Pacific Journal of Mathematics

THE BOCHNER FORMULA FOR ISOMETRIC IMMERSIONS

ALESSANDRO SAVO

Volume 272 No. 2 December 2014

PACIFIC JOURNAL OF MATHEMATICS
Vol. 272, No. 2, 2014
dx.doi.org/10.2140/pjm.2014.272.395

THE BOCHNER FORMULA FOR ISOMETRIC IMMERSIONS

ALESSANDRO SAVO

We study the Bochner formula for a manifold isometrically immersed into another and obtain a Gauss-type splitting of its curvature term. In fact, we prove that the curvature term in the Bochner formula is an operator that can be explicitly expressed in terms of the curvature operator of the ambient manifold and the extrinsic geometry (second fundamental form) of the immersion. Several applications of this splitting are given, namely, eigenvalue estimates for the Hodge Laplacian, vanishing results for the de Rham cohomology and rigidity of immersions of Kähler manifolds into negatively curved spaces.

1. Introduction

Let Σ^n be a Riemannian manifold of dimension n (all manifolds in this paper are connected, orientable and without boundary) and let ω be a differential p-form on Σ. The Bochner formula states

$$\Delta\omega = \nabla^\star\nabla\omega + \mathcal{B}^{[p]}\omega. \tag{1}$$

Here $\Delta = d\delta + \delta d$ is the Hodge Laplacian (δ being the adjoint of d), $\nabla^\star\nabla$ is the connection Laplacian, and

$$\mathcal{B}^{[p]} : \Lambda^p(\Sigma) \to \Lambda^p(\Sigma)$$

is a certain symmetric endomorphism of the bundle of p-forms. We call $\mathcal{B}^{[p]}$ the *Bochner curvature term* or simply the *Bochner operator*.

When Σ is compact, knowing that $\mathcal{B}^{[p]}$ is positive at each point implies that any harmonic p-form must vanish; then, by the Hodge–de Rham theorem, the de Rham cohomology $H^p(\Sigma, \mathbb{R})$ must vanish. More generally, a positive lower bound of the eigenvalues of $\mathcal{B}^{[p]}$ implies a positive lower bound of the first eigenvalue of the Hodge Laplacian acting on p-forms. All these facts are consequences of the well-known *Bochner method*, and will be recalled in Proposition 3.

It is then important to look for estimates of the eigenvalues of $\mathcal{B}^{[p]}$. It turns out that $\mathcal{B}^{[1]}$ is simply the Ricci tensor (acting on 1-forms); but for degrees $2 \le p \le n-2$

MSC2010: 58J50, 35P15.
Keywords: differential forms, Bochner formula, isometric immersions, vanishing theorems.

the operator $\mathcal{B}^{[p]}$ is more complicated and difficult to control. A breakthrough was obtained by Gallot and Meyer [1975]. They proved that, for all $p = 1, \dots, n-1$,

$$\mathcal{B}^{[p]} \geq p(n-p)\gamma_\Sigma, \tag{2}$$

where γ_Σ is a lower bound of the eigenvalues of the curvature operator of Σ.

In this paper, we study the Bochner curvature term of a manifold Σ^n isometrically immersed in a larger manifold M^{n+q} by a smooth map $f : \Sigma^n \to M^{n+q}$. The natural problem we address is to give a new expression of the Bochner curvature in terms of the extrinsic geometry of the immersion, which would eventually improve the estimate (2) in this important situation. It turns out that this is in fact possible. In Theorem 1 we prove a Gauss-type formula and show that $\mathcal{B}^{[p]}$ splits into the sum of two operators acting on $\Lambda^p(\Sigma)$:

$$\mathcal{B}^{[p]} = \mathcal{B}^{[p]}_{\mathrm{res}} + \mathcal{B}^{[p]}_{\mathrm{ext}}.$$

We then prove that $\mathcal{B}^{[p]}_{\mathrm{res}}$, which depends on the geometry of the ambient manifold M, is bounded below by the lowest eigenvalue of the curvature operator of M:

$$\mathcal{B}^{[p]}_{\mathrm{res}} \geq p(n-p)\gamma_M,$$

while the extrinsic part $\mathcal{B}^{[p]}_{\mathrm{ext}}$ is explicitly described in terms of the second fundamental form of the immersion. For example, if Σ has codimension one and S is the shape operator relative to any of the two choices of the unit normal vector field, then

$$\mathcal{B}^{[p]}_{\mathrm{ext}} = \operatorname{tr} S \cdot S^{[p]} - S^{[p]} \circ S^{[p]}, \tag{3}$$

where $S^{[p]} : \Lambda^p(\Sigma) \to \Lambda^p(\Sigma)$ is the self-adjoint extension of S, acting on the form ω by

$$S^{[p]}\omega(X_1, X_2, \dots, X_p) = \sum_{j=1}^{p} \omega(X_1, \dots, S(X_j), \dots, X_p) \tag{4}$$

for all tangent vectors $X_1, \dots, X_p$. In higher codimensions, one simply sums the expression (3) over the shape operators of an orthonormal basis of the normal bundle. We refer to Theorem 1 for the precise statement.

From (3) and (4) one sees that the eigenvalues of $\mathcal{B}^{[p]}_{\mathrm{ext}}$ can be estimated in terms of the eigenvalues of S (principal curvatures) and explicit bounds of the extrinsic part will follow.

Thus, one can bound the Bochner curvature in terms of the curvature operator of the ambient manifold M and the second fundamental form of the immersion. This can be done in different ways (see Section 4) and leads to a number of applications. In particular:

- We prove an extrinsic, sharp lower bound of the first eigenvalue of the Hodge Laplacian of Σ, assuming that Σ has codimension one in M (Theorem 7).
- We prove a vanishing theorem for the de Rham cohomology of Σ, assuming that the norm of the traceless second fundamental form of Σ is bounded above by a suitable function of the mean curvature (Theorem 8). The condition is sharp when the ambient manifold is a sphere, in which case we get a rigidity result for Clifford tori (Theorem 9).
- We prove a lower bound of the first eigenvalue of the Hodge Laplacian of Σ, assuming that the norm of its second fundamental form is bounded above by a suitable constant and that the ambient manifold is positively curved (Theorem 10). When the ambient manifold is a sphere this reproves a vanishing theorem for the de Rham cohomology of Σ due to Lawson and Simons; moreover the limit case leads to a rigidity result for a certain Clifford torus (Theorem 11).
- We prove that if Σ supports a nontrivial parallel p-form and admits an isometric immersion into a negatively curved space, then its mean curvature vector has norm bounded below by an explicit positive constant (Theorem 12). This has applications to immersions of Kähler manifolds (Corollary 13).
- We classify the compact hypersurfaces of $\mathbb{S}^{n+1}$ which support a parallel p-form (Theorem 4): if $n \geq 3$ they are in fact products of spheres (Clifford tori). This result is perhaps of independent interest, and is needed to prove the rigidity theorems above.

In conclusion, we hope that this new representation of the Bochner formula will be useful, and that it will lead to other interesting applications in submanifold geometry.

The paper is organized as follows. In Section 2 we state the main theorem; in Section 3 we state the main applications; these will be proved in Section 4, along with the explicit bounds on the Bochner operator. In Section 5 we prove the main theorem and in Section 6 we prove Theorem 4.

2. The Bochner curvature term

Let $f : \Sigma^n \to M^{n+q}$ be an isometric immersion with codimension $q \geq 1$. The second fundamental form L of f is defined by the relation

$$\nabla^M_X Y = \nabla_X Y + L(X, Y),$$

where $X, Y \in T\Sigma$ and ∇^M, ∇ denote the Levi-Civita connections in M and Σ, respectively. Note that $L(X, Y)$ is a vector normal to Σ, so that L takes values in the normal bundle of the immersion. If $\nu \in T^\perp\Sigma$ is a normal vector, we denote its associated shape operator by S_ν. It is the self-adjoint endomorphism of $T\Sigma$ defined on $X, Y \in T\Sigma$ by

$$\langle S_\nu(X), Y\rangle = \langle L(X,Y), \nu\rangle.$$

The mean curvature vector H is defined by $H = (1/n)\operatorname{tr} L$; so, for any orthonormal basis $(e_1, \dots, e_n)$ of $T\Sigma$, one has $nH = \sum_{j=1}^n L(e_j, e_j)$. The squared norm of the second fundamental form is denoted by $|S|^2$. Note that if $(\nu_1, \dots, \nu_q)$ is an orthonormal basis in the normal bundle, then

$$n^2|H|^2 = \sum_{i=1}^q (\operatorname{tr} S_{\nu_i})^2 \quad \text{and} \quad |S|^2 = \sum_{i=1}^q |S_{\nu_i}|^2.$$

Now extend S_ν to a self-adjoint operator acting on p-forms, $S_\nu^{[p]} : \Lambda^p(\Sigma) \to \Lambda^p(\Sigma)$, as in (4), and let

$$T_\nu^{[p]} \doteq (\operatorname{tr} S_\nu) S_\nu^{[p]} - S_\nu^{[p]} \circ S_\nu^{[p]}.$$

Introduce the self-adjoint endomorphism of $\Lambda^p(\Sigma)$

$$\mathcal{B}_{\text{ext}}^{[p]} = \sum_{j=1}^q T_{\nu_j}^{[p]} = \sum_{j=1}^q \big((\operatorname{tr} S_{\nu_j}) S_{\nu_j}^{[p]} - S_{\nu_j}^{[p]} \circ S_{\nu_j}^{[p]}\big). \tag{5}$$

Then, we have:

Theorem 1. *Let $f : \Sigma^n \to M^{n+q}$ be an isometric immersion. The Bochner operator acting on p-forms of Σ splits as*

$$\mathcal{B}^{[p]} = \mathcal{B}_{\text{res}}^{[p]} + \mathcal{B}_{\text{ext}}^{[p]},$$

where the operator $\mathcal{B}_{\text{ext}}^{[p]}$ is defined by (5), and the operator $\mathcal{B}_{\text{res}}^{[p]}$ satisfies the bounds

$$p(n-p)\gamma_M \le \mathcal{B}_{\text{res}}^{[p]} \le p(n-p)\Gamma_M,$$

where γ_M and Γ_M are respectively a lower and an upper bound on the curvature operator of M. If M has constant sectional curvature γ, then

$$\mathcal{B}_{\text{res}}^{[p]} = p(n-p)\gamma.$$

For the definition of $\mathcal{B}_{\text{res}}^{[p]}$ and the proof of Theorem 1, see Section 5. The notation "res" is chosen because $\mathcal{B}_{\text{res}}^{[p]}$ depends on the restriction of the curvature operator of M to the submanifold Σ.

Let us give a more explicit expression of $\mathcal{B}_{\text{ext}}^{[p]}$. In what follows, $S = S_\nu$ will be the shape operator associated to a given unit normal vector ν. We denote by I_p the set of p-multi-indices

$$I_p = \{\{j_1, \dots, j_p\} : 1 \le j_1 < \dots < j_p \le n\}.$$

Let $(e_1, \dots, e_n)$ be an orthonormal basis of $T\Sigma$ which diagonalizes S; then

$$S(e_j) = k_j e_j,$$

where $k_1, \dots, k_n$ are the principal curvatures. If $\alpha = \{j_1, \dots, j_p\} \in I_p$, we define

$$K_\alpha \doteq k_{j_1} + \cdots + k_{j_p} \tag{6}$$

and call it a *p-curvature* of S. Let $(\theta_1, \dots, \theta_n)$ be the dual basis of $(e_1, \dots, e_n)$; if $\alpha = \{j_1, \dots, j_p\} \in I_p$, consider the p-form

$$\Theta_\alpha \doteq \theta_{j_1} \wedge \cdots \wedge \theta_{j_p}.$$

Then $\{\Theta_\alpha\}_{\alpha \in I_p}$ is an orthonormal basis of $\Lambda^p(\Sigma)$. From the definition (4) one sees that S is extended as a derivation of $\Lambda^\star(\Sigma)$, hence Θ_α is an eigenform of $S^{[p]}$ associated to the eigenvalue K_α:

$$S^{[p]}\Theta_\alpha = K_\alpha \Theta_\alpha.$$

In turn, if $\alpha = \{j_1, \dots, j_p\} \in I_p$, let $\star\alpha \in I_{n-p}$ be the multi-index given by the complement of α in $\{1, \dots, n\}$:

$$\star\alpha \doteq \{1, \dots, n\} \setminus \{j_1, \dots, j_p\}.$$

Let $T^{[p]} = (\operatorname{tr} S)S^{[p]} - S^{[p]} \circ S^{[p]}$. Since $K_\alpha + K_{\star\alpha} = k_1 + \cdots + k_n$, we have

$$T^{[p]}\Theta_\alpha = \big((k_1 + \cdots + k_n)K_\alpha - K_\alpha^2\big)\Theta_\alpha = K_\alpha K_{\star\alpha}\Theta_\alpha$$

and then Θ_α is also an eigenform of $T^{[p]}$ associated to the eigenvalue $K_\alpha K_{\star\alpha}$. In conclusion, we have:

Lemma 2. *Let $S = S_\nu$ be the shape operator relative to a given unit normal vector $\nu \in T^\perp\Sigma$. Let $T^{[p]} : \Lambda^p(\Sigma) \to \Lambda^p(\Sigma)$ be the operator*

$$T^{[p]} = (\operatorname{tr} S)S^{[p]} - S^{[p]} \circ S^{[p]}.$$

Then the eigenvalues of $T^{[p]}$ are given by the $\binom{n}{p}$ numbers $K_\alpha K_{\star\alpha}$, where $\alpha \in I_p$ runs over the set of p-multi-indices and K_α are the p-curvatures defined in (6).

In particular, if Σ is a hypersurface of M^{n+1}, and S is the shape operator of Σ relative to any of the two choices of the unit normal vector, then $T^{[p]} = \mathcal{B}^{[p]}_{\text{ext}}$ and

$$\min_{\alpha \in I_p} K_\alpha K_{\star\alpha} \le \mathcal{B}^{[p]}_{\text{ext}} \le \max_{\alpha \in I_p} K_\alpha K_{\star\alpha}. \tag{7}$$

In higher codimensions, in order to estimate $\mathcal{B}^{[p]}_{\text{ext}}$ it is enough to estimate the p-curvatures of the shape operators S_{ν_j} for an orthonormal frame $(\nu_1, \dots, \nu_q)$ in the normal bundle of the immersion. It will then be possible to bound $\mathcal{B}^{[p]}_{\text{ext}}$ in terms of $|S|^2$ and $|H|^2$ (see Section 3).

Let us briefly explain why Theorem 1 improves the bound (2). For simplicity, assume that Σ^n is a hypersurface of $\mathbb{R}^{n+1}$, with principal curvatures $k_1, \dots, k_n$. It is known that the curvature operator of Σ has eigenvalues $\{k_i k_j : i \neq j\}$ (see also Section 5.2). Hence

$$\gamma_\Sigma = \inf\{k_i k_j : i \neq j\}.$$

Now observe that since $\mathcal{B}^{[p]}_{\text{res}} = 0$, we have $\mathcal{B}^{[p]} = \mathcal{B}^{[p]}_{\text{ext}}$; if $\alpha \in I_p$ is a multi-index, then $K_\alpha K_{\star\alpha}$ is a sum of $p(n-p)$ products of type $k_i k_j$ with $i \neq j$. Then (7) gives

$$\mathcal{B}^{[p]} \geq \min_{\alpha \in I_p} K_\alpha K_{\star\alpha} \geq p(n-p)\gamma_\Sigma,$$

the expression on the right being the lower bound in (2). Numerical examples show that the lower bound in (7) is often much better than (2).

2.1. ***The Bochner method.*** When the manifold Σ is compact, lower bounds of the Bochner curvature lead to lower bounds of the Hodge–Laplace spectrum. Here we recall the main facts. Let $\lambda_{1,p}(\Sigma)$ be the lowest eigenvalue of the Hodge–Laplace operator acting on p-forms of Σ. It is well known that

$$\lambda_{1,p}(\Sigma) = \inf\left\{ \frac{\int_\Sigma |d\omega|^2 + |\delta\omega|^2}{\int_\Sigma |\omega|^2} : \omega \in \Lambda^p(\Sigma) \setminus \{0\} \right\}. \tag{8}$$

As Σ is orientable and the quadratic form in (8) is invariant under the Poincaré duality induced by the Hodge $\star$-operator, we have $\lambda_{1,p} = \lambda_{1,n-p}$ and so we can assume that $p \leq n/2$. Clearly $\lambda_{1,p} = 0$ if and only if Σ supports a nontrivial harmonic p-form, in which case $H^p(\Sigma, \mathbb{R}) = H^{n-p}(\Sigma, \mathbb{R}) \neq 0$ (the Hodge–de Rham theorem).

The next proposition is well known and is often called the *Bochner method* (the estimate (iii) follows from a lower estimate of the energy of a form due to Gallot and Meyer [1975]). We state it in the following form for future reference.

Proposition 3. *Let Σ^n be a compact, orientable manifold, $1 \leq p \leq n/2$ and $\mathcal{B}^{[p]}$ the curvature term in the Bochner formula (1).*

(i) *If $\mathcal{B}^{[p]} \geq 0$ and the strict inequality holds at some point, then we have $H^p(\Sigma, \mathbb{R}) = H^{n-p}(\Sigma, \mathbb{R}) = 0$.*

(ii) *If $\mathcal{B}^{[p]} \geq 0$ and $H^p(\Sigma, \mathbb{R}) \neq 0$, then any harmonic p-form is parallel. In particular, Σ^n supports a parallel p-form.*

(iii) *If $\mathcal{B}^{[p]} \geq p(n-p)\Lambda$ for some $\Lambda > 0$, then*

$$\lambda_{1,p}(\Sigma) \geq p(n-p+1)\Lambda.$$

Proof. Let ω be a p-form. Taking the scalar product with ω on both sides of (1), we obtain

$$\langle \Delta\omega, \omega \rangle = |\nabla\omega|^2 + \langle \mathcal{B}^{[p]}\omega, \omega \rangle + \tfrac{1}{2}\Delta|\omega|^2. \tag{9}$$

Integrating on Σ (with respect to the canonical Riemannian measure), we get

$$\int_\Sigma \langle \Delta\omega, \omega\rangle = \int_\Sigma \bigl(|\nabla\omega|^2 + \langle \mathcal{B}^{[p]}\omega, \omega\rangle\bigr). \tag{10}$$

If we assume that $\mathcal{B}^{[p]} \geq 0$ and that ω is harmonic, we get

$$0 = \int_\Sigma \bigl(|\nabla\omega|^2 + \langle \mathcal{B}^{[p]}\omega, \omega\rangle\bigr) \geq 0,$$

which implies $|\nabla\omega| = 0$ and $\langle \mathcal{B}^{[p]}\omega, \omega\rangle = 0$ everywhere. It is well known that a harmonic form cannot vanish on an open set unless it is zero everywhere. If, at some point $x_0 \in \Sigma$, the strict inequality $\mathcal{B}^{[p]} > 0$ holds, we see that ω must vanish in a neighborhood of x_0, hence $\omega = 0$ everywhere. This proves (i). Assertion (ii) is immediate.

We now prove (iii). Let ω be a p-eigenform, so that $\Delta\omega = \lambda_{1,p}\omega$. By an inequality of Gallot and Meyer [1975] we have

$$|\nabla\omega|^2 \geq \frac{|d\omega|^2}{p+1} + \frac{|\delta\omega|^2}{n-p+1}.$$

Since $p \leq n/2$ we see that $p+1 \leq n-p+1$, hence

$$\int_\Sigma |\nabla\omega|^2 \geq \int_\Sigma \frac{|d\omega|^2 + |\delta\omega|^2}{n-p+1} = \frac{\lambda_{1,p}}{n-p+1}\int_\Sigma |\omega|^2.$$

Inserting this in (10) and using the lower bound on $\mathcal{B}^{[p]}$ we arrive easily at

$$\lambda_{1,p}\int_\Sigma |\omega|^2 \geq p(n-p+1)\Lambda \int_\Sigma |\omega|^2,$$

which gives the assertion. □

We already remarked that for $p = 1$, the Bochner operator $\mathcal{B}^1$ is simply given by the Ricci tensor acting on 1-forms. In particular, when $n = 2$, $\mathcal{B}^{[1]}$ is a scalar operator and is given by multiplication by the Gaussian curvature K_Σ of Σ: $\mathcal{B}^{[1]}\omega = K_\Sigma\omega$.

2.2. *Rigidity of Clifford tori.* For $p = 1, \dots, n-1$ and $r \in (0, 1)$, consider the manifold (Clifford torus)

$$T_{p,r} \doteq \mathbb{S}^p(r) \times \mathbb{S}^{n-p}\bigl(\sqrt{1-r^2}\bigr) \tag{11}$$

which is naturally isometrically embedded as a hypersurface in $\mathbb{S}^{n+1}$. Note that $T_{p,r}$ supports a parallel p-form, which is the pullback of the volume form of $\mathbb{S}^p(r)$ by the projection onto the first factor. We have the following rigidity theorem (when Σ is minimal, it reduces to [Colbois and Savo 2012, Theorem 10]).

Theorem 4. *Let Σ^n be a compact hypersurface of $\mathbb{S}^{n+1}$ supporting a (nontrivial) parallel p-form for some $p = 1, \dots, n-1$.*

(a) *If $n = 2$ and $p = 1$, then Σ is a flat 2-torus.*

(b) *If $n \geq 3$, then Σ is isometric to a Clifford torus $T_{p,r}$ for some $r \in (0, 1)$.*

For the proof, see Section 6. The interest in the theorem lies in the case $n \geq 3$. In fact, (a) holds for any compact, orientable surface supporting a parallel 1-form. We remark that there exist flat 2-tori in $\mathbb{S}^3$ which are not isometric to any Clifford torus $T_{1,r}$ (see [Weiner 1991] for a classification result).

Combining Theorem 4 with Proposition 3(ii), we obtain:

Corollary 5. *Let Σ^n be a compact hypersurface of $\mathbb{S}^{n+1}$ such that $H^p(\Sigma, \mathbb{R}) \neq 0$ and $\mathcal{B}^{[p]} \geq 0$ for some $p = 1, \dots, n-1$. Then Σ is a flat torus if $n = 2$ and, if $n \geq 3$, it is isometric to a Clifford torus $T_{p,r}$ for some $r \in (0, 1)$.*

We also record the following consequence.

Corollary 6. *Let Σ^n be a compact hypersurface of $\mathbb{S}^{n+1}$ having nonnegative sectional curvature. If $n \geq 3$, then either Σ is a homology sphere (i.e., $H^p(\Sigma, \mathbb{R}) = 0$ for all $p = 1, \dots, n-1$) or Σ is isometric to a Clifford torus.*

Proof. Let $(e_1, \dots, e_n)$ be an orthonormal basis of principal directions and let $k_1, \dots, k_n$ be the associated principal curvatures of Σ. Let $i \neq j$. By the Gauss formula $R(e_i, e_j, e_i, e_j) = 1 + k_i k_j$ and then, by our assumptions,

$$k_i k_j \geq -1.$$

If $\alpha \in I_p$ is any multi-index, we observe that $K_\alpha K_{\star\alpha}$ is a sum of $p(n-p)$ products of type $k_i k_j$ with $i \neq j$. Hence

$$K_\alpha K_{\star\alpha} \geq -p(n-p),$$

and by (7) we obtain $\mathcal{B}^{[p]}_{\text{ext}} \geq -p(n-p)$. As $\mathcal{B}^{[p]}_{\text{res}} = p(n-p)$ we conclude by Theorem 1 that $\mathcal{B}^{[p]} \geq 0$ for all $p = 1, \dots, n-1$. If $H^p(\Sigma, \mathbb{R}) \neq 0$ we see that Σ is a Clifford torus by Corollary 5. This completes the proof. $\square$

3. Applications

3.1. ***Applications in codimension one.*** Let Σ^n be a hypersurface of M^{n+1} and S its shape operator. Fix a point $x \in \Sigma$ and let $(k_1, \dots, k_n)$ be the principal curvatures of Σ at x. The (scalar) mean curvature is denoted by

$$H = \frac{1}{n}(k_1 + \cdots + k_n).$$

For each multi-index $\alpha = \{j_1, \dots, j_p\} \in I_p$, let $K_\alpha(x)$ be the corresponding p-curvature, as defined in (6). Set

$$\begin{cases} \beta_p(x) = \dfrac{1}{p(n-p)} \inf_{\alpha \in I_p} K_\alpha(x) K_{\star\alpha}(x), \\ \beta_p(\Sigma) = \inf_{x \in \Sigma} \beta_p(x). \end{cases} \tag{12}$$

From (7) we then see that at all points of Σ one has $\mathcal{B}^{[p]}_{\text{ext}} \geq p(n-p)\beta_p(\Sigma)$. If γ_M is a lower bound of the curvature operator of M^{n+1} then $\mathcal{B}^{[p]}_{\text{res}} \geq p(n-p)\gamma_M$, and then by Theorem 1

$$\mathcal{B}^{[p]} \geq p(n-p)(\gamma_M + \beta_p(\Sigma)). \tag{13}$$

From (13) and Proposition 3(iii) we immediately get the following estimate.

Theorem 7. *Let Σ^n be a compact hypersurface of M^{n+1}, a manifold with curvature operator bounded below by $\gamma_M \in \mathbb{R}$. Let $1 \leq p \leq n/2$. Then*

$$\lambda_{1,p}(\Sigma) \geq p(n-p+1)(\gamma_M + \beta_p(\Sigma)), \tag{14}$$

where $\beta_p(\Sigma)$ is defined by (12). If Σ is a geodesic sphere in a simply connected manifold of constant curvature γ_M, then equality holds.

To verify that the inequality is sharp, let $M^{n+1}(c)$ be the simply connected manifold of constant curvature $c = \gamma_M$. Then $M^{n+1}(c)$ is, respectively, the Euclidean space $\mathbb{R}^{n+1}$ when $c = 0$, the unit sphere $\mathbb{S}^{n+1}$ when $c = 1$ and the hyperbolic space $\mathbb{H}^{n+1}$ when $c = -1$. If Σ is a geodesic sphere in $M^{n+1}(c)$ then Σ is totally umbilical: $K_\alpha(x) = pH$, $K_{\star\alpha}(x) = (n-p)H$, where H is the mean curvature of Σ. Hence $\beta_p(x) = H^2$ and (14) becomes

$$\lambda_{1,p}(\Sigma) \geq p(n-p+1)(c + H^2).$$

On the other hand, Σ is known to be isometric to $\mathbb{S}^n(r)$, with $r = (c + H^2)^{-\frac{1}{2}}$. Therefore, by a well-known calculation in [Gallot and Meyer 1975],

$$\lambda_{1,p}(\Sigma) = p(n-p+1)(c + H^2).$$

This shows that Theorem 7 is sharp. Note also that the condition $\beta_p(\Sigma) > -\gamma_M$ implies $H^p(\Sigma, \mathbb{R}) = H^{n-p}(\Sigma, \mathbb{R}) = 0$. For an upper bound of $\lambda_{1,p}(\Sigma)$ when M is a sphere, see [Savo 2005].

We now compare this estimate with the results in [Raulot and Savo 2011]. Let us say that Σ is *p-convex* if we can choose an orientation of Σ so that all of its p-curvatures are nonnegative at every point: $K_\alpha(x) \geq 0$ for all $\alpha \in I_p$ and $x \in \Sigma$. This is equivalent to asking that the operator $S^{[p]}$ is nonnegative at every point.

Clearly, if Σ is p-convex, then it is q-convex for all $q \geq p$. In that case, let

$$\mathcal{K}_p = \frac{1}{p} \inf_{x \in \Sigma} \inf_{\alpha \in I_p} K_\alpha(x) \geq 0$$

be a lower bound of the *mean p-curvatures* (note that $\mathcal{K}_1$ is a lower bound of the principal curvatures, while $\mathcal{K}_n$ is a lower bound of the mean curvature). It is easy to prove that $\mathcal{K}_p \leq \mathcal{K}_q$ whenever $p \leq q$. Using a Reilly-type formula for differential forms, it is proved in [Raulot and Savo 2011] that if Σ is the boundary of a domain in a manifold M^{n+1} with nonnegative curvature operator ($\gamma_M \geq 0$), and if Σ is p-convex (for the orientation given by the inner unit normal), then

$$\lambda_{1,p}(\Sigma) \geq p(n-p+1)\mathcal{K}_p \mathcal{K}_{n-p+1} \tag{15}$$

with equality if and only if Σ is a sphere in $\mathbb{R}^{n+1}$. Numerical examples show that, in that situation, (14) is often better than (15). Moreover, (14) applies also in negative curvature, and for immersions which are not necessarily embeddings. For example, if M^{n+1} is the hyperbolic space $\mathbb{H}^{n+1}$, it is easy to see that the inequality gives a positive lower bound whenever $\mathcal{K}_p > 1$.

The next application is inspired by the following result of [Alencar and do Carmo 1994]. Assume that Σ^n is a compact hypersurface of $\mathbb{S}^{n+1}$ with constant mean curvature H. Let

$$\Phi = S - HI$$

be the *traceless* second fundamental form of Σ, and let $R(1, H)$ be the positive root of the polynomial

$$F_1(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(H^2+1). \tag{16}$$

Alencar and do Carmo [1994] proved that if $|\Phi| \leq R(1, H)$, then Σ is either totally umbilical ($|\Phi| = 0$) or a Clifford torus

$$T_{1,r} = \mathbb{S}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1-r^2})$$

with $r \geq \sqrt{1/n}$ (in which case $|\Phi| = R(1, H)$). To the best of our knowledge, there is still no similar characterization of the other Clifford tori $T_{p,r}$ for $2 \leq p \leq n-2$ among constant mean curvature hypersurfaces of the sphere.

We prove a vanishing result for the de Rham cohomology in degree p assuming that the norm of the traceless second fundamental form is bounded above by a suitable function of the mean curvature. When the ambient manifold is the sphere, this will lead to a topological version of the Alencar–do Carmo result, in which the assumption of constant mean curvature is replaced by an assumption of nontrivial cohomology in degree p.

More precisely, fix real numbers γ, H and an integer $1 \le p \le n/2$. Consider the polynomial function

$$F_p(x) = x^2 + \frac{n|n-2p|}{\sqrt{pn(n-p)}}|H|x - n(H^2+\gamma). \tag{17}$$

Note that $F_p(x)$ reduces to (16) when $p = \gamma = 1$. If $\gamma < 0$, we assume that $H^2 + \gamma \ge 0$. Then, the largest root of $F_p(x)$ is nonnegative, and will be denoted by $R(p, H)$. It is easy to check that for fixed H and γ one has

$$0 \le R(1, H) \le R(2, H) \le \cdots \le R\Big(\Big[\frac{n}{2}\Big], H\Big)$$

where $[n/2]$ is the largest integer less than or equal to $n/2$. If n is even, one has

$$R\Big(\frac{n}{2}, H\Big) = \sqrt{n(H^2+\gamma)}.$$

We then have the following result.

Theorem 8. *Let Σ^n be a compact hypersurface of M^{n+1}, a manifold with curvature operator bounded below by $\gamma \in \mathbb{R}$. We suppose $p \le n/2$ and $H^2 + \gamma \ge 0$. If*

$$|\Phi| \le R(p, H)$$

everywhere on Σ and strict inequality holds somewhere, then

$$H^k(\Sigma, \mathbb{R}) = 0 \quad \textit{for all } k = p, \ldots, n-p.$$

Here Φ is the traceless second fundamental form of Σ, and $R(p, H) \ge 0$ is the largest root of (17). In particular, if $|\Phi| \le R(1, H)$, with strict inequality somewhere, then Σ is a homology sphere.

The proof is given in Section 4.1. Now let Σ^n be a compact hypersurface of $\mathbb{S}^{n+1}$; we take $\gamma = 1$ and then consider the polynomial

$$F_p(x) = x^2 + \frac{n|n-2p|}{\sqrt{pn(n-p)}}|H|x - n(H^2+1). \tag{18}$$

We have the following rigidity result, which shows that the conditions in the previous theorem are sharp.

Theorem 9. *Let Σ^n be a compact hypersurface of $\mathbb{S}^{n+1}$, take $1 \le p \le n/2$ and denote by $R(p, H)$ the positive root of (18). We also assume $n \ge 3$.*

(a) *Let $1 \le p < n/2$. If*

$$|\Phi| \le R(p, H) \quad \textit{and} \quad H^p(\Sigma, \mathbb{R}) \ne 0,$$

then Σ is isometric to a Clifford torus $T_{p,r} = \mathbb{S}^p(r) \times \mathbb{S}^{n-p}\big(\sqrt{1-r^2}\big)$ for some $r \ge \sqrt{p/n}$.

(b) *Let n be even and $p = n/2$. If we assume that*

$$|\Phi|^2 \le n(1+H^2) \quad \textit{and} \quad H^p(\Sigma,\mathbb{R}) \neq 0,$$

then Σ is isometric to a Clifford torus $\mathbb{S}^p(r) \times \mathbb{S}^p\big(\sqrt{1-r^2}\big)$ for some $r \in (0,1)$.

For the proof, see Section 4.1.

3.2. ***Applications in arbitrary codimension.*** Lawson and Simons [1973] proved the following vanishing result. If Σ^n is a compact, immersed submanifold of a sphere $\mathbb{S}^{n+q}$ and if at all points of Σ one has

$$|S|^2 < \min\big\{p(n-p),\ 2\sqrt{p(n-p)}\big\}$$

for some $p = 1, \dots, n-1$, then $H^p(\Sigma,\mathbb{Z}) = H^{n-p}(\Sigma,\mathbb{Z}) = 0$. The proof depends on deep results of geometric measure theory, and used the fact that any integral homology class is represented by a stable current. Taking variations induced by suitable vector fields (namely, orthogonal projections to Σ of parallel vector fields on $\mathbb{R}^{n+2}$), one gets the stated result.

If we limit ourselves to real cohomology theory, we have another proof of this result by a completely different method (the Bochner method) as follows. Note that this also gives an explicit lower bound of the spectrum of the Hodge Laplacian and an associated rigidity result (Theorem 11).

Theorem 10. *Let Σ^n be a compact submanifold of M^{n+q}, a manifold with curvature operator bounded below by $\gamma_M > 0$, and let $1 \le p \le n/2$. If*

$$|S|^2 \le 2\gamma_M\sqrt{p(n-p)}$$

and strict inequality holds somewhere, then $H^k(\Sigma,\mathbb{R}) = 0$ for all $k = p, \dots, n-p$.

More generally, if

$$|S|^2 \le 2\gamma_M\sqrt{p(n-p)}(1-\Lambda)$$

for some $\Lambda \in (0,1]$, then $\lambda_{1,p}(\Sigma) \ge p(n-p+1)\gamma_M\Lambda$.

For the proof, see Section 4.3.

In codimension one the condition is sharp, and our approach gives the following rigidity result.

Theorem 11. *Let Σ^n be a compact hypersurface of $\mathbb{S}^{n+1}$, $n \ge 2$, such that*

$$|S|^2 \le 2\sqrt{p(n-p)} \quad \textit{and} \quad H^p(\Sigma,\mathbb{R}) \neq 0$$

for some $1 \le p \le n/2$. Then Σ is isometric to the Clifford torus $T_{p,r} = \mathbb{S}^p(r) \times \mathbb{S}^{n-p}\big(\sqrt{1-r^2}\big)$ for

$$r = \left(\frac{\sqrt{p}}{\sqrt{p}+\sqrt{n-p}}\right)^{\frac{1}{2}}.$$

In fact, one can check that the Clifford torus of Theorem 11 minimizes $|S|^2$ among the family of Clifford tori $\{T_{p,r} : r \in (0, 1)\}$. The proof is given in Section 4.4.

3.3. ***Submanifolds with a parallel p-form.*** In our final application, we study immersions of a manifold supporting a (nontrivial) parallel p-form (our estimates are local, and so we do not assume compactness). Noteworthy examples of such manifolds are given by:

- Riemannian products $N_1 \times N_2$ (having parallel forms in degrees $p = \dim N_1$, $\dim N_2$).
- Kähler manifolds.

In fact, the Kähler 2-form Ω is parallel. As all powers of Ω are nontrivial (and parallel), we see that a Kähler manifold supports nontrivial parallel forms in all even degrees.

It is well known that a Kähler manifold does not admit any minimal immersion into a hyperbolic space (see [Dajczer and Rodríguez 1986; El Soufi and Petit 2000]). More generally, in [Grosjean 2004], it is proved that a manifold Σ with a parallel p-form does not admit any minimal immersion into a manifold M if certain curvature conditions on Σ and M are met. We also refer to [Grosjean 2004] for other rigidity results on minimal immersions.

Our point of view is to observe that if ω is a parallel p-form, then $\langle \mathcal{B}^{[p]}\omega, \omega\rangle = 0$ everywhere on Σ. More generally, if ω is a harmonic p-form with constant length, then, from the Bochner formula (9),

$$\langle \mathcal{B}^{[p]}\omega, \omega\rangle = -|\nabla\omega|^2 \le 0.$$

Using the pointwise bounds on the eigenvalues of $\mathcal{B}^{[p]}$ derived in Section 4, we then obtain pointwise bounds for the extrinsic geometry of Σ. Precisely:

Theorem 12. *Let Σ^n be an immersed submanifold of M^{n+q} and let $p = 1, \dots, n-1$.*

(a) *If Σ supports a parallel p-form and M has curvature operator bounded above by $\Gamma_M < 0$ then*

$$|H|^2 \ge \frac{4p(n-p)}{n^2}|\Gamma_M| \quad \textit{and} \quad |S|^2 \ge 2|\Gamma_M|\sqrt{p(n-p)}$$

at all points of Σ.

(b) *If Σ supports a harmonic p-form of constant length* (*in particular*, *a parallel p-form*) *and M has curvature operator bounded below by $\gamma_M > 0$ then*

$$|S|^2 \ge 2\gamma_M\sqrt{p(n-p)}$$

at all points of Σ.

For the proof, see Section 4.5. Assertion (a) is sharp when $p = n/2$ (see below) and (b) is sharp for the Clifford torus of Theorem 11.

Now let Σ^{2m} be a Kähler manifold of complex dimension $m \geq 2$. Since Σ supports nontrivial parallel forms in all even degrees, we see that it supports a parallel form of degree m or $m-1$ depending on whether m is even or odd. Applying the previous theorem we immediately get the following estimates.

Corollary 13. *Let Σ^{2m} be a Kähler manifold of complex dimension $m \geq 2$ isometrically immersed in the Riemannian manifold M^{2m+q}.*

(a) *If M has curvature operator bounded above by $\Gamma_M < 0$, then at all points of Σ*

$$|H|^2 \geq \begin{cases} |\Gamma_M| & \text{if } m \text{ is even}, \\ \dfrac{m^2-1}{m^2}|\Gamma_M| & \text{if } m \text{ is odd}. \end{cases}$$

(b) *If M has curvature operator bounded below by $\gamma_M > 0$, then*

$$|S^2| \geq \begin{cases} 2m\gamma_M & \text{if } m \text{ is even}, \\ 2\gamma_M\sqrt{m^2-1} & \text{if } m \text{ is odd}. \end{cases}$$

We remark that if m is even and the ambient space is the hyperbolic space $\mathbb{H}^{2m+q}$ then (a) gives

$$|H|^2 \geq 1,$$

which is an equality when $\Sigma^{2m} = \mathbb{R}^{2m}$, embedded in $\mathbb{H}^{2m+1}$ as a horosphere.

4. Estimates of the Bochner operator

In this section we first estimate the extrinsic part of the Bochner operator, thanks to Lemma 2 and some elementary algebra. We then apply these estimates to prove the theorems of Section 3. Let Σ^n be a submanifold of the Riemannian manifold M^{n+q}. We start from the following algebraic lemma.

Lemma 14. *Let $S = S_\nu$ be the shape operator of Σ relative to a unit normal vector $\nu \in T^\perp\Sigma$, and let $T^{[p]} = (\operatorname{tr} S)S^{[p]} - S^{[p]} \circ S^{[p]}$. If $k_1, \dots, k_n$ are the eigenvalues of S, set*

$$nH = \sum_{j=1}^n k_j, \quad |S|^2 = \sum_{j=1}^n k_j^2, \quad |\Phi|^2 = \sum_{j=1}^n (k_j - H)^2. \tag{19}$$

Then the following inequalities for $T^{[p]}$ hold. Recall that if Σ has codimension one, then $T^{[p]} = \mathcal{B}^{[p]}_{\text{ext}}$.

(a) $-\dfrac{\sqrt{p(n-p)}}{2}|S|^2 \leq T^{[p]} \leq \dfrac{\sqrt{p(n-p)}}{2}|S|^2.$

(b) *If* $H=0$, *then*

$$-\frac{p(n-p)}{n}|S|^2 \le T^{[p]} \le 0.$$

(c) $\displaystyle \frac{n^2|H|^2}{4}-\frac{1}{4}\left(|n-2p||H|+\sqrt{\frac{4p(n-p)}{n}}|\Phi|\right)^2 \le T^{[p]} \le \frac{n^2|H|^2}{4}.$

(d) *If* n *is even and* $p=n/2$, *then*

$$T^{[p]} \ge \tfrac{1}{4}n^2|H|^2-\tfrac{1}{4}n|\Phi|^2.$$

Proof. We know from Lemma 2 that the eigenvalues of $T^{[p]}$ are given by $\lambda_\alpha = K_\alpha K_{\star\alpha}$, where α runs over the set of p-multi-indices. Then, it is enough to show the inequalities for any such eigenvalue. Fix a multi-index α. After reordering, we can assume that

$$\lambda_\alpha = (k_1+\cdots+k_p)(k_{p+1}+\cdots+k_n). \tag{20}$$

In conclusion, it is enough to show the given bounds for the product (20), given any set of real numbers $k_1,\ldots,k_n$ satisfying (19).

Proof of (a). We use the Schwarz inequality and the inequality $\sqrt{ab}\le(a+b)/2$ applied to $a=k_1^2+\cdots+k_p^2$, $b=k_{p+1}^2+\cdots+k_n^2$. We obtain

$$\begin{aligned}|\lambda_\alpha| &= |k_1+\cdots+k_p||k_{p+1}+\cdots+k_n|\\ &\le \sqrt{p(n-p)}\sqrt{k_1^2+\cdots+k_p^2}\sqrt{k_{p+1}^2+\cdots+k_n^2} \le \tfrac{1}{2}(\sqrt{p(n-p)})|S|^2,\end{aligned}$$

and (a) follows.

Proof of (b). Since $k_{p+1}+\cdots+k_n=-(k_1+\cdots+k_p)$, the Schwarz inequality yields

$$\begin{aligned}\lambda_\alpha &= -(k_1+\cdots+k_p)^2 \ge -p\big(k_1^2+\cdots+k_p^2\big),\\ \lambda_\alpha &= -(k_{p+1}+\ldots+k_n)^2 \ge -(n-p)\big(k_{p+1}^2+\cdots+k_n^2\big).\end{aligned}$$

Summing the two inequalities, we get

$$\frac{\lambda_\alpha}{p}+\frac{\lambda_\alpha}{n-p} \ge -|S|^2,$$

from which the lower bound follows. The upper bound is obvious.

Proof of (c). As $\sum_{j=1}^{n}(k_j-H)=0$, we see that, by the lower bound in (b),

$$((k_1-H)+\cdots+(k_p-H))((k_{p+1}-H)+\cdots+(k_n-H)) \ge -\frac{p(n-p)}{n}|\Phi|^2.$$

Hence

$$
\begin{aligned}
(21)\quad & -\frac{p(n-p)}{n}|\Phi|^2 \\
& \le (k_1+\cdots+k_p-pH)(k_{p+1}+\cdots+k_n-(n-p)H) \\
& = \lambda_\alpha + p(n-p)H^2 - pH(k_{p+1}+\cdots+k_n) - (n-p)H(k_1+\cdots+k_p).
\end{aligned}
$$

Substituting $k_{p+1}+\cdots+k_n = nH-(k_1+\cdots+k_p)$ in (21), we have

$$
(22)\qquad -\frac{p(n-p)}{n}|\Phi|^2 \le \lambda_\alpha - p^2H^2 - (n-2p)H(k_1+\cdots+k_p).
$$

Substituting $k_1+\cdots+k_p = nH-(k_{p+1}+\cdots+k_n)$ in (21), we also have

$$
(23)\qquad -\frac{p(n-p)}{n}|\Phi|^2 \le \lambda_\alpha - (n-p)^2H^2 + (n-2p)H(k_{p+1}+\cdots+k_n).
$$

We now sum (22) and (23) to obtain

$$
\begin{aligned}
(24)\quad & 2\lambda_\alpha - \big(p^2+(n-p)^2\big)H^2 + \frac{2p(n-p)}{n}|\Phi|^2 \\
& \ge (n-2p)H((k_1+\cdots+k_p)-(k_{p+1}+\cdots+k_n)) \\
& \ge -|n-2p||H||(k_1+\cdots+k_p)-(k_{p+1}+\cdots+k_n)|.
\end{aligned}
$$

Set $a=k_1+\cdots+k_p$, $b=k_{p+1}+\cdots+k_n$. As $|a-b|^2=(a+b)^2-4ab$, we see that

$$
n^2H^2-4\lambda_\alpha \ge 0,
$$

which is the upper bound in (c), and $|a-b|=\sqrt{n^2H^2-4\lambda_\alpha}$. Substituting in (24),

$$
(25)\quad 2\lambda_\alpha - \big(p^2+(n-p)^2\big)H^2 + \frac{2p(n-p)}{n}|\Phi|^2 \ge -|n-2p||H|\sqrt{n^2H^2-4\lambda_\alpha}.
$$

If we set $\delta=\sqrt{n^2H^2-4\lambda_\alpha}\ge 0$, then (25) takes the form

$$
\delta^2 - 2|n-2p||H|\delta + (n-2p)^2H^2 - \frac{4p(n-p)}{n}|\Phi|^2 \le 0,
$$

which implies

$$
\delta \le |n-2p||H| + \sqrt{\frac{4p(n-p)}{n}}\,|\Phi|.
$$

Recalling the definition of δ, one concludes that

$$
4\lambda_\alpha \ge n^2H^2 - \left(|n-2p||H| + \sqrt{\frac{4p(n-p)}{n}}\,|\Phi|\right)^2,
$$

which is the lower bound in (c). Finally, (d) is a particular case of (c). $\square$

4.1. ***Proofs of Theorems 8 and 9.*** Let Σ^n be a hypersurface of M^{n+1}, a manifold with curvature operator bounded below by $\gamma \in \mathbb{R}$. Let H be the mean curvature of Σ and Φ its traceless second fundamental form. Recall that

$$F_p(x) = x^2 + \frac{n|n-2p|}{\sqrt{pn(n-p)}}|H|x - n(H^2+\gamma).$$

The theorems are a result of the following bound for $\mathcal{B}^{[p]}$.

Proposition 15. *In the above notation, one has*

$$\mathcal{B}^{[p]} \geq -\frac{p(n-p)}{n} F_p(|\Phi|).$$

Proof. By Lemma 14(c) we have

$$\mathcal{B}^{[p]}_{\text{ext}} = T^{[p]} \geq \frac{n^2|H|^2}{4} - \frac{1}{4}\left(|n-2p||H| + \sqrt{\frac{4p(n-p)}{n}}|\Phi|\right)^2.$$

Then, as $\mathcal{B}^{[p]}_{\text{res}} \geq p(n-p)\gamma$ and $\mathcal{B}^{[p]} = \mathcal{B}^{[p]}_{\text{res}} + \mathcal{B}^{[p]}_{\text{ext}}$,

$$\begin{aligned}
4\mathcal{B}^{[p]} &\geq 4p(n-p)\gamma + 4\mathcal{B}^{[p]}_{\text{ext}} \\
&\geq 4p(n-p)\gamma + n^2|H|^2 - \left(|n-2p||H| + \sqrt{\frac{4p(n-p)}{n}}|\Phi|\right)^2 \\
&= 4p(n-p)\gamma + 4p(n-p)|H|^2 - \frac{4p(n-p)}{n}|\Phi|^2 \\
&\qquad\qquad - 2|n-2p|\sqrt{\frac{4p(n-p)}{n}}|\Phi||H| \\
&= \frac{4p(n-p)}{n}\left(n\big(\gamma + |H|^2\big) - |\Phi|^2 - \frac{n|n-2p|}{\sqrt{pn(n-p)}}|\Phi||H|\right) \\
&= -\frac{4p(n-p)}{n} F_p(|\Phi|)
\end{aligned}$$

and the assertion follows. □

Proof of Theorem 8. By assumption, $|\Phi| \leq R(p, H)$, hence $F_p(|\Phi|) \leq 0$ by the definition of $R(p, H)$. By Proposition 15 we see that

$$\mathcal{B}^{[p]} \geq 0.$$

As the inequality is strict somewhere, we can apply the Bochner method and conclude that $H^p(\Sigma, \mathbb{R}) = 0$. By Poincaré duality, one has also $H^{n-p}(\Sigma, \mathbb{R}) = 0$. Since $R(p, H)$ is nondecreasing in p, we see that $|\Phi| \leq R(k, H)$ for all $p \leq k \leq [n/2]$ and the conclusion follows. □

Proof of Theorem 9. Under the given assumptions, one has $\mathcal{B}^{[p]}\geq 0$ and $H^p(\Sigma,\mathbb{R})\neq 0$. Then, Σ is a Clifford torus by Corollary 5. Conversely, the Clifford torus $T_{p,r}$ obviously satisfies $H^p(T_{p,r})\neq 0$. Moreover, $T_{p,r}$ is known to have two distinct principal curvatures given (up to sign) by

$$\begin{cases} \lambda=\dfrac{\sqrt{1-r^2}}{r} & \text{with multiplicity } p,\\ \mu=-\dfrac{r}{\sqrt{1-r^2}} & \text{with multiplicity } n-p. \end{cases} \tag{26}$$

Therefore

$$|H|=\frac{|nr^2-p|}{nr\sqrt{1-r^2}}\quad\text{and}\quad|\Phi|=\sqrt{\frac{p(n-p)}{n}}\frac{1}{r\sqrt{1-r^2}}.$$

A straightforward calculation shows that if $p<n/2$ and $r^2\geq p/n$, then $F_p(|\Phi|)=0$, that is, $|\Phi|=R(p,H)$. If $p=n/2$ then $|\Phi|=R(p,H)=\sqrt{n(1+H^2)}$ for all $r\in(0,1)$. The proof is complete. □

4.2. *An estimate in higher codimensions.*

Proposition 16. *Let Σ^n be a submanifold of the manifold M^{n+q} having curvature operator bounded below by γ_M. Then*

$$\mathcal{B}^{[p]}\geq p(n-p)\biggl(\gamma_M-\frac{|S|^2}{2\sqrt{p(n-p)}}\biggr).$$

If M has curvature operator bounded above by Γ_M, then

$$\mathcal{B}^{[p]}\leq p(n-p)\biggl(\Gamma_M+\frac{|S|^2}{2\sqrt{p(n-p)}}\biggr)\quad\textit{and}\quad\mathcal{B}^{[p]}\leq p(n-p)\Gamma_M+\frac{n^2|H|^2}{4}.$$

Proof. Let $(\nu_1,\dots,\nu_q)$ be an orthonormal frame in the normal bundle. We know from the main theorem that

$$\mathcal{B}^{[p]}_{\mathrm{ext}}=\sum_{j=1}^q T^{[p]}_{\nu_j},\quad\text{where } T^{[p]}_{\nu_j}=(\operatorname{tr}S_{\nu_j})S^{[p]}_{\nu_j}-S^{[p]}_{\nu_j}\circ S^{[p]}_{\nu_j}.$$

If $\lambda_1\bigl(T^{[p]}_{\nu_j}\bigr)$ denotes the lowest eigenvalue of $T^{[p]}_{\nu_j}$, we see that $\mathcal{B}^{[p]}_{\mathrm{ext}}\geq\sum_{j=1}^q\lambda_1(T^{[p]}_{\nu_j})$. From Lemma 14(a) applied to $S=S_{\nu_j}$, we obtain

$$\lambda_1(T^{[p]}_{\nu_j})\geq-\frac{\sqrt{p(n-p)}}{2}|S_{\nu_j}|^2.$$

Summing over j, we get

$$\mathcal{B}^{[p]}_{\mathrm{ext}}\geq-\frac{\sqrt{p(n-p)}}{2}|S|^2.$$

From Theorem 1 and the above,

$$\begin{aligned}\mathcal{B}^{[p]} &= \mathcal{B}^{[p]}_{\mathrm{res}} + \mathcal{B}^{[p]}_{\mathrm{ext}} \\ &\geq p(n-p)\gamma_M - \frac{\sqrt{p(n-p)}}{2}|S|^2 \\ &= p(n-p)\biggl(\gamma_M - \frac{|S|^2}{2\sqrt{p(n-p)}}\biggr),\end{aligned}$$

as asserted. The other inequalities are proved similarly, using Lemma 14. □

4.3. ***Proof of Theorem 10.*** If

$$|S|^2 \leq 2\gamma_M \sqrt{p(n-p)}\,(1-\Lambda),$$

for some $\Lambda \in [0, 1]$, we see from the first estimate of Proposition 16 that

$$\mathcal{B}^{[p]} \geq p(n-p)\Lambda\gamma_M.$$

The assertions follow immediately from the Bochner method (Proposition 3). □

4.4. ***Proof of Theorem 11.*** Let $n \geq 3$. Together with Proposition 16, the assumptions give $\mathcal{B}^{[p]} \geq 0$; as $H^p(\Sigma, \mathbb{R}) \neq 0$, we get immediately that Σ must be a Clifford torus $T_{p,r}$ by Corollary 5. On the other hand, it is seen from (26) that the only Clifford torus satisfying $|S|^2 \leq 2\sqrt{p(n-p)}$ is the one corresponding to the stated value of r.

Now assume $n = 2$ and $p = 1$, so that $|S|^2 \leq 2$. We know that $\mathcal{B}^{[1]}$ is multiplication by the Gaussian curvature K_Σ of Σ. From the formula $4H^2 = |S|^2 + 2K_\Sigma - 2$ we obtain $K_\Sigma \geq 2H^2 \geq 0$. The assumption $H^1(\Sigma, \mathbb{R}) \neq 0$ and the Bochner formula force $K_\Sigma = 0$, hence Σ is a minimal flat torus. As such, it is isometric with $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ and the assertion follows. □

4.5. ***Proof of Theorem 12.*** Assume that ω is a parallel p-form. Then $\mathcal{B}^{[p]}\omega = 0$ identically and, by the last upper bound in Proposition 16,

$$0 = \langle \mathcal{B}^{[p]}\omega, \omega\rangle \leq \biggl(p(n-p)\Gamma_M + \frac{n^2|H|^2}{4}\biggr)|\omega|^2.$$

As $|\omega|$ is a positive constant, the assertion follows.

Now assume that ω is a harmonic p-form with constant length. From the Bochner formula (9) we see that $\langle \mathcal{B}^{[p]}\omega, \omega\rangle = -|\nabla\omega|^2 \leq 0$ at every point. Hence, applying Proposition 16,

$$0 \geq \langle \mathcal{B}^{[p]}\omega, \omega\rangle \geq p(n-p)\biggl(\gamma_M - \frac{|S|^2}{2\sqrt{p(n-p)}}\biggr)|\omega|^2,$$

which implies $|S|^2 \geq 2\gamma_M\sqrt{p(n-p)}$ everywhere, as asserted. □

5. Proof of the main theorem

Let Σ^n be a Riemannian manifold and R its curvature tensor, defined on tangent vectors X, Y by

$$R(X,Y) = -\nabla_X\nabla_Y + \nabla_Y\nabla_X + \nabla_{[X,Y]}. \tag{27}$$

The Bochner curvature term acting on p-forms is given by

$$\mathcal{B}^{[p]}\omega(X_1,\dots,X_p) = \sum_{j=1}^{n}\sum_{k=1}^{p}(-1)^k(R(e_j,X_k)\omega)(e_j,X_1,\dots,\hat{X}_k,\dots,X_p)$$

where $(e_1,\dots,e_n)$ is an orthonormal frame in $T\Sigma$ and $X_1,\dots,X_p$ are arbitrary tangent vectors. However, in our proof of Theorem 1, we follow the approach of [Petersen 1998], which uses the formalism of Clifford multiplication, and allows to express $\mathcal{B}^{[p]}$ directly in terms of the curvature operator (see Theorem 17 below). Our Theorem 1 follows from Theorem 17 and the splitting of the curvature operator induced by the Gauss formula. The next section relies on the exposition in [Petersen 1998, Section 7.4], which we follow closely; the only difference is the sign of the Riemann tensor.

5.1. ***The Bochner operator in the Clifford formalism.*** Let $\Lambda^\star(\Sigma)$ be the algebra of forms on Σ. Given $\theta\in\Lambda^1$ and $\omega\in\Lambda^p$, define their Clifford multiplication by

$$\begin{cases}\theta\cdot\omega = \theta\wedge\omega - i_{\theta^\#}\omega,\\ \omega\cdot\theta = (-1)^p(\theta\wedge\omega + i_{\theta^\#}\omega),\end{cases}$$

where $i_{\theta^\#}$ denotes interior multiplication of a form by $\theta^\#$, the dual vector field of θ. Note that by demanding that the product be bilinear and associative, the preceding equalities extend uniquely to define the Clifford multiplication of a p-form by a q-form. For 1-forms,

$$\begin{cases}\theta\cdot\theta = -|\theta|^2,\\ \theta_1\cdot\theta_2+\theta_2\cdot\theta_1 = -2\langle\theta_1,\theta_2\rangle,\end{cases} \tag{28}$$

hence orthogonal 1-forms anticommute; moreover, any two orthogonal forms satisfy

$$\omega_1\cdot\omega_2 = \omega_1\wedge\omega_2. \tag{29}$$

Define the bracket as usual: $[\omega_1,\omega_2]=\omega_1\cdot\omega_2-\omega_2\cdot\omega_1$. If θ is a 1-form and ψ is a 2-form, one checks that for all forms ω_1,ω_2

$$\begin{cases}\langle\theta\cdot\omega_1,\omega_2\rangle = -\langle\omega_1,\theta\cdot\omega_2\rangle,\\ \langle[\psi,\omega_1],\omega_2\rangle = -\langle\omega_1,[\psi,\omega_2]\rangle.\end{cases} \tag{30}$$

Fix an orthonormal frame $(e_1, \dots, e_n)$ with dual coframe $(\theta_1, \dots, \theta_n)$, and define the Dirac operator on forms $D : \Lambda^\star(\Sigma) \to \Lambda^\star(\Sigma)$ by

$$D\omega = \sum_{j=1}^{n} \theta_j \cdot \nabla_{e_j}\omega.$$

As $d\omega = \sum_{j=1}^{n} \theta_j \wedge \nabla_{e_j}\omega$ and $\delta\omega = -\sum_{j=1}^{n} i_{\theta_j^\#}\nabla_{e_j}\omega$, one sees that $D = d + \delta$, hence

$$D^2 = \Delta,$$

where Δ is the Laplacian on forms. Theorem 4.5 of [Petersen 1998] proves that

$$D^2\omega = \nabla^\star\nabla\omega + \frac{1}{2}\sum_{i,j=1}^{n} R(e_i, e_j)\omega \cdot \theta_i \cdot \theta_j,$$

$$D^2\omega = \nabla^\star\nabla\omega - \frac{1}{2}\sum_{i,j=1}^{n} \theta_i \cdot \theta_j \cdot R(e_i, e_j)\omega.$$

(As already observed, the change of sign in our formula is due to the opposite sign convention for the Riemann tensor adopted by Petersen.)

As $D^2\omega = \Delta\omega$, summing the two relations and dividing by 2 we then see

$$\mathcal{B}^{[p]}\omega = \frac{1}{4}\sum_{i,j=1}^{n} [R(e_i, e_j)\omega, \theta_i \cdot \theta_j]. \tag{31}$$

Now recall that the curvature operator $\mathcal{R} : \Lambda_2(\Sigma) \to \Lambda_2(\Sigma)$ is the self-adjoint operator uniquely determined by the formula

$$\langle \mathcal{R}(X \wedge Y), Z \wedge T\rangle = R(X, Y, Z, T) \doteq \langle R(X, Y)Z, T\rangle. \tag{32}$$

for all tangent vectors X, Y, Z, T. Then, we arrive at the following description of $\mathcal{B}^{[p]}$ in terms of $\mathcal{R}$.

Theorem 17. *Let Σ be a manifold, and let $\mathcal{B}^{[p]}$ be the Bochner operator acting on p-forms of Σ. At any point of Σ, fix any orthonormal basis $\{\xi_r\}$ of $\Lambda_2(\Sigma)$ (here $r = 1, \dots, \binom{n}{2}$) and let $\{\hat{\xi}_r\}$ be its dual basis. Then*

$$\langle \mathcal{B}^{[p]}\omega, \phi\rangle = \frac{1}{4}\sum_{r,s}\langle \mathcal{R}\xi_r, \xi_s\rangle\langle[\hat{\xi}_r, \omega], [\hat{\xi}_s, \phi]\rangle,$$

where the bracket is relative to Clifford multiplication and $\mathcal{R}$ is the curvature operator of Σ.

For the proof, we start from [Petersen 1998, Lemma 4.7], which gives

$$R(e_i, e_j)\omega = \frac{1}{4}\sum_{h,k=1}^{n} \langle R(e_h, e_k)e_i, e_j\rangle[\theta_h \cdot \theta_k, \omega].$$

By (31),

$$\mathcal{B}^{[p]}\omega = \frac{1}{4}\sum_{\substack{i<j\\ h<k}} \langle \mathcal{R}(e_h \wedge e_k), e_i \wedge e_j\rangle [[\theta_h \cdot \theta_k, \omega], \theta_i \cdot \theta_j]. \tag{33}$$

By the adjointness property (30), we see that if ϕ is another p-form,

$$\langle [[\theta_h \cdot \theta_k, \omega], \theta_i \cdot \theta_j],\ \phi\rangle = -\langle [\theta_i \cdot \theta_j, [\theta_h \cdot \theta_k, \omega]],\ \phi\rangle = \langle [\theta_h \cdot \theta_k, \omega],\ [\theta_i \cdot \theta_j, \phi]\rangle,$$

and then

$$\langle \mathcal{B}^{[p]}\omega, \phi\rangle = \frac{1}{4}\sum_{\substack{i<j\\ h<k}} \langle \mathcal{R}(e_h \wedge e_k), e_i \wedge e_j\rangle \langle [\theta_h \cdot \theta_k, \omega],\ [\theta_i \cdot \theta_j, \phi]\rangle.$$

This is the expression in the orthonormal basis $\{\xi_r\} = \{e_i \wedge e_j\}_{i<j}$ of $\Lambda_2(\Sigma)$. Obviously, the choice of the orthonormal basis is not important and the theorem follows. □

5.2. ***A splitting of the curvature operator.*** Assume that Σ^n is a submanifold of M^{n+q}. Let R be the Riemann tensor of Σ and R^M that of M. For $X, Y, Z, T \in T\Sigma$, the Gauss formula gives

$$R(X, Y, Z, T) = R^M(X, Y, Z, T) + R_{\text{ext}}(X, Y, Z, T),$$

where

$$R_{\text{ext}}(X, Y, Z, T) = \langle L(X, Z), L(Y, T)\rangle - \langle L(X, T), L(Y, Z)\rangle$$

and L is the second fundamental form. Accordingly, we can split the curvature operator $\mathcal{R}$ of Σ as the sum of two self-adjoint operators acting on $\Lambda_2(\Sigma)$,

$$\mathcal{R} = \mathcal{R}_{\text{res}} + \mathcal{R}_{\text{ext}},$$

which are respectively defined on decomposable elements $X \wedge Y$, $Z \wedge T$ in $\Lambda_2(\Sigma)$ by

$$\begin{cases} \langle \mathcal{R}_{\text{res}}(X \wedge Y), Z \wedge T\rangle = \langle \mathcal{R}^M(X \wedge Y), Z \wedge T\rangle, \\ \langle \mathcal{R}_{\text{ext}}(X \wedge Y), Z \wedge T\rangle = \langle L(X, Z), L(Y, T)\rangle - \langle L(X, T), L(Y, Z)\rangle. \end{cases} \tag{34}$$

Let γ_M be the lowest eigenvalue of $\mathcal{R}^M$. As $\langle \mathcal{R}_{\text{res}}\xi, \xi\rangle = \langle \mathcal{R}^M\xi, \xi\rangle \geq \gamma_M|\xi|^2$ for all $\xi \in \Lambda_2(\Sigma)$, we see that $\mathcal{R}_{\text{res}} \geq \gamma_M$. The same remark applies to the largest eigenvalue Γ_M of $\mathcal{R}^M$. Hence

$$\gamma_M \leq \mathcal{R}_{\text{res}} \leq \Gamma_M. \tag{35}$$

Now let $(\nu_1, \dots, \nu_q)$ be an orthonormal frame in the normal bundle of Σ. By the definition of S_ν, we can write $\mathcal{R}_{\text{ext}} = \sum_{j=1}^q \mathcal{R}^{(j)}_{\text{ext}}$, where

$$\langle \mathcal{R}^{(j)}_{\text{ext}}(X \wedge Y), Z \wedge T\rangle = \langle S_{\nu_j}(X), Z\rangle \langle S_{\nu_j}(Y), T\rangle - \langle S_{\nu_j}(X), T\rangle \langle S_{\nu_j}(Y), Z\rangle. \tag{36}$$

In conclusion, one has the splitting

$$\mathcal{R} = \mathcal{R}_{\text{res}} + \sum_{j=1}^{q} \mathcal{R}_{\text{ext}}^{(j)} \tag{37}$$

with $\mathcal{R}_{\text{res}}$ and $\mathcal{R}_{\text{ext}}^{(j)}$ respectively given by (34) and (36).

5.3. ***Algebraic lemma.*** The proof of Theorem 1 depends on the following algebraic fact. Let S be a self-adjoint endomorphism of $T\Sigma$ and consider the associated "curvature operator" $\mathcal{R}_S : \Lambda_2(\Sigma) \to \Lambda_2(\Sigma)$ uniquely determined by the formula

$$\langle \mathcal{R}_S(X \wedge Y), Z \wedge T\rangle = \langle S(X), Z\rangle\langle S(Y), T\rangle - \langle S(X), T\rangle\langle S(Y), Z\rangle \tag{38}$$

for all $X, Y, Z, T \in T\Sigma$. Clearly, $\mathcal{R}_S$ is self-adjoint. Introduce the self-adjoint operator $T_S^{[p]} : \Lambda^p(\Sigma) \to \Lambda^p(\Sigma)$ such that, on any pair of p-forms ω, ϕ,

$$\langle T_S^{[p]}\omega, \phi\rangle = \frac{1}{4}\sum_{r,s}\langle \mathcal{R}_S\xi_r, \xi_s\rangle\langle[\hat{\xi}_r, \omega], [\hat{\xi}_s, \phi]\rangle, \tag{39}$$

where $\{\xi_r\}$ is any fixed orthonormal basis of $\Lambda_2(\Sigma)$ and $\{\hat{\xi}_r\}$ is its dual basis.

Lemma 18. *In the above notation, the operator $T_S^{[p]}$ can be written as*

$$T_S^{[p]} = (\operatorname{tr} S)S^{[p]} - S^{[p]} \circ S^{[p]},$$

where $S^{[p]}$ is the self-adjoint extension of S to $\Lambda^p(\Sigma)$.

Proof. Let $(e_1, \dots, e_n)$ be an orthonormal basis which diagonalizes S, so that $S(e_j) = k_j e_j$ for all $j = 1, \dots, n$ and k_j are the associated eigenvalues. Let $(\theta_1, \dots, \theta_n)$ be its dual basis. We refer to the notation in the proof of Lemma 2. Denote by I_p the set of multi-indices $\{j_1, \dots, j_p\}$ with $j_1 < \cdots < j_p$. If $\alpha = \{j_1, \dots, j_p\}$, let

$$\Theta_\alpha = \theta_{j_1} \wedge \cdots \wedge \theta_{j_p} = \theta_{j_1} \cdots \theta_{j_p}$$

where the dots in the last term indicate Clifford multiplication. The set $\{\Theta_\alpha : \alpha \in I_p\}$ is then an orthonormal basis of $\Lambda^p(\Sigma)$. It is enough to show that

$$\begin{cases} \langle T^{[p]}\Theta_\alpha, \Theta_\beta\rangle = 0 & \text{if } \alpha \neq \beta, \\ \langle T^{[p]}\Theta_\alpha, \Theta_\alpha\rangle = K_\alpha K_{\star\alpha}. \end{cases}$$

In fact, in that case, each Θ_α is an eigenform of $T_S^{[p]}$ associated to the eigenvalue $K_\alpha K_{\star\alpha}$, and it is readily seen from the discussion in Lemma 2 that the operator $(\operatorname{tr} S)S^{[p]} - S^{[p]} \circ S^{[p]}$ is the only one having that property.

Observe from (38) that the 2-vector $e_i \wedge e_j$ with $i < j$ is an eigenvector of $\mathcal{R}_S$ with associated eigenvalue $k_i k_j$. The set $\{\xi_r\} = \{e_i \wedge e_j\}_{i<j}$ forms an orthonormal

basis of $\Lambda_2(\Sigma)$. Therefore, by the definition in (39),

$$\langle T_S^{[p]}\omega, \phi\rangle = \frac{1}{4}\sum_{i<j} k_i k_j \langle[\theta_i\cdot\theta_j, \omega], [\theta_i\cdot\theta_j, \phi]\rangle. \tag{40}$$

A straightforward calculation using (28) shows that, for any $\alpha \in I_p$,

$$[\theta_i\cdot\theta_j, \Theta_\alpha] = \begin{cases} 0 & \text{if } i, j \in \alpha, \\ 0 & \text{if } i, j \in \star\alpha, \\ 2\theta_i\cdot\theta_j\cdot\Theta_\alpha & \text{otherwise.} \end{cases} \tag{41}$$

Then $\langle[\theta_i\cdot\theta_j, \Theta_\alpha], [\theta_i\cdot\theta_j, \Theta_\beta]\rangle$ is either zero or is equal to

$$4\langle\theta_i\cdot\theta_j\cdot\Theta_\alpha, \theta_i\cdot\theta_j\cdot\Theta_\beta\rangle = 4\langle\Theta_\alpha, \Theta_\beta\rangle$$

because $\langle\theta\cdot\omega_1, \theta\cdot\omega_2\rangle = \langle\omega_1, \omega_2\rangle$ for any 1-form θ and p-forms ω_1, ω_2. In particular $\langle T^{[p]}\Theta_\alpha, \Theta_\beta\rangle = 0$ when $\alpha \neq \beta$. It remains to show that $\langle T^{[p]}\Theta_\alpha, \Theta_\alpha\rangle = K_\alpha K_{\star\alpha}$. After renumbering, we can assume that $\alpha = \{1, \dots, p\}$ so that $\star\alpha = \{p+1, \dots, n\}$. Then

$$|[\theta_i\cdot\theta_j, \Theta_\alpha]|^2 = \begin{cases} 4 & \text{if } i \le p,\ j \ge p+1, \\ 0 & \text{otherwise,} \end{cases}$$

so that, by (40),

$$\begin{aligned}\langle T^{[p]}\Theta_\alpha, \Theta_\alpha\rangle &= \frac{1}{4}\sum_{i<j} k_i k_j |[\theta_i\cdot\theta_j, \Theta_\alpha]|^2 \\ &= \sum_{\substack{i\le p \\ j\ge p+1}} k_i k_j = (k_1+\cdots+k_p)(k_{p+1}+\cdots+k_n) = K_\alpha K_{\star\alpha}\end{aligned}$$

as asserted. □

5.4. ***Proof of Theorem 1.*** Let $\{\xi_r\}$ be an orthonormal basis of $\Lambda_2(\Sigma)$ with dual basis $\{\hat{\xi}_r\}$, where r is an index running from 1 to $\binom{n}{2}$. By Theorem 17 and the splitting given in (37), we have

$$\mathcal{B}^{[p]} = \mathcal{B}^{[p]}_{\mathrm{res}} + \sum_{j=1}^{q} T^{[p]}_{\nu_j},$$

where, on given p-forms ω and ϕ,

$$\langle\mathcal{B}^{[p]}_{\mathrm{res}}\omega, \phi\rangle = \frac{1}{4}\sum_{r,s}\langle\mathcal{R}_{\mathrm{res}}\xi_r, \xi_s\rangle\langle[\hat{\xi}_r, \omega], [\hat{\xi}_s, \phi]\rangle,$$

$$\langle T^{[p]}_{\nu_j}\omega, \phi\rangle = \frac{1}{4}\sum_{r,s}\langle\mathcal{R}^{(j)}_{\mathrm{ext}}\xi_r, \xi_s\rangle\langle[\hat{\xi}_r, \omega], [\hat{\xi}_s, \omega]\rangle.$$

From Lemma 18 applied to $S = S_{\nu_j}$, we get directly that

$$T^{[p]}_{\nu_j} = (\operatorname{tr} S_{\nu_j}) S^{[p]}_{\nu_j} - S^{[p]}_{\nu_j} \circ S^{[p]}_{\nu_j}.$$

It remains to show the bounds on $\mathcal{B}^{[p]}_{\text{res}}$. Choose $\{\xi_r\}$ to be an orthonormal basis of eigenvectors of $\mathcal{R}_{\text{res}}$. We know from (35) that $\langle \mathcal{R}_{\text{res}} \xi_r, \xi_r \rangle \geq \gamma_M |\xi_r|^2 = \gamma_M$ for all r. Then, for any p-form ω,

$$\langle \mathcal{B}^{[p]}_{\text{res}} \omega, \omega \rangle = \frac{1}{4} \sum_r \langle \mathcal{R}_{\text{res}} \xi_r, \xi_r \rangle |[\hat{\xi}_r, \omega]|^2 \geq \frac{\gamma_M}{4} \sum_r |[\hat{\xi}_r, \omega]|^2.$$

Now, the right-hand side does not depend on the choice of the basis $\{\xi_r\}$ of $\Lambda_2(\Sigma)$; choosing the basis $\{\theta_i \cdot \theta_j\}_{i<j}$ relative to an orthonormal coframe $(\theta_1, \dots, \theta_n)$ of $T\Sigma$, we have

$$\frac{1}{4} \sum_r |[\hat{\xi}_r, \omega]|^2 = \frac{1}{4} \sum_{i<j} |[\theta_i \cdot \theta_j, \omega]|^2 = p(n-p)|\omega|^2,$$

which follows from Lemma 18 applied to $S = Id$, with eigenvalues k_j all equal to 1. Then $\mathcal{B}^{[p]}_{\text{res}} \geq p(n-p)\gamma_M$. The upper bound $\mathcal{B}^{[p]}_{\text{res}} \leq p(n-p)\Gamma_M$ is proved similarly. $\square$

6. Proof of Theorem 4

Let Σ^n be a compact hypersurface of $\mathbb{S}^{n+1}$ and let ω be a nontrivial parallel p-form on Σ, for some $p = 1, \dots, n-1$.

We first take care of the case $n = 2$, $p = 1$. As $\mathcal{B}^{[1]}\omega = K_\Sigma \omega$, we see immediately that $K_\Sigma = 0$, hence Σ is flat. As Σ is compact and orientable, Σ must be a flat torus.

We then assume $n \geq 3$. Let $(e_1, \dots, e_n)$ be a local orthonormal frame of principal directions associated to the principal curvatures $k_1, \dots, k_n$ on an open set U and let $(\theta_1, \dots, \theta_n)$ be its dual basis. The following facts have been proved in [Colbois and Savo 2012, Theorem 9]. As a consequence of the identity $R(e_i, e_j)\omega = 0$ one obtains, for all $i \neq j$,

$$(1 + k_i k_j)\Phi_{ij} = 0, \tag{42}$$

where Φ_{ij} is the p-form $\Phi_{ij} = \theta_j \wedge i_{e_i}\omega - \theta_i \wedge i_{e_j}\omega$. As ω is parallel, it never vanishes; as it is nontrivial we can assume, after renumbering the basis, that $\omega(e_1, \dots, e_p) \neq 0$ on U. This implies that for all $i \leq p$ and $j \geq p+1$ the form Φ_{ij} is nonzero, which forces $1 + k_i k_j = 0$. One quickly concludes that at each point, there are two principal curvatures, λ (with multiplicity p) and μ (with multiplicity $n-p$); that is,

$$\begin{cases} S(e_i) = \lambda e_i & \text{for } i = 1, \dots, p, \\ S(e_j) = \mu e_j & \text{for } j = p+1, \dots, n. \end{cases}$$

Moreover, one has, on U,

$$\lambda\mu = -1.$$

Now λ, μ are smooth functions on U. To prove Theorem 4 it is enough to show that $\nabla\lambda = \nabla\mu = 0$ on U. In fact, as U is arbitrary and Σ is connected, λ and μ will be constant on Σ. Therefore Σ is a compact isoparametric hypersurface with two principal curvatures, and by a well-known classification result it is isometric to a Clifford torus.

A result by T. Otsuki [1970] states that if Σ is a hypersurface in $\mathbb{S}^{n+1}$ such that the multiplicities of its principal curvatures are constant, then the distribution $D_\lambda = \{v \in T\Sigma : S(v) = \lambda v\}$ relative to a principal curvature λ is completely integrable. Moreover, if the multiplicity of λ is greater than one, then λ is constant on each of the integral leaves of the corresponding distribution. When there are only two principal curvatures (which is our case) this fact was also proved in [Ryan 1969, Proposition 2.3].

We first assume that $2 \le p \le n-2$. By what we have just said, on U we have

$$\begin{cases} \langle \nabla\lambda, e_i\rangle = 0 & \text{for } i = 1, \dots, p, \\ \langle \nabla\mu, e_j\rangle = 0 & \text{for } j = p+1, \dots, n. \end{cases} \tag{43}$$

Differentiating $\lambda\mu = -1$, we see that, on U,

$$\mu\nabla\lambda + \lambda\nabla\mu = 0. \tag{44}$$

Fix $i = 1, \dots, p$. As $\langle \nabla\lambda, e_i\rangle = 0$, we obtain from (44) that $\langle \lambda\nabla\mu, e_i\rangle = 0$; as $\lambda \neq 0$ we then have

$$\nabla\mu(e_i) = 0 \quad \text{for all } i = 1, \dots, p.$$

By (43) we see that $\nabla\mu = 0$ (hence $\nabla\lambda = 0$) on U.

We now assume that $p = 1$. Therefore

$$S(e_1) = \lambda e_1, \qquad S(e_j) = \mu e_j \quad \text{for } j = 2, \dots, n.$$

As $n \ge 3$, the multiplicity of μ is greater than one, and we have

$$\langle \nabla\mu, e_j\rangle = 0 \quad \text{for } j = 2, \dots, n.$$

By (44) we also have

$$\langle \nabla\lambda, e_j\rangle = 0 \quad \text{for } j = 2, \dots, n.$$

To prove the theorem, it then remains to show that $\langle \nabla\lambda, e_1\rangle = 0$.

From (42), we see that $(1 + k_i k_j)\Phi_{ij} = 0$, where $\Phi_{ij} = \omega(e_i)\theta_j - \omega(e_j)\theta_i$. Take $i, j \ge 2$ with $i \neq j$. As $1 + k_i k_j = 1 + \mu^2 \neq 0$, we must have $\Phi_{ij} = 0$. Hence

$$0 = \Phi_{ij}(e_j) = \omega(e_i) \quad \text{for all } i = 2, \dots, n.$$

As $\omega(e_1) \neq 0$, this gives

$$S^{[1]}\omega(e_1) = \lambda\omega(e_1) \quad \text{and} \quad S^{[1]}\omega(e_j) = 0 \quad \text{for all } j \geq 2.$$

This means that $S^{[1]}\omega = \lambda\omega$, and that the dual vector field X of ω is parallel and is a principal direction associated to λ:

$$S(X) = \lambda X.$$

As X has constant length, we can normalize so that $X = e_1$. We now compute $\operatorname{div}(S(X))$ in two ways. Since X is parallel, one has $\nabla_{e_j}(S(X)) = \nabla_{e_j} S(X)$. Then, by the Codazzi formula,

$$\operatorname{div}(S(X)) = \langle n\nabla H, X\rangle. \tag{45}$$

On the other hand, since $\operatorname{div} X = 0$,

$$\operatorname{div}(S(X)) = \operatorname{div}(\lambda X) = \langle \nabla\lambda, X\rangle. \tag{46}$$

Therefore $\langle n\nabla H - \nabla\lambda, X\rangle = 0$. Differentiating the identity $nH = \lambda - (n-1)/\lambda$, we obtain

$$\frac{n-1}{\lambda^2}\langle \nabla\lambda, X\rangle = 0 \quad \text{and} \quad \langle \nabla\lambda, e_1\rangle = \langle \nabla\lambda, X\rangle = 0.$$

The proof is complete. □

References

[Alencar and do Carmo 1994] H. Alencar and M. do Carmo, "Hypersurfaces with constant mean curvature in spheres", *Proc. Amer. Math. Soc.* **120**:4 (1994), 1223–1229. MR 94f:53108 Zbl 0802.53017

[Colbois and Savo 2012] B. Colbois and A. Savo, "Involutive isometries, eigenvalue bounds and a spectral property of Clifford tori", *Indiana Univ. Math. J.* **61**:1 (2012), 337–357. MR 3029401 Zbl 1269.58012

[Dajczer and Rodríguez 1986] M. Dajczer and L. Rodríguez, "Rigidity of real Kaehler submanifolds", *Duke Math. J.* **53**:1 (1986), 211–220. MR 87g:53089 Zbl 0599.53005

[El Soufi and Petit 2000] A. El Soufi and R. Petit, "Immersions minimales et immersions pluriharmoniques entre variétés Riemanniennes: résultants de non existence et de rigidité", *Ann. Inst. Fourier* (*Grenoble*) **50**:1 (2000), 235–256. MR 2001h:53085 Zbl 0945.53038

[Gallot and Meyer 1975] S. Gallot and D. Meyer, "Opérateur de courbure et Laplacien des formes différentielles d'une variété Riemannienne", *J. Math. Pures Appl.* (9) **54**:3 (1975), 259–284. MR 56 #13128 Zbl 0316.53036

[Grosjean 2004] J.-F. Grosjean, "Minimal submanifolds with a parallel or a harmonic p-form", *J. Geom. Phys.* **51**:2 (2004), 211–228. MR 2005h:53103 Zbl 1074.53050

[Lawson and Simons 1973] H. B. Lawson, Jr. and J. Simons, "On stable currents and their application to global problems in real and complex geometry", *Ann. of Math.* (2) **98** (1973), 427–450. MR 48 #2881 Zbl 0283.53049

[Otsuki 1970] T. Ôtsuki, "Minimal hypersurfaces in a Riemannian manifold of constant curvature", *Amer. J. Math.* **92** (1970), 145–173. MR 41 #9157 Zbl 0196.25102

[Petersen 1998] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics **171**, Springer, New York, 1998. MR 98m:53001 Zbl 0914.53001

[Raulot and Savo 2011] S. Raulot and A. Savo, "A Reilly formula and eigenvalue estimates for differential forms", *J. Geom. Anal.* **21**:3 (2011), 620–640. MR 2012j:58043 Zbl 1228.58018

[Ryan 1969] P. J. Ryan, "Homogeneity and some curvature conditions for hypersurfaces", *Tôhoku Math. J.* (2) **21** (1969), 363–388. MR 40 #6458 Zbl 0185.49904

[Savo 2005] A. Savo, "On the first Hodge eigenvalue of isometric immersions", *Proc. Amer. Math. Soc.* **133**:2 (2005), 587–594. MR 2005h:58053 Zbl 1054.58023

[Weiner 1991] J. L. Weiner, "Flat tori in S^3 and their Gauss maps", *Proc. London Math. Soc.* (3) **62**:1 (1991), 54–76. MR 92d:53057 Zbl 0724.53036

Received September 19, 2013. Revised April 7, 2014.

ALESSANDRO SAVO
DIPARTIMENTO SBAI, SEZIONE DI MATEMATICA
SAPIENZA UNIVERSITÀ DI ROMA
VIA ANTONIO SCARPA 16
I-00161 ROMA
ITALY

alessandro.savo@sbai.uniroma1.it

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2014 is US \$410/year for the electronic version, and \$535/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 272 No. 2 December 2014