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DOUBLE AND TRIPLE GIVENTAL'S J-FUNCTIONS FOR STABLE QUOTIENTS INVARIANTS

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We use mirror formulas for the stable quotients analogue of Givental's J-function for twisted projective invariants obtained in a previous paper to obtain mirror formulas for the analogues of the double and triple Givental's J-functions (with descendants at all marked points) in this setting. We then observe that the genus-0 stable quotients invariants need not satisfy the divisor, string, or dilaton relations of the Gromov–Witten theory, but they do possess the integrality properties of the genus-0 three-point Gromov–Witten invariants of Calabi–Yau manifolds. We also relate the stable quotients invariants to the BPS counts arising in Gromov–Witten theory and obtain mirror formulas for certain twisted Hurwitz numbers.

1. Introduction

Gromov–Witten invariants of projective varieties are counts of curves that are conjectured (and known in some cases) to possess a rich structure. In particular, so-called mirror formulas relate these symplectic invariants of a nonsingular variety X to complex-geometric invariants of the mirror family of X. In genus 0, this relation is often described by assembling two-point Gromov–Witten invariants (but

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without constraints on the second marked point) into a generating function, known as Givental's J-function, and expressing it in terms of an explicit hypergeometric series. The genus-0 Gromov–Witten invariants of a projective complete intersection X are equal to the twisted Gromov–Witten invariants of the ambient space associated to the direct sum of positive line bundles corresponding to X. The genus-0 mirror formula in Gromov–Witten theory extends to the twisted Gromov–Witten invariants associated with direct sums of line bundles over projective spaces; see [Elezi 2003; Givental 1996; Lian et al. 1999]. By [Cooper and Zinger 2014], the analogue of Givental's J-function for the twisted stable quotients invariants defined in [Marian et al. 2011] satisfies a simpler version of the mirror formula from Gromov–Witten theory. In this paper, we obtain mirror formulas for the stable quotients analogues of the double and triple Givental's J-functions for direct sums of line bundles. We use them to test the stable quotients invariants for the analogues of the standard properties satisfied by Gromov–Witten invariants. In the future, we intend to apply the methods of this paper to show that the stable quotients and Gromov–Witten invariants of projective complete intersections are related by a simple mirror transform, in all genera, but with at least one marked point.

1A. ***Stable quotients.*** The moduli spaces of stable quotients, $\overline{Q}_{g,m}(X,d)$, constructed in [Marian et al. 2011] and generalized in [Ciocan-Fontanine et al. 2014], provide an alternative to the moduli spaces of stable maps, $\overline{\mathfrak{M}}_{g,m}(X,d)$, for compactifying spaces of degree-d morphisms from genus-g nonsingular curves with m marked points to a projective variety X (with a choice of polarization). A stable tuple of quotients is a tuple

$$(\mathcal{C}, y_1, \dots, y_m; S_1 \subset \mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathcal{C}}, \dots, S_p \subset \mathbb{C}^{n_p} \otimes \mathcal{O}_{\mathcal{C}}), \tag{1-1}$$

where $\mathcal{C}$ is a connected (at worst) nodal curve, $y_1, \dots, y_m \in \mathcal{C}^*$ are distinct smooth points, and

$$S_1 \subset \mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathcal{C}}, \dots, S_p \subset \mathbb{C}^{n_p} \otimes \mathcal{O}_{\mathcal{C}}$$

are subsheaves such that the quotients $\mathbb{C}^{n_1} \otimes \mathcal{O}_{\mathcal{C}}/S_1, \dots, \mathbb{C}^{n_p} \otimes \mathcal{O}_{\mathcal{C}}/S_p$ are locally free at the nodes of $\mathcal{C}$ and the marked points $y_1, \dots, y_m$ and the $\mathbb{Q}$-line bundle

$$\omega_{\mathcal{C}}(y_1 + \dots + y_m) \otimes (\Lambda^{\mathrm{top}} S_1^*)^{\epsilon} \otimes \dots \otimes (\Lambda^{\mathrm{top}} S_p^*)^{\epsilon} \to \mathcal{C}$$

is ample for all $\epsilon \in \mathbb{Q}^+$; this implies that $2g+m \geq 2$.

In this paper, we are concerned only with the case $g=0$. For $m, d_1, \dots, d_p \in \mathbb{Z}^{\geq 0}$ and $n_1, \dots, n_p \in \mathbb{Z}^+$, the moduli space

$$\overline{Q}_{0,m}(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1}, (d_1, \dots, d_p)) \tag{1-2}$$

parameterizing the stable tuples of quotients as in (1-1) with $h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 0$, that is, $\mathcal{C}$ is a rational curve, $\mathrm{rk}(S_i) = 1$, and $\deg(S_i) = -d_i$, is a nonsingular irreducible

Deligne–Mumford stack and

$$\dim \overline{Q}_{0,m}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(d_1,\dots,d_p)) \\ = (d_1+1)n_1+\cdots+(d_p+1)n_p-p-3+m;$$

see [Cooper and Zinger 2014, Propositions 2.1, 2.2].

As in the case of stable maps, there are evaluation morphisms,

$$\mathrm{ev}_i : \overline{Q}_{0,m}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(d_1,\dots,d_p)) \to \mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},$$

$i=1,2,\dots,m$, corresponding to each marked point. There is also a universal curve

$$\pi:\mathcal{U}\to\overline{Q}_{0,m}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(d_1,\dots,d_p))$$

with m sections $\sigma_1,\dots,\sigma_m$ (given by the marked points) and p universal rank-1 subsheaves

$$\mathcal{S}_i\subset\mathbb{C}^{n_i}\otimes\mathcal{O}_{\mathcal{U}}.$$

In the case $p=1$, we will denote $\mathcal{S}_1$ by $\mathcal{S}$. For each $i=1,2,\dots,m$, let

$$\psi_i=-\pi_*(\sigma_i^2)\in H^2\big(\overline{Q}_{0,m}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(d_1,\dots,d_p))\big)$$

be the first chern class of the universal cotangent line bundle as usual.

The twisted invariants of projective spaces that we study in this paper are indexed by tuples $\mathbf{a}=(a_1,\dots,a_l)\in(\mathbb{Z}^*)^l$ of nonzero integers, with $l\in\mathbb{Z}^{\ge 0}$. For each such tuple $\mathbf{a}$, let

$$|\mathbf{a}|=\sum_{k=1}^{l}|a_k|,\quad \langle\mathbf{a}\rangle=\prod_{a_k>0}a_k\Big/\prod_{a_k<0}a_k,\quad \mathbf{a}!=\prod_{a_k>0}a_k!,\quad \mathbf{a}^{\mathbf{a}}=\prod_{k=1}^{l}a_k^{|a_k|},$$

$$\nu_n(\mathbf{a})=n-|\mathbf{a}|,\quad \ell^{\pm}(\mathbf{a})=|\{k:(\pm 1)a_k>0\}|,\quad \ell(\mathbf{a})=\ell^+(\mathbf{a})-\ell^-(\mathbf{a}).$$

If in addition $n,d\in\mathbb{Z}^+$, let

$$\mathcal{V}_{n;\mathbf{a}}^{(d)}=\bigoplus_{a_k>0}R^0\pi_*(\mathcal{S}^{*a_k})\oplus\bigoplus_{a_k<0}R^1\pi_*(\mathcal{S}^{*a_k})\to\overline{Q}_{0,m}(\mathbb{P}^{n-1},d), \tag{1-3}$$

where $\pi:\mathcal{U}\to\overline{Q}_{0,m}(\mathbb{P}^{n-1},d)$ is the universal curve and $m\ge 2$; these sheaves are locally free.

By [Ciocan-Fontanine et al. 2014, Theorem 4.5.2 and Proposition 6.2.3],

$$\mathrm{SQ}_{n;\mathbf{a}}^{d}(c_1,\dots,c_m)\equiv\int_{\overline{Q}_{0,m}(\mathbb{P}^{n-1},d)}\mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(d)})\prod_{i=1}^{m}\mathrm{ev}_i^*x^{c_i},\quad m\ge 2,\ d\in\mathbb{Z}^+,\ c_i\in\mathbb{Z}^{\ge 0},$$

where $x \in H^2(\mathbb{P}^{n-1})$ is the hyperplane class, are invariants of the total space $X_{n;\mathbf{a}}$ of the vector bundle

$$\bigoplus_{a_k<0} \mathcal{O}_{\mathbb{P}^{n-1}}(a_k)\big|_{X_{n;(a_k)_{a_k>0}}} \to X_{n;(a_k)_{a_k>0}}, \tag{1-4}$$

where $X_{n;(a_k)_{a_k>0}} \subset \mathbb{P}^{n-1}$ is a nonsingular complete intersection of multidegree $(a_k)_{a_k>0}$. If $\nu_n(\mathbf{a})=0$, that is, $X_{n;\mathbf{a}}$ is a Calabi–Yau complete intersection, let

$$\mathrm{GW}^{c_1,\dots,c_m}_{n;\mathbf{a}}(q) = \sum_{d=0}^{\infty} q^d\, \mathrm{SQ}^d_{n;\mathbf{a}}(c_1,\dots,c_m),$$

with $\mathrm{GW}^0_{n;\mathbf{a}}(\mathbf{c}) \equiv \langle \mathbf{a}\rangle$ if $|\mathbf{c}| = n-4-\ell(\mathbf{a})+m$ and 0 otherwise.

1B. ***SQ invariants and GW invariants.*** In Gromov–Witten theory, there is a natural evaluation morphism $\mathrm{ev}: \mathcal{U} \to \mathbb{P}^{n-1}$ from the universal curve

$$\pi: \mathcal{U} \to \overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1}, d).$$

If $n, d \in \mathbb{Z}^+$, the sheaf

$$\mathcal{V}^{(d)}_{n;\mathbf{a}} = \bigoplus_{a_k>0} R^0\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k) \oplus \bigoplus_{a_k<0} R^1\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k) \to \overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1}, d), \tag{1-5}$$

is locally free. It is well known that

$$\mathrm{GW}^d_{n;\mathbf{a}}(c_1,\dots,c_m) \equiv \int_{\overline{\mathfrak{M}}_{0,m}(\mathbb{P}^{n-1},d)} \mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a}}) \prod_{i=1}^{m} \mathrm{ev}_i^* x^{c_i}, \quad m, c_i \in \mathbb{Z}^{\ge 0}, d \in \mathbb{Z}^+,$$

are also invariants of $X_{n;\mathbf{a}}$. If $\nu_n(\mathbf{a})=0$ and $m \ge 2$, let

$$\mathrm{GW}^{c_1,\dots,c_m}_{n;\mathbf{a}}(Q) = \sum_{d=0}^{\infty} Q^d\, \mathrm{GW}^d_{n;\mathbf{a}}(c_1,\dots,c_m),$$

with $\mathrm{GW}^0_{n;\mathbf{a}}(\mathbf{c}) \equiv \langle \mathbf{a}\rangle$ if $|\mathbf{c}| = n-4-\ell(\mathbf{a})+m$ and 0 otherwise.

Stable quotients invariants and Gromov–Witten invariants are equal in many cases, but differ for many Calabi–Yau targets, as we now describe. Let

$$\dot{F}_{n;\mathbf{a}}(w,q) \equiv \sum_{d=0}^{\infty} q^d w^{\nu_n(\mathbf{a})d} \frac{\prod_{a_k>0}\prod_{r=1}^{a_k d}(a_k w + r)\prod_{a_k<0}\prod_{r=0}^{-a_k d-1}(a_k w - r)}{\prod_{r=1}^{d}((w+r)^n - w^n)} \in \mathbb{Q}(w)[\![q]\!], \tag{1-6}$$

$$\dot{I}_0(q) = \dot{F}_{n;\mathbf{a}}(0,q), \qquad J_{n;\mathbf{a}}(q) = \frac{1}{\dot{I}_0(q)} \frac{\partial \dot{F}_{n;\mathbf{a}}}{\partial w}\bigg|_{(0,q)}. \tag{1-7}$$

The term w^n in the denominator in (1-6) above is irrelevant for the purposes of the main formulas of Sections 1–3. Its introduction is related to the expansion (4-9), which is used in an essential way in the proof of (3-14) in Section 10.

Theorem 1. *Let $n, l \in \mathbb{Z}^+$ and $\mathbf{a} \in (\mathbb{Z}^*)^l$ be such that $\nu_n(\mathbf{a}) = 0$. If $m = 2, 3$ and $\mathbf{c} \in (\mathbb{Z}^{\geq 0})^m$, then*

$$d^{3-m}\mathrm{SQ}^d_{n;\mathbf{a}}(\mathbf{c}) \in \mathbb{Z} \quad \text{for all } d \in \mathbb{Z}, \tag{1-8}$$

$$\mathrm{GW}^{\mathbf{c}}_{n;\mathbf{a}}(Q) = \dot{I}_0(q)^{m-2}\,\mathrm{SQ}^{\mathbf{c}}_{n;\mathbf{a}}(q) - \delta_{m,2}\langle\mathbf{a}\rangle J_{n;\mathbf{a}}(q), \tag{1-9}$$

where $\delta_{m,2}$ is the Kronecker delta function and $Q = q\mathrm{e}^{J_{n;\mathbf{a}}(q)}$ is the mirror map. *Furthermore, the genus-0 three-marked stable quotients invariants of $X_{n;\mathbf{a}}$ satisfy the analogue of the dilaton equation of Gromov–Witten theory if and only if $\ell^-(\mathbf{a}) > 0$, and of the divisor and string relations if and only if $\ell^-(\mathbf{a}) > 1$.*

The relation (1-9) follows from the explicit mirror formulas for the stable quotients analogues of the double and triple Givental's J-functions provided by Theorem 2 in Section 2 and similar results in Gromov–Witten theory [Popa 2012; Zinger 2014]; see Section 2 for more details. By [Ciocan-Fontanine and Kim 2013, Theorem 1.2.2 and Corollaries 1.4.1, 1.4.2], (1-9) holds for $m > 3$ as well. As the mirror formulas of Theorem 2 relate SQ invariants to the hypergeometric series arising in the B-model of the mirror family without a change of variables, (1-9) illustrates the principle that the mirror map relating Gromov–Witten theory to the B-model reflects the choice of the curve-counting theory in the A-model and is not intrinsic to mirror symmetry itself.

The analogue of (1-8) for GW invariants is well known. By [McDuff and Salamon 2004, Proposition 7.3.2], the genus-0 GW invariants of a Calabi–Yau manifold with 3+ marked points are integer. The $m = 2$ case of (1-8) for GW invariants is implied by the $m = 3$ case and the divisor relation. The $m = 2, 3$ cases of (1-8) for SQ invariants follow from the $m = 2, 3$ cases of (1-8) for GW invariants and from (1-9), since $\dot{I}_0(q), Q(q) \in \mathbb{Z}[\![q]\!]$; the integrality of the coefficients of $Q(q)$ whenever $\ell^-(\mathbf{a}) = 0$ is a special case of [Krattenthaler and Rivoal 2010, Theorem 1].[1] Since (1-9) extends to $m > 3$ by [Ciocan-Fontanine and Kim 2013], so does (1-8), but without the d^{3-m} factors.

Since $\dot{I}_0(q) = 1$ if $\ell^-(\mathbf{a}) = 0$ and $J_{n;\mathbf{a}}(q) = 0$ if $\ell^-(\mathbf{a}) = 0, 1$, (1-9) gives the following corollary.

Corollary 1.1. *Let $n, l \in \mathbb{Z}^+$ and $\mathbf{a} \in (\mathbb{Z}^*)^l$ be such that $\nu_n(\mathbf{a}) = 0$ and $a_{k_1}, a_{k_2} < 0$ for some $k_1 \neq k_2$. If $m = 2, 3$ and $\mathbf{c} \in (\mathbb{Z}^{\geq 0})^m$, then*

$$\mathrm{GW}^{\mathbf{c}}_{n;\mathbf{a}}(Q) = \mathrm{SQ}^{\mathbf{c}}_{n;\mathbf{a}}(q).$$

[1] The integrality of the coefficients of $\dot{I}_0(q)$ and of $Q(q)$ in the cases $\ell^-(\mathbf{a}) > 0$ is immediate from their definitions.

d	$d\mathrm{GW}^d_{n;\mathbf{a}}(1, 1)$	$d\mathrm{SQ}^d_{n;\mathbf{a}}(1, 1)$
1	2875	6725
2	4876875	16482625
3	8564575000	44704818125
4	15517926796875	126533974065625
5	28663236110956000	366622331794131725
6	53621944306062201000	1078002594137326617625
7	101216230345800061125625	3201813567943782886368125
8	192323666400003538944396875	9579628267176528143932815625
9	367299732093982242625847031250	28820906443427523291443507328125
10	704288164978454714776724365580000	87086311562396929291553775833982625

Table 1. Some genus-0 GW and SQ invariants of a quintic threefold $X_{5;(5)}$.

*Furthermore, the genus-*0 *three-marked stable quotients invariants of* $X_{n;\mathbf{a}}$ *satisfy the analogue of the divisor*, *dilaton*, *and string equations of Gromov–Witten theory.*

By Theorem 2, the conclusions of Corollary 1.1 also apply to the descendant invariants. Stable quotients replacements for the divisor, string, or dilaton relations [Hori et al. 2003, Section 26.3] for an arbitrary Calabi–Yau complete intersection $X_{n;\mathbf{a}}$ are provided by (2-23), (2-24), and (2-25), respectively. For the sake of comparison, we list a few genus-0 SQ and GW invariants of the quintic threefold $X_{5,(5)} \subset \mathbb{P}^4$ in Table 1; these are obtained from (2-33) and (2-34), respectively.

1C. ***SQ invariants and BPS states.*** Using (1-9), the genus-0 two- and three-marked SQ invariants of a Calabi–Yau complete intersection threefold $X_{n;\mathbf{a}}$ can be expressed in terms of the BPS counts of GW theory. For example, by the $m = 2$ case of (1-9),

$$\mathrm{SQ}^{1,1}_{n;\mathbf{a}}(q) = \langle\mathbf{a}\rangle J_{n;\mathbf{a}}(q) - \sum_{d=1}^{\infty} \mathrm{BPS}^d_{n;\mathbf{a}}(1, 1) \ln\bigl(1 - q^d \mathrm{e}^{dJ_{n;\mathbf{a}}(q)}\bigr), \tag{1-10}$$

where $\mathrm{BPS}^d_{n;\mathbf{a}}(1, 1)$ are the genus-0 two-marked BPS counts for $X_{n;\mathbf{a}}$ defined by

$$\mathrm{GW}^{1,1}_{n;\mathbf{a}}(Q) = -\sum_{d=1}^{\infty} \mathrm{BPS}^d_{n;\mathbf{a}}(1, 1) \ln(1 - Q^d).$$

If all genus-0 curves in $X_{n;\mathbf{a}}$ of degree at most d were smooth and had normal bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, the number of degree-$d$ genus-0 curves in $X_{n;\mathbf{a}}$ would be $\mathrm{BPS}^d_{n;\mathbf{a}}(1, 1)$; see [Voisin 1996, Section 1].

Under the regularity assumption of the previous paragraph, the moduli space

$$\overline{Q}_{0,2}^{1,1}(X_{n;\mathbf{a}},d)\equiv\big\{u\in\overline{Q}_{0,2}(X_{n;\mathbf{a}},d):\ \mathrm{ev}_1(u)\in H_1,\mathrm{ev}_2(u)\in H_2\big\},$$

where $H_1,H_2\subset\mathbb{P}^{n-1}$ are generic hyperplanes, would split into the topological components:

- $\mathcal{Z}_0^{1,1}(d)$ of stable quotients with torsion of degree d and thus corresponding to a constant map to $H_1\cap H_2$;
- $\mathcal{Z}_C^{1,1}(d)$ of stable quotients with image in a genus-0 curve $C\subset X_{n;\mathbf{a}}$ of degree $d_C\le d$.

For $C\subset X_{n;\mathbf{a}}$ as above, $\mathcal{Z}_C^{1,1}(d)$ consists of the closed subspaces $\mathcal{Z}_{C;r}^{1,1}(d)$, with $r\in\mathbb{Z}^+$ and $d_Cr\le d$, whose generic element has torsion of degree $d-d_Cr$. We note that

$$\dim\mathcal{Z}_{C;r}^{1,1}(d)=2r-2+d-d_Cr+2=d-(d_C-2)r,$$

which implies that each $\mathcal{Z}_{C;r}^{1,1}(d)$ is an irreducible component if $d_C>1$. When $d_C=1$, $\mathcal{Z}_{C;r}^{1,1}(d)$ is contained in $\mathcal{Z}_{C;d}^{1,1}(d)$, but still gives rise to a separate contribution to $\mathrm{SQ}_{n;\mathbf{a}}^d(1,1)$, according to (1-10).

The number $\mathrm{SQ}_{n;\mathbf{a}}^d(2,0)$, which arises from the constrained moduli space

$$\overline{Q}_{0,2}^{2,0}(X_{n;\mathbf{a}},d)=\mathcal{Z}_0^{2,0}(d)=\mathcal{Z}_0^{1,1}(d),$$

is $\langle\mathbf{a}\rangle$ times the coefficient $[\![J_{n;\mathbf{a}}(q)]\!]_d$ of q^d in $J_{n;\mathbf{a}}(q)$; see [Cooper and Zinger 2014, Theorem 1]. The contribution of $\mathcal{Z}_0^{1,1}(d)$ to $\mathrm{SQ}_{n;\mathbf{a}}^d(2,0)$ is the same; this explains the first term on the right-hand side of (1-10). Under the above regularity assumption, (1-10) can be rewritten as

$$\mathrm{SQ}_{n;\mathbf{a}}^d(1,1)=\langle\mathbf{a}\rangle[\![J_{n;\mathbf{a}}(q)]\!]_d+\sum_C\sum_{r=1}^{\infty}\frac{1}{r}[\![\mathrm{e}^{J_{n;\mathbf{a}}(q)}]\!]_{d-rd_C},\tag{1-11}$$

where the outer sum is taken over all genus-0 curves $C\subset X_{n;\mathbf{a}}$. This suggests that the contribution of $\mathcal{Z}_{C;r}^{1,1}(d)$ to $\mathrm{SQ}_{n;\mathbf{a}}^d(1,1)$ is $\frac{1}{r}[\![\mathrm{e}^{J_{n;\mathbf{a}}(q)}]\!]_{d-rd_C}$. This contribution depends on the embedding into $\mathbb{P}^{n-1}$, which is as expected, given the nature of SQ invariants.

Since the embedding $C\to\mathbb{P}^{n-1}$ corresponds to an inclusion $\mathcal{O}_{\mathbb{P}^1}(-d_C)\to\mathbb{C}^n\otimes\mathcal{O}_{\mathbb{P}^1}$, each element of $\mathcal{Z}_{C;r}^{1,1}(d)$ corresponds to a tuple

$$(\mathcal{C},y_1,y_2;S\subset S'^{\otimes d_c},S'\subset\mathbb{C}^2\otimes\mathcal{O}_{\mathcal{C}}),\quad\text{where}$$
$$(\mathcal{C},y_1,y_2;S\subset\mathbb{C}^2\otimes\mathcal{O}_{\mathcal{C}})\in\overline{Q}_{0,2}(\mathbb{P}^1,d),\quad(\mathcal{C},y_1,y_2;S'\subset\mathbb{C}^2\otimes\mathcal{O}_{\mathcal{C}})\in\overline{Q}_{0,2}(\mathbb{P}^1,r).$$

This modular style definition readily extends to arbitrary genus, number of marked points, and dimension of projective space. The arising deformation-obstruction theory can be studied as in [Marian et al. 2011, Section 6].

1D. ***Outline of the paper.*** Theorem 1 is a direct consequence of Theorem 2 in Section 2, which in turn is the nonequivariant specialization of Theorem 3 in Theorem 3. We adapt the approaches of [Zinger 2009; Popa and Zinger 2014; Popa 2012] from Gromov–Witten theory, outlined in Sections 5 and 6, to show that certain equivariant two-point generating functions, including the stable quotients analogue of the double Givental's J-function, satisfy certain good properties which guarantee uniqueness. The proof that these generating functions satisfy the required properties follows principles similar to the proof of the analogous statements in [Zinger 2009; Popa and Zinger 2014; Popa 2012] and uses the localization theorem of [Atiyah and Bott 1984]; it is carried out in Sections 7 and 8.

This approach also implies that certain equivariant three-point generating functions, including the stable quotients analogue of the triple Givental's J-function, are determined by three-point primary (without ψ-classes) SQ invariants. In the Fano cases, that is, $\nu_n(\mathbf{a}) > 0$, enough of these invariants are essentially trivial for dimensional reasons to confirm Proposition 3.1 in these cases; see Corollary 9.1. However, there is no dimensional reason for the vanishing of these invariants to extend to the Calabi–Yau cases, that is, $\nu_n(\mathbf{a}) = 0$; thus, a different argument is needed in these cases. We employ the same kind of trick as used in [Cooper and Zinger 2014] to confirm mirror symmetry for the stable quotients analogue of Givental's J-function and essentially deduce the Calabi–Yau cases from the Fano cases. Specifically, we show that the equivariant three-point mirror formula of Proposition 3.1 is equivalent to the closed formula for twisted three-point Hurwitz numbers of Proposition 4.1, whenever $|\mathbf{a}| \le n$. In Section 9, we show that the validity of the latter does not depend n; since it holds whenever $|\mathbf{a}| < n$, it follows that it holds for all $\mathbf{a}$, and so the equivariant three-point mirror formula of Proposition 3.1 holds whenever $|\mathbf{a}| \le n$. Along with [Zinger 2014], Proposition 3.1 finally leads to the mirror formula for the stable quotients analogue of the triple Givental's J-function in Theorem 3; see Section 10.

The closed formulas for twisted Hurwitz numbers of Propositions 4.1 and 4.2 are among the key ingredients in computing the genus-1 twisted stable quotients invariants with 1 marked point. At the same time, this paper and [Zinger 2014] provide an approach to comparing the (equivariant) genus-g m-fold Givental's J-functions,

$$\sum_{d=0}^{\infty} q^d \{\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_m\}_* \left[\frac{e(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{(\hbar_1 - \psi_1) \cdots (\hbar_m - \psi_m)} \right] \in H^*(\mathbb{P}^{n-1})[\hbar_1^{-1}, \ldots, \hbar_m^{-1}][\![q]\!] \tag{1-12}$$

in the SQ and GW theories for all $g \ge 0$ and $m \ge 1$ with $2g + m \ge 2$. By Proposition 6.3 and Lemmas 6.5 and 6.6, in the genus-0 case the restrictions of these generating functions to insertions at only one marked point agree whenever $\nu_{\mathbf{a}} > 1$. In all cases, the approach of [Zinger 2014] can be adapted to show that (1-12) is a sum over (at least) trivalent m-marked graphs with coefficients that involve equivariant m'-pointed Hurwitz numbers with $m' \le m$, which are conversely completely determined by the stable quotients analogue of the m'-pointed Givental's J-function with insertions at only one marked point through relations that do not involve n. Since these relations hold whenever $\nu_n(\mathbf{a}) > 0$, they hold for all $\mathbf{a}$. We intend to clarify these points in a future paper.

The Gromov–Witten analogues of Theorem 2 and its equivariant counterpart, Theorem 3 in Section 3, extend to the so-called concavex vector bundles over products of projective spaces, that is, vector bundles of the form

$$\bigoplus_{k=1}^{l} \mathcal{O}_{\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1}}(a_{k;1},\dots,a_{k;p}) \to \mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},$$

where for each given $k = 1, 2, \dots, l$ either $a_{k;1}, \dots, a_{k;p} \in \mathbb{Z}^{\ge 0}$, with $a_{k;i} \ne 0$ for some i, or $a_{k;1}, \dots, a_{k;p} \in \mathbb{Z}^{-}$. The stable quotients analogue of these bundles are the sheaves

$$\bigoplus_{k=1}^{l} \mathcal{S}_1^{*a_{k;1}}\otimes\cdots\otimes\mathcal{S}_p^{*a_{k;p}} \to \mathcal{U} \to \overline{Q}_{0,2}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(d_1,\dots,d_p)) \tag{1-13}$$

with the same condition on $a_{k;i}$, where $\mathcal{S}_i \to \mathcal{U}$ is the universal subsheaf corresponding to the i-th factor. We will comment on the necessary modifications at each step of the proof.

2. Main theorem

We arrange stable quotients invariants with two and three marked points into generating functions in Section 2A and give explicit closed formulas for them in Section 2B. In Section 2C, we use these formulas to relate SQ and GW invariants, with descendants, and obtain replacements for the divisor, string, and dilaton relations for SQ invariants.

2A. *Givental's J-functions.* For computational purposes, it is convenient to define variations of the bundle (1-3) by

$$\begin{aligned} \dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)} &= \bigoplus_{a_k>0} R^0\pi_*(\mathcal{S}^{*a_k}(-\sigma_1)) \oplus \bigoplus_{a_k<0} R^1\pi_*(\mathcal{S}^{*a_k}(-\sigma_1)) \to \overline{Q}_{0,m}(\mathbb{P}^{n-1},d),\\ \ddot{\mathcal{V}}_{n;\mathbf{a}}^{(d)} &= \bigoplus_{a_k>0} R^0\pi_*(\mathcal{S}^{*a_k}(-\sigma_2)) \oplus \bigoplus_{a_k<0} R^1\pi_*(\mathcal{S}^{*a_k}(-\sigma_2)) \to \overline{Q}_{0,m}(\mathbb{P}^{n-1},d), \end{aligned} \tag{2-1}$$

where $n, d \in \mathbb{Z}^+$, $m \geq 2$, and $\pi : \mathcal{U} \to \overline{Q}_{0,m}(\mathbb{P}^{n-1}, d)$ is the universal curve; these sheaves are also locally free. Whenever $\nu_n(\mathbf{a}) \geq 0$, [Cooper and Zinger 2014, Theorem 1] provides an explicit closed formula for the stable quotients analogue of Givental's J-function: the power series

$$\dot{Z}_{n;\mathbf{a}}(x, \hbar, q) \equiv 1 + \sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{e(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{\hbar - \psi_1}\right] \in H^*(\mathbb{P}^{n-1})[\hbar^{-1}][\![q]\!], \tag{2-2}$$

where $\mathrm{ev}_1 : \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d) \to \mathbb{P}^{n-1}$ is as before and $x \in H^2(\mathbb{P}^{n-1})$ is the hyperplane class. In this paper, we obtain a closed formula for the power series

$$\ddot{Z}_{n;\mathbf{a}}(x, \hbar, q) \equiv 1 + \sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{e(\ddot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{\hbar - \psi_1}\right] \in H^*(\mathbb{P}^{n-1})[\hbar^{-1}][\![q]\!]; \tag{2-3}$$

see (2-26).

We also give explicit formulas for the stable quotients analogues of the double and triple Givental's J-functions, the power series

$$\begin{aligned}\dot{Z}^*_{n;\mathbf{a}}&(x_1, x_2, \hbar_1, \hbar_2, q)\\ &\equiv \sum_{d=1}^{\infty} q^d \{\mathrm{ev}_1 \times \mathrm{ev}_2\}_*\left[\frac{e(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)}\right] \in H^*(\mathbb{P}^{n-1})[\hbar_1^{-1}, \hbar_2^{-1}][\![q]\!],\end{aligned} \tag{2-4}$$

$$\begin{aligned}\dot{Z}^*_{n;\mathbf{a}}&(x_1, x_2, x_3, \hbar_1, \hbar_2, \hbar_3, q)\\ &\equiv \sum_{d=1}^{\infty} q^d \{\mathrm{ev}_1 \times \mathrm{ev}_2 \times \mathrm{ev}_3\}_*\left[\frac{e(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)(\hbar_3 - \psi_3)}\right],\end{aligned} \tag{2-5}$$

where $x_i = \pi_i^* x$ is the pullback of the hyperplane class in $\mathbb{P}^{n-1}$ by the i-th projection map and

$$\begin{aligned}\mathrm{ev}_1 \times \mathrm{ev}_2 &: \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d) \to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1},\\ \mathrm{ev}_1 \times \mathrm{ev}_2 \times \mathrm{ev}_3 &: \overline{Q}_{0,3}(\mathbb{P}^{n-1}, d) \to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}\end{aligned} \tag{2-6}$$

are the total evaluation maps. Let

$$\begin{aligned}\dot{Z}_{n;\mathbf{a}}&(x_1, x_2, \hbar_1, \hbar_2, q)\\ &= \Bigg(\frac{1}{\hbar_1 + \hbar_2} \sum_{\substack{s_1, s_2 \geq 0\\ s_1 + s_2 = n-1}} x_1^{s_1} x_2^{s_2}\Bigg) + \dot{Z}^*_{n;\mathbf{a}}(x_1, x_2, \hbar_1, \hbar_2, q),\\ \dot{Z}_{n;\mathbf{a}}&(x_1, x_2, x_3, \hbar_1, \hbar_2, \hbar_3, q)\\ &= \Bigg(\frac{1}{\hbar_1 \hbar_2 \hbar_3} \sum_{\substack{s_1, s_2, s_3 \geq 0\\ s_1, s_2, s_3 \leq n-1\\ s_1 + s_2 + s_3 = 2n-2}} x_1^{s_1} x_2^{s_2} x_3^{s_3}\Bigg) + \dot{Z}^*_{n;\mathbf{a}}(x_1, x_2, x_3, \hbar_1, \hbar_2, \hbar_3, q).\end{aligned} \tag{2-7}$$

For each $s\in\mathbb{Z}^{\ge 0}$, define

$$\begin{aligned}\dot Z^{(s)}_{n;\mathbf{a}}(x,\hbar,q)&\equiv x^s+\sum_{d=1}^{\infty}q^d\mathrm{ev}_{1*}\Big[\frac{e(\dot{\mathcal V}^{(d)}_{n;\mathbf{a}})\mathrm{ev}_2^*x^s}{\hbar-\psi_1}\Big]\in H^*(\mathbb{P}^{n-1})[\hbar^{-1}][\![q]\!],\\ \ddot Z^{(s)}_{n;\mathbf{a}}(x,\hbar,q)&\equiv x^s+\sum_{d=1}^{\infty}q^d\mathrm{ev}_{1*}\Big[\frac{e(\ddot{\mathcal V}^{(d)}_{n;\mathbf{a}})\mathrm{ev}_2^*x^s}{\hbar-\psi_1}\Big]\in H^*(\mathbb{P}^{n-1})[\hbar^{-1}][\![q]\!],\end{aligned}\tag{2-8}$$

where $\mathrm{ev}_1,\mathrm{ev}_2:\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)\to\mathbb{P}^{n-1}$. Thus, $\dot Z^{(0)}_{n;\mathbf{a}}=\dot Z_{n;\mathbf{a}}$, $\ddot Z^{(0)}_{n;\mathbf{a}}=\ddot Z_{n;\mathbf{a}}$, and

$$x^{\ell^+_-(\mathbf{a})}\dot Z^{(\ell^-_+(\mathbf{a})+s)}_{n;\mathbf{a}}(x,\hbar,q)=x^{\ell^-_+(\mathbf{a})}\ddot Z^{(\ell^+_-(\mathbf{a})+s)}_{n;\mathbf{a}}(x,\hbar,q)$$

for all $s\ge 0$, where

$$\ell^+_-(\mathbf{a})=\max(\ell(\mathbf{a}),0),\qquad \ell^-_+(\mathbf{a})=\max(-\ell(\mathbf{a}),0).$$

By Theorem 2 below, $\dot Z^{(s)}_{n;\mathbf{a}}$, $\ddot Z^{(s)}_{n;\mathbf{a}}$, and the stable quotients analogues of the double and triple Givental's J-functions, (2-4) and (2-5), are explicit transforms of Givental's J-function $\dot Z_{n;\mathbf{a}}$ and its "reflection" $\ddot Z_{n;\mathbf{a}}$; this transform depends only on $\mathbf{a}$ (and s in the first two cases).

2B. ***Mirror symmetry.*** Givental's J-function $\dot Z_{n;\mathbf{a}}$ and its "reflection" $\ddot Z_{n;\mathbf{a}}$ in the Gromov–Witten and stable quotients theories are described by the hypergeometric series (1-6) and

$$\ddot F_{n;\mathbf{a}}(w,q)\equiv\sum_{d=0}^{\infty}q^d w^{\nu_n(\mathbf{a})d}\frac{\prod_{a_k>0}\prod_{r=0}^{a_kd-1}(a_kw+r)\prod_{a_k<0}\prod_{r=1}^{-a_kd}(a_kw-r)}{\prod_{r=1}^{d}((w+r)^n-w^n)}\in\mathbb{Q}(w)[\![q]\!].\tag{2-9}$$

These are power series in q with constant term 1 whose coefficients are rational functions in w which are regular at $w=0$. We denote the subgroup of all such power series by $\mathcal{P}$ and define

$$\begin{aligned}&\mathbf{D}:\mathbb{Q}(w)[\![q]\!]\to\mathbb{Q}(w)[\![q]\!], &&\mathbf{M}:\mathcal{P}\to\mathcal{P}\quad\text{by}\\ &\mathbf{D}H(w,q)\equiv\Big\{1+\frac{q}{w}\frac{\mathrm{d}}{\mathrm{d}q}\Big\}H(w,q), &&\mathbf{M}H(w,q)\equiv\mathbf{D}\Big(\frac{H(w,q)}{H(0,q)}\Big);\end{aligned}\tag{2-10}$$

the operator $\mathbf{D}$ multiplies the coefficient of q^d by $(w+d)/w$. If $\nu_n(\mathbf{a})=0$ and $s\in\mathbb{Z}^{\ge 0}$, let

$$\dot I_s(q)\equiv\mathbf{M}^s\dot F_{n;\mathbf{a}}(0,q),\qquad \ddot I_s(q)\equiv\mathbf{M}^s\ddot F_{n;\mathbf{a}}(0,q).\tag{2-11}$$

For example, $\dot{I}_s(q)=1$ if $s<\ell^-(\mathbf{a})$, $\ddot{I}_s(q)=1$ if $s<\ell^+(\mathbf{a})$,

$$\dot{I}_{\ell^-(\mathbf{a})}(q)=\ddot{I}_{\ell^+(\mathbf{a})}(q)=\sum_{d=0}^{\infty}q^d\,\frac{\prod_{a_k>0}(a_kd)!\prod_{a_k<0}(-1)^{a_kd}(-a_kd)!}{(d!)^n}\quad\text{if }\ \nu_n(\mathbf{a})=0,$$

and more generally $\dot{I}_{s+\ell^-_+(\mathbf{a})}(q)=\ddot{I}_{s+\ell^+_-(\mathbf{a})}(q)$ for all $s\ge 0$. If $\nu_n(\mathbf{a})>0$, we set $\dot{I}_s(q),\ddot{I}_s(q)=1$.

It is also convenient to introduce

$$\text{(2-12)}\quad F_{n;\mathbf{a}}(w,q)\equiv\sum_{d=0}^{\infty}q^dw^{\nu_n(\mathbf{a})d}\,\frac{\prod_{a_k>0}\prod_{r=1}^{a_kd}(a_kw+r)\prod_{a_k<0}\prod_{r=1}^{-a_kd}(a_kw-r)}{\prod_{r=1}^{d}(w+r)^n}\in\mathbb{Q}(w)[\![q]\!]$$

and the associated power series $I_s(q)=\mathbf{M}^sF_{n;\mathbf{a}}(0,q)$ in the $\nu_n(\mathbf{a})=0$ case. In the case $0<\nu_n(\mathbf{a})<n$, we define $\mathrm{c}^{(d)}_{s,s'}\in\mathbb{Q}$ with $d,s,s'\ge 0$ by

$$\text{(2-13)}\qquad \sum_{d=0}^{\infty}\sum_{s'=0}^{\infty}\mathrm{c}^{(d)}_{s,s'}w^{s'}q^d\equiv w^s\mathbf{D}^sF_{n;\mathbf{a}}(w,q/w^{\nu_n(\mathbf{a})})$$
$$=w^s\mathbf{D}^{s+\ell^-(\mathbf{a})}\dot{F}_{n;\mathbf{a}}(w,q/w^{\nu_n(\mathbf{a})})$$
$$=w^s\mathbf{D}^{s+\ell^+(\mathbf{a})}\ddot{F}_{n;\mathbf{a}}(w,q/w^{\nu_n(\mathbf{a})}).$$

Since $\mathrm{c}^{(0)}_{s,s'}=\delta_{s,s'}$, the relations

$$\text{(2-14)}\qquad \sum_{\substack{d_1,d_2\ge 0\\ d_1+d_2=d}}\ \sum_{t=0}^{s-\nu_n(\mathbf{a})d_1}\tilde{\mathrm{c}}^{(d_1)}_{s,t}\mathrm{c}^{(d_2)}_{t,s'}=\delta_{d,0}\delta_{s,s'}\quad\text{for all }d,s'\in\mathbb{Z}^{\ge 0},\ s'\le s-\nu_n(\mathbf{a})d,$$

inductively define $\tilde{\mathrm{c}}^{(d)}_{s,s'}\in\mathbb{Q}$ in terms of the numbers $\tilde{\mathrm{c}}^{(d_1)}_{s,t}$ with $d_1<d$. For example, $\tilde{\mathrm{c}}^{(0)}_{s,s'}=\delta_{s,s'}$ and

$$\sum_{s'=0}^{s-\nu_n(\mathbf{a})}\tilde{\mathrm{c}}^{(1)}_{s,s'}w^{s'}+\prod_{k=1}^{l}a_k\,\frac{\prod_{a_k>0}\prod_{r=1}^{a_k-1}(a_kw+r)\prod_{a_k<0}\prod_{r=1}^{-a_k-1}(a_kw-r)}{(w+1)^{n-\ell^+(\mathbf{a})-\ell^-(\mathbf{a})-s}}\in w^{s-\nu_n(\mathbf{a})+1}\mathbb{Q}[\![w]\!].$$

If $s'<0$ or $\nu_n(\mathbf{a})=0,n$, we set $\tilde{\mathrm{c}}^{(d)}_{s,s'}=\delta_{d,0}\delta_{s,s'}$. The coefficients $\tilde{\mathrm{c}}^{(d)}_{s,s'}$ are used to express the power series (2-7) and (2-8) in terms of derivatives of the power series (2-2) and (2-3); see Theorem 2.

For $s_1, s_2, s_3, d \in \mathbb{Z}^{\geq 0}$ with $s_1, s_2, s_3 \leq n-1$, let

$$
\tilde{\mathfrak{c}}^{(d)}_{s_1,s_2,s_3} = \begin{cases} \Big[\!\!\Big[\big((1-\mathbf{a}^{\mathbf{a}}q)\dot{\mathbb{I}}^c_{s_1}(q)\ddot{\mathbb{I}}^c_{s_2}(q)\ddot{\mathbb{I}}^c_{s_3}(q)\big)^{-1} \Big]\!\!\Big]_d & \text{if } \nu_n(\mathbf{a}) = 0; \\ \displaystyle\sum_{\substack{d_0,d_1,d_2,d_3 \geq 0 \\ d_0+d_1+d_2+d_3=d}} (\mathbf{a}^{\mathbf{a}})^{d_0} \prod_{t=1}^{3} \tilde{\mathfrak{c}}^{(d_t)}_{\hat{s}_t-\ell_t(\mathbf{a}),\hat{s}_t-\nu_n(\mathbf{a})d_t-\ell_t(\mathbf{a})} & \text{if } \nu_n(\mathbf{a}) > 0; \end{cases} \tag{2-15}
$$

where

$$
\dot{\mathbb{I}}^c_s = \prod_{t=s+1}^{n-\ell^+(\mathbf{a})} \dot{I}_t, \quad \ddot{\mathbb{I}}^c_s = \prod_{t=s+1}^{n-\ell^-(\mathbf{a})} \ddot{I}_t, \quad \hat{s}_t = n-1-s_t, \tag{2-16}
$$

$$
\ell_t(\mathbf{a}) = \begin{cases} \ell^+(\mathbf{a}) & \text{if } t = 1; \\ \ell^-(\mathbf{a}) & \text{if } t = 2, 3; \end{cases}
$$

and $[\![f(q)]\!]_d$ is the coefficient of q^d of $f(q) \in \mathbb{Q}[\![q]\!]$. In particular, $\dot{\mathbb{I}}^c_s = 1$ if $s \geq n - \ell^+(\mathbf{a})$ and $\ddot{\mathbb{I}}^c_s = 1$ if $s \geq n - \ell^-(\mathbf{a})$. Since $I_t = \dot{I}_{t+\ell^-(\mathbf{a})} = \ddot{I}_{t+\ell^+(\mathbf{a})}$, we find that

$$
\dot{\mathbb{I}}^c_s(q) = (1-\mathbf{a}^{\mathbf{a}}q)^{-1} \text{if } s < \ell^-(\mathbf{a}), \quad \ddot{\mathbb{I}}^c_s(q) = (1-\mathbf{a}^{\mathbf{a}}q)^{-1} \text{if } s < \ell^+(\mathbf{a});
$$

see [Zinger 2014, Proposition 4.4]. This implies that

$$
\sum_{d=0}^{\infty} \tilde{\mathfrak{c}}^{(d)}_{s_1,s_2,s_3} q^d = 1 \ \text{ if } \nu_n(\mathbf{a}) = 0, \quad s_1+s_2+s_3 = 2n-2, \tag{2-17}
$$
$$
\min(s_1, s_2, s_3) < \ell^-(\mathbf{a}).
$$

We use this observation in Section 2C. Since $\tilde{\mathfrak{c}}^{(0)}_{s,s'} = \delta_{s,s'}$, $\tilde{\mathfrak{c}}^{(0)}_{s_1,s_2,s_3} = 1$.

Finally, we define $\mathfrak{D}^s \dot{Z}_{n;\mathbf{a}}(x,\hbar,q)$, $\mathfrak{D}^s \ddot{Z}_{n;\mathbf{a}}(x,\hbar,q) \in H^*(\mathbb{P}^{n-1})[\hbar][\![q]\!]$ for each $s \in \mathbb{Z}^+$ inductively by

$$
\begin{aligned}
\mathfrak{D}^0 \dot{Z}_{n;\mathbf{a}}(x,\hbar,q) &= \dot{Z}_{n;\mathbf{a}}(x,\hbar,q), \\
\mathfrak{D}^s \dot{Z}_{n;\mathbf{a}}(x,\hbar,q) &= \frac{1}{\dot{I}_s(q)} \Big\{ x + \hbar\, q \frac{\mathrm{d}}{\mathrm{d}q} \Big\} \mathfrak{D}^{s-1} \dot{Z}_{n;\mathbf{a}}(x,\hbar,q), \\
\mathfrak{D}^0 \ddot{Z}_{n;\mathbf{a}}(x,\hbar,q) &= \ddot{Z}_{n;\mathbf{a}}(x,\hbar,q), \\
\mathfrak{D}^s \ddot{Z}_{n;\mathbf{a}}(x,\hbar,q) &= \frac{1}{\ddot{I}_s(q)} \Big\{ x + \hbar\, q \frac{\mathrm{d}}{\mathrm{d}q} \Big\} \mathfrak{D}^{s-1} \ddot{Z}_{n;\mathbf{a}}(x,\hbar,q).
\end{aligned} \tag{2-18}
$$

The operator $\mathfrak{D}$ first multiplies the coefficient of q^d by $x+d\hbar$ and then renormalizes the power series so that the coefficient of x^s becomes 1 in the Calabi–Yau cases (there is no renormalization in the Fano cases).

Theorem 2. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, the stable quotients analogue of the double Givental's J-function satisfies*

$$\dot{Z}_{n;\mathbf{a}}(x_1, x_2, \hbar_1, \hbar_2, q) = \frac{1}{\hbar_1+\hbar_2} \sum_{\substack{s_1,s_2\geq 0\\ s_1+s_2=n-1}} \dot{Z}^{(s_1)}_{n;\mathbf{a}}(x_1, \hbar_1, q)\ddot{Z}^{(s_2)}_{n;\mathbf{a}}(x_2, \hbar_2, q). \tag{2-19}$$

If in addition $\nu_n(\mathbf{a}) \geq 0$,

$$\begin{aligned} \dot{Z}_{n;\mathbf{a}}&(x_1, x_2, x_3, \hbar_1, \hbar_2, \hbar_3, q) \\ &= \frac{1}{\hbar_1\hbar_2\hbar_3} \sum_{\substack{d,s_1,s_2,s_3\geq 0\\ s_1,s_2,s_3\leq n-1\\ s_1+s_2+s_3+\nu_n(\mathbf{a})d=2n-2}} \tilde{\mathfrak{c}}^{(d)}_{s_1,s_2,s_3} q^d \dot{Z}^{(s_1)}_{n;\mathbf{a}}(x_1, \hbar_1, q) \prod_{t=2}^{3} \ddot{Z}^{(s_t)}_{n;\mathbf{a}}(x_t, \hbar_t, q), \end{aligned} \tag{2-20}$$

$$\check{Z}^{(s)}_{n;\mathbf{a}}(x, \hbar, q) = \sum_{d=0}^{\infty} \sum_{s'=0}^{s-\nu_n(\mathbf{a})d} \tilde{\mathfrak{c}}^{(d)}_{s-\ell^*(\mathbf{a}),s'-\ell^*(\mathbf{a})} q^d \hbar^{s-\nu_n(\mathbf{a})d-s'} \mathfrak{D}^{s'} \check{Z}_{n;\mathbf{a}}(x, \hbar, q), \tag{2-21}$$

where $(\check{Z}, \ell^*) = (\dot{Z}, \ell^-), (\ddot{Z}, \ell^+)$.

2C. ***Some computations.*** The first identity in Theorem 2 also holds for the Gromov–Witten analogues of the generating series $\dot{Z}^*_{n;\mathbf{a}}$, $\dot{Z}^{(s)}_{n;\mathbf{a}}$, and $\ddot{Z}^{(s)}_{n;\mathbf{a}}$; see [Popa 2012, Theorem 1.2] for the general (toric) case. If $\nu_n(\mathbf{a}) \geq 2-\ell^-(\mathbf{a})$, the analogues of (2-20), (2-21), (2-26), and (2-27) hold in Gromov–Witten theory as well. Thus, in this case the double Givental's J-functions in Gromov–Witten and stable quotients theories agree. If $\nu_n(\mathbf{a})=1$ and $\ell^-(\mathbf{a})=0$, the analogue of (2-21) in Gromov–Witten theory holds with $\{x+\hbar q\,\mathrm{d}/\mathrm{d}q\}$ replaced by $\{\mathbf{a}!q+x+\hbar q\,\mathrm{d}/\mathrm{d}q\}$ in (2-18). Finally, if $\nu_n(\mathbf{a})=0$ and $\ell^-(\mathbf{a}) \leq 1$, the analogue of (2-21) in Gromov–Witten theory holds with

$$\mathfrak{D}^s \dot{Z}_{n;\mathbf{a}}(x, \hbar, Q) = \frac{\dot{I}_1(q)}{\dot{I}_s(q)} \Big\{x+\hbar Q\frac{\mathrm{d}}{\mathrm{d}Q}\Big\} \mathfrak{D}^{s-1}\dot{Z}_{n;\mathbf{a}}(x, \hbar, Q)$$
$$\mathfrak{D}^s \ddot{Z}_{n;\mathbf{a}}(x, \hbar, Q) = \frac{\dot{I}_1(q)}{\ddot{I}_s(q)} \Big\{x+\hbar Q\frac{\mathrm{d}}{\mathrm{d}Q}\Big\} \mathfrak{D}^{s-1}\ddot{Z}_{n;\mathbf{a}}(x, \hbar, Q)$$

for all $s \in \mathbb{Z}^+$, where $Q = q\mathrm{e}^{J_{n;\mathbf{a}}(q)}$. The same comparison applies to the equivariant version of Theorem 2, Theorem 3 in Section 3, and its Gromov–Witten analogue; see [Popa 2012, Theorem 4.1] for the general toric case. Thus, just as is the case for the standard Givental's J-function, the mirror formulas for the double Givental's J-function in the stable quotients theory are simpler versions of the mirror formulas for the double Givental's J-function in the Gromov–Witten theory. Furthermore, just as in Gromov–Witten theory, the generating functions $\dot{Z}^{(s)}_{n;\mathbf{a}}$, $\ddot{Z}^{(s)}_{n;\mathbf{a}}$, and $\dot{Z}^*_{n;\mathbf{a}}$ above do not change when the tuple $(a_1, \dots, a_l)$ is replaced by $(a_1, \dots, a_l, 1)$; this is consistent with [Ciocan-Fontanine et al. 2014, Proposition 4.6.1].

Comparing Theorem 2 and [Cooper and Zinger 2014, Equation (1.7)] with [Popa 2012, Theorem 1.2] and the $m=3$ case of [Zinger 2014, Theorem A], we find that

$$\begin{aligned}\text{(2-22)}\quad &\dot{Z}^{\mathrm{GW}}_{n;\mathbf{a}}(x_1,\dots,x_m,\hbar_1,\dots,\hbar_m,Q)\\ &\qquad=\dot{I}_0(q)^{m-2}\mathrm{e}^{-J_{n;\mathbf{a}}(q)(x_1/\hbar_1+\cdots+x_m/\hbar_m)}\dot{Z}_{n;\mathbf{a}}(x_1,\dots,x_m,\hbar_1,\dots,\hbar_m,q)\end{aligned}$$

with $Q=q\mathrm{e}^{J_{n;\mathbf{a}}(q)}$ as before and $m=2,3$; we intend to extend this comparison to $m>3$ in a future paper. The same relations hold between the generating series $Z_{n:\mathbf{a}}$ described below. For $m=2,3$ and $b_1,b_2,b_3,c_1,c_2,c_3\ge 0$, let

$$\mathrm{SQ}^0_{n;\mathbf{a}}(\tau_{b_1}c_1,\tau_{b_2}c_2,\tau_{b_3}c_3)=\begin{cases}\langle\mathbf{a}\rangle & \text{if } b_1,b_2,b_3=0,\ c_1+c_2+c_3=n-1-\ell(\mathbf{a});\\ 0 & \text{otherwise;}\end{cases}$$

$$\mathrm{SQ}^0_{n;\mathbf{a}}(\tau_{b_1}c_1,\tau_{b_2}c_2)=\begin{cases}\langle\mathbf{a}\rangle & \text{if } b_1,b_2=0,\ c_1+c_2=n-2-\ell(\mathbf{a});\\ \dfrac{\langle\mathbf{a}\rangle}{2} & \text{if } \{b_1,b_2\}=\{0,-1\},\ c_1+c_2=n-1-\ell(\mathbf{a});\\ 0 & \text{otherwise;}\end{cases}$$

$$\mathrm{SQ}^d_{n;\mathbf{a}}(\tau_{b_1}c_1,\dots,\tau_{b_m}c_m)=\int_{\bar{Q}_{0,m}(\mathbb{P}^{n-1},d)}\mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a}})\prod_{i=1}^{m}(\psi_i^{b_i}\mathrm{ev}_i^*x^{c_i})\quad\text{for all } d\in\mathbb{Z}^+,$$

$$\mathrm{SQ}^{c_1,\dots,c_m}_{n;\mathbf{a}}(q)_{b_1,\dots,b_m}=\sum_{d=0}^{\infty}q^d\,\mathrm{SQ}^d_{n;\mathbf{a}}(\tau_{b_1}c_1,\dots,\tau_{b_m}c_m).$$

For $m=3$, the degree-0 terms are as expected; for $m=2$, the degree-0 terms are chosen to insure the necessary recursivity and polynomiality properties, as outlined in Section 5. Since GW invariants satisfy the divisor, string, and dilaton relations, (2-22) leads to modified versions of these relations for SQ invariants:

$$\begin{aligned}\text{(2-23)}\quad &\dot{I}_0(q)\dot{I}_1(q)\mathrm{SQ}^{c_1,c_2,1}_{n;\mathbf{a}}(q)_{b_1,b_2,0}=\\ &\qquad q\frac{\mathrm{d}}{\mathrm{d}q}\mathrm{SQ}^{c_1,c_2}_{n;\mathbf{a}}(q)_{b_1,b_2}+\mathrm{SQ}^{c_1+1,c_2}_{n;\mathbf{a}}(q)_{b_1-1,b_2}+\mathrm{SQ}^{c_1,c_2+1}_{n;\mathbf{a}}(q)_{b_1,b_2-1},\end{aligned}$$

$$\text{(2-24)}\qquad \dot{I}_0\mathrm{SQ}^{c_1,c_2,0}_{n;\mathbf{a}}(q)_{b_1,b_2,0}=\mathrm{SQ}^{c_1,c_2}_{n;\mathbf{a}}(q)_{b_1-1,b_2}+\mathrm{SQ}^{c_1,c_2}_{n;\mathbf{a}}(q)_{b_1,b_2-1},$$

$$\text{(2-25)}\qquad \mathrm{SQ}^{c_1,c_2,0}_{n;\mathbf{a}}(q)_{b_1,b_2,1}=-J_{n;\mathbf{a}}(q)\mathrm{SQ}^{c_1,c_2,1}_{n;\mathbf{a}}(q)_{b_1,b_2,0}.$$

The discrepancy from the corresponding relations of GW invariants is exhibited by the power series $\dot{I}_0$ and $\dot{I}_1$ (or equivalently $J_{n;\mathbf{a}}(q)$).

By (3-12), (3-9), (1-6), and (2-9)

(2-26)

$$\dot{Z}_{n;\mathbf{a}}(x,\hbar,q)=\frac{\dot{F}_{n;\mathbf{a}}(x/h,q/x^{\nu_n(\mathbf{a})})}{\dot{I}_0(q)},\qquad \ddot{Z}_{n;\mathbf{a}}(x,\hbar,q)=\frac{\ddot{F}_{n;\mathbf{a}}(x/h,q/x^{\nu_n(\mathbf{a})})}{\ddot{I}_0(q)},$$

if $\nu_n(a) \geq 0$.[2] These two formulas explicitly determine the basic stable quotients invariants appearing in (2-2) and (2-3). For $s \in \mathbb{Z}^+$, define

$$\mathfrak{D}^0\dot{F}_{n;\mathbf{a}}(w,q) = \frac{\dot{F}_{n;\mathbf{a}}(w,q)}{\dot{I}_0(q)}, \quad \mathfrak{D}^s\dot{F}_{n;\mathbf{a}}(w,q) = \frac{1}{\dot{I}_s(q)}\Big\{1+\frac{q}{w}\frac{\mathrm{d}}{\mathrm{d}q}\Big\}\mathfrak{D}^{s-1}\dot{F}_{n;\mathbf{a}}(w,q),$$
$$\mathfrak{D}^0\ddot{F}_{n;\mathbf{a}}(w,q) = \frac{\ddot{F}_{n;\mathbf{a}}(w,q)}{\ddot{I}_0(q)}, \quad \mathfrak{D}^s\ddot{F}_{n;\mathbf{a}}(w,q) = \frac{1}{\ddot{I}_s(q)}\Big\{1+\frac{q}{w}\frac{\mathrm{d}}{\mathrm{d}q}\Big\}\mathfrak{D}^{s-1}\ddot{F}_{n;\mathbf{a}}(w,q).$$

Combining (2-26) with (2-19) and (2-21), we find that

$$\begin{aligned}\text{(2-27)}\quad &\dot{Z}_{n;\mathbf{a}}(x_1,x_2,\hbar_1,\hbar_2,q)\\ &= \frac{1}{\hbar_1+\hbar_2}\sum_{\substack{s_1,s_2\geq 0\\ s_1+s_2=n-1}} x_1^{s_1}\dot{F}^{(s_1)}_{n;\mathbf{a}}\Big(\frac{x_1}{\hbar_1},\frac{q}{x_1^{\nu_n(\mathbf{a})}}\Big)\cdot x_2^{s_2}\ddot{F}^{(s_2)}_{n;\mathbf{a}}\Big(\frac{x_2}{\hbar_2},\frac{q}{x_2^{\nu_n(\mathbf{a})}}\Big),\end{aligned}$$

where

$$\text{(2-28)}\qquad \check{F}^{(s)}_{n;\mathbf{a}}(w,q) = \sum_{d=0}^{\infty}\sum_{s'=0}^{s-\nu_n(\mathbf{a})d}\frac{\tilde{\mathrm{c}}^{(d)}_{s-\ell^*(\mathbf{a}),s'-\ell^*(\mathbf{a})}q^d}{w^{s-\nu_n(\mathbf{a})d-s'}}\,\mathfrak{D}^{s'}\check{F}_{n;\mathbf{a}}(w,q),$$

with $(\check{F},\ell^*) = (\dot{F},\ell^-), (\ddot{F},\ell^+)$.[3] Thus, (2-26) and Theorem 2 provide closed formulas for the twisted genus-0 two-point and three-point SQ invariants of projective spaces.

The equivariant versions of the generating functions $\dot{Z}_{n;\mathbf{a}}$ defined in (2-7) are ideally suited for further computations, such as of genus-0 invariants with more marked points and of positive-genus twisted invariants with at least one marked point. However, for the purposes of computing the genus-0 two-point and three-point invariants, it is more natural to consider the generating functions

(2-29)
$$Z^*_{n;\mathbf{a}}(x_1,x_2,\hbar_1,\hbar_2,q) \equiv \sum_{d=1}^{\infty}q^d\{\mathrm{ev}_1\times\mathrm{ev}_2\}_*\bigg[\frac{e(\mathcal{V}^{(d)}_{n;\mathbf{a}})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\bigg],$$
$$Z^*_{n;\mathbf{a}}(x_1,x_2,x_3,\hbar_1,\hbar_2,\hbar_3,q) \equiv \sum_{d=1}^{\infty}q^d\{\mathrm{ev}_1\times\mathrm{ev}_2\times\mathrm{ev}_3\}_*\bigg[\frac{e(\mathcal{V}^{(d)}_{n;\mathbf{a}})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)(\hbar_3-\psi_3)}\bigg],$$

[2]The right-hand sides of these expressions should be first simplified in $\mathbb{Q}(x,\hbar)[\![q]\!]$, eliminating division by x, and then projected to $H^*(\mathbb{P}^{n-1})[\hbar][\![q]\!]$.

[3]The right-hand side of (2-27) should be first simplified in $\mathbb{Q}(x_1,x_2,\hbar_1,\hbar_2)[\![q]\!]$, eliminating division by x_1 and x_2, and then projected to $H^*(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})[\hbar_1,\hbar_2][\![q]\!]$.

where $\mathcal{V}_{n;\mathbf{a}}^{(d)}$ is given by (1-3) and the evaluation maps are as in (2-6). In the case $\ell(\mathbf{a}) \geq 0$, (2-27) immediately gives

$$
\text{(2-30)}\quad Z_{n;\mathbf{a}}^{*}(x_1,x_2,\hbar_1,\hbar_2,q) = \frac{\langle\mathbf{a}\rangle x_1^{\ell(\mathbf{a})}}{\hbar_1+\hbar_2} \sum_{\substack{s_1,s_2\geq 0\\ s_1+s_2=n-1}} \left(-x_1^{s_1}x_2^{s_2}+x_1^{s_1}x_2^{s_2}\dot{F}_{n;\mathbf{a}}^{(s_1)}\left(\frac{x_1}{\hbar_1},\frac{q}{x_1^{\nu_n(\mathbf{a})}}\right)\cdot\ddot{F}_{n;\mathbf{a}}^{(s_2)}\left(\frac{x_2}{\hbar_2},\frac{q}{x_2^{\nu_n(\mathbf{a})}}\right)\right)
$$

and similarly for the three-point generating function in (2-29). In general, (3-28), (3-30), (3-15), the second identity in (3-12), (3-31), the middle identity in (3-13), and (2-28) give

$$
\begin{aligned}
\text{(2-31)}\quad Z_{n;\mathbf{a}}^{*}(x_1,x_2,\hbar_1,\hbar_2,q) = \frac{\langle\mathbf{a}\rangle}{\hbar_1+\hbar_2} \sum_{\substack{s_1,s_2\geq 0\\ s_1+s_2=n-1}} \Bigg(& x_1^{s_1}x_2^{s_2+\ell(\mathbf{a})}\dot{F}_{n;\mathbf{a}}^{(s_2)*}\left(\frac{x_2}{\hbar_2},\frac{q}{x_2^{\nu_n(\mathbf{a})}}\right) \\
& + x_1^{s_1+\ell(\mathbf{a})}x_2^{s_2}\dot{F}_{n;\mathbf{a}}^{(s_1)*}\left(\frac{x_1}{\hbar_1},\frac{q}{x_1^{\nu_n(\mathbf{a})}}\right)\ddot{F}_{n;\mathbf{a}}^{(s_2)}\left(\frac{x_2}{\hbar_2},\frac{q}{x_2^{\nu_n(\mathbf{a})}}\right)\Bigg),
\end{aligned}
$$

where $\dot{F}_{n;\mathbf{a}}^{(s)*}(w,q) \equiv \dot{F}_{n;\mathbf{a}}^{(s)}(w,q)-1$.[4]

An analogue of (2-31) for the three-point generating function in (2-29) can be similarly obtained from (3-29), the last identity in (3-13), and (2-17). In particular, in the Calabi–Yau case, $\nu_n(\mathbf{a})=0$,

$$
\begin{aligned}
\text{(2-32)}\quad & Z_{n;\mathbf{a}}^{*}(x_1,x_2,x_3,\hbar_1,\hbar_2,\hbar_3,q) \\
&= \frac{\langle\mathbf{a}\rangle}{\hbar_1\hbar_2\hbar_3}\Bigg\{ \sum_{\substack{s_1,s_2,s_3\geq 0\\ s_1,s_2,s_3\leq n-1\\ s_1+s_2+s_3=2n-2}} \Bigg(x_1^{s_1}x_2^{s_2}x_3^{s_3+\ell(\mathbf{a})}\dot{F}_{n;\mathbf{a}}^{(s_3)*}\left(\frac{x_3}{\hbar_3},q\right) \\
&\qquad + x_1^{s_1}x_2^{s_2+\ell(\mathbf{a})}x_3^{s_3}\dot{F}_{n;\mathbf{a}}^{(s_2)*}\left(\frac{x_2}{\hbar_2},q\right)\ddot{F}_{n;\mathbf{a}}^{(s_3)}\left(\frac{x_3}{\hbar_3},q\right) \\
&\qquad + x_1^{s_1+\ell(\mathbf{a})}x_2^{s_2}x_3^{s_3}\tilde{\mathfrak{c}}_{s_1,s_2,s_3}(q)\dot{F}_{n;\mathbf{a}}^{(s_1)*}\left(\frac{x_1}{\hbar_1},q\right)\prod_{t=2}^{3}\ddot{F}_{n;\mathbf{a}}^{(s_t)}\left(\frac{x_t}{\hbar_t},q\right)\Bigg) \\
&\qquad + \sum_{\substack{s_1\geq\ell^-(\mathbf{a}),\, s_2,s_3\geq 0\\ s_1,s_2,s_3\leq n-1\\ s_1+s_2+s_3=2n-2}} x_1^{s_1+\ell(\mathbf{a})}x_2^{s_2}x_3^{s_3}\tilde{\mathfrak{c}}_{s_1,s_2,s_3}^{*}(q)\prod_{t=2}^{3}\ddot{F}_{n;\mathbf{a}}^{(s_t)}\left(\frac{x_t}{\hbar_t},q\right)\Bigg\},
\end{aligned}
$$

where

$$
\tilde{\mathfrak{c}}_{s_1,s_2,s_3}(q) \equiv 1+\tilde{\mathfrak{c}}_{s_1,s_2,s_3}^{*}(q) = \frac{1}{(1-\mathbf{a}^{\mathbf{a}}q)\dot{\mathbb{I}}_{s_1}^{c}(q)\ddot{\mathbb{I}}_{s_2}^{c}(q)\ddot{\mathbb{I}}_{s_3}^{c}(q)}.
$$

[4] The right-hand side of (2-31) should be first simplified in $\mathbb{Q}(x_1,x_2,\hbar_1,\hbar_2)[\![q]\!]$, eliminating division by x_1 and x_2, and then projected to $H^*(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})[\hbar_1,\hbar_2][\![q]\!]$.

This presentation of the three-point formula eliminates division by x_1, even if $\ell(\mathbf{a})<0$, since $\dot{F}^{(s)*}_{n;\mathbf{a}}(w,q)$ is divisible by $w^{\ell^-(\mathbf{a})-s}$ for $s\le\ell^-(\mathbf{a})$.

In the Calabi–Yau case, that is, $\nu_n(\mathbf{a})=0$, we find that

(2-33)
$$\langle\mathbf{a}\rangle+q\frac{\mathrm{d}}{\mathrm{d}q}\mathrm{SQ}^{c_1,c_2}_{n;\mathbf{a}}(q)=\langle\mathbf{a}\rangle\dot{I}_{c_1+1}(q),\qquad \mathrm{SQ}^{c_1,c_2,c_3}_{n;\mathbf{a}}(q)=\frac{\langle\mathbf{a}\rangle}{(1-\mathbf{a}^{\mathbf{a}}q)\prod_{t=1}^{t=3}\prod_{c=0}^{c=c_t}\dot{I}_c(q)},$$

whenever $c_1,c_2,c_3\in\mathbb{Z}^{\ge0}$, $c_1+c_2=n-2-\ell(\mathbf{a})$ in the first equation, and $c_1+c_2+c_3=n-1-\ell(\mathbf{a})$ in the second equation. The $c_1=0$ case of (2-33) agrees with the $W/\!\!/G=X_{n;\mathbf{a}}$ case of [Ciocan-Fontanine and Kim 2013, Corollary 5.5.4(bc)]. By (2-33),

$$\begin{aligned}\max(c_1,c_2)\ge n-\ell^+(\mathbf{a})\quad&\Rightarrow\quad \mathrm{SQ}^d_{n;\mathbf{a}}(c_1,c_2)(q)=0\ \text{ for all } d\in\mathbb{Z}^+,\\ \max(c_1,c_2,c_3)\ge n-\ell^+(\mathbf{a})\quad&\Rightarrow\quad \mathrm{SQ}^d_{n;\mathbf{a}}(c_1,c_2,c_3)(q)=0\ \text{ for all } d\in\mathbb{Z}^+,\end{aligned}$$

as the case should be for intrinsic invariants of $X_{n;\mathbf{a}}$. On the other hand,

(2-34)
$$\begin{aligned}\langle\mathbf{a}\rangle+Q\frac{\mathrm{d}}{\mathrm{d}Q}\mathrm{GW}^{c_1,c_2}_{n;\mathbf{a}}(Q)&=\langle\mathbf{a}\rangle\frac{\dot{I}_{c_1+1}(q)}{\dot{I}_1(q)},\\ \mathrm{GW}^{c_1,c_2,c_3}_{n;\mathbf{a}}(Q)&=\frac{\langle\mathbf{a}\rangle\dot{I}_0(q)}{(1-\mathbf{a}^{\mathbf{a}}q)\prod_{t=1}^{t=3}\prod_{c=0}^{c=c_t}\dot{I}_c(q)},\end{aligned}$$

with the same assumptions on c_1,c_2,c_3 as in (2-33) and $Q=q\mathrm{e}^{J_{n;\mathbf{a}}(q)}$, as before; see [Popa and Zinger 2014, Equation (1.5)] and [Zinger 2014, Equation (1.7)], respectively.

In the case of products of projective spaces and concavex sheaves (1-13), the analogues of the above mirror formulas relate power series:

(2-35) $\check{F}_{n_1,\dots,n_p;\mathbf{a}}\in\mathbb{Q}(w)[\![q_1,\dots,q_p]\!]$,

(2-36) $\check{Z}^{(s_1,\dots,s_p)}_{n_1,\dots,n_p;\mathbf{a}}\in H^*(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1})[\hbar^{-1}][\![q_1,\dots,q_p]\!]$,

(2-37) $\check{Z}^*_{n_1,\dots,n_p;\mathbf{a}}\in H^*((\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1})^m)[\hbar_1^{-1},\dots,\hbar_m^{-1}][\![q_1,\dots,q_p]\!]$,

with $\check{F}$ and $\check{Z}$ denoting F, $\dot{F}$, $\ddot{F}$, Z, $\dot{Z}$, or $\ddot{Z}$ and $m=2,3$. The coefficients of $q_1^{d_1}\cdots q_p^{d_p}$ in (2-36) and (2-37) are defined by the same pushforwards as in (2-4), (2-5), (2-8), and (2-29), with the degree d of the stable quotients replaced by $(d_1,\dots,d_p)$ and x^s by $x_1^{s_1}\cdots x_p^{s_p}$. The coefficients of $q_1^{d_1}\cdots q_p^{d_p}$ in (2-35) are obtained from the coefficients in (1-6), (2-9), and (2-12) by replacing a_kd and a_kx by $a_{k;1}d_1+\cdots+a_{k;p}d_p$ and $a_{k;1}x_1+\cdots+a_{k;p}x_p$ in the numerator and taking the product of the denominators with $(n,x,d)=(n_i,x_i,d_i)$ for each $i=1,\dots,p$;

$$x_1,\dots,x_p\in H^*(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1})$$

now correspond to the pullbacks of the hyperplane classes by the projection maps. If $\ell^-(\mathbf{a})=0$, the analogue of (2-30) with $\langle\mathbf{a}\rangle x^{\ell(\mathbf{a})}$ replaced by the products of $a_{k;1}x_{1;1}+\cdots+a_{k;p}x_{1;p}$ and sums over pairs of p-tuples $(s_{1;1},\ldots,s_{1;p})$ and $(s_{2;1},\ldots,s_{2;p})$ with $s_{1;i}+s_{2;i}=n_i-1$ provides a closed formula for $Z^*_{n_1,\ldots,n_p;\mathbf{a}}$. In general, the relation (2-31) extends to this case by replacing $\langle\mathbf{a}\rangle x_i^{\ell(\mathbf{a})}$ by the products and ratios of the terms $a_{k;1}x_{i;1}+\cdots+a_{k;p}x_{i;p}$.

3. Equivariant mirror formulas

We begin this section by reviewing the equivariant setup used in [Zinger 2009; Popa and Zinger 2014; Cooper and Zinger 2014], closely following [Cooper and Zinger 2014, Section 3]. After defining equivariant versions of the generating functions $\dot{Z}^{(s)}_{n;\mathbf{a}}$, $\ddot{Z}^{(s)}_{n;\mathbf{a}}$, $\dot{Z}^*_{n;\mathbf{a}}$, and $Z^*_{n;\mathbf{a}}$ and of the hypergeometric series $\dot{F}_{n;\mathbf{a}}$ and $\ddot{F}_{n;\mathbf{a}}$, we state an equivariant version of Theorem 2; see Theorem 3 below. This theorem immediately implies Theorem 2. The proof of the two-point mirror formulas in Theorem 3 is outlined in Sections 5 and 6 and completed in Sections 7 and 8. We conclude this section with a specialization of the three-point formula of Theorem 3 in Proposition 3.1, which is proved in Section 9 and is a key step in the proof of the full three-point formula of Theorem 3 in Section 10.

3A. ***Equivariant setup.*** The quotient of the classifying space for the n-torus $\mathbb{T}$ is $B\mathbb{T}\equiv(\mathbb{P}^\infty)^n$. Thus, the group cohomology of $\mathbb{T}$ is

$$H^*_{\mathbb{T}}\equiv H^*(B\mathbb{T})=\mathbb{Q}[\alpha_1,\ldots,\alpha_n],$$

where $\alpha_i\equiv\pi_i^*c_1(\gamma^*)$, $\gamma\to\mathbb{P}^\infty$ is the tautological line bundle, and $\pi_i:(\mathbb{P}^\infty)^n\to\mathbb{P}^\infty$ is the projection to the i-th component. The field of fractions of $H^*_{\mathbb{T}}$ will be denoted by

$$\mathbb{Q}_\alpha\equiv\mathbb{Q}(\alpha_1,\ldots,\alpha_n).$$

We denote the equivariant $\mathbb{Q}$-cohomology of a topological space M with a $\mathbb{T}$-action by $H^*_{\mathbb{T}}(M)$. If the $\mathbb{T}$-action on M lifts to an action on a complex vector bundle $V\to M$, let $\mathbf{e}(V)\in H^*_{\mathbb{T}}(M)$ denote the equivariant Euler class of V. A continuous $\mathbb{T}$-equivariant map $f:M\to M'$ between two compact oriented manifolds induces a pushforward homomorphism

$$f_*:H^*_{\mathbb{T}}(M)\to H^*_{\mathbb{T}}(M').$$

The standard action of $\mathbb{T}$ on $\mathbb{C}^n$,

$$(\mathrm{e}^{\mathrm{i}\theta_1},\ldots,\mathrm{e}^{\mathrm{i}\theta_n})\cdot(z_1,\ldots,z_n)\equiv(\mathrm{e}^{\mathrm{i}\theta_1}z_1,\ldots,\mathrm{e}^{\mathrm{i}\theta_n}z_n),$$

descends to a $\mathbb{T}$-action on $\mathbb{P}^{n-1}$, which has n fixed points:

$$P_1=[1,0,\ldots,0],\quad P_2=[0,1,0,\ldots,0],\quad\ldots,\quad P_n=[0,\ldots,0,1]. \tag{3-1}$$

This standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ lifts to a natural $\mathbb{T}$-action on the tautological line bundle $\gamma \to \mathbb{P}^{n-1}$, since $\gamma \subset \mathbb{P}^{n-1} \times \mathbb{C}^n$ is preserved by the diagonal $\mathbb{T}$-action. With

$$\mathbf{x} \equiv \mathbf{e}(\gamma^*) \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})$$

denoting the equivariant hyperplane class, the equivariant cohomology of $\mathbb{P}^{n-1}$ is given by

$$H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) = \mathbb{Q}[\mathbf{x}, \alpha_1, \dots, \alpha_n] \big/ (\mathbf{x}-\alpha_1)\cdots(\mathbf{x}-\alpha_n). \tag{3-2}$$

Let $\mathbf{x}_t \in H_{\mathbb{T}}^*((\mathbb{P}^{n-1})^m)$ be the pullback of $\mathbf{x}$ by the t-th projection map.

The standard $\mathbb{T}$-representation on $\mathbb{C}^n$ (as well as any other representation) induces $\mathbb{T}$-actions on $\overline{Q}_{0,m}(\mathbb{P}^{n-1}, d)$, $\mathcal{U}$, $\mathcal{V}_{n;\mathbf{a}}^{(d)}$, $\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}$, and $\ddot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}$; see (1-3) and (2-1) for the notation. Thus, $\mathcal{V}_{n;\mathbf{a}}^{(d)}$, $\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}$, and $\ddot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}$ have well-defined equivariant Euler classes

$$\mathbf{e}\big(\mathcal{V}_{n;\mathbf{a}}^{(d)}\big), \mathbf{e}\big(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}\big), \mathbf{e}\big(\ddot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}\big) \in H_{\mathbb{T}}^*(\overline{Q}_{0,m}(\mathbb{P}^{n-1}, d)).$$

The universal cotangent line bundle for the i-th marked point also has a well-defined equivariant Euler class, which will still be denoted by ψ_i.

Similarly to (2-2) and (2-3), let

$$\begin{aligned}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) &\equiv 1 + \sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{\hbar - \psi_1}\right] \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar^{-1}, q]\!],\\ \ddot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) &\equiv 1 + \sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\ddot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{\hbar - \psi_1}\right] \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar^{-1}, q]\!].\end{aligned} \tag{3-3}$$

For each $s \in \mathbb{Z}^{\geq 0}$, let

$$\begin{aligned}\dot{\mathcal{Z}}_{n;\mathbf{a}}^{(s)}(\mathbf{x}, \hbar, q) &\equiv \mathbf{x}^s + \sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})\mathrm{ev}_2^*\mathbf{x}^s}{\hbar - \psi_1}\right] \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar^{-1}, q]\!],\\ \ddot{\mathcal{Z}}_{n;\mathbf{a}}^{(s)}(\mathbf{x}, \hbar, q) &\equiv \mathbf{x}^s + \sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\ddot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})\mathrm{ev}_2^*\mathbf{x}^s}{\hbar - \psi_1}\right] \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar^{-1}, q]\!].\end{aligned} \tag{3-4}$$

Similarly to (2-4) and (2-5), we define

$$\begin{aligned}&\dot{\mathcal{Z}}_{n;\mathbf{a}}^*(\mathbf{x}_1, \mathbf{x}_2, \hbar_1, \hbar_2, q)\\ &\quad\equiv \sum_{d=1}^{\infty} q^d \{\mathrm{ev}_1 \times \mathrm{ev}_2\}_*\left[\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)}\right] \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[\![\hbar_1^{-1}, \hbar_2^{-1}, q]\!],\end{aligned} \tag{3-5}$$

$$\begin{aligned}&\dot{\mathcal{Z}}_{n;\mathbf{a}}^*(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \hbar_1, \hbar_2, \hbar_3, q)\\ &\quad\equiv \sum_{d=1}^{\infty} q^d \{\mathrm{ev}_1 \times \mathrm{ev}_2 \times \mathrm{ev}_3\}_*\left[\frac{e(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)(\hbar_3 - \psi_3)}\right],\end{aligned} \tag{3-6}$$

with the total pushforwards by the total evaluation maps taken in equivariant cohomology. Similarly to (2-7), let

$$\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q)=\frac{\mathbf{PD}(\Delta^{(2)}_{\mathbb{P}^{n-1}})}{\hbar_1+\hbar_2}+\dot{\mathcal{Z}}^*_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q),\tag{3-7}$$
$$\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\hbar_1,\hbar_2,\hbar_3,q)=\frac{\mathbf{PD}(\Delta^{(3)}_{\mathbb{P}^{n-1}})}{\hbar_1\hbar_2\hbar_3}+\dot{\mathcal{Z}}^*_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\hbar_1,\hbar_2,\hbar_3,q),$$

where $\mathbf{PD}(\Delta^{(2)}_{\mathbb{P}^{n-1}})$ and $\mathbf{PD}(\Delta^{(3)}_{\mathbb{P}^{n-1}})$ are the equivariant Poincaré duals of the (small) diagonals in $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ and $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$, respectively.

The above Poincaré duals can be written as

(3-8)
$$\mathbf{PD}(\Delta^{(2)}_{\mathbb{P}^{n-1}})=\sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}}(-1)^r\mathbf{s}_r\mathbf{x}_1^{s_1}\mathbf{x}_2^{s_2},$$
$$\mathbf{PD}(\Delta^{(3)}_{\mathbb{P}^{n-1}})=\sum_{\substack{s_1,s_2,s_3,r\ge0\\ s_1+s_2+s_3+r=2n-2}}\mathbf{s}^{(2)}_r\mathbf{x}_1^{s_1}\mathbf{x}_2^{s_2}\mathbf{x}_3^{s_3}$$
$$=\sum_{\substack{s_1,s_2,s_3,r\ge0\\ s_1,s_2,s_3\le n-1\\ s_1+s_2+s_3+r=2n-2}}\sum_{\substack{r_0,r_1,r_2,r_3\ge0\\ r_1\le\hat{s}_1,r_2\le\hat{s}_2,r_3\le\hat{s}_3\\ r_0+r_1+r_2+r_3=r}}(-1)^{r_1+r_2+r_3}\eta_{r_0}\mathbf{s}_{r_1}\mathbf{s}_{r_2}\mathbf{s}_{r_3}\mathbf{x}_1^{s_1}\mathbf{x}_2^{s_2}\mathbf{x}_3^{s_3},$$

where $\mathbf{s}_r,\eta_r,\mathbf{s}^{(2)}_r\in\mathbb{Q}[\alpha_1,\dots,\alpha_n]$ are the r-th elementary symmetric polynomial in $\alpha_1,\dots,\alpha_n$, the sum of all degree-r monomials in $\alpha_1,\dots,\alpha_n$, and the degree-r term in $(1-\mathbf{s}_1+\mathbf{s}_2-\cdots)^2$, respectively. All three expressions for the Poincaré duals can be confirmed by pairing them with $\mathbf{x}_1^{t_1}\mathbf{x}_2^{t_2}$ and $\mathbf{x}_1^{t_1}\mathbf{x}_2^{t_2}\mathbf{x}_3^{t_3}$, with $t_1,t_2,t_3\le n-1$, and using the localization theorem of [Atiyah and Bott 1984] on $(\mathbb{P}^{n-1})^m$ and the residue theorem on S^2 to reduce the equivariant integrals of $\mathbf{x}^{s+t}$ on $\mathbb{P}^{n-1}$ to the polynomials η_r; these are the homogeneous polynomials in the power series expansion of $1/(1-\alpha_1)(1-\alpha_2)\cdots$. The coefficient of $\mathbf{x}_1^{s_1}\mathbf{x}_2^{s_2}\mathbf{x}_3^{s_3}$ in the second expression for $\mathbf{PD}(\Delta^{(3)}_{\mathbb{P}^{n-1}})$ is precisely $\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(0)$, with $\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}$ as in Theorem 3; see the end of Section 3B. This provides a direct check of the degree-0 term in (3-14).

3B. ***Equivariant mirror symmetry.*** The equivariant analogues of the power series in (1-6) and (2-9) are given by

$$\dot{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\equiv\sum_{d=0}^{\infty}q^d\frac{\prod_{a_k>0}\prod_{r=1}^{a_kd}(a_k\mathbf{x}+r\hbar)\prod_{a_k<0}\prod_{r=0}^{-a_kd-1}(a_k\mathbf{x}-r\hbar)}{\prod_{r=1}^{d}\Big(\prod_{k=1}^{n}(\mathbf{x}-\alpha_k+r\hbar)-\prod_{k=1}^{n}(\mathbf{x}-\alpha_k)\Big)}\tag{3-9}$$
$$\in\mathbb{Q}[\alpha_1,\dots,\alpha_n,\mathbf{x}][\![\hbar^{-1},q]\!],$$

$$\ddot{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\equiv\sum_{d=0}^{\infty}q^d\frac{\prod_{a_k>0}\prod_{r=0}^{a_kd-1}(a_k\mathbf{x}+r\hbar)\prod_{a_k<0}\prod_{r=1}^{-a_kd}(a_k\mathbf{x}-r\hbar)}{\prod_{r=1}^{d}\Big(\prod_{k=1}^{n}(\mathbf{x}-\alpha_k+r\hbar)-\prod_{k=1}^{n}(\mathbf{x}-\alpha_k)\Big)} \in\mathbb{Q}[\alpha_1,\dots,\alpha_n,\mathbf{x}][\![\hbar^{-1},q]\!].$$

The second products in the denominators above are irrelevant for the statements in this section, but are material to (4-9) and thus to the proof of (3-14) in this paper.

For each $s\in\mathbb{Z}^+$, we define $\mathfrak{D}^s\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$, $\mathfrak{D}^s\ddot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar^{-1},q]\!]$ inductively by

$$\begin{aligned}
&\mathfrak{D}^0\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q),\\
&\mathfrak{D}^s\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\frac{1}{\dot{I}_s(q)}\Big\{\mathbf{x}+\hbar q\frac{\mathrm{d}}{\mathrm{d}q}\Big\}\mathfrak{D}^{s-1}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q),\\
&\mathfrak{D}^0\ddot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\ddot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q),\\
&\mathfrak{D}^s\ddot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\frac{1}{\ddot{I}_s(q)}\Big\{\mathbf{x}+\hbar q\frac{\mathrm{d}}{\mathrm{d}q}\Big\}\mathfrak{D}^{s-1}\ddot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q).
\end{aligned}\tag{3-10}$$

The next theorem is the equivariant analogue of Theorem 2. It expresses the equivariant stable quotient invariants in (3-5) and (3-6) in terms of the basic equivariant stable quotient invariants in (3-3).

Theorem 3. *If $l\in\mathbb{Z}^{\ge0}$, $n\in\mathbb{Z}^+$, and $\mathbf{a}\in(\mathbb{Z}^*)^l$, then*

$$\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q) =\frac{1}{\hbar_1+\hbar_2}\sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}}(-1)^r\mathbf{s}_r\,\dot{\mathcal{Z}}^{(s_1)}_{n;\mathbf{a}}(\mathbf{x}_1,\hbar_1,q)\,\ddot{\mathcal{Z}}^{(s_2)}_{n;\mathbf{a}}(\mathbf{x}_2,\hbar_2,q),\tag{3-11}$$

where $\mathbf{s}_r\in\mathbb{Q}_\alpha$ is the r-th elementary symmetric polynomial in $\alpha_1,\dots,\alpha_n$. If in addition $\nu_n(\mathbf{a})\ge0$,

$$\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\frac{\dot{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)}{\dot{I}_0(q)},\qquad \ddot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\frac{\ddot{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)}{\ddot{I}_0(q)},\tag{3-12}$$

and there exist $\widetilde{\mathcal{C}}^{(r)}_{s,s'},\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}\in\mathbb{Q}[\alpha_1,\dots,\alpha_n][\![q]\!]$ with $s,s',s_1,s_2,s_3,r\in\mathbb{Z}^{\ge0}$ such that

$$\widetilde{\mathcal{C}}^{(r)}_{s,s'}(0)=\delta_{0,r}\delta_{s,s'},\tag{3-13}$$

$$[\![\widetilde{\mathcal{C}}^{(\nu_n(\mathbf{a})d)}_{s,s'}(q)\big|_{\alpha=0}]\!]_d=\tilde{\mathrm{c}}^{(d)}_{s,s'},\qquad [\![\widetilde{\mathcal{C}}^{(\nu_n(\mathbf{a})d)}_{s_1,s_2,s_3}(q)\big|_{\alpha=0}]\!]_d=\tilde{\mathrm{c}}^{(d)}_{s_1,s_2,s_3},$$

the coefficients of q^d in $\widetilde{\mathcal{C}}^{(r)}_{s,s'}(q)$ and $\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(q)$ are homogeneous symmetric polynomials in $\alpha_1,\alpha_2,\dots,\alpha_n$ of degree $r-\nu_n(\mathbf{a})d$, and

$$\begin{aligned}\text{(3-14)}\quad &\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\hbar_1,\hbar_2,\hbar_3,q)\\ &=\frac{1}{\hbar_1\hbar_2\hbar_3}\sum_{\substack{r,s_1,s_2,s_3\ge 0\\ s_1,s_2,s_3\le n-1\\ s_1+s_2+s_3+r=2n-2}}\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(q)\dot{\mathcal{Z}}^{(s_1)}_{n;\mathbf{a}}(\mathbf{x}_1,\hbar_1,q)\prod_{t=2}^{3}\ddot{\mathcal{Z}}^{(s_t)}_{n;\mathbf{a}}(\mathbf{x}_t,\hbar_t,q),\end{aligned}$$

$$\text{(3-15)}\qquad \check{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\sum_{r=0}^{s}\sum_{s'=0}^{s-r}\widetilde{\mathcal{C}}^{(r)}_{s-\ell^*(\mathbf{a}),s'-\ell^*(\mathbf{a})}(q)\,\hbar^{s-r-s'}\mathfrak{D}^{s'}\check{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q),$$

where $(\check{\mathcal{Z}},\ell^*)=(\dot{\mathcal{Z}},\ell^-),(\ddot{\mathcal{Z}},\ell^+)$.

Setting $\alpha=0$ in (3-11), (3-12), (3-14), and (3-15), we obtain (2-19), (2-26), (2-20) and (2-21), respectively.

We now completely describe the power series $\widetilde{\mathcal{C}}^{(r)}_{s,s'}$ of Theorem 3; it will be shown in Section 5 that they indeed satisfy (3-15). Let

$$\begin{aligned}\text{(3-16)}\quad &\mathfrak{D}^0\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\\ &=\frac{1}{I_0(q)}\sum_{d=0}^{\infty}q^d\frac{\prod_{a_k>0}\prod_{r=1}^{a_kd}(a_k\mathbf{x}+r\hbar)\prod_{a_k<0}\prod_{r=1}^{-a_kd}(a_k\mathbf{x}-r\hbar)}{\prod_{r=1}^{d}\prod_{k=1}^{n}(\mathbf{x}-\alpha_k+r\hbar)}.\end{aligned}$$

For $s\in\mathbb{Z}^+$, let

$$\begin{aligned}\text{(3-17)}\quad \mathfrak{D}^s\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)&=\frac{1}{I_s(q)}\Big\{\mathbf{x}+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\Big\}\mathfrak{D}^{s-1}\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\\ &\in\mathbf{x}^s+q\cdot\mathbb{Q}[\alpha_1,\dots,\alpha_n,\mathbf{x}][\hbar][\![\hbar^{-1},q]\!].\end{aligned}$$

Comparing with (2-12), we find that

$$\text{(3-18)}\qquad \mathfrak{D}^s\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\big|_{\alpha=0}=\mathbf{x}^s\mathfrak{D}^sF_{n;\mathbf{a}}(\mathbf{x}/\hbar,q/\mathbf{x}^{\nu_n(\mathbf{a})}),\quad\text{where}$$

$$\mathfrak{D}^0F_{n;\mathbf{a}}(w,q)=\frac{F_{n;\mathbf{a}}(w,q)}{I_0(q)},\qquad \mathfrak{D}^sF_{n;\mathbf{a}}(w,q)=\frac{1}{I_s(q)}\Big\{1+\frac{q}{w}\frac{\mathrm{d}}{\mathrm{d}q}\Big\}\mathfrak{D}^{s-1}F_{n;\mathbf{a}}(w,q)$$

for all $s\in\mathbb{Z}^+$. For $r,s,s'\ge 0$, define $\mathcal{C}^{(r)}_{s,s'}\in\mathbb{Q}[\alpha_1,\dots,\alpha_n][\![q]\!]$ by

$$\text{(3-19)}\qquad \hbar^s\sum_{s'=0}^{\infty}\sum_{r=0}^{s'}\mathcal{C}^{(r)}_{s,s'}(q)\mathbf{x}^{s'-r}\hbar^{-s'}=\mathfrak{D}^s\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q).$$

By (3-16), (3-17), and (3-19), the coefficient of q^d in $\mathcal{C}^{(r)}_{s,s'}$ is a degree-$r-\nu_n(\mathbf{a})d$ homogeneous symmetric polynomial in α. By (3-17) and (3-18),

$$\text{(3-20)}\qquad \mathcal{C}^{(0)}_{s,s}(q)=1,\quad \mathcal{C}^{(0)}_{s,s'}(q)=0\ \text{ for } s>s',\quad \mathcal{C}^{(r)}_{s,s'}(0)=\delta_{r,0}\delta_{s,s'}.$$

By the first two statements above, the relations

$$\sum_{\substack{r_1,r_2\ge 0\\ r_1+r_2=r}}\sum_{t=0}^{s-r_1}\tilde{\mathcal{C}}^{(r_1)}_{s,t}(q)\mathcal{C}^{(r_2)}_{t,s'-r_1}(q)=\delta_{r,0}\delta_{s,s'}\ \text{ for } r,s'\in\mathbb{Z}^{\ge 0},\ r\le s'\le s, \tag{3-21}$$

inductively define $\tilde{\mathcal{C}}^{(r)}_{s,s'-r}\in\mathbb{Q}[\alpha_1,\dots,\alpha_n][\![q]\!]$ with $r\le s'\le s$ in terms of the power series $\tilde{\mathcal{C}}^{(r_1)}_{s,t}$ with $r_1<r$ or $r_1=r$ and $t<s'-r$. By (3-20) and (3-21),

$$\tilde{\mathcal{C}}^{(0)}_{s,s'}=\delta_{s,s'},\qquad \tilde{\mathcal{C}}^{(r)}_{s,s'}(0)=\delta_{r,0}\delta_{s,s'},$$

and the coefficient of q^d in $\tilde{\mathcal{C}}^{(r)}_{s,s'}$ is a degree-$r-\nu_n(\mathbf{a})d$ homogeneous symmetric polynomial in α. If $s'<0$, we set $\tilde{\mathcal{C}}^{(r)}_{s,s'}=\delta_{r,0}\delta_{s,s'}$. If $\nu_n(\mathbf{a})>0$,

$$\mathcal{C}^{(\nu_n(\mathbf{a})d)}_{s,s'}\big|_{\alpha=0}=\mathrm{c}^{(d)}_{s,s'-\nu_n(\mathbf{a})d}q^d\ \text{ for all } s'\ge\nu_n(\mathbf{a})d$$

by (3-19), (3-18), and (2-13). Thus, setting $\alpha=0$ in (3-21) and comparing with (2-14) with s' replaced by $s'-\nu_n(\mathbf{a})d$, we obtain the second identity in (3-13).

We next completely describe the power series $\tilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}$ of Theorem 3; it will be shown in Section 10 that they indeed satisfy (3-14). For each $r\in\mathbb{Z}^{\ge 0}$, let $p_r,\mathcal{H}^{(r)}\in\mathbb{Q}[z_1,z_2,\dots]$ be such that

$$p_r(\alpha_1,\alpha_2,\dots)=\alpha_1^r+\alpha_2^r+\dots=\mathcal{H}^{(r)}(\mathbf{s}_1,\mathbf{s}_2,\dots). \tag{3-22}$$

For $r,\nu\in\mathbb{Z}^{\ge 0}$, we define $\mathcal{H}^{(r)}_\nu\in\mathbb{Q}[\mathbf{s}_1,\mathbf{s}_2,\dots][\![z]\!]$ by

$$\mathcal{H}^{(r)}_\nu(z)=\begin{cases}(1-z)^{-1} & \text{if } \nu=0, r=0;\\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{H}^{(r)}\big((1-z)^{-1}\mathbf{s}_1,(1-z)^{-1}\mathbf{s}_2,\dots\big) & \text{if } \nu=0, r\ge 1;\\ \frac{1}{r+\nu}\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{H}^{(r+\nu)}(\mathbf{s}_1,\dots,\mathbf{s}_{\nu-1},\mathbf{s}_\nu-(-1)^\nu z,\mathbf{s}_{\nu+1},\dots) & \text{if } \nu>0.\end{cases} \tag{3-23}$$

In particular, the coefficient of z^d in $\mathcal{H}^{(r)}_\nu(z)$ is a degree-$r-\nu d$ homogeneous symmetric polynomial in α,

$$\mathcal{H}^{(r)}_\nu(0)=\eta_r,\qquad [\![\mathcal{H}^{(\nu d)}_\nu(z)\big|_{\alpha=0}]\!]_d=1. \tag{3-24}$$

The second identity above follows from [Zinger 2014, Lemma B.3]. Using induction via Newton's identity [Artin 1991, page 577], the first identity in (3-24) can be reduced to

$$\sum_{t=0}^{r}(-1)^t\eta_{r-t}\mathbf{s}_t=0,\qquad \sum_{t=0}^{r}(-1)^t(r-t)\eta_{r-t}\mathbf{s}_t=p_r\ \text{ for all } r\in\mathbb{Z}^+;$$

these two identities are equivalent to

$$\frac{(1-\alpha_1 u)(1-\alpha_2 u)\cdots}{(1-\alpha_1 u)(1-\alpha_2 u)\cdots}=1,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{(1-\alpha_1 u)(1-\alpha_2 u)\cdots}{(1-\alpha_1 uz)(1-\alpha_2 uz)\cdots}\bigg|_{z=0}=\frac{\alpha_1 u}{1-\alpha_1 u}+\frac{\alpha_2 u}{1-\alpha_2 u}+\cdots$$

Let

$$\text{(3-25)}\quad \widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(q) = \sum_{\substack{r_0,r_1,r_2,r_3\ge 0\\ r_1\le \hat{s}_1, r_2\le\hat{s}_2, r_3\le\hat{s}_3\\ r_0+r_1+r_2+r_3=r}} \frac{\mathcal{H}^{(r_0)}_{\nu_n(\mathbf{a})}(\mathbf{a}^{\mathbf{a}}q)}{\dot{\mathbb{I}}^c_{s_1+r_1}(q)\ddot{\mathbb{I}}^c_{s_2+r_2}(q)\dot{\mathbb{I}}^c_{s_3+r_3}(q)}\ddot{\mathcal{C}}^{(r_1)}_{\hat{s}_1}(q)\dot{\mathcal{C}}^{(r_2)}_{\hat{s}_2}(q)\dot{\mathcal{C}}^{(r_3)}_{\hat{s}_3}(q),$$

where

$$\text{(3-26)}\qquad \check{\mathcal{C}}^{(r)}_s(q)=\sum_{\substack{r',r''\ge 0\\ r'+r''=r}}(-1)^{r'}\mathbf{s}_{r'}\check{\mathcal{C}}^{(r'')}_{s-r'-\ell^*(\mathbf{a}),s-r-\ell^*(\mathbf{a})}(q)$$

with $(\check{\mathcal{C}},\ell^*)=(\dot{\mathcal{C}},\ell^-),(\ddot{\mathcal{C}},\ell^+)$. Since the coefficients of q^d in $\mathcal{H}^{(r)}_\nu$ and in $\widetilde{\mathcal{C}}^{(r)}_{s,s'}$ are degree-$r-\nu_n(\mathbf{a})d$ homogeneous symmetric polynomials in α, the coefficient of q^d in $\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}$ is also a degree-$r-\nu_n(\mathbf{a})d$ homogeneous symmetric polynomial in α. The last identity in (3-13) follows from (3-25), the second identity in (3-24), the middle identity in (3-13), and (2-15).

3C. ***Related mirror formulas.*** Similarly to (2-29), we define

$$\mathcal{Z}^*_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q)\equiv\sum_{d=1}^{\infty}q^d\{\mathrm{ev}_1\times\mathrm{ev}_2\}_*\left[\frac{\mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a}})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\right],$$

$$\text{(3-27)}\quad \mathcal{Z}^*_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\hbar_1,\hbar_2,\hbar_3,q) \equiv\sum_{d=1}^{\infty}q^d\{\mathrm{ev}_1\times\mathrm{ev}_2\times\mathrm{ev}_3\}_*\left[\frac{e(\mathcal{V}^{(d)}_{n;\mathbf{a}})}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)(\hbar_3-\psi_3)}\right],$$

with the evaluation maps as in (2-6). For each $s\in\mathbb{Z}^{\ge 0}$, let

$$\mathcal{Z}^{(s)*}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\equiv\sum_{d=1}^{\infty}q^d\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a}})\mathrm{ev}_2^*\mathbf{x}^s}{\hbar-\psi_1}\right]\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar^{-1},q]\!],$$

$$\ddot{\mathcal{Z}}^{(s)*}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\equiv\sum_{d=1}^{\infty}q^d\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a}})\mathrm{ev}_2^*\mathbf{x}^s}{\hbar-\psi_1}\right]\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar^{-1},q]\!].$$

Since $\mathbf{x}_1, \mathbf{x}_2 \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}) \otimes_{\mathbb{Q}[\alpha_1,\dots,\alpha_n]} \mathbb{Q}_\alpha$ are invertible, the first equation in (3-8) gives

$$\begin{aligned}\langle\mathbf{a}\rangle \sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} &(-1)^r \mathbf{s}_r \mathbf{x}_1^{s_1} \ddot{\mathcal{Z}}^{(s_2)*}_{n;\mathbf{a}}(\mathbf{x}_2,\hbar,q)\\
&= \sum_{d=1}^{\infty} q^d \{\mathrm{id}\times \mathrm{ev}_1\}_* \left[\frac{\pi_2^*\mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a}})\,\{\mathrm{id}\times\mathrm{ev}_2\}^*(\mathbf{PD}(\Delta_{\mathbb{P}^{n-1}})\mathbf{x}_2^{-\ell(\mathbf{a})})}{\hbar-\psi_1}\right]\\
&= \sum_{d=1}^{\infty} q^d \{\mathrm{id}\times \mathrm{ev}_1\}_* \left[\frac{\pi_2^*\mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a}})\,\{\mathrm{id}\times\mathrm{ev}_2\}^*(\mathbf{PD}(\Delta_{\mathbb{P}^{n-1}})\mathbf{x}_1^{-\ell(\mathbf{a})})}{\hbar-\psi_1}\right]\\
&= \mathbf{x}_1^{-\ell(\mathbf{a})} \sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r \mathbf{s}_r \mathbf{x}_1^{s_1} \mathcal{Z}^{(s_2)*}_{n;\mathbf{a}}(\mathbf{x}_2,\hbar,q),\end{aligned}$$

where $\pi_2: \mathbb{P}^{n-1}\times\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)\to\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)$ is the projection map. Combining the last identity with (3-11), we obtain

$$\text{(3-28)}\quad \begin{aligned}&\mathcal{Z}^*_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q)\\
&\quad= \frac{1}{\hbar_1+\hbar_2}\sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r \mathbf{s}_r \Big(\mathbf{x}_1^{s_1}\mathcal{Z}^{(s_2)*}_{n;\mathbf{a}}(\mathbf{x}_2,\hbar_2,q)\\
&\qquad\qquad + \mathcal{Z}^{(s_1)*}_{n;\mathbf{a}}(\mathbf{x}_1,\hbar_1,q)\ddot{\mathcal{Z}}^{(s_2)}_{n;\mathbf{a}}(\mathbf{x}_2,\hbar_2,q)\Big).\end{aligned}$$

Similar reasoning gives

$$\text{(3-29)}\quad \begin{aligned}&\mathcal{Z}^*_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\hbar_1,\hbar_2,\hbar_3,q)\\
&\quad= \frac{1}{\hbar_1\hbar_2\hbar_3}\sum_{\substack{r,s_1,s_2,s_3\ge0\\ s_1,s_2,s_3\le n-1\\ s_1+s_2+s_3+r=2n-2}} \Big(\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(0)\mathbf{x}_1^{s_1}\mathbf{x}_2^{s_2}\mathcal{Z}^{(s_3)*}_{n;\mathbf{a}}(\mathbf{x}_3,\hbar_3,q)\\
&\qquad + \widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(0)\mathbf{x}_1^{s_1}\mathcal{Z}^{(s_2)*}_{n;\mathbf{a}}(\mathbf{x}_2,\hbar_2,q)\ddot{\mathcal{Z}}^{(s_3)}_{n;\mathbf{a}}(\mathbf{x}_3,\hbar_3,q)\\
&\qquad + \widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(0)\mathcal{Z}^{(s_1)*}_{n;\mathbf{a}}(\mathbf{x}_1,\hbar_1,q)\prod_{t=2}^{3}\ddot{\mathcal{Z}}^{(s_t)}_{n;\mathbf{a}}(\mathbf{x}_t,\hbar_t,q)\\
&\qquad + \langle\mathbf{a}\rangle\mathbf{x}_1^{\ell^-(\mathbf{a})}\widetilde{\mathcal{C}}^{(r)*}_{s_1,s_2,s_3}(q)\dot{\mathcal{Z}}^{(s_1)}_{n;\mathbf{a}}(\mathbf{x}_1,\hbar_1,q)\prod_{t=2}^{3}\ddot{\mathcal{Z}}^{(s_t)}_{n;\mathbf{a}}(\mathbf{x}_t,\hbar_t,q)\Big),\end{aligned}$$

where $\widetilde{\mathcal{C}}^{(r)*}_{s_1,s_2,s_3}(q)=\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(q)-\widetilde{\mathcal{C}}^{(r)}_{s_1,s_2,s_3}(0)$.

On the other hand, by (3-15) and the first identity in (3-12),

$$\text{(3-30)}\quad \begin{aligned}&\mathcal{Z}^{(s)*}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\\
&\quad= -\langle\mathbf{a}\rangle\mathbf{x}^{\ell(\mathbf{a})+s}+\langle\mathbf{a}\rangle\mathbf{x}^{\ell(\mathbf{a})}\sum_{r=0}^{s}\sum_{s'=0}^{s-r}\widetilde{\mathcal{C}}^{(r)}_{s-\ell^-(\mathbf{a}),s'-\ell^-(\mathbf{a})}(q)\hbar^{s-r-s'}\mathfrak{D}^{s'}\dot{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q),\end{aligned}$$

where

$$\mathfrak{D}^0\check{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\frac{\check{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)}{\check{I}_0(q)},$$
$$\mathfrak{D}^s\check{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\frac{1}{\check{I}_s(q)}\Big\{\mathbf{x}+\hbar\,q\frac{\mathrm{d}}{\mathrm{d}q}\Big\}\mathfrak{D}^{s-1}\check{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$$

for all $s\in\mathbb{Z}^+$ and $(\check{\mathcal{Y}},\check{I})=(\dot{\mathcal{Y}},\dot{I}),(\ddot{\mathcal{Y}},\ddot{I})$. By (3-9), (1-6), and (2-9),

(3-31) $$\mathfrak{D}^s\check{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\big|_{\alpha=0}=\mathbf{x}^s\mathfrak{D}^s\check{F}_{n;\mathbf{a}}(\mathbf{x}/\hbar,q/\mathbf{x}^{\nu_n(\mathbf{a})}),\quad\text{where}$$

$$\mathfrak{D}^0\check{F}_{n;\mathbf{a}}(w,q)=\frac{\check{F}_{n;\mathbf{a}}(w,q)}{\check{I}_0(q)},\quad\mathfrak{D}^s\check{F}_{n;\mathbf{a}}(w,q)=\frac{1}{\check{I}_s(q)}\Big\{1+\frac{q}{w}\frac{\mathrm{d}}{\mathrm{d}q}\Big\}\mathfrak{D}^{s-1}\check{F}_{n;\mathbf{a}}(w,q)$$

for all $s\in\mathbb{Z}^+$, with $(\check{\mathcal{Y}},\check{F},\check{I})=(\dot{\mathcal{Y}},\dot{F},\dot{I}),(\ddot{\mathcal{Y}},\ddot{F},\ddot{I})$. Simplifying the right-hand side of (3-30) in $\mathbb{Q}_\alpha(\mathbf{x},\hbar)[\![\hbar^{-1},q]\!]$ to eliminate division by $\mathbf{x}$ and setting $\alpha=0$, we obtain (2-31).

3D. ***Other three-point generating functions.*** The main step in the proof of the mirror formula (3-14) for the stable quotients analogue of the triple Givental's J-function involves determining a mirror formula for the generating function

(3-32) $$\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\mathbf{x},\hbar,q)\equiv1+\sum_{d=1}^{\infty}q^d\mathrm{ev}_{1*}\Bigg[\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}})}{\hbar-\psi_1}\Bigg]\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar^{-1},q]\!],$$

where $\mathrm{ev}_1:\overline{Q}_{0,3}(\mathbb{P}^{n-1},d)\to\mathbb{P}^{n-1}$ is the evaluation map at the first marked point; the meaning of the superscript $(\mathbf{0},\mathbf{1})$ is explained in (6-7). By (3-33), the SQ invariants do *not* satisfy the string relation [Hori et al. 2003, Section 26.3] in the pure Calabi–Yau cases, $\nu_n(\mathbf{a})=0$ and $\ell^-(\mathbf{a})=0$ (when $\dot{I}_0(q)\neq1$), even though the relevant forgetful morphism, $f_{2,3}$ below, is defined. Since in these cases the twisted invariants of $\mathbb{P}^{n-1}$ are intrinsic invariants of the corresponding complete intersection $X_{n;\mathbf{a}}$, this implies that the construction of virtual fundamental class in [Ciocan-Fontanine et al. 2014] does not respect the forgetful morphism

$$f_{2,3}:\overline{Q}_{0,3}(X_{n;\mathbf{a}},d)\to\overline{Q}_{0,2}(X_{n;\mathbf{a}},d),$$

at least in the Calabi–Yau cases.

Proposition 3.1. *If* $l\in\mathbb{Z}^{\geq0}$, $n\in\mathbb{Z}^+$, *and* $\mathbf{a}\in(\mathbb{Z}^*)^l$ *are such that* $\nu_n(\mathbf{a})\geq0$, *then*

(3-33) $$\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\mathbf{x},\hbar,q)=\hbar^{-1}\frac{\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)}{\dot{I}_0(q)}.$$

In principle, this proposition is contained in [Ciocan-Fontanine and Kim 2013, Corollary 1.4.1]. We give a direct proof, along the lines of [Cooper and Zinger 2014]. In the process of proving this proposition, we establish the mirror formula for equivariant Hurwitz numbers in Proposition 4.1. This in turn allows us to derive (3-14) from (3-11) and (3-15) following the approach of [Zinger 2014]; see Section 10.

Similarly to (3-32), let

$$\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}(\mathbf{x},\hbar,q) \equiv 1+\sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{f_{2,3}^*\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}})}{\hbar-\psi_1}\right] \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar^{-1},q]\!], \tag{3-34}$$

where $\mathrm{ev}_1 : \overline{Q}_{0,3}(\mathbb{P}^{n-1},d) \to \mathbb{P}^{n-1}$ is the evaluation map at the first marked point and

$$f_{2,3} : \overline{Q}_{0,3}(\mathbb{P}^{n-1},d) \to \overline{Q}_{0,2}(\mathbb{P}^{n-1},d)$$

is the forgetful morphism. By the proof of the string relation [Hori et al. 2003, Section 26.3],

$$\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}(\mathbf{x},\hbar,q) = \hbar^{-1}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q). \tag{3-35}$$

We use this identity to establish the mirror formula for Hurwitz numbers in Proposition 4.2.

As stated in Section 1, Theorem 3 generalizes to products of projective spaces and concavex sheaves (1-13). The relevant torus action is then the product of the actions on the components described above. If its weights are denoted by $\alpha_{i;j}$, with $i=1,\dots,p$ and $j=1,\dots,n_i$, the analogues of the above mirror formulas relate power series

$$\check{\mathcal{Y}}_{n_1,\dots,n_p;\mathbf{a}} \in \mathbb{Q}[\alpha_{1;1},\dots,\alpha_{p;n_p},\mathbf{x}_1,\dots,\mathbf{x}_p][\![\hbar^{-1},q_1,\dots,q_p]\!], \tag{3-36}$$

$$\check{\mathcal{Z}}^{(s_1,\dots,s_p)}_{n_1,\dots,n_p;\mathbf{a}} \in H^*_{\mathbb{T}}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1})[\![\hbar^{-1},q_1,\dots,q_p]\!], \tag{3-37}$$

$$\check{\mathcal{Z}}^{*}_{n_1,\dots,n_p;\mathbf{a}} \in H^*_{\mathbb{T}}((\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1})^m)[\![\hbar_1^{-1},\dots,\hbar_M^{-1},q_1,\dots,q_p]\!], \tag{3-38}$$

with $\check{\mathcal{Y}}$ and $\check{\mathcal{Z}}$ denoting $\mathcal{Y}$, $\dot{\mathcal{Y}}$, $\ddot{\mathcal{Y}}$, $\mathcal{Z}$, $\dot{\mathcal{Z}}$, or $\ddot{\mathcal{Z}}$ and $m=2,3$. The coefficients of $q_1^{d_1}\dots q_p^{d_p}$ in (3-37) and (3-38) are defined by the same pushforwards as in (3-4), (3-5), (3-6), and (3-27) with the degree d of the stable quotients replaced by $(d_1,\dots,d_p)$ and $\mathbf{x}^s$ by $\mathbf{x}_1^{s_1}\cdots\mathbf{x}_p^{s_p}$. The coefficients of $q_1^{d_1}\cdots q_p^{d_p}$ in (3-36) are obtained from the coefficients in (3-9) and (3-16) by replacing $a_k d$ and $a_k\mathbf{x}$ by $a_{k;1}d_1+\dots+a_{k;p}d_p$ and $a_{k;1}\mathbf{x}_1+\dots+a_{k;p}\mathbf{x}_p$ in the numerator and taking the product of the denominators with $(n,\mathbf{x},d)=(n_i,\mathbf{x}_i,d_i)$ for each $s=1,\dots,p$; in the i-th factor, α_k is also replaced by $\alpha_{i;k}$;

$$\mathbf{x}_1,\dots,\mathbf{x}_p \in H^*_{\mathbb{T}}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1})$$

now correspond to the pullbacks of the equivariant hyperplane classes by the projection maps. The statements of Theorem 3, (3-28), and (3-29) extend by replacing the symmetric polynomials by products of symmetric polynomials in the

p different sets of variables and $\langle\mathbf{a}\rangle\mathbf{x}^{\ell(\mathbf{a})}$ by the products and ratios of the terms $a_{k;1}\mathbf{x}_1+\cdots+a_{k;p}\mathbf{x}_p$; our proofs extend directly to this situation.

4. Equivariant twisted Hurwitz numbers

The fixed loci of the $\mathbb{T}$-action on $\overline{Q}_{0,m}(\mathbb{P}^{n-1},d)$ involve moduli spaces of weighted curves and certain vector bundles, which we describe in this section. As a corollary of the proof of Theorem 3, we obtain closed formulas for Euler classes of these vector bundles in some cases. These formulas, described in Propositions 4.1 and 4.2 below, are a key ingredient in computing the genus-1 stable quotients invariants.

A stable d-tuple of flecks on a quasistable m-marked curve is a tuple

$$(\mathcal{C}, y_1, \dots, y_m; \hat{y}_1, \dots, \hat{y}_d), \tag{4-1}$$

where $\mathcal{C}$ is a connected (at worst) nodal curve, $y_1,\dots,y_m\in\mathcal{C}^*$ are distinct smooth points, and $\hat{y}_1,\dots,\hat{y}_d\in\mathcal{C}^*-\{y_1,\dots,y_m\}$, such that the $\mathbb{Q}$-line bundle

$$\omega_{\mathcal{C}}(y_1+\cdots+y_m+\epsilon(\hat{y}_1+\cdots+\hat{y}_d))\to\mathcal{C}$$

is ample for all $\epsilon\in\mathbb{Q}^+$; this again implies that $2g+m\geq 2$. An isomorphism

$$\phi:(\mathcal{C}, y_1,\dots,y_m;\hat{y}_1,\dots,\hat{y}_d)\to(\mathcal{C}', y_1',\dots,y_m';\hat{y}_1',\dots,\hat{y}_d')$$

between curves with m marked points and d flecks is an isomorphism $\phi:\mathcal{C}\to\mathcal{C}'$ such that

$$\phi(y_i)=y_i' \text{ for all } i=1,\dots,m, \quad \phi(\hat{y}_j)=\hat{y}_j' \text{ for all } j=1,\dots,d.$$

The automorphism group of any stable curve with m marked points and d flecks is finite. For $g,m,d\in\mathbb{Z}^{\geq 0}$, the moduli space $\overline{\mathcal{M}}_{g,m|d}$ parameterizing the stable d-tuples of flecks as in (4-1) with $h^1(\mathcal{C},\mathcal{O}_{\mathcal{C}})=g$ is a nonsingular irreducible proper Deligne–Mumford stack; see [Cooper and Zinger 2014, Proposition 2.3]. If $m\geq m'\geq 2$, let

$$f_{m',m}:\overline{\mathcal{M}}_{0,m|d}\to\overline{\mathcal{M}}_{0,m'|d+m-m'},$$
$$(\mathcal{C},y_1,\dots,y_m;\hat{y}_1,\dots,\hat{y}_d)\mapsto(\mathcal{C}',y_1,\dots,y_{m'};\hat{y}_1,\dots,\hat{y}_d,y_{m'+1},\dots,y_m),$$

be the morphism converting the last $m-m'$ marked points into the last $m-m'$ flecks and contracting components of $\mathcal{C}$ if necessary.

Any tuple as in (4-1) induces a quasistable quotient

$$\mathcal{O}_{\mathcal{C}}(-\hat{y}_1-\cdots-\hat{y}_d)\subset\mathcal{O}_{\mathcal{C}}\equiv\mathbb{C}^1\otimes\mathcal{O}_{\mathcal{C}}.$$

For any ordered partition $d=d_1+\cdots+d_p$ with $d_1,\dots,d_p\in\mathbb{Z}^{\geq 0}$, this correspondence gives rise to a morphism

$$\overline{\mathcal{M}}_{g,m|d}\to\overline{Q}_{g,m}(\mathbb{P}^0\times\cdots\times\mathbb{P}^0,(d_1,\dots,d_p)).$$

In turn, this morphism induces an isomorphism

(4-2) $$\phi:\overline{\mathcal{M}}_{g,m|d}\big/\mathbb{S}_{d_1}\times\cdots\times\mathbb{S}_{d_p}\xrightarrow{\sim}\bar{Q}_{g,m}(\mathbb{P}^0\times\cdots\times\mathbb{P}^0,(d_1,\ldots,d_p)),$$

with the symmetric group $\mathbb{S}_{d_1}$ acting on $\overline{\mathcal{M}}_{g,m|d}$ by permuting the points $\hat{y}_1,\ldots,\hat{y}_{d_1}$, $\mathbb{S}_{d_2}$ acting on $\overline{\mathcal{M}}_{g,m|d}$ by permuting the points $\hat{y}_{d_1+1},\ldots,\hat{y}_{d_1+d_2}$, etc.

There is again a universal curve

$$\pi:\mathcal{U}\to\overline{\mathcal{M}}_{g,m|d}$$

with sections $\sigma_1,\ldots,\sigma_m$ and $\hat{\sigma}_1,\ldots\hat{\sigma}_d$. Let

$$\psi_i=-\pi_*(\sigma_i^2),\ \hat{\psi}_i=-\pi_*(\hat{\sigma}_i^2)\in H^2(\overline{\mathcal{M}}_{g,m|d})$$

be the first chern classes of the universal cotangent line bundles. For $m\ge 2$, $d',d\in\mathbb{Z}^+$ with $d'\le d$, and $\mathbf{r}\equiv(r_1,\ldots,r_{d'})\in(\mathbb{Z}^{\ge 0})^{d'}$, let

$$\mathcal{S}_{\mathbf{r}}=\mathcal{O}(-\hat{\sigma}_1-\cdots-\hat{\sigma}_{d-d'}-r_1\hat{\sigma}_{d-d'+1}-\cdots-r_{d'}\hat{\sigma}_d)\to\mathcal{U}\to\overline{\mathcal{M}}_{0,m|d}.$$

If $\beta\in H^2_{\mathbb{T}}$, denote by

(4-3) $$\mathcal{S}^*_{\mathbf{r}}(\beta)\to\mathcal{U}\to\overline{\mathcal{M}}_{0,m|d}$$

the sheaf $\mathcal{S}^*_{\mathbf{r}}$ with the $\mathbb{T}$-action so that

$$\mathbf{e}(\mathcal{S}^*_{\mathbf{r}}(\beta))=\beta\times 1+1\times e(\mathcal{S}^*_{\mathbf{r}})\in H^*_{\mathbb{T}}(\mathcal{U})=H^*_{\mathbb{T}}\otimes H^*(\mathcal{U}).$$

Similarly to (2-1), let

(4-4)
$$\dot{\mathcal{V}}'^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)=\bigoplus_{a_k>0}R^0\pi_*\big(\mathcal{S}^*_{\mathbf{r}}(\beta)^{a_k}(-\sigma_1)\big)\oplus\bigoplus_{a_k<0}R^1\pi_*\big(\mathcal{S}^*_{\mathbf{r}}(\beta)^{a_k}(-\sigma_1)\big)\to\overline{\mathcal{M}}_{0,m|d},$$
$$\ddot{\mathcal{V}}'^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)=\bigoplus_{a_k>0}R^0\pi_*\big(\mathcal{S}^*_{\mathbf{r}}(\beta)^{a_k}(-\sigma_2)\big)\oplus\bigoplus_{a_k<0}R^1\pi_*\big(\mathcal{S}^*_{\mathbf{r}}(\beta)^{a_k}(-\sigma_2)\big)\to\overline{\mathcal{M}}_{0,m|d},$$

where $\pi:\mathcal{U}\to\overline{\mathcal{M}}_{0,m|d}$ is the projection as before; these sheaves are locally free. If $m'\in\mathbb{Z}^+$, $2\le m'\le m$, and $\mathbf{r}\in(\mathbb{Z}^{\ge 0})^{m-m'}$, let

(4-5) $$\dot{\mathcal{V}}^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)=f^*_{m',m}\dot{\mathcal{V}}'^{(d)}_{\mathbf{a};\mathbf{r}}(\beta),\ \ddot{\mathcal{V}}^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)=f^*_{m',m}\ddot{\mathcal{V}}'^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)\to\overline{\mathcal{M}}_{0,m|d}.$$

In the case $m'=m$, we will denote the bundles $\dot{\mathcal{V}}^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)$ and $\ddot{\mathcal{V}}^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)$ by $\dot{\mathcal{V}}^{(d)}_{\mathbf{a}}(\beta)$ and $\ddot{\mathcal{V}}^{(d)}_{\mathbf{a}}(\beta)$, respectively.

The equivariant Euler classes of the bundles $\dot{\mathcal{V}}^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)$ and $\ddot{\mathcal{V}}^{(d)}_{\mathbf{a};\mathbf{r}}(\beta)$ enter into the localization computations in Sections 7–9. As a corollary of these computations, we obtain closed formulas for the Euler classes of these bundles in the case $m=3$; see Propositions 4.1 and 4.2 below. These formulas are a key ingredient in computing the genus-0 three-point and genus-1 SQ invariants.

If $f \in \mathbb{Q}_\alpha[\![q]\!]$ and $d \in \mathbb{Z}^{\ge 0}$, let $[\![f]\!]_{q;d} \in \mathbb{Q}_\alpha$ denote the coefficient of q^d in f. If $f = f(z)$ is a rational function in z and possibly some other variables, for any $z_0 \in \mathbb{P}^1 \supset \mathbb{C}$ let

$$\underset{z=z_0}{\mathfrak{R}} f(z) \equiv \frac{1}{2\pi \mathrm{i}} \oint f(z)\,\mathrm{d}z, \tag{4-6}$$

where the integral is taken over a positively oriented loop around $z = z_0$ with no other singular points of $f\,\mathrm{d}z$, denote the residue of the 1-form $f\,\mathrm{d}z$. If $z_1, \dots, z_k \in \mathbb{P}^1$ is any collection of points, let

$$\underset{z=z_1,\dots,z_k}{\mathfrak{R}} f(z) \equiv \sum_{i=1}^{i=k} \underset{z=z_i}{\mathfrak{R}} f(z) \tag{4-7}$$

be the sum of the corresponding residues.

For any variable $\mathbf{y}$ and $r \in \mathbb{Z}^{\ge 0}$, let $\mathbf{s}_r(\mathbf{y})$ denote the r-th elementary symmetric polynomial in $\{\mathbf{y} - \alpha_k\}$. We define power series $L_{n;\mathbf{a}}, \xi_{n;\mathbf{a}} \in \mathbb{Q}_\alpha[\mathbf{x}][\![q]\!]$ by

$$\begin{aligned} &L_{n;\mathbf{a}} \in \mathbf{x} + q\mathbb{Q}_\alpha[\mathbf{x}][\![q]\!], \quad &&\mathbf{s}_n(L_{n;\mathbf{a}}(\mathbf{x},q)) - q\mathbf{a}^{\mathbf{a}} L_{n;\mathbf{a}}(\mathbf{x},q)^{|\mathbf{a}|} = \mathbf{s}_n(\mathbf{x}),\\ &\xi_{n;\mathbf{a}} \in q\mathbb{Q}_\alpha[\mathbf{x}][\![q]\!], &&\mathbf{x} + q\frac{\mathrm{d}}{\mathrm{d}q}\xi_{n;\mathbf{a}}(\mathbf{x},q) = L_{n;\mathbf{a}}(\mathbf{x},q). \end{aligned} \tag{4-8}$$

By [Zinger 2014, Remark 4.5], the coefficients of the power series

$$\mathrm{e}^{-\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar}\, \dot{\mathcal{Y}}_{n;\mathbf{a}}(\alpha_i,\hbar,q) \in \mathbb{Q}_\alpha[\hbar][\![q]\!]$$

are regular at $\hbar = 0$. Thus, there is an expansion

$$\mathrm{e}^{-\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar}\, \dot{\mathcal{Y}}_{n;\mathbf{a}}(\alpha_i,\hbar,q) = \sum_{r=0}^{\infty} \dot{\Phi}^{(r)}_{n;\mathbf{a}}(\alpha_i,q)\hbar^r, \tag{4-9}$$

with $\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\mathbf{x},q) - 1, \dot{\Phi}^{(1)}_{n;\mathbf{a}}(\mathbf{x},q), \dot{\Phi}^{(2)}_{n;\mathbf{a}}(\mathbf{x},q), \dots \in q\mathbb{Q}_\alpha[\mathbf{x}][\![q]\!]$. Furthermore,

$$\begin{aligned} \dot{\Phi}^{(0)}_{n;\mathbf{a}}(\mathbf{x},q) &\\ = \biggl(&\frac{\mathbf{x}\cdot\mathbf{s}_{n-1}(\mathbf{x})}{L_{n;\mathbf{a}}(\mathbf{x},q)\,\mathbf{s}_{n-1}(L_{n;\mathbf{a}}(\mathbf{x},q)) - |\mathbf{a}|q\mathbf{a}^{\mathbf{a}}L_{n;\mathbf{a}}(\mathbf{x},q)^{|\mathbf{a}|}}\biggr)^{\frac{1}{2}} \biggl(\frac{L_{n;\mathbf{a}}(\mathbf{x},q)}{\mathbf{x}}\biggr)^{\frac{\ell(\mathbf{a})+1}{2}}.\text{[5]} \end{aligned} \tag{4-10}$$

Proposition 4.1. *If $l \in \mathbb{Z}^{\ge 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then for every $i = 1, \dots, n$*

$$\sum_{d=0}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,3|d}} \frac{\mathbf{e}\bigl(\dot{\mathcal{V}}^{(d)}_{\mathbf{a}}(\alpha_i)\bigr)}{\prod_{k\ne i}\mathbf{e}\bigl(\dot{\mathcal{V}}^{(d)}_1(\alpha_i-\alpha_k)\bigr)\,(\hbar_1-\psi_1)(\hbar_2-\psi_2)(\hbar_3-\psi_3)}$$

$$= \frac{\mathrm{e}^{\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_1 + \xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_2 + \xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_3}}{\hbar_1\hbar_2\hbar_3\,\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)} \in \mathbb{Q}_\alpha[\![\hbar_1^{-1},\hbar_2^{-1},\hbar_3^{-1},q]\!].$$

[5] Only the case $\ell^-(\mathbf{a}) = 0$ is explicitly considered in [Zinger 2014], but the argument is the same in all cases.

Proposition 4.2. *If $l \in \mathbb{Z}^{\ge 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then for every $i = 1, \dots, n$*

$$\sum_{b=0}^{\infty}\sum_{r=0}^{\infty}\sum_{d=0}^{\infty}\frac{q^d}{d!}\int_{\overline{\mathcal{M}}_{0,3|d}}\frac{\mathbf{e}\big(\dot{\mathcal{V}}_{\mathbf{a};r}^{(d)}(\alpha_i)\big)\psi_3^b\,\mathfrak{R}_{\hbar=0}\frac{(-1)^b}{\hbar^{b+1}}[\![\dot{\mathcal{Y}}_{n;\varnothing}(\alpha_i,\hbar,q)]\!]_{q;r}q^r}{\prod_{k\neq i}\mathbf{e}\big(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k)\big)\,(\hbar_1-\psi_1)(\hbar_2-\psi_2)}$$
$$=\frac{\mathrm{e}^{\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_1+\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_2}}{\hbar_1\hbar_2}\in\mathbb{Q}_\alpha[\![\hbar_1^{-1},\hbar_2^{-1},q]\!].$$

5. Outline of the proof of Theorem 3

The first identity in (3-12) is the subject of [Cooper and Zinger 2014, Theorem 3]. The proof of the remaining statements of Theorem 3 follows the same principle as the proof of [Popa and Zinger 2014, Theorem 4]; it is outlined below. However, its adaptation to the present situation requires a number of modifications. In particular, the twisted stable quotients invariants are not known to satisfy the analogue of the string relation of Gromov–Witten theory (in fact, by Proposition 3.1, in general they do not). This requires a direct proof of the key properties for the stable quotients analogue of double Givental's J-function described in Lemmas 6.5 and 6.6 below; in Gromov–Witten theory, these properties are deduced from the analogous properties for three-point invariants, which simplifies the argument. We thus describe the argument in detail.

Let $\mathbb{Q}_\alpha\lceil\!\lceil\hbar\rceil\!\rceil \equiv \mathbb{Q}_\alpha[\![\hbar^{-1}]\!]+\mathbb{Q}_\alpha[\hbar]$ denote the $\mathbb{Q}_\alpha$-algebra of Laurent series in $\hbar^{-1}$ (with finite principal part). We will view the $\mathbb{Q}_\alpha$-algebra $\mathbb{Q}_\alpha(\hbar)$ of rational functions in $\hbar$ with coefficients in $\mathbb{Q}_\alpha$ as a subalgebra of $\mathbb{Q}_\alpha\lceil\!\lceil\hbar\rceil\!\rceil$ via the embedding given by taking the Laurent series of rational functions at $\hbar^{-1}=0$. If

$$\mathcal{F}(\hbar,q)=\sum_{d=0}^{\infty}\sum_{r=-N_d}^{\infty}\mathcal{F}^{(r)}(d)\hbar^{-r}q^d\in\mathbb{Q}_\alpha\lceil\!\lceil\hbar\rceil\!\rceil[\![q]\!]$$

for some $N_d\in\mathbb{Z}$ and $\mathcal{F}^{(r)}(d)\in\mathbb{Q}_\alpha$, we define

$$\mathcal{F}(\hbar,q)\cong\sum_{d=0}^{\infty}\sum_{r=-N_d}^{p-1}\mathcal{F}^{(r)}(d)\hbar^{-r}\pmod{\hbar^{-p}},$$

that is we drop $\hbar^{-p}$ and higher powers of $\hbar^{-1}$, instead of higher powers of $\hbar$.

For $1\le i,j\le n$ with $i\neq j$ and $d\in\mathbb{Z}^+$, let

(5-1)
$$\dot{\mathfrak{C}}_i^j(d)\equiv\frac{\prod_{a_k>0}\prod_{r=1}^{a_kd}\big(a_k\alpha_i+r\,\frac{\alpha_j-\alpha_i}{d}\big)\prod_{a_k<0}\prod_{r=0}^{-a_kd-1}\big(a_k\alpha_i-r\,\frac{\alpha_j-\alpha_i}{d}\big)}{d\prod_{r=1}^{d}\prod_{k=1}^{n}{}_{(r,k)\neq(d,j)}\big(\alpha_i-\alpha_k+r\,\frac{\alpha_j-\alpha_i}{d}\big)}\in\mathbb{Q}_\alpha,$$
$$\ddot{\mathfrak{C}}_i^j(d)\equiv\frac{\prod_{a_k>0}\prod_{r=0}^{a_kd-1}\big(a_k\alpha_i+r\,\frac{\alpha_j-\alpha_i}{d}\big)\prod_{a_k<0}\prod_{r=1}^{-a_kd}\big(a_k\alpha_i-r\,\frac{\alpha_j-\alpha_i}{d}\big)}{d\prod_{r=1}^{d}\prod_{k=1}^{n}{}_{(r,k)\neq(d,j)}\big(\alpha_i-\alpha_k+r\,\frac{\alpha_j-\alpha_i}{d}\big)}\in\mathbb{Q}_\alpha.$$

We will follow the five steps in [Zinger 2009, Section 1.3] to verify (3-11), the second statement in (3-12), and (3-15):

(Ma) if $\mathcal{F}, \mathcal{F}' \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][\![q]\!]$,

$$\mathcal{F}(\mathbf{x}=\alpha_i, \hbar, q) \in \mathbb{Q}_\alpha(\hbar)[\![q]\!] \subset \mathbb{Q}_\alpha[\![\hbar]\!][\![q]\!] \quad \text{for all } i=1,2,\ldots,n,$$

$\mathcal{F}'$ is recursive in the sense of Definition 6.1, and $\mathcal{F}$ and $\mathcal{F}'$ satisfy a mutual polynomiality condition (MPC) of Definition 6.2, then the transforms of $\mathcal{F}'$ of Lemma 6.4 are also recursive and satisfy the same MPC with respect to $\mathcal{F}$;

(Mb) if $\mathcal{F}, \mathcal{F}' \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][\![q]\!]$,

$$\mathcal{F}(\mathbf{x}=\alpha_i, \hbar, q) \in \mathbb{Q}^*_\alpha + q\cdot\mathbb{Q}_\alpha(\hbar)[\![q]\!] \subset \mathbb{Q}_\alpha[\![\hbar]\!][\![q]\!] \quad \text{for all } i=1,2,\ldots,n,$$

$\mathcal{F}'$ is recursive in the sense of Definition 6.1, and $\mathcal{F}$ and $\mathcal{F}'$ satisfy a fixed MPC, then $\mathcal{F}'$ is determined by its "mod $\hbar^{-1}$ part";

(Mc) the two sides of the second identity in (3-12) and the $\ddot{\mathcal{Z}}$ case in (3-15) are $\ddot{\mathfrak{C}}$-recursive in the sense of Definition 6.1 with $\ddot{\mathfrak{C}}$ as in (5-1), while the two sides of the $\dot{\mathcal{Z}}$ case in (3-15) are $\dot{\mathfrak{C}}$-recursive in the sense of Definition 6.1 with $\dot{\mathfrak{C}}$ as in (5-1);

(Md) the two sides of each of the equations in (3-12) and (3-15) satisfy the same η-MPC (dependent on the equation) with respect to $\dot{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$;

(Me) the two sides of each of the four equations in (3-12) and (3-15), viewed as elements of $H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][\![q]\!]$, agree mod $\hbar^{-1}$.

The first two claims, (Ma) and (Mb), sum up Lemma 6.4 and Proposition 6.3, respectively. By Lemmas 6.5 and 6.6, the stable quotients generating functions $\dot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}$ and $\ddot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}$ are $\dot{\mathfrak{C}}$-recursive and $\ddot{\mathfrak{C}}$-recursive and satisfy MPCs with respect to $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$. Along with the first identity in (3-12), the latter implies that they satisfy MPCs with respect to $\dot{\mathcal{Y}}_{n;\mathbf{a}}$. It is immediate from (3-4) that

$$\dot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}(\mathbf{x},\hbar,q),\ \ddot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}(\mathbf{x},\hbar,q) \cong \mathbf{x}^s \pmod{\hbar^{-1}} \quad \text{for all } s\in\mathbb{Z}^{\geq 0}. \tag{5-2}$$

By the proof of the first identity in (3-12), as well as of its Gromov–Witten analogue, the power series $\dot{\mathcal{Y}}_{n;\mathbf{a}}$ is $\dot{\mathfrak{C}}$-recursive and satisfies the same MPC with respect to $\dot{\mathcal{Y}}_{n;\mathbf{a}}$ as $\dot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}$; see [Cooper and Zinger 2014, Lemma 5.4]. A nearly identical argument shows that the power series $\ddot{\mathcal{Y}}_{n;\mathbf{a}}$ is $\ddot{\mathfrak{C}}$-recursive and satisfies the same MPC with respect to $\dot{\mathcal{Y}}_{n;\mathbf{a}}$ as $\ddot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}$; see [Popa and Zinger 2014, Section 4.3] for the $\ell^-(\mathbf{a})=0$ case. Since

$$\ddot{\mathcal{Y}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q) \cong 1 \pmod{\hbar^{-1}},$$

this establishes the second identity in (3-12). Along with (3-12), the admissibility of transforms (i) and (ii) in Lemma 6.4 implies that both sides of the $\dot{\mathcal{Z}}$ equation in

(3-15) are $\dot{\mathfrak{C}}$-recursive and satisfy the same MPC with respect to $\dot{\mathcal{Y}}_{n;\mathbf{a}}$, no matter what the coefficients $\widetilde{\mathcal{C}}^{(r)}_{s,s'}$ are. Similarly, both sides of the $\ddot{\mathcal{Z}}$ equation in (3-15) are $\ddot{\mathfrak{C}}$-recursive and satisfy the same MPC with respect to $\dot{\mathcal{Y}}_{n;\mathbf{a}}$. By (3-10), (3-12), (3-9), (3-21), (3-19), (3-17), and (3-16),

$$\begin{aligned}&\sum_{r=0}^{s}\sum_{s'=0}^{s-r}\widetilde{\mathcal{C}}^{(r)}_{s-\ell^-(\mathbf{a}),s'-\ell^-(\mathbf{a})}(q)\,\hbar^{s-r-s'}\mathfrak{D}^{s'}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\cong\mathbf{x}^s \pmod{\hbar^{-1}},\\ &\sum_{r=0}^{s}\sum_{s'=0}^{s-r}\widetilde{\mathcal{C}}^{(r)}_{s-\ell^+(\mathbf{a}),s'-\ell^+(\mathbf{a})}(q)\,\hbar^{s-r-s'}\mathfrak{D}^{s'}\ddot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\cong\mathbf{x}^s \pmod{\hbar^{-1}}.^{6}\end{aligned}\tag{5-3}$$

Thus, (3-15) follows from (Mb).

The proof of (3-11) follows the same principle, which we apply to a multiple of (3-11). For each $i=1,2,\dots,n$, let

$$\phi_i\equiv\prod_{k\neq i}(\mathbf{x}-\alpha_k)\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}).\tag{5-4}$$

By [Atiyah and Bott 1984, localization theorem], ϕ_i is the equivariant Poincaré dual of the fixed point $P_i\in\mathbb{P}^{n-1}$; see [Zinger 2009, Section 3.1]. Since $\mathbf{x}|_{P_i}=\alpha_i$,

$$\begin{aligned}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\alpha_j,\hbar_1,\hbar_2,q)&=\int_{P_i\times P_j}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q)\\&=\int_{\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q)\phi_i\times\phi_j\\&=\frac{1}{\hbar_1+\hbar_2}\prod_{k\neq i}(\alpha_j-\alpha_k)+\sum_{d=1}^{\infty}q^d\int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}})\,\mathrm{ev}_1^*\phi_i\,\mathrm{ev}_2^*\phi_j}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)};\end{aligned}\tag{5-5}$$

the last equality holds by the defining property of the cohomology push-forward [Zinger 2009, Equation (3.11)]. By Lemmas 6.5 and 6.6, $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q)$ is $\dot{\mathfrak{C}}$-recursive and satisfies the same MPC as $\dot{\mathcal{Z}}_{n;\mathbf{a}}$ with respect to $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ for $(\mathbf{x},\hbar)=(\mathbf{x}_1,\hbar_1)$ and $\mathbf{x}_2=\alpha_j$ fixed.[7] It is also $\ddot{\mathfrak{C}}$-recursive and satisfies the same MPC as $\ddot{\mathcal{Z}}_{n;\mathbf{a}}$ with respect to $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ for $(\mathbf{x},\hbar)=(\mathbf{x}_2,\hbar_2)$ and $\mathbf{x}_1=\alpha_i$ fixed.

[6] The left-hand side of (3-21) with s replaced by $s-\ell^-(\mathbf{a})$ is the coefficient of $\hbar^s\mathbf{x}^{-r}(\mathbf{x}/\hbar)^{s'+\ell^-(\mathbf{a})}$ in the first identity in (5-3) if $s\ge\ell^-(\mathbf{a})$; The left-hand side of (3-21) with s replaced by $s-\ell^+(\mathbf{a})$ is the coefficient of $\hbar^s\mathbf{x}^{-r}(\mathbf{x}/\hbar)^{s'+\ell^+(\mathbf{a})}$ in the second identity in (5-3) if $s\ge\ell^+(\mathbf{a})$.

[7] In other words, the coefficient of every power of $\hbar_2^{-1}$ in $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\alpha_j,\hbar,\hbar_2,q)$ is $\dot{\mathfrak{C}}$-recursive and satisfies the same MPC as $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ with respect to $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$.

By (Ma) and (Mb), it is thus sufficient to compare

$$(\hbar_1+\hbar_2)\,\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}_1,\mathbf{x}_2,\hbar_1,\hbar_2,q) \quad \text{and} \qquad \text{(5-6)}$$
$$\sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r\mathbf{s}_r\,\dot{\mathcal{Z}}^{(s_1)}_{n;\mathbf{a}}(\mathbf{x}_1,\hbar_1,q)\,\ddot{\mathcal{Z}}^{(s_2)}_{n;\mathbf{a}}(\mathbf{x}_2,\hbar_2,q)$$

for all $\mathbf{x}_1=\alpha_i$ and $\mathbf{x}_2=\alpha_j$ with $i,j=1,2,\dots,n$ modulo $\hbar_1^{-1}$:

$$(\hbar_1+\hbar_2)\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\alpha_j,\hbar_1,\hbar_2,q) \cong \sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r\mathbf{s}_r\alpha_i^{s_1}\alpha_j^{s_2}+\sum_{d=1}^{\infty}q^d\int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}})\mathrm{ev}_1^*\phi_i\mathrm{ev}_2^*\phi_j}{\hbar_2-\psi_2};$$

$$\sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r\mathbf{s}_r\dot{\mathcal{Z}}^{(s_1)}_{n;\mathbf{a}}(\alpha_i,\hbar_1,q)\ddot{\mathcal{Z}}^{(s_2)}_{n;\mathbf{a}}(\alpha_j,\hbar_2,q) \cong \sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r\mathbf{s}_r\alpha_i^{s_1}\ddot{\mathcal{Z}}^{(s_2)}_{n;\mathbf{a}}(\alpha_j,\hbar_2,q).$$

In order to see that the two right-hand side power series are the same, it is sufficient to compare them modulo $\hbar_2^{-1}$:

$$\sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r\mathbf{s}_r\alpha_i^{s_1}\alpha_j^{s_2}+\sum_{d=1}^{\infty}q^d\int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}})\mathrm{ev}_1^*\phi_i\mathrm{ev}_2^*\phi_j}{\hbar_2-\psi_2} \cong \sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r\mathbf{s}_r\alpha_i^{s_1}\alpha_j^{s_2};$$

$$\sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r\mathbf{s}_r\alpha_i^{s_1}\ddot{\mathcal{Z}}^{(s_2)}_{n;\mathbf{a}}(\alpha_j,\hbar_2,q) \cong \sum_{\substack{s_1,s_2,r\ge0\\ s_1+s_2+r=n-1}} (-1)^r\mathbf{s}_r\alpha_i^{s_1}\alpha_j^{s_2}.$$

From this we conclude that the two expressions in (5-6) are the same; this proves (3-11).

By Proposition 6.3 and Lemmas 6.5 and 6.6, the stable quotients analogue of triple Givental's J-function is determined by the primary three-point SQ invariants. Since all such invariants are related to the corresponding GW invariants by [Ciocan-Fontanine and Kim 2013, Theorem 1.2.2 and Corollaries 1.4.1, 1.4.2], a version of (3-14) can be proved by comparing it to its GW analogue provided by [Zinger 2014, Theorem B]. We instead prove (3-14) directly in Section 10 by reducing the computation to the two-point formulas of Theorem 3 and the mirror formula for Hurwitz numbers in Propositions 4.1. In the process, we obtain a precise description of the equivariant structure coefficients appearing in (3-14), which is not done in [Zinger 2014].

6. Recursivity, polynomiality, and admissible transforms

This section describes the algebraic observations used in the proof of Theorem 3. It is based on [Zinger 2009, Sections 2.1, 2.2] and [Popa and Zinger 2014, Section 4.1]. Let

$$[n]=\{1,2,\dots,n\}.$$

Definition 6.1. Let $C\equiv(C_i^j(d))_{d,i,j\in\mathbb{Z}^+}$ be any collection of elements of $\mathbb{Q}_\alpha$. A power series $\mathcal{F}\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})\lceil\!\lceil\hbar\rceil\!\rceil[\![q]\!]$ is C-recursive if the following holds: if $d^*\in\mathbb{Z}^{\ge 0}$ is such that

$$[\![\mathcal{F}(\mathbf{x}=\alpha_i,\hbar,q)]\!]_{q;d^*-d}\in\mathbb{Q}_\alpha(\hbar)\subset\mathbb{Q}_\alpha\lceil\!\lceil\hbar\rceil\!\rceil\quad\text{for all } d\in[d^*],\ i\in[n],$$

and $[\![\mathcal{F}(\alpha_i,\hbar,q)]\!]_{q;d}$ is regular at $\hbar=(\alpha_i-\alpha_j)/d$ for all $d<d^*$ and $i\neq j$, then

$$[\![\mathcal{F}(\alpha_i,\hbar,q)]\!]_{q;d^*}-\sum_{d=1}^{d^*}\sum_{j\neq i}\frac{C_i^j(d)}{\hbar-(\alpha_j-\alpha_i)/d}[\![\mathcal{F}(\alpha_j,z,q)]\!]_{q;d^*-d}\big|_{z=(\alpha_j-\alpha_i)/d}\in\mathbb{Q}_\alpha[\hbar,\hbar^{-1}]\subset\mathbb{Q}_\alpha\lceil\!\lceil\hbar\rceil\!\rceil.\tag{6-1}$$

Thus, if $\mathcal{F}\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})\lceil\!\lceil\hbar\rceil\!\rceil[\![q]\!]$ is C-recursive, for any collection C, then

$$\mathcal{F}(\mathbf{x}=\alpha_i,\hbar,q)\in\mathbb{Q}_\alpha(\hbar)[\![q]\!]\subset\mathbb{Q}_\alpha\lceil\!\lceil\hbar\rceil\!\rceil[\![q]\!]\quad\text{for all } i\in[n],$$

as can be seen by induction on d, and

$$\mathcal{F}(\alpha_i,\hbar,q)=\sum_{d=0}^{\infty}\sum_{r=-N_d}^{N_d}\mathcal{F}_i^r(d)\hbar^r q^d+\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{C_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i)/d}\mathcal{F}(\alpha_j,(\alpha_j-\alpha_i)/d,q)\tag{6-2}$$

for all $i\in[n]$, for some $\mathcal{F}_i^r(d)\in\mathbb{Q}_\alpha$. The nominal issue with defining C-recursivity by (6-2), as is normally done, is that a priori the evaluation of $\mathcal{F}(\alpha_j,\hbar,q)$ at $\hbar=(\alpha_j-\alpha_i)/d$ need not be well defined, since $\mathcal{F}(\alpha_j,\hbar,q)$ is a power series with coefficients in $\mathbb{Q}_\alpha\lceil\!\lceil\hbar^{-1}\rceil\!\rceil$; a priori they may not converge anywhere. However, taking the coefficient of each power of q in (6-2) shows by induction on the degree d that this evaluation does make sense; this is the substance of Definition 6.1.

Definition 6.2. Let $\eta\in\mathbb{Q}_\alpha(\mathbf{x})$ be such that $\eta(\mathbf{x}=\alpha_i)\in\mathbb{Q}_\alpha$ is well defined and nonzero for every $i\in[n]$. For any $\mathcal{F}\equiv\mathcal{F}(\mathbf{x},\hbar,q),\mathcal{F}'\equiv\mathcal{F}'(\mathbf{x},\hbar,q)\in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})\lceil\!\lceil\hbar\rceil\!\rceil[\![q]\!]$, let

$$\Phi^\eta_{\mathcal{F},\mathcal{F}'}(\hbar,z,q)\equiv\sum_{i=1}^{n}\frac{\eta(\alpha_i)e^{\alpha_i z}}{\prod_{k\neq i}(\alpha_i-\alpha_k)}\mathcal{F}(\alpha_i,\hbar,qe^{\hbar z})\mathcal{F}'(\alpha_i,-\hbar,q)\in\mathbb{Q}_\alpha\lceil\!\lceil\hbar\rceil\!\rceil[\![z,q]\!].\tag{6-3}$$

If $\mathcal{F}, \mathcal{F}' \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][\![q]\!]$, the pair $(\mathcal{F}, \mathcal{F}')$ satisfies the η mutual polynomiality condition (η-MPC) if $\Phi^{\eta}_{\mathcal{F},\mathcal{F}'} \in \mathbb{Q}_\alpha[\hbar][\![z, q]\!]$.

If $\mathcal{F}, \mathcal{F}' \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][\![q]\!]$ and

$$\mathcal{F}(\mathbf{x}=\alpha_i, \hbar, q), \mathcal{F}'(\mathbf{x}=\alpha_i, \hbar, q) \in \mathbb{Q}_\alpha(\hbar)[\![q]\!] \quad \text{for all } i \in [n], \tag{6-4}$$

then the pair $(\mathcal{F}, \mathcal{F}')$ satisfies the η-MPC if and only if the pair $(\mathcal{F}', \mathcal{F})$ does; see [Zinger 2009, Lemma 2.2] for the $\eta = 1$, $\ell^+(\mathbf{a}) = 1$, $\ell^-(\mathbf{a}) = 0$ case (the proof readily carries over to the general case). Thus, if (6-4) holds, the statement that $\mathcal{F}$ and $\mathcal{F}'$ satisfy the MPC is unambiguous.

Proposition 6.3. *Let $\eta \in \mathbb{Q}_\alpha(\mathbf{x})$ be such that $\eta(\mathbf{x}=\alpha_i) \in \mathbb{Q}_\alpha$ is well defined and nonzero for every $i \in [n]$. If $\mathcal{F}, \mathcal{F}' \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][\![q]\!]$,*

$$\mathcal{F}(\mathbf{x}=\alpha_i, \hbar, q) \in \mathbb{Q}^*_\alpha + q \cdot \mathbb{Q}_\alpha(\hbar)[\![q]\!] \subset \mathbb{Q}_\alpha[\![\hbar]\!][\![q]\!] \quad \text{for all } i \in [n],$$

$\mathcal{F}'$ is recursive, and $\mathcal{F}$ and $\mathcal{F}'$ satisfy the η-MPC, then $\mathcal{F}' \cong 0 \pmod{\hbar^{-1}}$ if and only if $\mathcal{F}' = 0$.

This is essentially [Zinger 2009, Proposition 2.1], with the assumptions corrected in [Popa and Zinger 2014, Footnote 3]. The proof in [Zinger 2009], which treats the $\eta = 1$ case, readily extends to the general case; see also the paragraph following [Popa and Zinger 2014, Proposition 4.3].

Lemma 6.4. *Let $C \equiv (C^j_i(d))_{d,i,j\in\mathbb{Z}^+}$ be any collection of elements of $\mathbb{Q}_\alpha$ and $\eta \in \mathbb{Q}_\alpha(\mathbf{x})$ be such that $\eta(\mathbf{x}=\alpha_i) \in \mathbb{Q}_\alpha$ is well defined and nonzero for every $i \in [n]$. If $\mathcal{F}, \mathcal{F}' \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar]\!][\![q]\!]$,*

$$\mathcal{F}(\mathbf{x}=\alpha_i, \hbar, q) \in \mathbb{Q}_\alpha(\hbar)[\![q]\!] \subset \mathbb{Q}_\alpha[\![\hbar]\!][\![q]\!] \quad \text{for all } i \in [n],$$

$\mathcal{F}'$ is C-recursive (and satisfies the η-MPC with respect to $\mathcal{F}$), then

(i) *$\{\mathbf{x} + \hbar\, q\, \mathrm{d}/\mathrm{d}q\}\mathcal{F}'$ is C-recursive (and satisfies the η-MPC with respect to $\mathcal{F}$);*

(ii) *if $f \in \mathbb{Q}_\alpha[\hbar][\![q]\!]$, then $f\mathcal{F}'$ is C-recursive (and satisfies the η-MPC with respect to $\mathcal{F}$).*

This lemma is essentially contained in [Zinger 2009, Lemma 2.3]. The proof in [Zinger 2009], which treats the $\eta = 1$ case, readily extends to the general case; see also the paragraph following [Popa and Zinger 2014, Lemma 4.4].

The next two sections establish Lemmas 6.5 and 6.6 below, the $m = 2$ cases of which complete the proofs of (3-11), the second statement in (3-12), and (3-15). The $m = 3$ cases of these lemmas are used in the proof of Proposition 3.1 and 4.1 in Section 9. If $m \geq m' \geq 2$, let

$$f_{m',m} : \overline{Q}_{0,m}(\mathbb{P}^{n-1}, d) \to \overline{Q}_{0,m'}(\mathbb{P}^{n-1}, d) \tag{6-5}$$

denote the forgetful morphism dropping the last $m-m'$ points; this morphism is defined if $m'>2$ or $d>0$. With the bundles

$$\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}},\, \ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a}} \to \overline{Q}_{0,m'}(\mathbb{P}^{n-1},d)$$

defined by (2-1), let

$$\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'} = f^*_{m',m}\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}},\; \ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'} = f^*_{m',m}\ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a}} \to \overline{Q}_{0,m}(\mathbb{P}^{n-1},d). \tag{6-6}$$

For $\mathbf{b}\equiv(b_2,\dots,b_m)\in(\mathbb{Z}^{\ge 0})^{m-1}$ and $\varpi\equiv(\varpi_2,\dots,\varpi_m)\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})^{m-1}$, let

(6-7)

$$\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\mathbf{x},\hbar,q) \equiv \sum_{d=0}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'})}{\hbar-\psi_1}\prod_{j=2}^{j=m}(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j)\right] \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\hbar^{-1}][\![q]\!],$$

$$\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\mathbf{x},\hbar,q) \equiv \sum_{d=0}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'})}{\hbar-\psi_1}\prod_{j=2}^{j=m}(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j)\right] \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\hbar^{-1}][\![q]\!],$$

where $\mathrm{ev}_j:\overline{Q}_{0,m}(\mathbb{P}^{n-1},d)\to\mathbb{P}^{n-1}$ is the evaluation map at the j-th marked point and the degree-0 terms in the $m'=2$ case are defined by

$$\mathbf{e}(\dot{\mathcal{V}}^{(0)}_{n;\mathbf{a};2}),\mathbf{e}(\ddot{\mathcal{V}}^{(0)}_{n;\mathbf{a};2}) = 1 \qquad \text{if } m\ge 3,$$

$$\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\dot{\mathcal{V}}^{(0)}_{n;\mathbf{a};2})}{\hbar-\psi_1}(\psi_2^{b_2}\mathrm{ev}_2^*\varpi_2)\right],\mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\ddot{\mathcal{V}}^{(0)}_{n;\mathbf{a};2})}{\hbar-\psi_1}(\psi_2^{b_2}\mathrm{ev}_2^*\varpi_2)\right] = (-\hbar)^{b_2}\varpi_2 \quad \text{if } m=2.$$

Lemma 6.5. *Let $l\in\mathbb{Z}^{\ge 0}$, $m,m',n\in\mathbb{Z}^+$ with $m\ge m'\ge 2$, and $\mathbf{a}\in(\mathbb{Z}^*)^l$. For all $\mathbf{b}\in(\mathbb{Z}^{\ge 0})^{m-1}$ and $\varpi\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})^{m-1}$, the power series $\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ and $\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ defined by* (6-7) *are $\dot{\mathfrak{C}}$ and $\ddot{\mathfrak{C}}$-recursive, respectively.*

Lemma 6.6. *Let $l\in\mathbb{Z}^{\ge 0}$, $m,m',n\in\mathbb{Z}^+$ with $m\ge m'\ge 2$, $\mathbf{a}\in(\mathbb{Z}^*)^l$,*

$$\dot\eta(\mathbf{x}) = \langle\mathbf{a}\rangle\mathbf{x}^{\ell(\mathbf{a})}, \quad \ddot\eta(\mathbf{x}) = 1.$$

For all $\mathbf{b}\in(\mathbb{Z}^{\ge 0})^{m-1}$ and $\varpi\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})^{m-1}$, the power series

$$\hbar^{m-2}\,\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\mathbf{x},\hbar,q) \quad \textit{and} \quad \hbar^{m-2}\,\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\mathbf{x},\hbar,q)$$

satisfy the $\dot\eta$ and $\ddot\eta$-MPC, respectively, with respect to the power series $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ defined by (3-3).

By Lemma 6.5, the power series $\dot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}$ and $\ddot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}$ defined by (3-4) are $\dot{\mathfrak{C}}$- and $\ddot{\mathfrak{C}}$-recursive, respectively. Furthermore, the power series $\dot{\mathcal{Z}}_{n;\mathbf{a}}$ defined by (3-7) is $\dot{\mathfrak{C}}$-recursive for $(\mathbf{x},\hbar)=(\mathbf{x}_1,\hbar_1)$ and $\mathbf{x}_2=\alpha_j$ fixed and is $\ddot{\mathfrak{C}}$-recursive for $(\mathbf{x},\hbar)=(\mathbf{x}_2,\hbar_2)$ and $\mathbf{x}_1=\alpha_j$ fixed. By Lemma 6.6, $\dot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}$ and $\ddot{\mathcal{Z}}^{(s)}_{n;\mathbf{a}}$ satisfy the $\dot\eta$- and $\ddot\eta$-MPC, respectively, with respect to the power series $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ defined by (3-3).

Furthermore, the power series $\dot{\mathcal{Z}}_{n;\mathbf{a}}$ defined by (3-7) satisfies the $\dot{\eta}$-MPC with respect to $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ for $(\mathbf{x},\hbar)=(\mathbf{x}_1,\hbar_1)$ and $\mathbf{x}_2=\alpha_j$ fixed and the $\ddot{\eta}$-MPC with respect to $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ for $(\mathbf{x},\hbar)=(\mathbf{x}_2,\hbar_2)$ and $\mathbf{x}_1=\alpha_j$ fixed.

In the case of products of projective spaces and concavex sheaves (1-13), the above Definition 6.1 becomes inductive on the total degree $d_1+\cdots+d_p$ of $q_1^{d_1}\cdots q_p^{d_p}$. The power series $\mathcal{F}$ is evaluated at $(\mathbf{x}_1,\dots,\mathbf{x}_p)=(\alpha_{1;i_1},\dots,\alpha_{p;i_p})$ for the purposes of the C-recursivity condition (6-1) and (6-2). The relevant structure coefficients, extending (5-1), are given by

$$\dot{\mathfrak{C}}^j_{i_1\dots i_p}(s;d)\equiv\frac{\prod\limits_{a_{k;1}\ge0}\prod\limits_{r=1}^{a_{k;s}d}\Big(\sum\limits_{t=1}^{p}a_{k;t}\alpha_{t;i_t}+r\frac{\alpha_{s;j}-\alpha_{s;i_s}}{d}\Big)\prod\limits_{a_{k;1}<0}\prod\limits_{r=0}^{-a_{k;s}d-1}\Big(\sum\limits_{t=1}^{p}a_{k;t}\alpha_{t;i_t}-r\frac{\alpha_{s;j}-\alpha_{s;i_s}}{d}\Big)}{d\prod\limits_{\substack{r=1\\(r,k)\neq(d,j)}}^{d}\prod\limits_{k=1}^{n_s}\Big(\alpha_{s;i_s}-\alpha_{s;k}+r\frac{\alpha_{s;j}-\alpha_{s;i_s}}{d}\Big)},$$

$$\ddot{\mathfrak{C}}^j_{i_1\dots i_p}(s;d)\equiv\frac{\prod\limits_{a_{k;1}\ge0}\prod\limits_{r=0}^{a_{k;s}d-1}\Big(\sum\limits_{t=1}^{p}a_{k;t}\alpha_{t;i_t}+r\frac{\alpha_{s;j}-\alpha_{s;i_s}}{d}\Big)\prod\limits_{a_{k;1}<0}\prod\limits_{r=1}^{-a_{k;s}d}\Big(\sum\limits_{t=1}^{p}a_{k;t}\alpha_{t;i_t}-r\frac{\alpha_{s;j}-\alpha_{s;i_s}}{d}\Big)}{d\prod\limits_{\substack{r=1\\(r,k)\neq(d,j)}}^{d}\prod\limits_{k=1}^{n_s}\Big(\alpha_{s;i_s}-\alpha_{s;k}+r\frac{\alpha_{s;j}-\alpha_{s;i_s}}{d}\Big)},$$

with $s\in[p]$ and $j\neq i_s$. The double sums in these equations are then replaced by triple sums over $s\in[p]$, $j\in[n_s]-i_s$, and $d\in\mathbb{Z}^+$, and with $\mathcal{F}$ evaluated at

$$\mathbf{x}_t=\begin{cases}\alpha_{s;j} & \text{if } t=s;\\ \alpha_{t;i_t} & \text{if } t\neq s;\end{cases}\qquad z=\frac{\alpha_{s;j}-\alpha_{s;i_s}}{d}.$$

The secondary coefficients $\mathcal{F}^r_i(d)$ in (6-2) now become $\mathcal{F}^r_{i_1\dots i_p}(d_1,\dots,d_p)$, with $i_s\in[n_s]$ and $d_s\in\mathbb{Z}^{\ge0}$. In the analogue of Definition 6.2, $\eta\in R(\mathbf{x}_1,\dots,\mathbf{x}_p)$ is such that the evaluation of η at $(\alpha_{1;i_1},\dots,\alpha_{p;i_p})$ for all elements $(i_1,\dots,i_p)$ of $[n_1]\times\cdots\times[n_p]$ is well defined and not zero, $\Phi_{\mathcal{F}}$ is a power series in $z_1,\dots,z_p$ and $q_1,\dots,q_p$, the sum is taken over all elements $(i_1,\dots,i_p)$ of $[n_1]\times\cdots\times[n_p]$, the leading fraction is replaced by

$$\frac{\eta(\alpha_{1;i_1},\dots,\alpha_{p;i_p})\mathrm{e}^{\alpha_{1;i_1}z_1+\cdots+\alpha_{p;i_p}z_p}}{\prod_{s=1}^{p}\prod_{k\neq i_s}(\alpha_{s;i_s}-\alpha_{s;k})},$$

and the $q\mathrm{e}^{\hbar z}$-insertion in the first power series is replaced by $q_1\mathrm{e}^{\hbar z_1},\dots,q_p\mathrm{e}^{\hbar z_p}$. Lemma 6.6 holds with

$$\dot{\eta}(\mathbf{x}_1,\dots,\mathbf{x}_p)=\frac{\prod_{a_{k;1}\ge0}\sum_{s=1}^{p}a_{k;s}\mathbf{x}_s}{\prod_{a_{k;1}<0}\sum_{s=1}^{p}a_{k;s}\mathbf{x}_s}.$$

7. Recursivity for stable quotients

In this section, we use the classical localization theorem [Atiyah and Bott 1984] to show that the generating functions $\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ and $\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ defined in (6-7) are recursive. The argument is similar to the proof in [Cooper and Zinger 2014, Section 6] of recursivity for the generating function $\dot{\mathcal{Z}}_{n;\mathbf{a}}$ defined by (3-3), but requires some modifications.

If $\mathbb{T}$ acts smoothly on a smooth compact oriented manifold M, there is a well-defined integration-along-the-fiber homomorphism

$$\int_M : H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}$$

for the fiber bundle $BM \to B\mathbb{T}$. The classical localization theorem of [Atiyah and Bott 1984] relates it to integration along the fixed locus of the $\mathbb{T}$-action. The latter is a union of smooth compact orientable manifolds F; $\mathbb{T}$ acts on the normal bundle $\mathcal{N}F$ of each F. Once an orientation of F is chosen, there is a well-defined integration-along-the-fiber homomorphism

$$\int_F : H^*_{\mathbb{T}}(F) \to H^*_{\mathbb{T}}.$$

The localization theorem states that

$$\int_M \eta = \sum_F \int_F \frac{\eta|_F}{\mathbf{e}(\mathcal{N}F)} \in \mathbb{Q}_\alpha \quad \text{for all } \eta \in H^*_{\mathbb{T}}(M), \tag{7-1}$$

where the sum is taken over all components F of the fixed locus of $\mathbb{T}$. Part of the statement of (7-1) is that $\mathbf{e}(\mathcal{N}F)$ is invertible in $H^*_{\mathbb{T}}(F) \otimes_{\mathbb{Q}[\alpha_1,\dots,\alpha_n]} \mathbb{Q}_\alpha$. In the case of the standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$, (7-1) implies that

$$\eta|_{P_i} = \int_{\mathbb{P}^{n-1}} \eta\phi_i \in \mathbb{Q}_\alpha \tag{7-2}$$

for all $\eta \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$, $i = 1, 2, \dots, n$, with ϕ_i as in (5-4).

7A. ***Fixed locus data.*** The proof of Lemma 6.5 involves a localization computation on $\overline{Q}_{0,m}(\mathbb{P}^{n-1}, d)$. Thus, we need to describe the fixed loci of the $\mathbb{T}$-action on $\overline{Q}_{0,m}(\mathbb{P}^{n-1}, d)$, their normal bundles, and the restrictions of the relevant cohomology classes to these fixed loci.

As in the case of stable maps described in [Hori et al. 2003, Section 27.3], the fixed loci of the $\mathbb{T}$-action on $\overline{Q}_{0,m}(\mathbb{P}^{n-1}, d)$ are indexed by decorated graphs,

$$\Gamma = (\mathrm{Ver}, \mathrm{Edg}; \mu, \mathfrak{d}, \vartheta), \tag{7-3}$$

where (Ver, Edg) is a connected graph that has no loops, with Ver and Edg denoting its sets of vertices and edges, and

$$\mu : \mathrm{Ver} \to [n], \quad \mathfrak{d} : \mathrm{Ver} \sqcup \mathrm{Edg} \to \mathbb{Z}^{\geq 0}, \quad \text{and} \quad \vartheta : [m] \to \mathrm{Ver}$$

are maps such that

$$\text{(7-4)} \qquad \mu(v_1) \neq \mu(v_2) \ \text{ if } \{v_1, v_2\} \in \mathrm{Edg}, \quad \mathfrak{d}(e) \neq 0 \ \text{ for all } e \in \mathrm{Edg},$$
$$\mathrm{val}(v) \equiv |\vartheta^{-1}(v)| + |\{e \in \mathrm{Edg} : v \in e\}| + \mathfrak{d}(v) \geq 2 \ \text{ for all } v \in \mathrm{Ver}.$$

In Figure 1, the vertices of a decorated graph Γ are indicated by dots. The values of the map $(\mu, \mathfrak{d})$ on some of the vertices are indicated next to those vertices. Similarly, the values of the map $\mathfrak{d}$ on some of the edges are indicated next to them. The elements of the sets $[m]$ are shown in bold face; they are linked by line segments to their images under ϑ. By (7-4), no two consecutive vertices have the same first label and thus $j \neq i$.

With Γ as in (7-3), let

$$|\Gamma| \equiv \sum_{v \in \mathrm{Ver}} \mathfrak{d}(v) + \sum_{e \in \mathrm{Edg}} \mathfrak{d}(e)$$

be the degree of Γ. For each $v \in \mathrm{Ver}$, let

$$\mathrm{E}_v = \{e \in \mathrm{Edg} : v \in e\}$$

be the set of edges leaving from v. There is a unique partial order $\prec$ on Ver that has a unique minimal element $v_{\min}$ such that $v_{\min} = \vartheta(1)$ and $v \prec w$ if there exist distinct vertices $v_1, \ldots, v_k \in \mathrm{Ver}$ such that

$$v \in \{v_{\min}, v_1, \ldots, v_{k-1}\}, \quad w = v_k, \quad \text{and}$$
$$\{v_{\min}, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\} \in \mathrm{Edg},$$

in other words, v lies between $v_{\min}$ and w in (Ver, Edg). If $e = \{v_1, v_2\} \in \mathrm{Edg}$ is any edge in Γ with $v_1 \prec v_2$, let

$$\Gamma_e \equiv \big(\{v_1, v_2\}, \{e\}; \mu_e, \mathfrak{d}_e, \vartheta_e\big),$$

be the decorated graph as in (7-3) given by

$$\mu_e = \mu|_e, \quad \mathfrak{d}_e(e) = \mathfrak{d}(e), \mathfrak{d}_e|_e = 0, \quad \vartheta_e : \{1, 2\} \to e, \quad \vartheta_e(1) = v_1, \quad \vartheta_e(2) = v_2;$$

see Figure 2.

With $m' \leq m$ as in Lemmas 6.5 and 6.6, let

$$\mathrm{Ver}_{m'} = \big\{v \in \mathrm{Ver} : v \preceq \vartheta(i) \text{ for some } i \in [m']\big\},$$
$$\mathrm{Edg}_{m'} = \big\{\{v_1, v_2\} \in \mathrm{Edg} : v_1, v_2 \in \mathrm{Ver}_{m'}\big\}.$$

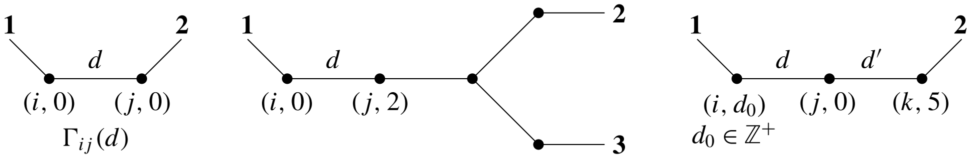

Figure 1. Two trees with $\mathrm{val}(v_{\min})=2$ and a tree with $\mathrm{val}(v_{\min})\ge 3$.

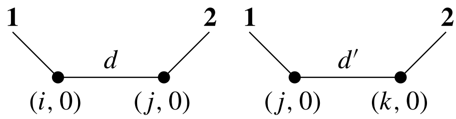

Figure 2. The subtrees corresponding to the edges of the last graph in Figure 1.

In particular, the graph $(\mathrm{Ver}_{m'},\mathrm{Edg}_{m'})$ is a tree; it is obtained from the original graph $(\mathrm{Ver},\mathrm{Edg})$ by discarding the branches that do not end at a vertex with a marked point labeled by $i\le m'$. For each $v\in\mathrm{Ver}_{m'}$, define

$$r_{m';v}:\mathrm{E}_v-\mathrm{Edg}_{m'}\to\mathbb{Z}^+\quad\text{by}\quad r_{m';v}(\{v,v'\})=\sum_{\substack{\{v_1,v_2\}\in\mathrm{Edg}\\ v'\preceq v_2}}\mathfrak{d}(\{v_1,v_2\})+\sum_{\substack{w\in\mathrm{Ver}\\ v'\preceq w}}\mathfrak{d}(w),$$

$$\mathfrak{d}_{m'}(v)=\mathfrak{d}(v)+\sum_{e\in\mathrm{E}_v-\mathrm{Edg}_{m'}}r_{m;'v}(e).$$

This construction increases the degree $\mathfrak{d}(v)$ of a vertex $v\in\mathrm{Ver}_{m'}$ by the total degree of all branches of Γ cut off at v to form the graph $(\mathrm{Ver}_{m'},\mathrm{Edg}_{m'})$. The motivation for this construction is described at the end of the next paragraph.

As is described in [Marian et al. 2011, Section 7.3], the fixed locus Q_Γ of $\overline{Q}_{0,m}(\mathbb{P}^{n-1},|\Gamma|)$ corresponding to a decorated graph Γ consists of the stable quotients

$$(\mathcal{C},y_1,\ldots,y_m;\,S\subset\mathbb{C}^n\otimes\mathcal{O}_{\mathcal{C}})$$

over quasistable rational m-marked curves that satisfy the following conditions. The components of $\mathcal{C}$ on which the corresponding quotient is torsion free are rational and correspond to the edges of Γ; the restriction of S to any such component corresponds to a morphism to $\mathbb{P}^{n-1}$ of the opposite degree to that of the subsheaf. Furthermore, if $e=\{v_1,v_2\}$ is an edge, the corresponding morphism f_e is a degree-$\mathfrak{d}(e)$ cover of the line

$$\mathbb{P}^1_{\mu(v_1),\mu(v_2)}\subset\mathbb{P}^{n-1}$$

passing through the fixed points $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$; it is ramified only over $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. In particular, f_e is unique up to isomorphism. The remaining components of $\mathcal{C}$ are indexed by the vertices $v\in\mathrm{Ver}$ of valence $\mathrm{val}(v)\ge 3$. The restriction of S to such a component $\mathcal{C}_v$ of $\mathcal{C}$ (or possibly a connected union of irreducible components) is a subsheaf of the trivial subsheaf $P_{\mu(v)} \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}_v}$ of degree $-\mathfrak{d}(v)$; thus, the induced morphism takes $\mathcal{C}_v$ to the fixed point $P_{\mu(v)} \in \mathbb{P}^{n-1}$. Each such component $\mathcal{C}_v$ also carries $|\vartheta^{-1}(v)| + |\mathrm{E}_v|$ marked points, corresponding to the marked points and/or the nodes of $\mathcal{C}$; we index these points by the set $\vartheta^{-1}(v) \sqcup \mathrm{E}_v$ in the canonical way. Thus, as stacks,

$$
\begin{aligned}
\mathcal{Q}_\Gamma &\approx \prod_{\substack{v\in\mathrm{Ver}\\ \mathrm{val}(v)\ge 3}} \overline{Q}_{0,|\vartheta^{-1}(v)|+|\mathrm{E}_v|}(\mathbb{P}^0, \mathfrak{d}(v)) \times \prod_{e\in\mathrm{Edg}} \mathcal{Q}_{\Gamma_e} \\
&\approx \prod_{\substack{v\in\mathrm{Ver}\\ \mathrm{val}(v)\ge 3}} \overline{\mathcal{M}}_{0,|\vartheta^{-1}(v)|+|\mathrm{E}_v||\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)} \times \prod_{e\in\mathrm{Edg}} \mathcal{Q}_{\Gamma_e} \\
&\approx \Bigg(\prod_{\substack{v\in\mathrm{Ver}\\ \mathrm{val}(v)\ge 3}} \overline{\mathcal{M}}_{0,|\vartheta^{-1}(v)|+|\mathrm{E}_v||\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)} \Bigg) \Big/ \prod_{e\in\mathrm{Edg}} \mathbb{Z}_{\mathfrak{d}(e)},
\end{aligned}
\tag{7-5}
$$

with each cyclic group $\mathbb{Z}_{\mathfrak{d}(e)}$ acting trivially. For example, in the case of the last diagram in Figure 1,

$$\mathcal{Q}_\Gamma \approx \big(\overline{\mathcal{M}}_{0,2|d_0}/\mathbb{S}_{d_0} \times \overline{\mathcal{M}}_{0,2|5}/\mathbb{S}_5\big)\big/\mathbb{Z}_d \times \mathbb{Z}_{d'}$$

is a fixed locus in $\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d_0+5+d+d')$. If $m' \le m$ is as in Lemmas 6.5 and 6.6, the morphism $f_{m',m}$ in (6-5) sends the locus $\mathcal{Q}_\Gamma$ of $\overline{Q}_{0,m}(\mathbb{P}^{n-1}, d)$ to (a subset of) the locus $\mathcal{Q}_{\Gamma_{m'}}$ of $\overline{Q}_{0,m'}(\mathbb{P}^{n-1}, d)$, where

$$\Gamma_{m'} = \big(\mathrm{Ver}_{m'}, \mathrm{Edg}_{m'}; \mu|_{\mathrm{Ver}_{m'}}, \mathfrak{d}_{m'}, \vartheta|_{[m']}\big),$$

as $f_{m',m}$ contracts the ends of the elements of $\overline{Q}_{0,m'}(\mathbb{P}^{n-1}, d)$ that do not carry any of the marked points indexed by the set $[m']$.

If $v \in \mathrm{Ver}$ and $\mathrm{val}(v) \ge 3$, for the purposes of definitions (4-4) and (4-5) we identify $[|\vartheta^{-1}(v)|+|\mathrm{E}_v|]$ with the set $\vartheta^{-1}(v) \sqcup \mathrm{E}_v$ indexing the marked points on $\mathcal{C}_v$ so that the element 1 in the former is identified with $1 \in [m]$ if $\vartheta(1) = v$ and with the unique edge $e_v^- = \{v_-, v\}$ with $v^- \prec v$ separating v from the marked point 1 otherwise. Similarly, if $v \preceq \vartheta(2)$, we associate the element 2 of $[|\vartheta^{-1}(v)|+|\mathrm{E}_v|]$ with $2 \in [m]$ if $\vartheta(2) = v$ and with the unique edge $e_v^+ = \{v, v_+\}$ with $v_+ \preceq \vartheta(2)$ separating v from the marked point 2 otherwise. Finally, if $m' \le m$ is as in Lemmas 6.5 and 6.6 and $v \in \mathrm{Ver}_{m'}$, we associate the $|\mathrm{E}_v - \mathrm{Edg}_{m'}|$ largest elements of $[\,|\vartheta^{-1}(v)|+|\mathrm{E}_v|\,]$ with the subset $\mathrm{E}_v - \mathrm{Edg}_{m'}$ of $\vartheta^{-1}(v) \sqcup \mathrm{E}_v$.

If Γ is a decorated graph as above and $e = \{v_1, v_2\} \in \mathrm{Edg}$ with $v_1 \prec v_2$, let

$$\pi_e : \mathcal{Q}_\Gamma \to \mathcal{Q}_{\Gamma_e} \subset \overline{Q}_{0,2}(\mathbb{P}^{n-1}, \mathfrak{d}(e))$$

be the projection in the decomposition (7-5) and

$$\omega_{e;v_1} = -\pi_e^*\psi_1,\ \omega_{e;v_2} = -\pi_e^*\psi_2 \in H^2(\mathcal{Q}_\Gamma).$$

Similarly, for each $v \in \mathrm{Ver}$ such that $\mathrm{val}(v) \geq 3$, let

$$\pi_v : Q_\Gamma \to \overline{\mathcal{M}}_{0,|\vartheta^{-1}(v)|+|\mathrm{E}_v||\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)}$$

be the corresponding projection and

$$\psi_{v;e} = \pi_v^* \psi_e \in H^2(Q_\Gamma) \quad \text{for all } v \in \mathrm{E}_v.$$

By [Hori et al. 2003, Section 27.2],

$$\omega_{e;v_i} = \frac{\alpha_{\mu(v_i)} - \alpha_{\mu(v_{3-i})}}{\mathfrak{d}(e)} \qquad i = 1, 2. \tag{7-6}$$

By [Marian et al. 2011, Section 7.4], the Euler class of the normal bundle of Q_Γ in $\overline{Q}_{0,m}(\mathbb{P}^{n-1}, |\Gamma|)$ is described by

$$\begin{aligned} &\frac{\mathbf{e}(\mathcal{N} Q_\Gamma)}{\mathbf{e}(T_{\mu(v_{\min})}\mathbb{P}^{n-1})} \\ &\quad = \prod_{\substack{v\in \mathrm{Ver}\\ \mathrm{val}(v)\geq 3}} \prod_{k\neq\mu(v)} \pi_v^* \mathbf{e}\big(\dot{\mathcal{V}}_1^{(\mathfrak{d}(v))}(\alpha_{\mu(v)}-\alpha_k)\big) \prod_{e\in\mathrm{Edg}} \pi_e^* \mathbf{e}\big(H^0(f_e^* T\mathbb{P}^n \otimes \mathcal{O}(-y_1))/\mathbb{C}\big) \\ &\qquad \times \prod_{\substack{v\in\mathrm{Ver}\\ \mathrm{val}(v)=2, \vartheta^{-1}(v)=\varnothing}} \Big(\sum_{e\in\mathrm{E}_v} \omega_{e;v}\Big) \prod_{\substack{v\in\mathrm{Ver}\\ \mathrm{val}(v)\geq 3}} \Big(\prod_{e\in \mathrm{E}_v} (\omega_{e;v} - \psi_{v;e})\Big), \end{aligned} \tag{7-7}$$

where $\mathbb{C} \subset H^0(f_e^* T\mathbb{P}^n \otimes \mathcal{O}(-y_1))$ denotes the trivial $\mathbb{T}$-representation. The terms on the first line correspond to the deformations of the sheaf without changing the domain, while the terms on the second line correspond to the deformations of the domain. By (6-6), (2-1), (4-4), and (4-5),

$$\begin{aligned} \mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a};m'}^{(|\Gamma|)})\big|_{Q_\Gamma} &= \prod_{\substack{v\in\mathrm{Ver}_{m'}\\ \mathrm{val}(v)\geq 3}} \pi_v^* \mathbf{e}\big(\dot{\mathcal{V}}_{\mathbf{a};r_{m';v}}^{(\mathfrak{d}(v))}(\alpha_{\mu(v)})\big) \cdot \prod_{e\in\mathrm{Edg}_{m'}} \pi_e^* \mathbf{e}\big(\dot{\mathcal{V}}_{n;\mathbf{a}}^{\mathfrak{d}(e)}\big), \\ \mathbf{e}(\ddot{\mathcal{V}}_{n;\mathbf{a};m'}^{(|\Gamma|)})\big|_{Q_\Gamma} &= \prod_{\substack{v\in\mathrm{Ver}_2\\ \mathrm{val}(v)\geq 3}} \pi_v^* \mathbf{e}\big(\ddot{\mathcal{V}}_{\mathbf{a};r_{m';v}}^{(\mathfrak{d}(v))}(\alpha_{\mu(v)})\big) \cdot \prod_{e\in\mathrm{Edg}_2} \pi_e^* \mathbf{e}\big(\ddot{\mathcal{V}}_{n;\mathbf{a}}^{\mathfrak{d}(e)}\big) \\ &\quad \times \prod_{\substack{v\in\mathrm{Ver}_{m'}-\mathrm{Ver}_2\\ \mathrm{val}(v)\geq 3}} \pi_v^* \mathbf{e}\big(\dot{\mathcal{V}}_{\mathbf{a};r_{m';v}}^{(\mathfrak{d}(v))}(\alpha_{\mu(v)})\big) \cdot \prod_{e\in\mathrm{Edg}_{m'}-\mathrm{Edg}_2} \pi_e^* \mathbf{e}\big(\dot{\mathcal{V}}_{n;\mathbf{a}}^{\mathfrak{d}(e)}\big). \end{aligned} \tag{7-8}$$

By [Hori et al. 2003, Section 27.2], for all $e = \{v_1, v_2\}$ with $v_1 \prec v_2$

$$\begin{aligned} \int_{Q_{\Gamma_e}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{\mathfrak{d}(e)})}{\mathbf{e}\big(H^0(f_e^* T\mathbb{P}^n \otimes \mathcal{O}(-y_1))/\mathbb{C}\big)} &= \dot{\mathfrak{C}}_{\mu(v_1)}^{\mu(v_2)}(\mathfrak{d}(e)), \\ \int_{Q_{\Gamma_e}} \frac{\mathbf{e}(\ddot{\mathcal{V}}_{n;\mathbf{a}}^{\mathfrak{d}(e)})}{\mathbf{e}\big(H^0(f_e^* T\mathbb{P}^n \otimes \mathcal{O}(-y_1))/\mathbb{C}\big)} &= \ddot{\mathfrak{C}}_{\mu(v_1)}^{\mu(v_2)}(\mathfrak{d}(e)), \end{aligned} \tag{7-9}$$

with $\dot{\mathfrak{C}}^{\mu(v_2)}_{\mu(v_1)}(\mathfrak{d}(e))$ and $\ddot{\mathfrak{C}}^{\mu(v_2)}_{\mu(v_1)}(\mathfrak{d}(e))$ given by (5-1).

7B. ***Proof of Lemma 6.5.*** We apply the localization theorem to

$$\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\alpha_i,\hbar,q)=\sum_{d=0}^{\infty}q^d\int_{\overline{Q}_{0,m}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\prod_{j=2}^{m}(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j),$$

$$\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\alpha_i,\hbar,q)=\sum_{d=0}^{\infty}q^d\int_{\overline{Q}_{0,m}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\prod_{j=2}^{m}(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j),\tag{7-10}$$

where ϕ_i is the equivariant Poincaré dual of the fixed point $P_i\in\mathbb{P}^{n-1}$, as in (5-4), and the degree-0 terms in the $m=2$ case are defined by

$$\int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},0)}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}(\psi_2^{b_2}\mathrm{ev}_2^*\varpi_2)\equiv(-\hbar)^{b_2}\varpi_2|_{P_i},$$

$$\int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},0)}\frac{\mathbf{e}(\ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}(\psi_2^{b_2}\mathrm{ev}_2^*\varpi_2)\equiv(-\hbar)^{b_2}\varpi_2|_{P_i}.$$

Since $\phi_i|_{P_j}=0$ unless $j=i$, a decorated graph as in (7-3) contributes to the two expressions in (7-10) only if the first marked point is attached to a vertex labeled i, that is, $\mu(v_{\min})=i$ for the smallest element $v_{\min}\in\mathrm{Ver}$. We show that, just as for Givental's J-function, the (d,j)-summand in (6-2) with $C=\dot{\mathfrak{C}},\ddot{\mathfrak{C}}$ and $\mathcal{F}=\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'},\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$, that is,

$$\frac{\dot{\mathfrak{C}}_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i/d)}\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\alpha_j,(\alpha_j-\alpha_i)/d,q)\quad\text{and}$$

$$\frac{\ddot{\mathfrak{C}}_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i/d)}\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\alpha_j,(\alpha_j-\alpha_i)/d,q),\tag{7-11}$$

respectively, is the sum over all graphs such that $\mu(v_{\min})=i$, that is, the first marked point is mapped to the fixed point $P_i\in\mathbb{P}^{n-1}$, $v_{\min}$ is a bivalent vertex, that is, $\mathfrak{d}(v_{\min})=0$, $\vartheta^{-1}(v_{\min})=\{1\}$, the only edge leaving this vertex is labeled d, and the other vertex of this edge is labeled j. We also show that the first sum on the right-hand side of (6-2) is the sum over all graphs such that $\mu(v_{\min})=i$ and $\mathrm{val}(v_{\min})\geq 3$.

If Γ is a decorated graph with $\mu(v_{\min})=i$ as above,

$$\mathrm{ev}_1^*\phi_i\big|_{Q_\Gamma}=\prod_{k\neq i}(\alpha_i-\alpha_k)=\mathbf{e}(T_{\mu(v_{\min})}\mathbb{P}^{n-1}).\tag{7-12}$$

Suppose in addition that $\mathrm{val}(v_{\min})=2$ and $|\mathrm{E}_{v_{\min}}|=1$. Let $v_1\equiv(v_{\min})_+$ be the immediate successor of $v_{\min}$ in Γ and $e_1=\{v_{\min},v_1\}$ be the edge leaving $v_{\min}$. If

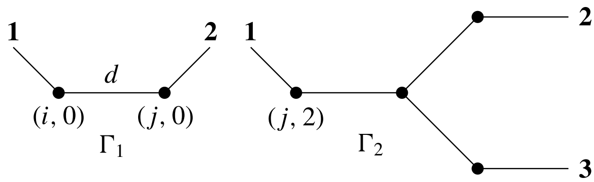

Figure 3. The two subgraphs of the second graph in Figure 1.

$|\mathrm{Edg}|>1$ or $\mathrm{val}(v_1)>2$, that is, Γ is not as in the first diagram in Figure 1, we break Γ at v_1 into two "subgraphs":

(i) $\Gamma_1=\Gamma_{e_1}$ consisting of the vertices $v_{\min}\prec v_1$, the edge $\{v_{\min},v_1\}$, with the $\mathfrak{d}$-value of 0 at both vertices, and a marked point at v and v_1;

(ii) Γ_2 consisting of all vertices, edges, and marked points of Γ, other than the vertex $v_{\min}$ and the edge $\{v_{\min},v_1\}$, and with the marked point 1 attached at v_1;

see Figure 3. By (7-5),

$$Q_\Gamma\approx Q_{\Gamma_1}\times Q_{\Gamma_2}. \tag{7-13}$$

Let $\pi_1,\pi_2: Q_\Gamma\to Q_{\Gamma_1},Q_{\Gamma_2}$ be the component projection maps. By (7-7) and (7-8),

$$\frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}=\pi_1^*\left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}\right)\cdot\pi_2^*\left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}{\mathbf{e}(T_{P_{\mu(v_1)}}\mathbb{P}^{n-1})}\right)\cdot(\omega_{e;v_1}-\pi_2^*\psi_1),$$

$$\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a};m'}\big)\big|_{Q_\Gamma}=\pi_1^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_1|)}_{n;\mathbf{a}}\big)\cdot\pi_2^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'}\big),$$

$$\mathbf{e}\big(\ddot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a};m'}\big)\big|_{Q_\Gamma}=\pi_1^*\mathbf{e}\big(\ddot{\mathcal{V}}^{(|\Gamma_1|)}_{n;\mathbf{a}}\big)\cdot\pi_2^*\mathbf{e}\big(\ddot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'}\big).$$

Combining the above splittings with (7-6), (7-9), and (7-12), we find that

$$\begin{aligned}
&q^{|\Gamma|}\int_{Q_\Gamma}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\prod_{j=2}^{j=m}(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j)\Big|_{Q_\Gamma}\frac{1}{\mathbf{e}(\mathcal{N}Q_\Gamma)}\\
&\quad=\frac{\dot{\mathfrak{C}}_i^{\mu(v_1)}(\mathfrak{d}(e_1))q^{\mathfrak{d}(e_1)}}{\hbar-(\alpha_{\mu(v_1)}-\alpha_i)/\mathfrak{d}(e_1)}\\
&\qquad\times\left(q^{|\Gamma_2|}\left\{\int_{Q_{\Gamma_2}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_{\mu(v_1)}}{\hbar-\psi_1}\prod_{j=2}^{j=m}(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j)\frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}\right\}\bigg|_{\hbar=\frac{\alpha_{\mu(v_1)}-\alpha_i}{\mathfrak{d}(e_1)}}\right);
\end{aligned}$$

the same identity holds with $\dot{\mathcal{V}}$ replaced by $\ddot{\mathcal{V}}$ and $\dot{\mathfrak{C}}_i^{\mu(v_1)}(\mathfrak{d}(e_1))$ by $\ddot{\mathfrak{C}}_i^{\mu(v_1)}(\mathfrak{d}(e_1))$. By the first equation in (7-10) with i replaced by $\mu(v_1)$ and the localization formula (7-1), the sum of the last factor above over all possibilities for Γ_2, with Γ_1 held fixed, is

$$\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}\big(\alpha_{\mu(v_1)},(\alpha_{\mu(v_1)}-\alpha_i)/\mathfrak{d}(e_1),q\big)-\delta_{m,2}\bigg(\frac{\alpha_i-\alpha_{\mu(v_1)}}{\mathfrak{d}(e_1)}\bigg)^{b_2}\varpi_2|_{P_{\mu(v_1)}};$$

if $\dot{\mathcal{V}}$ is replaced by $\ddot{\mathcal{V}}$, then the sum becomes

$$\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}\big(\alpha_{\mu(v_1)},(\alpha_{\mu(v_1)}-\alpha_i)/\mathfrak{d}(e_1),q\big)-\delta_{m,2}\bigg(\frac{\alpha_i-\alpha_{\mu(v_1)}}{\mathfrak{d}(e_1)}\bigg)^{b_2}\varpi_2|_{P_{\mu(v_1)}}.$$

In the $m=2$ case, the contributions of the one-edge graph $\Gamma_{i\mu(v_1)}(\mathfrak{d}(e_1))$ such as $\mathfrak{d}(v_1)=0$, as in the first diagram in Figure 1, to the two expressions in (7-10) are

$$\frac{\dot{\mathfrak{C}}_i^{\mu(v_1)}(\mathfrak{d}(e_1))q^{\mathfrak{d}(e_1)}}{\hbar_1-(\alpha_{\mu(v_1)}-\alpha_i)/\mathfrak{d}(e_1)}\bigg(\frac{\alpha_i-\alpha_{\mu(v_1)}}{\mathfrak{d}(e_1)}\bigg)^{b_2}\varpi_2|_{P_{\mu(v_1)}}\quad\text{and}$$
$$\frac{\ddot{\mathfrak{C}}_i^{\mu(v_1)}(\mathfrak{d}(e_1))q^{\mathfrak{d}(e_1)}}{\hbar_1-(\alpha_{\mu(v_1)}-\alpha_i)/\mathfrak{d}(e_1)}\bigg(\frac{\alpha_i-\alpha_{\mu(v_1)}}{\mathfrak{d}(e_1)}\bigg)^{b_2}\varpi_2|_{P_{\mu(v_1)}},$$

respectively. Thus, the contributions to the two expressions in (7-10) from all graphs Γ such that $\mathfrak{d}(v_{\min})=0$, $\mu(v_1)=j$, and $\mathfrak{d}(e_1)=d$ are given by (7-11), that is, they are the (d,j)-summands in the recursions (6-2) for $\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ and $\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$.

Suppose next that Γ is a graph such that $\mu(v_{\min})=i$ and $\mathrm{val}(v_{\min})\ge 3$. If $|\mathrm{Ver}|>1$, that is, Γ is not as in the first diagram in Figure 4, we break Γ at $v_{\min}$ into "subgraphs":

(i) Γ_0 consisting of the vertex $\{v_{\min}\}$ only, with the same μ and $\mathfrak{d}$-values as in Γ, with the same marked points as before, along with a marked point e for each edge $e\in\mathrm{E}_{v_{\min}}$ from $v_{\min}$;

(ii) for each $e\in\mathrm{E}_{v_{\min}}$, $\Gamma_{c;e}$ consisting of the branch of Γ beginning with the edge e at $v_{\min}$, with the $\mathfrak{d}$-value of $v_{\min}$ replaced by 0, and with one marked point at $v_{\min}$;

see Figure 4 and 8. By (7-5),

$$Q_\Gamma\approx Q_{\Gamma_0}\times\prod_{e\in\mathrm{E}_{v_{\min}}}Q_{\Gamma_{c;e}}=(\overline{\mathcal{M}}_{0,m_0|\mathfrak{d}(v_{\min})}/\mathbb{S}_{\mathfrak{d}(v_{\min})})\times\prod_{e\in\mathrm{E}_{v_{\min}}}Q_{\Gamma_{c;e}},\tag{7-14}$$
$$\text{where}\quad m_0=|\vartheta^{-1}(v_{\min})|+|\mathrm{E}_{v_{\min}}|.$$

Let $\pi_0,\pi_{c;e}$ be the component projection maps in (7-14). Since $\psi_1|_{Q_\Gamma}=\pi_0^*\psi_1$, $\mathbb{T}$ acts trivially on $\overline{\mathcal{M}}_{0,m_0|\mathfrak{d}(v_{\min})}$,

$$\psi_1=1\times\psi_1\in H^*_{\mathbb{T}}(\overline{\mathcal{M}}_{0,m_0|\mathfrak{d}(v_{\min})})=H^*_{\mathbb{T}}\otimes H^*(\overline{\mathcal{M}}_{0,m_0|\mathfrak{d}(v_{\min})}),$$

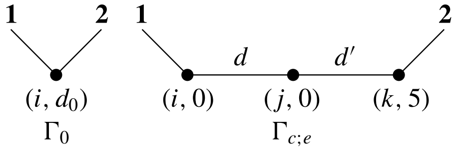

Figure 4. The two subgraphs of the last graph in Figure 1.

that is, $\mathbb{T}$ acts trivially on the universal cotangent line bundle for the first marked point on $\overline{\mathcal{M}}_{0,m_0|\mathfrak{d}(v_{\min})}$, and the dimension of $\overline{\mathcal{M}}_{0,m_0|\mathfrak{d}(v_{\min})}$ is $m_0+\mathfrak{d}(v_{\min})-3$,

$$\frac{1}{\hbar-\psi_1}\Big|_{Q_\Gamma}=\sum_{r=0}^{m_0+\mathfrak{d}(v_{\min})-3}\hbar^{-(r+1)}\pi_0^*\psi_1^r.$$

Since $m_0+\mathfrak{d}(v_{\min})\le m+|\Gamma|$ and Γ contributes to the coefficient of $q^{|\Gamma|}$ in (7-10), it follows that (6-2) holds with $\mathcal{F}$ replaced by $\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ and $\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ with $N_d=m+d-2$, $C_i^j(d)=\dot{\mathfrak{C}}_i^j(d)$ in the first case, and $C_i^j(d)=\ddot{\mathfrak{C}}_i^j(d)$ in the second case.

The argument in this section extends to products of projective spaces and concavex sheaves (1-13) as described in [Cooper and Zinger 2014, Section 6].

8. Polynomiality for stable quotients

In this section, we use the classical localization theorem [Atiyah and Bott 1984] to show that the generating functions $\hbar^{m-2}\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ and $\hbar^{m-2}\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ defined in (6-7) satisfy specific mutual polynomiality conditions of Definition 6.2 with respect to the generating function $\dot{\mathcal{Z}}_{n;\mathbf{a}}$ defined in (3-3). The argument is similar to the proof in [Cooper and Zinger 2014, Section 7] of self-polynomiality for the generating function $\dot{\mathcal{Z}}_{n;\mathbf{a}}$ defined in (3-3), but requires some modifications.

8A. ***Proof of Lemma 6.6.*** The proof involves applying the classical localization theorem [Atiyah and Bott 1984] with $(n+1)$-torus

$$\widetilde{\mathbb{T}}\equiv\mathbb{C}^*\times\mathbb{T},$$

where $\mathbb{T}=(\mathbb{C}^*)^n$ as before. We denote the weight of the standard action of the one-torus $\mathbb{C}^*$ on $\mathbb{C}$ by $\hbar$. Thus, by Section 3A,

$$H^*_{\mathbb{C}^*}\approx\mathbb{Q}[\hbar],\qquad H^*_{\widetilde{\mathbb{T}}}\approx\mathbb{Q}[\hbar,\alpha_1,\dots,\alpha_n].$$

Throughout this section, $V=\mathbb{C}\oplus\mathbb{C}$ denotes the representation of $\mathbb{C}^*$ with the weights 0 and $-\hbar$. The induced action on $\mathbb{P}V$ has two fixed points:

$$q_1\equiv[1,0],\qquad q_2\equiv[0,1].$$

With $\gamma_1\to\mathbb{P}V$ denoting the tautological line bundle,

$$\mathbf{e}(\gamma_1^*)\big|_{q_1}=0,\quad \mathbf{e}(\gamma_1^*)\big|_{q_2}=-\hbar,\quad \mathbf{e}(T_{q_1}\mathbb{P}V)=\hbar,\quad \mathbf{e}(T_{q_2}\mathbb{P}V)=-\hbar; \tag{8-1}$$

this follows from our definition of the weights in [Cooper and Zinger 2014, Section 3].

For each $d \in \mathbb{Z}^{\geq 0}$, the action of $\widetilde{\mathbb{T}}$ on $\mathbb{C}^n \otimes \mathrm{Sym}^d V^*$ induces an action on

$$\overline{\mathfrak{X}}_d \equiv \mathbb{P}(\mathbb{C}^n \otimes \mathrm{Sym}^d V^*).$$

It has $(d+1)n$ fixed points:

$$P_i(r) \equiv \big[\tilde{P}_i \otimes u^{d-r} v^r\big], \qquad i \in [n],\, r \in \{0\} \cup [d],$$

if (u, v) are the standard coordinates on V and $\tilde{P}_i \in \mathbb{C}^n$ is the i-th coordinate vector (so that $[\tilde{P}_i] = P_i \in \mathbb{P}^{n-1}$). Let

$$\Omega \equiv \mathbf{e}(\gamma^*) \in H^*_{\widetilde{\mathbb{T}}}(\overline{\mathfrak{X}}_d)$$

denote the equivariant hyperplane class.

For all $i \in [n]$ and $r \in \{0\} \cup [d]$,

$$\Omega|_{P_i(r)} = \alpha_i + r\hbar, \quad \mathbf{e}(T_{P_i(r)}\overline{\mathfrak{X}}_d) = \Bigg\{ \prod_{\substack{s=0 \\ (s,k)\neq(r,i)}}^{d} \prod_{k=1}^{n} (\Omega - \alpha_k - s\hbar) \Bigg\}\Bigg|_{\Omega=\alpha_i+r\hbar}. \tag{8-2}$$

Since

$$B\overline{\mathfrak{X}}_d = \mathbb{P}(B(\mathbb{C}^n \otimes \mathrm{Sym}^d V^*)) \to B\widetilde{\mathbb{T}} \quad \text{and}$$

$$c(B(\mathbb{C}^n \otimes \mathrm{Sym}^d V^*)) = \prod_{s=0}^{d} \prod_{k=1}^{n} (1 - (\alpha_k + s\hbar)) \in H^*(B\widetilde{\mathbb{T}}),$$

the $\widetilde{\mathbb{T}}$-equivariant cohomology of $\overline{\mathfrak{X}}_d$ is given by

$$\begin{aligned} H^*_{\widetilde{\mathbb{T}}}(\overline{\mathfrak{X}}_d) \equiv H^*(B\overline{\mathfrak{X}}_d) = H^*(B\widetilde{\mathbb{T}})[\Omega] \Big/ \prod_{s=0}^{d} \prod_{k=1}^{n} (\Omega - (\alpha_k + s\hbar)) \\ \approx \mathbb{Q}[\Omega, \hbar, \alpha_1, \dots, \alpha_n] \Big/ \prod_{s=0}^{d} \prod_{k=1}^{n} (\Omega - \alpha_k - s\hbar) \\ \subset \mathbb{Q}_\alpha[\hbar, \Omega] \Big/ \prod_{s=0}^{d} \prod_{k=1}^{n} (\Omega - \alpha_k - s\hbar). \end{aligned}$$

In particular, every element of $H^*_{\widetilde{\mathbb{T}}}(\overline{\mathfrak{X}}_d)$ is a polynomial in Ω with coefficients in $\mathbb{Q}_\alpha[\hbar]$ of degree at most $(d+1)n - 1$.

For each $d \in \mathbb{Z}^{\geq 0}$, let

(8-3)
$$\mathfrak{X}'_d = \big\{ b \in \overline{Q}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)) : \mathrm{ev}_1(b) \in q_1 \times \mathbb{P}^{n-1},\, \mathrm{ev}_2(b) \in q_2 \times \mathbb{P}^{n-1} \big\}.$$

A general element of b of $\mathfrak{X}'_d$ determines a morphism

$$(f, g) : \mathbb{P}^1 \to (\mathbb{P}V, \mathbb{P}^{n-1}),$$

up to an automorphism of the domain $\mathbb{P}^1$. Thus, the morphism

$$g \circ f^{-1} : \mathbb{P}V \to \mathbb{P}^{n-1}$$

is well defined and determines an element $\theta(b) \in \overline{\mathfrak{X}}_d$. By [Cooper and Zinger 2014, Section 7], this morphism extends to a $\widetilde{\mathbb{T}}$-equivariant morphism

$$\theta = \theta_d : \mathfrak{X}'_d \to \overline{\mathfrak{X}}_d.$$[8]

If $d \in \mathbb{Z}^+$, there is also a natural forgetful morphism

$$F : \mathfrak{X}'_d \to \overline{Q}_{0,m}(\mathbb{P}^{n-1}, d),$$

which drops the first sheaf in the pair and contracts one component of the domain if necessary. If in addition $m \geq m' \geq 2$, $f_{m',m}$ is as in (6-5), and $\mathcal{V}_{n;\mathbf{a}}^{(d)}$ is as in (1-3), let

$$\mathcal{V}_{n;\mathbf{a};m'}^{(d)} = f_{m',m}^* \mathcal{V}_{n;\mathbf{a}}^{(d)} \to \overline{Q}_{0,m}(\mathbb{P}^{n-1}, d).$$

From the usual short exact sequence for the restriction along σ_1, we find that

$$\mathbf{e}(\mathcal{V}_{n;\mathbf{a};m'}^{(d)}) = \langle \mathbf{a} \rangle \mathrm{ev}_1^* \mathbf{x}^{\ell(\mathbf{a})} \mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a};m'}^{(d)}) \in H_{\mathbb{T}}^*(\overline{Q}_{0,m}(\mathbb{P}^{n-1}, d)). \tag{8-4}$$

In the case $d = 0$, we set

$$F^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a};m'}^{(0)}) = \langle \mathbf{a} \rangle \mathrm{ev}_1^* (1 \times \mathbf{x}^{\ell(\mathbf{a})}) \in H^*\big(\overline{Q}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, 0))\big),$$
$$F^* \mathbf{e}(\ddot{\mathcal{V}}_{n;\mathbf{a};m'}^{(0)}) = 1 \in H^*\big(\overline{Q}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, 0))\big).$$

Lemma 8.1. *Let $l \in \mathbb{Z}^{\geq 0}$, $m, m', n \in \mathbb{Z}^+$ with $m \geq m' \geq 2$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$. With $\dot{\mathcal{Z}}_{n;\mathbf{a}}$, $\dot{\mathcal{Z}}_{n;\mathbf{a};m'}^{(\mathbf{b},\varpi)}$, $\ddot{\mathcal{Z}}_{n;\mathbf{a};m'}^{(\mathbf{b},\varpi)}$ as in (3-3) and (6-7),*

$$(-\hbar)^{m-2} \Phi^{\check{\eta}}_{\dot{\mathcal{Z}}_{n;\mathbf{a};m'}, \check{\mathcal{Z}}_{n;\mathbf{a};m'}^{(\mathbf{b},\varpi)}}(\hbar, z, q) = \sum_{d=0}^{\infty} q^d \int_{\mathfrak{X}'_d} \mathrm{e}^{(\theta^*\Omega)z} F^* \mathbf{e}(\check{\mathcal{V}}_{n;\mathbf{a};m'}^{(d)}) \, \psi_2^{b_2} \mathrm{ev}_2^* \varpi_2 \prod_{j=3}^{m} \psi_j^{b_j} \mathrm{ev}_j^* (\mathbf{e}(\gamma_1^*) \varpi_j). \tag{8-5}$$

with $(\check{\mathcal{Z}}, \check{\mathcal{V}}, \check{\eta}) = (\dot{\mathcal{Z}}, \mathcal{V}, \dot{\eta}), (\ddot{\mathcal{Z}}, \ddot{\mathcal{V}}, \ddot{\eta})$.

Since the right-hand sides of the above expressions lie in $H_{\mathbb{T}}^*[\![z, q]\!] \subset \mathbb{Q}_\alpha[\hbar][\![z, q]\!]$, this lemma is a more precise version of Lemma 6.6.

[8] This morphism is the composition of the morphism θ_d defined in [Cooper and Zinger 2014] in the $m = 2$ case with the forgetful morphism

$$\overline{Q}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)) \to \overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)).$$

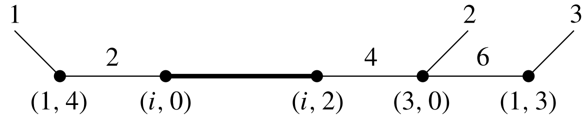

Figure 5. A graph representing a fixed locus in $\mathfrak{X}'_d$; $i \neq 1, 3$.

8B. ***Proof of Lemma 8.1.*** We apply the localization theorem of [Atiyah and Bott 1984] to the $\widetilde{\mathbb{T}}$-action on $\mathfrak{X}'_d$. We show that each fixed locus of the $\widetilde{\mathbb{T}}$-action on $\mathfrak{X}'_d$ contributing to the right-hand sides in (8-5) corresponds to a pair (Γ_1, Γ_2) of decorated graphs as in (7-3), with Γ_1 and Γ_2 contributing to the two generating functions in the subscript of the corresponding correlator Φ evaluated at $x = \alpha_i$ for some $i \in [n]$.

Similarly to Section 7, the fixed loci of the $\widetilde{\mathbb{T}}$-action on $\overline{Q}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (d', d))$ correspond to decorated graphs Γ with m marked points distributed between the ends of Γ. The map $\mathfrak{d}$ should now take values in pairs of nonnegative integers, indicating the degrees of the two subsheaves. The map μ should similarly take values in the pairs (i, j) with $i \in [2]$ and $j \in [n]$, indicating the fixed point (q_i, P_j) to which the vertex is mapped. The μ-values on consecutive vertices must differ by precisely one of the two components.

The situation for the $\widetilde{\mathbb{T}}$-action on

$$\mathfrak{X}'_d \subset \overline{Q}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d))$$

is simpler, however. There is a unique edge of positive $\mathbb{P}V$-degree; we draw it as a thick line in Figure 5. The first component of the value of $\mathfrak{d}$ on all other edges and on all vertices must be 0; so we drop it. The first component of the value of μ on the vertices changes only when the thick edge is crossed. Thus, we drop the first components of the vertex labels as well, with the convention that these components are 1 on the left side of the thick edge and 2 on the right. In particular, the vertices to the left of the thick edge (including the left endpoint) lie in $q_1 \times \mathbb{P}^{n-1}$ and the vertices to its right lie in $q_2 \times \mathbb{P}^{n-1}$. Thus, by (8-3), the marked point 1 is attached to a vertex to the left of the thick edge and the marked point 2 is attached to a vertex to the right. By the localization formula (7-1) and the first equation in (8-1), Γ does not contribute to the right-hand sides in (8-5) unless the marked points indexed by $j \geq 3$ are also attached to vertices to the right of the thick edge. Finally, the remaining, second component of μ takes the same value $i \in [n]$ on the two vertices of the thick edge.

Let $\mathcal{A}_i$ denote the set of graphs as above so that the μ-value on the two endpoints of the thick edge is labeled i; see Figure 5. We break each graph $\Gamma \in \mathcal{A}_i$ into three subgraphs:

(i) Γ_1 consisting of all vertices of Γ to the left of the thick edge, including its left vertex v_1 with its $\mathfrak{d}$-value, and a new marked point attached to v_1;

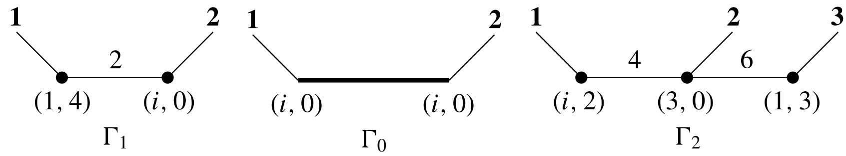

Figure 6. The three subgraphs of the graph in Figure 5.

(ii) Γ_0 consisting of the thick edge e_0, its two vertices v_1 and v_2, with $\mathfrak{d}$-values set to 0, and new marked points 1 and 2 attached to v_1 and v_2, respectively;

(iii) Γ_2 consisting of all vertices to the right of the thick edge, including its right vertex v_2 with its $\mathfrak{d}$-value, and a new marked point attached to v_2;

see Figure 6. From (7-5), we then obtain a splitting of the fixed locus in $\mathfrak{X}'_d$ corresponding to Γ:

(8-6)
$$\mathcal{Q}_\Gamma \approx \mathcal{Q}_{\Gamma_1} \times \mathcal{Q}_{\Gamma_0} \times \mathcal{Q}_{\Gamma_2} \subset \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, |\Gamma_1|) \times \overline{\mathcal{Q}}_{0,2}(\mathbb{P}V, 1) \times \overline{\mathcal{Q}}_{0,m}(\mathbb{P}^{n-1}, |\Gamma_2|).$$

The exceptional cases are $|\Gamma_1| = 0$ and $m = 2$, $|\Gamma_2| = 0$; the above isomorphism then holds with the corresponding component replaced by a point.

Let π_1, π_0, and π_2 denote the three component projection maps in (8-6). By (7-7),

(8-7)
$$\frac{\mathbf{e}(\mathcal{N}\mathcal{Q}_\Gamma)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} = \pi_1^*\left(\frac{\mathbf{e}(\mathcal{N}\mathcal{Q}_{\Gamma_1})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}\right)\cdot\pi_2^*\left(\frac{\mathbf{e}(\mathcal{N}\mathcal{Q}_{\Gamma_2})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}\right)\cdot(\omega_{e_0;v_1} - \pi_1^*\psi_2)(\omega_{e_0;v_2} - \pi_2^*\psi_1).$$

Since for every $j = m'+1, \dots, m$ the closest vertex of $\mathrm{Ver}_{m'}$ lies to the right of the thick edge, by (7-8) and (8-4),

(8-8)
$$\begin{aligned} F^*\mathbf{e}\big(\mathcal{V}^{(|\Gamma|)}_{n;\mathbf{a};m'}\big)\big|_{\mathcal{Q}_\Gamma} &= \dot{\eta}(\alpha_i)\pi_1^*\mathbf{e}\big(\ddot{\mathcal{V}}^{(|\Gamma_1|)}_{n;\mathbf{a}}\big)\pi_2^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'}\big),\\ F^*\mathbf{e}\big(\ddot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a};m'}\big)\big|_{\mathcal{Q}_\Gamma} &= \ddot{\eta}(\alpha_i)\pi_1^*\mathbf{e}\big(\ddot{\mathcal{V}}^{(|\Gamma_1|)}_{n;\mathbf{a}}\big)\pi_2^*\mathbf{e}\big(\ddot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'}\big).\end{aligned}$$

Since $\mathcal{Q}_{\Gamma_0}$ consists of a degree-1 map, by the last two identities in (8-1)

(8-9)
$$\omega_{e_0;v_1} = \hbar, \quad \omega_{e_0;v_2} = -\hbar.$$

The morphism θ takes the locus $\mathcal{Q}_\Gamma$ to a fixed point $P_k(r) \in \overline{\mathfrak{X}}_d$. It is immediate that $k = i$. By continuity considerations, $r = |\Gamma_1|$. Thus, by the first identity in (8-2),

(8-10)
$$\theta^*\Omega\big|_{\mathcal{Q}_\Gamma} = \alpha_i + |\Gamma_1|\hbar.$$

Combining (8-7)–(8-10) and the second equation in (8-1), we obtain

$$(8\text{-}11)\quad q^{|\Gamma|}\int_{Q_\Gamma}\frac{\mathrm{e}^{(\theta^*\Omega)z}F^*\mathbf{e}(\mathcal{V}^{(|\Gamma|)}_{n;\mathbf{a};m'})\,\psi_2^{b_2}\mathrm{ev}_2^*\varpi_2\prod_{j=3}^m\psi_j^{b_j}\mathrm{ev}_j^*(\mathbf{e}(\gamma_1^*)\varpi_j)}{\mathbf{e}(\mathcal{N}Q_\Gamma)}$$
$$=\frac{(-\hbar)^{m-2}\dot{\eta}(\alpha_i)\mathrm{e}^{\alpha_i z}}{\prod_{k\neq i}(\alpha_i-\alpha_k)}\left\{\mathrm{e}^{|\Gamma_1|\hbar z}q^{|\Gamma_1|}\int_{Q_{\Gamma_1}}\frac{\mathbf{e}(\ddot{\mathcal{V}}^{(|\Gamma_1|)}_{n;\mathbf{a}})\mathrm{ev}_2^*\phi_i}{\hbar-\psi_2}\Big|_{Q_{\Gamma_1}}\frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}\right\}$$
$$\times\left\{q^{|\Gamma_2|}\int_{Q_{\Gamma_2}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_i\prod_{j=2}^m(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j)}{(-\hbar)-\psi_1}\Big|_{Q_{\Gamma_2}}\frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}\right\}.$$

This identity remains valid with $|\Gamma_1|=0$ and/or $m=2$, $|\Gamma_2|=0$ if we set the corresponding integral to 1 or to $\hbar^{b_2}\varpi_2|_{P_i}$, respectively.

We now sum up the last identity over all $\Gamma\in\mathcal{A}_i$. This is the same as summing over all pairs (Γ_1,Γ_2) of decorated graphs such that

(1) Γ_1 is a 2-pointed graph of degree $d_1\geq 0$ such that the marked point 2 is attached to a vertex labeled i;

(2) Γ_2 is an m-pointed graph of degree $d_2\geq 0$ such that the marked point 1 is attached to a vertex labeled i.

By the localization formula (7-1) and symmetry,

$$1+\sum_{\Gamma_1}(q\mathrm{e}^{\hbar z})^{|\Gamma_1|}\left\{\int_{Q_{\Gamma_1}}\frac{\mathbf{e}(\ddot{\mathcal{V}}^{(|\Gamma_1|)}_{n;\mathbf{a}})\mathrm{ev}_2^*\phi_i}{(\hbar-\psi_2)\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}\right\}$$
$$=1+\sum_{d=1}^\infty(q\mathrm{e}^{\hbar z})^d\int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a}})\mathrm{ev}_2^*\phi_i}{\hbar-\psi_2}=\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q\mathrm{e}^{\hbar z});$$

$$\delta_{m,2}\hbar^{b_2}\varpi_2|_{P_i}+\sum_{\Gamma_2}q^{|\Gamma_2|}\left\{\int_{Q_{\Gamma_2}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_i\prod_{j=2}^m(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j)}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})(-\hbar-\psi_1)}\right\}$$
$$=\delta_{m,2}\hbar^{b_2}\varpi_2|_{P_i}+\sum_{d=\max(3-m,0)}^\infty q^d\int_{\overline{Q}_{0,m}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'})\mathrm{ev}_1^*\phi_i\prod_{j=2}^m(\psi_j^{b_j}\mathrm{ev}_j^*\varpi_j)}{(-\hbar-\psi_1)}$$
$$=\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\alpha_i,-\hbar,q).$$

Combining with this with (7-1), we obtain

$$\sum_{d=0}^\infty q^d\int_{\mathfrak{X}'_d}\mathrm{e}^{(\theta^*\Omega)z}F^*\mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a};m'})\,\psi_2^{b_2}\mathrm{ev}_2^*\varpi_2\prod_{j=3}^m\psi_j^{b_j}\mathrm{ev}_j^*(\mathbf{e}(\gamma_1^*)\varpi_j)$$
$$=(-\hbar)^{m-2}\sum_{i=1}^n\frac{\dot{\eta}(\alpha_i)\mathrm{e}^{\alpha_i z}}{\prod_{k\neq i}(\alpha_i-\alpha_k)}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q\mathrm{e}^{\hbar z})\,\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\alpha_i,-\hbar,q)$$
$$=(-\hbar)^{m-2}\Phi^{\dot{\eta}}_{\dot{\mathcal{Z}}_{n;\mathbf{a}},\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}}(\hbar,z,q),$$

as claimed in the $\dot{\mathcal{Z}}$ identity in (8-5).

From (8-7)–(8-10), we also find that (8-11) holds with $\mathcal{V}$ and $\dot{\mathcal{V}}$ replaced by $\ddot{\mathcal{V}}$ and $\dot{\eta}$ by $\ddot{\eta}$, with the same conventions in the $|\Gamma_1|=0$ and $m=2$, $|\Gamma_2|=0$ cases. We then sum up the resulting identity over all pairs (Γ_1,Γ_2) of decorated graphs as in the previous paragraph. The sum of the terms in the first curly brackets over all possibilities for Γ_1 is exactly the same as before, while the sum of the terms in the second curly brackets over all possibilities for Γ_2 is described by the same expression as before with $\dot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'}$ and $\dot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$ replaced by $\ddot{\mathcal{V}}^{(|\Gamma_2|)}_{n;\mathbf{a};m'}$ and $\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}$, respectively. Thus,

$$\begin{aligned}\sum_{d=0}^{\infty} q^d \int_{\mathfrak{X}'_d} \mathrm{e}^{(\theta^*\Omega)z} F^*\mathbf{e}(\ddot{\mathcal{V}}^{(d)}_{n;\mathbf{a};m'})\, \psi_2^{b_2}\mathrm{ev}_2^*\varpi_2 \prod_{j=3}^{m} \psi_j^{b_j}\mathrm{ev}_j^*(\mathbf{e}(\gamma_1^*)\varpi_j)\\ = (-\hbar)^{m-2}\sum_{i=1}^{n} \frac{\ddot{\eta}(\alpha_i)\mathrm{e}^{\alpha_i z}}{\prod_{k\neq i}(\alpha_i-\alpha_k)} \dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q\mathrm{e}^{\hbar z})\,\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}(\alpha_i,-\hbar,q)\\ = (-\hbar)^{m-2}\Phi^{\ddot{\eta}}_{\dot{\mathcal{Z}}_{n;\mathbf{a}},\ddot{\mathcal{Z}}^{(\mathbf{b},\varpi)}_{n;\mathbf{a};m'}}(\hbar,z,q),\end{aligned}$$

as claimed in the $\ddot{\mathcal{Z}}$ identity in (8-5).

In the case of products of projective spaces and concavex sheaves (1-13), the spaces

$$\overline{Q}_{0,m}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)) \quad\text{and}\quad \overline{\mathfrak{X}}_d=\mathbb{P}(\mathbb{C}^n\otimes\mathrm{Sym}^d V^*)$$

are replaced by

$$\begin{gathered}\overline{Q}_{0,m}(\mathbb{P}V\times\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(1,d_1,\dots,d_p)) \quad\text{and}\\ \mathbb{P}(\mathbb{C}^{n_1}\otimes\mathrm{Sym}^{d_1}V^*)\times\cdots\times\mathbb{P}(\mathbb{C}^{n_p}\otimes\mathrm{Sym}^{d_p}V^*),\end{gathered}$$

respectively. Lemma 8.1 extends to this situation by replacing z and q in (8-5) with $z_1,\dots,z_p$ and $q_1,\dots,q_p$, q^d with $q_1^{d_1}\cdots q_p^{d_p}$, $\mathfrak{X}'_d$ with $\mathfrak{X}'_{d_1,\dots,d_p}$, $\mathrm{e}^{(\theta^*\Omega z)}$ with $\mathrm{e}^{(\theta^*\Omega_1)z_1+\cdots+(\theta^*\Omega_p)z_p}$, and the indices d and n on the bundles $\mathcal{V}$, $\ddot{\mathcal{V}}$ with $(d_1,\dots,d_p)$ and $(n_1,\dots,n_p)$, and summing over $d_1,\dots,d_p\ge 0$ instead of $d\ge 0$. The vertices of the thick edge in Figure 5 are now labeled by a tuple $(i_1,\dots,i_p)$ with $i_s\in[n_s]$, as needed for the extension of Definition 6.2 described at the end of Section 6. The relation (8-10) becomes

$$\theta^*\Omega_s\big|_{Q_\Gamma}=\alpha_{s;i_s}+|\Gamma_1|_s\hbar,$$

where $|\Gamma_1|_s$ is the sum of the s-th components of the values of $\mathfrak{d}$ on the vertices and edges of Γ_1 (corresponding to the degree of the maps to $\mathbb{P}^{n_s-1}$). Otherwise, the proof is identical.

9. Stable quotients vs. Hurwitz numbers

Our proof of Propositions 4.1 and 4.2 that describe twisted Hurwitz numbers on $\overline{\mathcal{M}}_{0,3|d}$ is analogous to the proof of [Cooper and Zinger 2014, Theorem 4], which describes similar integrals on $\overline{\mathcal{M}}_{0,2|d}$. In particular, we show that it is sufficient to verify the statements of Propositions 4.1 and 4.2 for each fixed $\mathbf{a}$ and for all n sufficiently large (compared to $|\mathbf{a}|$). For $\nu_n(\mathbf{a})>0$, we obtain the statements of Propositions 4.1 and 4.2 by analyzing the secondary (middle) terms in the recursion (6-2) for the three-point generating functions $\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}$ and $\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}$ defined in (3-32) and (3-34), respectively. We also use (3-35) and (3-33). The latter is the string equation for stable quotients invariants; in Proposition 9.3, we show that it is equivalent to Proposition 4.2 whenever $\nu_n(\mathbf{a})\ge 0$. In Proposition 9.2, we show that (3-33) is equivalent to Proposition 4.1 whenever $\nu_n(\mathbf{a})\ge 0$. We confirm Proposition 4.1 whenever $\nu_n(\mathbf{a})>0$ using Proposition 6.3; see Corollary 9.1. Since it is sufficient to verify the statement of Proposition 4.1 with $\nu_n(\mathbf{a})>0$, the $\nu_n(\mathbf{a})=0$ case of Proposition 4.1 then concludes the proof of (3-33).

9A. ***Proof of Propositions 3.1, 4.1, and 4.2.*** With n and $\mathbf{a}$ as in Propositions 4.1 and 4.2 and $b_1,b_2,b_3,r\in\mathbb{Z}^{\ge 0}$, let

$$\mathcal{F}^{(b_1,b_2,b_3)}_{n;\mathbf{a}}(\alpha_i,q)=\sum_{d=0}^{\infty}\frac{q^d}{d!}\int_{\overline{\mathcal{M}}_{0,3|d}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{\mathbf{a}}(\alpha_i))\psi_1^{b_1}\psi_2^{b_2}\psi_3^{b_3}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}^{(d)}_1(\alpha_i-\alpha_k))},$$

$$\mathcal{F}^{(b_1,b_2,b_3)}_{n;\mathbf{a};r}(\alpha_i,q)=\sum_{d=0}^{\infty}\frac{q^d}{d!}\int_{\overline{\mathcal{M}}_{0,3|d}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{\mathbf{a};r}(\alpha_i))\psi_1^{b_1}\psi_2^{b_2}\psi_3^{b_3}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}^{(d)}_1(\alpha_i-\alpha_k))}.$$

By [Cooper and Zinger 2014, Remark 8.5],

$$\mathcal{F}^{(b_1,b_2,b_3)}_{n;\mathbf{a}}(\alpha_i,q)=\frac{\xi_{n;\mathbf{a}}(\alpha_i,q)^{b_1+b_2+b_3}}{b_1!\,b_2!\,b_3!}\mathcal{F}^{(0,0,0)}_{n;\mathbf{a}}(\alpha_i,q); \tag{9-1}$$

thus, it is sufficient to show that

$$\mathcal{F}^{(0,0,0)}_{n;\mathbf{a}}(\alpha_i,q)=\frac{1}{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)}. \tag{9-2}$$

By the same reasoning as in [Cooper and Zinger 2014, Remarks 8.4, 8.5],

$$\mathcal{F}^{(b_1,b_2,b_3)}_{n;\mathbf{a};r}(\alpha_i,q)=\frac{\xi_{n;\mathbf{a}}(\alpha_i,q)^{b_1+b_2}}{b_1!\,b_2!}\mathcal{F}^{(0,0,b_3)}_{n;\mathbf{a};r}(\alpha_i,q);$$

thus, it is sufficient to show that

$$\sum_{b=0}^{\infty}\sum_{r=0}^{\infty}\mathcal{F}^{(0,0,b)}_{n;\mathbf{a};r}(\alpha_i,q)\,\mathfrak{R}_{\hbar=0}\left\{\frac{(-1)^b}{\hbar^{b+1}}[\![\dot{\mathcal{Y}}_{n;\varnothing}(\alpha_i,\hbar,q)]\!]_{q;r}q^r\right\}=1. \tag{9-3}$$

Corollary 9.1. *Let $l \in \mathbb{Z}^{\ge 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$. If $\nu_n(\mathbf{a}) > 0$,*

$$\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\mathbf{x}, \hbar, q) = \hbar^{-1}\, \dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[\![\hbar^{-1}, q]\!].$$

Proof. By Lemma 6.4(ii) and Lemmas 6.5 and 6.6, the series $\hbar\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\mathbf{x}, \hbar, q)$ and $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$ are $\mathfrak{C}$-recursive and satisfy the $\dot{\eta}$-MPC with respect to $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$, no matter what n and $\mathbf{a}$ are. It is immediate that

$$\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \cong 1 \pmod{\hbar^{-1}}.$$

If $\nu_n(\mathbf{a}) > 0$ and $d \in \mathbb{Z}^+$,

$$\dim \overline{Q}_{0,3}(\mathbb{P}^{n-1}, d) - \mathrm{rk}\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a};3} = \nu_n(\mathbf{a})d + (n-1) > n-1 = \dim \mathbb{P}^{n-1}.$$

Thus,

$$\hbar\, \dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\mathbf{x}, \hbar, q) \cong 1 \pmod{\hbar^{-1}},$$

whenever $\nu_n(\mathbf{a}) > 0$. The claim now follows from Proposition 6.3. □

Proposition 9.2. *If $l \in \mathbb{Z}^{\ge 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $\nu_n(\mathbf{a}) \ge 0$, then*

$$\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\mathbf{x}, \hbar, q) = \hbar^{-1} \frac{\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)}{\dot{I}_0(q)} \in (H^*_{\mathbb{T}}(\mathbb{P}^{n-1}))[\![\hbar^{-1}, q]\!] \tag{9-4}$$

if and only if (9-2) *holds for all $i \in [n]$.*

Proposition 9.3. *If $l \in \mathbb{Z}^{\ge 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $\nu_n(\mathbf{a}) \ge n$, then*

$$\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}(\mathbf{x}, \hbar, q) = \hbar^{-1}\, \dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \in (H^*_{\mathbb{T}}(\mathbb{P}^{n-1}))[\![\hbar^{-1}, q]\!] \tag{9-5}$$

if and only if (9-3) *holds for all $i \in [n]$.*

For any $t, t' \in [d]$ with $t \neq t'$, let $\Delta_{tt'} \in H^2(\overline{\mathcal{M}}_{0,m|d})$ denote the class of the diagonal divisor

$$\big\{[\mathcal{C}, y_1, \dots, y_m; \hat{y}_1, \dots, \hat{y}_d] \in \overline{\mathcal{M}}_{g,m|d} : \hat{y}_t = \hat{y}_{t'}\big\}.$$

For any $t \in [d]$, let

$$\Delta_t = \sum_{t'>t} \Delta_{tt'}.$$

We denote by $\mathfrak{s}_1, \mathfrak{s}_2, \dots$ the elementary symmetric polynomials in

$$\{\beta_k\} = \big\{(\alpha_i - \alpha_k)^{-1} : k \neq i\big\}$$

for any given number of formal variables β_k. Let

$$A_{\mathbf{a}}(\alpha_i) = \prod_{a_k>0} (a_k^{a_k} \alpha_i^{a_k}) \prod_{a_k<0} (a_k^{-a_k} \alpha_i^{-a_k}), \quad A_{n;\mathbf{a}}(\alpha_i) = \frac{A_{\mathbf{a}}(\alpha_i)}{\prod_{k\neq i}(\alpha_i - \alpha_k)}.$$

Proof of (9-2). By (1) in the proof of [Cooper and Zinger 2014, Proposition 8.3],

$$\begin{aligned}
\text{(9-6)}\quad \frac{[\![\mathcal{F}^{(0,0,0)}_{n;\mathbf{a}}(\alpha_i,q)]\!]_{q;d}}{A^d_{n;\mathbf{a}}(\alpha_i)}
&= \int_{\overline{\mathcal{M}}_{0,3|d}} \frac{\prod\limits_{a_k>0}\prod\limits_{t=1}^{d}\prod\limits_{\lambda=1}^{a_k}\Big(1-\frac{\lambda\hat{\psi}_t}{a_k\alpha_i}+\frac{\Delta_t}{\alpha_i}\Big)\prod\limits_{a_k<0}\prod\limits_{t=1}^{d}\prod\limits_{\lambda=0}^{-a_k-1}\Big(1+\frac{\lambda\hat{\psi}_t}{a_k\alpha_i}+\frac{\Delta_t}{\alpha_i}\Big)}{\prod\limits_{k\neq i}\prod\limits_{t=1}^{d}\Big(1-\frac{\hat{\psi}_t}{\alpha_i-\alpha_k}+\frac{\Delta_t}{\alpha_i-\alpha_k}\Big)}\\
&= \mathcal{H}_{\mathbf{a};d}(\alpha_i^{-1},\mathfrak{s}_1,\dots,\mathfrak{s}_d)
\end{aligned}$$

for some $\mathcal{H}_{\mathbf{a};d}\in\mathbb{Q}[y,\mathfrak{s}_1,\dots,\mathfrak{s}_d]$ dependent only on $\mathbf{a}$ and d, but not on n.[9] Similarly, for any $d,d'\in\mathbb{Z}^{\geq 0}$ there exists $\dot{\mathcal{Y}}_{\mathbf{a};d,d'}\in\mathbb{Q}[y,\mathfrak{s}_1,\dots,\mathfrak{s}_{d'}]$, independent of n, such that

$$\text{(9-7)}\qquad [\![\hbar^d[\![\dot{\mathcal{Y}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)]\!]_{q;d}]\!]_{\hbar;d'} = A^d_{n;\mathbf{a}}(\alpha_i)\dot{\mathcal{Y}}_{\mathbf{a};d,d'}(y,\mathfrak{s}_1,\dots,\mathfrak{s}_{d'}).$$

Thus, by (4-9), there exist $\xi_{\mathbf{a};d},\dot{\Phi}^{(0)}_{\mathbf{a};d}\in\mathbb{Q}[y,\mathfrak{s}_1,\dots,\mathfrak{s}_d]$, independent of n, such that

$$\begin{aligned}
[\![\xi_{n;\mathbf{a}}(\alpha_i,q)]\!]_{q;d} &\equiv \mathop{\mathfrak{R}}_{\hbar=0}[\![\log\dot{\mathcal{Y}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)]\!]_{q;d} = A^d_{n;\mathbf{a}}(\alpha_i)\xi_{\mathbf{a};d}(\alpha_i^{-1},\mathfrak{s}_1,\dots,\mathfrak{s}_{d-1}),\\
[\![\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)]\!]_{q;d} &\equiv \mathop{\mathfrak{R}}_{\hbar=0}\frac{1}{\hbar}[\![\mathrm{e}^{-\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar}\dot{\mathcal{Y}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)]\!]_{q;d}\\
&= A^d_{n;\mathbf{a}}(\alpha_i)\dot{\Phi}^{(0)}_{\mathbf{a};d}(\alpha_i^{-1},\mathfrak{s}_1,\dots,\mathfrak{s}_d).
\end{aligned}$$

We conclude that (9-2) is equivalent to

$$\sum_{\substack{d_1,d_2\geq 0\\ d_1+d_2=d}}\mathcal{H}_{\mathbf{a};d_1}\dot{\Phi}^{(0)}_{\mathbf{a};d_2}=\delta_{d,0}\quad\text{for all } d\in\mathbb{Z}^{\geq 0}.$$

By Corollary 9.1 and Proposition 9.2, these relations hold whenever $\nu_n(\mathbf{a})>0$; since they do not involve n, they thus hold for all pairs $(n,\mathbf{a})$. □

Proof of (9-3). For $t\in[d+1]$ and $r\in\mathbb{Z}^{\geq 0}$, we define $\hat{\psi}'_t,\Delta'_{t;r}\in H^2(\overline{\mathfrak{M}}_{0,3|d})$ by

$$\hat{\psi}'_t=f^*_{2;3}\hat{\psi}_t,\qquad \Delta_{t;r}=f^*_{2;3}\Delta_t+\begin{cases}(r-1)f^*_{2;3}\Delta_{t,d+1} & \text{if } t\leq d;\\ 0 & \text{if } t=d+1.\end{cases}$$

[9] Whatever polynomial works for $n>d$ works for all n; this can be seen by setting the extra β_k's to 0.

Similarly to (1) in the proof of [Cooper and Zinger 2014, Proposition 8.3],

$$a_k > 0 \;\Rightarrow\; \mathbf{e}(\dot{\mathcal{V}}_{a_k;r}^{(d)}(\alpha_i)) = \prod_{t=1}^{d}\prod_{\lambda=1}^{a_k}(a_k\alpha_i - \lambda\hat{\psi}_t' + a_k\Delta_{t;r}')\cdot\prod_{\lambda=1}^{ra_k}(a_k\alpha_i - \lambda\hat{\psi}_{d+1}');$$

$$a_k < 0 \;\Rightarrow\; \mathbf{e}(\dot{\mathcal{V}}_{a_k;r}^{(d)}(\alpha_i)) = \prod_{t=1}^{d}\prod_{\lambda=0}^{-a_k-1}(a_k\alpha_i + \lambda\hat{\psi}_t' + a_k\Delta_{t;r}')\cdot\prod_{\lambda=0}^{-ra_k-1}(a_k\alpha_i + \lambda\hat{\psi}_{d+1}').$$

Thus, similarly to (9-6),

$$\frac{[\![\mathcal{F}_{n;\mathbf{a};r}^{(0,0,b)}(\alpha_i,q)]\!]_{q;d}}{A_{n;\mathbf{a}}(\alpha_i)^d A_{\mathbf{a}}(\alpha_i)^r} = \mathcal{H}_{\mathbf{a};r;d}^{(b)}(\alpha_i^{-1},\mathfrak{s}_1,\dots,\mathfrak{s}_d)$$

for some $\mathcal{H}_{\mathbf{a};r;d}^{(b)} \in \mathbb{Q}[y,\mathfrak{s}_1,\dots,\mathfrak{s}_d]$ dependent only on $\mathbf{a}$, r, b, and d, but not on n. Thus, by (9-7) with $\mathbf{a}=\varnothing$, (9-3) is equivalent to

$$\sum_{\substack{d_1,d_2\ge 0\\ d_1+d_2=d}}\sum_{b=0}^{\infty}(-1)^b\mathcal{H}_{\mathbf{a};d_2;d_1}^{(b)}\dot{\mathcal{Y}}_{\varnothing;d_2,d_2+b} = \delta_{d,0} \quad \text{for all } d\in\mathbb{Z}^{\ge 0}.$$

By (3-35) and Proposition 9.3, these relations hold whenever $\nu_n(\mathbf{a})\ge 0$; since they do not involve n, they thus hold for all pairs $(n,\mathbf{a})$. □

9B. ***Proof of Proposition 9.2.*** We study the secondary (middle) terms in the recursions (6-2) for

$$\widetilde{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q) \equiv \hbar^{-1}\frac{\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)}{\dot{I}_0(q)} \quad\text{and}\quad \dot{\mathcal{Z}}_{n;\mathbf{a};3}^{(\mathbf{0},\mathbf{1})}(\mathbf{x},\hbar,q).$$

We show that (9-4) implies (9-2) by considering the $r=-1$ coefficients in these recursions. Conversely, if (9-2) holds, we show that the $r=-1$ coefficients in these recursions are described in the same degree-recursive way in terms of the corresponding power series; Proposition 6.3 and Lemma 6.5 then imply that $\dot{\mathcal{Z}}_{n;\mathbf{a};3}^{(\mathbf{0},\mathbf{1})} = \widetilde{\mathcal{Z}}_{n;\mathbf{a}}$.[10]

By Lemmas 6.4 and 6.5,

(9-8)

$$\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q) = \sum_{d=0}^{\infty}\sum_{r=0}^{N_d-1}\{\dot{\mathcal{Z}}_{n;\mathbf{a}}\}_i^r(d)\hbar^{-r}q^d + \sum_{d=1}^{\infty}\sum_{j\ne i}\frac{\dot{\mathfrak{C}}_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i)/d}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_j,(\alpha_j-\alpha_i)/d,q),$$

[10] The same argument, with slightly more notation, can be used to show that all secondary coefficients are described in the same degree-recursive way, thus bypassing Proposition 6.3 and Lemma 6.5.

$$\widetilde{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)$$
$$=\sum_{d=0}^{\infty}\sum_{r=1}^{N_d}\{\widetilde{\mathcal{Z}}_{n;\mathbf{a}}\}_i^r(d)\hbar^{-r}q^d+\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{\dot{\mathfrak{C}}_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i)/d}\widetilde{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_j,(\alpha_j-\alpha_i)/d,q),$$

for some $N_d\in\mathbb{Z}^+$ and $\{\dot{\mathcal{Z}}_{n;\mathbf{a}}\}_i^r(d)$, $\{\widetilde{\mathcal{Z}}_{n;\mathbf{a}}\}_i^r(d)\in\mathbb{Q}_\alpha$. It is immediate that

$$\begin{aligned}\dot{I}_0(q)\sum_{d=0}^{\infty}\{\widetilde{\mathcal{Z}}_{n;\mathbf{a}}\}_i^1(d)q^d-\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}_{n;\mathbf{a}}\}_i^0(d)q^d\\
&=-\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{\dot{\mathfrak{C}}_i^j(d)q^d}{(\alpha_j-\alpha_i)/d}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_j,(\alpha_j-\alpha_i)/d,q)\\
&=-\sum_{d=1}^{\infty}\sum_{j\neq i}\underset{\hbar=(\alpha_j-\alpha_i)/d}{\mathfrak{R}}\{\hbar^{-1}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\}\\
&=\underset{\hbar=0,\infty}{\mathfrak{R}}\{\hbar^{-1}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\}=\underset{\hbar=0}{\mathfrak{R}}\{\hbar^{-1}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\}-1;\end{aligned}$$

the first and second equalities above follow from the first equation in (9-8), while the third from the residue theorem on $\mathbb{P}^1$ and (9-8) again, which implies that the coefficients of q^d in $\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)$ are regular in $\hbar$ away from $\hbar=(\alpha_j-\alpha_i)/d$ with $d\in\mathbb{Z}^+$ and $j\neq i$ and $\hbar=0,\infty$. Combining the last identity with the first statement in (3-12), and (4-9), we obtain

$$\sum_{d=0}^{\infty}\{\widetilde{\mathcal{Z}}_{n;\mathbf{a}}\}_i^1(d)q^d=\frac{\dot{\Phi}_{n;\mathbf{a}}^{(0)}(q)}{\dot{I}_0(q)^2}-\sum_{b=1}^{\infty}\frac{\xi_{n;\mathbf{a}}(q)^b}{b!}\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{(-1)^b}{\hbar^b}\widetilde{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\right\}.\tag{9-9}$$

By Lemma 6.5,

$$\dot{\mathcal{Z}}_{n;\mathbf{a};3}^{(\mathbf{0},\mathbf{1})}(\alpha_i,\hbar,q)$$
$$=\sum_{d=0}^{\infty}\sum_{r=1}^{N_d}\{\dot{\mathcal{Z}}_{n;\mathbf{a};3}^{(\mathbf{0},\mathbf{1})}\}_i^r(d)\hbar^{-r}q^d+\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{\dot{\mathfrak{C}}_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i)/d}\dot{\mathcal{Z}}_{n;\mathbf{a};3}^{(\mathbf{0},\mathbf{1})}(\alpha_j,(\alpha_j-\alpha_i)/d,q),$$

for some $N_d\in\mathbb{Z}^+$ and $\{\dot{\mathcal{Z}}_{n;\mathbf{a};3}^{(\mathbf{0},\mathbf{1})}\}_i^r(d)\in\mathbb{Q}_\alpha$. By Section 7B, the secondary coefficients $\{\dot{\mathcal{Z}}_{n;\mathbf{a};3}^{(\mathbf{0},\mathbf{1})}\}_i^r(d)$ arise from the contributions of decorated graphs Γ as in (7-3) such that the vertex $v_{\min}$ to which the first marked point is attached is of valence 3 or higher. In this case, there are four types of such graphs, as shown in Figure 7:

(i) single-vertex graphs;

(ii) graphs with either marked point 2 or 3, but not both, attached to $v_{\min}$, that is, $|\vartheta^{-1}(v_{\min})|=2$;

(iii) graphs with two edges leaving $v_{\min}$, that is, $|\mathrm{E}_{v_{\min}}|=2$;

(iv) graphs with $|\vartheta^{-1}(v_{\min})|$, $|\mathrm{E}_{v_{\min}}|=1$, but $\mathfrak{d}(v_{\min})>0$.

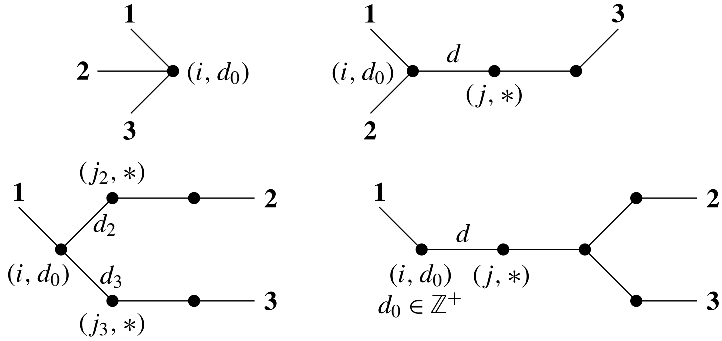

Figure 7. The four types of graphs determining the secondary coefficients $\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}\}^r_i(d)$.

By (7-7), (7-8), and (7-12), the contribution of the graphs of type (i) to the sum $\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}\}^1_i(d)q^d$ is

$$\sum_{d=0}^{\infty}\frac{q^d}{d!}\int_{\overline{\mathcal{M}}_{0,3|d}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{\mathbf{a}}(\alpha_i))}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}^{(d)}_1(\alpha_i-\alpha_k))}=\mathcal{F}^{(0,0,0)}_{n;\mathbf{a}}(\alpha_i,q). \tag{9-10}$$

In the three remaining cases, we split each decorated graph Γ into subgraphs as on page 485; see Figure 8. Let π_0, $\pi_{c;e}$ denote the projection maps in the decomposition (7-14). By (7-7) and (7-8),

(9-11)
$$\frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}=\prod_{k\neq i}\pi_0^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_0|)}_1(\alpha_i-\alpha_k)\big)\cdot\prod_{e\in\mathrm{E}_{v_{\min}}}\left(\pi^*_{c;e}\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_{c;e}})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}(\omega_{e;v_{\min}}-\pi_0^*\psi_e)\right),$$
$$\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a}}\big)\big|_{Q_\Gamma}=\pi_0^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_0|)}_{\mathbf{a}}(\alpha_i)\big)\cdot\prod_{e\in\mathrm{E}_{v_{\min}}}\pi^*_{c;e}\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_{c;e}|)}_{n;\mathbf{a}}\big).$$

Thus, the contribution of Γ to $\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}\}^1_i(d)q^d$ is

$$\begin{aligned}(9\text{-}12)\quad q^{|\Gamma|}\int_{Q_\Gamma}&\frac{\mathbf{e}(\dot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a}})\mathrm{ev}_1^*\phi_i|_{Q_\Gamma}}{\mathbf{e}(\mathcal{N}Q_\Gamma)}\\ &=\sum_{\mathbf{b}\in(\mathbb{Z}^{\geq 0})^{\mathrm{E}_{v_{\min}}}}\left(\frac{q^{d_0}}{d_0!}\int_{\overline{\mathcal{M}}_{0,m_0|d_0}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d_0)}_{\mathbf{a}}(\alpha_i))\prod_{e\in\mathrm{E}_{v_{\min}}}\psi_e^{b_e}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}^{(d_0)}_1(\alpha_i-\alpha_k))}\right.\\ &\qquad\left.\times\prod_{e\in\mathrm{E}_{v_{\min}}}q^{|\Gamma_{c;e}|}\omega_{e;v_{\min}}^{-(b_e+1)}\int_{Q_{\Gamma_{c;e}}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(|\Gamma_{c;e}|)}_{n;\mathbf{a}})\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}Q_{\Gamma_{c;e}})}\right),\end{aligned}$$

where $m_0=|\vartheta^{-1}(v_{\min})|+|\mathrm{E}_{v_{\min}}|$ (which equals 3 if Γ is of type (ii) or (iii), or 2 if Γ is of type (iv) above) and $d_0=\mathfrak{d}(v_{\min})$.

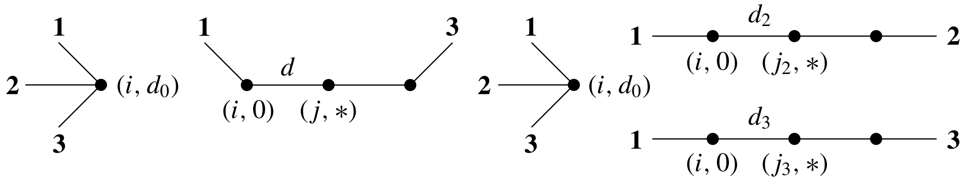

Figure 8. The subgraphs of the second and third graphs in Figure 7.

We now sum up (9-12) over all possibilities for Γ of each of the three types. For each $e\in \mathrm{E}_{v_{\min}}$, let $v_e\in \mathrm{Ver}$ denote the vertex of e other than $v_{\min}$. By (7-6) and Section 7B, the sum of the factor corresponding to $e\in \mathrm{E}_{v_{\min}}$ over all possibilities for Γ_e with $\mathfrak{d}(e)=d_e$ and $\mu(v_e)=j_e$ fixed is

$$(-1)^{b_e+1}\mathop{\mathfrak{R}}_{\hbar=(\alpha_{j_e}-\alpha_i/d_e)}\big\{\hbar^{-(b_e+1)}\dot{\mathcal{Z}}(\alpha_i,\hbar,q)\big\},$$

where $\dot{\mathcal{Z}}=\dot{\mathcal{Z}}_{n;\mathbf{a}}$ in cases (ii) and (iii) and $\dot{\mathcal{Z}}=\dot{\mathcal{Z}}^{(\mathbf{1},\mathbf{0})}_{n;\mathbf{a};3}$ in case (iv). Thus, by the residue theorem on $\mathbb{P}^1$ and Lemma 6.5, the sum of the factors corresponding to $e\in \mathrm{E}_{v_{\min}}$ over all possibilities for Γ_e is

(9-13)
$$(-1)^{b_e}\mathop{\mathfrak{R}}_{\hbar=0,\infty}\left\{\frac{\dot{\mathcal{Z}}(\alpha_i,\hbar,q)}{\hbar^{b_e+1}}\right\}=(-1)^{b_e}\mathop{\mathfrak{R}}_{\hbar=0}\left\{\frac{\dot{\mathcal{Z}}(\alpha_i,\hbar,q)}{\hbar^{b_e+1}}\right\}-\begin{cases}\delta_{b_e,0} & \text{in cases (ii), (iii);}\\ 0 & \text{in case (iv).}\end{cases}$$

Combining (9-12) and (9-13) with (9-1), the first equation in (3-12), and (4-9), we find that the contribution to $\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}\}^1_i(d)q^d$ from all graphs Γ of types (ii) and (iii) above is given by

$$\begin{aligned}\mathcal{F}^{(0,0,0)}_{n;\mathbf{a}}(\alpha_i,q)\sum_{\mathbf{b}\in(\mathbb{Z}^{\ge 0})^{\mathrm{E}_{v_{\min}}}}\frac{(-\xi_{n;\mathbf{a}}(\alpha_i,q))^{|\mathbf{b}|}}{\mathbf{b}!}\prod_{e\in\mathrm{E}_{v_{\min}}}\left(\mathop{\mathfrak{R}}_{\hbar=0}\left\{\frac{1}{\hbar^{b_e+1}}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\right\}-\delta_{b_e,0}\right)\\
=\mathcal{F}^{(0,0,0)}_{n;\mathbf{a}}(\alpha_i,q)\left(\mathop{\mathfrak{R}}_{\hbar=0}\left\{\frac{1}{\hbar}\mathrm{e}^{-\frac{\xi_{n;\mathbf{a}}(\alpha_i,q)}{\hbar}}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\right\}-1\right)^{|\mathrm{E}_{v_{\min}}|}\\
=\mathcal{F}^{(0,0,0)}_{n;\mathbf{a}}(\alpha_i,q)\left(\frac{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)}{\dot{I}_0(q)}-1\right)^{|\mathrm{E}_{v_{\min}}|},\end{aligned}$$

with $|\mathrm{E}_{v_{\min}}|=1$ in (ii) and $|\mathrm{E}_{v_{\min}}|=2$ in (iii). Using [Cooper and Zinger 2014, Theorem 4] instead of (9-1), we find that the contribution to $\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}\}^1_i(d)q^d$ from all graphs Γ of type (iv) above is given by

$$\begin{aligned}-\sum_{b=0}^{\infty}\frac{(-\xi_{n;\mathbf{a}}(\alpha_i,q))^{b+1}}{(b+1)!}\mathop{\mathfrak{R}}_{\hbar=0}\left\{\frac{1}{\hbar^{b+1}}\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\alpha_i,\hbar,q)\right\}\\
=-\sum_{b=1}^{\infty}\frac{\xi_{n;\mathbf{a}}(\alpha_i,q)^b}{b!}\mathop{\mathfrak{R}}_{\hbar=1}\left\{\frac{(-1)^b}{\hbar^b}\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\alpha_i,\hbar,q)\right\}.\end{aligned}$$

Putting this all together and taking into account that there are two flavors of type (ii) graphs, we conclude that

$$\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}\}^1_i(d)q^d = \mathcal{F}^{(0,0,0)}_{n;\mathbf{a}}(\alpha_i,q)\frac{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)^2}{\dot{I}_0(q)^2} - \sum_{b=1}^{\infty}\frac{\xi_{n;\mathbf{a}}(\alpha_i,q)^b}{b!}\mathfrak{R}_{\hbar=0}\left\{\frac{(-1)^b}{\hbar^b}\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}(\alpha_i,\hbar,q)\right\}. \tag{9-14}$$

This is the same degree-recursive relation as (9-9) if and only if (9-2) holds.

9C. ***Proof of Proposition 9.3.*** We next apply the same argument to the power series

$$\widetilde{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q) \equiv \hbar^{-1}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\mathbf{x},\hbar,q) \quad\text{and}\quad \dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}(\mathbf{x},\hbar,q).$$

In this case, (9-9) becomes

$$\sum_{d=0}^{\infty}\{\widetilde{\mathcal{Z}}_{n;\mathbf{a}}\}^1_i(d)q^d = \frac{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(q)}{\dot{I}_0(q)} - \sum_{b=1}^{\infty}\frac{\xi_{n;\mathbf{a}}(q)^b}{b!}\mathfrak{R}_{\hbar=0}\left\{\frac{(-1)^b}{\hbar^b}\widetilde{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\right\}. \tag{9-15}$$

The graphs contributing to $\{\dot{\mathcal{Z}}_{n;\mathbf{a}}\}^r_i(d)$ are the same as before, as are the decomposition (7-14) and the first splitting in (9-11). However, the second splitting in (9-11) changes. For graphs Γ of type (i) and (ii) with $\vartheta(3)=v_{\min}$, it becomes

$$\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a};2}\big)\big|_{\mathcal{Q}_\Gamma} = \pi_0^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_0|)}_{\mathbf{a};0}(\alpha_i)\big)\cdot\pi_{c;e}^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_{c;e}|)}_{n;\mathbf{a}}\big)$$

with the second factor being 1 for the graphs of type (i) and $e\in\mathrm{E}_{v_{\min}}$ denoting the unique element for the graphs of type (ii). For graphs of type (ii) with $\vartheta(2)=v_{\min}$, graphs of type (iii), and graphs of type (iv), it becomes

$$\begin{aligned}
\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a};2}\big)\big|_{\mathcal{Q}_\Gamma} &= \pi_0^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_0|)}_{\mathbf{a};|\Gamma_{c;e}|}(\alpha_i)\big),\\
\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a};2}\big)\big|_{\mathcal{Q}_\Gamma} &= \pi_0^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_0|)}_{\mathbf{a};|\Gamma_{c;e_3}|}(\alpha_i)\big)\cdot\pi_{c;e_2}^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_{c;e_2}|)}_{n;\mathbf{a}}\big),\\
\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma|)}_{n;\mathbf{a};2}\big)\big|_{\mathcal{Q}_\Gamma} &= \pi_0^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_0|)}_{\mathbf{a}}(\alpha_i)\big)\cdot\pi_{c;e}^*\mathbf{e}\big(\dot{\mathcal{V}}^{(|\Gamma_{c;e}|)}_{n;\mathbf{a};2}\big),
\end{aligned}$$

respectively.

Thus, like (9-10), the contribution of the graphs of type (i) to $\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}\}^1_i(d)q^d$ is

$$\sum_{d=0}^{\infty}\frac{q^d}{d!}\int_{\overline{\mathcal{M}}_{0,3|d}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{\mathbf{a};0}(\alpha_i))}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}^{(d)}_1(\alpha_i-\alpha_k))} = \sum_{b=0}^{\infty}\mathcal{F}^{(0,0,b)}_{n;\mathbf{a};0}(\alpha_i,q)\,\mathfrak{R}_{\hbar=0}\left\{\frac{(-1)^b}{\hbar^{b+1}}[\![\dot{\mathcal{Z}}_{n;\varnothing}(\alpha_i,\hbar,q)]\!]_{q;0}q^0\right\}.$$

Similarly to (9-13), the sum of the factor corresponding to an edge $e \in \mathrm{E}_{v_{\min}}$ in the analogue of (9-12) over all possibilities for Γ_e is

$$(-1)^{b_e}\begin{cases} \underset{\hbar=0}{\mathfrak{R}}\left\{\frac{\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)}{\hbar^{b_e+1}}\right\}-\delta_{b_e,0} & \text{in case (ii) with } \vartheta(3)=v_{\min}\text{, (iii) with } e=e_2;\\ \underset{\hbar=0}{\mathfrak{R}}\left\{\frac{\dot{\mathcal{Z}}_{n;\varnothing}(\alpha_i,\hbar,q)}{\hbar^{b_e+1}}\right\}-\delta_{b_e,0} & \text{in case (ii) with } \vartheta(2)=v_{\min}\text{, (iii) with } e=e_3;\\ \underset{\hbar=0}{\mathfrak{R}}\left\{\frac{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}(\alpha_i,\hbar,q)}{\hbar^{b_e+1}}\right\} & \text{in case (iv).}\end{cases}$$

Thus, the contribution to $\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};3}\}_i^1(d)q^d$ from all graphs Γ of types (ii) with $\vartheta(3)=v_{\min}$ and $\vartheta(2)=v_{\min}$ is

$$\mathcal{F}^{(0,0,0)}_{n;\mathbf{a};0}(\alpha_i,q)\sum_{b=0}^{\infty}\frac{(-\xi_{n;\mathbf{a}}(\alpha_i,q))^b}{b!}\left(\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)}{\hbar^{b+1}}\right\}-\delta_{b,0}\right)$$
$$=\left(\frac{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)}{\dot{I}_0(q)}-1\right)\mathcal{F}^{(0,0,0)}_{n;\mathbf{a};0}(\alpha_i,q)$$

and

$$\sum_{b=0}^{\infty}\sum_{r=1}^{\infty}\mathcal{F}^{(0,0,b)}_{n;\mathbf{a};r}(\alpha_i,q)\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{(-1)^b}{\hbar^{b+1}}[\![\dot{\mathcal{Z}}_{n;\varnothing}(\alpha_i,\hbar,q)]\!]_{q;r}q^r\right\},$$

respectively. Similarly, the contribution from all graphs Γ of type (iii) is

$$\sum_{b_2,b_3\ge 0}^{\infty}\sum_{r=1}^{\infty}\mathcal{F}^{(0,0,b_3)}_{n;\mathbf{a};r}(\alpha_i,q)\left(\frac{(-\xi_{n;\mathbf{a}}(\alpha_i,q))^{b_2}}{b_2!}\left(\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\hbar,q)}{\hbar^{b_2+1}}\right\}-\delta_{b_2,0}\right)\right.$$
$$\left.\times\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{(-1)^{b_3}}{\hbar^{b_3+1}}[\![\dot{\mathcal{Z}}_{n;\varnothing}(\alpha_i,\hbar,q)]\!]_{q;r}q^r\right\}\right)$$
$$=\left(\frac{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)}{\dot{I}_0(q)}-1\right)\sum_{b=0}^{\infty}\sum_{r=1}^{\infty}\mathcal{F}^{(0,0,b)}_{n;\mathbf{a};r}(\alpha_i,q)\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{(-1)^b}{\hbar^{b+1}}[\![\dot{\mathcal{Z}}_{n;\varnothing}(\alpha_i,\hbar,q)]\!]_{q;r}q^r\right\}.$$

Finally, the contribution from all graphs Γ of type (iv) is given by

$$-\sum_{b=1}^{\infty}\frac{\xi_{n;\mathbf{a}}(\alpha_i,q)^b}{b!}\underset{\hbar=1}{\mathfrak{R}}\left\{\frac{(-1)^b}{\hbar^b}\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}(\alpha_i,\hbar,q)\right\}.$$

Putting this all together and using the first equation in (3-12), but now with $\mathbf{a}=\varnothing$ and thus $\dot{I}_0=1$, we conclude that

$$\sum_{d=0}^{\infty}\{\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}\}^1_i(d)q^d$$
$$= \frac{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)}{\dot{I}_0(q)}\sum_{b=0}^{\infty}\sum_{r=0}^{\infty}\mathcal{F}^{(0,0,b)}_{n;\mathbf{a};r}(\alpha_i,q)\,\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{(-1)^b}{\hbar^{b+1}}[\![\dot{\mathcal{Y}}_{n;\varnothing}(\alpha_i,\hbar,q)]\!]_{q;r}q^r\right\}$$
$$-\sum_{b=1}^{\infty}\frac{\xi_{n;\mathbf{a}}(\alpha_i,q)^b}{b!}\,\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{(-1)^b}{\hbar^b}\dot{\mathcal{Z}}^{(\mathbf{0},\mathbf{1})}_{n;\mathbf{a};2}(\alpha_i,\hbar,q)\right\}.$$

This is the same degree-recursive relation as (9-15) if and only if (9-3) holds.

10. Proof of (3-14)

The equivariant cohomology of $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ is given by

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})$$
$$=\mathbb{Q}[\alpha_1,\dots,\alpha_n,\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]\Big/\left\{\prod_{k=1}^{n}(\mathbf{x}_1-\alpha_k),\prod_{k=1}^{n}(\mathbf{x}_2-\alpha_k),\prod_{k=1}^{n}(\mathbf{x}_3-\alpha_k)\right\}.$$

Thus, by the defining property of the cohomology pushforward [Zinger 2009, Equation (3.11)], the three-point power series $\dot{\mathcal{Z}}_{n;\mathbf{a}}$ in (3-3) is completely determined by the n^3 power series

(10-1) $$\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\hbar_1,\hbar_2,\hbar_3,q)$$
$$=\sum_{d=0}^{\infty}q^d\int_{\overline{Q}_{0,3}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\dot{\mathcal{V}}^d_{n;\mathbf{a}})\,\mathrm{ev}_1^*\phi_{i_1}\,\mathrm{ev}_2^*\phi_{i_2}\,\mathrm{ev}_3^*\phi_{i_3}}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)(\hbar_3-\psi_3)}.$$

The localization formula (7-1) reduces this expression to a sum over decorated trees as in Section 7. Each of these trees has a unique special vertex v_0: the vertex where the branches from the three marked points come together (one or more of the marked points may be attached to this vertex). We compute this sum by breaking each such tree Γ at v_0 into up to 4 "subgraphs":

(i) Γ_0 consisting of the vertex v_0 only, with 3 marked points and with the same μ and $\mathfrak{d}$-values as in Γ;

(ii) for each marked point $t=1,2,3$ of Γ with $\vartheta(t)\neq v_0$, Γ_t consisting of the branch of Γ running between the vertices $\vartheta(t)$ and v_0, with the $\mathfrak{d}$-value of v_0 replaced by 0 and with one new marked point attached to v_0;

see Figure 9. The contribution of the vertex graphs (i) is accounted for by the Hurwitz numbers of Proposition 4.1, while the contribution of each of the strands is accounted for by the SQ analogue of the double Givental's J-function computed by (3-11), (3-12), and (3-15). Putting these contributions together, we will obtain (3-14).

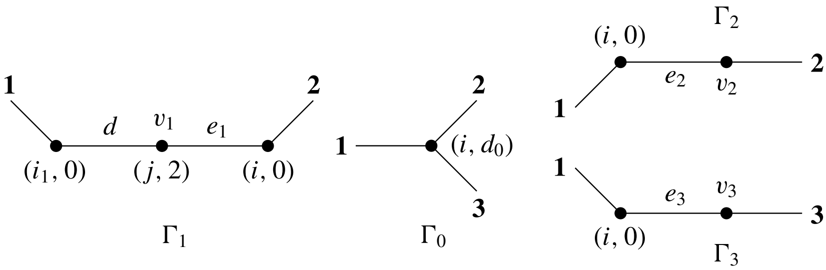

Figure 9. The four subgraphs of the second graph in Figure 1, with label i replaced by i_1.

Let $i=\mu(v_0)$ and $d_0=\mathfrak{d}(v_0)$. For each $t=1,2,3$ with $\vartheta(t)\neq v_0$, let $e_t=\{v_0,v_t\}$ be the edge leaving v_0 in the direction of $\vartheta(t)$. By (7-5),

$$Q_\Gamma \approx Q_{\Gamma_0}\times\prod_{t=1}^{3}Q_{\Gamma_t}=(\overline{\mathcal{M}}_{0,3|d_0}/\mathbb{S}_{d_0})\times\prod_{t=1}^{3}Q_{\Gamma_t}, \tag{10-2}$$

where the t-th factor is defined to be a point if $\vartheta(t)=v_0$. Let $\pi_0,\dots,\pi_3$ be the component projection maps in (10-2). By (7-7) and (7-8),

(10-3)

$$\frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}=\prod_{k\neq i}\pi_0^*\mathbf{e}\big(\dot{\mathcal{V}}_1^{(d_0)}(\alpha_i-\alpha_k)\big)\cdot\prod_{t=1}^{3}\bigg(\pi_t^*\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_t})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}(\omega_{e_t;v_0}-\pi_0^*\psi_t)\bigg),$$

$$\mathbf{e}\big(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma|)}\big)\big|_{Q_\Gamma}=\pi_0^*\mathbf{e}\big(\dot{\mathcal{V}}_{\mathbf{a}}^{(d_0)}(\alpha_i)\big)\cdot\prod_{t=1}^{3}\pi_t^*\mathbf{e}\big(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_t|)}\big),$$

with the t-factor defined to be 1 if $\vartheta(t)=v_0$. Thus, the contribution of Γ to (10-1) is

(10-4)

$$\frac{1}{\prod_{k\neq i}(\alpha_i-\alpha_k)}\sum_{b_1,b_2,b_3\ge 0}\bigg(\frac{q^{d_0}}{d_0!}\int_{\overline{\mathcal{M}}_{0,3|d_0}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d_0)}(\alpha_i))\prod_{t=1}^{3}\psi_t^{b_t}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d_0)}(\alpha_i-\alpha_k))}$$
$$\times q^{|\Gamma_1|}\omega_{e_1;v_0}^{-(b_1+1)}\int_{Q_{\Gamma_1}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_1|)})\mathrm{ev}_1^*\phi_{i_1}\mathrm{ev}_2^*\phi_i}{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})(\hbar_1-\psi_1)}\prod_{t=2}^{3}q^{|\Gamma_t|}\omega_{e_t;v_0}^{-(b_t+1)}\int_{Q_{\Gamma_t}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_t|)})\mathrm{ev}_1^*\phi_i\,\mathrm{ev}_t^*\phi_{i_t}}{\mathbf{e}(\mathcal{N}Q_{\Gamma_t})(\hbar_t-\psi_t)}\bigg),$$

where the t-th factor on the second line is defined to be $\hbar_t^{-(b_t+1)}$ if $\vartheta(t)=v_0$.

We next sum up (10-4) over all possibilities for Γ. Let

$$\dot{\mathcal{Z}}_i(\hbar,\alpha_{i_t},\hbar_t,q)=\begin{cases}\dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_{i_1},\alpha_i,\hbar_1,\hbar,q) & \text{if } t=1;\\ \dot{\mathcal{Z}}_{n;\mathbf{a}}(\alpha_i,\alpha_{i_t},\hbar,\hbar_t,q) & \text{if } t=2,3.\end{cases}$$

By (7-6) and Section 7B, the sum of the factor in (10-4) corresponding to each $t=1,2,3$ over all possibilities for Γ_t with $\mathfrak{d}(e_t)=d_t$ and $\mu(v_t)=j_t$ fixed is

$$(-1)^{b_t+1}\underset{\hbar=(\alpha_{j_t}-\alpha_i)/d_t}{\mathfrak{R}}\big\{\hbar^{-(b_t+1)}\dot{\mathcal{Z}}_i(\hbar,\alpha_{i_t},\hbar_t,q)\big\}.$$

Thus, by the residue theorem on $\mathbb{P}^1$ and Lemma 6.5, the sum of the factor in (10-4) corresponding to each $t=1,2,3$ over all possibilities for Γ_t nontrivial is

$$(-1)^{b_t}\underset{\hbar=0,\infty,-\hbar_t}{\mathfrak{R}}\left\{\frac{\dot{\mathcal{Z}}_i(\hbar,\alpha_{i_t},\hbar_t,q)}{\hbar^{b_t+1}}\right\} = (-1)^{b_t}\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{\mathcal{Z}_i(\hbar,\alpha_{i_t},\hbar_t,q)}{\hbar^{b_t+1}}\right\}-\hbar_t^{-(b_t+1)}\prod_{k\neq i}(\alpha_{i_t}-\alpha_k).$$

Since the last term above is the contribution from the trivial subgraph Γ_t, the sum of the factor in (10-4) corresponding to each $t=1,2,3$ over all possibilities for Γ_t with $\mu(v_0)=i$ fixed is

$$\sum_{\Gamma_t}\big[t\text{-factor in (10-4)}\big]=(-1)^{b_t}\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{\dot{\mathcal{Z}}_i(\hbar,\alpha_{i_t},\hbar_t,q)}{\hbar^{b_t+1}}\right\}; \tag{10-5}$$

this takes into account the graphs Γ with $\vartheta(t)=i$.

By (10-4), (10-5), and Proposition 4.1,

$$\begin{aligned}\dot{\mathcal{Z}}_{n;\mathbf{a}}&(\alpha_{i_1},\alpha_{i_2},\alpha_{i_3},\hbar_1,\hbar_2,\hbar_3,q)\\ &=\sum_{i=1}^{n}\frac{1}{\mathbf{s}_{n-1}(\alpha_i)\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)}\prod_{t=1}^{3}\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{1}{\hbar}\mathrm{e}^{-\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar}\,\dot{\mathcal{Z}}_i(\hbar,\alpha_{i_t},\hbar_t,q)\right\}.\end{aligned}\tag{10-6}$$

By (3-11), (3-15), (3-12), and (4-9),

$$\begin{aligned}\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{1}{\hbar}\mathrm{e}^{-\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar}\,\dot{\mathcal{Z}}_i(\hbar,\alpha_{i_t},\hbar_t,q)\right\}&=\sum_{\substack{s'_t,s_t,r'_t\ge0\\ s'_t+s_t+r'_t=n-1}}\Bigg((-1)^{r'_t}\mathbf{s}_{r'_t}\\ &\times\sum_{r''_t=0}^{s'_t}\widetilde{\mathcal{C}}^{(r''_t)}_{s'_t-\ell^-(\mathbf{a}),s'_t-r''_t-\ell^-(\mathbf{a})}(q)\frac{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)L_{n;\mathbf{a}}(\alpha_i,q)^{s'_t-r''_t}}{I_0(q)\cdots I_{s'_t-r''_t}(q)}\ddot{\mathcal{Z}}^{(s_t)}_{n;\mathbf{a}}(\alpha_{i_t},\hbar_t,q)\Bigg)\end{aligned}$$

for $t=2,3$. Combining this with (3-26), [Popa 2013, Proposition 4.4], and (2-16), we find that

$$\begin{aligned}\underset{\hbar=0}{\mathfrak{R}}&\left\{\frac{1}{\hbar}\mathrm{e}^{-\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar}\,\dot{\mathcal{Z}}_i(\hbar,\alpha_{i_t},\hbar_t,q)\right\}\\ &=\sum_{s_t=0}^{n-1}\sum_{r_t=0}^{\hat{s}_t}\dot{\mathcal{C}}^{(r_t)}_{\hat{s}_t}(q)\frac{\dot{\Phi}^{(0)}_{n;\mathbf{a}}(\alpha_i,q)L_{n;\mathbf{a}}(\alpha_i,q)^{\hat{s}_t-r_t}}{\ddot{\mathbb{I}}^{c}_{s_t+r_t}(q)}\ddot{\mathcal{Z}}^{(s_t)}_{n;\mathbf{a}}(\alpha_{i_t},\hbar_t,q)\end{aligned}\tag{10-7}$$

for $t = 2, 3$. By the same reasoning,

$$\mathfrak{R}_{\hbar=0}\left\{\frac{1}{\hbar}\mathrm{e}^{-\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar}\,\dot{\mathcal{Z}}_i(\hbar,\alpha_{i_1},\hbar_1,q)\right\} = \sum_{s_1=0}^{n-1}\sum_{r_1=0}^{\hat{s}_1}\ddot{\mathcal{C}}_{\hat{s}_1}^{(r_1)}(q)\frac{\ddot{\Phi}_{n;\mathbf{a}}^{(0)}(\alpha_i,q)L_{n;\mathbf{a}}(\alpha_i,q)^{\hat{s}_1-r_1}}{\mathbb{I}_{s_1+r_1}^{c}(q)}\dot{\mathcal{Z}}_{n;\mathbf{a}}^{(s_1)}(\alpha_{i_1},\hbar_1,q), \tag{10-8}$$

where

$$\ddot{\Phi}_{n;\mathbf{a}}^{(0)}(\alpha_i,q) = \left(\frac{L_{n;\mathbf{a}}(\alpha_i,q)}{\alpha_i}\right)^{-\ell(\mathbf{a})}\dot{\Phi}_{n;\mathbf{a}}^{(0)}(\alpha_i,q). \tag{10-9}$$

On the other hand, by (4-10) and (4-8),

$$\begin{aligned}\sum_{i=1}^{n}\frac{\dot{\Phi}_{n;\mathbf{a}}^{(0)}(\alpha_i,q)^3L_{n;\mathbf{a}}(\alpha_i,q)^s}{\mathbf{s}_{n-1}(\alpha_i)\dot{\Phi}_{n;\mathbf{a}}^{(0)}(\alpha_i,q)}\left(\frac{L_{n;\mathbf{a}}(\alpha_i,q)}{\alpha_i}\right)^{-\ell(\mathbf{a})} &= \frac{1}{\mathbf{a}^{\mathbf{a}}}\sum_{i=1}^{n}L_{n;\mathbf{a}}(\alpha_i,q)^{s-|\mathbf{a}|}\frac{\mathrm{d}L}{\mathrm{d}q}\\ &= \frac{1}{\mathbf{a}^{\mathbf{a}}}\frac{\mathrm{d}}{\mathrm{d}q}\begin{cases}\ln\prod_{i=1}^{n}L_{n;\mathbf{a}}(\alpha_i,q) & \text{if } s=|\mathbf{a}|-1;\\ \frac{1}{s+1-|\mathbf{a}|}\sum_{i=1}^{n}L_{n;\mathbf{a}}(\alpha_i,q)^{s+1-|\mathbf{a}|} & \text{otherwise.}\end{cases}\end{aligned}$$

The collection $\{L_{n;\mathbf{a}}(\alpha_i,q)^{-1}\}$ is the set of n roots $\mathbf{y}$ of the equation

$$1-\mathbf{s}_1\mathbf{y}+\cdots+(-1)^n\mathbf{s}_n\mathbf{y}^n-\mathbf{a}^{\mathbf{a}}q\mathbf{y}^{\nu_n(\mathbf{a})}=0.$$

Thus, if $s\geq 0$ and $s+1<|\mathbf{a}|$,

$$\frac{\mathrm{d}}{\mathrm{d}q}\sum_{i=1}^{n}L_{n;\mathbf{a}}(\alpha_i,q)^{s+1-|\mathbf{a}|} = \frac{\mathrm{d}}{\mathrm{d}q}\mathcal{H}^{(|\mathbf{a}|-s-1)}\left(-\frac{\mathbf{s}_{n-1}}{\mathbf{s}_n},\frac{\mathbf{s}_{n-2}}{\mathbf{s}_n},\ldots,(-1)^{|\mathbf{a}|-s-1}\frac{\mathbf{s}_{\nu_n(\mathbf{a})+s+1}}{\mathbf{s}_n}\right)=0,$$

where $\mathcal{H}^{(r)}$ is as in (3-22). If $|\mathbf{a}|=n$, $\{L_{n;\mathbf{a}}(\alpha_i,q)\}$ is the set of n roots $\mathbf{y}$ of the equation

$$\mathbf{y}^n-(1-\mathbf{a}^{\mathbf{a}}q)^{-1}\mathbf{s}_1\mathbf{y}^{n-1}+(1-\mathbf{a}^{\mathbf{a}}q)^{-1}\mathbf{s}_2\mathbf{y}^{n-2}-\cdots+(-1)^n(1-\mathbf{a}^{\mathbf{a}}q)^{-1}\mathbf{s}_n=0.$$

Thus, if $s+1\leq|\mathbf{a}|=n$,

$$\sum_{i=1}^{n}L_{n;\mathbf{a}}(\alpha_i,q)^{s-|\mathbf{a}|}\frac{\mathrm{d}L}{\mathrm{d}q}=\mathbf{a}^{\mathbf{a}}\mathcal{H}_{\nu_n(\mathbf{a})}^{(s+1-n)}(\mathbf{a}^{\mathbf{a}}q), \tag{10-10}$$

where $\mathcal{H}_\nu^{(r)}$ is as in (3-23). If $|\mathbf{a}| < n$, $\{L_{n;\mathbf{a}}(\alpha_i, q)\}$ is the set of n roots $\mathbf{y}$ of the equation

$$\mathbf{y}^n - \mathbf{s}_1\mathbf{y}^{n-1} + \cdots + (-1)^{\nu_n(\mathbf{a})-1}\mathbf{s}_{\nu_n(\mathbf{a})-1}\mathbf{y}^{|\mathbf{a}|+1} + (-1)^{\nu_n(\mathbf{a})}\big(\mathbf{s}_{\nu_n(\mathbf{a})} - (-1)^{\nu_n(\mathbf{a})}\mathbf{a}^{\mathbf{a}}q\big)\mathbf{y}^{|\mathbf{a}|} + (-1)^{\nu_n(\mathbf{a})+1}\mathbf{s}_{\nu_n(\mathbf{a})+1}\mathbf{y}^{|\mathbf{a}|-1} + \cdots + (-1)^n\mathbf{s}_n = 0.$$

Thus, if $s+1 \le |\mathbf{a}| < n$, (10-10) still holds. Combining the equations in this paragraph, we find that

$$\sum_{i=1}^{n} \frac{\dot{\Phi}_{n;\mathbf{a}}^{(0)}(\alpha_i, q)^3 L_{n;\mathbf{a}}(\alpha_i, q)^s}{\mathbf{s}_{n-1}(\alpha_i)\dot{\Phi}_{n;\mathbf{a}}^{(0)}(\alpha_i, q)}\left(\frac{L_{n;\mathbf{a}}(\alpha_i, q)}{\alpha_i}\right)^{-\ell(\mathbf{a})} = \begin{cases} \mathcal{H}_{\nu_n(\mathbf{a})}^{(s+1-n)}(\mathbf{a}^{\mathbf{a}}q) & \text{if } s \ge n-1; \\ 0 & \text{if } 0 \le s < n-1. \end{cases}$$

Combining this with (10-6)–(10-9) and (3-25), we obtain (3-14).

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