Delta, k)^\vee$ is isomorphic to $u(\Delta, k)^\sigma$ if and only if Δ^\vee is isomorphic to Δ^σ . The result is then a consequence of [Kable 2004, Theorem 7] and of [Anandavardhanan et al. 2004, Corollary 1.6]. \square

The general case. First, we notice that the class of σ -induced unitary irreducible representations of G_n is contained in the class of σ -distinguished representations.

Proposition 2.11. *For $n \geq 1$, let π be an irreducible unitary representation of G_n which is σ -induced. Then it is σ -distinguished.*

Proof. Let Δ be a discrete series of G_m with $m \geq 1$, let k be a positive integer, and let α be a real number. Then the representations $v_K^\alpha u(\Delta, k) \times ((v_K^\alpha u(\Delta, k))^\vee)^\sigma$ and $\pi(u(\Delta, k), \alpha) \times (\pi(u(\Delta, k), \alpha)^\vee)^\sigma$ are σ -distinguished according to the main theorem of [Blanc and Delorme 2008]. But as a product of σ -distinguished representations is σ -distinguished according to [Flicker 1992, Proposition 26], it follows from Remark 1.15 that if π is σ -induced, then it is indeed σ -distinguished. \square

It remains to prove the converse to obtain the main result of this paper. First, we make the following obvious but useful observation.

Lemma 2.12. *Let $\pi = u(\Delta_1, k_1) \times \cdots \times u(\Delta_r, k_r) \times \pi(u(\Delta_{r+1}, k_{r+1}), \alpha_{r+1}) \times \cdots \times \pi(u(\Delta_t, k_t), \alpha_t)$ be an irreducible unitary representation of G_n with Δ_i discrete series and real numbers α_i in $(0, 1/2)$. If the integers k_i satisfy $k_i \geq 2$, then π is σ -induced if and only if its highest shifted derivative $\pi^{[-]}$ is σ -induced.*

Proof. With the notations of the statement, according to Lemma 1.5 and Proposition 1.11, the representation $\pi^{[-]}$ is equal to the product

$$u(\Delta_1, k_1 - 1) \times \cdots \times u(\Delta_r, k_r - 1) \\ \times \pi(u(\Delta_{r+1}, k_{r+1} - 1), \alpha_{r+1}) \times \cdots \times \pi(u(\Delta_t, k_t - 1), \alpha_t).$$

Now it is clear that π is σ -self-dual if and only if $\pi^{[-]}$ is and that a representation $u(\Delta, k)$ (with Δ unitary) occurs with odd multiplicity in π if and only if $u(\Delta, k - 1)$ occurs with odd multiplicity in $\pi^{[-]}$. The result now follows from the fact that a Speh representation $u(\Delta, k)$ with $k \geq 2$ is σ -distinguished if and only if $u(\Delta, k - 1)$ is σ -distinguished, thanks to Corollary 2.9. \square

Theorem 2.13. *If π is an irreducible unitary representation of G_n for $n \geq 1$, then π is σ -distinguished if and only if it is σ -induced.*

Proof. One direction is Proposition 2.11. Hence, it remains to show that when π is σ -distinguished, it is σ -induced. To do this, we first write π under the form $\pi_1 \times \pi_2$, where π_1 is an irreducible unitary representation of G_{n_1} for some $n_1 \geq 0$ which is a product of the form described in the statement of Lemma 2.12 (i.e., the k_i are ≥ 2) and π_2 is generic unitary of G_{n_2} for $n_2 \geq 0$ (i.e., if you write it as a standard product in Tadić's classification, all the k_i are equal to 1). Notice that π_1 and π_2 , and hence n_1 and n_2 , are uniquely determined by π . We now prove the statement by induction on n_1 .

The case $n_1 = 0$ is true thanks to Theorem 1.16. We thus suppose that n_1 is positive, in which case it is necessarily at least 2 by definition of the representation π_1 (the integers k_i occurring in its definition being at least 2), and we suppose that the statement to prove is true for any irreducible unitary representation $\pi' = \pi'_1 \times \pi'_2$ with $n'_1 < n_1$. By hypothesis, the representation π is σ -distinguished, and hence the representation $\pi^{[-]} = \pi_1^{[-]} \times \pi_2^{[-]}$ is σ -distinguished as well thanks to Proposition 2.4. Then, by induction hypothesis, the representation $\pi_1^{[-]}$ must be σ -induced (because if one writes $\pi' = \pi'_1 \times \pi'_2$ under the form $\pi'_1 \times \pi'_2$, then we have $n'_1 < n_1$). This implies that the representation π_1 is σ -induced as well according to Lemma 2.12. In particular, it is σ -distinguished by Proposition 2.11. Then, we notice that the representation π_2 is $P_{n_2}^\sigma$ -distinguished according to Proposition 2.4, as $\pi_2^{[-]}$ is the trivial character of G_0 . We can now apply Proposition 2.6 and conclude that π_2 is σ -distinguished, thus σ -induced thanks to Theorem 1.16. This finally implies that π is σ -induced as well. \square

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
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| | |
|--|-----|
| Monoids of modules and arithmetic of direct-sum decompositions | 257 |
| NICHOLAS R. BAETH and ALFRED GEROLDINGER | |
| On the torsion anomalous conjecture in CM abelian varieties | 321 |
| SARA CHECCOLI and EVELINA VIADA | |
| Eigenvalue estimate and compactness for closed f -minimal surfaces | 347 |
| XU CHENG, TITO MEJIA and DETANG ZHOU | |
| Lefschetz numbers of symplectic involutions on arithmetic groups | 369 |
| STEFFEN KIONKE | |
| Categorification of a parabolic Hecke module via sheaves on moment graphs | 415 |
| MARTINA LANINI | |
| Unitary representations of $GL(n, K)$ distinguished by a Galois involution for a p -adic field K | 445 |
| NADIR MATRINGE | |
| On f -biharmonic maps and f -biharmonic submanifolds | 461 |
| YE-LIN OU | |
| Unitary principal series of split orthogonal groups | 479 |
| ALESSANDRA PANTANO, ANNEGRET PAUL and SUSANA SALAMANCA RIBA | |



0030-8730(201410)271:2;1-6