**A NOTE ON FLUX INTEGRALS
OVER SMOOTH REGULAR DOMAINS**

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We provide new bounds on a flux integral over the portion of the boundary of one regular domain contained inside a second regular domain, based on properties of the second domain rather than the first one. This bound is amenable to numerical computation of a flux through the boundary of a domain, for example, when there is a large variation in the normal vector near a point. We present applications of this result to occupational measures and two-dimensional differential equations, including a new proof that all minimal invariant sets in the plane are trivial.

1. Introduction

A *regular domain* in $\mathbb{R}^d$ is a closed, embedded d-dimensional smooth submanifold with boundary, such as a closed ball or a closed half-space. (Throughout this paper, *smooth* means infinitely differentiable.) If $D \subset \mathbb{R}^d$ is a regular domain, its interior $\mathring{D}$ is an open subset of $\mathbb{R}^d$, and its boundary ∂D is a closed, embedded, codimension-1 smooth submanifold (without boundary) which is the common topological boundary of the open sets $\mathring{D}$ and $\mathbb{R}^d \smallsetminus D$. For this reason, the boundary of a regular domain is often called a *space-separating hypersurface*. The Jordan–Brouwer separation theorem (see, for example, [Guillemin and Pollack 1974, p. 89]) shows that if $S \subset \mathbb{R}^d$ is any compact, connected, embedded hypersurface, then the complement of S has two connected components, one bounded (the *interior* of S) and another unbounded (the *exterior* of S), with S as their common boundary; thus $S \cup \operatorname{Int} S$ and $S \cup \operatorname{Ext} S$ are both regular domains. But in general, the boundary of a regular domain need not be connected (for example, an annulus in the plane).

Surface integrals computing the flux through boundaries of regular domains are ubiquitous in physics and engineering. We present two bounds for surface integrals on a portion of the boundary of one domain contained inside a second domain. The results are presented for regular domains in Euclidean space for simplicity, but Theorems 1.1 and 1.2 extend to regular domains in Riemannian manifolds. See Theorem 3.3. For more details about the notation in these theorems, see Section 2.

Theorem 1.1. *Suppose $D_1, D_2 \subset \mathbb{R}^d$ are regular domains, such that $D_1 \cap D_2$ is compact and D_2 has finite volume and surface area. Suppose f is a smooth vector field defined on a neighborhood of D_2 such that $|f|$ and $|\nabla \cdot f|$ are bounded. Then the absolute value of the flux of f across the portion of ∂D_1 inside D_2 satisfies*

$$(1\text{-}1) \qquad \left| \int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA \right| \leq \operatorname{Area}(\partial D_2) \| f \|_\infty + \operatorname{Vol}(D_2) \| \nabla \cdot f \|_\infty.$$

When the vector field is divergence-free, we have the following much better bound.

Theorem 1.2. *Suppose $D_1, D_2 \subset \mathbb{R}^d$ are regular domains with compact intersection and finite surface areas, and f is a smooth bounded vector field on $\mathbb{R}^d$ satisfying $\nabla \cdot f \equiv 0$. Then*

$$(1\text{-}2) \qquad \left| \int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA \right| \leq \tfrac{1}{2} \operatorname{Area}(\partial D_2) \| f \|_\infty.$$

A surprising corollary to Theorem 1.2 bounds the integral of the normal vector of the portion of a hypersurface contained inside a second regular domain.

Corollary 1.3. *Suppose $D_1, D_2 \subset \mathbb{R}^d$ are regular domains with compact intersection and finite surface areas. The following inequality holds*:

$$(1\text{-}3) \qquad \left| \int_{\partial D_1 \cap D_2} \boldsymbol{n}_{\partial D_1} \, dA \right| \leq \tfrac{1}{2} \operatorname{Area}(\partial D_2).$$

When D_2 is convex we have the following alternative bound, which is an improvement in some cases.

Theorem 1.4. *Suppose $D_1, D_2 \subset \mathbb{R}^d$ are regular domains. If D_2 is compact and convex with diameter δ, then*

$$(1\text{-}4) \qquad \left| \int_{\partial D_1 \cap D_2} \boldsymbol{n}_{\partial D_1} \, dA \right| \leq \tfrac{1}{2} \operatorname{Vol}(B^{d-1}(\delta/2)),$$

where $B^{d-1}(\delta/2)$ denotes the ball in $\mathbb{R}^{d-1}$ of radius $\delta/2$.

The significance of these results is that, although the integration is with respect to the portion of ∂D_1 inside D_2, which might have arbitrarily large surface area (see Figure 1), the bound depends only on D_2. This is due to the cancellations of the normal vector that occur in hypersurfaces that bound regular domains, and would not hold for images of general immersions of codimension 1 (see Example 4.2).

Theorem 1.1 is applicable to the numerical computation of the flux on the surface of a regular domain when there is a large variation of the normal vector near a point, resulting in a large surface area contained in a region of small volume. Indeed, the flux over the problematic part can be estimated by finding a domain containing it, avoiding direct computation. We provide an application of Corollary 1.3 in
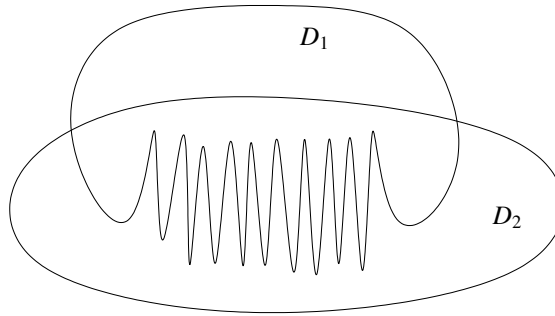
Figure 1. The setup for Theorems 1.1 and 1.2.

Section 5, for limits of sequences of regular domains with surface area increasing without bound; there we use the bound to show that in the limit, the average normal vector, say in a ball, is zero. Such a result is applied in the case $d = 2$, in [Artstein and Bright 2010], to obtain a new Poincaré–Bendixson type result for planar infinite-horizon optimal control.

Corollary 1.3 generalizes a previous result, for $d = 2$, established by Artstein and Bright [2010; 2013]. This topological result has proved fruitful in applications, providing new Poincaré–Bendixson type results, in an optimal-control setting [Artstein and Bright 2010; Bright 2012], and in the context of dynamics with no differentiability assumptions by Bright [2012]. The proofs of the planar result in [Artstein and Bright 2010; 2013] employ a dynamical argument, which is similar to the one used in the textbook proof of the Poincaré–Bendixson theorem. In this paper, we generalize the results to boundaries of open sets, restricting ourselves in this presentation to regular domains; however the results hold for more general sets and vector fields. The results in their fullest generality for nonsmooth domains and fluxes are presented in [Bright and Torres 2014].

Remark 1.5. The requirement that $D_1 \cap D_2$ be compact is essential, as it implies that $\partial D_1 \cap D_2$ is compact, so that the integrals in (1-1)–(1-3) are finite.

Remark 1.6. Theorem 1.1 can be extended, by replacing the smooth vector field f with a smooth matrix-valued function Π, using the induced norm.

Remark 1.7. For simplicity, Theorem 1.2 is stated under the assumption that f is defined on all of $\mathbb{R}^d$; but as the proof will show, if D_2 has finite volume it is only necessary that f be defined on some neighborhood of D_2.

The structure of this paper is as follows. Section 2 presents notation and lemmas used in the paper. In Section 3 we prove Theorems 1.1 and 1.2, and describe how our results extend to regular domains in a Riemannian manifold. In Section 4 we prove Corollary 1.3 and Theorem 1.4, and also provide examples showing the tightness of the bound. In the last section we provide three applications of Corollary 1.3: an

application to limits of sequences of regular domains; an extension when $d = 2$; and a simplified proof of a theorem on invariant sets for dynamical systems.

2. Notation and lemmas

Throughout this paper, we denote the characteristic function of a set $A \subset \mathbb{R}^d$ by χ_A. The d-dimensional volume is denoted by $\mathrm{Vol}(A)$, and the $(d-1)$-dimensional surface area of its boundary by $\mathrm{Area}(\partial A)$. Given two submanifolds $S_1, S_2 \subset \mathbb{R}^d$, the notation $S_1 \pitchfork S_2$ means that S_1 and S_2 intersect transversally. The Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$, and the supremum norm on functions by $\|\cdot\|_\infty$. The divergence of a smooth vector field $f = (f^1, f^2, \ldots, f^d)$ at the point $x = (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d$ is denoted by

$$\nabla \cdot f(x) = \frac{\partial}{\partial x^1} f^1(x) + \frac{\partial}{\partial x^2} f^2(x) + \cdots + \frac{\partial}{\partial x^d} f^d(x).$$

The following is a simple lemma we need for the proof of the main theorems.

Lemma 2.1. *Suppose (X, μ) is a measure space, $U, V \subset X$, and U has finite measure. For every real-valued function $f \in L^\infty(X)$, we have*

$$\left| \int_{U \smallsetminus V} f(x)\mu(dx) \right| \leq \frac{1}{2} \left(\mu(U)\|f\|_\infty + \left| \int_U f(x)\mu(dx) \right| \right),$$

$$\left| \int_{U \cap V} f(x)\mu(dx) \right| \leq \frac{1}{2} \left(\mu(U)\|f\|_\infty + \left| \int_U f(x)\mu(dx) \right| \right).$$

Proof. The first inequality follows from the triangle inequality:

$$\int_U \left| f(x) \right| \mu(dx) \geq \left| \int_{U \smallsetminus V} f(x)\mu(dx) \right| + \left| \int_{U \cap V} f(x)\mu(dx) \right|$$

$$= \left| \int_{U \smallsetminus V} f(x)\mu(dx) \right| + \left| \int_{U \smallsetminus V} f(x)\mu(dx) - \int_U f(x)\mu(dx) \right|$$

$$\geq 2 \left| \int_{U \smallsetminus V} f(x)\mu(dx) \right| - \left| \int_U f(x)\mu(dx) \right|.$$

The second inequality follows by replacing V with $X \smallsetminus V$. □

The proofs of the main theorems are based on the divergence theorem for certain domains in $\mathbb{R}^d$. Let us say a *regular domain with corners* in $\mathbb{R}^d$ is a closed subset $D \subset \mathbb{R}^d$ such that for each point $p \in D$, there exist an open set $U \subset \mathbb{R}^d$ containing p and a smooth coordinate chart $\varphi : U \to \mathbb{R}^d$ such that $\varphi(U \cap D)$ is the intersection of $\varphi(U)$ with $\overline{\mathbb{R}}^d_+ = \{x \in \mathbb{R}^d \mid x^1 \geq 0, \ldots, x^d \geq 0\}$. Some typical examples are closed simplices and rectangular solids. Every regular domain is a regular domain with corners, and a regular domain with corners is a d-dimensional smooth manifold with corners in the sense defined in [Lee 2013].

Here is the version of the divergence theorem we will use.

Lemma 2.2. *Suppose $D \subset \mathbb{R}^d$ is a regular domain with corners, having finite volume and surface area. If f is a smooth vector field defined on D such that both $|f|$ and $|\nabla \cdot f|$ are bounded, then*

$$\int_{\partial D} f \cdot \boldsymbol{n}_{\partial D} \, dA = \int_D \nabla \cdot f \, dV.$$

Proof. If D is compact, or more generally if f is compactly supported, this follows immediately from Stokes's theorem applied to the $(d-1)$-form $f \lrcorner (dx^1 \wedge \cdots \wedge dx^d)$, where $\lrcorner$ denotes interior multiplication. (For Stokes's theorem on manifolds with corners, see, for example, [Lee 2013, Theorem 16.25, p. 419].) In the general case, we argue as follows. Let $\varphi : [0, \infty) \to [0, 1]$ be a smooth function that is equal to 1 on $\left[0, \frac{1}{2}\right]$ and supported in $[0, 1]$, and for each $r > 0$ let $\varphi_r(x) = \varphi(|x|^2/r^2)$. Then the vector field $\varphi_r f$ is compactly supported, so the divergence theorem implies

$$(2\text{-}1) \qquad \int_{\partial D} \varphi_r f \cdot \boldsymbol{n}_{\partial D} \, dA = \int_D \nabla \cdot (\varphi_r f) \, dV.$$

As $r \to \infty$, the integral on the left-hand side of (2-1) converges to $\int_{\partial D} f \cdot \boldsymbol{n}_{\partial D} \, dA$ by the dominated convergence theorem. On the other hand, for each $r > 0$,

$$|\nabla \cdot (\varphi_r f)(x)| = \left| \varphi_r(x) \nabla \cdot f(x) + \frac{2}{r^2} \sum_{i=1}^d \varphi'\left(\frac{|x|^2}{r^2}\right) x^i f^i(x) \right|$$

$$\leq \|\nabla \cdot f\|_\infty + \frac{2}{r} \|\varphi'\|_\infty \|f\|_\infty,$$

because $|x| \leq r$ on the support of $\varphi'(|x|^2/r^2)$. Since $\nabla \cdot (\varphi_r f)$ converges pointwise to $\nabla \cdot f$ and D has finite volume, it follows from the dominated convergence theorem that the right-hand side of (2-1) converges to $\int_D \nabla \cdot f \, dV$. $\square$

The next proposition is used in the proof of the main theorems.

Proposition 2.3. *Suppose D_1 and D_2 are regular domains in $\mathbb{R}^d$, with $D_1 \cap D_2$ compact and with D_2 of finite volume and surface area. Suppose further that f is a smooth bounded vector field defined on a neighborhood of D_2. There exists a sequence of regular domains $D_{2,i}$ such that $\partial D_{2,i}$ is transverse to ∂D_1, and the following limits hold as $i \to \infty$:*

(a) $\mathrm{Vol}(D_{2,i}) \to \mathrm{Vol}(D_2)$;

(b) $\mathrm{Area}(\partial D_{2,i}) \to \mathrm{Area}(\partial D_2)$;

(c) $\int_{\partial D_{2,i}} f \cdot \boldsymbol{n}_{\partial D_{2,i}} \, dA \to \int_{\partial D_2} f \cdot \boldsymbol{n}_{\partial D_2} \, dA$.

The domains can be chosen so that $D_{2,i}$ is either a decreasing sequence of domains whose intersection is D_2, or an increasing sequence of domains whose union is $\mathring{D}_2$.
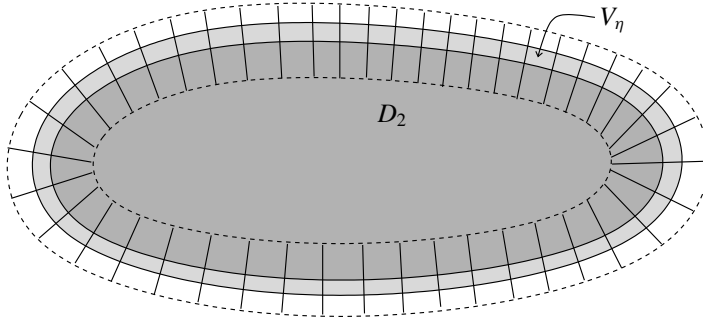
Figure 2. Defining a domain D_2^η containing D_2.

Proof. As a smooth embedded hypersurface, ∂D_2 has a tubular neighborhood N, and there exists a smooth embedding $E : \partial D_2 \times (-\delta, \delta) \to N$ such that $E(\cdot, 0)$ is the identity on ∂D_2. It can be chosen such that $E(\partial D_2 \times (0, \delta)) \cap D_2 = \varnothing$ and $E(\partial D_2 \times (-\delta, 0]) \subset D_2$.

Let $W \subset \mathbb{R}^d$ be a precompact neighborhood of $D_1 \cap D_2$ contained in the set on which f is defined, and let $\varphi : \mathbb{R}^d \to [0, 1]$ be a smooth compactly supported function that is equal to 1 on $\overline{W}$. For each η such that $\delta > \eta > 0$, define

$$V_\eta = \{E(x, s) : 0 \le s \le \eta\varphi(x)\},$$
$$D_2^\eta = D_2 \cup V_\eta.$$

(See Figure 2.) Then D_2^η is a regular domain containing D_2, which agrees with D_2 outside the support of φ. Its boundary ∂D_2^η is the image of the embedding $\iota_\eta : \partial D_2 \to \mathbb{R}^d$ given by $\iota_\eta(x) = E(x, \eta\varphi(x))$, which is equal to the inclusion map $\partial D_2 \hookrightarrow \mathbb{R}^d$ outside supp φ. The map E has full rank in $(\partial D_2 \cap W) \times (-\delta, \delta)$, and $\varphi \equiv 1$ there, so by the parametric transversality theorem (see, for example, [Lee 2013, Theorem 6.35, p. 145]), ∂D_2^η is transverse to ∂D_1 for almost every $\eta \in (-\delta, \delta)$.

Now let η_i be a sequence of positive numbers that decreases to zero, chosen so that $\partial D_2^{\eta_i}$ is transverse to ∂D_1 for each i, and set $D_{2,i} = D_2^{\eta_i}$. Then $D_{2,i}$ decreases to D_2 and $\mathrm{Vol}(D_{2,i})$ decreases to $\mathrm{Vol}(D_2)$. Moreover, because the embeddings ι_{η_i} converge uniformly with all derivatives to the inclusion map $\partial D_2 \hookrightarrow \mathbb{R}^d$, the surface area of $\partial D_{2,i}$ converges to that of ∂D_2. Further, the function $\boldsymbol{n}_{\partial D_{2,i}} \circ \iota_{\eta_i} : \partial D_2 \to \mathbb{R}^d$ converges to $\boldsymbol{n}_{\partial D_2}$. Combining these two arguments, we conclude that (c) is satisfied.

To obtain a sequence of domains that increase to $\mathring{D}_2$, we proceed instead as follows. For each η such that $-\delta < \eta < 0$, define

$$V_\eta = \{E(x, s) \mid \eta\varphi(x) < s \le 0\}, \quad D_2^\eta = D_2 \smallsetminus V_\eta.$$

In this case, we can choose a sequence of negative numbers η_i increasing to zero such that $\partial D_2^{\eta_i}$ is transverse to ∂D_1. The rest of the proof proceeds as before. $\quad\square$

3. Proof of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. We start with a more general result that implies both theorems; first, we prove it when the boundaries of the domains intersect transversally, then, employing an approximation argument, we prove the general case.

Theorem 3.1. *Suppose D_1 and D_2 are two regular domains in $\mathbb{R}^d$, such that $D_1 \cap D_2$ is compact and D_2 has finite volume and surface area. Let f be a smooth vector field defined on a neighborhood of D_2 such that both $|f|$ and $|\nabla \cdot f|$ are bounded. The absolute value of the flux of f across the portion of ∂D_1 inside D_2 satisfies the following bound*:

$$
(3\text{-}1) \quad \left| \int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA \right|
$$
$$
\leq \frac{1}{2} \left(\text{Area}(\partial D_2) \| f \|_\infty + \left| \int_{\partial D_2} f \cdot \boldsymbol{n}_{\partial D_2} \, dA \right| + \text{Vol}(D_2) \| \nabla \cdot f \|_\infty + \left| \int_{D_2} \nabla \cdot f \, dV \right| \right).
$$

The same estimate holds when $\partial D_1 \cap D_2$ is replaced by $\partial D_1 \cap \mathring{D}_2$ on the left-hand side.

Proposition 3.2. *Theorem 3.1 holds when $\partial D_1 \pitchfork \partial D_2$.*

Proof. Note that $\partial(D_1 \cap D_2)$ is compact, and

$$
(3\text{-}2) \qquad \partial(D_1 \cap D_2) = (\partial D_1 \cap D_2) \cup (D_1 \cap \partial D_2).
$$

Adding and subtracting $\int_{\partial D_2 \cap D_1} f \cdot \boldsymbol{n}_{\partial D_2} \, dA$, we obtain

$$
\int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA = \int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA + \int_{\partial D_2 \cap D_1} f \cdot \boldsymbol{n}_{\partial D_2} \, dA - \int_{\partial D_2 \cap D_1} f \cdot \boldsymbol{n}_{\partial D_2} \, dA
$$
$$
= \int_{\partial(D_1 \cap D_2)} f \cdot \boldsymbol{n}_{\partial(D_1 \cap D_2)} \, dA - \int_{\partial D_2 \cap D_1} f \cdot \boldsymbol{n}_{\partial D_2} \, dA,
$$

since $\partial D_1 \cap \partial D_2$ is a smooth $(d-2)$-dimensional submanifold and thus has zero $(d-1)$-dimensional area.

The assumption $\partial D_1 \pitchfork \partial D_2$ implies that $D_1 \cap D_2$ is a smooth manifold with corners. To see this, we just need to show that each point is contained in the domain of an appropriate smooth coordinate chart. For points not in $\partial D_1 \cap \partial D_2$, this follows easily from the fact that D_1 and D_2 are regular domains. If $x \in \partial D_1 \cap \partial D_2$, we can find a local defining function u^1 for D_1 such that D_1 is locally given by the equation $u^1 \geq 0$; and similarly we can find a local defining function u^2 for D_2. The assumption $\partial D_1 \pitchfork \partial D_2$ ensures that du^1 and du^2 are linearly independent at x. Thus we can find smooth functions $u^3, \ldots, u^d$ such that $(u^1, \ldots, u^d)$ form the required local coordinates in a neighborhood of x.

Applying the divergence theorem, we get

$$\int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA = \int_{D_2 \cap D_1} \nabla \cdot f \, dV - \int_{\partial D_2 \cap D_1} f \cdot \boldsymbol{n}_{\partial D_2} \, dA.$$

Applying Lemma 2.1 to both terms on the right hand side completes the proof for $\partial D_1 \cap D_2$. The result for $\partial D_1 \cap \mathring{D}_2$ is immediate in this case, because $\partial D_1 \cap \partial D_2$ has zero surface area. □

Proof of Theorem 3.1. Let $D_{2,i}$ be a sequence of regular domains decreasing to D_2 and satisfying the conclusions of Proposition 2.3. By Proposition 3.2, for every i we have that $\left| \int_{\partial D_1 \cap D_{2,i}} f \cdot \boldsymbol{n}_{\partial D_1} \, dA \right|$ is bounded by

$$\frac{1}{2} \left(\text{Area}(\partial D_{2,i}) \| f \|_\infty + \left| \int_{\partial D_{2,i}} f \cdot \boldsymbol{n}_{\partial D_{2,i}} \, dA \right| + \text{Vol}(D_{2,i}) \| \nabla \cdot f \|_\infty + \left| \int_{D_{2,i}} \nabla \cdot f \, dV \right| \right).$$

Proposition 2.3 shows that the first three terms above converge to the first three terms on the right-hand side of (3-1). To complete the proof, we use the facts that the sets $D_{2,i}$ decrease to D_2 and the compact sets $\partial D_1 \cap D_{2,i}$ decrease to $\partial D_1 \cap D_2$ as i goes to infinity, and thus the Lebesgue dominated convergence theorem yields

$$\lim_{i \to \infty} \left| \int_{D_{2,i}} \nabla \cdot f \, dV \right| = \left| \int_{D_2} \nabla \cdot f \, dV \right|$$

and

$$\lim_{i \to \infty} \left| \int_{\partial D_1 \cap D_{2,i}} f \cdot \boldsymbol{n}_{\partial D_1} \, dA \right| = \left| \int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA \right|.$$

This completes the proof for $\partial D_1 \cap D_2$.

To prove the estimate for $\partial D_1 \cap \mathring{D}_2$, we use the same argument, but with $D_{2,i}$ chosen to increase to $\mathring{D}_2$. Because ∂D_2 has d-dimensional measure zero, we have $\int_{\mathring{D}_2} \nabla \cdot f \, dV = \int_{D_2} \nabla \cdot f \, dV$, and the result follows. □

Proof of Theorem 1.1. Inequality (1-1) follows immediately from (3-1) and obvious estimates for the integrals. □

Proof of Theorem 1.2. We first assume that $\text{Vol}(D_2) < \infty$, so that (3-1) holds. In this case, the last two terms in (3-1) are zero because $\nabla \cdot f = 0$, and the second term is zero by the divergence theorem.

Now consider the case in which D_2 has infinite volume. Let D_2' denote the closure of $\mathbb{R}^d \smallsetminus D_2$, which is a regular domain with interior $\mathring{D}_2' = \mathbb{R}^d \smallsetminus D_2$. Because $\text{Area}(\partial D_2') = \text{Area}(\partial D_2) < \infty$, the isoperimetric inequality (see [De Giorgi 1953]) implies that D_2' has finite volume. If D_1 also has finite volume, the divergence theorem gives

$$\int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA + \int_{\partial D_1 \cap \mathring{D}_2'} f \cdot \boldsymbol{n}_{\partial D_1} \, dA = \int_{\partial D_1} f \cdot \boldsymbol{n}_{\partial D_1} \, dA = \int_{D_1} \nabla \cdot f \, dV = 0,$$

and (1-2) follows from Theorem 3.1 applied to the second term on the left-hand side above. On the other hand, if $\mathrm{Vol}(D_1) = \infty$, we let D_1' be the closure of $\mathbb{R}^d \smallsetminus D_1$ (which has finite volume), and apply the above argument with D_1' in place of D_1. $\square$

To conclude this section, we explain what modifications need to be made to Theorems 1.1 and 1.2 and their proofs to adapt them to the case of regular domains in Riemannian manifolds.

Suppose M is a d-dimensional smooth Riemannian manifold with Riemannian metric g and volume density dV_g. (If M is oriented, dV_g can be interpreted as a differential d-form; but otherwise it needs to be interpreted as a density. See [Lee 2013, pp. 427–434] for basic properties of densities.) A *regular domain* $D \subset M$ is defined just as in the case $M = \mathbb{R}^d$. If $D \subset M$ is a regular domain, it has a uniquely defined outward unit normal vector field $\boldsymbol{n}_{\partial D}$. For any such domain, we let $\tilde{g}$ denote the induced Riemannian metric on ∂D, and let $dA_{\tilde{g}}$ denote its volume density.

For any smooth vector field f defined on an open subset of M, the *divergence* of f, denoted by $\nabla \cdot f$, is defined as follows. If M is oriented, then $\nabla \cdot f$ is the unique vector field that satisfies $(\nabla \cdot f)\, dV_g = d(f \lrcorner dV_g)$. On a nonorientable manifold, we define it locally by choosing an orientation and using the same formula; because $\nabla \cdot f$ is unchanged when the orientation is reversed, it is globally defined. The divergence theorem then holds in exactly the same form for smooth d-dimensional submanifolds with corners in M. Moreover, any compact smooth embedded hypersurface in M has a tubular neighborhood in M. (See [Bredon 1993, Theorem 11.14, p. 100] for a proof. Although the proof there is for manifolds embedded in Euclidean space, it follows from the Whitney embedding theorem that it applies to all smooth manifolds.)

Using these facts, the proof of the following theorem is carried out exactly like the proofs of Theorems 1.1 and 1.2. To avoid complications, we restrict to the case in which D_2 is compact.

Theorem 3.3. *If D_1 and D_2 are regular domains in a Riemannian manifold (M, g) with D_2 compact, and f is a smooth vector field defined on a neighborhood of D_2, then the conclusions of Theorems 1.1 and 1.2 hold, namely,*

$$\left| \int_{\partial D_1 \cap D_2} \langle f, \boldsymbol{n}_{\partial D_1} \rangle_g \, dA_{\tilde{g}} \right| \leq \mathrm{Area}(\partial D_2)\|f\|_\infty + \mathrm{Vol}(D_2)\|\nabla \cdot f\|_\infty,$$

and if $\nabla \cdot f \equiv 0$,

$$\left| \int_{\partial D_1 \cap D_2} \langle f, \boldsymbol{n}_{\partial D_1} \rangle_g \, dA_{\tilde{g}} \right| \leq \tfrac{1}{2} \mathrm{Area}(\partial D_2)\|f\|_\infty.$$

4. Bounding integrals of normal fields

In this section, we prove Corollary 1.3 and Theorem 1.4. We also provide examples on the tightness of the bound.

Proof of Corollary 1.3. Let $v = \int_{\partial D_1 \cap D_2} \boldsymbol{n}_{\partial D_1} \, dA$. If $|v| = 0$ there is nothing to prove, so we assume that $|v| > 0$, and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be the constant vector field $f \equiv v/|v|$. Clearly, $|v| = v \cdot v/|v| = \int_{\partial D_1 \cap D_2} f \cdot \boldsymbol{n}_{\partial D_1} \, dA$. Now, since $\nabla \cdot f \equiv 0$ and $\|f\|_\infty = 1$, the proof follows from Theorem 1.2. $\qquad \square$

To prove Theorem 1.4, we begin with a lemma.

Lemma 4.1. *Suppose $D \subset \mathbb{R}^d$ is a compact convex regular domain with diameter δ and C is any measurable subset of ∂D. Then for any unit vector $v \in \mathbb{R}^d$, we have*

$$(4\text{-}1) \qquad \int_C v \cdot \boldsymbol{n}_{\partial D} \, dA \leq \tfrac{1}{2} \operatorname{Vol}(B^{d-1}(\delta/2)).$$

Proof. First consider the case $v = e_d = (0, \ldots, 0, 1)$. After applying a translation, we can assume that D is contained in the set where $x^d \geq 0$. Its boundary is the union of the three subsets ∂D_+, ∂D_0, and ∂D_-, defined as the subsets of ∂D where $v \cdot \boldsymbol{n}_{\partial D}$ is positive, zero, or negative, respectively.

Now, let A be the following subset of $\mathbb{R}^d$:

$$A = \{(x^1, \ldots, x^{d-1}, tx^d) \mid (x^1, \ldots, x^d) \in D, \ 0 \leq t \leq 1\}.$$

Then A is a compact convex set, and its boundary is the union of the three subsets ∂A_+, ∂A_0, and ∂A_-, defined in the same way as above. (See Figure 3.)

The fact that D is convex ensures that $\partial A_+ = \partial D_+$, $\partial A_0 \supset \partial D_0$, and ∂A_- is contained in the hyperplane where $x^d = 0$. Moreover, A is a C^1 manifold with corners. (Its boundary might not be smooth at points where ∂A_0 meets $\overline{\partial A}_+$, but it is at least C^1 there.)
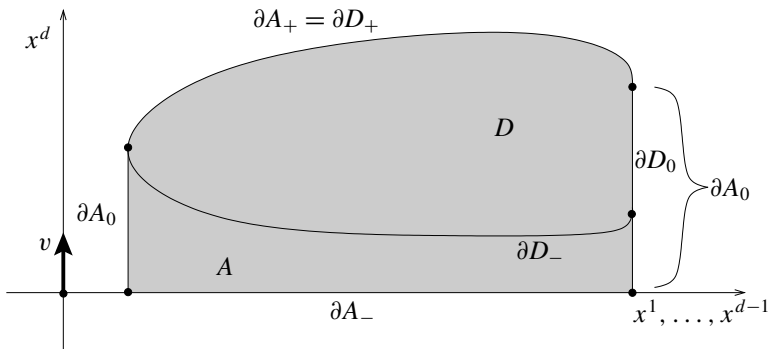
Figure 3. Proof of Lemma 4.1.

Using the fact that $v \cdot \boldsymbol{n}_{\partial D} < 0$ on ∂D_- and $v \cdot \boldsymbol{n}_{\partial D} = 0$ on ∂D_0, we compute

$$\int_C v \cdot \boldsymbol{n}_{\partial D} \, dA = \int_{C \cap \partial D_+} v \cdot \boldsymbol{n}_{\partial D} \, dA + \int_{C \cap \partial D_0} v \cdot \boldsymbol{n}_{\partial D} \, dA + \int_{C \cap \partial D_-} v \cdot \boldsymbol{n}_{\partial D} \, dA$$

$$\leq \int_{C \cap \partial D_+} v \cdot \boldsymbol{n}_{\partial D} \, dA \leq \int_{\partial D_+} v \cdot \boldsymbol{n}_{\partial D} \, dA$$

$$= \int_{\partial A_+} v \cdot \boldsymbol{n}_{\partial A} \, dA = - \int_{\partial A_-} v \cdot \boldsymbol{n}_{\partial A} \, dA,$$

where in the last line we have used the divergence theorem for the vector field $f \equiv v$ and the fact that $v \cdot \boldsymbol{n}_{\partial A} = 0$ on ∂A_0. Since $\boldsymbol{n}_{\partial A} = -v$ on ∂A_-, the last integral is equal to the area of ∂A_-. Since ∂A_- is contained in a $(d-1)$-dimensional ball of radius $\delta/2$, the result follows.

Finally, for the case of a general unit vector v, we just apply a rotation to D and apply the above argument. $\qquad \square$

Proof of Theorem 1.4. Let D_1 and D_2 be as in the statement of the theorem. If $\int_{\partial D_1 \cap D_2} \boldsymbol{n}_{\partial D_1} \, dA = 0$, there is nothing to prove, so assume the integral is nonzero, and let v be the unit vector in the direction of $\int_{\partial D_1 \cap D_2} \boldsymbol{n}_{\partial D_1} \, dA$. Then

$$\left| \int_{\partial D_1 \cap D_2} \boldsymbol{n}_{\partial D_1} \, dA \right| = v \cdot \int_{\partial D_1 \cap D_2} \boldsymbol{n}_{\partial D_1} \, dA$$

$$= \int_{\partial D_1 \cap D_2} v \cdot \boldsymbol{n}_{\partial D_1} \, dA,$$

and the result follows from Lemma 4.1. $\qquad \square$

The following examples demonstrate the tightness of the bound for nonconvex sets, as well as the necessity of the condition that the hypersurface be the boundary of a regular domain.

Example 4.2. The main theorem explicitly uses the divergence theorem, which is applied to space-separating hypersurfaces. In fact, the bounds do not apply for images of general smooth immersions. To construct a counterexample in the plane (i.e., for $d = 2$), start with a smooth Jordan curve in the plane, then cover it m times, with small perturbations, making the integral on the left-hand side of (1-3) roughly m times as large, while the right-hand side is fixed because it depends only on ∂D_2. Clearly, whenever the left-hand side of (1-3) is not zero, we can choose m large enough that the inequality does not hold.

Example 4.3. To see that the bound obtained in Corollary 1.3 is tight, and cannot be replaced by a bound based only on the diameter of D_2 when D_2 is not convex, we consider comb-shaped subsets of $\mathbb{R}^d$, for $d \geq 2$, generated in the following manner. Fix $n > 2$, and let D_n be a closed nonsmooth comb-shaped set defined as
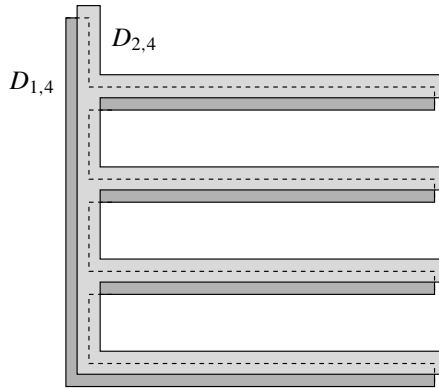
Figure 4. The domains of Example 4.3 in the case $n = 4$ (before smoothing).

the union of the following rectangles:

$$R_{i,n} = \{(x^1, \ldots, x^d) \in [0, 1]^d \mid i/n \leq x^2 \leq i/n + 1/n^2\}, \quad 0 \leq i \leq n - 1;$$

$$R_{n,n} = \{(x^1, \ldots, x^d) \in [0, 1]^d \mid 0 \leq x^1 \leq 1/n^2\}.$$

Applying a small perturbation we then smooth its corners, and set $D_{1,n}$ accordingly. Let $D_{2,n}$ be the translation of $D_{1,n}$ by the vector $(1/(2n^2), 1/(2n^2), 0, \ldots, 0) \in \mathbb{R}^d$. (See Figure 4.) By our construction, the surface area of each set $\partial D_{1,n}$ or $\partial D_{2,n}$ is roughly $2n + 2$, and the fraction of the boundary area where the normal vector of $\partial D_{1,n}$ is parallel to the x^2-axis is roughly $n/(n + 1)$, approaching 1 when n is large. Notice that by the choice of $D_{2,n}$, when we integrate the normal vector in the portion of $\partial D_{1,n}$ inside $D_{2,n}$ we capture only the part pointing in the positive direction of the x^2-axis. This shows that the integral of the normal vector has magnitude of roughly n, approaching half the surface area when we take n to infinity.

5. Applications: limits of hypersurfaces & planar results

In this section we provide two applications of Corollary 1.3, extending previous planar results in [Artstein and Bright 2010; 2013]. The first is for limits of regular domains whose surface areas increase without bound. The second is an application in the planar case.

Corollary 1.3 bounds the normal vector of the boundary of a regular domain in a second regular domain, by the surface area of the boundary of the second domain, and completely disregarding the surface area of the original hypersurface. This is now applied to surfaces with increasing surface area, establishing a new result on the limit.

We will denote by $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ the unit $(d-1)$-sphere. For every hypersurface we define a corresponding probability measure using the following notation:

Definition 5.1. Suppose $S \subset \mathbb{R}^d$ is a smooth hypersurface endowed with a unit normal vector field $\boldsymbol{n}_S$. We define the *empirical measure* $\mu \in P(\mathbb{R}^d \times \mathbb{S}^{d-1})$ corresponding to S by

$$\mu(U \times V) = \frac{1}{\text{Area}(S)} \int_{S \cap U} \chi_V(\boldsymbol{n}_S) \, dA,$$

for all open sets $U \subset \mathbb{R}^d$ and $V \subset \mathbb{S}^{d-1}$.

A useful property of empirical measures is that, if $f : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}$ is continuous, we have

$$\frac{1}{\text{Area}(S)} \int_S f(x, \boldsymbol{n}_S) \, dA = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} f(x, n) \mu(dx, dn).$$

We endow the set of probability measures $P(\mathbb{R}^d \times \mathbb{S}^{d-1})$ with the weak topology, namely, a sequence of measures $\mu_1, \mu_2, \ldots \in P(\mathbb{R}^d \times \mathbb{S}^{d-1})$ converges to a measure $\mu_0 \in P(\mathbb{R}^d \times \mathbb{S}^{d-1})$ if for every bounded continuous function $g(x, n)$,

$$\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} g(x, n) \mu_0(dx, dn) = \lim_{i \to \infty} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} g(x, n) \mu_i(dx, dn).$$

Another tool we need for the next theorem is disintegration of measures. Given a probability measure $\mu \in P(\mathbb{R}^d \times \mathbb{S}^{d-1})$, we define its marginal measure, $p(dx)$, as the projection on $\mathbb{R}^d$, namely, $p(A) = \mu(A \times \mathbb{S}^{d-1})$ for every measurable set $A \subset \mathbb{R}^d$. Also, we denote the measure valued function $\mu^x(dn)$, the disintegration of μ with respect to p, for p-almost every x. With this notation, for every pair of measurable sets $U \subset \mathbb{R}^d$ and $V \subset \mathbb{S}^{d-1}$, we have that

$$\mu(U \times V) = \int_U \mu^x(V) p(dx).$$

We now state the main result regarding the limits of regular domains.

Theorem 5.2. *Let $D_1, D_2, \ldots \subset \mathbb{R}^d$ be a sequence of compact regular domains, such that the surface areas of their boundaries increases to infinity. If the empirical measures $\mu_1, \mu_2, \ldots$, corresponding to the sequence $\partial D_1, \partial D_2, \ldots$, converge weakly to μ_0, then*

$$h(x) = \int_{\mathbb{S}^{d-1}} n \, \mu_0^x(x)(dn) = 0$$

for p_0-almost every x, where $\mu_0(dx, dn) = p_0(dx)\mu_0^x(dn)$ is the disintegration of μ_0 with respect to its projection p_0.

Proof. Let $B = B(x, r) \subset \mathbb{R}^d$ be a ball centered at x with radius $r > 0$. By the definition of the empirical measures and by Corollary 1.3,

$$\left| \int_{B \times \mathbb{S}^{d-1}} n \, d\mu_i(dx, dn) \right| = \left| \frac{1}{\text{Area}(\partial D_i)} \int_{\partial D_i \cap B} \boldsymbol{n}_{\partial D_i} \, dA \right| \le \frac{\text{Area}(\partial B)}{2 \, \text{Area}(\partial D_i)}.$$

Weak convergence of measures and the dominated convergence theorem imply that

$$\left| \int_{B \times \mathbb{S}^{d-1}} n \, d\mu_0(dx, dn) \right| = \lim_{i \to \infty} \left| \int_{B \times \mathbb{S}^{d-1}} n \, d\mu_i(dx, dn) \right|$$

$$\leq \lim_{i \to \infty} \frac{\text{Area}(\partial B)}{2 \, \text{Area}(\partial D_i)} = 0$$

for a set of values of $r > 0$ of full measure for which $\mu_i(\partial B(x, r) \times \mathbb{S}^{d-1}) = 0$, for all $i = 0, 1, 2, \ldots$. Using the disintegration notation we obtain that

$$\left| \int_{B \times \mathbb{S}^{d-1}} n \, d\mu_0(dx, dn) \right| = \left| \int_B \left(\int_{\mathbb{S}^{d-1}} n \, \mu_0^x(dn) \right) p_0(dx) \right|$$

$$= \int_B h(x) p_0(dx) = 0$$

for almost every ball B. If the measure $p_0(dx)$ is Lebesgue measure, by the Lebesgue differentiation theorem we have $h(x) = 0$ almost everywhere. The Lebesgue–Besicovitch differentiation theorem extends this result to Radon measures (see, for example, [Evans and Gariepy 1992, p. 43]). $\qquad \square$

Remark 5.3. Theorem 5.2 requires the convergence of the empirical measures. When the domains in the sequence are contained in some compact set K, the compactness of the space $K \times \mathbb{S}^{d-1}$ implies the compactness of $P(K \times \mathbb{S}^{d-1})$, and, therefore, the existence of a converging subsequence [Billingsley 1999, p. 72].

In two dimensions, our result extends as follows.

Corollary 5.4. *Suppose* $x_1 : [0, \tau_1] \to \mathbb{R}^2$ *is a parametrized smooth Jordan curve and* $D_2 \subset \mathbb{R}^2$ *is a regular domain. If the length of* ∂D_2 *is* L_2, *then*

$$\left| \int_0^{\tau_1} \chi_{D_2}(x_1(t)) \frac{d}{dt} x_1(t) \, dt \right| \leq \frac{L_2}{2}.$$

Proof. Let T_1 and N_1 be the unit tangent and normal vectors of x_1. Using the arc-length parametrization, we have that

$$\left| \int_0^{\tau_1} \chi_{D_2}(x_1(t)) \frac{d}{dt} x_1(t) \, dt \right| = \left| \int_0^{L_1} \chi_{D_2}(x_1(s)) T_1(s) \, ds \right|,$$

where L_1 is the length of x_1. Expressing the tangent vector in terms of the normal vector, we reduce the previous expression to

$$\left| \int_0^{L_1} \chi_{D_2}(x_1(s)) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} N_1(s) \, ds \right| = \left| \int_0^{L_1} \chi_{D_2}(x_1(s)) N_1(s) \, ds \right|,$$

as the rotation matrix is orthogonal. Applying Corollary 1.3 completes the proof. $\square$

For our final application, we consider an ordinary differential equation in the plane defined by

$$\text{(5-1)} \qquad \frac{dx}{dt} = f(x),$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field (generally assumed at least Lipschitz continuous). An *invariant set* for f is a subset of $\mathbb{R}^2$ that is invariant under the forward flow of f and a *minimal set* is a nonempty closed invariant set that is minimal with respect to inclusions. A *trivial minimal set* is a set that is the image of either a stationary solution or a periodic solution.

We present a new short proof of the following well-known result.

Theorem 5.5. *Suppose f is a smooth vector field on $\mathbb{R}^2$. Then every minimal set for f is trivial.*

The textbook proof of this theorem (see [Verhulst 1996]) relies on the Poincaré–Bendixson theorem, and employs dynamical arguments. Here we present a simpler proof based on the divergence theorem, and specifically on Corollary 1.3. Note that the divergence theorem was used by Bendixson in the proof of the Bendixson criterion, which verifies that no periodic solutions exist.

Our proof uses the following well-known lemmas.

Lemma 5.6. *Suppose $\Omega \subset \mathbb{R}^2$ is a minimal set for (5-1) and $x^* : [0, \infty) \to \mathbb{R}^2$ is a solution to (5-1) with trajectory contained in Ω. For every $y_0 \in \Omega$, $s \in [0, \infty)$, and $\delta > 0$, there exists $t > s$ such that $|x^*(t) - y_0| < \delta$.*

Proof. Suppose the lemma does not hold for some y_0, s, and δ. Then the curve $y^*(t) = x^*(s + t)$ is a solution to (5-1) with trajectory contained in $\Omega \smallsetminus B(y_0, \delta)$ for a suitable $\delta > 0$, in contradiction to the minimality of Ω. $\qquad\square$

The next lemma follows easily from Sard's theorem.

Lemma 5.7. *Suppose $I \subset \mathbb{R}$ is a compact interval and $g : I \to \mathbb{R}$ is smooth. Then for almost every $r \in \mathbb{R}$, the set $g^{-1}(r) = \{t \in I \mid g(t) = r\}$ is finite.*

Proof of Theorem 5.5. Clearly, Ω is a singleton if and only if it contains a point $y \in \Omega$ such that $f(y) = 0$, so we may assume henceforth that f does not vanish in Ω and Ω contains more than one point. Choose $D > 0$ such that $\Omega \smallsetminus B(x^*(0), 3D) \neq \varnothing$. We construct sequences of real numbers $\{\delta_i\}$ and $\{t_i\}$, and a sequence of simple closed curves $\{\gamma_i\}$, as follows. Set $\delta_0 = D$, and let t_0 be the first time where x^* meets $\partial B(x^*(0), \delta_0)$. For $i = 1, 2, \ldots$ do the following:

(a) Choose $\delta_i < \delta_{i-1}/2$ small enough that $|x^*(0) - x^*(t)| > \delta_i$ for all $t \in [t_0, t_{i-1}]$.

(b) Let t_i be the first time after t_0 where the curve x^* meets $\partial B(x^*(0), \delta_i)$. (Here we use Lemma 5.6.)
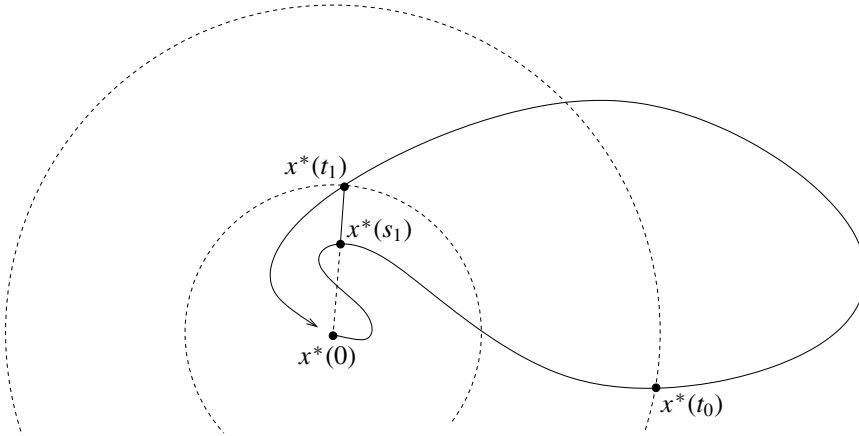
Figure 5. Proof of Theorem 5.5.

(c) Starting from $x^*(t_i)$, follow the line connecting it to $x^*(0)$, until that line first meets a point in $x^*([0, t_0])$. Let $x^*(s_i)$ be this point. (See Figure 5.)

(d) Let γ_i be the parametrized piecewise smooth curve obtained by following the curve x^* in the interval $[s_i, t_i]$, and then the line connecting its endpoints with unit speed.

Note that t_i is an increasing sequence and that the uniqueness of the solution with respect to the initial condition implies that every γ_i is a Jordan curve. Suppose first that the sequence $\{t_i\}$ is bounded above. Then $t_i \to t^* \in \mathbb{R}^+$ and $x^*(t_i) \to x^*(t^*)$. According to our construction, $|x^*(0) - x^*(t_i)| = \delta_i < 2^{-i}D$ for every i. Hence by continuity $x^*(t^*) = x^*(0)$, and x^* is periodic. By the minimality of Ω, the image of x^* is Ω.

The only remaining possibility is $t_i \nearrow \infty$. Fix $y_0 \in \Omega$ such that $|y_0 - x^*(0)| > 2D$. By Lemma 5.7, there exists arbitrarily small $r_0 < D$ such that the set

$$\{t \in [0, s] \mid |x^*(t) - y_0| = r_0\}$$

is finite for every $s > 0$. (This follows from the fact that $g(t) = |x^*(t) - y_0|^2$ is a smooth function of t.) Note that this implies that the portion of γ_i in $B_0 = B(y_0, r_0)$ is part of the trajectory x^*, and that for every i the Jordan curve γ_i intersects ∂B_0 at a finite number of points.

For every i, we let D_i denote the domain consisting of the Jordan curve γ_i together with its interior. Although D_i is not a regular domain, it is a regular domain with two corner points, which are outside of $\bar{B}_0$, and it is easy to see that Corollary 1.3 can be applied to $\partial D_i \cap B_0$. Thus by Corollary 5.4,

$$\left| \int_{\{t \le t_i \mid x^*(t) \in B_0\}} \frac{d}{dt} x^*(t)\, dt \right| = \left| \int_{\{t \le t_i \mid x^*(t) \in B_0\}} f(x^*(t))\, dt \right| \le \pi r_0.$$

Because Ω is minimal, Lemma 5.6 implies that the set $\{t \mid x^*(t) \in B_0\}$ has infinite measure. This implies that 0 is contained in the convex hull of the set

$$\{f(y) \mid y \in \bar{B}_0\}.$$

The radius r_0 can be chosen arbitrary small; therefore, the continuity of f implies that $f(y_0) = 0$, in contradiction. $\qquad\square$

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