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We consider f-biharmonic maps, the extrema of the f-bienergy functional. We prove that an f-biharmonic map from a compact Riemannian manifold into a nonpositively curved manifold with constant f-bienergy density is a harmonic map; that any f-biharmonic function on a compact manifold is constant; and that the inversion in the sphere S^{m-1} is a proper f-biharmonic conformal diffeomorphism for $m \geq 3$. We derive equations for f-biharmonic submanifolds (that is, submanifolds whose defining isometric immersions are f-biharmonic maps) and prove that a surface in a manifold (N^n, h) is an f-biharmonic surface if and only if it can be biharmonically conformally immersed into (N^n, h). We also give a complete classification of f-biharmonic curves in three-dimensional Euclidean space. Examples are given of proper f-biharmonic maps and f-biharmonic surfaces and curves.

1. Harmonic, biharmonic, f-harmonic, and f-biharmonic maps

All objects in this paper, including manifolds, tensor fields, and maps, are assumed smooth unless stated otherwise.

We recall the key definitions, focusing on maps on compact Riemannian manifolds M. (For noncompact M, the relevant functionals are integrals over fixed compact domains $K \subset M$, and the criticality conditions must hold for all K.)

Harmonic maps. Harmonic maps are critical points of the energy functional for maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g.$$

The Euler–Lagrange equation gives the harmonic map equation [Eells and Sampson 1964]

$$\tau(\phi) := \mathrm{Trace}_g \nabla d\phi = 0,$$

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where $\tau(\phi) = \mathrm{Trace}_g \nabla d\phi$ is called the tension field of the map ϕ.

Biharmonic maps. Biharmonic maps are critical points of the bienergy functional for maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g.$$

The Euler–Lagrange equation of this functional gives the biharmonic map equation [Jiang 1986b], namely the vanishing of the *bitension field $\tau_2(\phi)$ of ϕ*:

$$\tau_2(\phi) := \mathrm{Trace}_g(\nabla^\phi \nabla^\phi - \nabla^\phi_{\nabla^M})\tau(\phi) - \mathrm{Trace}_g R^N(d\phi, \tau(\phi))d\phi = 0.$$

Here R^N is the curvature operator of (N, h), defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

f-harmonic maps. f-harmonic maps are critical points of the f-energy functional for maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_f(\phi) = \frac{1}{2} \int_M f|d\phi|^2 v_g.$$

Here f is a fixed function $M \to (0, \infty)$. The Euler–Lagrange equation gives the f-harmonic map equation [Course 2004; Ouakkas et al. 2010]

$$\tau_f(\phi) := f\tau(\phi) + d\phi(\mathrm{grad}\, f) = 0.$$

We call $\tau_f(\phi)$ the f-tension field of the map ϕ.

f-biharmonic maps. f-biharmonic maps are critical points of the f-bienergy functional for maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_{2,f}(\phi) = \frac{1}{2} \int_M f|\tau(\phi)|^2 v_g.$$

The Euler–Lagrange equation gives the f-biharmonic map equation [Lu 2013]

$$\tau_{2,f}(\phi) := f\tau_2(\phi) + (\Delta f)\tau(\phi) + 2\nabla^\phi_{\mathrm{grad}\, f}\tau(\phi) = 0.$$

Bi-f-harmonic maps. Bi-f-harmonic maps are critical points of the bi-f-energy functional for maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E^2_f(\phi) = \frac{1}{2} \int_M |\tau_f(\phi)|^2 v_g. \tag{1}$$

The Euler–Lagrange equation gives the bi-f-harmonic map equation [Ouakkas et al. 2010]

$$\tau^2_f(\phi) := fJ^\phi(\tau_f(\phi)) - \nabla^\phi_{\mathrm{grad}\, f}\tau_f(\phi) = 0,$$

where J^ϕ is the Jacobi operator of the map, defined by

$$J^\phi(X) = -(\mathrm{Trace}_g \nabla^\phi \nabla^\phi X - \nabla^\phi_{\nabla^M} X - R^N(d\phi, X)d\phi).$$

Remark. Ouakkas et al. [2010] used the name "f-biharmonic maps" for the critical points of the functional (1). We think that it is more reasonable to call them "bi-f-harmonic maps" as parallel to "biharmonic maps".

We have the following obvious inclusions among the various types of harmonic maps:

$$\{\text{harmonic}\} \subset \{\text{biharmonic}\} \subset \{f\text{-biharmonic}\},$$

$$\{\text{harmonic}\} \subset \{f\text{-harmonic}\} \subset \{\text{bi-}f\text{-harmonic}\}.$$

From now on we will call an f-biharmonic map which is neither harmonic nor biharmonic a *proper f-biharmonic map*.

Harmonic maps as a generalization of important concepts of geodesics, minimal surfaces, and harmonic functions have been studied extensively with tremendous progress in the past 40-plus years. There is voluminous literature about the beautiful theory, important applications, and interesting links of harmonic maps to other areas of mathematics and theoretical physics including nonlinear partial differential equations, holomorphic maps in several complex variables, the theory of stochastic processes, liquid crystals in materials science, and the nonlinear field theory.

The study of biharmonic maps was proposed in [Eells and Lemaire 1983] and Jiang [1986a; 1986b; 1987] made the first serious study on these maps by using the first and second variational formulas of the bienergy functional and specializing on the biharmonic isometric immersions which nowadays are called biharmonic submanifolds. Very interestingly, the concept of biharmonic submanifolds was also introduced in a different way by B. Y. Chen [1991] in his program of understanding the finite-type submanifolds in Euclidean spaces. Since 2000, biharmonic maps have been receiving a growing attention and have become a popular subject of study with great progress. For some recent geometric study of general biharmonic maps see [Baird and Kamissoko 2003; Montaldo and Oniciuc 2006; Ou 2006; 2012b; Balmuş et al. 2007; Ouakkas 2008; Baird et al. 2010; Ou and Lu 2013; Nakauchi et al. 2014; Wang et al. 2014] and the references therein. For some recent study of biharmonic submanifolds see [Jiang 1986a; 1987; Dimitrić 1992; Chen and Ishikawa 1998; Caddeo et al. 2001; 2002; Balmuş et al. 2008; 2013; Ou 2010; Ou and Wang 2011; Ou and Tang 2012; Alías et al. 2013; Chen and Munteanu 2013; Liang and Ou 2013; Nakauchi and Urakawa 2013] and the references therein. For biharmonic conformal immersions and submersions see [Baird et al. 2008; Ou 2009; 2012a; Loubeau and Ou 2010; Wang and Ou 2011] and the references therein.

Lu [2013] introduced f-biharmonic maps and calculated the first variation to obtain the f-biharmonic map equation and the equation for the f-biharmonic conformal maps between the same dimensional manifolds. In this paper, we study some basic properties of f-biharmonic maps and introduce the concept of f-biharmonic submanifolds. We prove that an f-biharmonic map from a compact Riemannian manifold into a nonpositively curved manifold with constant f-bienergy density is a harmonic map (Theorem 2.4); any f-biharmonic function on a compact manifold is constant (Corollary 2.6); and that the inversion in sphere S^{m-1} is a proper f-biharmonic conformal diffeomorphism for $m \geq 3$ (Proposition 2.9). We derive f-biharmonic submanifolds equations (Theorem 3.2 and Corollary 3.4) and prove that a surface in a manifold (N^n, h) is an f-biharmonic surface if and only if it can be biharmonically conformally immersed into (N^n, h) (Corollary 3.6). We also give a complete classification of f-biharmonic curves in three-dimensional Euclidean spaces (Theorem 4.4) according to which proper f-biharmonic curves are some special subclasses of planar curves or general helices in $\mathbb{R}^3$. Many examples of proper f-biharmonic maps and f-biharmonic surfaces and curves are given.

2. Some properties and examples of f-biharmonic maps

As mentioned, f-biharmonic maps are critical points of the f-bienergy functional for maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_{2,f}(\phi) = \frac{1}{2} \int_M f |\tau(\phi)|^2 v_g.$$

The following theorem was proved in [Lu 2013]. We give a brief outline of the proof for completeness, but note that our notation is different from Lu's.

Theorem 2.1. *A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is an f-biharmonic map if and only if*

$$\tau_{2,f}(\phi) := f\tau_2(\phi) + (\Delta f)\tau(\phi) + 2\nabla^{\phi}_{\operatorname{grad} f}\tau(\phi) = 0, \tag{2}$$

where $\tau(\phi)$ and $\tau_2(\phi)$ are the tension and the bitension fields of ϕ respectively.

Proof. Since f is fixed, we can use the standard method (see, e.g., [Baird and Kamissoko 2003; Jiang 1986b]) of calculating the first variation of the bienergy functional to obtain

$$\begin{aligned}\frac{\partial}{\partial t} E_{2,f}(\phi_t)\Big|_{t=0} &= \frac{1}{2} \int_M f \Big\{ \frac{\partial}{\partial t} \langle \tau(\phi_t), \tau(\phi_t) \rangle \Big\}_{t=0} v_g \\ &= -\int_M f \langle \tau(\phi), J^{\phi}(V) \rangle v_g \\ &= \int_M \langle f\tau(\phi), \operatorname{Trace}_g \nabla^{\phi}\nabla^{\phi} V - \nabla^{\phi}_{\nabla^M} V - R^N(d\phi, V)d\phi \rangle v_g.\end{aligned}$$

Using the symmetry property of the curvature tensor and the divergence theorem we can switch the positions of V and $f\tau(\phi)$ to have

$$\frac{\partial}{\partial t}E_{2,f}(\phi_t)\Big|_{t=0} = -\int_M \langle V, J^{\phi}(f\tau(\phi))\rangle v_g.$$

It follows that ϕ is an f-biharmonic map if and only if the f-bitension field vanishes identically, i.e., $\tau_{2,f}(\phi) = -J^{\phi}(f\tau(\phi)) \equiv 0$. Finally, using [Ou 2006, (7)], we have

$$\begin{aligned}\tau_{2,f}(\phi) &= -J^{\phi}(f\tau(\phi)) = -\{fJ^{\phi}(\tau(\phi)) - (\Delta f)\tau(\phi) - 2\nabla^{\phi}_{\operatorname{grad} f}\tau(\phi)\}\\ &= f\tau_2(\phi) + (\Delta f)\tau(\phi) + 2\nabla^{\phi}_{\operatorname{grad} f}\tau(\phi),\end{aligned}$$

from which the f-biharmonic map equation (2) follows. □

It is well known that for $m \neq 2$, the harmonicity and f-harmonicity of a map $\phi : (M^m, g) \to (N^n, h)$ are related via a conformal change of the domain metric. More precisely:

Proposition 2.2 [Lichnerowicz 1969]. *A map $\phi : (M^m, g) \to (N^n, h)$ with $m \neq 2$ is f-harmonic if and only if $\phi : (M^m, f^{\frac{2}{m-2}}g) \to (N^n, h)$ is a harmonic map.*

In general, this does not generalize to the case of the relationship between biharmonicity and f-biharmonicity, but very interestingly, we have:

Theorem 2.3. *A map $\phi : (M^2, g) \to (N^n, h)$ is an f-biharmonic map if and only if $\phi : (M^2, f^{-1}g) \to (N^n, h)$ is a biharmonic map.*

Proof. On the one hand, we notice that the map $\phi : (M^2, g) \to (N^n, h)$ is an f-biharmonic map if and only if

$$f\tau_2(\phi, g) + (\Delta f)\tau(\phi, g) + 2\nabla^{\phi}_{\operatorname{grad} f}\tau(\phi, g) = 0, \tag{3}$$

which is equivalent to

$$\tau_2(\phi, g) + (\Delta \ln f + |\operatorname{grad}\ln f|^2)\tau(\phi, g) + 2\nabla^{\phi}_{\operatorname{grad}\ln f}\tau(\phi, g) = 0. \tag{4}$$

On the other hand, by [Ou 2009, Corollary 1], the relationship between the bitension field $\tau_2(\phi, g)$ and that of the map $\phi : (M^2, \bar{g} = F^{-2}g) \to (N^n, h)$ is given by

$$\tau_2(\phi, \bar{g}) = F^4\big\{\tau^2(\phi, g) + 2(\Delta \ln F + 2\,|\operatorname{grad}\ln F|^2)\tau(\phi, g) + 4\nabla^{\phi}_{\operatorname{grad}\ln F}\,\tau(\phi, g)\big\},$$

which is equivalent to

$$\tau_2(\phi, \bar{g}) = F^4\big\{\tau^2(\phi, g) + (\Delta \ln F^2 + |\operatorname{grad}\ln F^2|^2)\tau(\phi, g) + 2\nabla^{\phi}_{\operatorname{grad}\ln F^2}\,\tau(\phi, g)\big\}.$$

It follows that the map $\phi : (M^2, \bar{g} = F^{-2}g) \to (N^n, h)$ is biharmonic if and only if

$$\tau_2(\phi, g) + (\Delta\ln F^2 + |\operatorname{grad}\ln F^2|^2)\tau(\phi, g) + 2\nabla^{\phi}_{\operatorname{grad}\ln F^2}\tau(\phi, g) = 0. \tag{5}$$

Substituting $F^2 = f$ into (5) yields (4). Hence the map $\phi : (M^2, g) \to (N^n, h)$ is f-biharmonic if and only if $\phi : (M^m, f^{-1}g) \to (N^n, h)$ is biharmonic. □

Theorem 2.4. *Any f-biharmonic map $\phi : (M^m, g) \to (N^n, h)$ from a compact Riemannian manifold into a nonpositively curved manifold with constant f-bienergy density (i.e., $f|\tau(\phi)|^2 = C$) is a harmonic map.*

Proof. A straightforward computation gives

$$
\begin{aligned}
(6)\quad & \Delta\big(\tfrac{1}{2} f|\tau(\phi)|^2\big) \\
&= \tfrac{1}{2}\Delta\langle f^{\frac{1}{2}}\tau(\phi), f^{\frac{1}{2}}\tau(\phi)\rangle \\
&= (\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^{M}_{e_i}e_i})\langle f^{\frac{1}{2}}\tau(\phi), f^{\frac{1}{2}}\tau(\phi)\rangle \\
&= \langle\nabla^{\phi}_{e_i} f^{\frac{1}{2}}\tau(\phi), \nabla^{\phi}_{e_i} f^{\frac{1}{2}}\tau(\phi)\rangle + \langle(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^{M}_{e_i}e_i}) f^{\frac{1}{2}}\tau(\phi), f^{\frac{1}{2}}\tau(\phi)\rangle \\
&= \langle\nabla^{\phi}_{e_i} f^{\frac{1}{2}}\tau(\phi), \nabla^{\phi}_{e_i} f^{\frac{1}{2}}\tau(\phi)\rangle + f\langle(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^{M}_{e_i}e_i})\tau(\phi), \tau(\phi)\rangle \\
&\qquad + f^{\frac{1}{2}}(\Delta f^{\frac{1}{2}})|\tau(\phi)|^2 + 2 f^{\frac{1}{2}}\langle\nabla^{\phi}_{\operatorname{grad} f^{\frac{1}{2}}}\tau(\phi), \tau(\phi)\rangle.
\end{aligned}
$$

Since ϕ is assumed to be f-biharmonic we have

$$
\begin{aligned}
(7)\quad & f\langle(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^{M}_{e_i}e_i})\tau(\phi), \tau(\phi)\rangle \\
&= \langle f R^N(d\phi(e_i), \tau(\phi))d\phi(e_i) - (\Delta f)\tau(\phi) - 2\nabla^{\phi}_{\operatorname{grad} f}\tau(\phi), \tau(\phi)\rangle \\
&= f\langle R^N(d\phi(e_i), \tau(\phi))d\phi(e_i), \tau(\phi)\rangle - (\Delta f)|\tau(\phi)|^2 - 2\langle\nabla^{\phi}_{\operatorname{grad} f}\tau(\phi), \tau(\phi)\rangle.
\end{aligned}
$$

Substituting (7) into (6) and simplifying the result gives

$$
\Delta\big(\tfrac{1}{2} f|\tau(\phi)|^2\big) = f|\nabla^{\phi}_{e_i}\tau(\phi)|^2 - f R^N(d\phi(e_i), \tau(\phi), d\phi(e_i), \tau(\phi)) - \tfrac{1}{2}(\Delta f)|\tau(\phi)|^2.
$$

This, together with the assumptions that $f|\tau(\phi)|^2 = C$, $f > 0$, and

$$
R^N(d\phi(e_i), \tau(\phi), d\phi(e_i), \tau(\phi)) \le 0, \tag{8}
$$

allows us to conclude that f is a subharmonic function on the compact manifold (M, g) and hence f is a constant function. It follows that the f-biharmonic map ϕ is actually a biharmonic map from a compact manifold into a nonpositively curved manifold, and thus a harmonic map by a theorem in [Jiang 1986b]. □

Remark. There are many harmonic maps between spheres with constant energy density (called eigenmaps). As our Theorem 2.4 implies that there is no proper f-biharmonic maps from a compact manifold into a nonpositively curved manifold with constant f-bienergy density, it would be interesting to know if there is any proper f-biharmonic map between spheres with constant f-bienergy density.

Proposition 2.5. *A function $u : (M, g) \to \mathbb{R}$ is f-biharmonic if and only if*

$$
f\Delta^2 u + (\Delta f)\Delta u + 2g(\operatorname{grad} f, \operatorname{grad} \Delta u) = 0, \quad \textit{or, equivalently}, \tag{9}
$$

$$
\Delta(f\Delta u) = 0, \tag{10}
$$

where $\Delta^2 u = \Delta(\Delta u)$ denotes the bi-Laplacian of u. In other words, a function u is an f-biharmonic function if and only if the product $f\Delta u$ is a harmonic function. In particular, a quasiharmonic function u (i.e., a function $u : (M, g) \to \mathbb{R}$ with $\Delta u = \text{constant} \neq 0$) is an f-biharmonic function if and only if $f : (M, g) \to \mathbb{R}$ is a harmonic function.

Proof. A straightforward computation gives the tension and the bitension fields of $u : (M, g) \to \mathbb{R}$ as

$$\tau(u) = (\Delta u)\frac{\partial}{\partial t} \quad \text{and} \quad \tau_2(u) = (\Delta^2 u)\frac{\partial}{\partial t}. \tag{11}$$

Substituting these into f-biharmonic map equation (2) and performing a further computation we obtain the f-biharmonic function equation (9). The last statement thus follows. □

Corollary 2.6. *Any f-biharmonic function on a compact manifold (M, g) is a constant function.*

Proof. By Proposition 2.5, u is an f-biharmonic function if and only if $f\Delta u$ is a harmonic function. By the well-known fact that any harmonic function on a compact manifold is constant we have $f\Delta u = C$, and hence

$$\Delta u = \frac{C}{f} \tag{12}$$

since $f > 0$ by our assumption. If $C = 0$, then we have $\Delta u = 0$ and hence u is a harmonic function, so u is a constant function in this case. If $C \neq 0$, then (12) implies that u is either a subharmonic or a superharmonic function since f has a fixed sign with $f > 0$. Again, the well-known fact that a subharmonic or superharmonic function on a compact manifold is constant implies that u is constant. This completes the proof of the corollary. □

Example 1. Let $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be the function $f(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$ and let $u : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be the function given by $u(x, y, z) = x^2 + y^2 + z^2$. It is easily checked that $\Delta f = 0$, $\Delta u = 6$ and $\Delta^2 u = 0$ and hence f and u satisfy (9). So, $u(x, y, z)$ is an f-biharmonic function on $\mathbb{R}^3 \setminus \{0\}$ for $f(x, y, z)$. Clearly, this f-biharmonic function u is not a harmonic function.

Example 2. Let $f, u : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be the functions defined by $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ and $u(x, y, z) = x/(x^2 + y^2 + z^2)$. Then we can check (see also Proposition 2.9) that u is a proper f-biharmonic function which is neither harmonic nor biharmonic.

Corollary 2.7. *Let $f, u : \mathbb{R} \to \mathbb{R}$ be two functions with $f(x) > 0$ for all $x \in \mathbb{R}$. Then u is an f-biharmonic function if and only if*

$$u(x) = \iint \frac{Ax + B}{f}\, dx\, dx + Cx + D, \tag{13}$$

where A, B, C, D are arbitrary constants. In particular:

(I) *For $f(x) = 1 + x^2$, a function $u : \mathbb{R} \to \mathbb{R}$ is f-biharmonic if and only if $u(x) = \frac{1}{2}(Ax - B)\ln(1+x^2) + (Bx + A)\arctan x + (C - A)x + D$, where A, B, C, D are constants.*

(II) *For $f(x) = e^{-x}$, a function $u : \mathbb{R} \to \mathbb{R}$ is f-biharmonic if and only if $u(x) = (Ax - 2A + B)e^x + Cx + D$, where A, B, C, D are constants.*

Proof. In this case, the f-biharmonic equation (10) reduces to $(f u'')'' = 0$ which has solution (13). Finally, statements (I) and (II) are obtained by elementary integrations. □

Remark. It is easily checked that for $A \neq 0$, $B \neq 0$ the function

$$u(x) = (Ax - 2A + B)e^x + Cx + D$$

is neither a harmonic nor a biharmonic function, so it provides many examples of proper f-biharmonic functions.

Theorem 2.8. *Any f-biharmonic map $\phi : (M^m, g) \to \mathbb{R}^n$ from a compact manifold into a Euclidean space is a constant map.*

Proof. Since the target manifold is a Euclidean space, the curvature is zero. If we write $\phi : (M^m, g) \to \mathbb{R}^n$ as $\phi(p) = (\phi^1(p), \phi^2(p), \ldots, \phi^n(p))$, then we can easily check that

$$\begin{aligned}
\tau(\phi) &= (\Delta\phi^1, \Delta\phi^2, \ldots, \Delta\phi^n),\\
\tau_2(\phi) &= (\Delta^2\phi^1, \Delta^2\phi^2, \ldots, \Delta^2\phi^n),\\
\nabla^{\phi}_{\operatorname{grad} f}\tau(\phi) &= (\nabla^{\phi}_{\operatorname{grad} f}\Delta\phi^1, \nabla^{\phi}_{\operatorname{grad} f}\Delta\phi^2, \ldots, \nabla^{\phi}_{\operatorname{grad} f}\Delta\phi^n).
\end{aligned}$$

It follows that the f-biharmonic map equation for ϕ becomes

$$f\Delta^2\phi^\alpha + (\Delta f)\Delta\phi^\alpha + 2g(\operatorname{grad} f, \operatorname{grad}\Delta\phi^\alpha) = 0, \quad \alpha = 1, 2, \ldots, n.$$

In other words, a map $\phi : (M^m, g) \to \mathbb{R}^n$ from a manifold into a Euclidean space is an f-biharmonic map if and only if each of its component functions is an f-biharmonic function. From this and Corollary 2.6, which states that any f-biharmonic function on a compact manifold is constant, we obtain the theorem. □

Proposition 2.9. *The map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$ with $\phi(x) = x/|x|^p$ is an f-biharmonic map for $f(x) = |x|^k$ if and only if (i) $p = 0$, or (ii) $p = m$, or (iii) $k = p + 2$, or (iv) $k = p + 2 - m$. In particular, for $m \geq 3$, the inversion in sphere S^{m-1}, $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$ with $\phi(x) = x/|x|^2$ is a proper f-biharmonic map for $f(x) = |x|^4$. When $m \neq 4$, this inversion is also a proper f-biharmonic map for $f(x) = |x|^{4-m}$.*

Proof. As we have seen in the proof of Theorem 2.8, a map into a Euclidean space is an f-biharmonic map if and only if each of its component functions is an f-biharmonic function. So, $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$ with $\phi(x) = x/|x|^p$ is f-biharmonic if and only if the function $u : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ with $u(x) = x^i|x|^{-p}$ is an f-biharmonic function for any $i = 1, 2, \dots, m$. This, by Proposition 2.5, is equivalent to the product $f\Delta u$ being a harmonic function. Using the formula $\triangle^{\mathbb{R}^m}(|x|^\alpha) = \alpha(\alpha-2+m)|x|^{\alpha-2}$ and a straightforward computation we have

$$\begin{aligned}\triangle^{\mathbb{R}^m} u &= \triangle^{\mathbb{R}^m}(x^i|x|^{-p}) = x^i \triangle^{\mathbb{R}^m}|x|^{-p} + 2\langle \operatorname{grad} x^i, \operatorname{grad}|x|^{-p}\rangle \\ &= p(p-m)x^i|x|^{-p-2}.\end{aligned}$$

For $f(x) = |x|^k$, we have

$$\begin{aligned}\triangle^{\mathbb{R}^m}(f\triangle^{\mathbb{R}^m} u) &= p(p-m)\triangle^{\mathbb{R}^m}(x^i|x|^{k-p-2}) \\ &= p(p-m)[x^i\triangle^{\mathbb{R}^m}|x|^{k-p-2} + 2\langle \operatorname{grad} x^i, \operatorname{grad}|x|^{k-p-2}\rangle] \\ &= p(p-m)(k-p-2)(k-p+m-2)x^i|x|^{k-p-4}.\end{aligned}$$

It follows that $u(x) = x^i|x|^{-p}$ is an f-biharmonic function with $f = |x|^k$ if and only if $p(p-m)(k-p-2)(k-p+m-2) = 0$. Solving this equation we have (i) $p = 0$, or (ii) $p = m$, or (iii) $k = p+2$, or (iv) $k = p+2-m$, from which the proposition follows. □

Remark. (A) One can check (see also [Balmuş et al. 2007]) that for the cases (i) $p = 0$ and (ii) $p = m$, the maps $\phi = x/|x|^p$ are actually harmonic maps. We know that in these cases these maps are f-biharmonic for any f. For $k = 0$ we have $f(x) = 1$ and hence f-biharmonicity reduces to biharmonicity. In this case, (iii) and (iv) imply that $\phi = x/|x|^p$ is a proper biharmonic map if and only if $p = -2$, or $p = m-2$. Note that the case $p = -2$ was missed in the list of [ibid., Remark 5.8].

(B) For $p \neq 0$, m, and $k \neq 0$, the maps in cases (iii) and (iv) provide infinitely many examples of proper f-biharmonic maps (i.e., which are neither harmonic nor biharmonic maps).

(C) It is well known that the inversion in sphere S^{m-1}, $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$, $\phi(x) = x/|x|^2$ is a conformal map between the same dimensional Euclidean spaces. Note that the f-biharmonic map equation for conformal maps between the same dimensional spaces was derived in [Lu 2013], however, not a single example of such maps was found. Our Proposition 2.9 shows that there are infinitely many proper f-biharmonic conformal diffeomorphisms and all but one of which are proper f-biharmonic for at least two different choices of f functions. For a study of biharmonic diffeomorphisms see [Baird et al. 2008].

3. f-biharmonic submanifolds

Definition 3.1. A submanifold in a Riemannian manifold is called an *f-biharmonic submanifold* if the isometric immersion defining the submanifold is an f-biharmonic map.

From the definition and the relationships among harmonic, biharmonic and f-biharmonic maps we have the inclusions

$$\{\text{minimal}\} \subset \{\text{biharmonic}\} \subset \{f\text{-biharmonic}\}.$$

From now on we will call an f-biharmonic submanifold a *proper f-biharmonic submanifold* if it is neither a minimal nor a biharmonic submanifold.

Theorem 3.2. *Let $\phi : M^m \to N^{m+1}$ be an isometric immersion of codimension one with mean curvature vector $\eta = H\xi$. Then φ is an f-biharmonic if and only if*

$$\begin{cases} \Delta H - H|A|^2 + H\,\mathrm{Ric}^N(\xi,\xi) + H(\Delta f)/f + 2(\mathrm{grad}\ln f)H = 0, \\ 2A(\mathrm{grad}\,H) + \frac{m}{2}\,\mathrm{grad}\,H^2 - 2H(\mathrm{Ric}^N(\xi))^\top + 2HA(\mathrm{grad}\ln f) = 0, \end{cases} \tag{14}$$

where $\mathrm{Ric}^N : T_qN \to T_qN$ denotes the Ricci operator of the ambient space defined by $\langle \mathrm{Ric}^N(Z), W\rangle = \mathrm{Ric}^N(Z, W)$; A is the shape operator of the hypersurface with respect to the unit normal vector ξ; and Δ, grad are the Laplace and the gradient operator of the hypersurface respectively.

Proof. It is well known that the tension field of the hypersurface is given by

$$\tau(\phi) = mH\xi. \tag{15}$$

From [Ou 2010, Theorem 2.1] we have the bitension field of the hypersurface:

$$\begin{aligned}\tau_2(\phi) = m(\Delta H - H|A|^2 + H\,\mathrm{Ric}^N(\xi,\xi))\xi& \\ -m\big(2A(\mathrm{grad}\,H) + \tfrac{m}{2}(\mathrm{grad}\,H^2) - 2H(\mathrm{Ric}(\xi))^\top\big)&.\end{aligned} \tag{16}$$

To compute the term $\nabla^\phi_{\mathrm{grad}\,f}\tau(\phi)$, we choose a local orthonormal frame $\{e_i\}_{i=1,\dots,m}$ on M so that $\{d\phi(e_1), \dots, d\phi(e_m), \xi\}$ forms an adapted orthonormal frame of the ambient space defined on the hypersurface. Identifying $d\phi(X) = X$, $\nabla^\phi_X W = \nabla^N_X W$ we have

$$\nabla^\phi_{\mathrm{grad}\,f}\tau(\phi) = m\nabla^N_{\mathrm{grad}\,f}H\xi = m\{[(\mathrm{grad}\,f)H]\xi - A(\mathrm{grad}\,f)\}. \tag{17}$$

Substituting (15), (16) and (17) into the f-biharmonic map equation (2) and simplifying the result we obtain the theorem. □

Corollary 3.3. *A hypersurface $\phi : M^m \to N^{m+1}(C)$ in a space form of constant sectional curvature C is f-biharmonic if and only if its mean curvature function H*

satisfies the equation

$$\begin{cases} \Delta H - H|A|^2 + mCH + H(\Delta f)/f + 2(\operatorname{grad} \ln f)H = 0, \\ 2A(\operatorname{grad} H) + \frac{1}{2}m \operatorname{grad} H^2 + 2HA(\operatorname{grad} \ln f) = 0. \end{cases} \tag{18}$$

Similarly:

Corollary 3.4. *A submanifold $\phi : M^m \to N^n(C)$ in a space form of constant sectional curvature C is f-biharmonic if and only if its mean curvature vector H satisfies the equation*

$$\begin{cases} \Delta^\perp H - (\Delta f/f)H - 2\nabla^\perp_{\operatorname{grad} \ln f} H + \operatorname{Trace} B(-, A_H -) + CmH = 0, \\ 2 \operatorname{Trace} A_{\nabla^\perp_{(-)} H}(-) + \frac{1}{2}m \operatorname{grad}(|H|^2) + 2A_H(\operatorname{grad} \ln f) = 0, \end{cases}$$

where $\Delta^\perp H = -\operatorname{Trace}(\nabla^\perp)^2 H$.

Corollary 3.5. *A compact nonzero constant mean curvature f-biharmonic hypersurface $\phi : M^m \to S^{m+1}$ in a sphere with $|A|^2 =$ constant is biharmonic.*

Proof. Substituting $H = \text{constant} \neq 0$ into the f-biharmonic hypersurface equation (18) we have

$$\begin{cases} \Delta f = (|A|^2 - m) f, \\ A(\operatorname{grad} \ln f) = 0. \end{cases} \tag{19}$$

If $|A|^2$ is constant, we have either $|A|^2 - m = 0$, in which case the first equation of (19) implies that f is a harmonic function, or $|A|^2 - m \neq 0$. In the latter case, the first line of (19) implies that f is either subharmonic or superharmonic since $f > 0$. Since M is compact, the well-known fact that any harmonic (subharmonic or superharmonic) function on a compact manifold is constant implies that f is a constant function. Thus, an f-biharmonic hypersurface is actually biharmonic. □

For classification of biharmonic submanifolds with parallel mean curvature vector and $|A|^2 =$ constant in sphere see [Balmuş et al. 2013].

In Euclidean space $\mathbb{R}^3$, any biharmonic surface is minimal (see, e.g., [Jiang 1987; Chen and Ishikawa 1998]), so there are no proper biharmonic surfaces. The first question we ask is: Are there proper f-biharmonic surfaces in $\mathbb{R}^3$? We will show that there are infinitely many. We achieve this by using a link between f-biharmonic surfaces and biharmonic conformal immersions of surfaces in a three-manifold. For the study of biharmonic conformal immersions of surfaces in three-manifolds we refer the reader to [Ou 2009; 2012a]. We recall that a surface (i.e., an isometric immersion) $\phi : M^2 \to (N^3, h)$ is said to admit a biharmonic conformal immersion into a three-manifold (N^3, h) if there exists a function $\lambda : M^2 \to (0, \infty)$ such that the conformal immersion $\phi : (M^2, \lambda^{-2}\phi^* h) \to (N^3, h)$ is biharmonic map. In this case, we also say that the surface $\phi : M^2 \to (N^3, h)$ can be biharmonically conformally immersed into the three-manifold (N^3, h) with conformal factor λ.

Corollary 3.6. (i) *A surface $\phi : M^2 \to (N^3, h)$ in a three-manifold is f-biharmonic if and only if the conformal immersion*

$$\phi : (M^2, f^{-1}\phi^* h) \to (N^3, h)$$

is a biharmonic map, i.e., the surface can be biharmonically conformally immersed into (N^3, h) with conformal factor $\lambda = f^{\frac{1}{2}}$.

(ii) *The circular cylinder $\phi : D = \{(\theta, z) \in (0, 2\pi) \times \mathbb{R}\} \to (\mathbb{R}^3, \delta_0)$ with $\phi(\theta, z) = (R\cos\theta, R\sin\theta, z)$ is an f-biharmonic surface for any function f from the family $f = \big(C_2 e^{\pm z/R} - C_1 C_2^{-1} R^2 e^{\mp z/R}\big)/2$, where C_1, C_2 are constants.*

Proof. Statement (i) follows from the definition of an f-biharmonic surface and Theorem 2.3, whilst (ii) is obtained by using (i) and [Ou 2009, Proposition 2]. □

4. f-biharmonic curves

Another special case of f-biharmonic maps is an f-biharmonic curve.

Lemma 4.1. *A curve $\gamma : (a, b) \to (N^m, g)$ parametrized by arclength is an f-biharmonic curve with a function $f : (a, b) \to (0, \infty)$ if and only if*

$$f\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma' - R^N(\gamma', \nabla^N_{\gamma'}\gamma')\gamma'\big) + 2f'\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma' + f''\nabla^N_{\gamma'}\gamma' = 0. \tag{20}$$

Proof. Let $\gamma = \gamma(s)$ be parametrized by arclength. Then $e_1 = \partial/\partial s$ is an orthonormal frame on $((a, b), ds^2)$ and $d\gamma(e_1) = d\gamma(\partial/\partial s) = \gamma'$. Thus, the tension field of the curve is given by $\tau(\gamma) = \nabla^{\gamma}_{e_1} d\gamma(e_1) = \nabla^N_{\gamma'}\gamma'$. It is also easy to see that for a function $f : (a, b) \to (0, \infty)$, $\Delta f = f''$ and $\nabla^{\gamma}_{\operatorname{grad} f}\tau(\gamma) = f'\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma'$. Substituting these into the f-biharmonic map equation gives the lemma. □

Theorem 4.2. *A curve $\gamma : (a, b) \to N^n(C)$ parametrized by arclength in an n-dimensional space form is a proper f-biharmonic curve if and only if one of the following cases happens*:

(i) $\kappa_2 = 0$, $f = c_1\kappa_1^{-\frac{3}{2}}$ *and the curvature κ_1 solves the ODE*

$$3\kappa_1'^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2(\kappa_1^2 - C);$$

(ii) $\kappa_2 \neq 0$, $\kappa_3 = 0$, $\kappa_2/\kappa_1 = c_3$, $f = c_1\kappa_1^{-\frac{3}{2}}$, *and the curvature κ_1 solves the ODE*

$$3\kappa_1'^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^2[(1 + c_3^2)\kappa_1^2 - C].$$

Proof. Let $\gamma : (a, b) \to N^n(C)$ be a curve with arclength parametrization. Let $\{F_i, i = 1, 2, \ldots, n\}$ be the Frenet frame along the curve $\gamma(s)$, which is obtained as the orthonormalization of the n-tuple $\big\{\nabla^{(k)}_{\partial/\partial s} d\gamma(\partial/\partial s) \mid k = 1, 2, \ldots, n\big\}$. Then we have the following Frenet formula (see, e.g., [Laugwitz 1965]) along the curve:

$$\begin{cases} \nabla^{\gamma}_{\partial/\partial s} F_1 = \kappa_1 F_2, \\ \nabla^{\gamma}_{\partial/\partial s} F_i = -\kappa_{i-1} F_{i-1} + \kappa_i F_{i+1} \qquad \text{for } \ i = 2, 3, \dots, n-1, \\ \nabla^{\gamma}_{\partial/\partial s} F_n = -\kappa_{n-1} F_{n-1}, \end{cases}$$

where $\{\kappa_1, \kappa_2, \dots, \kappa_{n-1}\}$ are the curvatures of the curve γ.

Using this formula and a straightforward computation one finds the tension and the bitension fields of the curve given by

$$\tau(\gamma) = \nabla^N_{\gamma'} \gamma' = \kappa_1 F_2,$$
$$\nabla^N_{\gamma'} \nabla^N_{\gamma'} \gamma' = -\kappa_1^2 F_1 + \kappa_1' F_2 + \kappa_1 \kappa_2 F_3,$$
$$\tau_2(\gamma) = -3\kappa_1 \kappa_1' F_1 + (\kappa_1'' - \kappa_1 \kappa_2^2 - \kappa_1^3 + \kappa_1 C) F_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') F_3 + \kappa_1 \kappa_2 \kappa_3 F_4.$$

Substituting these into the f-biharmonic curve equation (20) and comparing the coefficients of both sides we have

$$\begin{cases} -3\kappa_1 \kappa_1' - 2\kappa_1^2 f'/f = 0, \\ \kappa_1'' - \kappa_1 \kappa_2^2 - \kappa_1^3 + \kappa_1 C + \kappa_1 f''/f + 2\kappa_1' f'/f = 0, \\ 2\kappa_1' \kappa_2 + \kappa_1 \kappa_2' + 2\kappa_1 \kappa_2 f'/f = 0, \\ \kappa_1 \kappa_2 \kappa_3 = 0. \end{cases} \tag{21}$$

It is easy to see that if $\kappa_1 = \text{constant} \neq 0$, then the first equation of (21) implies that f is constant and the curve γ is biharmonic. Also, if $\kappa_2 = \text{constant} \neq 0$, then the first and the third equations imply that f is constant and hence the curve γ is biharmonic again.

Now, if $\kappa_2 = 0$, then the f-biharmonic curve equation (21) is equivalent to

$$\begin{cases} 3\kappa_1'/\kappa_1 + 2f'/f = 0, \\ \kappa_1''/\kappa_1 - \kappa_1^2 + C + f''/f + 2(\kappa_1'/\kappa_1)(f'/f) = 0. \end{cases} \tag{22}$$

Integrating the first equation of (22) and substituting the result in to the second we obtain the statements in case (i).

Finally, if $\kappa_1 \neq \text{constant}$ and $\kappa_2 \neq \text{constant}$, then the system (21) is equivalent to

$$\begin{cases} f^2 \kappa_1^3 = c_1^2, \\ (f\kappa_1)'' = f\kappa_1(\kappa_2^2 + \kappa_1^2 - C), \\ f^2 \kappa^2 \kappa_2 = c_2, \\ \kappa_3 = 0. \end{cases} \tag{23}$$

Solving the first equation of (23) we obtain $f = c_1 \kappa_1^{-\frac{3}{2}}$. Substituting the first equation into the third one we obtain $\kappa_2/\kappa_1 = c_3$. Finally, substituting $\kappa_2/\kappa_1 = c_3$ and $f\kappa_1 = c_1 \kappa_1^{-\frac{1}{2}}$ into the second equation we obtain the results stated in case (ii). This completes the proof of the theorem. □

From the proof of Theorem 4.2 we have:

Corollary 4.3. *A curve $\gamma : (a, b) \to N^n(C)$ parametrized by arclength in an n-dimensional space form with constant geodesic curvature is biharmonic.*

It is known [Dimitrić 1992] that any biharmonic curve in a Euclidean space is a geodesic. It would be interesting to know if there is any proper f-biharmonic curve in a Euclidean space. Our next theorem gives a complete classification of proper f-biharmonic curves in $\mathbb{R}^3$ which, together with the fundamental theorem for curves in $\mathbb{R}^3$, can be used to produce many examples of proper f-biharmonic curves in a three-dimensional Euclidean space.

Theorem 4.4. *A curve $\gamma : (a, b) \to \mathbb{R}^3$ parametrized by arclength in a three-dimensional Euclidean space is a proper f-biharmonic curve if and only if*

(i) *γ is a planar curve with $\tau(s)=0$, $\kappa(s)=4c_2/(16+(c_2s+c_3)^2)$, and $f=c_1\kappa^{-\frac{3}{2}}$, where $c_1 > 0$, $c_2 > 0$, and c_3 are constants, or*

(ii) *γ is a general helix with $\kappa(s) = 4c_2/(16(1+c^2) + (c_2s+c_3)^2)$, $\tau/\kappa = c$, and $f = c_1\kappa^{-\frac{3}{2}}$, where $c \neq 0$, $c_1 > 0$, $c_2 > 0$, and c_3 are constants.*

Proof. For the arclength-parametrized curve $\gamma : (a, b) \to \mathbb{R}^3$, we have the curvature $\kappa = \kappa_1$ and the torsion $\tau = \kappa_2$. Applying Theorem 4.2 with $C = 0$ we conclude that the curve γ is a proper f-biharmonic curve if and only if

(i) $\tau = 0$, $f = c_1\kappa^{-\frac{3}{2}}$ and the curvature κ solves the ODE

$$3\kappa'^2 - 2\kappa\kappa'' = 4\kappa^4, \quad \text{or}$$

(ii) $\tau \neq 0$, $\tau/\kappa = c$, $f = c_1\kappa_1^{-\frac{3}{2}}$, and the curvature κ solves the ODE

$$3\kappa'^2 - 2\kappa\kappa'' = 4(1+c^2)\kappa^4.$$

Solving the ODEs in each case and noting that $\tau = 0$ means the curve is planar and $\tau/\kappa = c$ means the curve is a general helix (Lancret's theorem; see, e.g., [Barros 1997]) we obtain the theorem. □

Remark. (A) Recall that the fundamental theorem for curves in $\mathbb{R}^3$ states that for any given functions $p, q : [s_0, s_1] \to \mathbb{R}$ with $p(s) > 0$ for all $s \in [s_0, s_1]$, there exists a unique (up to a rigid motion) curve in $\mathbb{R}^3$ whose curvature and torsion take on the prescribed functions $\kappa(s) = p(s)$, $\tau(s) = q(s)$. This, together with our Theorem 4.4, implies that there are many examples of proper f-biharmonic curves in $\mathbb{R}^3$.

(B) Our classification theorem also implies that proper f-biharmonic curves in $\mathbb{R}^3$ must be special subclasses of planar curves or general helices in $\mathbb{R}^3$. As the following example shows that there are general helices which are not proper f-biharmonic curves.

Example 3. The general helix $\gamma : I \to \mathbb{R}^3$ with $\gamma(s) = \left(\frac{2}{3}(1+\frac{s}{2})^{\frac{3}{2}}, \frac{2}{3}(1-\frac{s}{2})^{\frac{3}{2}}, \frac{s}{\sqrt{2}}\right)$ is never an f-biharmonic curve for any function f.

In fact, one can easily check that $|\gamma'(s)| = 1$ so s is an arclength parameter for the curve. A straightforward computation gives $\kappa(s) = \tau(s) = 1/(2\sqrt{2}\sqrt{4-s^2})$. So, the curve is indeed a general helix with $\tau/\kappa = 1$. Since the curvature is not of the form given in case (ii) of Theorem 4.4 we conclude that the helix is never an f-biharmonic curve for any f.

Finally, we give an example of a proper f-biharmonic planar curve to close this section.

Example 4. The planar curve $\gamma(s) = (4\ln|\sqrt{16+s^2}+s|, \sqrt{16+s^2}\,)$ is a proper f-biharmonic curve.

In fact, we can check that

$$\gamma'(s) = \left(\frac{4}{\sqrt{16+s^2}}, \frac{s}{\sqrt{16+s^2}}\right) \quad \text{and} \quad |\gamma'(s)| = 1.$$

So s is the arclength parameter of the curve. In this case, we have the curvature $\kappa(s) = |\gamma''(s)| = 4/(16+s^2)$ and, by case (i) of Theorem 4.4, the curve γ is a proper f-biharmonic curve with $f = 8c_1(16+s^2)^{\frac{3}{2}}$ for some constant $c_1 > 0$.

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