Pacific Journal of Mathematics

Volume 273 No. 2 February 2015

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.
STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA
UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2015 is US $420/year for the electronic version, and $570/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2015 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.257

TAUT FOLIATIONS IN SURFACE BUNDLES WITH MULTIPLE BOUNDARY COMPONENTS

TEJAS KALELKAR AND RACHEL ROBERTS

Let M be a fibered 3-manifold with multiple boundary components. We show that the fiber structure of M transforms to closely related transversely oriented taut foliations realizing all rational multislopes in some open neighborhood of the multislope of the fiber. Each such foliation extends to a taut foliation in the closed 3-manifold obtained by Dehn filling along its boundary multislope. The existence of these foliations implies that certain contact structures are weakly symplectically fillable.

1. Introduction

Any closed, orientable 3-manifold can be realized by Dehn filling a 3-manifold which is fibered over S^1 [Alexander 1920; Myers 1978]. In other words, any closed oriented 3-manifold can be realized by Dehn filling a 3-manifold M_0, where M_0 has the form of a mapping torus

$$M_0 = S \times [0, 1]/\sim,$$

where S is a compact orientable surface with nonempty boundary and $\sim$ is an equivalence relation given by $(x, 1) \sim (h(x), 0)$ for some orientation-preserving homeomorphism $h : S \to S$ which fixes the components of ∂S setwise. Although we shall not appeal to this fact in this paper, it is interesting to note that it is possible to assume that h is pseudo-Anosov [Colin and Honda 2008] and hence M_0 is hyperbolic [Thurston 1988]. It is also possible to assume that S has positive genus. Any nonorientable closed 3-manifold admits a double cover of this form.

Taut codimension-one foliations are topological objects which have proved very useful in the study of 3-manifolds. The problem of determining when a 3-manifold contains a taut foliation appears to be a very difficult one. A complete classification exists for Seifert fibered manifolds [Brittenham 1993; Eisenbud et al. 1981; Jankins and Neumann 1985; Naimi 1994], but relatively little is known for the case of hyperbolic 3-manifolds. There are many partial results demonstrating existence

MSC2010: 57M50.
Keywords: Dehn filling, taut foliation, fibered 3-manifold, contact structure, open book decomposition.

(see, for example, [Calegari and Dunfield 2003; Delman and Roberts 1999; Gabai 1983; 1987a; 1987b; Li 2002; Li and Roberts 2013; Roberts 1995; 2001a; 2001b]) and partial results demonstrating nonexistence [Jun 2004; Kronheimer and Mrowka 2007; Kronheimer et al. 2007; Roberts et al. 2003]. In this paper, we investigate the existence of taut codimension-one foliations in closed orientable 3-manifolds by first constructing taut codimension-one foliations in corresponding mapping tori M_0. In contrast with [Roberts 2001a; 2001b], we consider the case that the boundary of M_0 is not connected. We obtain the following results. Precise definitions will follow in Section 2.

Theorem 1.1. *Given an orientable, fibered compact* 3*-manifold, a fibration with fiber surface of positive genus can be modified to yield transversely oriented taut foliations realizing a neighborhood of rational boundary multislopes about the boundary multislope of the fibration.*

As an immediate corollary for closed manifolds we therefore have:

Corollary 1.2. *Let $M = \widehat{M_0}(r^j)$ be the closed manifold obtained from M_0 by Dehn filling M_0 along the multicurve with rational multislope $(r^j)_{j=1}^k$. When (r^j) is sufficiently close to the multislope of the fibration, M admits a transversely oriented taut foliation.*

Dehn filling M_0 along the slope of the fiber gives a mapping torus of a closed surface with the fibration as the obvious taut foliation. This corollary shows that Dehn filling M_0 along slopes sufficiently close to the multislope of the fiber also gives a closed manifold with a taut foliation.

When the surgery coefficients r^j are all meridians, the description of M as a Dehn filling of M_0 gives an open book decomposition (S, h) of M. The foliations of Corollary 1.2 can be approximated by a pair of contact structures, one positive and one negative, both naturally related to the contact structure $\xi_{(S,h)}$ compatible with the open book decomposition (S, h) [Eliashberg and Thurston 1998; Kazez and Roberts 2014]. It follows that the contact structure $\xi_{(S,h)}$ is weakly symplectically fillable.

Corollary 1.3. *Let M have open book decomposition (S, h). Then M is obtained by Dehn filling M_0 along the multicurve with rational multislope $(r^j)_{j=1}^k$, where the r^j are all meridians. When (r^j) is sufficiently close to the multislope of the fibration, $\xi_{(S,h)}$ is weakly symplectically fillable and hence universally tight.*

It is natural to ask whether the qualifier "sufficiently close" can be made precise.

Honda, Kazez, and Matić [Honda et al. 2008] proved that when an open book with connected binding has monodromy with fractional Dehn twist coefficient c at least one, it supports a contact structure which is close to a coorientable taut foliation. Note that $c \geq 1$ is sufficient but not always necessary to guarantee that $\xi_{(S,h)}$ is close to a coorientable taut foliation.

For an open book with multiple binding components, there is no such global lower bound on the fractional Dehn twist coefficients sufficient to guarantee that $\xi_{(S,h)}$ is close to a coorientable taut foliation. This was shown by Baldwin and Etnyre [2013], who constructed a sequence of open books with arbitrarily large fractional Dehn twist coefficients and disconnected bindings that support contact structures which are not deformations of a taut foliation. So we cannot expect to obtain a neighborhood around the slope of the fiber which would satisfy our criteria of "sufficiently close" for every open book decomposition. At the end of the paper, in Section 4, we explicitly compute a neighborhood around the multislope of the fiber realizable by our construction for the Baldwin–Etnyre examples.

2. Preliminaries

In this section we introduce basic definitions and fix conventions used in the rest of the paper.

Foliations. Roughly speaking, a codimension-one foliation $\mathscr{F}$ of a 3-manifold M is a disjoint union of injectively immersed surfaces such that $(M, \mathscr{F})$ looks locally like $(\mathbb{R}^3, \mathbb{R}^2 \times \mathbb{R})$.

Definition 2.1. Let M be a closed C^∞ 3-manifold and let r be a nonnegative integer or infinity. A *C^r codimension-one foliation $\mathscr{F}$* of (or in) M is a union of disjoint connected surfaces L_i, called the *leaves of $\mathscr{F}$*, in M such that

(1) $\bigcup_i L_i = M$, and
(2) there exists a C^r atlas $\mathscr{A}$ on M which contains all C^∞ charts and with respect to which $\mathscr{F}$ satisfies the following local product structure: for every $p \in M$, there exists a coordinate chart $(U, (x, y, z))$ in $\mathscr{A}$ about p such that $U \approx \mathbb{R}^3$ and the restriction of $\mathscr{F}$ to U is the union of planes given by $z =$ constant.

When $r = 0$, we require also that the tangent plane field $T\mathscr{F}$ be C^0.

A *taut* foliation [Gabai 1983] is a codimension-one foliation of a 3-manifold for which there exists a transverse simple closed curve that has nonempty intersection with each leaf of the foliation. Although every 3-manifold contains a codimension-one foliation [Lickorish 1965; Novikov 1965; Wood 1969], it is not true that every 3-manifold contains a codimension-one taut foliation. In fact, the existence of a taut foliation in a closed orientable 3-manifold has important topological consequences for the manifold. For example, if M is a closed, orientable 3-manifold that has a taut foliation with no sphere leaves then M is covered by $\mathbb{R}^3$ [Palmeira 1978], M is irreducible [Rosenberg 1968] and has an infinite fundamental group [Haefliger 1962]. In fact, its fundamental group acts nontrivially on interesting 1-dimensional objects (see, for example, [Thurston 1998; Calegari and Dunfield 2003; Palmeira

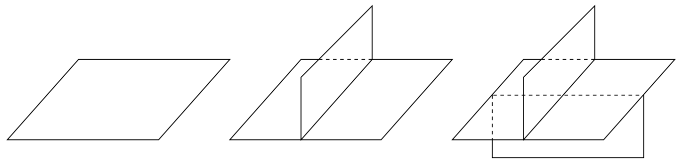

Figure 1. Local model of a standard spine.

1978; Roberts et al. 2003]). Moreover, taut foliations can be perturbed to weakly symplectically fillable contact structures [Eliashberg and Thurston 1998] and hence can be used to obtain Heegaard–Floer information [Ozsváth and Szabó 2004].

Multislopes. Let F be a compact oriented surface of positive genus and with nonempty boundary consisting of k components. Let h be an orientation-preserving homeomorphism of F which fixes each boundary component pointwise. Let

$$M = F \times I/(x, 1) \sim (h(x), 0),$$

and denote the k (toral) boundary components of ∂M by $T^1, T^2, \dots, T^k$.

We use the given surface bundle structure on M to fix a coordinate system on each of the boundary tori, as follows. (See, for example, [Rolfsen 1976, Section 9.G] for a definition and description of this coordinate system.) For each j we choose as longitude $\lambda^j = \partial F \cap T^j$, with orientation inherited from the orientation of F. For each j, we then fix as meridian μ_j an oriented simple closed curve dual to λ_j. Although, as described in [Kazez and Roberts 2014; Roberts 2001b], it is possible to use the homeomorphism h to uniquely specify such simple closed curves μ_j, we choose not to do so in this paper, as all theorem statements are independent of the choice of meridional multislope.

We say a taut foliation $\mathcal{F}$ in M *realizes boundary multislope* $(m^j)_{j=1}^k$ if for each j, $1 \le j \le k$, $\mathcal{F} \cap T^j$ is a foliation of T^j of slope m^j in the chosen coordinate system of T^j.

Spines and branched surfaces.

Definition 2.2. A *standard spine* [Casler 1965] is a space X locally modeled on one of the spaces of Figure 1. The *critical locus* of X is the 1-complex of points of X where the spine is not locally a manifold.

Definition 2.3. A *branched surface* (see [Williams 1974] and [Oertel 1984; 1988]) is a space B locally modeled on the spaces of Figure 2. The *branching locus* L of B is the 1-complex of points of B where B is not locally a manifold. The components of $B \setminus L$ are called the *sectors* of B. The points where L is not locally a manifold are called *double points* of L.

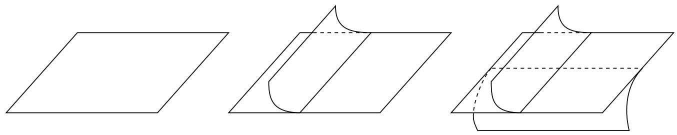

Figure 2. Local model of a branched surface.

A standard spine X together with an orientation in a neighborhood of the critical locus determines a branched surface B in the sense illustrated in Figure 3.

Example 2.4. Let $F_0 := F \times \{0\}$ be a fiber of $M = F \times I/(x, 1) \sim (h(x), 0)$. Let α_i, $1 \le i \le k$, be pairwise disjoint, properly embedded arcs in F_0, and set $D_i = \alpha_i \times I$ in M. Isotope the image arcs $h(\alpha_i)$ as necessary so that the intersection $(\bigcup \alpha_i) \cap (\bigcup_i h(\alpha_i))$ is transverse and minimal. Assign an orientation to F and to each D_i. Then $X = F_0 \cup \bigcup_i D_i$ is a transversely oriented spine. We will denote by $B = \langle F; \bigcup_i D_i \rangle$ the transversely oriented branched surface associated with X.

Similarly, $\langle \bigcup_i F_i; \bigcup_{i,j} D_i^j \rangle$ will denote the transversely oriented branched surface associated to the transversely oriented standard spine

$$X = F_0 \cup F_1 \cup \cdots \cup F_{n-1} \cup \bigcup_{i,j} D_i^j,$$

where $F_i = F \times \{i/n\}$ and $D_i^j = \alpha_i^j \times [i/n, (i+1)/n]$ for some set of arcs α_i^j properly embedded in F so that the intersection $\big(\bigcup_j \alpha_{i-1}^j\big) \cap \big(\bigcup_j \alpha_i^j\big)$ is transverse and minimal.

Definition 2.5. A lamination carried by a branched surface B in M is a closed subset λ of an I-fibered regular neighborhood $N(B)$ of B, such that λ is a disjoint union of injectively immersed 2-manifolds (called leaves) that intersect the I-fibers of $N(B)$ transversely.

Laminar branched surfaces. Li [2002; 2003] introduced the fundamental notions of sink disk and half sink disk.

Definition 2.6 [Li 2002; 2003]. Let B be a branched surface in a 3-manifold M. Let L be the branching locus of B and let X denote the union of double points of L. Associate to each component of $L \setminus X$ a vector (in B) pointing in the direction of the cusp. A *sink disk* is a disk branch sector D of B for which the branch direction

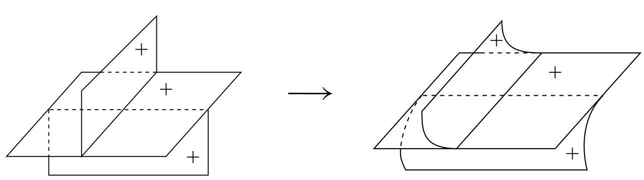

Figure 3. Oriented spine to oriented branched surface.

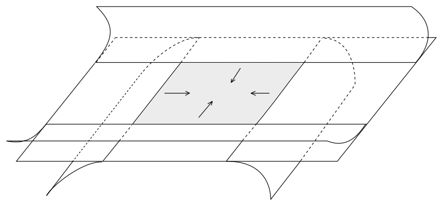

Figure 4. A sink disk.

of each component of $(L \setminus X) \cap \overline{D}$ points into D (as shown in Figure 4). A *half sink disk* is a sink disk which has nonempty intersection with ∂M.

Sink disks and half sink disks play a key role in Li's notion of laminar branched surface.

Definition 2.7 [Li 2002, Definition 1.3]. Let D_1 and D_2 be the two disk components of the horizontal boundary of a $D^2 \times I$ region in $M \setminus \operatorname{int} N(B)$. If the projection $\pi : N(B) \to B$ restricted to the interior of $D_1 \cup D_2$ is injective, that is, the intersection of any I-fiber of $N(B)$ with $\operatorname{int} D_1 \cup \operatorname{int} D_2$ is either empty or a single point, then we say that $\pi(D_1 \cup D_2)$ forms a *trivial bubble* in B.

Definition 2.8 [Li 2002, Definition 1.4]. A branched surface B in a closed 3-manifold M is called a *laminar* branched surface if it satisfies the following conditions:

(1) $\partial_h N(B)$ is incompressible in $M \setminus \operatorname{int} N(B)$, no component of $\partial_h N(B)$ is a sphere and $M \setminus B$ is irreducible.

(2) There is no monogon in $M \setminus \operatorname{int} N(B)$, that is, no disk $D \subset M \setminus \operatorname{int} N(B)$ with $\partial D = D \cap N(B) = \alpha \cup \beta$, where $\alpha \subset \partial_v N(B)$ is in an interval fiber of $\partial_v N(B)$ and $\beta \subset \partial_h N(B)$

(3) There is no Reeb component; that is, B does not carry a torus that bounds a solid torus in M.

(4) B has no trivial bubbles.

(5) B has no sink disk or half sink disk.

Gabai and Oertel [1989] introduced essential branched surfaces and proved that any lamination fully carried by an essential branched surface is an essential lamination and, conversely, any essential lamination is fully carried by an essential branched surface. In practice, to check if a manifold has an essential lamination, the tricky part often is to verify that a candidate branched surface does in fact fully carry a lamination. Li [2002] uses laminar branched surfaces to relax this requirement and prove the following:

Theorem 2.9 [Li 2002, Theorem 1]. *Suppose that M is a closed and orientable 3-manifold. Then*:

(1) *Every laminar branched surface in M fully carries an essential lamination.*

(2) *Any essential lamination in M that is not a lamination by planes is fully carried by a laminar branched surface.*

Li [2003] noticed that if a branched surface has no half sink disk, then it can be arbitrarily split in a neighborhood of its boundary train track without introducing any sink disk (or half sink disk). He was therefore able to conclude the following.

Theorem 2.10 [Li 2003, Theorem 2.2]. *Let M be an irreducible and orientable 3-manifold whose boundary is a union of incompressible tori. Suppose B is a laminar branched surface and $\partial M \setminus \partial B$ is a union of bigons. Then, for any multislope $(s_1, \dots, s_k) \in (\mathbb{Q} \cup \{\infty\})^k$ that can be realized by the train track ∂B, if B does not carry a torus that bounds a solid torus in $\widehat{M}(s_1, \dots, s_k)$, then B fully carries a lamination $\lambda_{(s_1,\dots,s_k)}$ whose boundary consists of the multislope $(s_1, \dots, s_k)$, and $\lambda_{(s_1,\dots,s_k)}$ can be extended to an essential lamination in $\widehat{M}(s_1, \dots, s_k)$.*

We note that Li stated Theorem 2.2 only for the case that ∂M is connected. However, as Li has observed and is easily seen, his proof extends immediately to the case that ∂M consists of multiple toral boundary components. Key is the fact that splitting B open, to a branched surface B' say, in a neighborhood of its boundary, so that $\partial B'$ consists of multislopes $(s_1, \dots, s_k)$, does not introduce sink disks. Therefore, capping B' off to $\widehat{B'}$ yields a laminar branched surface in $\widehat{M}(s_1, \dots, s_k)$.

Good oriented sequence of arcs. In this section we introduce some definitions that will be used in the rest of the paper.

Definition 2.11. Let $(\alpha^1, \dots, \alpha^k)$ be a tuple of pairwise disjoint simple arcs properly embedded in F with $\partial\alpha^j \subset T^j$. Such a tuple will be called *parallel* if $F \setminus \{\alpha^1, \dots, \alpha^k\}$ has k components, $k-1$ of which are annuli $\{A^j\}$ with ∂A^j containing $\{\alpha^j, \alpha^{j+1}\}$ and one of which is a surface S of genus $g-1$ with ∂S containing $\{\alpha^1, \alpha^k\}$. Furthermore, all α^j are oriented in parallel, that is, the orientation of ∂A^j agrees with $\{-\alpha^j, \alpha^{j+1}\}$ and the orientation of ∂S agrees with $\{-\alpha^k, \alpha^1\}$. Note that, in particular, each α^j is nonseparating. See Figure 5 for an example of a parallel tuple.

Definition 2.12. A pair of tuples $(\alpha^i)_{i=1,\dots,k}$ and $(\beta^j)_{j=1,\dots,k}$ will be called *good* if both are parallel tuples and α^i and β^j have exactly one (interior) point of intersection when $i \neq j$, while α^i is disjoint from β^j when $i = j$.

A sequence of parallel tuples

$$\sigma = ((\alpha_0^1, \alpha_0^2, \dots, \alpha_0^k), (\alpha_1^1, \alpha_1^2, \dots, \alpha_1^k), \dots, (\alpha_n^1, \alpha_n^2, \dots, \alpha_n^k)),$$

also shortened to

$$((\alpha_0^j), (\alpha_1^j), \dots, (\alpha_n^j))$$

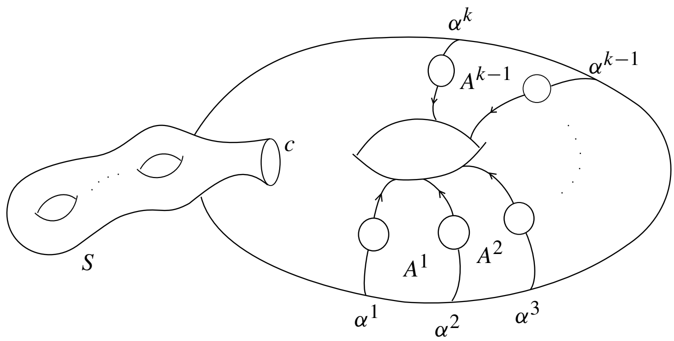

Figure 5. A parallel tuple (α^i) on the surface F.

or

$$(\alpha_0^j) \xrightarrow{\sigma} (\alpha_n^j),$$

will be called *good* if, for each fixed j, $1 \le j \le k$, the pair $((\alpha_i^j), (\alpha_{i+1}^j))$ is good.

Definition 2.13. We say a good pair $((\alpha^j), (\beta^j))$ is *positively oriented* if for each $j \in \{1, \dots, k\}$ a neighborhood of the j-th boundary component in F is as shown on the right in Figure 6. Similarly, we say a good pair $((\alpha^j), (\beta^j))$ is *negatively oriented* if for each $j \in \{1, \dots, k\}$ a neighborhood of the j-th boundary component in F is as shown on the left in Figure 6.

We say a good sequence $\sigma = ((\alpha_0^j), (\alpha_1^j), \dots, (\alpha_n^j))$ is positively oriented if each pair $((\alpha_i^j), (\alpha_{i+1}^j))$ is positively oriented. Similarly, $\sigma = ((\alpha_1^j), (\alpha_2^j), \dots, (\alpha_n^j))$ is negatively oriented if each pair $((\alpha_i^j), (\alpha_{i+1}^j))$ is negatively oriented. We say the sequence σ is oriented if it is positively or negatively oriented. See Figure 7 for an example of a negatively oriented good pair in F.

Preferred generators. Let

$$\mathcal{H}_{g,k} = \{\eta_1, \eta_2, \dots, \eta_{2g-2+k,}, \gamma_{12}, \gamma_{24}, \gamma_{46}, \gamma_{68}, \dots,$$
$$\gamma_{2g-4,2g-2}, \beta, \beta_1, \beta_2, \dots, \beta_{g-1}, \delta_1, \delta_2, \dots, \delta_{k-1}\}$$

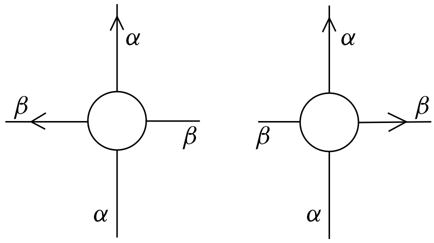

Figure 6. Left: a negatively oriented pair of arcs (α, β). Right: a positively oriented pair of arcs (α, β).

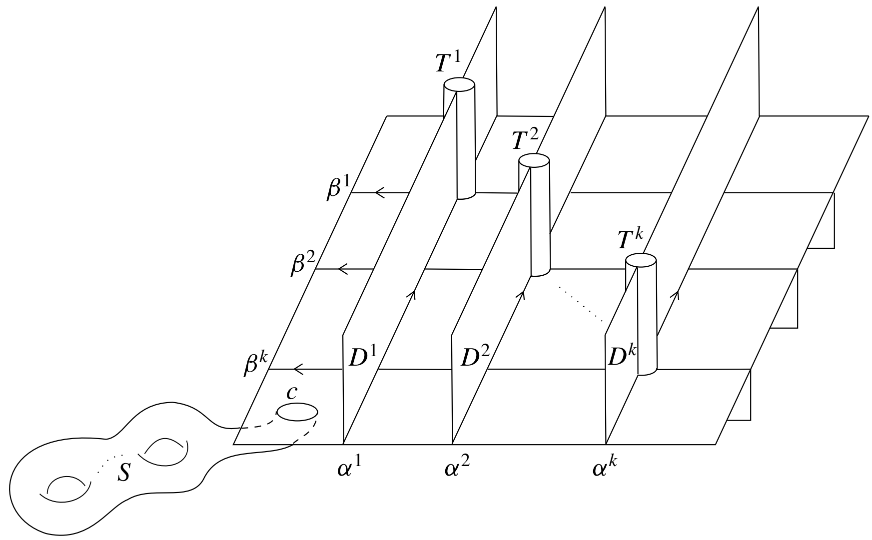

Figure 7. Neighborhood of F with a good negatively oriented pair $((\alpha^j), (\beta^j))$ in the oriented spine X.

be the curves on F as shown in Figure 8. Then by [Gervais 2001, Proposition 1 and Theorem 1] the mapping class group $\mathrm{MCG}(F, \partial F)$ of F (fixing boundary) is generated by Dehn twists about curves in $\mathcal{H}_{g,k}$.

Theorem (Gervais). The mapping class group $\mathrm{MCG}(F, \partial F)$ of F is generated by Dehn twists about the curves in $\mathcal{H}_{g,k}$.

As Dehn twists about δ_i are isotopic to the identity via an isotopy that does not fix the boundary, we have the following obvious corollary:

Corollary 2.14. *The mapping class group* $\mathrm{MCG}(F)$ *of* F (*not fixing the boundary pointwise*) *is generated by Dehn twists about the curves in*

$$\mathcal{H}'_{g,k} = \mathcal{H}_{g,k} \setminus \{\delta_1, \dots, \delta_{k-1}\}.$$

3. Main theorem

Definition 3.1. Let $(\alpha^1, \dots, \alpha^k)$ be a parallel tuple in F. Orient F so that the normal vector $\hat{n}$ induced by the orientation of M points in the direction of increasing $t \in [0, 1]$. Let $D^j = \alpha^j \times [0, 1]$ in M_h with the orientation induced by orientations of α^j and F; that is, if v^j is tangent to α^j then $(v^j, \hat{n})$ gives the orientation of D^j. Let $X = F \cup \bigcup_j D^j$ be an oriented standard spine and $B_\alpha = \langle F; \bigcup_j D^j \rangle$ the transversely oriented branched surface associated with X.

Notice that the multislope of the fibration is $\bar{0}$. In order to prove Theorem 1.1, we shall prove the following:

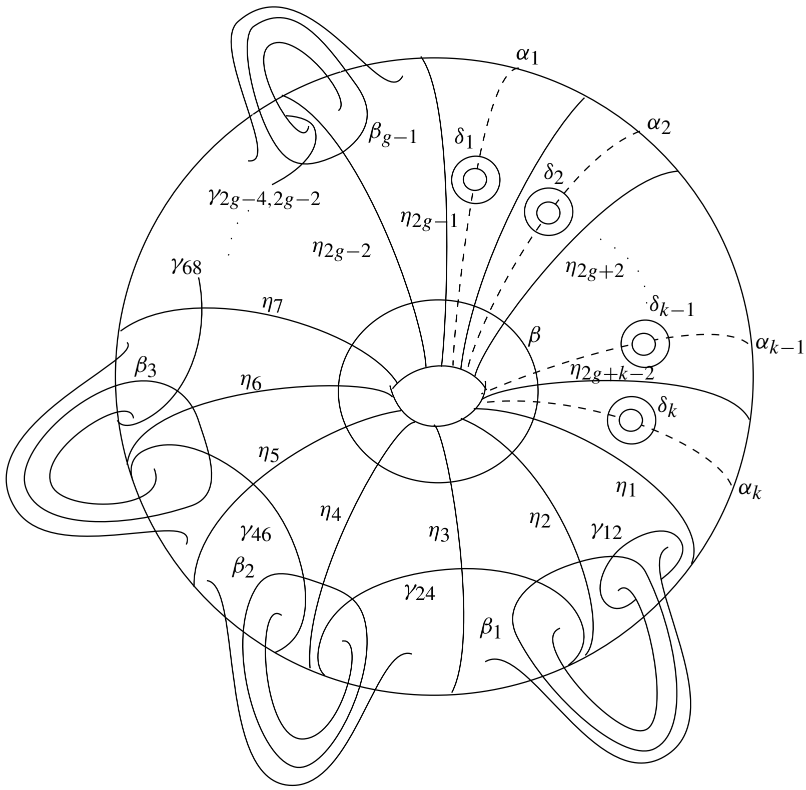

Figure 8. Generators of the mapping class group.

Theorem 3.2. *There is an open neighborhood $\mathcal{U}$ of $\bar{0} \in \mathbb{R}^k$ such that, for each point $(m^1, \ldots, m^k) \in \mathcal{U} \cap \mathbb{Q}^k$, there exists a lamination carried by B_α with boundary multislope (m^j). These laminations extend to taut foliations which also intersect the boundary in foliations with multislope (m^j).*

This gives us the following corollary for closed manifolds.

Corollary 3.3. *Let $\widehat{M}(r^j)$ denote the closed manifold obtained from M by a Dehn filling along a multicurve with rational multislope $(r^j)_{j=1}^k$. For each tuple (r^j) in $\mathcal{U} \cap \mathbb{Q}^k$, the closed manifold $\widehat{M}(r^j)$ also has a transversely oriented taut foliation.*

We outline the proof of Theorem 3.2 with details worked out in the lemmas.

Proof. In Lemma 3.4 we show that there is a good positively oriented sequence $(\alpha_0^j) \to (h^{-1}(\alpha_0^j))$, or equivalently from $(h(\alpha_n^j)) \to (\alpha_n^j)$. In Lemma 3.6 we show that whenever there exists such a positive sequence there is a splitting of the branched surface B_α to a branched surface B_σ that is laminar and that therefore carries laminations realizing every multislope in some open neighborhood of $\bar{0} \in \mathbb{R}^k$. Finally, in Lemma 3.8 we show that these laminations extend to taut foliations on all of M. $\square$

Lemma 3.4. *Let (α^j) be a parallel tuple in F and let $h \in \mathrm{Aut}^+(F)$. Then there is a good positively oriented sequence $(\alpha^j) \xrightarrow{\sigma} (h(\alpha^j))$.*

Proof. By Corollary 2.14 to Gervais's theorem, $h \sim h_m h_{m-1} \cdots h_2 h_1$ for twists h_i about curves in $\mathcal{H}'_{g,k}$. Set $h' = h_m h_{m-1} \cdots h_2 h_1$, and notice that $M_h = M_{h'}$.

By changing the handle decomposition of F as necessary, we may assume that the parallel tuple (α^j) is as shown in Figure 8. Let b denote the Dehn twist about $\beta \in \mathcal{H}'_{g,k}$. Notice that any h_i in the factorization of h' is either b, b^{-1} or a twist about a curve disjoint from all components of α^j. Thus $((\alpha^j), (h_i(\alpha^j))$ is either a good positive pair, a good negative pair, or a pair of equal tuples.

Now, if $((\alpha^j), (\beta^j))$ is a good pair then so is $((h_i(\alpha^j)), (h_i(\beta^j)))$; therefore, each of the pairs

$$\begin{aligned}
&\big((\alpha^j), h_m(\alpha^j)\big),\\
&\big((h_m(\alpha^j)), (h_m h_{m-1}(\alpha^j))\big),\\
&\big((h_m h_{m-1}(\alpha^j)), (h_m h_{m-1} h_{m-2}(\alpha^j))\big),\\
&\qquad\vdots\\
&\big((h_m h_{m-1} \cdots h_2(\alpha^j)), (h_m h_{m-1} \cdots h_2 h_1(\alpha^j) = h(\alpha^j))\big),
\end{aligned}$$

is either a good oriented pair or a pair of equal tuples.

If at least one of the h_i is b or b^{-1} then, ignoring the equal tuples, we get a good oriented sequence $((\alpha^j_0), (\alpha^j_1), \ldots, (\alpha^j_{n-1}), (\alpha^j_n) = h((\alpha^j_0)))$ or $(\alpha^j) \xrightarrow{\sigma} (h(\alpha^j))$ as required. The length of this sequence is equal to the number of times h_i equals b or b^{-1}, that is, $n = n_+ + n_-$, where n_+ is the sum of the positive powers of b in this expression of h' and n_- is the magnitude of the sum of negative powers of b.

If none of the h_i are Dehn twists about β then $(\alpha^j) = (h(\alpha^j))$. In this case, $\sigma = ((\alpha^j), (b(\alpha^j)), (b^{-1}b(\alpha^j) = (\alpha^j)))$ is a good oriented sequence.

If $((\alpha^j), (\beta^j))$ is a positively oriented good pair then $((\alpha^j), (-\beta^j), (-\alpha^j), (\beta^j))$ is a negatively oriented good sequence. Performing n_- such substitutions, we get a positively oriented good sequence $(\alpha^j) \xrightarrow{\sigma} (h(\alpha^j))$. □

Definition 3.5. Let $\sigma = (h(\alpha^j_n) = \alpha^j_0, \alpha^j_1, \ldots, \alpha^j_{n-1}, \alpha^j_n)$ be a good oriented sequence. Let $F_i = F \times \{i/n\}$ for $0 \le i < n$ and let $D^j_i = \alpha^j_i \times [i/n, (i+1)/n]$, for $0 \le i < n$, in M_h. Let

$$X = \big(\textstyle\bigcup_i F_i\big) \cup \big(\bigcup_{i,j} D^j_i\big),$$

and orient F_i and D^j_i as in Definition 3.1. Define

$$B_\sigma = \big\langle \textstyle\bigcup_i F_i ; \bigcup_{i,j} D^j_i \big\rangle$$

as the associated branched surface. Figure 7 shows the neighborhood of F in X, while Figure 9 shows a neighborhood of F in the associated branched surface.

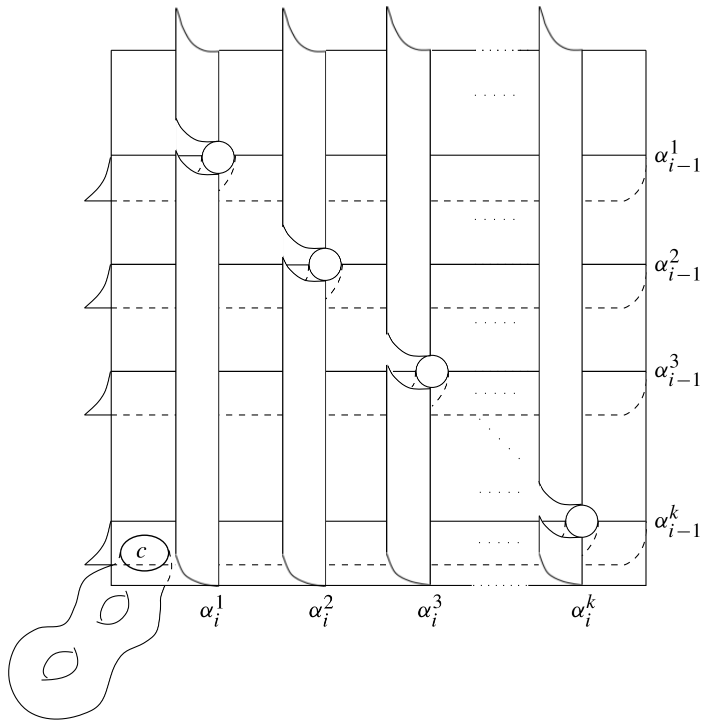

Figure 9. A neighborhood of one of the fibers in the branched surface B. The small circles along the diagonal represent longitudes of the boundary tori. The vertical subarcs of the boundaries of the vertical disk sectors lie on these boundary tori. Compare with Figure 7.

Lemma 3.6. *Let $\sigma = (h(\alpha^j_n) = \alpha^j_0, \alpha^j_1, \ldots, \alpha^j_{n-1}, \alpha^j_n)$ be a good oriented sequence in F and B_σ the associated branched surface in M_h. Then B_σ has no sink disk or half sink disk.*

Proof. As the sequence σ is good and oriented for each fixed i, the tuple of arcs (α^j_i) is parallel and $|\alpha^j_i \cap \alpha^k_{i-1}| = \delta^k_j$, so a neighborhood of F_i in B_σ is as shown in Figure 9.

The sectors of B_σ consist of disks $D^j_i = \alpha^j_i \times [i/n, (i+1)/n]$ and components of $F_i \setminus \{\alpha^j_i \cup \alpha^j_{i-1}\}_{j=1,\ldots,k}$. As F_{i-1} and F_i both have a coorientation in the direction of increasing t for $(x, t) \in M_h$, so for any orientation of D^j_i, ∂D^j_i is the union of two arcs in ∂M_h, together with one arc with the direction of the cusp pointing into the disk and one arc with the direction of the cusp pointing outwards. Similarly, as α^j_i and α^{j+1}_i are oriented in parallel, each disk component of $F_i \setminus \{\alpha^j_i, \alpha^j_{i-1}\}_{j=1,\ldots,k}$ has a boundary arc with cusp direction pointing outwards. Therefore, no branch sector in B_σ is a sink disk or a half sink disk. □

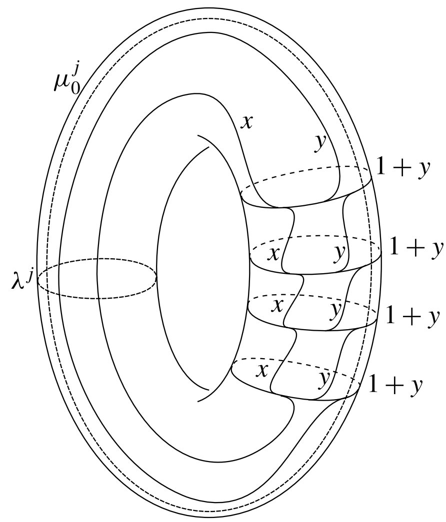

Figure 10. The weighted boundary train track when $n = 4$.

Remark 3.7. Notice that $B_\sigma = \langle \bigcup_i F_i; \bigcup_{i,j} D_i^j \rangle$ is a splitting (see [Oertel 1988]) of the original branched surface $B_\alpha = \langle F; \bigcup_j D^j \rangle$ and, equivalently, B_σ collapses to B_α. So, in particular, laminations carried by B_σ are also carried by B_α.

Now consider the train tracks $\tau^j = B_\sigma \cap T^j$. Focus on one of the τ^j. Recall that we fixed a coordinate system (λ^j, μ^j) on T^j. For simplicity of exposition, we now make a second choice μ_0^j of meridian. This choice is dictated by the form of τ^j; namely, we choose μ_0^j to be disjoint from the disks D_i^j so that τ^j has the form shown in Figure 10. Notice that there is a change of coordinates homeomorphism taking slopes in terms of the coordinate system (λ^j, μ_0^j) to slopes in terms of the coordinate system (λ^j, μ^j). Since λ^j is unchanged, this homeomorphism takes an open interval about 0 to an open interval about 0. Assign to τ^j the measure determined by weights x, y shown in Figure 10. In terms of the coordinate system (λ^j, μ_0^j), τ^j carries all slopes realizable by

$$\frac{x-y}{n(1+y)}$$

for some $x, y > 0$. Therefore, in terms of the coordinate system (λ^j, μ_0^j), τ^j carries all slopes in $(-1/n, \infty)$. Converting to the coordinate system (λ^j, μ^j), τ^j carries all slopes in some open neighborhood of 0. Repeat for all j. By Theorem 2.10, we see that the branched surface B_σ carries laminations $\lambda_{(\bar{x},\bar{y})}$ realizing multislopes

$$\left(\frac{x_1-y_1}{n(1+y_1)}, \frac{x_2-y_2}{n(1+y_2)}, \dots, \frac{x_k-y_k}{n(1+y_k)}\right)$$

for any strictly positive values of $x_1, \dots, x_k, y_1, \dots, y_k$, and hence realizing all rational multislopes in some open neighborhood of $\bar{0} \in \mathbb{R}^k$.

Lemma 3.8. *Suppose the weights $\bar{x}$, $\bar{y}$ are distinct and have strictly positive coordinates. Then each lamination $\lambda_{(\bar{x},\bar{y})}$ contains only noncompact leaves. Furthermore, each lamination $\lambda_{(\bar{x},\bar{y})}$ extends to a taut foliation $\mathscr{F}_{(\bar{x},\bar{y})}$, which realizes the same multislope.*

Proof. Suppose that $\lambda_{(\bar{x},\bar{y})}$ contains a compact leaf L. Such a leaf determines a transversely invariant measure on B given by counting intersections with L.

Now focus on any i, j, where $0 \le i, j < n$. By considering, for example, a simple closed curve in F_i parallel to the arc α_i^j, we see that there is an oriented simple closed curve in F_i which intersects the branching locus of B_σ exactly k times and which has orientation consistent with the branched locus. Since this is true for all possible i, j, it follows that the only transversely invariant measure B can support is the one with all weights on the branches D_i^j necessarily 0. But this means that $\lambda_{(\bar{x},\bar{y})}$ realizes multislope $\bar{0}$ and hence that $\bar{x} = \bar{y}$.

The complementary regions to the lamination $\lambda_{(\bar{x},\bar{y})}$ are product regions. Filling these up with product fibrations, we get the required foliation $\mathscr{F}_{(\bar{x},\bar{y})}$, which also has no compact leaves and is therefore taut. □

4. Example

As discussed in the introduction, an open book with connected binding and monodromy with fractional Dehn twist coefficient more than one supports a contact structure which is the deformation of a coorientable taut foliation [Honda et al. 2008]. However, for open books with disconnected binding there is no such universal lower bound on the fractional Dehn twist coefficient. This was illustrated by Baldwin and Etnyre [2013] who constructed a sequence of open books with arbitrarily large fractional Dehn twist coefficients and disconnected binding that support contact structures which are not deformations of a taut foliation. This shows, in particular, that there is no global neighborhood about the multislope of the fiber of a surface bundle such that Dehn filling along rational slopes in that neighborhood produces closed manifolds with taut foliations.

The notion of “sufficiently close” in Corollary 1.2 can, however, be bounded below for a given manifold. Deleting a neighborhood of the binding in the Baldwin–Etnyre examples gives a surface bundle, and using the techniques developed in the previous sections we now calculate a neighborhood of multislopes realized by taut foliations around the multislope of the fiber in this fibration. In particular, we observe that this neighborhood does not contain the meridional multislope. So Dehn filling along these slopes does not give a taut foliation of the sequence of Baldwin–Etnyre manifolds, as is to be expected.

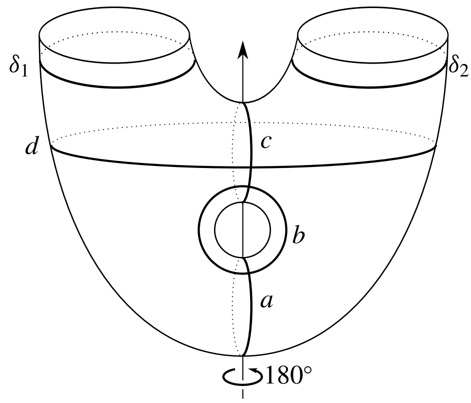

Figure 11. The Baldwin–Etnyre examples.

The following is a description of the Baldwin–Etnyre examples [2013]. Let T denote the genus one surface with two boundary components, B_1 and B_2. Let ψ be the diffeomorphisms of T given by the product of Dehn twists,

$$\psi = D_a D_b^{-1} D_c D_d^{-1},$$

where a, b, c and d are the curves shown in Figure 11 (reproduced from Figure 1 of [Baldwin and Etnyre 2013]). Then ψ is pseudo-Anosov by a well-known construction of Penner [1988]. We define

$$\psi_{n,k_1,k_2} = D_{\delta_1}^{k_1} D_{\delta_2}^{k_2} \psi^n,$$

where δ_1 and δ_2 are curves parallel to the boundary components B_1 and B_2 of T.

Let M_{n,k_1,k_2} be the open book (T, ψ_{n,k_1,k_2}). Let $N(B_1)$, $N(B_2)$ be regular neighborhoods of B_1 and B_2 in M_{n,k_1,k_2} and let $M'_{n,k_1,k_2} = M_{n,k_1,k_2} \setminus (N(B_1) \cup N(B_2))$. Let λ_1, λ_2 be the closed curves in $T \cap \partial M'_{n,k_1,k_2}$ represented by B_1, B_2, with induced orientation. The monodromy ψ_{n,k_1,k_2} is freely isotopic to the pseudo-Anosov map ψ^n. Let μ_1, μ_2 be the suspension flow of a point in λ_1 and λ_2, respectively, under the monodromy ψ^n. As ψ^n is the identity on ∂T, $\mu_i = p_i \times S^1$ in $\partial M'_{n,k_1,k_2} = (B_1 \times S^1) \cup (B_2 \times S^1)$ for $p_i \in \lambda_i$.

We use these pairs of dual curves (λ_1, μ_1) and (λ_2, μ_2) as coordinates to calculate the slope of curves on the boundary tori of M'_{n,k_1,k_2}, as detailed in Section 2.

If D_1 is the meridional disk of a regular neighborhood $N(B_1)$ of B_1 in M_{n,k_1,k_2}, then $\partial D_1 = \mu_1$. Similarly, for D_2 a meridional disk of a regular neighborhood of B_2 in M_{n,k_1,k_2}, $\partial D_2 = \mu_2$.

In order to express the monodromy of the surface bundle in terms of the Gervais generators we use the pseudo-Anosov monodromy $\psi^n = \psi_{n,0,0}$ which is freely isotopic to ψ_{n,k_1,k_2}, with the observation that Dehn filling $M'_{n,0,0}$ along slopes $-1/k_1$

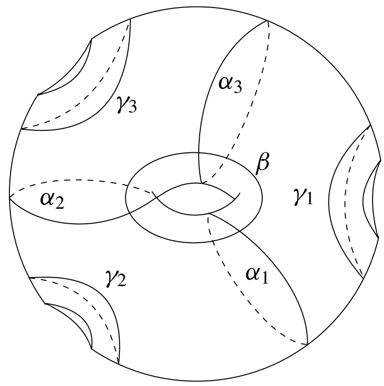

Figure 12. The Gervais star-relation.

and $-1/k_2$ gives the manifold M_{n,k_1,k_1}. So for $M'_{n,0,0}$ we have slope$(\partial D_1) = -1/k_1$, slope$(\partial D_2) = -1/k_2$.

As shown in Theorem 1.16 of [Baldwin and Etnyre 2013], for any $N > 0$ there exist $n, k_1 > N$ such that the corresponding open book in $M_{n,k_1,n}$ has a compatible contact structure that is not a deformation of the tangent bundle of a taut foliation. We shall now show that the slope $-1/n$ lies outside the interval of perturbation that gives slopes of taut foliations via our construction. Hence, the manifolds $M_{n,k_1,n}$ cannot be obtained by capping off the taut foliations realized by our interval of boundary slopes around the fibration.

To obtain the branched surface required in our construction in the previous sections we need a good sequence of arcs $\alpha^j \to \phi^{-1}(\alpha^j)$, where $\phi = \psi^n$, $j = 1, 2$. These arcs are used to construct product disks which we then smooth along copies of the fiber surface to get the required branched surface.

Following the method outlined in Lemma 3.4, we need to express ϕ^{-1} in terms of the Gervais generators. The curves a, b and c correspond to the generating curves η_1, β and η_2 among the Gervais generators, as can be seen in Figure 8. We now need to express the curve d in terms of these generating curves.

Definition 4.1. Let $S_{g,n}$ be a surface of genus g and n boundary components. Consider a subsurface of $S_{g,n}$ homeomorphic to $S_{1,3}$. Then for curves α_i, β, γ_i as shown in Figure 12 (reproduced from Figure 2 of [Gervais 2001]), the star-relation is

$$(D_{\alpha_1} D_{\alpha_2} D_{\alpha_3} D_{\beta})^3 = D_{\gamma_1} D_{\gamma_2} D_{\gamma_3},$$

where D represents Dehn-twist along the corresponding curves.

Let S be the component of $T \setminus d$ which is homeomorphic to a once-punctured torus. Let $\gamma_1 = d$ and γ_2, γ_3 be curves bounding disjoint disks D_1 and D_2 in S so that $S \setminus (D_1 \cup D_2)$ is homeomorphic to $S_{1,3}$. As γ_2, γ_3 are trivial in T, $\gamma_1 = d$ and $\alpha_1 = \alpha_2 = \alpha_3 = a$, so the star relation reduces to $D_d = (D_a^3 D_b)^3$.

Hence, the monodromy ψ in terms of the Gervais generators is the word $\psi = D_a D_b^{-1} D_c (D_a^3 D_b)^{-3}$, which gives us $\psi^{-1} = D_a^3 D_b D_a^3 D_b D_a^3 D_b D_c^{-1} D_b D_a^{-1}$. Take arcs α_1, α_2 as shown in Figure 8, where $a = \eta_1$, $b = \beta$ and $c = \eta_2$. Then, as $(\alpha_j, D_b(\alpha_j))$ is a negatively oriented pair and $\alpha_j = D_a(\alpha_j)$, $\alpha_j = D_c(\alpha_j)$ so we have a negatively oriented good sequence $(\alpha_1, \alpha_2) \to (\psi^{-1}(\alpha_1), \psi^{-1}(\alpha_2))$ obtained by taking the sequence of arcs

$$\sigma = \big(\alpha_j,\ D_a^3 D_b(\alpha_j),\ D_a^3 D_b D_a^3 D_b(\alpha_j),\ D_a^3 D_b D_a^3 D_b D_a^3 D_b(\alpha_j),\\ D_a^3 D_b D_a^3 D_b D_a^3 D_b D_c^{-1} D_b D_a^{-1}(\alpha_j) = \psi^{-1}(\alpha_j)\big) \quad \text{for } j = 1, 2.$$

Let B_σ be the branched surface corresponding to this good oriented sequence, as in Definition 3.5. The weighted train track $\tau_\sigma = B_\sigma \cap \partial M'_{n,0,0}$ on the boundary tori is as shown in Figure 10.

The slope of this measured boundary lamination is $(x - y)/(4(1 + y))$, so the interval of slopes that are realized by taut foliations is $\big(-\frac{1}{4}, \infty\big)$.

When the monodromy is ψ^n (instead of ψ), by a similar argument, we get the slope of the measured lamination on the boundary as $(x - y)/(4n(1 + y))$ so that the interval of slopes realized by taut foliations is $(-1/4n, \infty)$. And we observe that the point $(-1/k_1, -1/n)$ does not lie in $(-1/4n, \infty) \times (-1/4n, \infty)$; that is, the taut foliations from our construction cannot be capped off to give a taut foliation of the Baldwin–Etnyre examples.

References

[Alexander 1920] J. W. Alexander, "Note on Riemann spaces", *Bull. Amer. Math. Soc.* **26**:8 (1920), 370–372. MR 1560318 JFM 47.0529.02

[Baldwin and Etnyre 2013] J. A. Baldwin and J. B. Etnyre, "Admissible transverse surgery does not preserve tightness", *Math. Ann.* **357**:2 (2013), 441–468. MR 3096514 Zbl 06228509

[Brittenham 1993] M. Brittenham, "Essential laminations in Seifert-fibered spaces", *Topology* **32**:1 (1993), 61–85. MR 94c:57027 Zbl 0791.57013

[Calegari and Dunfield 2003] D. Calegari and N. M. Dunfield, "Laminations and groups of homeomorphisms of the circle", *Invent. Math.* **152**:1 (2003), 149–204. MR 2005a:57013 Zbl 1025.57018

[Casler 1965] B. G. Casler, "An imbedding theorem for connected 3-manifolds with boundary", *Proc. Amer. Math. Soc.* **16** (1965), 559–566. MR 31 #2730 Zbl 0129.15801

[Colin and Honda 2008] V. Colin and K. Honda, "Stabilizing the monodromy of an open book decomposition", *Geom. Dedicata* **132** (2008), 95–103. MR 2008m:57041 Zbl 1151.57017

[Delman and Roberts 1999] C. Delman and R. Roberts, "Alternating knots satisfy Strong Property P", *Comment. Math. Helv.* **74**:3 (1999), 376–397. MR 2001g:57009 Zbl 0949.57002

[Eisenbud et al. 1981] D. Eisenbud, U. Hirsch, and W. Neumann, "Transverse foliations of Seifert bundles and self-homeomorphism of the circle", *Comment. Math. Helv.* **56**:4 (1981), 638–660. MR 83j:57016 Zbl 0516.57015

[Eliashberg and Thurston 1998] Y. M. Eliashberg and W. P. Thurston, *Confoliations*, University Lecture Series **13**, Amer. Math. Soc., Providence, RI, 1998. MR 98m:53042 Zbl 0893.53001

[Gabai 1983] D. Gabai, "Foliations and the topology of 3-manifolds", *J. Differential Geom.* **18**:3 (1983), 445–503. MR 86a:57009 Zbl 0533.57013

[Gabai 1987a] D. Gabai, "Foliations and the topology of 3-manifolds, II", *J. Differential Geom.* **26**:3 (1987), 461–478. MR 89a:57014a Zbl 0627.57012

[Gabai 1987b] D. Gabai, "Foliations and the topology of 3-manifolds, III", *J. Differential Geom.* **26**:3 (1987), 479–536. MR 89a:57014b Zbl 0639.57008

[Gabai and Oertel 1989] D. Gabai and U. Oertel, "Essential laminations in 3-manifolds", *Ann. of Math.* (2) **130**:1 (1989), 41–73. MR 90h:57012 Zbl 0685.57007

[Gervais 2001] S. Gervais, "A finite presentation of the mapping class group of a punctured surface", *Topology* **40**:4 (2001), 703–725. MR 2002m:57025 Zbl 0992.57013

[Haefliger 1962] A. Haefliger, "Variétés feuilletées", *Ann. Scuola Norm. Sup. Pisa* (3) **16** (1962), 367–397. MR 32 #6487 Zbl 0122.40702

[Honda et al. 2008] K. Honda, W. H. Kazez, and G. Matić, "Right-veering diffeomorphisms of compact surfaces with boundary, II", *Geom. Topol.* **12**:4 (2008), 2057–2094. MR 2009i:57057 Zbl 1170.57013

[Jankins and Neumann 1985] M. Jankins and W. D. Neumann, "Rotation numbers of products of circle homeomorphisms", *Math. Ann.* **271**:3 (1985), 381–400. MR 86g:58082 Zbl 0543.57019

[Jun 2004] J. Jun, "$(-2, 3, 7)$-pretzel knot and Reebless foliation", *Topology Appl.* **145**:1-3 (2004), 209–232. MR 2005g:57015 Zbl 1090.57014

[Kazez and Roberts 2014] W. H. Kazez and R. Roberts, "Approximating C^0-foliations", preprint, 2014. To appear in *Interactions between low-dimensional topology and mapping class groups* (Bonn, 2013), Geometry and Topology Monographs **19**, Math. Sciences Publishers, Berkeley, 2015. arXiv 1404.5919v2

[Kronheimer and Mrowka 2007] P. Kronheimer and T. Mrowka, *Monopoles and three-manifolds*, New Mathematical Monographs **10**, Cambridge University Press, 2007. MR 2009f:57049 Zbl 1158.57002

[Kronheimer et al. 2007] P. Kronheimer, T. Mrowka, P. Ozsváth, and Z. Szabó, "Monopoles and lens space surgeries", *Ann. of Math.* (2) **165**:2 (2007), 457–546. MR 2008b:57037 Zbl 1204.57038

[Li 2002] T. Li, "Laminar branched surfaces in 3-manifolds", *Geom. Topol.* **6** (2002), 153–194. MR 2003h:57019 Zbl 1067.57011

[Li 2003] T. Li, "Boundary train tracks of laminar branched surfaces", pp. 269–285 in *Topology and geometry of manifolds* (Athens, GA, 2001), edited by G. Matić and C. McCrory, Proc. Sympos. Pure Math. **71**, Amer. Math. Soc., Providence, RI, 2003. MR 2005h:57028 Zbl 1043.57006

[Li and Roberts 2013] T. Li and R. Roberts, "Taut foliations in knot complements", preprint, 2013. To appear in *Pac. J. Math.* arXiv 1211.3066

[Lickorish 1965] W. B. R. Lickorish, "A foliation for 3-manifolds", *Ann. of Math.* (2) **82** (1965), 414–420. MR 32 #6488 Zbl 0142.41104

[Myers 1978] R. Myers, "Open book decompositions of 3-manifolds", *Proc. Amer. Math. Soc.* **72**:2 (1978), 397–402. MR 80a:57004 Zbl 0395.57002

[Naimi 1994] R. Naimi, "Foliations transverse to fibers of Seifert manifolds", *Comment. Math. Helv.* **69**:1 (1994), 155–162. MR 94m:57060 Zbl 0797.55009

[Novikov 1965] S. P. Novikov, "The topology of foliations", *Trudy Moskov. Mat. Obšč.* **14** (1965), 248–278. In Russian; translated in *Trans. Moscow Math. Soc.* **14** (1965), 268–304. MR 34 #824 Zbl 0247.57006

[Oertel 1984] U. Oertel, "Incompressible branched surfaces", *Invent. Math.* **76**:3 (1984), 385–410. MR 86a:57007 Zbl 0539.57006

[Oertel 1988] U. Oertel, "Measured laminations in 3-manifolds", *Trans. Amer. Math. Soc.* **305**:2 (1988), 531–573. MR 89d:57011 Zbl 0652.57006

[Ozsváth and Szabó 2004] P. Ozsváth and Z. Szabó, "Holomorphic disks and genus bounds", *Geom. Topol.* **8** (2004), 311–334. MR 2004m:57024 Zbl 1056.57020

[Palmeira 1978] C. F. B. Palmeira, "Open manifolds foliated by planes", *Ann. Math.* (2) **107**:1 (1978), 109–131. MR 58 #18490 Zbl 0382.57010

[Penner 1988] R. C. Penner, "A construction of pseudo–Anosov homeomorphisms", *Trans. Amer. Math. Soc.* **310**:1 (1988), 179–197. MR 89k:57026 Zbl 0706.57008

[Roberts 1995] R. Roberts, "Constructing taut foliations", *Comment. Math. Helv.* **70**:4 (1995), 516–545. MR 96j:57032 Zbl 0855.57009

[Roberts 2001a] R. Roberts, "Taut foliations in punctured surface bundles, I", *Proc. London Math. Soc.* (3) **82**:3 (2001), 747–768. MR 2003a:57040 Zbl 1034.57017

[Roberts 2001b] R. Roberts, "Taut foliations in punctured surface bundles, II", *Proc. London Math. Soc.* (3) **83**:2 (2001), 443–471. MR 2003j:57016 Zbl 1034.57018

[Roberts et al. 2003] R. Roberts, J. Shareshian, and M. Stein, "Infinitely many hyperbolic 3-manifolds which contain no Reebless foliation", *J. Amer. Math. Soc.* **16**:3 (2003), 639–679. MR 2004e:57023 Zbl 1012.57022

[Rolfsen 1976] D. Rolfsen, *Knots and links*, Mathematics Lecture Series **7**, Publish or Perish, Berkeley, CA, 1976. MR 58 #24236 Zbl 0339.55004

[Rosenberg 1968] H. Rosenberg, "Foliations by planes", *Topology* **7** (1968), 131–138. MR 37 #3595 Zbl 0157.30504

[Thurston 1988] W. P. Thurston, "On the geometry and dynamics of diffeomorphisms of surfaces", *Bull. Amer. Math. Soc.* (*N.S.*) **19**:2 (1988), 417–431. MR 89k:57023 Zbl 0674.57008

[Thurston 1998] W. P. Thurston, "Three-manifolds, foliations and circles, II", Unfinished manuscript, 1998.

[Williams 1974] R. F. Williams, "Expanding attractors", *Inst. Hautes Études Sci. Publ. Math.* 43 (1974), 169–203. MR 50 #1289 Zbl 0279.58013

[Wood 1969] J. W. Wood, "Foliations on 3-manifolds", *Ann. of Math.* (2) **89** (1969), 336–358. MR 40 #2123 Zbl 0176.21402

Received September 15, 2013. Revised September 23, 2013.

TEJAS KALELKAR
DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY IN ST. LOUIS
1 BROOKINGS DRIVE
ST. LOUIS, MO 63130
UNITED STATES
tejas@math.wustl.edu

RACHEL ROBERTS
DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY IN ST. LOUIS
1 BROOKINGS DRIVE
ST. LOUIS, MO 63130
UNITED STATES
roberts@math.wustl.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.277

SOME RESULTS ON ARCHIMEDEAN RANKIN–SELBERG INTEGRALS

JINGSONG CHAI

We use a notion of derivatives of smooth representations of moderate growth of GL(n, $\mathbb{R}$) and exceptional poles to study local Rankin–Selberg integrals. We obtain various results which are archimedean analogs of p-adic results obtained by Cogdell and Piatetski-Shapiro.

1. Introduction

Let F be a p-adic field, π a smooth admissible representation of $\mathrm{GL}(n, F)$. J. Bernstein and A. Zelevinsky [1977] defined the notion of derivatives for π, denoted by $\pi^{(k)}$, $n \geq k \geq 0$, which is a useful tool to study properties of π.

If π' is another smooth admissible representation of $\mathrm{GL}(n, F)$, when both π and π' are generic with associated Whittaker models $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \psi^{-1})$, where ψ is a fixed nontrivial additive character of F, we have the following local Rankin–Selberg integrals:

$$I(s, W, W', \Phi) = \int_{N_n \backslash \mathrm{GL}_n} W(g)W'(g)\Phi(\epsilon_n g)|\det g|^s \, dg$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \bar{\psi})$, $\Phi \in \mathcal{S}(F^n)$ a Schwartz function, s a complex number, and $\epsilon_n = (0, 0, \dots, 1) \in F^n$.

By the work of H. Jacquet, J. Shalika and Piatetski-Shapiro [1983], these integrals converge in some right half-plane of s, and have a meromorphic continuation to the whole plane. Suppose s_0 is a pole with the expansion

$$I(s, W, W', \Phi) = \frac{B_{s_0}(W, W', \Phi)}{(q^s - q^{s_0})^d} + \cdots .$$

Note that the Schwartz function space $\mathcal{S}(F^n)$ has a filtration

$$0 \subset \mathcal{S}^0 \subset \mathcal{S}(F^n),$$

MSC2010: primary 11F70; secondary 22E46.
Keywords: archimedean derivatives, exceptional poles, Rankin–Selberg integrals.

where $\mathcal{S}^0 = \{\Phi \in \mathcal{S}(F^n) : \Phi(0) = 0\}$. Cogdell and Piatetski-Shapiro [≥ 2015] defined s_0 to be an exceptional pole if the leading coefficient $B_{s_0}(W, W', \Phi)$ vanishes identically on $\mathcal{S}^0$, and used it together with derivatives to analyze the poles of local Rankin–Selberg integrals. As a consequence, they can compute the local L-factor for a pair of generic representations on general linear groups in terms of L-functions of the inducing datum.

It is interesting to see if there is an analogous theory for $\mathrm{GL}(n, \mathbb{R})$, and there is in fact some work in this direction; for example, [Chang and Cogdell 1999]. In this paper, we will take one more step towards such an archimedean theory, based on results in that reference. There are a couple of difficulties in the archimedean case. First of all, we need an appropriate theory of "derivatives". In a recent preprint, A. Aizenbud, D. Gourevitch and S. Sahi [Aizenbud et al. 2012] defined the derivatives for smooth representations of moderate growth on $\mathrm{GL}(n, \mathbb{R})$ as the inverse limit of certain coinvariants. But this seems complicated for our applications to local Rankin–Selberg integrals.

Here we simply take the naive analog of p-adic derivatives as our archimedean derivatives. It is a component in the $\mathfrak{n}$-homology, where $\mathfrak{n}$ is the nilradical of some parabolic subalgebra. The advantages of this definition are that it is relatively easier to deal with, and compatible with Rankin–Selberg integrals. But it is also interesting to see if one can relate the derivatives defined in [ibid.] to integrals $I(s, W, W', \Phi)$ in some way.

For the exceptional poles, the situation again is a little more complicated. The leading coefficients in the expansion of $I(s, W, W', \Phi)$ at a pole will involve a finite-dimensional representation of $\mathrm{GL}(n, \mathbb{R})$, due to the nature of the differences between Schwartz functions on $\mathbb{R}$ and the p-adic field F. To be more precise, the Schwartz function space $\mathcal{S}_n = \mathcal{S}_n(\mathbb{R}^n)$ has a natural filtration. Let

$$\mathcal{S}_n^m = \{f \in \mathcal{S} : f \text{ vanishes to order at least } m \text{ at zero}\}.$$

Then each $\mathcal{S}_n^m$ is a closed subspace, and we have a filtration

$$\mathcal{S}_n = \mathcal{S}_n^0 \supset \mathcal{S}_n^1 \supset \cdots \supset \mathcal{S}_n^m \supset \cdots ,$$

where $\mathcal{S}_n^m / \mathcal{S}_n^{m+1}$ is isomorphic to the space of homogeneous polynomials on $\mathbb{R}^n$ of degree m, denoted as E_n^m — a finite-dimensional representation of $\mathrm{GL}(n, \mathbb{R})$.

At a pole s_0, $I(s, W, W', \Phi)$ has an expansion

$$I(s, W, W', \Phi) = \frac{B_{s_0}(W, W', \Phi)}{(q^s - q^{s_0})^d} + \cdots ,$$

and we say s_0 is an exceptional pole of type 1 and level m if B_{s_0} vanishes identically on $\mathcal{S}^{m+1}$, but not on $\mathcal{S}^m$.

In general, we say s_0 is an exceptional pole of type 2 and level m, for π and π', if there exists a continuous trilinear form

$$l : V \times V' \times E_n^m \to \mathbb{C}$$

such that, for $g \in \mathrm{GL}(n, \mathbb{R})$,

$$l(g \cdot W, g \cdot W', g \cdot \Phi_n) = |\det g|^{-s_0} l(W, W', \Phi_n).$$

It follows that an exceptional pole of type 1 is also of type 2.

We can now state our main results. We say π is in general position as in [Chang and Cogdell 1999] (or see Section 2 for more details). We refer to page 294 for a definition of depth of exceptional poles of type 1.

Theorem. *Let π and π' be irreducible generic Casselman–Wallach representations of* $\mathrm{GL}_n(\mathbb{R})$ *and* $\mathrm{GL}_m(\mathbb{R})$ *in general position.*

Case $m = n$: *Any pole of the Rankin–Selberg integrals for π and π' is an exceptional pole of type* 2 *for a pair of components of* $\pi^{(k)}$ *and* $\pi'^{(k)}$, $0 \le k \le n-1$. *On the other hand, any exceptional pole of type* 1 *of depth* 0 *for a pair of components of* $\pi^{(k)}$ *and* $\pi'^{(k)}$, $0 \le k \le n-1$, *is a pole of the Rankin–Selberg integrals of π and π'.*

Case $m < n$: *Any pole of the Rankin–Selberg integrals for π and π' is an exceptional pole of type* 2 *for a pair of components of* $\pi^{(n-k)}$ *and* $\pi'^{(m-k)}$, $1 \le k \le m$. *On the other hand, any exceptional pole of type* 1 *of depth* 0 *for a pair of components of* $\pi^{(n-k)}$ *and* $\pi'^{(m-k)}$, $1 \le k \le m$, *is a pole of the Rankin–Selberg integrals of π and π'.* □

The first remark is that these are not the exact archimedean analog we are seeking. We expect that the poles of Rankin–Selberg integrals are exactly exceptional poles of type 1 for pairs of components of derivatives of π and π'. A missing point here is that we haven't obtained the asymptotic results analogous to those in [Cogdell and Piatetski-Shapiro $\geq$ 2015, Section 1.4]; this will be addressed in the future.

We also remark here that the same ideas and techniques of this paper can also be applied to local exterior square L-integrals in [Jacquet and Shalika 1990]; this will appear in a forthcoming paper.

The paper is organized as follows. In Section 2 we review some preliminaries. In Section 3 we define the derivatives and obtain some basic properties. Section 4 is devoted to the study of exceptional poles. We obtain the main results in Section 5 for $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$, and in Section 6 we discuss the case $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_m(\mathbb{R})$, $m < n$.

2. Notations and preliminaries

In this section, we introduce some notations and results needed in this paper.

Let $G_n = \mathrm{GL}_n(\mathbb{R})$ be the general linear group of invertible $n \times n$ matrices over $\mathbb{R}$, and $K = K_n = O(n)$ be the orthogonal subgroup of G_n, which is a maximal compact subgroup of G_n. We use $\mathfrak{g} = \mathfrak{g}_n$, $\mathfrak{k} = \mathfrak{k}_n$ to denote the complexified Lie algebras of G_n and K_n respectively. Let N_n be the upper triangular unipotent subgroup of G_n. Fix ψ as the additive character of $\mathbb{R}$ given by $\psi(x) = \exp(2\pi\sqrt{-1}x)$, and define a character on N_n, still denoted as ψ, by

$$\psi(u) = \psi\Big(\sum_i u_{i,i+1}\Big),$$

where $u = (u_{ij}) \in N_n$. Let μ be the differential of ψ; then μ is a linear form on $\mathfrak{n}_n$, the Lie algebra of N_n, vanishing on $[\mathfrak{n}_n, \mathfrak{n}_n]$.

A smooth representation (π, V) is called generic if it admits a nontrivial Whittaker functional. A Whittaker functional Λ with respect to μ on (π, V) is a continuous linear functional on V satisfying

$$\Lambda(\pi(X)v) = \mu(X)\Lambda(v)$$

for all $X \in \mathfrak{n}_n$, $v \in V$.

If π is generic, let Λ be the Whittaker functional on π, and for any $v \in V$ define a function $W_v : G_n \to \mathbb{C}$ by $W_v(g) = \Lambda(\pi(g)v)$. Then W_v is called the Whittaker function on G_n corresponding to v, and the space $\mathcal{W}(\pi, \psi) = \{W_v : v \in V\}$ is called the Whittaker model of π.

Throughout the paper, we will work with smooth representations of moderate growth. Suppose V is a Fréchet space. A smooth representation (π, V) is called a representation of moderate growth if, for every seminorm ρ on V, there exists a positive integer N and a seminorm ν such that for every $g \in G_n$, $v \in V$, we have

$$|\pi(g)v|_\rho \le \|g\|^N |v|_\nu,$$

where $\|g\| = \mathrm{Tr}(g\,{}^tg) + \mathrm{Tr}(g^{-1}g^\iota)$ and $g^\iota = {}^tg^{-1}$. If in addition every irreducible representation of K has finite multiplicity in π, we will say π is admissible.

We have the following important result of Casselman and Wallach.

Theorem 2.1. *For any finitely generated admissible $(\mathfrak{g}, K)$-module W, there exists exactly one smooth representation of moderate growth on a Fréchet space V, up to canonical topological isomorphism, such that the underlying $(\mathfrak{g}, K)$-module V_K is isomorphic to W. Moreover, the assignment $W \to V$ is an exact functor from the category of finitely generated admissible modules to the category of smooth admissible finitely generated Fréchet representations of moderate growth.*

Proof. See, for example, [Wallach 1992, Chapter 12]. □

Remark. We refer to V in this theorem as the completion or globalization of W, and we refer to smooth admissible finitely generated Fréchet representations of moderate growth (π, V) as Casselman–Wallach representations.

For irreducible Casselman–Wallach representations, by results of J. Shalika [1974], there exists at most one Whittaker functional with respect to a given nontrivial ψ, unique up to a scalar.

For a given smooth representation V of G_n, and a nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, we use $H_0(\mathfrak{n}, V)$ to denote the quotient of V by the closure of the subspace spanned by $\{X \cdot v : X \in \mathfrak{n}, v \in V\}$. When W is a $(\mathfrak{g}, K)$-module, use $H_0(\mathfrak{n}, W)$ to denote $W/\mathfrak{n}W$. Similarly, if N is a unipotent subgroup of G_n, denote by $H_0(N, V)$ the quotient of V by the closure of the subspace spanned by vectors $\{\pi(u)v - v : u \in N, v \in V\}$.

If (π, V) is an irreducible Casselman–Wallach representation of G_n, V_K denotes its K-finite vectors. If P is a standard parabolic subgroup of G_n, denote its Levi decomposition by $P = MN$, where N is the unipotent subgroup of P and M is the Levi component. Let $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{n}$ be their complexified Lie algebras, respectively. It is a result of B. Casselman that $H_0(\mathfrak{n}, V_K)$ is nonzero. By results of Stafford and N. Wallach it is an admissible $(\mathfrak{m}, K \cap M)$ module. Moreover, it is finitely generated over $U(\mathfrak{m})$; here $U(\mathfrak{m})$ denotes the universal enveloping algebra of $\mathfrak{m}$. See, for example, [Borel and Wallach 2000] for more details.

For $H_0(\mathfrak{n}, V)$, M acts naturally on this quotient, which is also a Fréchet space. This gives a smooth representation of M, which is also of moderate growth.

Naturally $H_0(\mathfrak{n}, V_K)$ embeds into $H_0(\mathfrak{n}, V)$, sending $v + \mathfrak{n}V_K$ to $v + \overline{\mathfrak{n}V}$ for any $v \in V_K$. Moreover, we have the following.

Proposition 2.2. *$H_0(\mathfrak{n}, V)$ is a Casselman–Wallach representation of M, and its $K \cap M$-finite vectors are exactly $H_0(\mathfrak{n}, V_K)$; so it is the completion of $H_0(\mathfrak{n}, V_K)$.*

Proof. The image of the embedding $H_0(\mathfrak{n}, V_K) \to H_0(\mathfrak{n}, V)$ is a $(\mathfrak{m}, K \cap M)$-module, and is dense in $H_0(\mathfrak{n}, V)$. Hence $H_0(\mathfrak{n}, V_K)$ can be identified with the underlying $(\mathfrak{m}, K \cap M)$-module of $H_0(\mathfrak{n}, V)$. As $H_0(\mathfrak{n}, V_K)$ is nonzero, finitely generated and admissible, so is $H_0(\mathfrak{n}, V)$. Hence $H_0(\mathfrak{n}, V)$ is the completion of $H_0(\mathfrak{n}, V_K)$. □

Remark. According to an unpublished result of B. Casselman, $H_*(\mathfrak{n}, V)$ is the completion of $H_*(\mathfrak{n}, V_K)$; see [Bunke and Olbrich 1997, Theorem 1.5].

For any two smooth representations of moderate growth (π, V) and (ρ, W) of G_n and G_m, respectively, denote by $(\pi \widehat{\otimes} \rho, V \widehat{\otimes} W)$ the complete projective tensor product. It is also a smooth representation of moderate growth on $G_n \times G_m$.

Now if π (or π') is an irreducible admissible representation of G_n (or G_m), by the local Langlands correspondence π (or π') corresponds to an n- (or m-) dimensional semisimple representation of the Weil group $W_{\mathbb{R}}$, denoted as ρ (or ρ'). Now consider the tensor product $\rho \otimes \rho'$, which defines a semisimple representation of $W_{\mathbb{R}}$ with dimension mn. Then one can associate a local L-factor, denoted by $L(s, \pi \times \pi')$, to $\rho \otimes \rho'$, which is a product of gamma functions. For more details, see, for example, [Knapp 1994].

For any $W_v \in \mathcal{W}(\pi, \psi)$, define $\widetilde{W}_v(g) = W_v(\omega_n g^\iota)$, where

$$\omega_n = \begin{pmatrix} 0 & \cdots & 1 \\ & \iddots & \\ 1 & \cdots & 0 \end{pmatrix}$$

and $g^\iota = {}^t g^{-1}$. Then by [Jacquet and Shalika 1981], it is known that $\{\widetilde{W}_v : v \in V\}$ is a Whittaker model for $\tilde{\pi}$ with respect to $\bar{\psi}$, the contragredient of π.

To introduce the local Rankin–Selberg integrals, assume (π, V) and (π', V') are generic irreducible Casselman–Wallach representations of G_n and G_m, respectively, with Whittaker models $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \bar{\psi})$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions on $\mathbb{R}^n$.

If $m = n$, set

$$I(s, W, W', \Phi) = \int_{N_n \backslash G_n} W(g) W'(g) \Phi(\epsilon_n g) |\det g|^s \, dg$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \bar{\psi})$, $\Phi \in \mathcal{S}(\mathbb{R}^n)$, and $\epsilon_n = (0, 0, \ldots, 1) \in \mathbb{R}^n$.

If $n > m$, set

$$I(s, W, W') = \int_{N_m \backslash G_m} W \begin{pmatrix} g & 0 \\ 0 & I_{n-m} \end{pmatrix} W'(g) |\det g|^{s - \frac{n-m}{2}} \, dg.$$

In general, for $0 \le j \le n - m - 1$, set

$$I_j(s, W, W') = \int_{M(m \times j, \mathbb{R})} \int_{N_m \backslash G_m} W \begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{n-m-j} \end{pmatrix} W'(g) |\det g|^{s - \frac{n-m}{2}} \, dg \, dX.$$

The following theorem is due to Jacquet and Shalika; see, for example, [Jacquet 2009].

Theorem 2.3. (1) *These integrals converge for* $\mathrm{Re}(s) \gg 0$.

(2) *Each integral has a meromorphic continuation to all* $s \in \mathbb{C}$, *which is a holomorphic multiple of* $L(s, \pi \times \pi')$.

(3) *The following functional equations are satisfied*:

$$I_j(1 - s, \widetilde{W}, \widetilde{W}') = \omega'(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) I_{n-m-1-j}(s, W, W')$$

and

$$I(1 - s, \widetilde{W}, \widetilde{W}', \widehat{\Phi}) = \omega'(-1)^{n-1} I(s, W, W', \Phi),$$

where $\widehat{\Phi}$ is the Fourier transform of Φ, given by

$$\widehat{\Phi}(X) = \int \Phi(Y)\psi(-\operatorname{Tr}({}^tXY))\,dY. \qquad \square$$

Now we recall some results from [Cogdell and Piatetski-Shapiro 2004] which are essential in Section 4, which studies exceptional poles.

The first result is an extension of the Dixmier–Malliavin theorem. Let (π, V) be a continuous representation of G_n on a Fréchet space V. Still use π to denote the smooth representation of G_n induced from π on the smooth vectors V^∞ of V.

Proposition 2.4 [Cogdell and Piatetski-Shapiro 2004, Proposition 1.1]. *Let $v_k \to v_0$ be a convergent sequence in V^∞. Then there exists a finite set of functions $f_j \in \mathscr{C}_c^\infty(G_n)$ and a collection of vectors $v_{k,j} \in V^\infty$ such that $v_k = \sum_j \pi(f_j)v_{k,j}$ for all $k \geq 0$, and such that $v_{k,j} \to v_{0,j}$ as $k \to \infty$ for each j.* $\square$

The second result is about the continuity of archimedean Rankin–Selberg integrals. Let (π, V) and (π', V') be generic irreducible Casselman–Wallach representations of G_n and G_m, respectively. For $W \in \mathscr{W}(\pi, \psi)$, $W' \in \mathscr{W}(\pi', \psi^{-1})$, $\Phi \in \mathscr{S}(\mathbb{R}^n)$, we have:

Theorem 2.5. *The linear functionals*

$$\Lambda_s = \frac{I(s, W, W')}{L(\pi \times \pi')}, \qquad n > m,$$
$$\Lambda_s = \frac{I(s, W, W', \Phi)}{L(\pi \times \pi')}, \quad n = m,$$

are uniformly continuous in s on compact sets with respect to the topologies involved.

Proof. See [Cogdell and Piatetski-Shapiro 2004, Theorem 1.1]. $\square$

Remark. As noted in [Cogdell and Piatetski-Shapiro 2004], here we claim the result is true for all s.

To end this section, let's explain irreducible representations in general position, following [Chang and Cogdell 1999]. Let $P = MN$ be a parabolic subgroup of G_n, with $M = G_1^p \times G_2^q$ and $p + 2q = n$. Write C_2 for the cyclic group $\{\pm 1\}$, $G_1 \simeq \mathbb{R}_{>0} \times C_2$ and $G_2 \simeq \mathbb{R}_{>0} \times \mathrm{SL}_2^\pm$, where $\mathrm{SL}_2^\pm$ stands for the subgroup of G_2 consisting of matrices with determinant ± 1. So $M = (\mathbb{R}_{>0})^{p+q} \times C_2^p \times (\mathrm{SL}_2^\pm)^q$.

Let T_m be the discrete series of $\mathrm{SL}_2^\pm$ with parameter $m \in \mathbb{Z}_{>0}$. We will use notation $(s_1, \ldots, s_p)$ to denote the character on $(\mathbb{R}_{>0})^p$ sending $(x_1, \ldots, x_p)$ to $\prod_{i=1}^p x_i^{s_i}$. And let ϵ be a character on C_2. Then form the tensor product

$$\sigma = (s_1, \ldots, s_p, 2t_1, \ldots, 2t_q) \otimes (\epsilon_1 \otimes \cdots \otimes \epsilon_p \otimes T_{m_1} \otimes \cdots \otimes T_{m_q}).$$

This is a representation on M, and then we get the normalized parabolic induced representation $\mathrm{Ind}(\sigma)$. We say $\pi = \mathrm{Ind}(\sigma)$ is a representation in general position if

$$s_i, t_j, s_i - s_j \notin \mathbb{Z} \ \text{ for } i \neq j, \quad t_i - t_j \notin \tfrac{1}{2}\mathbb{Z} \ \text{ for } i \neq j, \quad s_i - t_j \notin \tfrac{1}{2}\mathbb{Z}.$$

It is known that these induced representations are irreducible and generic, see [Chang and Cogdell 1999] for more information.

3. Archimedean derivatives

In this section we introduce archimedean derivatives. First we need more notation. For any $1 \le l \le n$, let U_{n-l+1} be the unipotent radical of the standard parabolic subgroup associated to the partition $(n-l, 1, \dots, 1)$, that is, the subgroup of N_n consisting of matrices having the form

$$\begin{pmatrix} I_{n-l} & x \\ 0 & u \end{pmatrix},$$

where x is a $(n-l) \times l$ matrix and $u \in N_l$ is an upper triangular matrix with 1 on the diagonal. Note that $U_1 = N_n$. Denote by $\mathfrak{u}_{n-l+1}$ the corresponding Lie algebras. Define a linear form μ_{n-l+1} on each $\mathfrak{u}_{n-l+1}$ by

$$\mu_{n-l+1}(X) = \mu(X_{n-l+1,n-l+2} + \cdots + X_{n-1,n}).$$

Now let (π, V) be a Casselman–Wallach representation of G_n, V_K its underlying $(\mathfrak{g}, K)$-module. For $1 \le l \le n$, let V_l be the closure of the subspace spanned by $\{X \cdot v - \mu_{n-l+1}(X)v : v \in V, X \in \mathfrak{u}_{n-l+1}\}$.

Definition. For each integer $0 \le l \le n$, we define the l-th derivative of π, denoted by $(\pi^{(l)}, V^{(l)})$, as follows:

(1) If $l = 0$, put $(\pi^{(0)}, V^{(0)}) = (\pi, V)$.

(2) If $1 \le l \le n$, put $V^{(l)} = V/V_l$, and define the action $\pi^{(l)}$ by

$$\pi^{(l)}(g) \cdot (v + V_l) = |\det g|^{-l/2} \pi(g)v + V_l \quad \text{for any } g \in G_{n-l}.$$

To continue, we need more notation. Use $P_{n-l,l}$ to denote the standard parabolic subgroup of G_n associated to the partition $(n-l, l)$ of n. It has Levi decomposition $P_{n-l,l} = M_{n-l,l} N_{n-l,l}$, with Levi component $M_{n-l,l}$ isomorphic to $G_{n-l} \times G_l$ and unipotent part $N_{n-l,l}$. Let $\mathfrak{p}_{n-l,l}$, $\mathfrak{m}_{n-l,l}$ and $\mathfrak{n}_{n-l,l}$ be their complexified Lie algebras, respectively.

Note that we have the decomposition

$$\begin{pmatrix} I_{n-l} & x \\ 0 & v \end{pmatrix} = \begin{pmatrix} I_{n-l} & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} I_{n-l} & x \\ 0 & I_l \end{pmatrix},$$

so we can write $U_{n-l+1} = V_{n-l+1} N_{n-l,l}$, where V_{n-l+1} is the standard unipotent subgroup N_l of G_l embedded in G_n in the right lower corner.

Let $\mathfrak{v}_{n-l+1}$ be the complexified Lie algebra of V_{n-l+1}. Note that the character μ_{n-l+1} is trivial on $\mathfrak{n}_{n-l,l}$. Let Y_l be the closure of the space spanned by

$$\{X \cdot \bar{v} - \mu_{n-l+1}(X)\bar{v} : X \in \mathfrak{v}_{n-l+1}, \bar{v} \in H_0(\mathfrak{n}_{n-l,l}, V)\}.$$

Since $V^{(l)} = V/V_l$ and

$$V_l = \overline{\{X \cdot v - \mu_{n-l+1}(X)v : v \in V, X \in \mathfrak{u}_{n-l+1}\}},$$

note that $\mathfrak{u}_{n-l+1} = \mathfrak{v}_{n-l+1} + \mathfrak{n}_{n-l,l}$. Then

$$\begin{aligned} &H_0(\mathfrak{n}_{n-l,l}, V)/Y_l \\ &\quad = (V/\overline{\mathfrak{n}_{n-l,l}V})\big/\big(\overline{\{X \cdot v - \mu_{n-l+1}(X)v : X\in\mathfrak{u}_{n-l+1}, v\in V\}}/\overline{\mathfrak{n}_{n-l,l}V}\big) = V/V_l. \end{aligned}$$

Thus, we have verified the following proposition.

Proposition 3.1. $\qquad V^{(l)} = H_0(\mathfrak{n}_{n-l,l}, V)/Y_l.$ □

The following result states that the derivatives $\pi^{(l)}$ belong to a nice class of representations.

Proposition 3.2. *For each l, $\pi^{(l)}$ is a Casselman–Wallach representation of G_{n-l}.*

Proof. This follows from the fact that the $\mathfrak{n}$-homology $H_0(\mathfrak{n}, V)$ is admissible. □

Now assume (π, V) is an irreducible smooth admissible generic representation of moderate growth on G_n in general position as in Section 2. Denote by V_K its K-finite vectors, which is an irreducible admissible $(\mathfrak{g}, K)$-module. For the rest of this section, unless otherwise stated, we will drop the subscript for the standard upper triangular parabolic subgroup $P = MN$ associated with the partition $(n-k, k)$ of n, to simplify notation.

By [Chang and Cogdell 1999, Theorem 4.2] the $\mathfrak{n}$-homology $V_K/\mathfrak{n}V_K$ is nonzero and is a semisimple $(\mathfrak{m}, K \cap M)$-module. By Proposition 2.2, $V/\overline{\mathfrak{n}V}$ is the smooth completion of $V_K/\mathfrak{n}V_K$. It follows that $V/\overline{\mathfrak{n}V}$ is also semisimple, so we can write

$$V/\overline{\mathfrak{n}V} = \bigoplus_{i=1}^{r} A_i,$$

where each A_i is an irreducible smooth admissible representation of moderate growth on M and hence, by results of D. Gourevitch and A. Kemarsky [2013], isomorphic to $\rho_i \widehat{\otimes} \sigma_i$, where each ρ_i and σ_i are irreducible smooth representations of moderate growth on G_{n-k} and G_k, respectively. Note that it is possible to have $A_i \cong A_j$ for $i \neq j$. We use $\rho_{i,K}$ and $\sigma_{i,K}$ to denote the representations on the underlying K-finite modules. Let p_i be the natural projection from $V_K/\mathfrak{n}V_K$ onto

$\rho_{i,K} \otimes \sigma_{i,K}$, and also be the projection from $V/\overline{\mathfrak{n}V}$ onto $\rho_i \widehat{\otimes} \sigma_i$. We will also use p to denote the projections $V \to V/\overline{\mathfrak{n}V}$.

Lemma 3.3. *For each i, ρ_i and σ_i are generic representations.*

Proof. This follows from [Chang and Cogdell 1999, Theorem 4.2]. See Remarks 4.3 there. ☐

Denote by $\mathcal{W}(\rho_i, \psi)$ the Whittaker model for ρ_i.

Proposition 3.4. *For every $W_i \in \mathcal{W}(\rho_i, \psi)$ and every $\Phi \in \mathcal{S}(\mathbb{R}^{n-k})$, there is a Whittaker function $W_v \in \mathcal{W}(\pi, \psi)$ such that*

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = W_i(g)\Phi(\epsilon_{n-k}g).$$

Proof. The projection p_i from $V_K/\mathfrak{n}V_K$ onto $\rho_{i,K} \otimes \sigma_{i,K}$ induces an injective intertwining map $V_K \to \mathrm{Ind}(|\det|^{-k/2}\rho_{i,K} \otimes |\det|^{(n-k)/2}\sigma_{i,K})$. This extends to an injective map

$$V \to \mathrm{Ind}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i).$$

Denote by Q its quotient; we have a short exact sequence of smooth representations of moderate growth

$$0 \longrightarrow V \longrightarrow \mathrm{Ind}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i) \longrightarrow Q \longrightarrow 0. \tag{1}$$

The underlying $(\mathfrak{g}, K)$-modules also form a short exact sequence

$$0 \longrightarrow V_K \longrightarrow \mathrm{Ind}(|\det|^{-k/2}\rho_{i,K} \otimes |\det|^{(n-k)/2}\sigma_{i,K}) \longrightarrow Q_K \longrightarrow 0. \tag{2}$$

By taking the dual (contragredient representation) of the short exact sequence (2), we have

$$0 \longrightarrow Q_K^* \longrightarrow \big(\mathrm{Ind}(|\det|^{-k/2}\rho_{i,K} \otimes |\det|^{(n-k)/2}\sigma_{i,K})\big)^* \longrightarrow V_K^* \longrightarrow 0.$$

By [Wallach 1988, Lemma 4.5.2], we have

$$0 \longrightarrow Q_K^* \longrightarrow \mathrm{Ind}\big((|\det|^{-k/2}\rho_{i,K})^* \otimes (|\det|^{(n-k)/2}\sigma_{i,K})^*\big) \longrightarrow V_K^* \longrightarrow 0,$$

which induces a short exact sequence for their smooth completions:

$$0 \longrightarrow Q^* \longrightarrow \mathrm{Ind}\big((|\det|^{-k/2}\rho_i)^* \widehat{\otimes} (|\det|^{(n-k)/2}\sigma_i)^*\big) \longrightarrow V^* \longrightarrow 0. \tag{3}$$

Now for any representation (τ, U), define representation (τ^s, U) by $\tau^s(g) \cdot u = \tau({}^t g^{-1}) \cdot u$ for any $g \in G_n$, $u \in U$; then τ^s is isomorphic to τ^* when τ is irreducible, by [Aizenbud et al. 2008, Theorem 2.4.2]. Note that we are working in the same space, but simply changing the action. So if we have a short exact sequence

$$0 \longrightarrow (\tau_1, U_1) \longrightarrow (\tau_2, U_2) \longrightarrow (\tau_3, U_3) \longrightarrow 0,$$

applying the operation 's', we then have a new exact sequence

$$0 \longrightarrow (\tau_1^s, U_1) \longrightarrow (\tau_2^s, U_2) \longrightarrow (\tau_3^s, U_3) \longrightarrow 0.$$

Now apply operation 's' to the sequence (3); then we have

$$0 \longrightarrow (Q^*)^s \longrightarrow \big(\mathrm{Ind}((|\det|^{-k/2}\rho_i)^* \widehat{\otimes} (|\det|^{(n-k)/2}\sigma_i)^*)\big)^s \longrightarrow (V^*)^s \longrightarrow 0. \tag{4}$$

It follows that the sequence (4) becomes

$$0 \longrightarrow (Q^*)^s \longrightarrow \mathrm{Ind}_{P'}^{G_n}\big((|\det|^{-k/2}\rho_i)^{*s} \widehat{\otimes} (|\det|^{(n-k)/2}\sigma_i)^{*s}\big) \longrightarrow (V^*)^s \longrightarrow 0.$$

Since π, ρ_i and σ_i are irreducible, the above is

$$0 \longrightarrow (Q^*)^s \longrightarrow \mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i) \longrightarrow V \longrightarrow 0.$$

Let Λ be the unique (up to a constant) continuous Whittaker functional on V. Composed with the projection

$$\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i) \longrightarrow V,$$

we get a nontrivial continuous Whittaker functional Λ' on

$$\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i).$$

By the last conclusion of [Wallach 1992, Theorem 15.4.1], there is a linear bijection between the space of Whittaker functionals on

$$\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i)$$

and the space of Whittaker functionals on

$$|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i.$$

By Lemma 3.3, the latter space has dimension 1, thus there is a unique (up to a constant) continuous Whittaker functional on $\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i)$, and it must be Λ'. Then we can conclude that the space of Whittaker functions $\mathcal{W}(\pi, V)$ for π and that for $\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i)$, are the same.

So in order to prove the existence of W_v in $\mathcal{W}(\pi, V)$ as in the proposition, it suffices to find some Whittaker function for $\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \widehat{\otimes} |\det|^{(n-k)/2}\sigma_i)$ with the required property. Now this follows from [Jacquet 2009, Proposition 14.1], which finishes the proof. □

Corollary 3.5. *For every Whittaker function W_i in any irreducible component of $\pi^{(k)}$, and any Schwartz function Φ on $\mathbb{R}^{n-k}$, we can always find some $W_v \in \mathcal{W}(\pi, \psi)$ such that*

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = W_i(g)\Phi(\epsilon_{n-k}g)|\det g|^{k/2}.$$

Proof. This follows from the fact that $\pi^{(k)}$ is isomorphic to $|\det|^{-k/2}\bigoplus_i \rho_i$. $\square$

4. Exceptional poles

In this section, we will introduce two types of exceptional poles and discuss their basic properties. Set

$$\mathcal{S}_n^m=\{f\in\mathcal{S}: f \text{ vanishes to order at least } m \text{ at zero}\};$$

then we have a filtration of closed subspaces for the Schwartz function space $\mathcal{S}_n=\mathcal{S}_n(\mathbb{R}^n)$:

$$\mathcal{S}_n=\mathcal{S}_n^0\supset\mathcal{S}_n^1\supset\cdots\supset\mathcal{S}_n^m\supset\cdots.$$

$\mathcal{S}_n^m/\mathcal{S}_n^{m+1}$ is isomorphic to the space of homogeneous polynomials on $\mathbb{R}^n$ of degree m, denoted by E_n^m. The group G_n acts on $\mathcal{S}_n$ from the right, which preserves this filtration, and therefore induces an action on E_n^m.

Let π and π' be irreducible generic Casselman–Wallach representations on G_n. The Rankin–Selberg integrals for π and π', are given by

$$I(s,W,W',\Phi)=\int_{N_n\backslash G_n} W(g)W'(g)\Phi(\epsilon_n g)|\det g|^s\,dg$$

for $W\in\mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\psi^{-1})$, $\Phi\in\mathcal{S}$, where $\epsilon_n=(0,0,\dots,1)\in\mathbb{R}^n$, $s\in\mathbb{C}$. By Theorem 2.3, these integrals converge when s is in some right half-plane, and have a meromorphic continuation to the whole complex plane.

For any integer $1\le k\le n$, for $v\in\pi$, $v'\in\pi'$ and $\Phi\in\mathcal{S}_k$, we define the following family of integrals:

$$I_k(s,W_v,W_{v'},\Phi)=\int_{N_k\backslash G_k} W_v\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix} W_{v'}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}\Phi(\epsilon_k g)|\det g|^{s-n+k}\,dg.$$

Lemma 4.1. *The integrals I_k belong to the space of Rankin–Selberg integrals for π and π'.*

Proof. This follows from [Jacquet 2009, Proposition 6.1 and Lemma 14.1]. $\square$

Thus it follows that I_k converges when $\mathrm{Re}(s)$ is large and has a meromorphic continuation to the whole complex plane. Suppose s_0 is a pole of order d for the integral $I_k(s,W,W',\Phi)$, with Laurent expansion

$$I_k(s,W,W',\Phi)=\frac{B_{s_0,k}(W,W',\Phi)}{(s-s_0)^d}+\cdots,$$

where $B_{s_0,k}(W,W',\Phi)$ is a trilinear form on $V\times V'\times\mathcal{S}_k$ satisfying the following invariance property:

$$B_{s_0,k}(g\cdot W,g\cdot W',g\cdot\Phi)=|\det g|^{-s_0+n-k}B_{s_0,k}(W,W',\Phi)$$

for any $g \in G_k$, $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, $\Phi \in \mathcal{S}_k$.

Proposition 4.2. *The trilinear form $B_{s_0,k}$ is continuous with respect to the topologies involved.*

Proof. When $k = n$, the continuity of $B_{s_0,n}$ follows from Theorem 2.5. When $k < n$, we will use the constructions in the proof of [Jacquet 2009, Lemma 14.1] to prove the continuity.

Now suppose $v_l \to v$, $v'_l \to v'$ and $\Phi_l \to \Phi$; then write

$$I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \frac{B_{s_0,k}(v_l, v'_l, \Phi_l)}{(s - s_0)^d} + \cdots$$

and

$$I_k(s, W_v, W_{v'}, \Phi) = \frac{B_{s_0,k}(v, v', \Phi)}{(s - s_0)^d} + \cdots .$$

Then we want to show that $B_{s_0,k}(v_l, v'_l, \Phi_l) \to B_{s_0,k}(v, v', \Phi)$ as $l \to \infty$.

Let Ψ_l and Ψ be Schwartz functions on $\mathbb{R}^k$ whose Fourier transforms are given by $\widehat{\Psi}_l = \Phi_l$, $\widehat{\Psi} = \Phi$. Since Fourier transform is a topological isomorphism on Schwartz function space, it follows that $\Psi_l \to \Psi$. Now we set

$$u_l = \int \pi \begin{pmatrix} I_k & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} v_l \Psi_l(x)\, dx$$

and

$$u = \int \pi \begin{pmatrix} I_k & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} v \Psi(x)\, dx.$$

Claim 1. *If f is a Schwartz function on $\mathbb{R}^k$, the map $(f, v) \mapsto \pi(f)v$ is a continuous map from $V \times \mathcal{S}_k$ to V, where*

$$\pi(f)v = \int_{\mathbb{R}^k} f(x)\pi(x)v\, dx.$$

Proof of Claim 1. Suppose $f_l \to f$ in $\mathcal{S}_k$, $v_l \to v$ in V. We want to show that $\pi(f_l)v_l \to \pi(f)v$.

Because (π, V) is of moderate growth, for any seminorm $|\cdot|_1$ on V there exists a seminorm $|\cdot|_2$ on V, a positive integer N_0, and a positive number C, such that for any $v \in V$ and $x \in \mathbb{R}^k$, we have $|\pi(x)v|_1 \le C(1 + \|x\|^2)^{N_0}|v|_2$. Here we identify x with

$$\begin{pmatrix} I_k & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} \in G_n,$$

and $\|x\|$ denotes the usual Euclidean norm of x. Then we have

$$\begin{aligned}|\pi(f_l)v_l-\pi(f)v|_1 &\le |\pi(f_l)v_l-\pi(f)v_l|_1+|\pi(f)v_l-\pi(f)v|_1\\ &\le \int |f_l(x)-f(x)\|\pi(x)v_l|_1\,dx+\int |f(x)\|\pi(x)(v_l-v)|_1\,dx\\ &\le C|v_l|_2\int |f_l(x)-f(x)|(1+\|x\|^2)^{N_0}\,dx\\ &\qquad\qquad +C|v_l-v|_2\int |f(x)|(1+\|x\|^2)^{N_0}\,dx.\end{aligned}$$

Since $v_l\to v$, $|v_l|_2$ is bounded for any l, and $|v_l-v|_2\to 0$ as $l\to\infty$. Because $f_l\to f$ in $\mathcal{S}_k$,

$$\int |f_l(x)-f(x)|(1+\|x\|^2)^{N_0}\,dx\to 0 \quad \text{as} \quad l\to\infty.$$

Hence $\pi(f_l)v_l\to\pi(f)v$ as $l\to\infty$, which proves the claim. □

So, by Claim 1, $u_l\to u$. And by the first conclusion of [Jacquet 2009, Proposition 6.1], we have

$$W_{u_l}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}=W_{v_l}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}\Phi_l(\epsilon_k g)$$

and

$$W_{u}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}=W_{v}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}\Phi(\epsilon_k g).$$

Thus

$$I_k(s,W_{v_l},W_{v'_l},\Phi_l)=\int W_{u_l}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}W_{v'_l}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}|\det g|^{s-n+k}\,dg$$

and

$$I_k(s,W_{v},W_{v'},\Phi)=\int W_{u}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}W_{v'}\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}|\det g|^{s-n+k}\,dg.$$

We will view $w_l=u_l\otimes v'_l$ as an element in $\sigma=\pi\widehat{\otimes}\pi'$; consequently $W_{w_l}(g)=W_{u_l}(g)W_{v'_l}(g)\in\mathcal{W}(\pi\widehat{\otimes}\pi',\psi\otimes\psi^{-1})$, and we have

$$I_k(s,W_{v_l},W_{v'_l},\Phi_l)=\int W_{w_l}\left(\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix},\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix}\right)|\det g|^{s-n+k}\,dg. \tag{5}$$

Similarly, write $w = u \otimes v' \in \sigma = \pi \widehat{\otimes} \pi'$; then we have $w_l \to w$ and

$$I_k(s, W_v, W_{v'}, \Phi) = \int W_w \left(\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} \right) |\det g|^{s-n+k}\, dg. \tag{6}$$

Now by Proposition 2.4[1] applied to the group $\mathbb{R}^k \times \mathbb{R}^\times$, there exists a finite set of functions $f_j(x, h) \in \mathscr{C}_c^\infty(\mathbb{R}^k \times \mathbb{R}^\times)$ and vectors $w_{l,j} \in \pi \widehat{\otimes} \pi'$ with $l \geq 0$ such that

$$w_l = \sum_j \sigma(f_j) w_{l,j} \quad \text{for all } l \geq 1, \quad w = \sum_j \sigma(f_j) w_{0,j},$$

and $w_{l,j} \to w_{0,j}$ for each j.

More precisely, we can write

$$w_l = \sum_j \int \sigma \left(\begin{pmatrix} a^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} a^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} \right) w_{l,j}\, f_j(x,h)\, dx\, d^\times h$$

and

$$w = \sum_j \int \sigma \left(\begin{pmatrix} a^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} a^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} \right) w_{0,j}\, f_j(x,h)\, dx\, d^\times h,$$

where $a(h) = \operatorname{diag}(h, 1, \ldots, 1)$.

Then the integrals (5) and (6) now become

$$\begin{aligned} &I_k(s, W_{v_l}, W_{v_l'}, \Phi_l) \\ &\quad = \sum_j \int W_{w_{l,j}} \left(\begin{pmatrix} ga^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} ga^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} \right) \\ &\qquad\qquad \times f_j(x, h) |\det g|^{s-n+k}\, dg\, dx\, d^\times h, \end{aligned}$$

and

$$\begin{aligned} &I_k(s, W_v, W_{v'}, \Phi) \\ &\quad = \sum_j \int W_{w_{0,j}} \left(\begin{pmatrix} ga^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} ga^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} \right) \\ &\qquad\qquad \times f_j(x, h) |\det g|^{s-n+k}\, dg\, dx\, d^\times h. \end{aligned}$$

[1] There is a change of topology for the convergence in Proposition 2.4 in general, but in our special case considered here, the topologies involved are the same.

Make the change of variable $ga^{-1}(h) \to g$; we have the integrals

$$(7)\quad I_k(s, W_{v_l}, W_{v_l'}, \Phi_l) = \sum_j \int W_{w_{l,j}}\left(\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}\right) \times f_j(x,h)|\det g|^{s-n+k}|h|^{s+1-n+k}\,dg\,dx\,d^\times h$$

and

$$(8)\quad I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int W_{w_{0,j}}\left(\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}\right) \times f_j(x,h)|\det g|^{s-n+k}|h|^{s+1-n+k}\,dg\,dx\,d^\times h.$$

Now we will view $f_j(x,h)$ as Schwartz functions on $\mathbb{R}^{k+1}$ which vanish on $\mathbb{R}^k \times \{0\}$. Then let

$$e_{l,j} = \int \sigma\left(\begin{pmatrix} I_{k+1} & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-2}\end{pmatrix}, \begin{pmatrix} I_{k+1} & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-2}\end{pmatrix}\right) w_{l,j} f_j(y)\,dy$$

and

$$e_{0,j} = \int \sigma\left(\begin{pmatrix} I_{k+1} & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-2}\end{pmatrix}, \begin{pmatrix} I_{k+1} & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-2}\end{pmatrix}\right) w_{0,j} f_j(y)\,dy,$$

where $y = (x,h) \in \mathbb{R}^{k+1}$.

Thus it follows that, $e_{l,j} \to e_{0,j}$ for each j, and we have

$$(9)\quad I_k(s, W_{v_l}, W_{v_l'}, \Phi_l) = \sum_j \int W_{e_{l,j}}\left(\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}\right) \times |\det g|^{s-n+k}|h|^{s+1-n+k}\,dg\,dx\,d^\times h$$

and

$$(10)\quad I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int W_{e_{0,j}}\left(\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}\right) \times |\det g|^{s-n+k}|h|^{s+1-n+k}\,dg\,dx\,d^\times h.$$

As in [Jacquet 2009, Lemma 14.1],

$$f \to \int f \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} dx |\det g|^{-1} \, d^{\times}h$$

gives an invariant measure on $N_{k+1} \backslash G_{k+1}$. Thus, we can rewrite these integrals as

$$(11) \quad I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \sum_j \int_{N_{k+1} \backslash G_{k+1}} W_{e_{l,j}} \left(\begin{pmatrix} g & 0 \\ 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & I_{n-k-1} \end{pmatrix} \right) \times |\det g|^{s+1-n+k} \, dg$$

and

$$(12) \quad I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int_{N_{k+1} \backslash G_{k+1}} W_{e_{0,j}} \left(\begin{pmatrix} g & 0 \\ 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & I_{n-k-1} \end{pmatrix} \right) \times |\det g|^{s+1-n+k} \, dg,$$

which are the same type integrals as (5) and (6) belonging to I_{k+1}.

So by induction, we may assume $k = n-1$ in the integrals (5) and (6); then integrals (7) and (8) now become

$$(13) \quad I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \sum_j \int W_{w_{l,j}} \left(\begin{pmatrix} g & 0 \\ x & h \end{pmatrix}, \begin{pmatrix} g & 0 \\ x & h \end{pmatrix} \right) f_j(x, h) |\det g|^{s-1} |h|^s \, dg \, dx \, d^{\times}h$$

and

$$(14) \quad I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int W_{w_{0,j}} \left(\begin{pmatrix} g & 0 \\ x & h \end{pmatrix}, \begin{pmatrix} g & 0 \\ x & h \end{pmatrix} \right) f_j(x, h) |\det g|^{s-1} |h|^s \, dg \, dx \, d^{\times}h.$$

Write

$$g' = \begin{pmatrix} g & 0 \\ x & h \end{pmatrix} \in G_n,$$

and view $f_j(x, h)$ as Schwartz functions on $\mathbb{R}^n$ which vanish on $\mathbb{R}^{n-1} \times \{0\}$; then the above integrals now become

$$I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \sum_j \int W_{w_{l,j}}(g', g') f_j(\epsilon_n g') |\det g'|^s |\det g'|^{-1} \, dg' \, dx \, d^{\times}h$$

and

$$I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int W_{w_{0,j}}(g', g') f_j(\epsilon_n g') |\det g'|^s |\det g'|^{-1} \, dg' \, dx \, d^\times h.$$

Again, as in [Jacquet 2009, Lemma 14.1],

$$f \mapsto \int f(g') \, dx |\det g'|^{-1} \, d^\times h$$

gives an invariant measure on $N_n \setminus G_n$. We can rewrite the above integrals as

$$I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \sum_j \int_{G_n} W_{w_{l,j}}(g', g') f_j(\epsilon_n g') |\det g'|^s \, dg' \tag{15}$$

and

$$I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int_{G_n} W_{w_{0,j}}(g', g') f_j(\epsilon_n g') |\det g'|^s \, dg'. \tag{16}$$

It follows that

$$B_{s_0,k}(v_l, v'_l, \Phi_l) = \sum_j B_{s_0,n}(w_{l,j}, f_j),$$

and similarly

$$B_{s_0,k}(v, v', \Phi) = \sum_j B_{s_0,n}(w_{0,j}, f_j).$$

Since $w_{l,j} \to w_{0,j}$ for each j, and the form $B_{s_0,n}$ is continuous, we conclude that $B_{s_0,k}$ is continuous. This completes the proof. □

Definition. We say a pole s_0 is an exceptional pole of type 1, with level m and depth $n-k$, if the corresponding $B_{s_0,k}$ is zero on $\mathcal{S}_k^{m+1}$, but not identically zero on $\mathcal{S}_k^m$. In this case, we also say s_0 is an exceptional pole for the integrals $I_k(s, W_v, W_{v'}, \Phi)$.

Remark. If s_0 is an exceptional pole of order m, then B_{s_0} defines a continuous linear form on $V \times V' \times E_k^m$ such that, for any $g \in G_k$,

$$B_{s_0,k}(g \cdot W, g \cdot W', g \cdot \Phi) = |\det g|^{-s_0+n-k} B_{s_0,k}(W, W', \Phi).$$

Definition. We say a complex number s_0 is an exceptional pole of type 2, with level m, for π and π', if there exists a continuous trilinear form

$$l : V \times V' \times E_n^m \to \mathbb{C}$$

such that for $g \in G_n$,

$$l(g \cdot W, g \cdot W', g \cdot \Phi_n) = |\det g|^{-s_0} l(W, W', \Phi_n).$$

Remark. It follows that an exceptional pole of type 1 with level m and depth 0 is also of type 2 with level m.

Next we want to relate the exceptional poles for the integrals I_k to the exceptional poles of type 2 for the components of $\pi^{(n-k)}$ and $\pi'^{(n-k)}$.

Lemma 4.3. *If $X = (X_{ij}) \in \mathfrak{n}_{k,n-k}$, then there exists a linear form P_X on $\mathbb{R}^k$ such that for any $v \in V$ we have*

$$W_{\pi(X)\cdot v}\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} = P_X(\epsilon_k g) W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix}.$$

Proof. First, it is easy to see that

$$\begin{aligned} W_{\pi(X)\cdot v}\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} &= \frac{d}{dt}\Big|_{t=0} W_v\left(\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix}\begin{pmatrix} I_k & tX \\ 0 & I_{n-k}\end{pmatrix}\right) \\ &= 2\pi\sqrt{-1}\sum_{j=1}^{k} g_{kj} X_{j,k+1} W_v\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix}. \end{aligned}$$

So define a linear form $P_X(a_1, \ldots, a_k) = 2\pi\sqrt{-1}\sum_{j=1}^{k} X_{j,k+1} a_j$ on $\mathbb{R}^k$; then $P_X(\epsilon_k g) = 2\pi\sqrt{-1}\sum_{j=1}^{k} g_{kj} X_{j,k+1}$, which proves the lemma. □

Proposition 4.4. *Let s_0 be an exceptional pole of level m for the integrals I_k; then the continuous trilinear form $B_{s_0,k}$ defines a continuous trilinear form on $V/\overline{\mathfrak{n}V} \times V'/\overline{\mathfrak{n}V'} \times E_k^m$.*

Proof. It suffices to show that the form $B_{s_0,k}$ vanishes on $\overline{\mathfrak{n}V}$ and $\overline{\mathfrak{n}V'}$ when restricted to $\mathcal{S}_k^m$.

For any $W_{\pi(X)\cdot v}$, $X \in \mathfrak{n}$, any $W_{v'}$ and any $\Phi \in \mathcal{S}_k^m$, by Lemma 4.3 we have

$$W_{\pi(X)\cdot v}\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} = P_X(\epsilon_k g) W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix}$$

for some linear form P_X on $\mathbb{R}^k$.

It follows that

$$\begin{aligned} &I_k(s, W_{\pi(X)\cdot v}, W_{v'}, \Phi) \\ &\quad= \int W_{\pi(X)\cdot v}\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} W_{v'}\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} \Phi(\epsilon_k g)|\det g|^{s-k+n}\, dg \\ &\quad= \int_{N_k\backslash G_k} W_v\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} W_{v'}\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} \Psi(\epsilon_k g)|\det g|^{s-k+n}\, dg, \end{aligned}$$

where $\Psi_k(\epsilon_k g) = P_X(\epsilon_k g)\Phi(\epsilon_k g)$.

Since $\Phi \in \mathcal{S}_k^m$, thus $\Psi = P_X \Phi \in \mathcal{S}_k^{m+1}$. Note that s_0 is an exceptional pole with level m, so

$$B_{s_0,k}(W_{\pi(X)\cdot v}, W_{v'}, \Phi) = B_{s_0,k}(W_{\cdot v}, W_{v'}, \Psi_k) = 0.$$

Similarly, $B_{s_0,k}$ vanishes when $v' \in \overline{\mathfrak{n}V'}$. Thus the proposition follows. $\square$

Theorem 4.5. *If s_0 is an exceptional pole of type 1 with level m and depth $n-k$, then s_0 is an exceptional pole of type 2 with level m for some components of $\pi^{(n-k)}$ and $\pi'^{(n-k)}$.*

Proof. Note that we have the decompositions

$$V/\overline{\mathfrak{n}V} = \bigoplus_i (\rho_i, A_i)\widehat{\otimes}(\sigma_i, B_i)$$

and

$$V'/\overline{\mathfrak{n}V'} = \bigoplus_i (\rho'_i, A'_i)\widehat{\otimes}(\sigma'_i, B'_i).$$

By Proposition 4.4, if s_0 is an exceptional pole of level m for I_k, $B_{s_0,k}$ defines a nontrivial continuous trilinear form on $V/\overline{\mathfrak{n}V} \times V'/\overline{\mathfrak{n}V'} \times E_k^m$. Thus it has to be nontrivial on some components

$$B_{s_0,k} : (\rho_i, A_i)\widehat{\otimes}(\sigma_i, B_i) \times (\rho'_j, A'_j)\widehat{\otimes}(\sigma'_j, B'_j) \times E_k^m \to \mathbb{C},$$

which implies it is also nontrivial on the subspace $A_i \otimes B_i \times A'_i \otimes B'_i \times E_k^m$.

Now fix $v_2 \in B_i$, $v'_2 \in B'_i$, so that $B_{s_0,k}$ is nontrivial on $A_i \otimes v_2 \times A'_i \otimes v'_2 \times E_k^m$. Then the restriction of $B_{s_0,k}$ to this subspace induces a nontrivial continuous trilinear form, still denoted as $B_{s_0,k}$, on $A_i \times A'_i \times E_k^m$, with

$$B_{s_0,k}(g \cdot v_1, g \cdot v'_1, g.\Phi) = |\det g|^{-s_0+n-k} B_{s_0,k}(v_1, v'_1, \Phi)$$

for any $v_1 \in A_i$, $v'_1 \in A'_i$, $\Phi_i \in E_k^m$ and $g \in G_k$. Note that $|\det|^{(n-k)/2}\rho_i$ is a component for $\pi^{(n-k)}$, thus we have proved the theorem. $\square$

5. Rankin–Selberg integrals: $G_n \times G_n$

Suppose a pole s_0 is not exceptional for the integrals I_n, and that we have the Laurent expansion

$$I_n(s, W, W', \Phi) = \frac{B_{s_0}(W, W', \Phi)}{(s-s_0)^d} + \cdots,$$

and B_{s_0} is continuous on $V \times V' \times E_n^m$ with the invariance property

$$B_{s_0,n}(g \cdot W, g \cdot W', g \cdot \Phi) = |\det g|^{-s_0} B_{s_0,n}(W, W', \Phi).$$

Since s_0 is not exceptional, for any integer m, we can find some $\Phi \in \mathcal{S}^m$ such that the form $B_{s_0,n}(W, W', \Phi)$ is nonzero for some choices of W and W'. Because of the continuity of $B_{s_0,n}$, we may further assume W and W' are both K_n-finite. By Iwasawa decomposition, we have

$$I_n(s, W, W', \Phi) = \int_{K_n}\int_{N_n\backslash P_n} W(pk)W'(pk)|\det p|^{s-1}\int_{\mathbb{R}^\times} \omega(a)\omega'(a)|a|^{ns}\Phi(\epsilon_n ak)\, d^\times a\, dp\, dk.$$

Take $\{W_i\}$ to be some base vectors in the K-span subspace of W, and we write $W(gk) = \sum_i f_i(k)W_i(g)$, where f_i are continuous functions on K. Similarly, write $W'(gk) = \sum_i f_i'(k)W_i'(g)$, where $\{W_j'\}$ are some base vectors of the K-span subspace of W', and f_i' are continuous functions on K. Now $I(s, W, W', \Phi)$ equals

$$\sum_{i,j}\int_{N_n\backslash P_n} W_i(p)W_j(p)|\det p|^{s-1}\int_{\mathbb{R}^\times}\omega(a)\omega'(a)|a|^{ns} \times \int_K f_i(k)f_j'(k)\Phi(\epsilon_n ak)\, dk\, d^\times a\, dp.$$

Lemma 5.1. *For any continuous function $f(k)$ on K, the function*

$$\Psi(a) = \int_K f(k)\Phi(\epsilon_n ak)\, dk$$

belongs to $\mathcal{S}^m(\mathbb{R})$ if Φ is in $\mathcal{S}^m(\mathbb{R}^n)$.

Proof. We will only check that $\Psi(a)$ vanishes at least to order m around 0; other verifications are routine and will be omitted. Since Φ vanishes at 0 at least to order m, by [Trèves 1967, Theorem 38.1] there exists a homogeneous polynomial $P(x_1, \dots, x_n)$ of degree m such that the Taylor expansion of Φ at 0 has the form

$$\Phi(x_1, \dots, x_n) = P(x_1, \dots, x_n) + \cdots.$$

Then

$$\Psi(a) = \int_K f(k)\Phi(\epsilon_n ak)dk = \int_K f(k)P(\epsilon_n ak)\, dk + \cdots = a^m\int_K f(k)P(\epsilon_n k)\, dk + \cdots.$$

This shows that $\Psi(a)$ vanishes at least to order m at 0, which finishes the proof. $\square$

Lemma 5.2. *If $\Phi \in \mathcal{S}^m(\mathbb{R})$ for some $m > 0$, then as a function of $s \in \mathbb{C}$, the function*

$$\int_0^\infty a^s\Phi(a)\, d^\times a$$

is holomorphic in the half-plane $\mathrm{Re}(s) > -m$.

Proof. Since Φ is a Schwartz function, the integral

$$\int_\epsilon^\infty a^s\Phi(a)\, d^\times a$$

is holomorphic in s, when ϵ is away from 0.

In a neighborhood of 0, when $\mathrm{Re}(s) > -m$ and $\Phi \in \mathcal{S}^m(\mathbb{R})$, the function $a^s\Phi(a)$ is continuous. Thus

$$\int_0^{\epsilon} a^s \Phi(a)\, d^{\times}a$$

is also holomorphic in s. □

By Lemma 5.1, as a function of a, the integral

$$\int_K f_i(k) f_j'(k)\Phi(\epsilon a k)\, dk$$

belongs to $\mathcal{S}_n^m(\mathbb{R})$, and by Lemma 5.2, when we choose m large enough, the function

$$\int_{\mathbb{R}^{\times}} \omega(a)\omega'(a)|a|^{ns} \int_K f_i(k) f_j'(k)\Phi(\epsilon a k)\, d^{\times}a\, dk$$

is holomorphic in the half-plane containing s_0. Hence the pole s_0 has to occur in the sum

$$\sum_{i,j} \int_{N_n \backslash P_n} W_i(p) W_j'(p) |\det p|^{s-1}\, dp,$$

and we may assume one of the terms

$$\int_{N_n \backslash P_n} W_i(p) W_j'(p) |\det p|^{s-1}\, dp$$

contains the pole s_0. But this integral descends to the integral

$$\int_{N_{n-1} \backslash G_{n-1}} W_i \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W_j' \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} |\det g|^{s-1}\, dg$$

on $N_{n-1} \backslash G_{n-1}$.

Each $W_v\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ can be written as a finite sum

$$\sum_i W_i \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \Phi_i(\epsilon_{n-1} g)$$

for some functions $W_i \in \mathcal{W}(\pi, \psi)$ and Schwartz functions Φ_i on $\mathbb{R}^{n-1}$. Thus the above integral becomes

$$\sum_i \int_{N_{n-1} \backslash G_{n-1}} W_i \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W' \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \Phi_i(\epsilon_{n-1} g) |\det g|^{s-1}\, dg,$$

which are integrals belonging to I_{n-1}. So we have the following corollary.

Proposition 5.3. *If a pole s_0 of I_n of order d is not exceptional of type* 1, *then it occurs as a pole of order d for the integrals I_{n-1}.*

In general, we have the following reduction result.

Proposition 5.4. *If a pole s_0 of I_k is not an exceptional pole for the integrals I_k, then it is a pole for I_{k-1}.*

Proof. By [Jacquet and Shalika 1990, Proposition 2], there exists a finite set of functions $\{\xi\}$ on $(\mathbb{R}^\times)^k$, which have the form $\xi(z_1,\dots,z_k)=\prod_{j=1}^k \chi_j(z_j)(\log|z_j|)^{n_j}$, where χ_j is a character on $\mathbb{R}^\times$, and Schwartz functions ϕ_ξ on $\mathbb{R}^k\times O(n)$, such that

$$W_v(\alpha x)=\sum_\xi \xi(a_1,\dots,a_k)\phi_\xi(a_1,\dots,a_k,x),$$

where $x\in O(n)$ and

$$\alpha=\operatorname{diag}(a_1\cdots a_k,a_2\cdots a_k,\dots,a_{k-1}a_k,a_k),$$

which will be viewed as

$$\operatorname{diag}(a_1\cdots a_k,a_2\cdots a_k,\dots,a_{k-1}a_k,a_k,1,\dots,1)\in G_n.$$

Since ϕ_ξ is a Schwartz function, for each x, it has a Taylor expansion around 0,

$$\phi_\xi(a_1,\dots,a_k,x)=f(x)P_\xi(a_1,\dots,a_k)+\cdots,$$

where $f(x)$ is some continuous function of x, and P_ξ denotes the sum of leading coefficients in the Taylor expansion, which is a polynomial in $a_1,\dots,a_k$.

It follows that, around 0, we can write

$$W_v(\alpha x)=\sum_\xi\{f(x)\xi(a_1,\dots,a_k)P_\xi(a_1,\dots,a_k)+\cdots\}. \tag{17}$$

Similarly, around 0, we have

$$W_{v'}(\alpha x)=\sum_{\xi'}\{f'(x)\xi'(a_1,\dots,a_k)P_{\xi'}(a_1,\dots,a_k)+\cdots\}. \tag{18}$$

By Iwasawa decomposition, we have

$$I_k=\int W_v\begin{pmatrix}pax&0\\0&I_{n-k}\end{pmatrix}W_{v'}\begin{pmatrix}pax&0\\0&I_{n-k}\end{pmatrix}\\ \times\Phi(\epsilon_k ax)|\det p|^{s-n+k-1}|a|^{k(s-n+k)}\,dp\,dx\,d^\times a,$$

with $p\in N_k\setminus P_k$, where P_k is the mirabolic subgroup in G_k, $x\in O(k)$, $a\in\mathbb{R}^\times$.

Note that $N_k\setminus P_k=N_{k-1}\setminus G_{k-1}$, so we can write $pax=n_{k-1}\alpha yx$ for some $n_{k-1}\in N_{k-1}$,

$$\alpha=\operatorname{diag}(a_1\cdots a_{k-1}a,\dots,a_{k-1}a,a,1,\dots,1),$$

and $y\in O(k-1)$.

Thus by (17), around 0 we have

$$W_v(pax) = \psi(n_k) \sum_{\xi} \{ f(yx)\xi(a_1, \dots, a_{k-1}, a) P_\xi(a_1, \dots, a_{k-1}, a) + \cdots \}$$

and

$$W_{v'}(pax) = \psi^{-1}(n_k) \sum_{\xi'} \{ f'(yx)\xi'(a_1, \dots, a_{k-1}, a) P_{\xi'}(a_1, \dots, a_{k-1}, a) + \cdots \}.$$

Note that the poles of I_k are caused by the integration around 0, and in a neighborhood of 0, the integral is

$$\sum_{\xi,\xi'} \int f(yx) f'(yx)\, dy\, dx \int (\xi P_\xi \xi' P_{\xi'})(a_1, \dots, a_{k-1}, a) \\ \times \Phi(\epsilon_k a x) |a|^{k(s-n+k)} |a_1|^{c_1} \cdots |a_{k-1}|^{c_k}\, d^\times a \cdots d^\times a_{k-1} + \cdots,$$

where $c_1, \dots, c_{k-1}$ are some complex numbers depending on s.

First, since s_0 is a pole for this integral and $O(k)$, $O(k-1)$ are compact, it follows that this pole occurs as a pole for the integral with respect to the variables $a_1, \dots, a_{k_1}, a$, and the integrals with respect to x, y are nonzero.

Since s_0 is not an exceptional pole, we can choose the Schwartz function Φ so that the integral on a in the above expression is holomorphic in a region containing s_0. Thus the pole is caused by the integration with respect to the variables $a_1, \dots, a_{k-1}$. This implies that the integral

$$\int W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k+1} \end{pmatrix} W'_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k+1} \end{pmatrix} |\det g|^{s-n+k-1}\, dg$$

has the pole s_0. This integral belongs to the integrals I_{k-1}, and the proposition follows. □

Corollary 5.5. *Any pole of the Rankin–Selberg integrals I_n for π and π' is an exceptional pole of type* 2 *for some components of $\pi^{(k)}$ and $\pi'^{(k)}$*, $0 \le k < n$.

For the other direction, suppose σ and σ' are a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$ respectively.

Proposition 5.6. *Any Rankin–Selberg integral of σ and σ' can be written as a sum of Rankin–Selberg integrals of π and π'.*

Proof. For any $W_{v_1} \in \mathcal{W}(\sigma, \psi)$, $W_{v'_1} \in \mathcal{W}(\sigma', \psi^{-1})$, and $\Phi \in \mathcal{S}_{n-k}$, we have the Rankin–Selberg integral for σ and σ':

$$I(s, W_{v_1}, W_{v'_1}, \Phi) = \int_{N_{n-k} \backslash G_{n-k}} W_{v_1}(g) W_{v'_1}(g) \Phi(\epsilon_{n-k} g) |\det g|^s\, dg.$$

By Corollary 3.5, there exists some $W_v \in \mathcal{W}(\pi, \psi)$ such that

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = W_{v_1}(g)\Phi(\epsilon_{n-k} g)|\det g|^{k/2}.$$

Thus, the above integral is

$$I(s, W_{v_1}, W_{v_1'}, \Phi) = \int_{N_{n-k}\backslash G_{n-k}} W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W_{v_1'}(g)|\det g|^{s-k/2}\, dg,$$

which can be written as

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = \sum_j W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} \Phi_j(\epsilon_{n-k} g)$$

with $W_j \in \mathcal{W}(\pi, \psi)$, and Schwartz functions Φ_j on $\mathbb{R}^{n-k}$. So we have

$$I(s, W_{v_1}, W_{v_1'}, \Phi) = \sum_j \int W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W_{v_1'}(g)\Phi_j(\epsilon_{n-k} g)|\det g|^{s-k/2}\, dg.$$

Using Corollary 3.5 again, we have

$$I(s, W_{v_1}, W_{v_1'}, \Phi) = \sum_j \int W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W_j' \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} |\det g|^{s-k}\, dg$$

for some $W_j' \in \mathcal{W}(\pi', \psi^{-1})$. Then by [Jacquet 2009, Lemma 14.1], each integral on the right side can be written as a Rankin–Selberg integral for π and π'. Thus the proposition follows. □

Corollary 5.7. *Any exceptional pole of type* 1 *of depth* 0 *for Rankin–Selberg integrals of σ and σ' is a pole of the Rankin–Selberg integrals I_n for π and π'.*

Summarizing the above, we obtain the main result of this section.

Theorem 5.8. *Let π and π' be irreducible generic Casselman–Wallach representations of G_n in general position. Then any pole of the Rankin–Selberg integrals for π and π' is an exceptional pole of type* 2 *for a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$*, $0 \le k \le n-1$. *On the other hand, any exceptional pole of type* 1 *of depth* 0 *for a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$*, $0 \le k \le n-1$, *is a pole of the Rankin–Selberg integrals of π and π'.*

6. Case $G_n \times G_m$, $m < n$

This section is devoted to the case $G_n \times G_m$, $m < n$, using the same ideas and techniques as in the previous section. We will indicate the necessary changes and omit details.

Now suppose π and π' are generic irreducible Casselman–Wallach representations of G_n and G_m in general position, respectively. Let $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\pi',\psi^{-1})$ be their Whittaker models. The family of integrals is given by

$$I(s,W,W')=\int_{N_m\backslash G_m} W\begin{pmatrix} g & 0\\ 0 & I_{n-m}\end{pmatrix} W'(g)|\det g|^{s-\frac{n-m}{2}}\,dg,$$

and for $1\le j\le n-m-1$

$$\begin{aligned}&I^j(s,W,W')\\ &\quad=\int_{M(m\times j,\mathbb{R})}\int_{N_m\backslash G_m} W\begin{pmatrix} g & 0 & 0\\ X & I_j & 0\\ 0 & 0 & I_{n-m-j}\end{pmatrix} W'(g)|\det g|^{s-\frac{n-m}{2}}\,dg\,dX,\end{aligned}$$

with $W\in\mathcal{W}(\pi,\psi)$ and $W'\in\mathcal{W}(\pi',\psi^{-1})$. We will only consider the integrals $I(s,W,W')$ since they have the same poles with the same multiplicities as $I^j(s,W,W')$ for each j.

For each $1\le k\le m$, let Φ be a Schwartz function on $\mathbb{R}^k$, and introduce

$$\begin{aligned}&I_k(s,W,W',\Phi)\\ &\quad=\int_{N_k\backslash G_k} W\begin{pmatrix} g & 0\\ 0 & I_{n-k}\end{pmatrix} W'\begin{pmatrix} g & 0\\ 0 & I_{m-k}\end{pmatrix}\Phi(\epsilon_k g)|\det g|^{s-\frac{n+m}{2}+k}\,dg.\end{aligned}$$

By [Jacquet 2009, Lemma 14.1], the integrals I_k belong to the family I_m, which implies that they are convergent when $\mathrm{Re}(s)$ is large, and have meromorphic continuations to the whole plane.

At a pole s_0 for $I_k(s,W,W',\Phi)$, we have an expansion

$$I_k(s,W,W',\Phi)=\frac{B_{s_0,k}(W,W',\Phi)}{(s-s_0)^d}+\cdots,$$

where $B_{s_0,k}(W,W',\Phi)$ is a trilinear form on $V\times V'\times\mathcal{S}_k$ satisfying the following invariance property: for any $g\in G_k$,

$$B_{s_0,k}(g\cdot W,g\cdot W',g\cdot\Phi)=|\det g|^{-s_0+\frac{n+m}{2}-k}B_{s_0,k}(W,W',\Phi).$$

Similar to Proposition 4.2, we can show $B_{s_0,k}$ is continuous.

Definition. We say a pole s_0 is an exceptional pole of type 1, with level l and depth $m-k$, if the corresponding $B_{s_0,k}$ is zero on $\mathcal{S}_k^{l+1}$, but not identically zero on $\mathcal{S}_k^l$. In this case, we also say s_0 is an exceptional pole for the integrals $I_k(s,W_v,W_{v'},\Phi)$.

Definition. We say a complex number s_0 is an exceptional pole of type 2, with level l, for π and π', if there exists a continuous trilinear form

$$l: V\times V'\times E_k^l\to\mathbb{C}$$

such that

$$l(g \cdot W, g \cdot W', g \cdot \Phi) = |\det g|^{-s_0 + \frac{n-m}{2}} l(W, W', \Phi).$$

Remark. If s_0 is an exceptional pole of type 1 with level l and depth 0, then s_0 is also an exceptional pole of type 2 with level l for π and π'.

Along the same lines, we have the following theorem.

Theorem 6.1. *If s_0 is an exceptional pole of type* 1 *with level l and depth $m-k$, then s_0 is an exceptional pole of type* 2 *with level l for some components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$.*

The main reduction step is the following analog to Proposition 5.4, with essentially the same proof.

Proposition 6.2. *If a pole s_0 of I_k is not an exceptional pole for these integrals, then it is a pole of I_{k-1}.*

As a corollary, we have:

Corollary 6.3. *Any pole of the Rankin–Selberg integrals I_m for π and π' is an exceptional pole of type* 2 *for some components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $0 \le k < m$.*

A converse statement is also true.

Proposition 6.4. *Any exceptional pole of type* 1 *of depth* 0 *for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$ is a pole of the Rankin–Selberg integrals I_n for π and π'.*

The main result of this section is the following.

Theorem 6.5. *Let π and π' be irreducible generic Casselman–Wallach representations of G_n and G_m in general position. Then any pole of the Rankin–Selberg integrals for π and π' is an exceptional pole of type* 2 *for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $1 \le k \le m$. On the other hand, any exceptional pole of type* 1 *of depth* 0 *for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $1 \le k \le m$, is a pole of the Rankin–Selberg integrals of π and π'.*

Acknowledgements

The results of this paper consist of the main part of the author's thesis at the Ohio State University. The author would like to thank his advisor Professor Cogdell for many helpful discussions and constant encouragement. The author also would like to thank Professor Stanton for carefully reading the draft and for his suggestions. During the final preparation of this paper, the author was supported by the Young Teacher Program of Hunan University.

References

[Aizenbud et al. 2008] A. Aizenbud, D. Gourevitch, and E. Sayag, "$(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a Gelfand pair for any local field F", *Compos. Math.* **144**:6 (2008), 1504–1524. MR 2009k:22022 Zbl 1157.22004

[Aizenbud et al. 2012] A. Aizenbud, D. Gourevitch, and S. Sahi, "Derivatives for smooth representations of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$", preprint, 2012. arXiv 1109.4374

[Bernstein and Zelevinsky 1977] I. N. Bernstein and A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, I", *Ann. Sci. École Norm. Sup.* (4) **10**:4 (1977), 441–472. MR 58 #28310 Zbl 0412.22015

[Borel and Wallach 2000] A. Borel and N. R. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd ed., Mathematical Surveys and Monographs **67**, American Mathematical Society, Providence, RI, 2000. MR 2000j:22015 Zbl 0980.22015

[Bunke and Olbrich 1997] U. Bunke and M. Olbrich, "Cohomological properties of the canonical globalizations of Harish-Chandra modules: consequences of theorems of Kashiwara–Schmid, Casselman, and Schneider–Stuhler", *Ann. Global Anal. Geom.* **15**:5 (1997), 401–418. MR 99b:22025 Zbl 0890.22004

[Chang and Cogdell 1999] J.-T. Chang and J. W. Cogdell, "$\mathfrak{n}$-homology of generic representations for $\mathrm{GL}(N)$", *Proc. Amer. Math. Soc.* **127**:4 (1999), 1251–1256. MR 99f:22021 Zbl 0911.22011

[Cogdell and Piatetski-Shapiro 2004] J. W. Cogdell and I. I. Piatetski-Shapiro, "Remarks on Rankin–Selberg convolutions", pp. 255–278 in *Contributions to automorphic forms, geometry, and number theory* (Baltimore, MD, 2002), edited by H. Hida et al., Johns Hopkins University Press, Baltimore, MD, 2004. MR 2005d:11075 Zbl 1080.11038

[Cogdell and Piatetski-Shapiro $\geq$ 2015] J. W. Cogdell and I. I. Piatetski-Shapiro, "Derivatives and L functions for $\mathrm{GL}(n)$", in *The Heritage of B. Moishezon*, edited by M. Teicher. To appear.

[Gourevitch and Kemarsky 2013] D. Gourevitch and A. Kemarsky, "Irreducible representations of a product of real reductive groups", *Journal of Lie Theory* **23**:4 (2013), 1005–1010.

[Jacquet 2009] H. Jacquet, "Archimedean Rankin–Selberg integrals", pp. 57–172 in *Automorphic forms and L-functions, II: Local aspects*, edited by D. Ginzburg et al., Contemp. Math. **489**, American Mathematical Society, Providence, RI, 2009. MR 2011a:11103 Zbl 1250.11051

[Jacquet and Shalika 1981] H. Jacquet and J. A. Shalika, "On Euler products and the classification of automorphic forms, II", *Amer. J. Math.* **103**:4 (1981), 777–815. MR 82m:10050b Zbl 0491.10020

[Jacquet and Shalika 1990] H. Jacquet and J. A. Shalika, "Exterior square L-functions", pp. 143–226 in *Automorphic forms, Shimura varieties, and L-functions* (Ann Arbor, MI, 1988), vol. 2, edited by L. Clozel and J. S. Milne, Perspect. Math. **11**, Academic Press, Boston, 1990. MR 91g:11050 Zbl 0695.10025

[Jacquet et al. 1983] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, "Rankin-Selberg convolutions", *Amer. J. Math.* **105**:2 (1983), 367–464. MR 85g:11044

[Knapp 1994] A. W. Knapp, "Local Langlands correspondence: the Archimedean case", pp. 393–410 in *Motives* (Seattle, WA, 1991), vol. 2, edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, American Mathematical Society, Providence, RI, 1994. MR 95d:11066 Zbl 0811.11071

[Shalika 1974] J. A. Shalika, "The multiplicity one theorem for GL_n", *Ann. of Math.* (2) **100** (1974), 171–193. MR 50 #545 Zbl 0316.12010

[Trèves 1967] F. Trèves, *Topological vector spaces, distributions and kernels*, Pure and Applied Mathematics **25**, Academic Press, New York, 1967. MR 37 #726 Zbl 0171.10402

[Wallach 1988] N. R. Wallach, *Real reductive groups*, vol. 1, Pure and Applied Mathematics **132**, Academic Press, Boston, 1988. MR 89i:22029 Zbl 0666.22002

[Wallach 1992] N. R. Wallach, *Real reductive groups*, vol. 2, Pure and Applied Mathematics **132**, Academic Press, Boston, 1992. MR 93m:22018 Zbl 0785.22001

Received October 21, 2013. Revised October 31, 2013.

JINGSONG CHAI
COLLEGE OF MATHEMATICS AND ECONOMETRICS
HUNAN UNIVERSITY
CHANGSHA, HUNAN 410082
CHINA
chaijingsong@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.307

PRESCRIBING THE BOUNDARY GEODESIC CURVATURE ON A COMPACT SCALAR-FLAT RIEMANN SURFACE VIA A FLOW METHOD

ZHANG HONG

We study the problem of prescribing the boundary geodesic curvature on a compact scalar-flat Riemann surface. We use the negative gradient flow method. We prove the global existence and the convergence of the flow as time goes to infinity under sufficient conditions on the prescribed function.

1. Introduction

Let (M, g_0) be a compact Riemann surface with boundary equipped with a scalar-flat metric g_0. Given a function f on ∂M, does there exist a scalar-flat metric g which is pointwise conformal to g_0, i.e., a $g = e^{2u} g_0$ such that f is the geodesic curvature of ∂M under the metric g? This problem is equivalent to solving the boundary value problem

$$\begin{cases} \Delta_{g_0} u = 0 & \text{in } M, \\ \partial_n u + k_0 = f e^u & \text{on } \partial M, \end{cases} \tag{1-1}$$

where ∂_n is the outward-pointing normal derivative operator with respect to g_0 and k_0 is the geodesic curvature of ∂M under the metric g_0. We may assume without loss of generality that k_0 is a constant since there always exists such a metric in the conformal class of g_0. Let us first derive necessary conditions for (1-1) to have a solution. By integrating (1-1), we obtain

$$\int_{\partial M} f e^u \, ds_{g_0} = k_0 \mathscr{L}(\partial M), \tag{1-2}$$

where $\mathscr{L}(\partial M) = \mathscr{L}(\partial M, g_0)$ is the arc length of ∂M, and

$$\int_{\partial M} f \, ds_{g_0} = -\int_M |\nabla_{g_0} u|^2 e^{-u} \, dA_{g_0} + k_0 \int_{\partial M} e^{-u} \, ds_{g_0}. \tag{1-3}$$

MSC2010: primary 53C44; secondary 35J65.
Keywords: prescribed geodesic curvature, negative gradient flow.

Depending on the sign of k_0, with the help of (1-2) and (1-3), we conclude that the geodesic curvature candidate f should satisfy the conditions

$$\text{(1-4)}\qquad \begin{cases} \text{(i)}\ \max\limits_{x\in\partial M} f(x)>0 & \text{when } k_0>0,\\ \text{(ii)}\ \max\limits_{x\in\partial M} f(x)>0\ \text{ and }\ \displaystyle\int_{\partial M} f\,ds_{g_0}<0 & \text{when } k_0=0,\\ \text{(iii)}\ \displaystyle\int_{\partial M} f(x)\,ds_{g_0}<0 & \text{when } k_0<0.\end{cases}$$

Remark 1.1. Notice that for $k_0=0$, it is not hard to see that if (1-1) has a solution, then either $f\equiv 0$ or $\max_{x\in\partial M} f(x)>0$ and $\int_{\partial M} f\,ds_{g_0}<0$. However, we do not include the case $f\equiv 0$ in (ii). This is because (1-1) becomes trivial in that case. Hence, we only consider the case $f\not\equiv 0$ in this paper.

To the author's knowledge, there are very few papers concerned with sufficient conditions on f for the existence of a solution to problem (1-1). Cherrier [1984] studied the regularity issue for (1-1), and for $k_0=0$ he showed that if condition (ii) in (1-4) holds then the equation has a nontrivial solution. This implies that condition (ii) is necessary and sufficient for (1-1) to have a nontrivial solution. Kazdan and Warner [1975] found the similar condition in the prescribed Gaussian curvature problem on Riemann surfaces without boundary. For $k_0<0$, Ho [2011] proved that (1-1) has a solution provided the prescribed function f is strictly negative by using a flow method. He considered the evolution problem

$$\text{(1-5)}\qquad \begin{cases} \dfrac{\partial g}{\partial t}=(\alpha(t)f-k)g & \text{in } \partial M,\\ K=0 & \text{on } M,\end{cases}$$

where $\alpha(t)=2\pi\chi(M)/\big(\int_{\partial M} f\,ds_g\big)$, and k and K are the geodesic curvature and Gaussian curvature of the time metric $g(t)$. Such a flow has been used in many works; see for instance [Brendle 2002a; 2003; Struwe 2005; Malchiodi and Struwe 2006; Chen and Xu 2012] and the literature therein. When $k_0>0$ and $M=D$ (the unit disc in the plane), Liu and Huang [2005] showed that there exists a solution of (1-1) if f possesses some kind of symmetries, while for a more general smooth function, Chang and Liu [1996] obtained an existence result through the Morse theory method.

In this paper, we will use the negative gradient flow introduced in [Baird et al. 2004; 2006] to investigate the problem of prescribing the geodesic curvature when the candidate curvature function f is not necessarily of constant sign. This gradient flow will be different from (1-5). To be precise, it is introduced in the following way. Motivated by [Chang and Liu 1996], we consider the functional

$$J(u)=\int_{\partial M}\frac{1}{2}\partial_n u\cdot u+k_0u\,ds_{g_0}$$

on the Sobolev space $H := \{u \in H^1(M) : \Delta_{g_0} u = 0 \text{ in } M\}$ under the constraint

$$u \in X := \left\{ u \in H : L(u) := \int_{\partial M} e^u f \, ds_{g_0} = k_0 \mathcal{L}(\partial M) \right\}.$$

Note that the set X is not empty, thanks to the conditions in (1-4). From the Moser–Trudinger inequality with boundary [Li and Liu 2005, Theorem A]

$$\int_{\partial M} e^u \, ds_{g_0} \le C \exp\left\{ \frac{1}{4\pi} \int_M |\nabla u|^2 \, dA_{g_0} + \frac{1}{\mathcal{L}(\partial M)} \int_{\partial M} u \, ds_{g_0} \right\}, \tag{1-6}$$

where the constant C depends on M and g_0, it follows that L is well-defined on H. Since H is restricted to the set of harmonic functions, we may assume that H is equipped with the scalar product

$$\langle u, v \rangle = \int_{\partial M} \partial_n u \cdot v + u \cdot v \, ds_{g_0},$$

for $u, v \in H$. Hence the associated norm on H is given by

$$\|u\|^2 = \int_{\partial M} \partial_n u \cdot u + u^2 \, ds_{g_0}.$$

The functionals J and L are analytic, and their gradients are given by

$$\langle \nabla J(u), \phi \rangle = \int_{\partial M} (\partial_n u + k_0) \phi \, ds_{g_0} \quad \text{for all } \phi \in H, \tag{1-7}$$

which implies that

$$\nabla J(u) = (\partial_n + I)^{-1} (\partial_n u + k_0),$$

and

$$\langle \nabla L(u), \phi \rangle = \int_{\partial M} e^u f \phi \, ds_{g_0} \quad \text{for all } \phi \in H, \tag{1-8}$$

which implies that

$$\nabla L(u) = (\partial_n + I)^{-1} (e^u f),$$

where I is the identity transformation.

Since $\nabla L(u) \neq 0$ for all $u \in X$ by the hypothesis (1-4), the set X is a regular hypersurface of H. A unit normal field at a point u in X is given by

$$N(u) = \frac{\nabla L(u)}{\|\nabla L(u)\|}. \tag{1-9}$$

The gradient of the functional J with respect to the hypersurface X is thus defined by

$$\nabla^X J(u) = \nabla J(u) - \langle \nabla J(u), N(u) \rangle N(u). \tag{1-10}$$

Then the negative gradient flow of J with respect to the hypersurface X is

$$\begin{cases} \partial_t u = -\nabla^X J(u), \\ u(0) = u_0 \in X. \end{cases} \tag{1-11}$$

If the flow (1-11) exists for all time and converges at infinity, then the limit function u_∞ produces a solution of (1-1) and so defines a metric of geodesic curvature f. In this paper, we will show the long-time existence of a solution of (1-11) and its convergence as $t \to \infty$ under sufficient conditions on the prescribed function f. We also describe the asymptotic behavior of the flow at infinity.

2. Statement of the results

We will first show the long-time existence of the solution to (1-11).

Theorem 2.1. *Let (M, g_0) be a compact scalar-flat Riemann surface with boundary and let $f \in C^0(\partial M)$ satisfy the appropriate condition in* (1-4). *Then for any $u_0 \in X$, there exists a unique global solution $u \in C^\infty([0, \infty[, H)$ of* (1-11). *In addition, the energy identity*

$$\int_0^t \|\partial_\tau u(\tau)\|^2 \, d\tau + J(u(t)) = J(u_0), \tag{2-1}$$

holds for all $t \geq 0$.

We will study the convergence of the global solution depending on the sign of k_0. When $k_0 > 0$, we only consider the case $M = D$, the unit disc. Let $u_0 \in X$ and $u : [0, \infty[\to X$ be the solution of (1-11) obtained in Theorem 2.1.

Theorem 2.2. *Suppose that $k_0 = 0$. Let $f \in C^0(\partial M)$ satisfy the conditions*

$$\max_{x \in \partial M} f(x) > 0 \quad \text{and} \quad \int_{\partial M} f(x) \, ds_{g_0} < 0;$$

then u converges in H as $t \to \infty$ to a function $u_\infty \in H \cap C^\alpha(\partial M)$ with the property that the function $v_\infty = u_\infty + \lambda$ is a solution of

$$\begin{cases} \Delta_{g_0} v_\infty = 0 & \text{in } M, \\ \partial_n v_\infty + k_0 = f e^{v_\infty} & \text{on } \partial M, \end{cases}$$

for some constant λ. Moreover, there exist two constants $\beta, \delta > 0$ such that

$$\|u(t) - u_\infty\| \leq \beta (1 + t)^{-\delta}$$

for all $t \geq 0$.

Corollary 2.3. *Suppose that $k_0 = 0$. Let $f \in C^0(\partial M)$ satisfy the conditions $\max_{x \in \partial M} f(x) > 0$ and $\int_{\partial M} f(x) \, ds_{g_0} < 0$; then there exists a metric conformal to g_0 with associated geodesic curvature f.*

For the negative case, we have:

Theorem 2.4. *Suppose that $k_0 < 0$. Let $f \in C^0(\partial M)$ satisfy the condition $\int_{\partial M} f(x)\, ds_{g_0} < 0$; then there exists a positive constant $\bar{C}$ depending only on the function $f^-(x) = \max(-f(x), 0)$, g_0 and M, such that if u_0 satisfies*

$$e^{\xi \|u_0\|^2} \max_{x \in \partial M} f(x) \le \bar{C}, \tag{2-2}$$

where $\xi > 1$ is a constant depending only on g_0 and M, then u converges in H as $t \to \infty$ to a solution $u_\infty \in H \cap C^\alpha(\partial M)$ of (1-1). *Moreover, there exist two constants $\beta, \delta > 0$ such that*

$$\|u(t) - u_\infty\| \le \beta(1+t)^{-\delta}$$

for all $t \ge 0$. In particular, if $f \le 0$, then u converges in H as $t \to \infty$ to a solution $u_\infty \in H \cap C^\alpha(\partial M)$ of (1-1) *and $\|u(t) - u_\infty\| \le \beta(1+t)^{-\delta}$ for all $t \ge 0$.*

Corollary 2.5. *Suppose that $k_0 < 0$. Let $f \in C^0(\partial M)$ satisfy the condition $\int_{\partial M} f(x)\, ds_{g_0} < 0$. There exists a positive constant $\bar{C}$ depending only on the function f^-, g_0 and M, such that if f satisfies*

$$\max_{x \in \partial M} f(x) \le \bar{C},$$

then (1-1) *admits a solution $u \in H \cap C^\alpha(\partial M)$. In particular, if $f \le 0$, then* (1-1) *admits a solution $u \in H \cap C^\alpha(\partial M)$.*

We now consider the positive case. In this case, we assume that $M = D$, the unit disc. Suppose that the function f is invariant under a group G of isometries of $\partial D = S^1$ (f is a G-invariant function). Then we can establish the convergence.

Recall that a function on S^1 is said to be G-invariant if it satisfies

$$f(\sigma x) = f(x) \quad \text{for all } x \in S^1 \text{ and } \sigma \in G.$$

Let Σ denote the set of fixed points of G, that is,

$$\Sigma = \{x \in S^1 : \sigma x = x \text{ for all } \sigma \in G\};$$

we have the following result:

Theorem 2.6. *Let $f \in C^0(\partial M)$ be a function invariant under a group G of isometries of S^1 with $\max_{x \in S^1} f(x) > 0$, and let $u_0 \in X$ also be invariant under G. If either*

(i) $\Sigma = \varnothing$, *or*

(ii) $\max_{p \in \Sigma} f(p) \le e^{-J(u_0)/2\pi}$,

then u converges in H as $t \to \infty$ to a G-invariant solution $u_\infty \in H \cap C^\alpha(S^1)$ of (1-1). *Moreover, there exist two constants $\beta, \delta > 0$ such that*

$$\|u(t) - u_\infty\| \le \beta(1+t)^{-\delta}$$

for all $t \ge 0$.

Let $a \in D$ and denote by Φ_a the Möbius transformation given by

$$\Phi_a = \frac{z+a}{\bar{a}z+1}.$$

For a suitable choice of the initial data u_0, we have the following:

Corollary 2.7. *Let $f \in C^0(S^1)$ be a function with $\max_{x\in S^1} f(x) > 0$ which is invariant under a group of isometries of S^1. If either*

(i) $\Sigma = \varnothing$, *or*

(ii) *there exists $a_0 \in \Sigma$ such that*

$$\max_{p\in\Sigma} f(p) \leq \max\left(0, \fint_{S^1} f \circ \Phi_{a_0}\, ds_{g_0}\right), \tag{2-3}$$

then (1-6) *admits a G-invariant solution $u \in H \cap C^\alpha(S^1)$. In particular, if*

$$\max_{p\in\Sigma} f(p) \leq \max\left(0, \fint_{S^1} f\, ds_{g_0}\right), \tag{2-4}$$

then (1-6) *admits a G-invariant solution $u \in H \cap C^\alpha(S^1)$.*

3. Long-time existence

In this section, we first show that the solution of the flow (1-11) is well-defined on $[0, \infty[$. Then we show the convergence of the flow under the assumption of uniform boundedness of the conformal factor u. To do so, we will first prove:

Lemma 3.1. *The linear mapping $(\partial_n + I)^{-1} : L^2(\partial M) \to H$ is compact.*

Proof. Let S be a bounded set in $L^2(\partial M)$. Then there exists a sequence $(\phi_i)_i \subset S$ that weakly converges to a function ϕ_∞ in $L^2(\partial M)$. Define $u_i = (\partial_n + I)^{-1}\phi_i$ and $u_\infty = (\partial_n + I)^{-1}\phi_\infty$. We then have

$$(\partial_n + I)(u_i - u_\infty) = \phi_i - \phi_\infty.$$

Hence, u_i weakly converges to u_∞ in H. By the compact embedding $H \hookrightarrow L^2(\partial M)$, u_i strongly converges to u_∞ in $L^2(\partial M)$. Now, a simple calculation, Hölder's inequality and boundedness of ϕ_i and ϕ_∞ in $L^2(\partial M)$ yield

$$\begin{aligned}\|u_i - u_\infty\|^2 &= \int_{\partial M} (\partial_n + I)(u_i - u_\infty) \cdot (u_i - u_\infty)\, ds_{g_0} \\ &= \int_{\partial M} (\phi_i - \phi_\infty) \cdot (u_i - u_\infty)\, ds_{g_0} \leq C\|u_i - u_\infty\|_{L^2(\partial M)} \to 0.\end{aligned}$$

Hence, $(\partial_n + I)^{-1}\phi_i$ strongly converges to $(\partial_n + I)^{-1}\phi_\infty$ in H. This implies that $(\partial_n + I)^{-1}(S)$ is relatively compact in H. Therefore, the linear mapping $(\partial_n + I)^{-1}$ is compact. $\square$

Proof of Theorem 2.1. Since the functionals J and L are C^∞ on H and $\nabla L(u) \neq 0$ for all $u \in H$, it follows that $\nabla^X J$ is C^∞ on H, and the short-time existence follows from the classical Cauchy–Lipschitz theorem. We now extend this short-time solution to $[0, \infty[$.

Since $\nabla J(u) = -(\partial_n + I)^{-1}(u - k_0) + u$ and $(\partial_n + I)^{-1}$ is a bounded linear map by Lemma 3.1, it follows that

$$\|\partial_t u\| = \|\nabla^X J(u)\| \leq \|\nabla J(u)\| \leq C_0\|u\| + C_0.$$

From the inequality above, we deduce that, for all $t < T$,

$$\|u(t)\| \leq (\|u_0\| + 1)e^{C_0 T},$$

which ensures that the solution u is globally defined on $[0, \infty[$.

Now, using (1-9)–(1-11) we can obtain

$$\begin{aligned} \frac{dJ(u)}{dt} &= \langle \nabla J(u), \partial_t u\rangle = \langle \nabla J(u), -\nabla^X J(u)\rangle \\ &= -\|\nabla^X J(u)\|^2 = -\|\partial_t u\|^2. \end{aligned} \tag{3-1}$$

Integrating the equality above from 0 to t yields the energy identity (2-1), which completes the proof. □

Next, we wish to establish convergence at infinity under the assumption of uniform boundedness of the global solution u in H. For this we will prove:

Lemma 3.2. *Let $u : [0, \infty[\to H$ be the solution of* (1-11). *If u satisfies*

$$\|u(t)\| \leq C \tag{3-2}$$

for all $t > 0$, where C is a positive constant independent of t, then $u(t)$ converges in H as $t \to \infty$ to a function $u_\infty \in H \cap C^\alpha(\partial M)$ $(0 < \alpha < 1)$. If $k_0 \neq 0$, then u_∞ is a solution of (1-1). *If $k_0 = 0$, then the function $v_\infty = u_\infty + \lambda$ is a solution of*

$$e^{-v_\infty}(\partial_n v_\infty + k_0) = f,$$

for some constant λ. Moreover, there exist two constants $\beta, \delta > 0$ such that

$$\|u(t) - u_\infty\| \leq \beta(1 + t)^{-\delta}$$

for all $t \geq 0$.

Proof. The energy identity (2-1) and (3-2) imply that

$$\int_0^\infty \|\partial_t u\|^2\, dt \leq J(u_0) + c \sup_t \|u(t)\| \leq J(u_0) + C_1,$$

where $C_1 > 0$ is a constant depending on M, g_0 and the constant C in (3-2). Thus, there exists a sequence $t_k \to \infty$ such that

$$\|\partial_t u(t_k)\| = \|\nabla^X J(u(t_k))\| \to 0. \tag{3-3}$$

From (3-2), we have $\|u(t_k)\| \le C$; hence there exist a function $u_\infty \in H$ and a subsequence of t_k (again denoted by t_k), such that

$$\begin{cases} u(t_k) \to u_\infty & \text{weakly in } H, \\ u(t_k) \to u_\infty & \text{strongly in } L^2(\partial M). \end{cases} \tag{3-4}$$

It follows from (3-2) and (1-6) that for all $p \in \mathrm{R}$, there exists a positive constant $C(p)$ such that

$$\int_{\partial M} e^{pu(t_k)}\, ds_{g_0} \le C(p). \tag{3-5}$$

A straightforward computation from (3-4) and (3-5) shows that for all $p \ge 1$,

$$\lim_{k\to\infty} \|fe^{u(t_k)} - fe^{u_\infty}\|_{L^p(\partial M)} = 0. \tag{3-6}$$

Since $u(t_k) \in X$, which means that

$$\int_{\partial M} fe^{u(t_k)}\, ds_{g_0} = k_0 \mathscr{L}(\partial M),$$

we conclude from (3-6) that $u_\infty \in X$.

Next, we show that $\nabla^X J(u_\infty) = 0$. Recall that

$$\nabla^X J(u(t)) = \nabla J(u(t)) - \big\langle \nabla J(u(t)), \nabla L(u(t))\big\rangle \frac{\nabla L(u(t))}{\|\nabla L(u(t))\|^2} \tag{3-7}$$

with

$$\nabla L(u(t)) = (\partial_n + I)^{-1}(fe^{u(t)}) \tag{3-8}$$

and

$$\nabla J(u(t)) = -[(\partial_n + I)^{-1} - I]u + (\partial_n + I)^{-1} k_0 \tag{3-9}$$

Since $(\partial_n + I)^{-1} : L^2(\partial M) \to H$ is compact by Lemma 3.1, we deduce from (3-4) and (3-6), as well as from (3-7)–(3-9) above, that $\nabla^X J(u(t_k))$ weakly converges in H to $\nabla^X J(u_\infty)$. It follows from (3-3) that $\nabla^X J(u_\infty) = 0$. Therefore,

$$(\partial_n + I)^{-1}(\partial_n u_\infty + k_0) = \eta(u_\infty)(\partial_n + I)^{-1}(fe^{u_\infty}),$$

where $\eta(u_\infty)$ is a constant. Hence,

$$\partial_n u_\infty + k_0 = \eta(u_\infty) fe^{u_\infty}. \tag{3-10}$$

From (3-6) , (3-10) and [Brendle 2002b, Lemma 3.2], it follows that $\|\nabla u_\infty\|_{L^p(\partial M)} \le C$. Since $u_\infty \in H$, we have, by the Sobolev embedding theorem, $u_\infty \in L^p$ for all $2 \le p < \infty$. Hence $u_\infty \in W^{1,p}(\partial M)$ with $2 \le p < \infty$. By the Sobolev embedding theorem again, we obtain $u_\infty \in H \cap C^\alpha(\partial M)$ for all $0 < \alpha < 1$.

Suppose that $k_0 \ne 0$; then since $u_\infty \in X$, by integrating (3-10), we deduce that $\eta(u_\infty) = 1$ and u_∞ is a solution of (1-1). On the other hand, for $k_0 = 0$, if $\eta(u_\infty) = 0$, then $\partial_n u_\infty = 0$ and hence u_∞ is a constant, contradicting (1-4) and the fact proved above that $u_\infty \in X$; if $\eta(u_\infty) < 0$, then $v_\infty = u_\infty + \log(-\eta(u_\infty))$ is a solution of $e^{-v_\infty}\partial_n v_\infty = -f$. However, by integrating this equation, one has $\int_M f\, ds_{g_0} = \int_M e^{-v_\infty}|\nabla v_\infty|^2\, ds_{g_0} > 0$, contradicting (1-4). Hence, the only possibility is $\eta(u_\infty) > 0$, and for this case, one can see that $v_\infty = u_\infty + \log \eta(u_\infty)$ is a solution of $e^{-v_\infty}(\partial_n v_\infty + k_0) = f$.

In order to prove the asymptotic behavior of the flow, we need to show that

$$\lim_{k\to\infty} \|u(t_k) - u_\infty\| = 0. \tag{3-11}$$

Since $\nabla^X J(u_\infty) = 0$, it follows from (3-7) and (3-9) that

$$\|u(t_k) - u_\infty\| \le \|\nabla^X J(u(t_k))\| + \|(\partial_n + I)^{-1}(u(t_k) - u_\infty)\| \\ + C(\|(\partial_n + I)^{-1}(f e^{u_k} - f e^{u_\infty})\| + |\eta(u(t_k)) - \eta(u_\infty)|).$$

At this point, (3-11) follows from (3-3), Lemma 3.1 and (3-6).

Finally, we will end the proof of the lemma by showing that there exist two constants $\beta, \delta > 0$ such that for all $t \ge 0$,

$$\|u(t) - u_\infty\| \le \beta(1+t)^{-\delta}. \tag{3-12}$$

Before doing this, we will cite a version of the Łojasiewicz–Simon inequality:

Lemma 3.3 [Baird et al. 2004]. *Let X be an analytic manifold modeled on a Hilbert space $\mathcal{H}$ and suppose that $\mathcal{J} : X \to R$ is an analytic function on a neighborhood of a point $\tilde u \in X$ satisfying*:

(i) $\nabla \mathcal{J}(\tilde u) = 0$.

(ii) $\nabla^2 \mathcal{J}(\tilde u) : T_{\tilde u}X \to T_{\tilde u}X$ *is a Fredholm operator.*

Here, $\nabla \mathcal{J}$ denotes the gradient in X of $\mathcal{J}$ and we consider the second derivative $\nabla^2 \mathcal{J}(\tilde u)$ as a linear map $\nabla^2 \mathcal{J}(\tilde u) : T_{\tilde u}X \to T_{\tilde u}X$ by using the inner product on $T_{\tilde u}X$. Then there exist constants $\mu > 0$ and $0 < \theta < \frac{1}{2}$ such that if $u \in B(\tilde u, \mu)$ (the geodesic ball of radius μ centered on $\tilde u$), we have

$$\|\nabla \mathcal{J}(u)\| \ge |\mathcal{J}(u) - \mathcal{J}(\tilde u)|^{1-\theta}.$$

Now we apply Lemma 3.3 to the functional J in a neighborhood of the point u_∞.

Since L is an analytic function on H, X is an analytic manifold. Moreover, $J : X \subset H \to R$ is analytic and $\nabla^X J(u_\infty) = 0$. Let $\Pi_{u_\infty} : H \to T_{u_\infty} X$ be the projection onto $T_{u_\infty} X$. From (3-7)–(3-9), it follows that for all $v \in T_{u_\infty} X$,

$$\nabla^2 J(u_\infty)(v) = (I + \Pi_{u_\infty} A)(v),$$

where $A : H \to H$ is defined by

$$A(v) = -(\partial_n + I)^{-1}(v) - \left\langle \nabla J(u_\infty), \frac{\nabla L(u_\infty)}{\|\nabla L(u_\infty)\|^2} \right\rangle (\partial_n + I)^{-1}(f e^{u_\infty} v) \\ + \left\langle \nabla J(u_\infty), \frac{\nabla L(u_\infty)}{\|\nabla L(u_\infty)\|^2} \right\rangle \left\langle v, \frac{\nabla \|\nabla L(u_\infty)\|}{\|\nabla L(u_\infty)\|} \right\rangle \nabla L(u_\infty).$$

It is not difficult to check that A is a compact operator since $(\partial_n + I)^{-1}$ is a compact operator. Since Π_{u_∞} is a continuous map, it follows that $\Pi_{u_\infty} A$ is also compact. Hence, we conclude that $\nabla^2 J(u_\infty)$ is a Fredholm operator. It follows from Lemma 3.3 that there exist constants $\mu > 0$ and $0 < \theta < \frac{1}{2}$ such that if $\|u(t) - u_\infty\| < \mu$, then

$$\|\nabla^X J(u(t))\| \geq (J(u(t)) - J(u_\infty))^{1-\theta}. \tag{3-13}$$

We may assume that $J(u(t) - J(u_\infty)) > 0$ for all $t \geq 0$. Otherwise, if there exists $\tilde{t} \geq 0$ such that $J(u(\tilde{t})) = J(u_\infty)$, then since J is nonincreasing and the solution of (1-1) is unique, it follows that $u(t) \equiv u_\infty$ for all $t \geq \tilde{t}$. Therefore the solution is stationary and the estimate (3-12) is trivial. In view of (3-1), we have

$$-\frac{dJ(u(t))}{dt} = \|\nabla^X J(u(t))\| \|\partial_t u(t)\|. \tag{3-14}$$

From (3-11), we deduce that for all $\epsilon > 0$, there exists $N > 0$ such that

$$\|u(t_n) - u_\infty\| \leq \frac{\epsilon}{2} \quad \text{and} \quad \frac{1}{\theta}(J(u(t_n)) - J(u_\infty))^\theta \leq \frac{\epsilon}{2} \qquad \text{for all } n \geq N.$$

Let $\epsilon = \frac{1}{2}\mu$ and $t^* = \sup\{t \geq t_N : \|u(\tau) - u_\infty\| < \mu \text{ for all } \tau \in [t_N, t]\}$. Suppose that $t^* < \infty$. It follows from (3-13) and (3-14) that

$$-\frac{d}{dt}[(J(u(t)) - J(u_\infty))^\theta] \geq \theta \|\partial_t u(t)\| \tag{3-15}$$

for all $t \in [t_N, t^*]$. Integrating (3-15) and using the monotonicity of J yields

$$\|u(t^*) - u(t_N)\| \leq \int_{t_N}^{t^*} \|\partial_\tau u(\tau)\| \, d\tau \leq \frac{1}{\theta}(J(u(t_N)) - J(u_\infty))^\theta < \frac{\epsilon}{2}.$$

Recalling that $\epsilon = \frac{1}{2}\mu$, we have

$$\|u(t^*) - u_\infty\| \leq \|u(t^*) - u(t_N)\| + \|u(t_N) - u_\infty\| < \frac{\mu}{2},$$

which contradicts the definition of t^*. Hence $t^* = \infty$. This implies that estimate (3-13) holds for all $t \geq t_N$. Now, set $h(t) = J(u(t)) - J(u_\infty)$. Then from (3-13) and (3-14) again, it follows that

$$-\frac{dh(t)}{dt} = \|\nabla^X J(u(t))\|^2 \geq h^{2(1-\theta)}(t),$$

or, equivalently,

$$\frac{d}{dt}h^{2\theta-1}(t) \geq (1-2\theta),$$

for all $t \geq t_N$. Since $0 < \theta < \frac{1}{2}$, we can deduce that

$$h(t) \leq (h^{2\theta-1}(t_N) + (1-2\theta)(t-t_N))^{1/(2\theta-1)} \leq Ct^{-\delta'}, \tag{3-16}$$

where $\delta' = 1/(1-2\theta)$ and C are positive constants. We fix $t > t_N$ and integrate (3-15) from t to t_n (with n sufficiently large) to obtain, by estimate (3-16),

$$\|u(t) - u(t_n)\| \leq \int_t^{t_n} \|\partial_\tau u(\tau)\|\, d\tau \leq \frac{1}{\theta}(J(u(t)) - J(u_\infty))^\theta = \frac{1}{\theta}h^\theta(t) \leq \frac{1}{\theta}Ct^{-\theta\delta'}.$$

By letting $n \to \infty$, we obtain

$$\|u(t) - u_\infty\| \leq \frac{1}{\theta}Ct^{-\theta\delta'}$$

for all $t > t_N$. However, for $t \leq t_N$, $\|u(t) - u_\infty\|$ is bounded, so there exist two positive constants $\delta = \theta\delta'$ and β with

$$\|u(t) - u_\infty\| \leq \beta(1+t)^{-\delta}$$

for all $t \geq 0$. □

4. Convergence

In this section, we will apply Lemma 3.2 to obtain convergence. We thus only need to prove uniform boundedness in H of the global solution $u : [0, \infty) \to H$ in Theorem 2.1.

Proof of Theorem 2.2. Suppose that $k_0 = 0$. Writing $\mathbf{1}$ for the constant function, in view of (1-7) and (1-8) we have for $u \in X$ that

$$\langle \nabla J(u), \mathbf{1}\rangle = \int_{\partial M} \partial_n u \cdot \mathbf{1}\, ds_{g_0} = 0 \quad \text{and} \quad \langle \nabla L(u), \mathbf{1}\rangle = \int_{\partial M} e^u f \cdot \mathbf{1}\, ds_{g_0} = 0.$$

From this it follows that

$$0 = \langle \partial_t u, \mathbf{1}\rangle = \int_{\partial M} (\partial_n \partial_t u + \partial_t u)\mathbf{1}\, ds_{g_0},$$

which implies that

$$\partial_t \int_{\partial M} u(t)\, ds_{g_0} = 0.$$

Hence,

$$\int_{\partial M} u(t,\cdot)\, ds_{g_0} = \int_{\partial M} u_0\, ds_{g_0}. \tag{4-1}$$

From now on, we set $\bar{u} = (1/\mathscr{L}(\partial M)) \int_{\partial M} u\, ds_{g_0}$. Since $k_0 = 0$, the energy identity (2-1) yields

$$J(u) = \frac{1}{2}\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} \le J(u_0). \tag{4-2}$$

In order to show that $\|u(t)\| \le C$, it remains to bound $\int_{\partial M} u^2\, ds_{g_0}$. By Poincaré's inequality, we have

$$\|u-\bar{u}\|^2_{L^2} \le \lambda_1^{-1} \int_{\partial M} \partial_n u \cdot u\, ds_{g_0}, \tag{4-3}$$

where λ_1 is the first nonzero Steklov eigenvalue. From (4-3) and (4-1), it follows that

$$\begin{aligned}\int_{\partial M} u^2\, ds_{g_0} &\le \lambda_1^{-1}\int_{\partial M} \partial_n u\cdot u\, ds_{g_0} + \mathscr{L}(\partial M)\bar{u}^2\\ &= \lambda_1^{-1}\int_{\partial M} \partial_n u\cdot u\, ds_{g_0} + \mathscr{L}(\partial M)\overline{u_0}^2.\end{aligned}$$

Hence, we deduce from (4-2) that $\int_{\partial M} u^2\, ds_{g_0}$ is bounded. □

Proof of Theorem 2.4. Suppose that $k_0 < 0$; without loss of generality we assume that $k_0 = -1$. We first prove that the solution u satisfies a nonconcentration lemma:

Lemma 4.1. *Let K be a measurable subset of ∂M with $\mathscr{L}(K) > 0$. Then there exist a constant $\alpha > 1$ depending on M and g_0 and a constant $C_K > 1$ depending on M, g_0 and $\mathscr{L}(K)$ such that*

$$\int_{\partial M} e^u\, ds_{g_0} \le C_K e^{\alpha\|u_0\|^2} \max\biggl(\biggl(\int_K e^u\, ds_{g_0}\biggr)^{\alpha}, 1\biggr).$$

Proof. Step 1. We claim that there exists a positive constant C depending on M and g_0 such that, for any measurable subset K of M with $\mathscr{L}(K) > 0$, we have

$$\int_{\partial M} u\, ds_{g_0} \le |J(u_0)| + \frac{C}{\mathscr{L}(K)} + \frac{2\sqrt{2}\mathscr{L}(\partial M)}{\mathscr{L}(K)} \max\biggl(\int_K u\, ds_{g_0}, 0\biggr). \tag{4-4}$$

Fix $t > 0$. Suppose that $\int_{\partial M} u\, ds_{g_0} > 0$, otherwise estimate (4-4) is trivial. By the energy identity (2-1), we have

$$\frac{1}{2}\int_{\partial M} \partial_n u\cdot u\, ds_{g_0} \le J(u_0) + \int_{\partial M} u\, ds_{g_0}. \tag{4-5}$$

It follows from (4-5) and (4-3) that

$$\int_{\partial M} u^2\,ds_{g_0} \le \frac{2}{\lambda_1} J(u_0) + \frac{2}{\lambda_1}\int_{\partial M} u\,ds_{g_0} + \frac{1}{\mathscr{L}(\partial M)}\biggl(\int_{\partial M} u\,ds_{g_0}\biggr)^2. \tag{4-6}$$

Now, we consider the following two cases:

Case (i): $\int_K u(t)\,ds_{g_0} \le 0$. Then

$$\biggl(\int_{\partial M} u\,ds_{g_0}\biggr)^2 \le \biggl(\int_{K^c} u\,ds_{g_0}\biggr)^2 \le \mathscr{L}(K^c)\int_{\partial M} u^2\,ds_{g_0},$$

where K^c denotes the compliment of K in ∂M. Plugging this inequality into (4-6) yields

$$\frac{\mathscr{L}(K)}{\mathscr{L}(\partial M)}\int_{\partial M} u^2\,ds_{g_0} \le \frac{2}{\lambda_1} J(u_0) + \frac{2}{\lambda_1}\int_{\partial M} u\,ds_{g_0}. \tag{4-7}$$

On the other hand, by Young's inequality, we have

$$\biggl|\int_{\partial M} u\,ds_{g_0}\biggr| \le \epsilon\int_{\partial M} u^2\,ds_{g_0} + (4\epsilon)^{-1}\mathscr{L}(\partial M).$$

Taking $\epsilon = \lambda_1\mathscr{L}(K)/(4\mathscr{L}(\partial M))$ and substituting into (4-7) gives

$$\int_{\partial M} u^2\,ds_{g_0} \le \frac{4\mathscr{L}(\partial M)}{\lambda_1\mathscr{L}(K)} J(u_0) + \frac{2\mathscr{L}^3(\partial M)}{\lambda_1^2\mathscr{L}^2(K)}. \tag{4-8}$$

Since

$$\biggl(\int_{\partial M} u\,ds_{g_0}\biggr)^2 \le \mathscr{L}(\partial M)\int_{\partial M} u^2\,ds_{g_0},$$

it follows from (4-8) that

$$\begin{aligned}\biggl(\int_{\partial M} u\,ds_{g_0}\biggr)^2 &\le \frac{4\mathscr{L}^2(\partial M)}{\lambda_1\mathscr{L}(K)} J(u_0) + \frac{2\mathscr{L}^4(\partial M)}{\lambda_1^2\mathscr{L}^2(K)}\\ &\le |J(u_0)|^2 + \frac{4\mathscr{L}^4(\partial M)}{\lambda_1^2\mathscr{L}^2(K)} + \frac{2\mathscr{L}^4(\partial M)}{\lambda_1^2\mathscr{L}^2(K)}.\end{aligned}$$

Therefore,

$$\int_{\partial M} u\,ds_{g_0} \le |J(u_0)| + \frac{C}{\mathscr{L}(K)},$$

where C is a constant depending on $\mathscr{L}(\partial M)$ and g_0. This establishes case (i).

Case (ii): $\int_K u\,ds_{g_0} > 0$. Rewrite (4-6) as

$$\begin{aligned}\int_{\partial M} u^2\,ds_{g_0} &\le \frac{2}{\lambda_1} J(u_0) + \frac{2}{\lambda_1}\int_{\partial M} u\,ds_{g_0}\\ &\quad + \frac{1}{\mathscr{L}(\partial M)}\biggl\{\biggl(\int_K u\,ds_{g_0}\biggr)^2 + \biggl(\int_{K^c} u\,ds_{g_0}\biggr)^2 + 2\biggl(\int_K u\,ds_{g_0}\biggr)\biggl(\int_{K^c} u\,ds_{g_0}\biggr)\biggr\}.\end{aligned}$$

By Young's inequality and the fact that

$$\left(\int_{K^c} u\,ds_{g_0}\right)^2 \le \mathcal{L}(K^c)\int_{\partial M} u^2\,ds_{g_0},$$

we have

$$2\left(\int_K u\,ds_{g_0}\right)\left(\int_{K^c} u\,ds_{g_0}\right) \le \frac{2\mathcal{L}(K^c)}{\mathcal{L}(K)}\left(\int_K u\,ds_{g_0}\right)^2 + \frac{\mathcal{L}(K)}{2}\int_{\partial M} u^2\,ds_{g_0}.$$

Hence we arrive at

$$\frac{\mathcal{L}(K)}{2\mathcal{L}(\partial M)}\int_{\partial M} u^2\,ds_{g_0} \le \frac{2}{\lambda_1}J(u_0) + \frac{2}{\lambda_1}\int_{\partial M} u\,ds_{g_0} + \frac{2}{\mathcal{L}(K)}\left(\int_K u\,ds_{g_0}\right)^2.$$

By Young's inequality again, we obtain

$$\left|\int_{\partial M} u\,ds_{g_0}\right| \le \frac{\lambda_1\mathcal{L}(K)}{8\mathcal{L}(\partial M)}\int_{\partial M} u^2\,ds_{g_0} + \frac{2\mathcal{L}^2(\partial M)}{\lambda_1\mathcal{L}(K)}.$$

Therefore,

$$\int_{\partial M} u^2\,ds_{g_0} \le \frac{8\mathcal{L}(\partial M)}{\lambda_1\mathcal{L}(K)}J(u_0) + \frac{16\mathcal{L}^3(\partial M)}{\lambda_1^2\mathcal{L}^2(K)} + \frac{8\mathcal{L}(\partial M)}{\mathcal{L}^2(K)}\left(\int_K u\,ds_{g_0}\right)^2.$$

Since

$$\left(\int_{\partial M} u\,ds_{g_0}\right)^2 \le \mathcal{L}(\partial M)\left(\int_{\partial M} u^2\,ds_{g_0}\right),$$

it follows that

$$\left(\int_{\partial M} u\,ds_{g_0}\right)^2 \le \frac{8\mathcal{L}^2(\partial M)}{\lambda_1\mathcal{L}(K)}|J(u_0)| + \frac{16\mathcal{L}^4(\partial M)}{\lambda_1^2\mathcal{L}^2(K)} + \frac{8\mathcal{L}^2(\partial M)}{\mathcal{L}^2(K)}\left(\int_K u\,ds_{g_0}\right)^2,$$

which implies that

$$\int_{\partial M} u\,ds_{g_0} \le |J(u_0)| + \frac{C}{\mathcal{L}(K)} + \frac{2\sqrt{2}\mathcal{L}(\partial M)}{\mathcal{L}(K)}\int_K u\,ds_{g_0},$$

for a constant C depending on M and g_0. This establishes (4-4).

Step 2. We are in position to establish the lemma using the result in Step 1.

The energy identity (2-1) yields

$$\begin{aligned}\frac{1}{2}\int_{\partial M} \partial_n u\cdot u\,ds_{g_0} &\le J(u_0) + \int_{\partial M} u\,ds_{g_0}\\ &= J(u_0) + \mathcal{L}(\partial M)\bar{u} + \int_{\partial M}(u-\bar{u})\,ds_{g_0}.\end{aligned}$$

From the Young and Poincaré inequalities, it follows that

$$\begin{aligned}\frac{1}{2}\int_{\partial M}\partial_n u\cdot u\,ds_{g_0} &\le J(u_0)+\mathscr{L}(\partial M)\bar{u}+\frac{1}{4\epsilon}\mathscr{L}(\partial M)+\epsilon\int_{\partial M}(u-\bar{u})^2\,ds_{g_0}\\ &\le J(u_0)+\mathscr{L}(\partial M)\bar{u}+\frac{1}{4\epsilon}\mathscr{L}(\partial M)+\frac{\epsilon}{\lambda_1}\int_{\partial M}\partial_n u\cdot u\,ds_{g_0}.\end{aligned}$$

Taking $\epsilon=\frac{1}{4}\lambda_1$ gives

$$\int_{\partial M}\partial_n u\cdot u\,ds_{g_0}\le 4J(u_0)+4\mathscr{L}(\partial M)\bar{u}+\frac{4}{\lambda_1}\mathscr{L}(\partial M).$$

Using the inequality (1-6), we deduce that

$$\int_{\partial M}e^u\,ds_{g_0}\le C\exp\left\{\frac{J(u_0)}{\pi}+\frac{\mathscr{L}(\partial M)}{\pi\lambda_1}+\left(\frac{1}{\mathscr{L}(\partial M)}+\frac{1}{\pi}\right)\int_{\partial M}u\,ds_{g_0}\right\}. \tag{4-9}$$

Notice that we have

$$\begin{aligned}J(u_0)&=\int_{\partial M}\left(\frac{1}{2}\partial_n u_0-1\right)u_0\,ds_{g_0}\\ &\le\int_{\partial M}\partial_n u_0\cdot u_0\,ds_{g_0}+\int_{\partial M}u_0^2\,ds_{g_0}+\mathscr{L}(\partial M)\\ &=\|u_0\|^2+\mathscr{L}(\partial M).\end{aligned}$$

Plugging this inequality into (4-9) yields

$$\int_{\partial M}e^u\,ds_{g_0}\le C\exp\left\{\frac{\|u_0\|^2}{\pi}+B\int_{\partial M}u\,ds_{g_0}\right\},$$

where B and C are positive constants depending on M and g_0.

It follows from (4-4) that

$$\int_{\partial M}e^u\,ds_{g_0}\le C'_K\exp\left\{A_1\|u_0\|^2+\frac{B_1}{\mathscr{L}(K)}\max\left(\int_K u\,ds_{g_0},0\right)\right\},$$

where A_1, B_1 depend on M and g_0 and C'_K is a positive constant depending on M, $\mathscr{L}(K)$ and g_0. Moreover, we set $\alpha=\max(A_1,B_1)+1$. Then we have

$$\int_{\partial M}e^u\,ds_{g_0}\le C'_K\exp\left\{\alpha\|u_0\|^2+\frac{\alpha}{\mathscr{L}(K)}\max\left(\int_K u\,ds_{g_0},0\right)\right\}. \tag{4-10}$$

By Jensen's inequality, we have

$$\exp\left\{\frac{1}{\mathscr{L}(K)}\int_K u\,ds_{g_0}\right\}\le\frac{1}{\mathscr{L}(K)}\int_K e^u\,ds_{g_0}.$$

Hence

$$\exp\left\{\frac{\alpha}{\mathscr{L}(K)}\int_K u\,ds_{g_0}\right\}\le\left(\frac{1}{\mathscr{L}(K)}\int_K e^u\,ds_{g_0}\right)^{\alpha}.$$

But

$$\exp\left\{\frac{\alpha}{\mathscr{L}(K)}\max\left(\int_K u\,ds_{g_0},0\right)\right\}=\max\left(\exp\left\{\frac{\alpha}{\mathscr{L}(K)}\int_K u\,ds_{g_0}\right\},1\right)$$
$$\le\max\left(\left(\frac{1}{\mathscr{L}(K)}\int_K e^u\,ds_{g_0}\right)^{\alpha},1\right).$$

Inequality (4-10) thus implies that

$$\int_{\partial M} e^u\,ds_{g_0}\le C'_K e^{\alpha\|u_0\|^2}\max\left(\left(\frac{1}{\mathscr{L}(K)}\right)^{\alpha},1\right)\times\max\left(\left(\int_K e^u\,ds_{g_0}\right)^{\alpha},1\right)$$
$$\le C_K e^{\alpha\|u_0\|^2}\max\left(\left(\int_K e^u\,ds_{g_0}\right)^{\alpha},1\right),$$

where C_K is a constant depending on $\mathscr{L}(K)$, M and g_0, which we can suppose to be greater than 1. This completes the proof. □

The estimate of Lemma 4.1 will allow us to uniformly bound $\int_{\partial M} e^u\,ds_{g_0}$. Let

$$f^+=\max(f,0)\quad\text{and}\quad K=\left\{x\in\partial M: f(x)\le\tfrac{1}{2}\min_{x\in\partial M} f(x)\right\}.$$

Notice that we have $u_0\in X$. Then

$$\mathscr{L}(\partial M)=\int_{\partial M}-fe^{u_0}\,ds_{g_0}=\int_{\partial M}f^-e^{u_0}\,ds_{g_0}-\int_{\partial M}f^+e^{u_0}\,ds_{g_0}$$

implies that

$$\frac{\mathscr{L}(\partial M)}{-\min_{x\in\partial M}f(x)}\le\int_{\partial M}e^{u_0}\,ds_{g_0}. \tag{4-11}$$

However,

$$\int_{\partial M}u_0\,ds_{g_0}\le\int_{\partial M}u_0^2\,ds_{g_0}+\mathscr{L}(\partial M),$$

which, together with inequality (1-6), implies that

$$\begin{aligned}\int_{\partial M}e^{u_0}\,ds_{g_0}&\le C_1\exp\left\{C_1\left(\int_{\partial M}\partial_n u_0\cdot u_0\,ds_{g_0}+\int_{\partial M}u_0^2\,ds_{g_0}\right)\right\}\\&=C_1e^{C_1\|u_0\|^2},\end{aligned}\tag{4-12}$$

where C_1, which we may assume to be greater than 1, is a constant depending on M and g_0. Hence,

$$\frac{\mathscr{L}(\partial M)}{-\min_{x\in\partial M}f(x)}\le C_1e^{C_1\|u_0\|^2}. \tag{4-13}$$

Now set $\gamma = C_K(8C_1)^\alpha e^{(C_1+1)\alpha\|u_0\|^2}$ ($C_K > 1$ and $\alpha > 1$ are constants in Lemma 4.1). Suppose that condition (2-2) of Theorem 2.4,

$$e^{\xi\|u_0\|^2}\max_{x\in\partial M} f(x) \le \overline{C},$$

holds, with $\overline{C} = -\min_{x\in\partial M} f(x)/(8^\alpha C_K C_1^{\alpha-1})$ and $\xi = \alpha(C_1+1) - C_1$. We wish to show that

$$\int_{\partial M} e^{u(t)}\, ds_{g_0} \le 2\gamma \quad \text{for all } t \ge 0. \tag{4-14}$$

Let

$$I = \left\{ t \ge 0 : \int_{\partial M} e^{u(\tau)}\, ds_{g_0} \le 2\gamma \text{ for all } \tau \in [0, t] \right\}.$$

From (4-12), it follows that $0 \in I$. Let $T = \sup I$. Suppose $T < \infty$. Then by continuity of the map $t \to \int_{\partial M} e^{u(t)}\, ds_{g_0}$, we have

$$\int_{\partial M} e^{u(T)}\, ds_{g_0} = 2\gamma. \tag{4-15}$$

We consider two cases:

Case (i): $$\int_{\partial M} f^+ e^{u(T)}\, ds_{g_0} \le \frac{1}{2}\int_{\partial M} f^- e^{u(T)}\, ds_{g_0}.$$

Using the fact that $u(T) \in X$, we get

$$\int_{\partial M} f^- e^{u(T)}\, ds_{g_0} \le -2\int_{\partial M} f e^{u(T)}\, ds_{g_0} = 2\mathcal{L}(\partial M). \tag{4-16}$$

Since $f^-(x) \ge \frac{1}{2}(-\min_{x\in\partial M} f(x))$ for all $x \in K$, it follows from (4-16) and (4-11) that

$$\int_K e^{u(T)}\, ds_{g_0} \le \frac{4\mathcal{L}(\partial M)}{-\min_{x\in\partial M} f(x)} \le 4C_1 e^{C_1\|u_0\|^2}.$$

We thus deduce from Lemma 4.1 that

$$\begin{aligned}\int_{\partial M} e^{u(T)}\, ds_{g_0} &\le C_K e^{\alpha\|u_0\|^2}\max\left(\left(\int_K e^{u(T)}\, ds_{g_0}\right)^\alpha, 1\right)\\ &\le C_K e^{\alpha\|u_0\|^2}\max((2C_1)^\alpha e^{\alpha C_1\|u_0\|^2}, 1)\\ &= C_K(2C_1)^\alpha e^{(C_1+1)\alpha\|u_0\|^2} < \gamma,\end{aligned}$$

which contradicts (4-15).

Case (ii): $$\int_{\partial M} f^+ e^{u(T)}\, ds_{g_0} > \frac{1}{2}\int_{\partial M} f^- e^{u(T)}\, ds_{g_0}.$$

Since $f^-(x) \geq \frac{1}{2}(-\min_{x\in\partial M} f(x))$ for all $x \in K$, it follows from (4-15) that

$$\begin{aligned} -\frac{\min_{x\in\partial M} f(x)}{2}\int_K e^{u(T)}\, ds_{g_0} &\leq \int_{\partial M} f^- e^{u(T)}\, ds_{g_0} \\ &\leq 2\int_{\partial M} f^+ e^{u(T)}\, ds_{g_0} \leq 4\gamma \max_{x\in\partial M} f(x). \end{aligned}$$

Then condition (2-2) of Theorem 2.4 implies that

$$\int_K e^{u(T)}\, ds_{g_0} \leq 8\frac{\gamma \max_{x\in\partial M} f(x)}{-\min_{x\in\partial M} f(x)} \leq 8C_1 e^{C_1\|u_0\|^2}.$$

As before, by Lemma 4.1, we have

$$\int_{\partial M} e^{u(T)}\, ds_{g_0} \leq \gamma,$$

which contradicts (4-15) again. We thus conclude that (4-14) holds.

Now from Jensen's inequality, (4-14) implies that

(4-17) $$\bar{u} = \frac{1}{\mathscr{L}(\partial M)}\int_{\partial M} u(t)\, ds_{g_0} \leq C,$$

where C is a constant depending on M, g_0, f and u_0. The energy identity gives

(4-18) $$\frac{1}{2}\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} - \int_{\partial M} (u-\bar{u})\, ds_{g_0} - \mathscr{L}(\partial M)\bar{u} \leq J(u_0).$$

On the other hand, Young's inequality gives

(4-19) $$\left|\int_{\partial M} (u-\bar{u})\, ds_{g_0}\right| \leq \epsilon\|u-\bar{u}\|^2_{L^2} + \frac{1}{4\epsilon}\mathscr{L}(\partial M).$$

Setting $\epsilon = \frac{1}{4}\lambda_1$ and using Poincaré's inequality, we deduce from (4-18) and (4-19) that

(4-20) $$\frac{1}{4}\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} - \mathscr{L}(\partial M)\bar{u} \leq J(u_0) + \frac{1}{\lambda_1}\mathscr{L}(\partial M).$$

Using (4-17), we have

$$\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} \leq C.$$

To show that $\|u(t)\| \leq C$, it remains to bound $\int_{\partial M} u^2\, ds_{g_0}$. From (4-20), it follows that

$$-\mathscr{L}(\partial M)\bar{u} \leq C;$$

hence, we deduce that $\bar{u} \geq C$. Combining this with (4-17) yields

$$|\bar{u}| \leq C.$$

Now Poincaré's inequality implies that

$$\|u-\bar{u}\|_{L^2}^2 \leq \frac{1}{\lambda_1} \int_{\partial M} \partial_n u \cdot u \, ds_{g_0} \leq C.$$

Thus, $\int_{\partial M} u^2 \, ds_{g_0} \leq C$. □

Proofs of Theorem 2.6 and Corollary 2.7. Suppose that $k_0 > 0$. In this case, we consider $M = D$, the unit disc. Then $k_0 = 1$. Hence, if $u \in X$, we have $\int_{\partial M} f e^u \, ds_{g_0} = 2\pi$.

Proof of Theorem 2.6. Let $v_a = u \circ \Phi_a + \log|\Phi_a'|$, where Φ_a is the Möbius transformation. From [Chang and Liu 1996, Theorem 2.1] and (2-1), it follows that

$$J(v_a) = J(u) \leq J(u_0), \tag{4-21}$$

and since $u \in X$, we have

$$\int_{S^1} f \circ \Phi_a e^{v_a} \, ds_{g_0} = \int_{S^1} f e^u \, ds_{g_0} = 2\pi. \tag{4-22}$$

From (4-22), we deduce that

$$\int_{S^1} e^{v_a} \, ds_{g_0} \geq \frac{2\pi}{\max_{x \in S^1} f(x)}. \tag{4-23}$$

It is well-known that for all $t > 0$, there exists $a(t) \in D$ such that

$$\int_{S^1} x_i e^{v_{a(t)}} \, ds_{g_0} = 0 \quad \text{for } i = 1, 2. \tag{4-24}$$

Set $v(t) = v_{a(t)}$ and $\Phi(t) = \Phi_{a(t)}$. From now on, we assume that C is a constant only depending on u_0 and $\sup_{x \in S^1} f(x)$. In view of (4-23) and (4-24), it follows from the Osgood–Phillips–Sarnak inequality (see [Osgood et al. 1988]) that

$$C \leq ⨍_{S^1} e^{v(t)} \, ds_{g_0} \leq \exp\left\{ ⨍_{S^1} \left(\frac{1}{4} \partial_n v(t) + 1\right) v(t) \, ds_{g_0} \right\}. \tag{4-25}$$

It follows from (4-21) and (4-25) that

$$\frac{1}{2} \int_{S^1} \partial_n v(t) \cdot v(t) \, ds_{g_0} + \int_{S^1} v(t) \, ds_{g_0} \leq C, \tag{4-26}$$

and

$$\frac{1}{4} \int_{S^1} \partial_n v(t) \cdot v(t) \, ds_{g_0} + \int_{S^1} v(t) \, ds_{g_0} \geq C. \tag{4-27}$$

By taking the difference between (4-26) and (4-27), we obtain

$$\int_{S^1} \partial_n v(t) \cdot v(t)\, ds_{g_0} \le C. \tag{4-28}$$

Now, combining (4-26) and (4-27) yields

$$\left| \int_{S^1} v(t)\, ds_{g_0} \right| \le C. \tag{4-29}$$

Therefore, by the Lebedev–Milin inequality (see [Chang and Liu 1996, (1.12)]), we deduce from (4-28) and (4-29) that for all $p > 1$,

$$\int_{S^1} e^{|pv(t)|}\, ds_{g_0} \le C(p). \tag{4-30}$$

It follows from (4-30) that

$$\int_{S^1} v^2(t)\, ds_{g_0} \le C,$$

which, together with (4-28), implies that

$$\|v(t)\| \le C. \tag{4-31}$$

Next, we wish to prove that u is uniformly bounded in H. To do so, we first establish the following lemma.

Lemma 4.2. *Either*:

(i) *there exists a constant* C *such that* $\|u(t)\| \le C$; *or*,

(ii) *there exists a sequence* $t_n \to \infty$ *and a point* $a_\infty \in S^1$ *such that for all* $r > 0$,

$$\lim_{n\to\infty} \int_{\mathcal{S}(a_\infty, r)} f e^{u(t_n)}\, ds_{g_0} = 2\pi, \tag{4-32}$$

where $\mathcal{S}(a_\infty, r)$ *is an arc in* S^1 *centered at* a_∞ *and with radius* r. *Moreover, for all* $q \in S^1 \backslash \{a_\infty\}$ *and all* $0 < r < \operatorname{dist}(q, a_\infty)$, *we have*

$$\lim_{n\to\infty} \int_{\mathcal{S}(q,r)} f e^{u(t_n)}\, ds_{g_0} = 0.$$

Proof. There are two possibilities:

Case (i): $\limsup_{t\to\infty} |a(t)| < 1$. Then we have for all $t \ge 0$ that $0 < C_1 \le |\Phi'| \le C_2$. Hence, it follows from (4-31) that

$$\int_{S^1} |u(t)|\, ds_{g_0} \le C. \tag{4-33}$$

Combining (4-33) with the energy identity (2-1) yields

$$\int_{S^1} \partial_n u(t) \cdot u(t)\, ds_{g_0} \le C. \tag{4-34}$$

Hence, using Poincaré's inequality, we deduce from (4-33) and (4-34) that

$$\|u(t)\| \le C.$$

Case (ii): there exist a sequence $t_n \to \infty$ and $a_\infty \in S^1$ such that $a(t_n) \to a_\infty$. From the estimate (4-31), it follows that there exist a subsequence of t_n, still denoted by t_n, and a function $v_\infty \in H$, such that

$$\begin{cases} v(t_n) \to v_\infty & \text{weakly in } H, \\ v(t_n) \to v_\infty & \text{strongly in } L^2. \end{cases}$$

Let $r > 0$ and set $K_n = (\Phi(t_n))^{-1}(\mathcal{S}(a_\infty, r))$. Then we have

$$\left| \int_{S^1} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} - \int_{K_n} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} \right| \\ \le \max_{x \in S^1} f(x) \left(\mathcal{L}(K_n^c) \int_{S^1} e^{|2v(t_n)|}\, ds_{g_0} \right)^{\frac{1}{2}}.$$

Since $\lim_{t\to\infty} \Phi(t_n)(x) = a_\infty$ a.e., it follows that $\lim_{n\to\infty} \mathcal{L}(K_n) = 2\pi$. Thus, we deduce from (4-30) that

$$\begin{aligned} \int_{\mathcal{S}(a_\infty, r)} f e^{u(t_n)}\, ds_{g_0} &= \int_{K_n} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} \\ &= \int_{S^1} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} + \epsilon_n, \end{aligned} \tag{4-35}$$

with $\lim\limits_{n\to\infty} \epsilon_n = 0$. In view of (4-22), we have

$$\int_{S^1} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} = 2\pi.$$

Therefore, it follows from (4-35) that (4-32) holds. □

Now, we suppose that $u(t) \ne u_0$ for all $t > 0$ (otherwise the solution is stationary and the convergence is obvious). Since u_0 is G-invariant, by using the uniqueness of the solution u, it is not hard to conclude that u is also G-invariant. Again from the uniqueness of u, we can see from the energy identity (2-1) that

$$J(u(t)) < J(u(t')) \quad \text{for } t > t'. \tag{4-36}$$

Case (i): $\Sigma=\varnothing$. Suppose that u is not uniformly bounded in H. So from Lemma 4.2, there exists a point $a_\infty \in S^1$ satisfying (4-32) for all $r>0$. Since $\Sigma=\varnothing$, there exists $\sigma \in G$ such that $\sigma(a_\infty) \neq a_\infty$. Now, for all $r>0$, we have

$$\lim_{n\to\infty} \int_{\mathcal{I}(\sigma(a_\infty),r)} f e^{u(t_n)}\, ds_{g_0} = \lim_{n\to\infty} \int_{\mathcal{I}(a_\infty,r)} f e^{u(t_n)}\, ds_{g_0} = 2\pi,$$

which contradicts Lemma 4.2(ii).

Case (ii): $\Sigma \neq \varnothing$. Suppose that u is not uniformly bounded in H. So from Lemma 4.2, there exists a point $a_\infty \in S^1$ satisfying (4-32) for all $r>0$. If $a_\infty \notin \Sigma$, then in the same way as in case (i) above, we arrive at a contradiction. Otherwise, we have for all $r>0$ that

$$\begin{aligned}\int_{\mathcal{I}(a_\infty,r)} f e^{u(t_n)}\, ds_{g_0} &\le \max_{x\in\mathcal{I}(a_\infty,r)} f(x) \int_{\mathcal{I}(a_\infty,r)} e^{u(t_n)}\, ds_{g_0}\\ &\le \max\Bigl(\max_{x\in\mathcal{I}(a_\infty,r)} f(x),0\Bigr) \int_{\mathcal{I}(a_\infty,r)} e^{u(t_n)}\, ds_{g_0}\\ &\le \max\Bigl(\max_{x\in\mathcal{I}(a_\infty,r)} f(x),0\Bigr) \int_{S^1} e^{u(t_n)}\, ds_{g_0}.\end{aligned} \tag{4-37}$$

Now, we may write the Lebedev–Milin inequality as

$$\fint_{S^1} e^{u(t_n)}\, ds_{g_0} \le e^{J(u(t_n))/2\pi},$$

which, together with (4-36), yields

$$\fint_{S^1} e^{u(t_n)}\, ds_{g_0} \le e^{J(u(t_n))/2\pi} \le e^{J(u_0)/2\pi}. \tag{4-38}$$

Plugging (4-38) into (4-37) and letting $n\to\infty$, we obtain

$$2\pi \le 2\pi \max\Bigl(\max_{x\in\mathcal{I}(a_\infty,r)} f(x),0\Bigr) e^{J(u_0)/2\pi}. \tag{4-39}$$

Estimate (4-39) implies that $f(a_\infty)>0$ so that

$$1 \le f(a_\infty) e^{J(u_0)/2\pi}.$$

Hence

$$f(a_\infty) > e^{-J(u_0)/2\pi},$$

which contradicts assumption (ii) of the theorem. This establishes Theorem 2.6. $\square$

Proof of Corollary 2.7. If $\Sigma = \varnothing$, then the result of Corollary 2.7 is a direct consequence of Theorem 2.6. Suppose now that $\Sigma \neq \varnothing$ and let f satisfy inequality (2-3): if $\int_{S^1} f \circ \Phi_{a_0} \, ds_{g_0} \leq 0$, then $\sup_{p\in\Sigma} f(p) \leq 0$; so condition (ii) of Theorem 2.6 is satisfied. Otherwise, if $\int_{S^1} f \circ \Phi_{a_0} \, ds_{g_0} > 0$, we let $u^* = \log |\Phi'_{a_0}|$. Then we have $J(u^*) = 0$ (see [Chang and Liu 1996]). Now set $u_0 = u^* + C$, where C is a constant satisfying

$$e^C \int_{S^1} f \circ \Phi_{a_0} \, ds_{g_0} = 2\pi.$$

This implies that $u_0 \in X$. Since $a_0 \in \Sigma$, it is not difficult to see that u_0 is G-invariant. Hence we conclude that condition (ii) of Theorem 2.6 is equivalent to

$$\max_{p\in\Sigma} f(p) \leq \fint_{S^1} f \circ \Phi_{a_0} \, ds_{g_0}.$$

This completes the proof of Corollary 2.7. □

Acknowledgements

The author would like to thank his academic advisor, Professor Xu Xingwang, for reading the manuscript and providing valuable comments, and for his constant support and encouragement. The author also would like to thank the anonymous referee for the critical comments.

References

[Baird et al. 2004] P. Baird, A. Fardoun, and R. Regbaoui, "The evolution of the scalar curvature of a surface to a prescribed function", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **3**:1 (2004), 17–38. MR 2005e:53100 Zbl 1170.58306

[Baird et al. 2006] P. Baird, A. Fardoun, and R. Regbaoui, "Q-curvature flow on 4-manifolds", *Calc. Var. Partial Differential Equations* **27**:1 (2006), 75–104. MR 2007d:53107 Zbl 1094.53024

[Brendle 2002a] S. Brendle, "Curvature flows on surfaces with boundary", *Math. Ann.* **324**:3 (2002), 491–519. MR 2003j:53103 Zbl 1024.53045

[Brendle 2002b] S. Brendle, "A generalization of the Yamabe flow for manifolds with boundary", *Asian J. Math.* **6**:4 (2002), 625–644. MR 2003m:53052 Zbl 1039.53035

[Brendle 2003] S. Brendle, "Global existence and convergence for a higher order flow in conformal geometry", *Ann. of Math.* (2) **158**:1 (2003), 323–343. MR 2004e:53098 Zbl 1042.53016

[Chang and Liu 1996] K. C. Chang and J. Q. Liu, "A prescribing geodesic curvature problem", *Math. Z.* **223**:2 (1996), 343–365. MR 97h:53043 Zbl 0865.53033

[Chen and Xu 2012] X. Chen and X. Xu, "The scalar curvature flow on S^n—perturbation theorem revisited", *Invent. Math.* **187**:2 (2012), 395–506. MR 2885623 Zbl 1275.53056

[Cherrier 1984] P. Cherrier, "Problèmes de Neumann non linéaires sur les variétés Riemanniennes", *J. Funct. Anal.* **57**:2 (1984), 154–206. MR 86c:58154 Zbl 0552.58032

[Ho 2011] P. T. Ho, "Prescribed curvature flow on surfaces", *Indiana Univ. Math. J.* **60**:5 (2011), 1517–1542. MR 2996999 Zbl 06122677

[Kazdan and Warner 1975] J. L. Kazdan and F. W. Warner, "Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures", *Ann. of Math.* (2) **101** (1975), 317–331. MR 51 #11349 Zbl 0297.53020

[Li and Liu 2005] Y. Li and P. Liu, "A Moser–Trudinger inequality on the boundary of a compact Riemann surface", *Math. Z.* **250**:2 (2005), 363–386. MR 2007b:58036 Zbl 1070.26017

[Liu and Huang 2005] P. Liu and W. Huang, "On prescribing geodesic curvature on D^2", *Nonlinear Anal.* **60**:3 (2005), 465–473. MR 2005h:53057 Zbl 1086.53054

[Malchiodi and Struwe 2006] A. Malchiodi and M. Struwe, "Q-curvature flow on S^4", *J. Differential Geom.* **73**:1 (2006), 1–44. MR 2007c:53087 Zbl 1099.53034

[Osgood et al. 1988] B. Osgood, R. Phillips, and P. Sarnak, "Extremals of determinants of Laplacians", *J. Funct. Anal.* **80**:1 (1988), 148–211. MR 90d:58159 Zbl 0653.53022

[Struwe 2005] M. Struwe, "A flow approach to Nirenberg's problem", *Duke Math. J.* **128**:1 (2005), 19–64. MR 2005m:53125 Zbl 1087.53034

Received November 19, 2013. Revised July 30, 2014.

HONG ZHANG
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE, 119076
SINGAPORE

hzhang@nus.edu.sg

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.331

-1-PHENOMENA FOR THE PLURI χ_y-GENUS AND ELLIPTIC GENUS

PING LI

Several independent articles have observed that the Hirzebruch χ_y-genus has an important feature, which we call -1-phenomenon and which tells us that the coefficients of the Taylor expansion of the χ_y-genus at $y = -1$ have explicit expressions. Hirzebruch's original χ_y-genus can be extended towards two directions: the pluri-case and the case of elliptic genus. This paper contains two parts, in which we investigate the -1-phenomena in these two generalized cases and show that in each case there exists a -1-phenomenon in a suitable sense. Our main results in the first part have an application, which states that all characteristic numbers (Chern numbers and Pontrjagin numbers) on manifolds can be expressed, in a very explicit way, in terms of some rational linear combination of indices of some elliptic operators. This gives an analytic interpretation of characteristic numbers and affirmatively answers a question posed by the author several years ago. The second part contains our attempt to generalize this -1-phenomenon to the elliptic genus, a modern version of the χ_y-genus. We first extend the elliptic genus of an almost-complex manifold to a twisted version where an extra complex vector bundle is involved, and show that it is a weak Jacobi form under some assumptions. A suitable manipulation on the theory of Jacobi forms will produce new modular forms from this weak Jacobi form, and thus much arithmetic information related to the underlying manifold can be obtained, in which the -1-phenomenon of the original χ_y-genus is hidden.

The author was partially supported by the National Natural Science Foundation of China (grants 11101308 and 11471247) and the Fundamental Research Funds for the Central Universities.

MSC2010: primary 58J20; secondary 58J26, 11F11, 11F50.

Keywords: χ_y-genus, pluri-genus, elliptic genus, elliptic operator, characteristic number, Jacobi form, modular form.

1. Introduction

The Hirzebruch χ_y-genus and its -1-phenomenon. In his highly influential book, Hirzebruch [1966] defined a polynomial with integral coefficients $\chi_y(M)$ given a projective manifold M, which encodes the information of indices of Dolbeault complexes and is now called the *Hirzebruch χ_y-genus*. After the discovery of the general index theorem due to Atiyah and Singer, we know that $\chi_y(\,\cdot\,)$ can be defined on compact almost-complex manifolds and computed in terms of Chern numbers as follows.

Suppose (M^{2d}, J) is a compact connected almost-complex manifold with an almost-complex structure J. The choice of an almost Hermitian metric on M enables us to define the Hodge star operator $*$ and the formal adjoint $\bar{\partial}^* = - * \bar{\partial} *$ of the $\bar{\partial}$-operator. For each pair $0 \le p, q \le d$, we denote by

$$\Omega^{p,q}(M) := \Gamma(\Lambda^p T^*M \otimes \Lambda^q \overline{T^*M})$$

the complex vector space which consists of smooth complex-valued (p,q)-forms. Here T^*M is the dual of the holomorphic tangent bundle TM in the sense of J. Then for each $0 \le p \le d$, we have the Dolbeault-type elliptic differential operator

$$\bigoplus_{q \text{ even}} \Omega^{p,q}(M) \xrightarrow{(\bar{\partial}+\bar{\partial}^*)|_p} \bigoplus_{q \text{ odd}} \Omega^{p,q}(M),$$

whose index is denoted by $\chi^p(M)$ in the notation of [Hirzebruch 1966]. Then the Hirzebruch χ_y-genus of M is nothing but the generating function of the indices $\chi^p(M)$ $(0 \le p \le d)$:

$$\chi_y(M) := \sum_{p=0}^{d} \chi^p(M) \cdot y^p.$$

Let us denote by $x_1, \dots, x_d$ the formal Chern roots of TM. This means that the i-th elementary symmetric polynomial of $x_1, \dots, x_d$ represents the i-th Chern class c_i of TM. Then the general form of the Hirzebruch–Riemann–Roch theorem (first proved by Hirzebruch [1966] for projective manifolds, and in the general case by Atiyah and Singer [1968]) tells us that

$$\chi_y(M) = \int_M \prod_{i=1}^{d} \frac{x_i(1+ye^{-x_i})}{1-e^{-x_i}}. \tag{1-1}$$

Among other things, the Hirzebruch χ_y-genus has an important feature, which we call the "-1-*phenomenon*" and has been noticed, implicitly or explicitly, in several independent articles [Narasimhan and Ramanan 1975; Libgober and Wood 1990; Salamon 1996]. This -1-phenomenon says that at $y = -1$, the coefficients of the

Taylor expansion of $\chi_y(M)$ have explicit expressions. To be more precise, if we write

$$\chi_y(M) =: \sum_{i=0}^{d} a_i(M)\cdot(y+1)^i, \tag{1-2}$$

then these $a_i(M)$ can be given explicit expressions in terms of Chern numbers of (M^{2d}, J) as

$$\begin{aligned} a_0(M) &= c_d,\\ a_1(M) &= -\tfrac{1}{2}dc_d,\\ a_2(M) &= \ \ \tfrac{1}{12}\big[\tfrac{1}{2}d(3d-5)c_d + c_1c_{d-1}\big],\\ a_3(M) &= -\tfrac{1}{24}\big[\tfrac{1}{2}d(d-2)(d-3)c_d + (d-2)c_1c_{d-1}\big],\\ &\ \,\vdots \end{aligned} \tag{1-3}$$

By definition, these $a_i(M)$ are integers. Thus, immediate consequences of their expressions include divisibility properties of Chern numbers. The derivation of these expressions is direct, i.e., by expanding the right-hand side of (1-1) at $y = -1$ and expressing the coefficients in terms of elementary symmetric polynomials of $x_1, \ldots, x_d$. The calculations of a_0 and a_1 are quite easy. The calculation of a_2 appears implicitly in [Narasimhan and Ramanan 1975, p. 18] and explicitly in [Libgober and Wood 1990, p. 141–143]. Narasimhan and Ramanan used a_2 to give a topological restriction on some moduli spaces of stable vector bundles on smooth projective varieties. Libgober and Wood used a_2 to prove the uniqueness of the complex structure on Kähler manifolds of certain homotopy types. Inspired by [Narasimhan and Ramanan 1975], Salamon applied a_2 [1996, Corollary 3.4] to obtain a restriction on the Betti numbers of hyper-Kähler manifolds [ibid., Theorem 4.1]. The expressions of a_3 and a_4 are also included in [ibid., p. 145]. Hirzebruch [1999] used a_1, a_2 and a_3 to obtain a divisibility result on the Euler characteristic of those almost-complex manifolds where $c_1c_{d-1} = 0$. In particular, those almost-complex manifolds with $c_1 = 0$ satisfy this property.

Pluri-χ_y-genus. Some acquaintance with index theory will lead to the observation that $\chi_y(M)$ is the index of the Todd operator (whose index is the Todd genus)

$$\Omega^{0,\mathrm{even}}(M) \xrightarrow{(\bar\partial+\bar\partial^*)|_0} \Omega^{0,\mathrm{odd}}(M) \tag{1-4}$$

twisted by $\Omega_y(M)$, with

$$\Omega_y(M) := \sum_{p=0}^{d} \Lambda^p(T^*M)\cdot y^p \in K(M)[y],$$

where $\Lambda^p(\cdot)$ and $K(\cdot)$ denote the p-th exterior power and K-group. Therefore $\chi_y(M)$ can be rewritten as

$$\chi_y(M) = \mathrm{Ind}\big((\bar\partial + \bar\partial^*)|_0 \otimes \Omega_y(M)\big) =: \chi(M, \Omega_y(M)).$$

Here, for simplicity we denote by the standard notation $\chi(M, (\cdot))$ the index of the Todd operator (1-4) twisted by an element $(\cdot) \in K(M)$.

We can also consider, for an arbitrarily fixed positive integer g, the *pluri χ_y-genus* $\chi_{\underline{y}}(M)$ by using sufficiently many forms of the type

$$\begin{aligned}\Omega_{\underline{y}}(M) &:= \sum_{0\le p_1,\dots,p_g\le d} \Lambda^{p_1}(T^*M)\otimes\cdots\otimes\Lambda^{p_g}(T^*M)\cdot y_1^{p_1}\cdots y_g^{p_g} \\ &= \Omega_{y_1}(M)\otimes\cdots\otimes\Omega_{y_g}(M) \in K(M)[y_1,\dots,y_g]\end{aligned} \tag{1-5}$$

to twist $(\bar\partial + \bar\partial^*)|_0$, i.e.,

$$\chi_{\underline{y}}(M) := \mathrm{Ind}\big((\bar\partial + \bar\partial^*)|_0 \otimes \Omega_{\underline{y}}(M)\big) = \chi(M, \Omega_{\underline{y}}(M)),$$

which specializes to Hirzebruch's original χ_y-genus when $g = 1$.

Inspired by the above-mentioned -1-phenomenon of the χ_y-genus, we may ask what the coefficients look like if we expand $\chi_{\underline{y}}(M)$ at $y_1 = \cdots = y_g = -1$. Our first main observation in this article is that the coefficients of $(y+1)^{p_1}\cdots(y+1)^{p_g}$ in $\chi_{\underline{y}}(M)$ can be divided into three parts, which is our main result in Section 3 (Theorem 2.2). Moreover, we can do a similar manipulation for signature operator on closed smooth oriented manifolds, and their coefficients also have a similar feature (Theorem 2.3). A direct corollary of these two theorems is that any Chern number of (M^{2d}, J) or any Pontrjagin number of a closed smooth oriented manifold can be written explicitly as a rational linear combination of indices of some elliptic operators, which provides an analytic interpretation of characteristic numbers and answers [Li 2011, Question 1.1] affirmatively.

Elliptic genus. Elliptic genera of oriented differentiable manifolds and almost-complex manifolds were first constructed by Ochanine, Landweber, Stong and Hirzebruch in a topological way; Witten gave it a geometric interpretation, in which they can be viewed as the loop space analogues of the Hirzebruch L-genus and χ_y-genus (see [Landweber 1988] and the references therein). The most remarkable property of elliptic genera is their rigidity for spin manifolds and almost-complex Calabi–Yau manifolds (in the very weak sense that c_1 vanishes up to torsion, i.e., $c_1 = 0 \in H^2(M, \mathbb{R})$), which was conjectured by Witten and generalizes the famous rigidity property of the original L-genus, $\hat{A}$-genus [Atiyah and Hirzebruch 1970] and χ_y-genus [Lusztig 1971]. The first rigorous proof was presented in [Bott and Taubes 1989; Taubes 1989]. A quite simple, unified and enlightening proof was discovered by Liu [1996], in which modular invariance of the four classical

Jacobi theta functions and their various transformation laws play key roles. Later on, this modular invariance property, its various remarkable extensions and relation with vertex operator algebra were established by Liu and his coauthors from various perspectives [Liu 1995a; 1995b; Liu and Ma 2000; Liu et al. 2001; 2003; Han and Zhang 2004; Dong et al. 2005; Chen and Han 2009; Chen et al. 2011; Han et al. 2012; Han and Liu 2014].

We are concerned in this paper with the elliptic genus of almost-complex manifolds. The elliptic genus of a compact, almost-complex manifold (M^{2d}, J), which we denote by $\mathrm{Ell}(M, \tau, z)$, is defined as a function of two variables $(\tau, z) \in \mathbb{H} \times \mathbb{C}$, where $\mathbb{H}$ is the upper half plane. To be more precise, $\mathrm{Ell}(M, \tau, z)$ is defined to be the index of the Todd operator (1-4) twisted by

$$y^{-d/2} \bigotimes_{n\geq 1} (\Lambda_{-yq^{n-1}} T^* \otimes \Lambda_{-y^{-1}q^n} T \otimes \mathrm{S}_{q^n} T^* \otimes \mathrm{S}_{q^n} T) =: \mathrm{E}_{q,y},$$

i.e., $\mathrm{Ell}(M, \tau, z) := \chi(M, \mathrm{E}_{q,y})$, where $q = e^{2\pi\sqrt{-1}\tau}$, $y = e^{2\pi\sqrt{-1}z}$ and T (resp. T^*) is the holomorphic (resp. dual of the holomorphic) tangent bundle of M in the sense of J. Here, for any complex vector bundle W,

$$\Lambda_t(W) := \bigoplus_{i\geq 0} \Lambda^i(W) \quad \text{and} \quad S_t(W) := \bigoplus_{i\geq 0} S^i(W)$$

denote the generating series of the exterior and symmetric powers of W, respectively.

According to the Atiyah–Singer index theorem, we have

$$\begin{aligned}\mathrm{Ell}(M, \tau, z)& \\ &= \int_M \mathrm{td}(M) \cdot \mathrm{ch}(\mathrm{E}_{q,y}) \\ &= y^{-\frac{1}{2}d} \chi_{-y}(M) + q \cdot [y^{-\frac{1}{2}d} \chi_{-y}(M, T^*(1-y) + T(1-y^{-1}))] + q^2 \cdot (\cdots),\end{aligned}$$

where

$$\mathrm{td}(M) := \prod_{i=1}^{d} \frac{x_i}{1 - e^{-x_i}}$$

is the Todd class of M and $\mathrm{ch}(\,\cdot\,)$ is the Chern character.

Thus, the elliptic genus $\mathrm{Ell}(M, \tau, z)$ can be viewed as a generalization of the Hirzebruch χ_y-genus, in the sense that the q^0-term of the Fourier expansion of $\mathrm{Ell}(M, \tau, z)$ is essentially $\chi_y(M)$. If (M^{2d}, J) is Calabi–Yau, the coefficients of q-expansion of $\mathrm{Ell}(M, \tau, z)$ are rigid for arbitrary y [Liu 1996, Theorem B]. Moreover, in this case, $\mathrm{Ell}(M, \tau, z)$ itself is a weak Jacobi form of weight 0 and index $\frac{1}{2}d$ [Gritsenko 1999b, Proposition 1.2; Borisov and Libgober 2000, Theorem 2.2].

As we have mentioned above, the elliptic genus $\mathrm{Ell}(M, \tau, z)$ can be viewed as a generalization of $\chi_y(M)$, and also has a rigidity property when M is Calabi–Yau. So we may ask in the Calabi–Yau case whether $\mathrm{Ell}(M, \tau, z)$ has some kind of arithmetic phenomenon which extends the original -1-phenomenon of $\chi_y(M)$. Note that, strictly speaking, $\mathrm{Ell}(M, \tau, z)$ is a generalization of $\chi_{-y}(M)$ rather than $\chi_y(M)$, as the q^0-term of $\mathrm{Ell}(M, \tau, z)$ is $y^{-d/2}\chi_{-y}(M)$. So if there exists some kind of phenomenon which extends the original -1-phenomenon of $\chi_y(M)$, the parameter $y = e^{2\pi\sqrt{-1}z}$ should correspond to 1 rather than -1. Thus the variable z should correspond to 0. Indeed, there does exist such a kind of generalization, which depends on some arithmetic properties of Jacobi forms and has been implicitly used by Gritsenko [1999b]. Our aim in Section 3 is twofold. On the one hand, given a compact almost-complex manifold (M^{2d}, J) and a rank-l complex vector bundle W over it, we construct a generalized elliptic genus $\mathrm{Ell}(M, W, \tau, z)$, which is defined to be the index of the Todd operator (1-4) twisted by

$$\left[\prod_{i=1}^{\infty}(1-q^i)\right]^{2(d-l)} \cdot y^{-l/2} \bigotimes_{n\geq 1} (\Lambda_{-yq^{n-1}} W^* \otimes \Lambda_{-y^{-1}q^n} W \otimes \mathrm{S}_{q^n} T^* \otimes \mathrm{S}_{q^n} T),$$

and show that it is a weak Jacobi form of weight $d-l$ and index $\frac{1}{2}l$ if the first Pontrjagin classes $p_1(M)$ equals $p_1(W)$ and the first Chern class $c_1(W)$ is 0 in $H^*(M, \mathbb{R})$. On the other hand, we highlight a well-known manipulation in Jacobi forms to obtain modular forms from $\mathrm{Ell}(M, W, \tau, z)$, whose arithmetic information will in turn give geometric results on M and W. Some examples are given to illustrate this observation.

2. -1-phenomenon of the pluri-χ_y-genus

Statements of the main results related to the pluri-χ_y-genus. Let (M^{2n}, J) (resp. X^{2n}) be a compact almost-complex manifold of complex dimension n (resp. smooth, closed oriented manifold of real dimension $2n$). As before, we use $(\bar\partial + \bar\partial^*)|_0$ to denote the Todd operator on (M^{2n}, J), whose index is the Todd genus of M. We denote by D the signature operator on X, whose index is the signature of X^{2n} [Atiyah and Singer 1968, Section 6]. By definition $\mathrm{Ind}(D)$ is zero unless n is even.

Let W be a complex vector bundle over M or X. By means of a connection on W, the elliptic operator $(\bar\partial + \bar\partial^*)|_0$ and D can be extended to a new elliptic operator $((\bar\partial+\bar\partial^*)|_0)\otimes W$ and $D\otimes W$, whose indices via the Atiyah–Singer index theorem are

$$\begin{aligned}\chi(M, W) = \mathrm{Ind}\big(((\bar\partial + \bar\partial^*)|_0) \otimes W\big) &= \int_M [\mathrm{td}(M)\cdot \mathrm{ch}(W)] \\ &= \int_M \left[\prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \cdot \mathrm{ch}(W)\right]\end{aligned}$$

and

$$\operatorname{Ind}((D\otimes W))=\int_X\Big[\Big(\prod_{i=1}^{n}\frac{x_i}{\tanh(x_i/2)}\Big)\cdot\operatorname{ch}(W)\Big]$$

respectively. Here we use the i-th elementary symmetric polynomial of $x_1,\dots,x_n$ (resp. $x_1^2,\dots,x_n^2$) to denote the i-th Chern class (resp. Pontrjagin class) of (M^{2n},J) (resp. X^{2n}).

Definition 2.1. For an arbitrary fixed positive integer g, we define

$$\begin{aligned}\Omega_{\underline{y}}(M)&:=\sum_{0\le p_1,\dots,p_g\le n}\Lambda^{p_1}(T^*M)\otimes\cdots\otimes\Lambda^{p_g}(T^*M)\cdot y_1^{p_1}\cdots y_g^{p_g}\\&=\Omega_{y_1}(M)\otimes\cdots\otimes\Omega_{y_g}(M)\in K(M)[y_1,\dots,y_g],\\ \Omega^{\mathbb{R}}_{\underline{y}}(X)&:=\sum_{0\le p_1,\dots,p_g\le 2n}\Lambda^{p_1}(T^*_{\mathbb{C}}X)\otimes\cdots\otimes\Lambda^{p_g}(T^*_{\mathbb{C}}X)\cdot y_1^{p_1}\cdots y_g^{p_g}\\&=\Omega^{\mathbb{R}}_{y_1}(X)\otimes\cdots\otimes\Omega^{\mathbb{R}}_{y_g}(X)\in(KO(X)\otimes\mathbb{C})[y_1,\dots,y_g],\end{aligned}$$

where

$$\Omega^{\mathbb{R}}_{y}(X):=\sum_{p=0}^{2n}\Lambda^p(T^*_{\mathbb{C}}X)\cdot y^p$$

and $T^*_{\mathbb{C}}X$ is the dual of the complexified tangent bundle of X, and

$$\begin{aligned}\chi_{\underline{y}}(M)&:=\sum_{0\le p_1,\dots,p_g\le n}\operatorname{Ind}\big[(\bar\partial+\bar\partial^*)|_0\otimes(\Lambda^{p_1}(T^*M)\otimes\cdots\otimes\Lambda^{p_g}(T^*M))\big]\\&\qquad\qquad\cdot y_1^{p_1}\cdots y_g^{p_g}\\&=\int_M\Big[\prod_{i=1}^{n}\frac{x_i}{1-e^{-x_i}}\cdot\operatorname{ch}(\Omega_{\underline{y}}(M))\Big]\\ D_{\underline{y}}(X)&:=\sum_{0\le p_1,\dots,p_g\le 2n}\operatorname{Ind}\big[D\otimes(\Lambda^{p_1}(T^*_{\mathbb{C}}X)\otimes\cdots\otimes\Lambda^{p_g}(T^*_{\mathbb{C}}X))\big]\cdot y_1^{p_1}\cdots y_g^{p_g}\\&=\int_X\Big[\Big(\prod_{i=1}^{n}\frac{x_i}{\tanh(x_i/2)}\Big)\cdot\operatorname{ch}(\Omega^{\mathbb{R}}_{\underline{y}}(X))\Big].\end{aligned}$$

Our main result in this section is:

Theorem 2.2. *The coefficient of* $(1+y_1)^{n-q_1}\cdots(1+y_g)^{n-q_g}$ *in* $\chi_{\underline{y}}(M)$ *is equal to*

$$\begin{cases}0 & \text{if } \sum_{i=1}^{g}q_i>n,\\ \displaystyle\int_M\prod_{i=1}^{g}c_{q_i}(M) & \text{if } \sum_{i=1}^{g}q_i=n,\\ \text{a rational linear combination of Chern numbers of } M & \text{if } \sum_{i=1}^{g}q_i<n.\end{cases}$$

We have a similar result for smooth manifolds.

Theorem 2.3. *If n is even, the coefficient of*

$$(1+y_1)^{2(n-q_1)}\cdots(1+y_g)^{2(n-q_g)}$$

in $D_{\underline{y}}(X)$ is equal to

$$\begin{cases} 0 & \text{if } \sum_{i=1}^{g} q_i > \frac{1}{2}n, \\ (-1)^{n/2}\cdot 2^n \cdot \displaystyle\int_X \prod_{i=1}^{g} p_{q_i}(X) & \text{if } \sum_{i=1}^{g} q_i = \frac{1}{2}n, \\ \text{a rational linear combination of Pontrjagin numbers of } X & \text{if } \sum_{i=1}^{g} q_i < \frac{1}{2}n, \end{cases}$$

where $p_i(X)$ is the i-th Pontrjagin class of X.

Clearly, a direct corollary of this theorem is the following result, which gives an affirmative answer to [Li 2011, Question 1.1].

Corollary 2.4. *Any Chern number (resp. Pontrjagin number) on a compact almost-complex manifold (resp. compact smooth manifold) can be expressed in an explicit way in terms of the indices of some elliptic differential operators over this manifold.*

Proofs of Theorems 2.2 and 2.3. Abusing notation, we use $c_q(\cdots)$ to denote both the q-th Chern class of an almost-complex manifold and the q-th elementary symmetric polynomial of the variables in the bracket.

The proofs of Theorems 2.2 and 2.3 depend on the following lemma:

Lemma 2.5. *If we assign each x_i $(1 \le i \le n)$ the same degree, then we have:*

(1) *the coefficient of $(1+y)^{n-q}$ $(0 \le q \le n)$ in $\prod_{i=1}^{n}(1+ye^{-x_i})$ is*

$$c_q(x_1,\dots,x_n) + \text{higher-degree terms};$$

(2) *the coefficient of $(1+y)^{2(n-q)}$ $(0 \le q \le n)$ in $\prod_{i=1}^{n}(1+ye^{-x_i})(1+ye^{x_i})$ is*

$$(-1)^q c_q(x_1^2,\dots,x_n^2) + \text{higher-degree terms}.$$

Proof. We have

$$\prod_{i=1}^{n}(1+ye^{-x_i}) = \prod_{i=1}^{n}[(1-e^{-x_i})+(1+y)e^{-x_i}] = e^{-c_1}\prod_{i=1}^{n}[(e^{x_i}-1)+(1+y)].$$

Thus the coefficient of $(1+y)^{n-q}$ in $\prod_{i=1}^{n}(1+ye^{-x_i})$ is

$$e^{-c_1}\cdot c_q(e^{x_1}-1,\dots,e^{x_n}-1) = c_q(x_1,\dots,x_n) + \text{higher-degree terms}.$$

Similarly,

$$\prod_{i=1}^{n}(1+ye^{-x_i})(1+ye^{x_i})=\prod_{i=1}^{n}[(e^{x_i}-1)+(1+y)][(e^{-x_i}-1)+(1+y)],$$

and the coefficient of $(1+y)^{2n-q}$ is

$$\begin{aligned}c_q(e^{x_1}-1,\dots,e^{x_n}-1,e^{-x_1}-1,\dots,e^{-x_n}-1)\\ =c_q(x_1,\dots,x_n,-x_1,\dots,-x_n)+\text{higher-degree terms.}\end{aligned}$$

Note that

$$c_q(x_1,\dots,x_n,-x_1,\dots,-x_n)=\begin{cases}0 & \text{if } q \text{ is odd,}\\ (-1)^{q/2}c_{q/2}(x_1^2,\dots,x_n^2) & \text{if } q \text{ is even.}\end{cases}$$

This gives the desired property. □

Now we can prove Theorems 2.2 and 2.3.

Proof. If we use $x_1,\dots,x_n$ (resp. $x_1,\dots,x_n,-x_1,\dots,-x_n$) to denote the formal Chern roots of TM (resp. $T_{\mathbb{C}}X$), then we have (see [Hirzebruch et al. 1992, p. 11])

$$\operatorname{ch}(\Omega_{\underline{y}}(M))=\prod_{j=1}^{g}\Bigl[\prod_{i=1}^{n}(1+y_je^{-x_i})\Bigr]$$

and

$$\operatorname{ch}(\Omega_{\underline{y}}^{\mathbb{R}}(X))=\prod_{j=1}^{g}\Bigl[\prod_{i=1}^{n}(1+y_je^{-x_i})(1+y_je^{x_i})\Bigr].$$

Thus,

$$\begin{aligned}\chi_{\underline{y}}(M)&=\int_M\Bigl[\Bigl(\prod_{i=1}^{n}\frac{x_i}{1-e^{-x_i}}\Bigr)\cdot\operatorname{ch}(\Omega_{\underline{y}}(M))\Bigr]\\ &=\int_M\Bigl\{\Bigl(\prod_{i=1}^{n}\frac{x_i}{1-e^{-x_i}}\Bigr)\cdot\prod_{j=1}^{g}\Bigl[\prod_{i=1}^{n}(1+y_je^{-x_i})\Bigr]\Bigr\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ind}\bigl(D_{\underline{y}}^{\mathbb{R}}(X)\bigr)&=\int_X\Bigl[\Bigl(\prod_{i=1}^{n}\frac{x_i}{\tanh(x_i/2)}\Bigr)\cdot\operatorname{ch}(\Omega_{\underline{y}}^{\mathbb{R}}(X))\Bigr]\\ &=\int_X\Bigl\{\Bigl(\prod_{i=1}^{n}\frac{x_i}{\tanh(x_i/2)}\Bigr)\cdot\prod_{j=1}^{g}\Bigl[\prod_{i=1}^{n}(1+y_je^{-x_i})(1+y_je^{x_i})\Bigr]\Bigr\}.\end{aligned}$$

Note that the constant terms of

$$\frac{x_i}{1-e^{-x_i}} = 1 + \cdots \quad \text{and} \quad \frac{x_i}{\tanh(x_i/2)} = \frac{x_i(1+e^{-x_i})}{1-e^{-x_i}} = 2 + \cdots$$

are 1 and 2 respectively. So by Lemma 2.5, when considering the Taylor expansions of $\mathrm{Ind}(D_{\underline{y}}(M))$ and $\mathrm{Ind}(D^{\mathbb{R}}_{\underline{y}}(X))$ at $y_1 = \cdots = y_g = -1$, the coefficients before the terms $(1+y_1)^{n-q_1} \cdots (1+y_g)^{n-q_g}$ and $(1+y_1)^{2(n-q_1)} \cdots (1+y_g)^{2(n-q_g)}$ are

$$\int_M \Big\{(1 + \text{higher-degree terms}) \cdot \prod_{j=1}^{g} [c_{q_j}(x_1, \ldots, x_n) + \text{higher-degree terms}]\Big\}$$
$$= \int_M \prod_{i=1}^{g} c_{q_i}(M) + \int_M (\text{higher-degree terms})$$

and

$$\int_X \Big\{(2^n + \text{higher-degree terms}) \cdot \prod_{j=1}^{g} [(-1)^{q_j} c_{q_j}(x_1^2, \ldots, x_n^2) + \text{higher degree terms}]\Big\}$$
$$= 2^n \cdot (-1)^{\sum_{j=1}^{g} q_j} \int_X \prod_{j=1}^{g} p_{q_i}(X) + \int_X (\text{higher degree terms}),$$

respectively, which give the desired proofs of Theorems 2.2 and 2.3. $\square$

3. The generalized elliptic genus and its −1-phenomenon

The generalized elliptic genus of almost-complex manifolds. In this subsection, we extend the original definition of the elliptic genus of almost-complex manifolds by considering an extra complex vector bundle and showing that it is a weak Jacobi form. As before, let (M^{2d}, J) be a compact almost-complex manifold and W a rank-l complex vector bundle over it.

Definition 3.1. The generalized elliptic genus of (M^{2d}, J) with respect to W, which we denote by $\mathrm{Ell}(M, W, \tau, z)$, is defined to be the index of the Todd operator

$$\Omega^{0,\mathrm{even}}(M) \xrightarrow{(\bar\partial+\bar\partial^*)|_0} \Omega^{0,\mathrm{odd}}(M)$$

twisted by

$$c^{2(d-l)} \cdot y^{-l/2} \bigotimes_{n\geq 1} (\Lambda_{-yq^{n-1}} W^* \otimes \Lambda_{-y^{-1}q^n} W \otimes \mathrm{S}_{q^n} T^* \otimes \mathrm{S}_{q^n} T) =: \mathrm{E}(W, q, y),$$

where

$$q = e^{2\pi\sqrt{-1}\tau}, \quad y = e^{2\pi\sqrt{-1}z},$$

and for simplicity $c := \prod_{i=1}^{\infty}(1-q^i)$. If $W = T$, this definition degenerates to the original elliptic genus.

Our first observation in this section is the following, which extends [Gritsenko 1999b, Proposition 1.2; Borisov and Libgober 2000, Theorem 2.2], in which $W = T$.

Theorem 3.2. *The generalized elliptic genus* $\mathrm{Ell}(M, W, \tau, z)$ *is a weak Jacobi form of weight $d - l$ and index $\frac{1}{2}l$ provided that the first Pontrjagin classes $p_1(M)$ equals $p_1(W)$ and the first Chern class $c_1(W)$ is* 0 *in $H^*(M, \mathbb{R})$.*

Remark 3.3. (1) A two-variable function $\varphi(\tau, z)$ for $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ is called a *weak Jacobi form of weight k and index m* for $k \in \mathbb{Z}$ and $m \in \mathbb{Z}/2$ if it is a holomorphic function with respect to the two variables τ and z, has no negative powers of q in its Fourier expansion in terms of $q^i y^j$ and satisfies some transformation laws involving k and m; the precise definition can be found in [Eichler and Zagier 1985, p. 9, p. 104]. There, only the integral indices are considered. However, with minor modifications of inserting a character, this notion can be easily extended to the case where the index is allowed to be a half-integer (see [Gritsenko 1999b, p. 102]).

(2) Motivated by his ingenious proof of the rigidity theorem, Liu constructed a two-variable function for (M, J) and W and showed that it is a weak Jacobi form under some assumptions, and the original Witten theorem exactly corresponds to the case where the index is equal to zero [Liu 1995b, Theorem 3, Corollary 3.1]. This construction later was generalized to the family case by Liu and Ma [2000, Theorem 3.1]. So our theorem has a similar flavor to their work.

(3) Gritsenko [1999a, Theorem 1.2] further extended the original elliptic genus to another case where an extra complex bundle is involved. But his construction is different from ours as it is still of weight zero.

The Atiyah–Singer index theorem tells us that

$$\mathrm{Ell}(M, W, \tau, z) = \int_M \mathrm{td}(M) \cdot \mathrm{ch}(\mathrm{E}(W, q, y)).$$

In particular, if J is integrable, $\mathrm{Ell}(M, W, \tau, z)$ is the holomorphic Euler characteristic of the (virtual) bundle $\mathrm{E}(W, q, y)$.

Let us recall one of the Jacobi-theta series [Chandrasekharan 1985, Chapter 5]:

$$\begin{aligned}\theta(\tau, z) &:= \sum_{n\in\mathbb{Z}} (-1)^n q^{(n+1/2)^2/2} y^{n+1/2} \\ &= 2cq^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n y)(1 - q^n y^{-1}) \\ &= 2cq^{1/8} \sinh(\pi\sqrt{-1}z) \prod_{n=1}^{\infty} (1 - q^n y)(1 - q^n y^{-1}) \\ &= 2cq^{1/8} \sinh(\pi\sqrt{-1}z) \prod_{n=1}^{\infty} (1 - q^n e^{2\pi\sqrt{-1}z})(1 - q^n e^{-2\pi\sqrt{-1}z}).\end{aligned}$$

The following lemma says that $\mathrm{Ell}(M, W, \tau, z)$ can be expressed in terms of $\theta(\tau, z)$.

Lemma 3.4. *If we denote by $2\pi\sqrt{-1}x_i$ $(1 \le i \le d)$ and $2\pi\sqrt{-1}w_i$ $(1 \le i \le l)$ the Chern roots of TM and W, respectively, then we have*

$$\begin{aligned}
&\mathrm{Ell}(M, W, \tau, z)\\
&\quad = \int_M \left[\exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \cdot (\eta(\tau))^{3(d-l)} \cdot \prod_{i=1}^{d} \frac{2\pi\sqrt{-1}x_i}{\theta(\tau, x_i)} \cdot \prod_{j=1}^{l} \theta(\tau, w_j - z)\right],
\end{aligned}$$

where

$$\eta(\tau) := q^{1/24} \cdot c = q^{1/24} \prod_{i=1}^{\infty} (1 - q^i)$$

is the famous Dedekind eta function. In particular, $\mathrm{Ell}(M, W, \tau, z)$ *is a holomorphic function with respect to the two variables τ and z and has no negative powers of q in its Fourier expansion.*

Proof. We have

$$\begin{aligned}
&\mathrm{ch}(\mathrm{E}(W, q, y))\\
&\quad = c^{2(d-l)} y^{-l/2} \prod_{j=1}^{l} (1 - y e^{-2\pi\sqrt{-1}w_j})\\
&\qquad\qquad \times \prod_{n=1}^{\infty} \frac{\prod_{j=1}^{l} (1 - y q^n e^{-2\pi\sqrt{-1}w_j})(1 - y^{-1} q^n e^{2\pi\sqrt{-1}w_j})}{\prod_{i=1}^{d} (1 - q^n e^{-2\pi\sqrt{-1}x_i})(1 - q^n e^{2\pi\sqrt{-1}x_i})}\\
&\quad = c^{2(d-l)} y^{-l/2} \prod_{j=1}^{l} (1 - y e^{-2\pi\sqrt{-1}w_j})\\
&\qquad\qquad \times \prod_{j=1}^{l} \frac{\theta(\tau, w_j - z)}{2cq^{1/8} \sinh(\pi\sqrt{-1}(w_j - z))} \prod_{i=1}^{d} \frac{2cq^{1/8} \sinh(\pi\sqrt{-1}x_i)}{\theta(\tau, x_i)}\\
&\quad = \exp\left(\frac{c_1(M) - c_1(W)}{2}\right) \cdot (\eta(\tau))^{3(d-l)} \cdot \prod_{i=1}^{d} \frac{1 - e^{-2\pi\sqrt{-1}x_i}}{\theta(\tau, x_i)} \cdot \prod_{j=1}^{l} \theta(\tau, w_j - z).
\end{aligned}$$

The last equality is due to the fact that

$$c_1(M) = \sum_{i=1}^{d} 2\pi\sqrt{-1}x_i \quad \text{and} \quad c_1(W) = \sum_{j=1}^{l} 2\pi\sqrt{-1}w_j.$$

Therefore,

$$
\begin{aligned}
&\mathrm{Ell}(M, W, \tau, z)\\
&= \int_M \mathrm{td}(M)\cdot \mathrm{ch}(\mathrm{E}(W,q,y)) = \int_M \prod_{i=1}^{d} \frac{2\pi\sqrt{-1}x_i}{1-e^{-2\pi\sqrt{-1}x_i}}\cdot \mathrm{ch}(\mathrm{E}(W,q,y))\\
&= \int_M \biggl[\exp\biggl(\frac{c_1(M)-c_1(W)}{2}\biggr)\cdot(\eta(\tau))^{3(d-l)}\cdot \prod_{i=1}^{d}\frac{2\pi\sqrt{-1}x_i}{\theta(\tau,x_i)}\cdot \prod_{j=1}^{l}\theta(\tau, w_j - z)\biggr].
\end{aligned}
$$

The holomorphicity of $\mathrm{Ell}(M, W, \tau, z)$ is now clear from this expression, as the Jacobi theta function $\theta(\tau, z)$ only has zeroes of order 1 along $z = m_1 + m_2\tau$ ($m_1, m_2 \in \mathbb{Z}$) [Chandrasekharan 1985, p. 59]. Also it is obvious from this expression that $\mathrm{Ell}(M, W, \tau, z)$ has no negative powers of q in its Fourier expansion. □

Proof of Theorem 3.2. $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$
\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.
$$

To verify that $\mathrm{Ell}(M, W, \tau, z)$ satisfies the required transformation laws, it suffices to show the four identities

$$\mathrm{Ell}(M, W, \tau+1, z) = \mathrm{Ell}(M, W, \tau, z), \tag{3-1}$$

$$\mathrm{Ell}(M, W, \tau, z+1) = (-1)^l\,\mathrm{Ell}(M, W, \tau, z), \tag{3-2}$$

$$\mathrm{Ell}(M, W, \tau, z+\tau) = (-1)^l \exp\bigl(-\pi\sqrt{-1}\,l(\tau+2z)\bigr)\,\mathrm{Ell}(M, W, \tau, z), \tag{3-3}$$

$$\mathrm{Ell}(M, W, -1/\tau, z/\tau) = \tau^{d-l}\exp(\pi\sqrt{-1}\,lz^2/\tau)\,\mathrm{Ell}(M, W, \tau, z). \tag{3-4}$$

For Dedekind eta function $\eta(\tau)$ and Jacobi theta function $\theta(\tau, z)$ we have transformation laws [Chandrasekharan 1985]:

$$
\begin{aligned}
\eta^3\Bigl(-\frac{1}{\tau}\Bigr) &= \Bigl(\frac{\tau}{\sqrt{-1}}\Bigr)^{3/2}\eta^3(\tau),\\
\eta^3(\tau+1) &= \exp\Bigl(\frac{\pi\sqrt{-1}}{4}\Bigr)\eta^3(\tau),\\
\theta(\tau, z+1) &= -\theta(\tau, z),\\
\theta(\tau, z+\tau) &= -q^{-1/2}\exp(-2\pi\sqrt{-1}z)\theta(\tau, z),\\
\theta(\tau+1, z) &= \exp\Bigl(\frac{\pi\sqrt{-1}}{4}\Bigr)\theta(\tau, z),\\
\theta\Bigl(-\frac{1}{\tau}, z\Bigr) &= -\sqrt{-1}\Bigl(\frac{\tau}{\sqrt{-1}}\Bigr)^{1/2}\exp(\pi\sqrt{-1}\tau z^2)\theta(\tau, \tau z).
\end{aligned}
$$

The first three identities (3-1)–(3-3) are easy to verify by using the transformation laws above. Here we only need to check (3-4) carefully. Indeed,

$$
\begin{aligned}
\text{(3-5)} \qquad \prod_{i=1}^{d} \theta\Big(-\frac{1}{\tau}, x_i\Big) &= \prod_{i=1}^{d} -\sqrt{-1}\Big(\frac{\tau}{\sqrt{-1}}\Big)^{1/2} \exp(\pi\sqrt{-1}\tau x_i^2)\theta(\tau, \tau x_i) \\
&= \exp\Big(\frac{\tau p_1(M)}{4\pi\sqrt{-1}}\Big) \prod_{i=1}^{d} -\sqrt{-1}\Big(\frac{\tau}{\sqrt{-1}}\Big)^{1/2} \theta(\tau, \tau x_i).
\end{aligned}
$$

Here, we use the assumption that

$$p_1(M) = \sum_{i=1}^{d} (2\pi\sqrt{-1}x_i)^2.$$

Similarly,

$$
\begin{aligned}
\text{(3-6)} \quad & \prod_{j=1}^{l} \theta\Big(-\frac{1}{\tau}, w_i - \frac{z}{\tau}\Big) \\
&= \prod_{j=1}^{l} -\sqrt{-1}(\frac{\tau}{\sqrt{-1}})^{1/2} \exp\Big(\pi\sqrt{-1}\tau\Big(w_j - \frac{z}{\tau}\Big)^2\Big)\theta(\tau, \tau w_j - z) \\
&= \exp\Big(\frac{\tau p_1(W)}{4\pi\sqrt{-1}} + \frac{\pi\sqrt{-1}lz^2}{\tau}\Big) \prod_{j=1}^{l} -\sqrt{-1}\Big(\frac{\tau}{\sqrt{-1}}\Big)^{1/2} \theta(\tau, \tau w_j - z).
\end{aligned}
$$

In the last equality, we used the assumption that

$$c_1(W) = \sum_{j=1}^{l} 2\pi\sqrt{-1}w_j = 0.$$

Combining the transformation law of $\eta(\tau)$, (3-5), (3-6) and the fact that $p_1(M) = p_1(W)$ leads to

$$
\begin{aligned}
&\mathrm{Ell}\Big(M, W, -\frac{1}{\tau}, \frac{z}{\tau}\Big) \\
&= \int_M \Big[\exp\Big(\frac{c_1(M) - c_1(W)}{2}\Big)\Big(\eta\Big(-\frac{1}{\tau}\Big)\Big)^{3(d-l)} \\
&\qquad\qquad \times \prod_{i=1}^{d} \frac{2\pi\sqrt{-1}x_i}{\theta(-1/\tau, x_i)} \prod_{j=1}^{l} \theta\Big(-\frac{1}{\tau}, w_j - \frac{z}{\tau}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \tau^{d-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \\
&\qquad \times \int_M \left[\exp\left(\frac{c_1(M)-c_1(W)}{2}\right)(\eta(\tau))^{3(d-l)} \prod_{i=1}^{d} \frac{2\pi\sqrt{-1}x_i}{\theta(\tau,\tau x_i)} \prod_{j=1}^{l} \theta(\tau,\tau w_j - z)\right] \\
&= \tau^{-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \\
&\qquad \times \int_M \left[\exp\left(\frac{c_1(M)-c_1(W)}{2}\right)(\eta(\tau))^{3(d-l)} \prod_{i=1}^{d} \frac{2\pi\sqrt{-1}(\tau x_i)}{\theta(\tau,\tau x_i)} \prod_{j=1}^{l} \theta(\tau,\tau w_j - z)\right] \\
&= \tau^{d-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \\
&\qquad \times \int_M \left[\exp\left(\frac{c_1(M)-c_1(W)}{2}\right)(\eta(\tau))^{3(d-l)} \prod_{i=1}^{d} \frac{2\pi\sqrt{-1}x_i}{\theta(\tau,x_i)} \prod_{j=1}^{l} \theta(\tau,w_j - z)\right] \\
&= \tau^{d-l} \exp\left(\frac{\pi\sqrt{-1}lz^2}{\tau}\right) \mathrm{Ell}(M,W,\tau,z)
\end{aligned}
$$

The penultimate equality is due to the fact that in the integrand we are only concerned with the homogeneous part of degree d ($\deg(x_i) = \deg(w_j) = 1$). This completes the proof of Theorem 3.2. $\square$

Algebraic preliminaries. Before discussing the arithmetic properties of the generalized elliptic genus $\mathrm{Ell}(M, W, \tau, z)$, we need to review a well-known manipulation in algebraic number theory which helps derive modular forms from Jacobi forms.

Recall that the *Eisenstein series* $G_{2k}(\tau)$ are defined to be

$$G_{2k}(\tau) := -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) \cdot q^n,$$

[Hirzebruch et al. 1992, p. 131], where

$$\sigma_k(n) := \sum_{\substack{m>0 \\ m \mid n}} m^k$$

and the B_{2k} are the Bernoulli numbers.

These $G_{2k}(\tau)$ carry rich arithmetic information. It is well-known that $G_{2k}(\tau)$ ($k \geq 2$) are modular forms of weight $2k$ over the full modular group $\mathrm{SL}_2(\mathbb{Z})$ and the whole graded ring of modular forms over $\mathrm{SL}_2(\mathbb{Z})$ are generated by $G_4(\tau)$ and $G_6(\tau)$. However, $G_2(\tau)$ is *not* a modular form but a *quasimodular form*, as it transforms as [Hirzebruch et al. 1992, p. 138]

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2G_2(\tau)-\frac{c(c\tau+d)}{4\pi\sqrt{-1}}\quad\text{for all }\begin{pmatrix}a&b\\c&d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}).\tag{3-7}$$

The next proposition, which is a well-known fact in algebraic number theory and has been used implicitly by Gritsenko in the proof of [1999b, Lemma 1.6], provides us with a method for deriving modular forms from Jacobi forms.

Proposition 3.5. *Suppose a function $\varphi(\tau, z):\ \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ satisfies*

$$\varphi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^k\exp\left(\frac{2\pi\sqrt{-1}mcz^2}{c\tau+d}\right)\cdot\varphi(\tau,z),\tag{3-8}$$

for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$, i.e., $\varphi(\tau,z)$ transforms like a Jacobi form of weight k and index m.

Then, if we define

$$\Phi(\tau,z):=\exp(-8\pi^2mG_2(\tau)z^2)\varphi(\tau,z),$$

we have

$$\Phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^k\Phi(\tau,z).\tag{3-9}$$

This means that if we set

$$\Phi(\tau,z)=:\sum_{n\in\mathbb{Z}}a_n(\tau)\cdot z^n,$$

then

$$a_n\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{k+n}a_n(\tau).$$

In particular, if $\varphi(\tau,z)$ is a weak Jacobi form of weight k and index m, then these $a_n(\tau)$ are modular forms of weight $k+n$ over $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Equation (3-9) can be verified directly by using the assumption (3-8) and the transformation law (3-7). If, moreover, $\varphi(\tau,z)$ is a weak Jacobi form, then $\varphi(\tau,z)$ and thus $\Phi(\tau,z)$ are holomorphic and have no negative powers of q when considering their Fourier expansions in terms of q and y. This implies that these $a_n(\tau)$ are also holomorphic and have no negative powers of q when considering the Fourier expansions of q, which gives the desired proof. □

With the assumptions in Theorem 3.2 understood, we know that $\mathrm{Ell}(M,W,\tau,z)$ is a weak Jacobi form of weight $d-l$ and index $\frac{1}{2}l$. Then Proposition 3.5 tells us:

Proposition 3.6. *The series $a_n(M, W, \tau)$ determined by*

$$\exp[l \cdot G_2(\tau) \cdot (2\pi\sqrt{-1}z)^2] \cdot \mathrm{Ell}(M, W, \tau, z) =: \sum_{n\geq 0} a_n(M, W, \tau) \cdot (2\pi\sqrt{-1}z)^n$$

are modular forms of weight $d-l+n$ over $\mathrm{SL}_2(\mathbb{Z})$. Furthermore, the first three series of $a_n(M, W, \tau)$ are of the form

$$\begin{aligned} a_0(M, W, \tau) &= \chi(M, \Lambda_{-1}W^*) \\ &\qquad + q \cdot \chi(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)) + q^2 \cdot (\cdots), \\ a_1(M, W, \tau) &= \sum_{p=0}^{l} (-1)^p \Big(p - \frac{l}{2}\Big) \chi(M, \Lambda^p W^*) + q \cdot (\cdots), \\ a_2(M, W, \tau) &= -\frac{l}{24}\chi(M, \Lambda_{-1}W^*) + \frac{1}{2}\sum_{p=0}^{l}(-1)^p \Big(p - \frac{l}{2}\Big)^2 \chi(M, \Lambda^p W^*) + q \cdot (\cdots). \end{aligned}$$

Proof. The first statement is a direct application of Proposition 3.5 as $\mathrm{Ell}(M, W, \tau, z)$ is a weak Jacobi form of weight $d-l$ and index $\frac{1}{2}l$. For the second one, if we set

$$\exp[lG_2(\tau)(2\pi\sqrt{-1}z)^2] =: A_0(y) + A_1(y) \cdot q + (\cdots) \cdot q^2$$

and

$$\mathrm{Ell}(M, W, \tau, z) =: B_0(y) + B_1(y) \cdot q + (\cdots) \cdot q^2,$$

we can easily deduce from their explicit expressions that

$$\begin{aligned} A_0(y) &= \exp\Big[-\frac{l}{24}(2\pi\sqrt{-1}z)^2\Big] = 1 - \frac{l}{24}(2\pi\sqrt{-1}z)^2 + \cdots, \\ A_1(y) &= l(2\pi\sqrt{-1}z)^2 - \frac{l^2}{24}(2\pi\sqrt{-1}z)^4 + \cdots, \\ B_0(y) &= \sum_{p=0}^{l}(-1)^p \chi(M, \Lambda^p W^*) y^{p-l/2} \\ &= \sum_{p=0}^{l}(-1)^p \chi(M, \Lambda^p W^*) \\ &\qquad \times \Big[1 + \Big(p - \frac{l}{2}\Big)(2\pi\sqrt{-1}z) + \frac{1}{2}\Big(p - \frac{l}{2}\Big)^2 (2\pi\sqrt{-1}z)^2 + \cdots\Big], \\ B_1(y) &= \chi\big(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)\big) + 2\pi\sqrt{-1}z(\cdots). \end{aligned}$$

Note that

$$\sum_{n\geq 0} a_n(M, W, \tau)(2\pi\sqrt{-1}z)^n = A_0(y)B_0(y) + [A_0(y)B_1(y) + A_1(y)B_0(y)]q + \cdots;$$

then it is easy to deduce the expressions in Proposition 3.6 in terms of those of $A_0(y)$, $A_1(y)$, $B_0(y)$ and $B_1(y)$. □

−1-phenomenon of the generalized elliptic genus. Here, using Proposition 3.6, presented in the last subsection, we investigate the arithmetic information of the generalized elliptic genus $\mathrm{Ell}(M, W, \tau, z)$, which can be viewed as an appropriate -1-phenomenon of $\mathrm{Ell}(M, W, \tau, z)$.

We will present one proposition and two examples related to $a_2(M, W, \tau)$, $a_0(M, W, \tau)$ and $a_1(M, W, \tau)$, respectively, to illustrate an appropriate -1-phenomenon of the generalized elliptic genus $\mathrm{Ell}(M, W, \tau, z)$.

Our next proposition related to $a_2(M, W, \tau)$ gives the "reason" why these $a_n(M, W, \tau)$ should be the -1-phenomenon of $\mathrm{Ell}(M, W, \tau, z)$.

Proposition 3.7. *$a_2(M, W, \tau)$ is a modular form of weight $d-l+2$ over $\mathrm{SL}_2(\mathbb{Z})$ provided that $p_1(M) = p_1(W)$ and $c_1(W) = 0$ in $H^*(M, \mathbb{R})$. Consequently, if either* (i) *$d-l$ is odd, or* (ii) *$d \le l$ but $d-l \ne -2$, we have*

$$\sum_{p=0}^{l}(-1)^p\Big(p-\frac{l}{2}\Big)^2\chi(M, \Lambda^p W^*) = \frac{l}{12}\chi(M, \Lambda_{-1} W^*). \tag{3-10}$$

Moreover, if $W = T$ and $c_1(M) = 0$ in $H^*(M, \mathbb{R})$, (3-10) *is nothing but the original -1-phenomenon of the Hirzebruch χ_y-genus.*

Proof. If either (i) $d-l$ is odd or (ii) $d \le l$ but $d-l \ne -2$, $a_2(M, W, \tau)$ is a modular form over $\mathrm{SL}_2(\mathbb{Z})$ whose weight is either (i) odd or (ii) no more than 2 but not zero. This means $a_2(M, W, \tau) \equiv 0$; then its expression in Proposition 3.6 gives (3-10).

If $W = T$, then the right-hand side of (3-10) is

$$\frac{d}{12}\chi(M, \Lambda_{-1} T^*) = \frac{d}{12}\chi_y(M)\Big|_{y=-1} = \frac{d}{12}c_d(M).$$

However, the left-hand side of (3-10) is

$$\begin{aligned}
&\sum_{p=0}^{d}(-1)^p\Big(p-\frac{d}{2}\Big)^2\chi^p(M)\\
&\quad= \sum_{p=0}^{d}(-1)^p\Big[2\cdot\frac{p(p-1)}{2}+(1-d)p+\frac{d^2}{4}\Big]\chi^p(M)\\
&\quad= 2a_2(M)-(1-d)a_1(M)+\frac{d^2}{4}a_0(M)\\
&\quad= \frac{d(3d-5)}{12}c_d(M)+\frac{(1-d)d}{2}c_d(M)+\frac{d^2}{4}c_d(M) \quad \text{(via (1-3) and } c_1(M)=0)\\
&\quad= \frac{d}{12}c_d(M) = \text{the right-hand side of (3-10).} \qquad \square
\end{aligned}$$

The next two examples, related to $a_0(M, W, \tau)$ and $a_1(M, W, \tau)$, give much arithmetic information about M and W.

Example 3.8. By Proposition 3.6, we know that $a_0(M, W, \tau)$ is a modular form of weight $d-l$ over $\mathrm{SL}_2(\mathbb{Z})$ provided that $p_1(M) = p_1(W)$ and $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Consequently:

(1) If either $d-l$ is odd or $d-l \le 2$ but is nonzero, we have

$$\chi(M, \Lambda_{-1}W^*) = \chi\big(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)\big) = 0.$$

(2) If $d-l = 4$, $a_0(M, W, \tau)$ is proportional to the Eisenstein series

$$G_4(\tau) = -\frac{B_4}{8} + q + \cdots = \frac{1}{240} + q + \cdots ,$$

and so

$$\chi\big(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)\big) = 240\chi(M, \Lambda_{-1}W^*).$$

(3) If $d-l = 6$, $a_0(M, W, \tau)$ is proportional to the Eisenstein series

$$G_6(\tau) = -\frac{B_6}{12} + q + \cdots = -\frac{1}{504} + q + \cdots ,$$

and so

$$\chi(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)) = -504\chi(M, \Lambda_{-1}W^*).$$

(4) If $d-l = 8$, $a_0(M, W, \tau)$ is proportional to

$$[G_4(\tau)]^2 = \left[\frac{1}{240} + q + \cdots\right]^2 = \frac{1}{240^2} + \frac{1}{120}q + \cdots ,$$

and so

$$\chi\big(M, \Lambda_{-1}W^* \otimes (-2(d-l) - W - W^* + T + T^*)\big) = 480\chi(M, \Lambda_{-1}W^*).$$

Example 3.9. By Proposition 3.6, we know that $a_1(M, W, \tau)$ is a modular form of weight $d-l+1$ over $\mathrm{SL}_2(\mathbb{Z})$ provided that $p_1(M) = p_1(W)$ and $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Consequently, if either $d-l$ is even or $d-l \le 1$ but $d-l \ne -1$, we have

$$\sum_{p=0}^{l} (-1)^p \Big(p - \frac{l}{2}\Big)\chi(M, \Lambda^p W^*) = 0.$$

Acknowledgements

The first part of this paper was inspired by my fruitful discussions with George Thompson via email in 2010 and 2011. The second part was suggested to me by Fei Han when I visited him at the National University of Singapore in 2012. This paper was initiated when I was holding a JSPS Postdoctoral Fellowship for Foreign Researchers at Waseda University in Japan with the help of Martin Guest. To all of them I would like to express my sincere thanks.

References

[Atiyah and Hirzebruch 1970] M. F. Atiyah and F. Hirzebruch, "Spin-manifolds and group actions", pp. 18–28 in *Essays on topology and related topics: mémoires dédiés à Georges de Rham)*, edited by A. Haefliger and R. Narasimhan, Springer, New York, 1970. MR 43 #4064 Zbl 0193.52401

[Atiyah and Singer 1968] M. F. Atiyah and I. M. Singer, "The index of elliptic operators, III", *Ann. of Math.* (2) **87** (1968), 546–604. MR 38 #5245 Zbl 0164.24301

[Borisov and Libgober 2000] L. A. Borisov and A. Libgober, "Elliptic genera of toric varieties and applications to mirror symmetry", *Invent. Math.* **140**:2 (2000), 453–485. MR 2001j:58037 Zbl 0958.14033

[Bott and Taubes 1989] R. Bott and C. Taubes, "On the rigidity theorems of Witten", *J. Amer. Math. Soc.* **2**:1 (1989), 137–186. MR 89k:58270 Zbl 0667.57009

[Chandrasekharan 1985] K. Chandrasekharan, *Elliptic functions*, Grundlehren der Mathematischen Wissenschaften **281**, Springer, Berlin, 1985. MR 87e:11058 Zbl 0575.33001

[Chen and Han 2009] Q. Chen and F. Han, "Modular invariance and twisted cancellations of characteristic numbers", *Trans. Amer. Math. Soc.* **361**:3 (2009), 1463–1493. MR 2010a:58029 Zbl 1168.58009

[Chen et al. 2011] Q. Chen, F. Han, and W. Zhang, "Generalized Witten genus and vanishing theorems", *J. Differential Geom.* **88**:1 (2011), 1–40. MR 2012j:58024 Zbl 1252.58011

[Dong et al. 2005] C. Dong, K. Liu, and X. Ma, "Elliptic genus and vertex operator algebras", *Pure Appl. Math. Q.* **1**:4 (2005), 791–815. MR 2007f:58028 Zbl 1144.58014

[Eichler and Zagier 1985] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55**, Birkhäuser, Boston, 1985. MR 86j:11043 Zbl 0554.10018

[Gritsenko 1999a] V. Gritsenko, "Complex vector bundles and Jacobi forms", *Sūrikaisekikenkyūsho Kōkyūroku* 1103 (1999), 71–85. MR 2001a:11082 Zbl 1015.11015 arXiv math/9906191v1

[Gritsenko 1999b] V. Gritsenko, "Elliptic genus of Calabi–Yau manifolds and Jacobi and Siegel modular forms", *Algebra i Analiz* **11**:5 (1999), 100–125. MR 2001i:11051 Zbl 1101.14308 arXiv math/9906190v1

[Han and Liu 2014] F. Han and K. Liu, "Gravitational anomaly cancellation and modular invariance", *Algebr. Geom. Topol.* **14**:1 (2014), 91–113. MR 3158754 Zbl 1291.53061

[Han and Zhang 2004] F. Han and W. Zhang, "Modular invariance, characteristic numbers and η invariants", *J. Differential Geom.* **67**:2 (2004), 257–288. MR 2006k:58038 Zbl 1132.58014

[Han et al. 2012] F. Han, K. Liu, and W. Zhang, "Modular forms and generalized anomaly cancellation formulas", *J. Geom. Phys.* **62**:5 (2012), 1038–1053. MR 2901844 Zbl 06031298

[Hirzebruch 1966] F. Hirzebruch, *Topological methods in algebraic geometry*, 3rd ed., Springer, Berlin, 1966. MR 96c:57002 Zbl 0138.42001

[Hirzebruch 1999] F. Hirzebruch, "On the Euler characteristic of manifolds with $c_1 = 0$: a letter to V. Gritsenko", *Algebra i Analiz* **11**:5 (1999), 126–129. MR 2001h:32042 Zbl 0972.14014

[Hirzebruch et al. 1992] F. Hirzebruch, T. Berger, and R. Jung, *Manifolds and modular forms*, Aspects of Mathematics **E20**, Vieweg, Braunschweig, 1992. MR 94d:57001 Zbl 0767.57014

[Landweber 1988] P. S. Landweber (editor), *Elliptic curves and modular forms in algebraic topology* (Princeton, NJ, 1986), Lecture Notes in Math. **1326**, Springer, Berlin, 1988. MR 91a:57021 Zbl 0642.00007

[Li 2011] P. Li, "Chern numbers and the indices of some elliptic differential operators", *Pacific J. Math.* **251**:1 (2011), 173–182. MR 2012f:58044 Zbl 1223.57021

[Libgober and Wood 1990] A. S. Libgober and J. W. Wood, "Uniqueness of the complex structure on Kähler manifolds of certain homotopy types", *J. Differential Geom.* **32**:1 (1990), 139–154. MR 91g:32039 Zbl 0711.53052

[Liu 1995a] K. Liu, "Modular invariance and characteristic numbers", *Comm. Math. Phys.* **174**:1 (1995), 29–42. MR 96m:57034 Zbl 0867.57021

[Liu 1995b] K. Liu, "On modular invariance and rigidity theorems", *J. Differential Geom.* **41**:2 (1995), 343–396. MR 96f:58152 Zbl 0836.57024

[Liu 1996] K. Liu, "On elliptic genera and theta-functions", *Topology* **35**:3 (1996), 617–640. MR 97h:58149 Zbl 0858.57034

[Liu and Ma 2000] K. Liu and X. Ma, "On family rigidity theorems, I", *Duke Math. J.* **102**:3 (2000), 451–474. MR 2001j:58038 Zbl 0988.58010

[Liu et al. 2001] K. Liu, X. Ma, and W. Zhang, "On elliptic genera and foliations", *Math. Res. Lett.* **8**:3 (2001), 361–376. MR 2002h:58039 Zbl 0986.57023

[Liu et al. 2003] K. Liu, X. Ma, and W. Zhang, "Rigidity and vanishing theorems in K-theory", *Comm. Anal. Geom.* **11**:1 (2003), 121–180. arXiv math/9912108 arXiv math/0001014. MR 2004g:58028 Zbl 1087.58014

[Lusztig 1971] G. Lusztig, "Remarks on the holomorphic Lefschetz numbers", pp. 193–204 in *Analyse globale* (Montréal, QC, 1969), Sém. Math. Supérieures **42**, Presses de l'Université de Montréal, Montréal, QC, 1971. MR 52 #1718 Zbl 0249.58013

[Narasimhan and Ramanan 1975] M. S. Narasimhan and S. Ramanan, "Generalised Prym varieties as fixed points", *J. Indian Math. Soc.* (*N.S.*) **39** (1975), 1–19. MR 54 #12777 Zbl 0422.14018

[Salamon 1996] S. M. Salamon, "On the cohomology of Kähler and hyper-Kähler manifolds", *Topology* **35**:1 (1996), 137–155. MR 97f:32042 Zbl 0854.58004

[Taubes 1989] C. H. Taubes, "S^1 actions and elliptic genera", *Comm. Math. Phys.* **122**:3 (1989), 455–526. MR 90f:58167 Zbl 0683.58043

Received January 5, 2014. Revised July 19, 2014.

PING LI
DEPARTMENT OF MATHEMATICS
TONGJI UNIVERSITY
SHANGHAI, 200092
CHINA
pingli@tongji.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.353

ON THE GEOMETRY OF PRÜFER INTERSECTIONS OF VALUATION RINGS

BRUCE OLBERDING

Let F be a field, let D be a subring of F and let Z be an irreducible subspace of the space of all valuation rings between D and F that have quotient field F. Then Z is a locally ringed space whose ring of global sections is $A = \bigcap_{V \in Z} V$. All rings between D and F that are integrally closed in F arise in such a way. Motivated by applications in areas such as multiplicative ideal theory and real algebraic geometry, a number of authors have formulated criteria for when A is a Prüfer domain. We give geometric criteria for when A is a Prüfer domain that reduce this issue to questions of prime avoidance. These criteria, which unify and extend a variety of different results in the literature, are framed in terms of morphisms of Z into the projective line $\mathbb{P}^1_D$.

1. Introduction

A subring V of a field F is a *valuation ring of F* if for each nonzero $x \in F$, x or x^{-1} is in V; equivalently, the ideals of V are linearly ordered by inclusion and V has quotient field F. Although the ideal theory of valuation rings is straightforward, an intersection of valuation rings in F can be quite complicated. Indeed, by a theorem of Krull [Matsumura 1980, Theorem 10.4], every integrally closed subring of F is an intersection of valuation rings of F. In this article, we describe a geometrical approach to determining when an intersection A of valuation rings of F is a *Prüfer domain*, meaning that for each prime ideal P of A, the localization A_P is a valuation ring of F. Whether an intersection of valuation rings is Prüfer is of consequence in multiplicative ideal theory, where Prüfer domains are of central importance, and real algebraic geometry, where the real holomorphy ring is a Prüfer domain that expresses properties of fields involving sums of squares; see the discussion below. Over the past eighty years, Prüfer domains have been extensively studied from ideal-theoretic, homological and module-theoretic points of view; see, for example, [Fontana et al. 1997; Fuchs and Salce 2001; Gilmer 1968; Knebusch and Zhang 2002; Larsen and McCarthy 1971].

MSC2010: primary 13F05, 13F30; secondary 13B22, 14A15.
Keywords: Prüfer domain, valuation ring, Zariski–Riemann space.

Throughout the paper F denotes a field, D is a subring of F that need not have quotient field F, and Z is a subspace of the *Zariski–Riemann space* $\mathfrak{X}$ of F/D, the space of all valuation rings of F that contain D. The topology on $\mathfrak{X}$ is given by declaring the basic open sets to be those of the form $\{V \in \mathfrak{X} : t_1, \dots, t_n \in V\}$, where $t_1, \dots, t_n \in F$. We assume for technical convenience that $F \in Z$. With this notation fixed, the focus of this article is the *holomorphy ring*[1] $A = \bigcap_{V \in Z} V$ of the subspace Z. Such a ring is integrally closed in F, and, as noted above, every ring between D and F that is integrally closed in F occurs as the holomorphy ring of a subspace of $\mathfrak{X}$. In general it is difficult to determine the structure of A from properties of Z, topological or otherwise; see [Olberding 2007; 2008; 2011], where the emphasis is on the case in which D is a two-dimensional Noetherian domain with quotient field F. In this direction, there are a number of results that are concerned with when the holomorphy ring A is a Prüfer domain with quotient field F. Geometrically, this is equivalent to $\mathrm{Spec}(A)$ being an affine scheme in $\mathfrak{X}$. Moreover, by virtue of the valuative criterion for properness, A is a Prüfer domain with quotient field F if and only if there are no nontrivial proper birational morphisms into the scheme $\mathrm{Spec}(A)$, an observation that motivates Temkin and Tyomkin's notion [2013] of Prüfer algebraic spaces.

We show in this article that the morphisms of Z (viewed as a locally ringed space) into the projective line $\mathbb{P}^1_D$ determine whether the holomorphy ring A of Z is a Prüfer domain. A goal in doing so is to provide a unifying explanation for an interesting variety of results in the literature. By way of motivation, and because we will refer to them later, we recall these results here.

(1) Perhaps the earliest result in this direction is due to Nagata [1962, (11.11)]: When Z is finite, the holomorphy ring A of Z is a Prüfer domain with quotient field F.

(2) Gilmer [1969, Theorem 2.2] shows that when f is a nonconstant monic polynomial over D having no root in F and each valuation ring in Z contains the set $S := \{1/f(t) : t \in F\}$, then A is a Prüfer domain with torsion Picard group and quotient field F. Rush [2001, Theorem 1.4] has since generalized this by allowing the polynomial f to vary with the choice of t, but at the (necessary) expense of requiring the rational functions in S to have certain numerators other than 1. Gilmer was motivated by a special case of this theorem due to Dress [1965], which states that when the field F is formally real (meaning that -1 is not a sum of squares), then the subring of F generated by $\{(1+t^2)^{-1} : t \in F\}$ is a Prüfer domain with quotient field F whose set of valuation overrings is precisely the set of valuation rings of F

[1] This terminology is due to Roquette [1973, p. 362]. Viewing Z as consisting of places rather than valuation rings, the elements of A are precisely the elements of F that have no poles (i.e., do not have value infinity) at the places in Z.

for which -1 is not a square in the residue field. In the literature of real algebraic geometry, the Prüfer domain thus constructed is the *real holomorphy ring* of F/D. The fact that such rings are Prüfer has a number of interesting consequences for real algebraic geometry and sums of powers of elements of F; see, for example, [Becker 1982; Schülting 1982]. These rings are also the only known source of Prüfer domains having finitely generated ideals that cannot be generated by two elements, as was shown by Schülting [1979] and Swan [1984]; the related literature on this aspect of holomorphy rings is discussed in [Olberding and Roitman 2006]. The notion of existential closure leads to more general results on Prüfer holomorphy rings in function fields. For references on this generalization, see [Olberding 2006].

(3) Roquette [1973, Theorem 1] proves that when there exists a nonconstant monic polynomial $f \in A[T]$ which has no root in the residue field of V for each valuation ring $V \in Z$ (i.e., the residue fields are "uniformly algebraically non-closed"), A is a Prüfer domain with torsion Picard group and quotient field F. Roquette [1973, p. 362] developed these ideas as a general explanation for his principal ideal theorem, which states that the ring of totally p-integral elements of a formally p-adic field is a *Bézout domain*; that is, every finitely generated ideal is principal. In particular, if there is a bound on the size of the residue fields of the valuation rings in Z, then A is a Bézout domain [Roquette 1973, Theorem 3]. Motivated by just such a situation, Loper [1994] independently proved similar results in order to apply them to the ring of integer-valued polynomials of a domain R with quotient field F : $\mathrm{Int}(R) = \{g(T) \in F[T] : g(R) \subseteq R\}$.

(4) In [Olberding and Roitman 2006] it is shown that if the holomorphy ring A of Z contains a field of cardinality greater than that of Z, then A is a Bézout domain.

In this article we offer a geometric explanation for these results that reduces all the arguments to a question of homogeneous prime avoidance in the projective line $\mathbb{P}^1_D := \mathrm{Proj}(D[T_0, T_1])$. Nagata's theorem in (1) reduces to the observation that a finite set of points of $\mathbb{P}^1_D$ is contained in an affine open subset of $\mathbb{P}^1_D$. The example in (4) is explained similarly by showing that a "small" enough set of points in $\mathbb{P}^1_D$ is contained in an affine open set. And finally, in cases (2) and (3), the condition on the residue fields guarantees that the image of each D-morphism $Z \to \mathbb{P}^1_D$ is contained in the open affine subset $(\mathbb{P}^1_D)_g$, where g is the homogenization of f.

To frame things geometrically, we view Z as a locally ringed space. Its structure sheaf $\mathbb{O}_Z$ is defined for each nonempty open subset U of Z by $\mathbb{O}_Z(U) = \bigcap_{V \in Z} V$, while the ring of sections of the empty set is defined to be the trivial ring with $0 = 1$; thus $\mathbb{O}_Z$ is the *holomorphy sheaf* of Z. The restriction maps on $\mathbb{O}_Z$ off the empty set are simply set inclusion, and the stalks of $\mathbb{O}_Z$ are the valuation rings in Z. The standing assumption that F is one of the valuation rings in Z guarantees that Z is an irreducible space; irreducibility in turn guarantees that $\mathbb{O}_Z$ is a sheaf. (Note that since we are interested in the ring $A = \bigcap_{V \in Z} V$, the assumption $F \in Z$ is no limitation.) When considering irreducible subspaces Y of $\mathfrak{X}$, we similarly treat Y as a locally ringed space with structure sheaf defined in this way.

By a *morphism* we always mean a morphism in the category of locally ringed spaces. If X and Y are locally ringed spaces with fixed morphisms $\alpha : X \to \mathrm{Spec}(D)$ and $\beta : Y \to \mathrm{Spec}(D)$, then a morphism $\phi : X \to Y$ is a *D-morphism* if $\alpha = \beta \circ \phi$. A scheme X is a *D-scheme* if a morphism $\phi : X \to \mathrm{Spec}(D)$ is fixed. There is a morphism $\delta = (d, d^{\#}) : Z \to \mathrm{Spec}(D)$ defined by letting d be the continuous map that sends a valuation ring in Z to its center in D, and by letting $d^{\#} : \mathbb{O}_{\mathrm{Spec}(D)} \to d_* \mathbb{O}_Z$ be the sheaf morphism defined for each open subset U of $\mathrm{Spec}(D)$ by the set inclusion $d^{\#}_U : \mathbb{O}_{\mathrm{Spec}(D)}(U) \to \mathbb{O}_Z(d^{-1}(U))$. Thus when considering D-morphisms from Z to X, with X a D-scheme, we always assume that the structure morphism $Z \to \mathrm{Spec}(D)$ is the one defined above.

2. Morphisms into projective space

In this section we describe the D-morphisms of Z into projective space by proving an analogue of the fact that morphisms from schemes into projective space are determined by invertible sheaves. Our main technical device in describing such morphisms is the notion of a projective model, as defined in [Zariski and Samuel 1975, Chapter VI, §17]. Let $t_0, \ldots, t_n$ be nonzero elements of F, and for each $i = 0, 1, \ldots, n$, define $D_i = D[t_0/t_i, \ldots, t_n/t_i]$ and $U_i = \mathrm{Spec}(D_i)$. Then the *projective model of F/D defined by* $t_0, \ldots, t_n$ is

$$X = \{(D_i)_P : P \in \mathrm{Spec}(D_i),\ i = 0, 1, \ldots, n\}.$$

The projective model X is a topological space whose basic open sets are of the form $\{R \in X : u_0, \ldots, u_m \in R\}$, where $u_0, \ldots, u_m \in F$, and which is covered by the open subsets $\{(D_i)_P : P \in U_i\}$, $i = 0, 1, \ldots, n$. Define a sheaf $\mathbb{O}_X$ of rings on X for each nonempty open subset U of X by $\mathbb{O}_X(U) = \bigcap_{R \in U} R$, and let the ring of sections of the empty set be the trivial ring with $0 = 1$. Since X is irreducible, $\mathbb{O}_X$ is a sheaf and hence $(X, \mathbb{O}_X)$ is a scheme, and in light of the following remark, it is a projective scheme.

Remark 2.1. *If X is a projective model defined by $n + 1$ elements, then there is a closed immersion $X \to \mathbb{P}^n_D$.* For let X be the projective model defined by $t_0, \ldots, t_n \in F$. For each $i = 0, 1, \ldots, n$, let $b_i : D[T_0/T_i, \ldots, T_n/T_i] \to D_i$ be the D-algebra homomorphism that sends T_j/T_i to t_j/t_i, and let

$$a_i : \mathrm{Spec}(D_i) \to \mathrm{Spec}(D[T_0/T_i, \ldots, T_n/T_i])$$

be the induced continuous map of topological spaces. Then the scheme morphisms $(a_i, b_i) : \mathrm{Spec}(D_i) \to \mathrm{Spec}(D[T_0/T_i, \ldots, T_n/T_i])$ glue together to a morphism

$\phi : X \to \mathbb{P}^n_D$ [Hartshorne 1977, p. 88], which, by virtue of the way it is constructed, is a closed immersion [de Jong et al. 2005–, Lemma 01QO].

Let $t_0, \dots, t_n$ be nonzero elements of F, and let X be the projective model of F/D defined by $t_0, \dots, t_n$. For each valuation ring V in Z, there exists $i = 0, 1, \dots, n$ such that $t_j/t_i \in V$ for all j, and it follows that each valuation ring V in Z *dominates* a unique local ring R in the model X, meaning that $R \subseteq V$ and the maximal ideal of R is contained in the maximal ideal of V. The *domination morphism* $\delta = (d, d^\#) : Z \to X$ is defined by letting d be the continuous map that sends a valuation ring in Z to the local ring in X that it dominates, and by letting $d^\# : \mathcal{O}_X \to d_*\mathcal{O}_Z$ be the sheaf morphism defined for each open subset U of Z by the set inclusion $d^\#_U : \mathcal{O}_X(U) \to \mathcal{O}_Z(d^{-1}(U))$.

Let $\gamma : X \to \mathbb{P}^n_D$ be the closed immersion defined in Remark 2.1, and let $\delta : Z \to X$ denote the domination morphism. Then we say that the D-morphism $\gamma \circ \delta$ is the *morphism defined by* $t_0, \dots, t_n$. We show in Proposition 2.3 that each D-morphism $Z \to \mathbb{P}^n_D$ arises in this way. Our standing assumption $F \in Z$ is used in a strong way here, in that the proposition relies on a lemma which shows that the D-morphisms from Z into projective space are calibrated by the inclusion morphism $\mathrm{Spec}(F) \to Z$.

Lemma 2.2. *Let* $\iota : \mathrm{Spec}(F) \to Z$ *be the canonical morphism, let* $\phi = (f, f^\#) : Z \to X$ *and* $\gamma = (g, g^\#) : Z \to X$ *be morphisms of locally ringed spaces, where* X *is a separated scheme, and let* $\eta = f(F)$. *Then* $\phi = \gamma$ *if and only if* $\phi \circ \iota = \gamma \circ \iota$ *if and only if* $\eta = f(F) = g(F)$ *and* $f^\#_\eta = g^\#_\eta$.

Proof. Suppose that $\eta = f(F) = g(F)$ and $f^\#_\eta = g^\#_\eta$. Let U be an affine open subset of X containing η, and let $Y = f^{-1}(U)$. Then Y is a locally ringed space with structure sheaf $\mathcal{O}_Y$ defined for each open set W in Y by $\mathcal{O}_Y(W) = \mathcal{O}_Z(W)$. We claim that $\phi|_Y = \gamma|_Y$. Since U is affine and Y is a locally ringed space, the morphisms $\phi|_Y$ and $\gamma|_Y$ are equal if and only if $f^\#_U = g^\#_U$ [Holme 2012, Theorem 10.8, p. 200]. Now since $\mathcal{O}_Z(Y) \subseteq \mathcal{O}_{Z,F} = F$ and the restriction maps on the sheaf $\mathcal{O}_Z$ are set inclusions, for each $s \in \mathcal{O}_X(U)$ we have $f^\#_U(s) = f^\#_\eta(s) = g^\#_\eta(s) = g^\#_U(s)$. Thus $f^\#_U = g^\#_U$, and hence $\phi|_Y = \gamma|_Y$. Finally, let $\{U_i\}$ be the collection of all affine open subsets of X that contain η. Then $\{f^{-1}(U_i)\}$ is a cover of Z, and we have shown that ϕ and γ restrict to the same morphism on each of these open sets, so we conclude that $\phi = \gamma$. It is straightforward to verify that $\phi \circ \iota = \gamma \circ \iota$ if and only if $f(F) = g(F)$ and $f^\#_\eta = g^\#_\eta$, so the lemma follows. □

Proposition 2.3. *If* $\phi : Z \to \mathbb{P}^n_D$ *is a* D-*morphism, then there exist* $t_0, \dots, t_n \in F$ *such that* ϕ *is defined by* $t_0, \dots, t_n$.

Proof. Write $\phi = (f, f^\#)$, let $\eta = f(F)$, and let $S = \mathbb{P}^n_D = \mathrm{Proj}(D[T_0, \dots, T_n])$. For each $i = 0, \dots, n$, let U_i be the open affine set S_{T_i}, so that $S = U_0 \cup \cdots \cup U_n$. Let

$\alpha = (a, a^\#) : \mathrm{Spec}(F) \to S$ be the composition of ϕ with the canonical morphism $\mathrm{Spec}(F) \to Z$, and note that for each i, we have $a^\#_{U_i}(s) = f^\#_{S,\eta}(s)$ for all $s \in \mathbb{O}_S(U_i)$. Since α is a morphism of schemes into projective n-space over D, there exist $t_0, \ldots, t_n \in F$ such that $f^\#_{U_i}(T_j/T_i) = t_j/t_i$ for each i, j; see the proof of Theorem II.7.1 of [Hartshorne 1977, p. 150]. Let X be the projective model of F/D defined by $t_0, \ldots, t_n$. Then $t_0, \ldots, t_n$ can be viewed as global sections of an invertible sheaf on X that is the image of the twisting sheaf $\mathbb{O}(1)$ of S. By the theorem just cited and its proof, there is then a unique D-morphism $\gamma = (g, g^\#) : X \to S$ such that $g^\#_U = f^\#_U$ for each open set U of S and $g : X \to S$ is the continuous map that for each $i = 0, \ldots, n$ sends the equivalence class of a prime ideal P in $\mathrm{Spec}(D[t_0/t_i, \ldots, t_n/t_i]) \subseteq X$ to the equivalence class of the prime ideal $(f^\#_{U_i})^{-1}(P)$ in $U_i = \mathrm{Spec}(D[T_0/T_i, \ldots, T_n/T_i])$. Then, with $\delta = (d, d^\#) : Z \to X$ the domination morphism, $\gamma \circ \delta : Z \to S$ is a D-morphism. Moreover, $g(d(F)) = g(F) = \eta = f(F)$ and (viewing F as a point in both X and Z), $(d^\# \circ g^\#)_F = d^\#_F \circ g^\#_\eta = f^\#_\eta$. Therefore, by Lemma 2.2, $\phi = \gamma \circ \delta$. $\square$

Corollary 2.4. *Every D-morphism $\phi : Z \to \mathbb{P}^n_D$ lifts to a unique D-morphism $\widetilde{\phi} : \mathfrak{X} \to \mathbb{P}^n_D$.*

Proof. Let $\phi : Z \to \mathbb{P}^n_D$ be a D-morphism. Then by Proposition 2.3, there exist a projective model X of F/D and a D-morphism $\gamma : X \to \mathbb{P}^n_D$ such that $\phi = \gamma \circ \delta|_Z$, where $\delta : Z \to X$ is the domination map. Since X is a projective model of F/D, each valuation ring in $\mathfrak{X}$ dominates X, and hence $\delta : Z \to X$ extends to the domination morphism $\widetilde{\delta} : \mathfrak{X} \to X$. Thus $\widetilde{\phi} = \gamma \circ \widetilde{\delta}$ lifts ϕ. If there is another morphism $\psi : \mathfrak{X} \to \mathbb{P}^n_D$ that lifts ϕ, then with $\iota : \mathrm{Spec}(F) \to Z$ the canonical morphism, $\psi \circ \iota = \phi \circ \iota = \widetilde{\phi} \circ \iota$, so that by Lemma 2.2, $\psi = \widetilde{\phi}$. $\square$

Remark 2.5. By Lemma 2.2, the D-morphisms $Z \to \mathbb{P}^n_D$ are determined by their composition with the morphism $\mathrm{Spec}(F) \to \mathbb{P}^n_D$. Conversely, by Corollary 2.4, each D-morphism $\mathrm{Spec}(F) \to Z$ lifts to a unique morphism $Z \to \mathfrak{X}$. Thus the D-morphisms $Z \to \mathbb{P}^n_D$ are in one-to-one correspondence with the F-valued points of $\mathbb{P}^n_D$.

3. A geometrical characterization of Prüfer domains

We show in this section that if Z has the property that the image of every D-morphism $Z \to \mathbb{P}^1_D$ of locally ringed spaces factors through an affine scheme, then the holomorphy ring A of Z is a Prüfer domain. A special case in which this is satisfied is when there is a homogeneous polynomial $f(T_0, T_1)$ of positive degree d such that the image of each such morphism is contained in $(\mathbb{P}^1_D)_f$. In this case, we show that the Prüfer domain A has torsion Picard group.

Theorem 3.1. *The ring $A = \bigcap_{V \in Z} V$ is a Prüfer domain with quotient field F if and only if every D-morphism $Z \to \mathbb{P}^1_D$ factors through an affine scheme.*

Proof. Suppose A is a Prüfer domain, and let $\phi : Z \to \mathbb{P}^1_D$ be a D-morphism. By Proposition 2.3, there exist a projective model X of F/D and a D-morphism $\gamma : X \to \mathbb{P}^1_D$ such that $\phi = \gamma \circ \delta$, where $\delta : Z \to X$ is the domination morphism. Since A is a Prüfer domain with quotient field F, every localization of A is a valuation domain and hence dominates a local ring in X. Since every valuation ring in Z contains A, it follows that ϕ factors through the affine scheme $\mathrm{Spec}(A)$.

Conversely, suppose that every D-morphism $Z \to \mathbb{P}^1_D$ factors through an affine scheme. Let P be a prime ideal of A. To prove that A_P is a valuation domain with quotient field F, it suffices to show that for each $0 \neq t \in F$ we have $t \in A_P$ or $t^{-1} \in A_P$. Let $0 \neq t \in F$, and let X be the projective model of F/D defined by $1, t$. Then by Remark 2.1, there is a closed immersion of X into $\mathbb{P}^1_D$. Let $\phi = (f, f^\#) : Z \to \mathbb{P}^1_D$ be the D-morphism that results from composing this closed immersion with the domination morphism $Z \to X$. In particular, with $\nu = f(F)$, we have $f^\#_\nu(T_1/T_0) = t$ and $f^\#_\nu(T_0/T_1) = t^{-1}$.

By assumption, there are a ring R and D-morphisms $\delta = (d, d^\#) : Z \to \mathrm{Spec}(R)$ and $\gamma = (g, g^\#) : \mathrm{Spec}(R) \to \mathbb{P}^1_D$ such that $\phi = \gamma \circ \delta$. By replacing R with its image in F under $d^\#_\eta$, where $\eta = d(F)$, we may assume by Lemma 2.2 that R is a subring of F and that δ is the domination morphism. Then since R is the ring of global sections of $\mathrm{Spec}(R)$ and A is the ring of global sections of Z, it follows that $R \subseteq A$, and hence $Q = R \cap P$ is a prime ideal of R. Let $x = g(Q)$. Then $x \in (\mathbb{P}^1_D)_{T_0}$ or $x \in (\mathbb{P}^1_D)_{T_1}$. In the former case, $f^\#_x(T_1/T_0) = t$, and in the latter, $f^\#_x(T_0/T_1) = t^{-1}$. But $f^\# = d^\# \circ g^\#$ and $d^\#$ restricts on each nonempty open subset of $\mathrm{Spec}(R)$ to the inclusion mapping, so either $x \in (\mathbb{P}^1_D)_{T_0}$, so that $t = f^\#_x(T_1/T_0) = g^\#_x(T_1/T_0) \in R_Q \subseteq A_P$, or $x \in (\mathbb{P}^1_D)_{T_1}$, so that $t^{-1} = f^\#_x(T_0/T_1) = g^\#_x(T_0/T_1) \in R_Q \subseteq A_P$. This proves that A is a Prüfer domain with quotient field F. □

Nagata's theorem discussed in (1) follows then from prime avoidance:

Corollary 3.2 [Nagata 1962, (11.11)]. *If Z is a finite set, then A is a Prüfer domain with quotient field F.*

Proof. Let $\phi : Z \to \mathbb{P}^1_D$ be a D-morphism. Then the image of ϕ in $\mathbb{P}^1_D$ is finite, so by homogeneous prime avoidance [Bruns and Herzog 1993, Lemma 1.5.10], there exists a homogeneous polynomial f (necessarily of positive degree) in the irrelevant ideal (T_0, T_1) of $D[T_0, T_1]$ such that f is not in the union of the finitely many homogeneous prime ideals corresponding to the image of Z in $\mathbb{P}^1_D$; i.e., the image of ϕ is contained in $(\mathbb{P}^1_D)_f$. This subset is affine [Eisenbud and Harris 2000, Exercise III.10, p. 99], so by Theorem 3.1, A is a Prüfer domain with quotient field F. □

In fact, when Z is finite, A is a Bézout domain: If M is a maximal ideal of A, then A_M is a valuation domain, but since Z is finite, $A_M = \bigcap_{V \in Z} VA_M$, which, since A_M is a valuation domain, forces $A_M = V$ for some $V \in Z$. Therefore, A has only finitely many maximal ideals, so that every invertible ideal is principal, and hence A is a Bézout domain.

In Theorem 3.5, we give a criterion for when A is a Prüfer domain with torsion Picard group. In this case, each D-morphism $Z \to \mathbb{P}^1_D$ not only factors through an affine scheme, but has image in an affine open subscheme of $\mathbb{P}^1_D$. For lack of a precise reference, we note the following standard observation.

Lemma 3.3. *Let X be a projective model of F/D defined by $t_0, \dots, t_n \in F$, and let $f(T_0, \dots, T_n) \in D[T_0, \dots, T_n]$ be homogeneous of positive degree d such that $f(t_0, \dots, t_n) \neq 0$. Let*

$$R = \{0\} \cup \left\{ \frac{h(t_0, \dots, t_n)}{f(t_0, \dots, t_n)^e} : e \geq 0 \textit{ and } h \textit{ is a homogeneous form of degree } de \right\}.$$

Then $\{R_P : P \in \operatorname{Spec}(R)\}$ is an open affine subset of X.

Proof. Let $S = \mathbb{P}^n_D$. Then S_f is an open affine subset of S [Eisenbud and Harris 2000, Exercise III.10, p. 99]. By Remark 2.1, there is a closed immersion $\gamma = (g, g^{\#}) : X \to S$ such that with $\eta = g(F)$, we have $g^{\#}_\eta(T_j/T_i) = t_j/t_i$ for each i, j. Since S_f is an open affine subset of S and γ is a closed immersion, $g^{-1}(S_f)$ is an open affine subset of X whose ring of sections is $g^{\#}_\eta(\mathbb{O}_S(S_f))$ [de Jong et al. 2005–, Lemma 01IN]. Now $\mathbb{O}_S(S_f)$ is the ring consisting of 0 and the rational functions of the form h/f^e, where $e > 0$ and h is a homogeneous form of degree de. Moreover, for such a rational function, since $f(t_0, \dots, t_n) \neq 0$, we have that $f(T_0, \dots, T_n)$ is a unit in $\mathbb{O}_{S,\eta}$ and

$$g^{\#}_\eta\left(\frac{h(T_0, \dots, T_n)}{f(T_0, \dots, T_n)^e}\right) = \frac{h(t_0, \dots, t_n)}{f(t_0, \dots, t_n)^e} \in R.$$

Thus $g^{\#}_\eta(\mathbb{O}_S(S_f)) = R$, which proves the lemma. □

Lemma 3.4. *Let $t_0, t_1, \dots, t_n$ be nonzero elements of F, and let f be a homogeneous polynomial in $D[T_0, \dots, T_n]$ of positive degree d. Then the following are equivalent.*

(1) $t_0^d, \dots, t_n^d \in f(t_0, \dots, t_n)A$.

(2) $(t_0, \dots, t_n)^d A = f(t_0, \dots, t_n)A$.

(3) *The image of the morphism $Z \to \mathbb{P}^n_D$ defined by $t_0, \dots, t_n$ is in $(\mathbb{P}^n_D)_f$.*

Proof. Let $u = f(t_0, \dots, t_n)$. First we claim that (1) implies (2). If $V \in Z$, then there is i such that t_i divides in V each of $t_0, \dots, t_n$. It follows that when $\sum_i e_i = d$ for nonnegative integers e_i, we have $t_0^{e_0}t_1^{e_1}\dots t_n^{e_n} \in t_i^d V$. Thus by (1), $t_0^{e_0}t_1^{e_1}\dots t_n^{e_n} \in uV$, so that $t_0^{e_0}t_1^{e_1}\dots t_n^{e_n} \in uA$. Statement (2) now follows.

To see that (2) implies (3), let $\gamma = (g, g^\#) : Z \to \mathbb{P}^n_D$ be the morphism defined by $t_0, \dots, t_n$. By (2), $u = f(t_0, \dots, t_n)$ is nonzero. Define

$$R = \{0\} \cup \left\{ \frac{h(t_0, \dots, t_n)}{u^e} : e \geq 0 \text{ and } h \text{ is a homogeneous form of degree } de \right\},$$
$$S = \{0\} \cup \left\{ \frac{h(T_0, \dots, T_n)}{f(T_0, \dots, T_n)^e} : e \geq 0 \text{ and } h \text{ is a homogeneous form of degree } de \right\},$$

so that $(\mathbb{P}^n_D)_f = \operatorname{Spec}(S)$. Let $\alpha = (a, a^\#) : \operatorname{Spec}(R) \to \operatorname{Spec}(S)$ be the morphism induced by the ring homomorphism $a^\# : S \to R$ given by evaluation at $t_0, \dots, t_n$. Observe that $R \subseteq A$, since if h is a homogeneous form in $D[T_0, \dots, T_n]$ of degree de, then by (2), $h(t_0, \dots, t_n) \in (t_0, \dots, t_n)^{de}A = u^eA$, so that $R \subseteq A$. Now let $\beta : Z \to \operatorname{Spec}(R)$ be the induced domination morphism. We claim that $\gamma = \alpha \circ \beta$. Indeed, by Lemma 3.3, $\operatorname{Spec}(R)$ is an affine submodel of the projective model X of F/D defined by $t_0, \dots, t_n$, and γ factors through X. Since β is the domination mapping, it follows that $\gamma = \alpha \circ \beta$, and hence the image of γ is contained in $\operatorname{Spec}(S) = (\mathbb{P}^n_D)_f$.

Finally, to see that (3) implies (1), let $U = (\mathbb{P}^n_D)_f$ and let $\gamma = (g, g^\#) : Z \to \mathbb{P}^n_D$ be the morphism defined by $t_0, \dots, t_n$. By (3), $Z \subseteq g^{-1}(U)$, so S, the ring of sections of U, is mapped via $g^\#_U$ into the holomorphy ring A of Z. But the image of $g^\#_U$ is R, so $R \subseteq A$, and hence every element of F of the form t_i^d/u is an element of A, from which (1) follows. □

Theorem 3.5. *The ring $A = \bigcap_{V \in Z} V$ is a Prüfer domain with torsion Picard group and quotient field F if and only if for each A-morphism $\phi : Z \to \mathbb{P}^1_A$ there is a homogeneous polynomial $f \in A[T_0, T_1]$ of positive degree such that the image of ϕ is in $(\mathbb{P}^1_A)_f$.*

Proof. The choice of the subring D of F was arbitrary, so for the sake of this proof we may assume without loss of generality that $D = A$ and apply then the preceding results to A. Suppose that for each A-morphism $\phi : Z \to \mathbb{P}^1_A$ there exists a homogeneous polynomial $f \in A[T_0, T_1]$ of positive degree such that the image of ϕ is in the affine subset $(\mathbb{P}^1_A)_f$. By Theorem 3.1, A is a Prüfer domain with quotient field F. Thus, to prove that A has torsion Picard group, it suffices to show that for each two-generated ideal $(t_0, t_1)A$ of A, there exists $e > 0$ such that $(t_0, t_1)^eA$ is a principal ideal (see, for example, the proof of Theorem 2.2 of [Gilmer 1969]). Let $t_0, t_1 \in F$, and let $\phi : Z \to \mathbb{P}^1_A$ be the morphism defined by t_0, t_1. Then by assumption, there exists a homogeneous polynomial $f \in A[T_0, T_1]$ of positive degree d such that the image of Z in $\mathbb{P}^1_A$ is contained in $(\mathbb{P}^1_A)_f$. Thus by Lemma 3.4, $(t_0, t_1)^dA$ is a principal ideal.

Conversely, let $\phi : Z \to \mathbb{P}^1_A$ be an A-morphism. Then by Proposition 2.3, there exist $t_0, t_1 \in F$ such that ϕ is defined by t_0, t_1. Since A has torsion Picard group and quotient field F, there exists $d > 0$ such that $(t_0, t_1)^d A = uA$ for some $u \in (t_0, t_1)^d A$. Since u is an element of $(t_0, t_1)^d A$, there exists a homogeneous polynomial $f \in A[T_0, T_1]$ of degree d such that $f(t_0, t_1) = u$, and hence by Lemma 3.4, the image of the morphism ϕ is contained in $(\mathbb{P}^1_A)_f$. □

For applications such as those discussed in (2) and (3) of the introduction, one needs to work with D-morphisms into the projective line over D, rather than A. This involves a change of base, but causes no difficulties when verifying that A is a Prüfer domain. However, the converse of Theorem 3.5 (which is not needed in the applications in (2) and (3) of the introduction) is lost in the base change.

Corollary 3.6. *If for each D-morphism $\phi : Z \to \mathbb{P}^1_D$ there exists a homogeneous polynomial $f \in D[T_0, T_1]$ of positive degree such that the image of Z is contained in $(\mathbb{P}^1_D)_f$, then A is a Prüfer domain with torsion Picard group and quotient field F.*

Proof. Let $\phi : Z \to \mathbb{P}^1_A$ be a D-morphism, and let $\alpha : \mathbb{P}^1_A \to \mathbb{P}^1_D$ be the change of base morphism. By assumption, there exists a homogeneous polynomial $f \in D[T_0, T_1]$ such that the image of $\alpha \circ \phi$ is contained in $(\mathbb{P}^1_D)_f$. Then the image of ϕ is contained in $(\mathbb{P}^1_A)_f$, and the corollary follows from Theorem 3.1. □

Let n be a positive integer. An abelian group G is an *n-group* if each element of G has finite order and this order is divisible by only such primes that also appear as factors of n. If A is a Prüfer domain with quotient field F, then the Picard group of A is an n-group if and only if for each $t \in F$ there exists $k > 0$ such that $(A + tA)^{n^k}$ is a principal fractional ideal of A [Roquette 1973, Lemma 1].

Remark 3.7. If each homogeneous polynomial f arising as in the statement of the corollary can be chosen with degree $\leq n$ (n fixed), then the Picard group of the Prüfer domain A is an n-group. For when $t \in F$ and $\phi : Z \to \mathbb{P}^1_D$ is the D-morphism defined by $1, t$, then with f the polynomial of degree $\leq n$ given by the corollary, Lemma 3.4 shows that $(A + tA)^n$ is a principal fractional ideal of A. In particular, when for each D-morphism $\phi : Z \to \mathbb{P}^1_D$ there exists a linear homogeneous polynomial $f \in A[T_0, T_1]$ such that the image of ϕ is contained in $(\mathbb{P}^1_A)_f$, the ring A is a Bézout domain with quotient field F.

The next corollary is a stronger version of statement (4) in the introduction.

Corollary 3.8. *If D is a local domain and Z has cardinality less than that of the residue field of D, then A is a Bézout domain with quotient field F.*

Proof. Let $\phi : Z \to \mathbb{P}^1_D$ be a D-morphism. For each $P \in \mathrm{Proj}(D[T_0, T_1])$, let $\Delta_P = \{d \in D : T_0 + dT_1 \in P\}$. Then all the elements of Δ_P have the same image in the residue field of D. Indeed, if $d_1, d_2 \in \Delta_P$, then

$$(d_1 - d_2)T_1 = (T_0 + d_1 T_1) - (T_0 + d_2 T_1) \in P.$$

If $T_1 \in P$, then since $T_0 + d_1 T_1 \in P$, this forces $(T_0, T_1) \subseteq P$, a contradiction to the fact that $P \in \mathrm{Proj}(D[T_0, T_1])$. Therefore, $T_1 \notin P$, so that $d_1 - d_2 \in P \cap D \subseteq \mathfrak{m} :=$ maximal ideal of D, which shows that all the elements of Δ_P have the same image in the residue field of D. Let X denote the image of ϕ in $\mathbb{P}^1_D$. Then since $|X| < |D/\mathfrak{m}|$, there exists $d \in D \setminus \bigcup_{P \in X} \Delta_P$, and hence $f(T_0, T_1) := T_0 + dT_1 \notin P$ for all $P \in X$. Thus the image of ϕ is in $(\mathbb{P}^1_D)_f$, and by Corollary 3.6 and Remark 3.7, A is a Bézout domain with quotient field F. □

The following corollary is a small improvement of a theorem of Rush [2001, Theorem 1.4]. Whereas the theorem of Rush requires that $1, t, t^2, \dots, t^{d_t} \in f_t(t)A$, we need only that $1, t^{d_t} \in f_t(t)A$.

Corollary 3.9. *The ring A is a Prüfer domain with torsion Picard group and quotient field F if and only if for each $0 \neq t \in F$, there is a polynomial $f_t(T) \in A[T]$ of positive degree d_t such that $1, t^{d_t} \in f_t(t)A$.*

Proof. If A is a Prüfer domain with torsion Picard group and quotient field F, then for each $0 \neq t \in F$, there is $d_t > 0$ such that $(1, t)^{d_t} A$ is a principal fractional ideal of A. Since A is a Prüfer domain, local verification shows that $(1, t)^{d_t} A = (1, t^{d_t})A$, and it follows that there is a polynomial $f_t(T) \in A[T]$ of positive degree d_t such that $1, t^{d_t} \in f_t(t)A$.

To prove the converse, we use Theorem 3.5. Let $\phi : Z \to \mathbb{P}^1_D$ be a D-morphism. Then by Proposition 2.3, there exists $0 \neq t \in F$ such that ϕ is defined by $1, t$. By assumption, there is a polynomial $f_t(T) \in A[T]$ of positive degree d_t such that $1, t^{d_t} \in f_t(t)A$. Set $g_t(T_0, T_1) = f_t(T_0/T_1)T_1^{d_t}$, so that $g_t(T_0, T_1)$ is a homogeneous form of positive degree. Then $1, t^{d_t} \in g_t(t, 1)A$, and by Lemma 3.4, the image of ϕ is in $(\mathbb{P}^1_A)_g$. By Theorem 3.5, A is a Prüfer domain with torsion Picard group and quotient field F. □

Rush [2001, Theorem 2.2] proves that when f is a monic polynomial of positive degree in $A[T]$, then (a) $\{1/f(t) : t \in F\} \subseteq A$ if and only if (b) the image of f in $(V/\mathfrak{M}_V)[T]$ has no root in $V/\mathfrak{M}_V$ for each $V \in Z$ if and only if (c) A is a Prüfer domain and $f(a)$ is a unit in A for each $a \in A$. As Rush points out, Gilmer's theorem discussed in (2) of the introduction follows quickly from the equivalence of (a) and (b) and Corollary 3.9; see the discussion on pp. 314–315 of [Rush 2001]. Similarly, the results of Loper and Roquette described in (3) of the introduction also follow from Corollary 3.9 and the equivalence of (a) and (b). Thus all the constructions in (1)–(4) of the introduction are recovered by the results in this section.

4. The case where D is a local ring

This section focuses on the case where D is a local ring that is integrally closed in F. (By a local ring, we mean a ring that has a unique maximal ideal; in particular, we do not require local rings to be Noetherian.) In such a case, as is noted in the proof of Theorem 4.2, every proper subset of closed points of $\mathbb{P}^1_D$ is contained in an affine open subset of $\mathbb{P}^1_D$, a fact which leads to a stronger result than could be obtained in the last section. To prove the theorem, we need a coset version of homogeneous prime avoidance. The proof of the lemma follows Gabber, Liu and Lorenzini [Gabber et al. 2013] but involves a slight modification to permit cosets.

Lemma 4.1. (cf. [Gabber et al. 2013, Lemma 4.11]) *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring, and let $P_1, \dots, P_n$ be incomparable homogeneous prime ideals not containing R_1. Let $I = \bigoplus_{i=0}^{\infty} I_i$ be a homogeneous ideal of R such that $I \not\subseteq P_i$ for each $i = 1, \dots, n$. Then there exists $e_0 > 0$ such that for all $e \geq e_0$ and $r_1, \dots, r_n \in R$, $I_e \not\subseteq \bigcup_{i=1}^{n} (P_i + r_i)$.*

Proof. The proof is by induction on n. For the case $n = 1$, let s be a homogeneous element in $I \setminus P_1$, let $e_0 = \deg s$, let $e \geq e_0$ and let $t \in R_1 \setminus P_1$. Suppose that $r_1 \in R$ and $I_e \subseteq P_1 + r_1$. Then since $0 \in I_e$, this forces $r_1 \in P_1$ and hence $st^{e-e_0} \in I_e \subseteq P_1$, a contradiction to the fact that neither s nor t is in P_1. Thus $I_e \not\subseteq P_1 + r_1$. Next, let $n > 1$, and suppose that the lemma holds for $n - 1$. Then since the P_i are incomparable, $IP_1 \dots P_{n-1} \not\subseteq P_n$, and by the case $n = 1$, there exists $f_0 > 0$ such that for all $f \geq f_0$ and $r_n \in R$ we have $(IP_1 \dots P_{n-1})_f \not\subseteq (P_n + r_n)$. Also, by the induction hypothesis, there exists $g_0 > 0$ such that for all $g \geq g_0$ and $r_1, \dots, r_{n-1} \in R$ we have $(IP_n)_g \not\subseteq \bigcup_{i=1}^{n-1} (P_i + r_i)$. Let $e_0 = \max\{f_0, g_0\}$, let $e \geq e_0$ and let $r_1, \dots, r_n \in R$. Then in light of the above considerations, we may choose $a \in (IP_1 \dots P_{n-1})_e \setminus (P_n + r_n)$ and $b \in (IP_n)_e \setminus \bigcup_{i=1}^{n-1} (P_i + r_i)$. Then $a + b \in I_e \setminus \bigcup_{i=1}^{n} (P_i + r_i)$. □

Theorem 4.2. *Suppose D is local and integrally closed in F and only finitely many valuation rings in Z do not dominate D. If no D-morphism $Z \to \mathbb{P}^1_D$ has every closed point of $\mathbb{P}^1_D$ in its image, then $A = \bigcap_{V \in Z} V$ is a Prüfer domain with torsion Picard group and quotient field F.*

Proof. Let $S = D[T_0, T_1]$. By Corollary 3.6, it suffices to show that for each D-morphism $\phi : Z \to \mathbb{P}^1_D$, there is a homogeneous polynomial $f \in S$ of positive degree such that the image of ϕ is in $(\mathbb{P}^1_D)_f$. To this end, let $\phi : Z \to \mathbb{P}^1_D$ be a D-morphism. By assumption, there is a closed point $x \in \mathbb{P}^1_D$ not in the image of ϕ. Let $\pi : \mathbb{P}^1_D \to \operatorname{Spec}(D)$ be the structure morphism. Since π is a proper morphism, π is closed and hence $\pi(x)$ is a closed point in $\operatorname{Spec}(D)$. Thus since D is local, $\pi(x)$ is the maximal ideal $\mathfrak{m}$ of D. Let $\Bbbk$ be the residue field of D. Then, with Q the homogeneous prime ideal in S corresponding to x, we must have $\mathfrak{m} \subseteq Q$, and hence $\mathrm{Proj}(\Bbbk[T_0, T_1])$ is isomorphic to a closed subset of $\mathbb{P}^1_D$ containing Q. Since a homogeneous prime ideal in $\mathrm{Proj}(\Bbbk[T_0, T_1])$ is generated by a homogeneous polynomial in $\Bbbk[T_0, T_1]$, it follows that there is a homogeneous polynomial $g \in S$ of positive degree d such that $Q = (\mathfrak{m}, g)S$. Since, as noted above, every prime ideal in $\mathbb{P}^1_D = \mathrm{Proj}(S)$ corresponding to a closed point in $\mathbb{P}^1_D$ contains $\mathfrak{m}$, it follows that every closed point in $\mathbb{P}^1_D$ distinct from x is contained in $(\mathbb{P}^1_D)_g$. Thus if every valuation ring in Z other than F dominates D, then the image of ϕ is contained in $(\mathbb{P}^1_D)_g$, which proves the theorem.

It remains to consider the case where Z also contains, in addition to the valuation ring F, valuation rings $V_1, \ldots, V_n$ that are not centered on the maximal ideal $\mathfrak{m}$ of D. Let $P_1, \ldots, P_n$ be the homogeneous prime ideals of S that are the images under ϕ of $V_1, \ldots, V_n$, respectively. Let $I = \mathfrak{m}S$. No V_i dominates D, so since ϕ is a morphism of locally ringed spaces, $I \not\subseteq P_i$ for all $i = 1, \ldots, n$. We may assume $P_1, \ldots, P_k$ are the prime ideals that are maximal in the set $\{P_1, \ldots, P_n\}$. Then by Lemma 4.1, there exists $e > 0$ such that $I_{de} \not\subseteq \bigcup_{i=1}^k (P_i + g^e)$. Let h be a homogeneous element in $I_{de} \setminus \bigcup_{i=1}^k (P_i + g^e)$. Since $P_1, \ldots, P_k$ are maximal in $\{P_1, \ldots, P_n\}$, it follows that $h \in I_{de} \setminus \bigcup_{i=1}^n (P_i + g^e)$. Set $f = h - g^e$. Then $f \notin P_i$ for all i. In particular, $f \neq 0$, and hence f is homogeneous of degree de. Since $f \notin P_1 \cup \cdots \cup P_n$, we have $P_1, \ldots, P_n \in (\mathbb{P}^1_D)_f$.

Finally we show that every closed point of $\mathbb{P}^1_D$ distinct from x is in $(\mathbb{P}^1_D)_f$. Let L be a prime ideal in $\mathrm{Proj}(S)$ corresponding to a closed point distinct from x. Then $L \neq Q$, and to finish the proof, we need only show that $f \notin L$. As noted above, $\mathfrak{m} \subseteq L$, so if $f \in L$, then since $h \in \mathfrak{m}S$, we have $g^e \in L$. But then $Q = (\mathfrak{m}, g)S \subseteq L$, forcing $Q = L$ since Q is maximal in $\mathrm{Proj}(S)$. This contradiction implies that $f \notin L$, and hence every closed point of $\mathbb{P}^1_D$ distinct from x is in $(\mathbb{P}^1_D)_f$, which completes the proof. □

Remark 4.3. When the valuation rings in Z do not dominate D, the theorem can still be applied if there exists $Y \subseteq \mathfrak{X}$ containing F such that (a) each valuation ring in Y other than F dominates D, (b) each valuation ring in Z specializes to a valuation ring in Y, and (c) no D-morphism $\phi : Y \to \mathbb{P}^1_D$ has every closed point in its image. For by the theorem, the holomorphy ring of Y is a Prüfer domain with torsion Picard group and quotient field F. As an overring of the holomorphy ring of Y, the holomorphy ring of Z has these same properties also.

The following corollary shows how the theorem can be used to prove that real holomorphy rings can be intersected with finitely many nondominating valuation rings and the result remains a Prüfer domain with quotient field F. In general an intersection of a Prüfer domain and a valuation domain need not be a Prüfer domain. For example, when D is a two-dimensional local Noetherian UFD with quotient field F and f is an irreducible element of D, then D_f is a PID and $D_{(f)}$

is a valuation ring, but $D = D_f \cap D_{(f)}$, so that the intersection is not Prüfer. This example can be modified to show more generally that for this choice of D, there exist quasicompact schemes in $\mathfrak{X}$ that are not affine.

Corollary 4.4. *Suppose D is essentially of finite type over a real-closed field and that F and the residue field of D are formally real. Let H be the real holomorphy ring of F/D. Then for any valuation rings $V_1, \dots, V_n \in \mathfrak{X}$ not dominating D, the ring $H \cap V_1 \cap \cdots \cap V_n$ is a Prüfer domain with torsion Picard group and quotient field F.*

Proof. Each formally real valuation ring in $\mathfrak{X}$ specializes to a formally real valuation ring dominating D (this can be deduced, for example, from Theorem 23 of [Kuhlmann 2004]). Let Y be the set of all the formally real valuation rings dominating D, let $Z = Y \cup \{F, V_1, \dots, V_n\}$, and let $\phi : Z \to \mathbb{P}^1_D$ be a D-morphism. Then the image of Y under ϕ is contained in $(\mathbb{P}^1_D)_f$, where $f(T_0, T_1) = T_0^2 + T_1^2$. Because $V_1, \dots, V_n$ do not dominate D, they are not mapped by ϕ to closed points of $\mathbb{P}^1_D$. Thus the corollary follows from Theorem 4.2. □

We include the last corollary as more of a curiosity than an application. Suppose that D has quotient field F. A valuation ring V in $\mathfrak{X}$ *admits local uniformization* if there exists a projective model X of F/D such that V dominates a regular local ring in X. Thus if Spec(D) has a resolution of singularities, then every valuation ring in $\mathfrak{X}$ admits local uniformization. If D is essentially of finite type over a field k of characteristic 0, then D has a resolution of singularities by the theorem of Hironaka, but when k has positive characteristic, it is not known in general whether local uniformization holds in dimension greater than 3; see, for example, [Cutkosky 2004; Temkin 2013].

Corollary 4.5. *Suppose that D is a quasiexcellent integrally closed local Noetherian domain with quotient field F. If there exists a valuation ring in $\mathfrak{X}$ that dominates D but does not admit local uniformization, and Y consists of all such valuation rings, then the holomorphy ring of Y is a Prüfer domain with torsion Picard group.*

Proof. Let $Z = Y \cup \{F\}$, and let $\phi : Z \to \mathbb{P}^1_D$ be a D-morphism. Then by Proposition 2.3, ϕ factors through a projective model X of F/D. Since Y is nonempty, the projective model X has a singularity, and thus since D is quasiexcellent, the singular points of X are contained in a proper nonempty closed subset of X. In particular, there are closed points of X that are not in the image of the domination map $Z \to X$, and hence there are closed points of $\mathbb{P}^1_D$ that are not in the image of ϕ. Therefore, by Theorem 4.2, A is a Prüfer domain with torsion Picard group and quotient field F. □

In particular, all the valuation rings that dominate D and do not admit local uniformization lie in an affine scheme in $\mathfrak{X}$.

References

[Becker 1982] E. Becker, “The real holomorphy ring and sums of $2n$-th powers”, pp. 139–181 in *Géométrie algébrique réelle et formes quadratiques* (Rennes, 1981), edited by J.-L. Colliot-Thèléne et al., Lecture Notes in Math. **959**, Springer, Berlin, 1982. MR 84g:12032 Zbl 0508.12020

[Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1993. MR 95h:13020 Zbl 0788.13005

[Cutkosky 2004] S. D. Cutkosky, *Resolution of singularities*, Graduate Studies in Mathematics **63**, American Mathematical Society, Providence, RI, 2004. MR 2005d:14022 Zbl 1076.14005

[Dress 1965] A. Dress, “Lotschnittebenen mit halbierbarem rechtem Winkel”, *Arch. Math.* (*Basel*) **16** (1965), 388–392. MR 33 #1781 Zbl 0137.40204

[Eisenbud and Harris 2000] D. Eisenbud and J. Harris, *The geometry of schemes*, Graduate Texts in Mathematics **197**, Springer, New York, 2000. MR 2001d:14002 Zbl 0960.14002

[Fontana et al. 1997] M. Fontana, J. A. Huckaba, and I. J. Papick, *Prüfer domains*, Monographs and Textbooks in Pure and Applied Mathematics **203**, Marcel Dekker, New York, 1997. MR 98d:13021 Zbl 0861.13006

[Fuchs and Salce 2001] L. Fuchs and L. Salce, *Modules over non-Noetherian domains*, Mathematical Surveys and Monographs **84**, American Mathematical Society, Providence, RI, 2001. MR 2001i:13002 Zbl 0973.13001

[Gabber et al. 2013] O. Gabber, Q. Liu, and D. Lorenzini, “The index of an algebraic variety”, *Invent. Math.* **192**:3 (2013), 567–626. MR 3049930 Zbl 1268.13009

[Gilmer 1968] R. W. Gilmer, *Multiplicative ideal theory*, Queen’s Papers in Pure and Applied Mathematics **12**, Queen’s University, Kingston, ON, 1968. MR 37 #5198 Zbl 0155.36402

[Gilmer 1969] R. W. Gilmer, “Two constructions of Prüfer domains”, *J. Reine Angew. Math.* **239–240** (1969), 153–162. MR 41 #1710 Zbl 0184.29102

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977. MR 57 #3116 Zbl 0367.14001

[Holme 2012] A. Holme, *A royal road to algebraic geometry*, Springer, Berlin, 2012. MR 2858123 Zbl 1237.14001

[de Jong et al. 2005–] A. J. de Jong et al., “The Stacks project”, electronic reference, 2005–, http://stacks.math.columbia.edu.

[Knebusch and Zhang 2002] M. Knebusch and D. Zhang, *Manis valuations and Prüfer extensions, I: A new chapter in commutative algebra*, Lecture Notes in Math. **1791**, Springer, Berlin, 2002. MR 2003j:13002 Zbl 1033.13001

[Kuhlmann 2004] F.-V. Kuhlmann, “Places of algebraic function fields in arbitrary characteristic”, *Adv. Math.* **188**:2 (2004), 399–424. MR 2005h:12014 Zbl 1134.12304

[Larsen and McCarthy 1971] M. D. Larsen and P. J. McCarthy, *Multiplicative theory of ideals*, Pure and Applied Mathematics **43**, Academic Press, New York, 1971. MR 54 #2629 Zbl 0237.13002

[Loper 1994] K. A. Loper, “On Prüfer non-D-rings”, *J. Pure Appl. Algebra* **96**:3 (1994), 271–278. MR 95k:13024 Zbl 0840.13009

[Matsumura 1980] H. Matsumura, *Commutative ring theory*, Kyoritsu Shuppan, Tokyo, 1980. In Japanese; translated in Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1986. 2nd ed. in 1989. MR 90i:13001 Zbl 0603.13001

[Nagata 1962] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics **13**, Wiley, New York, 1962. MR 27 #5790 Zbl 0123.03402

[Olberding 2006] B. Olberding, "Holomorphy rings of function fields", pp. 331–347 in *Multiplicative ideal theory in commutative algebra*, edited by J. W. Brewer et al., Springer, New York, 2006. MR 2008f:13035 Zbl 1118.13022

[Olberding 2007] B. Olberding, "Irredundant intersections of valuation overrings of two-dimensional Noetherian domains", *J. Algebra* **318**:2 (2007), 834–855. MR 2009b:13057 Zbl 1137.13003

[Olberding 2008] B. Olberding, "Overrings of two-dimensional Noetherian domains representable by Noetherian spaces of valuation rings", *J. Pure Appl. Algebra* **212**:7 (2008), 1797–1821. MR 2009c:13008 Zbl 1140.13006

[Olberding 2011] B. Olberding, "Intersections of valuation overrings of two-dimensional Noetherian domains", pp. 335–361 in *Commutative algebra: Noetherian and non-Noetherian perspectives*, edited by M. Fontana et al., Springer, New York, 2011. MR 2012b:13014 Zbl 1225.13010

[Olberding and Roitman 2006] B. Olberding and M. Roitman, "The minimal number of generators of an invertible ideal", pp. 349–367 in *Multiplicative ideal theory in commutative algebra*, edited by J. W. Brewer et al., Springer, New York, 2006. MR 2007i:13021 Zbl 1118.13019

[Roquette 1973] P. Roquette, "Principal ideal theorems for holomorphy rings in fields", *J. Reine Angew. Math.* **262–263** (1973), 361–374. MR 48 #3950 Zbl 0271.13009

[Rush 2001] D. E. Rush, "Bezout domains with stable range 1", *J. Pure Appl. Algebra* **158**:2-3 (2001), 309–324. MR 2002k:13034 Zbl 1094.13531

[Schülting 1979] H.-W. Schülting, "Über die Erzeugendenanzahl invertierbarer Ideale in Prüferringen", *Comm. Algebra* **7**:13 (1979), 1331–1349. MR 81j:12016 Zbl 0432.13010

[Schülting 1982] H.-W. Schülting, "Real holomorphy rings in real algebraic geometry", pp. 433–442 in *Géométrie algébrique réelle et formes quadratiques* (Rennes, 1981), edited by J.-L. Colliot-Thèléne et al., Lecture Notes in Math. **959**, Springer, Berlin, 1982. MR 84d:14016 Zbl 0498.14013

[Swan 1984] R. G. Swan, "n-generator ideals in Prüfer domains", *Pacific J. Math.* **111**:2 (1984), 433–446. MR 85k:13019 Zbl 0546.13008

[Temkin 2013] M. Temkin, "Inseparable local uniformization", *J. Algebra* **373** (2013), 65–119. MR 2995017 Zbl 1276.14021

[Temkin and Tyomkin 2013] M. Temkin and I. Tyomkin, "Prüfer algebraic spaces", preprint, 2013. arXiv 1101.3199

[Zariski and Samuel 1975] O. Zariski and P. Samuel, *Commutative algebra*, vol. II, Graduate Texts in Mathematics **29**, Springer, New York, 1975. MR 52 #10706 Zbl 0313.13001

Received January 22, 2014. Revised July 26, 2014.

BRUCE OLBERDING
DEPARTMENT OF MATHEMATICAL SCIENCES
NEW MEXICO STATE UNIVERSITY
LAS CRUCES, NM 88003-8001
UNITED STATES
olberdin@nmsu.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.369

NON-KÄHLER EXPANDING RICCI SOLITONS, EINSTEIN METRICS, AND EXOTIC CONE STRUCTURES

MARIA BUZANO, ANDREW S. DANCER,
MICHAEL GALLAUGHER AND MCKENZIE WANG

We consider complete multiple warped product type Riemannian metrics on manifolds of the form $\mathbb{R}^2 \times M_2 \times \cdots \times M_r$, where $r \geq 2$ and M_i are arbitrary closed Einstein spaces with positive scalar curvature. We construct on these spaces a family of non-Kähler, non-Einstein, expanding gradient Ricci solitons with conical asymptotics as well as a family of Einstein metrics with negative scalar curvature. The 2-dimensional Euclidean space factor allows us to obtain homeomorphic but not diffeomorphic examples which have analogous cone structure behaviour at infinity. We also produce numerical evidence for complete expanding solitons on the vector bundles whose sphere bundles are the twistor or Sp(1) bundles over quaternionic projective space.

0. Introduction

In [Buzano et al. 2013] we constructed complete steady gradient Ricci soliton structures (including Ricci-flat metrics) on manifolds of the form $\mathbb{R}^2 \times M_2 \times \cdots \times M_r$, where M_i, $2 \leq i \leq r$, are arbitrary closed Einstein manifolds with positive scalar curvature. We also produced numerical solutions of the steady gradient Ricci soliton equation on certain nontrivial $\mathbb{R}^3$ and $\mathbb{R}^4$ bundles over quaternionic projective spaces. In the current paper we will present the analogous results for the case of *expanding* solitons on the same underlying manifolds.

Recall that a gradient Ricci soliton is a manifold M together with a smooth Riemannian metric g and a smooth function u, called the soliton potential, which give a solution to the equation

$$\mathrm{Ric}(g) + \mathrm{Hess}(u) + \frac{\epsilon}{2}\, g = 0 \tag{0.1}$$

for some constant ϵ. The soliton is then called expanding, steady, or shrinking according to whether ϵ is greater than, equal to, or less than zero.

M. Wang is partially supported by NSERC Grant No. OPG0009421.
MSC2010: primary 53C25; secondary 53C44.
Keywords: expanders, gradient Ricci solitons, Einstein metrics, exotic structures.

A gradient Ricci soliton is called *complete* if the metric g is complete. The completeness of the vector field ∇u follows from that of the metric; see [Zhang 2009]. If the metric of a gradient Ricci soliton is Einstein, then either $\mathrm{Hess}\, u = 0$ (i.e., ∇u is parallel) or we are in the case of the Gaussian soliton; see [Petersen and Wylie 2009; Pigola et al. 2011].

At present most examples of non-Kählerian expanding solitons arise from left-invariant metrics on nilpotent and solvable Lie groups (resp. *nilsolitons*, *solvsolitons*), as a result of work by J. Lauret [2001; 2011], M. Jablonski [2013], and many others (see the survey [Lauret 2009]). These expanders are however not of gradient type, i.e., they satisfy the more general equation

$$\mathrm{Ric}(g) + \frac{1}{2}\,\mathsf{L}_X g + \frac{\epsilon}{2}\, g = 0, \tag{0.2}$$

where X is a vector field on M and L denotes Lie differentiation.

A large class of complete, non-Einstein, non-Kählerian expanders of gradient type (with dimension ≥ 3) consists of an r-parameter family of solutions to (0.1) on $\mathbb{R}^{k+1} \times M_2 \times \cdots \times M_r$ where $k > 1$ and M_i are positive Einstein manifolds. The special case $r = 1$ (i.e., no M_i) is due to R. Bryant [2005] and the solitons have positive sectional curvature. The $r = 2$ case is due to Gastel and Kronz [2004], who adapted Böhm's construction of complete Einstein metrics with negative scalar curvature to the soliton case. The case of arbitrary r was treated in [Dancer and Wang 2009a] via a generalization of the dynamical system studied by Bryant. The soliton metrics in this family are all of multiple warped product type. In other words, the manifold is thought of as being foliated by hypersurfaces of the form $S^k \times M_2 \times \cdots \times M_r$ each equipped with a product metric depending smoothly on a real parameter t. As $k \geq 2$ in these works, the hypersurfaces and the asymptotic cones have finite fundamental group.

More recently, Schulze and Simon [2013] constructed expanding gradient Ricci solitons with nonnegative curvature operator in arbitrary dimensions by studying the scaling limits of the Ricci flow on complete open Riemannian manifolds with nonnegative bounded curvature operator and positive asymptotic volume ratio.

As pointed out in [Buzano et al. 2013], the situation of multiple warped products on nonnegative Einstein manifolds is rather special because of the automatic lower bound on the scalar curvature of the hypersurfaces. This leads, in the case where all factors have positive scalar curvature, $k > 1$, to definiteness of certain energy functionals occurring in the analysis of the dynamical system arising from (0.1), and hence to coercive estimates on the flow. In the present case, where one factor is a circle, i.e., $k = 1$, we can pass, as in [Buzano et al. 2013], to a subsystem where coercivity holds, and this is enough for the analysis to proceed. The new solitons obtained, like those of [Dancer and Wang 2009a], have conical asymptotics and are not of Kähler type (Theorem 2.14). We note that the lowest-dimensional solitons we obtain form a 2-parameter family on $\mathbb{R}^2 \times S^2$. The special case $r = 1$ was analysed earlier by the physicists Gutperle, Headrick, Minwalla and Schomerus [Gutperle et al. 2003].

As in [Buzano et al. 2013], we also obtain a family of solutions to our soliton equations that yield complete Einstein metrics of negative scalar curvature (Theorem 3.1). These are analogous to the metrics discovered by Böhm [1999]. Recall that for Böhm's construction the fact that the hyperbolic cone over the product Einstein metric on the hypersurface acts as an attractor plays an important role in the convergence proof for the Einstein trajectories. When $k = 1$, however, no product metric on the hypersurface can be Einstein with positive scalar curvature, so the hyperbolic cone construction cannot be exploited directly. It turns out that the analysis of the soliton case already contains most of the analysis required for the Einstein case. The new Einstein metrics we obtain have exponential volume growth.

The fact that $k = 1$ (rather than $k > 1$) allows for some new phenomena displayed by the asymptotic cones of some of our expander and Einstein examples. This is a consequence of the striking observation of Kwasik and Schultz [2002] that for an exotic sphere Σ and the standard sphere S of the same dimension, $\mathbb{R}^2 \times \Sigma$ is not diffeomorphic to $\mathbb{R}^2 \times \mathsf{S}$, but if we replace $\mathbb{R}^2$ by $\mathbb{R}^3$ in the products the resulting spaces do become diffeomorphic. In fact, the open cones $\mathbb{R}_+ \times S^1 \times \Sigma$ and $\mathbb{R}_+ \times S^1 \times \mathsf{S}$ are also homeomorphic but not diffeomorphic. As a result, we obtain examples of pairs of expanders and negative Einstein manifolds whose asymptotic cones are also homeomorphic but not diffeomorphic. These results are described in greater detail at the end of Section 3; see Corollary 3.2 and Proposition 3.3.

To make further progress in the search for expanders, we need to consider more complicated hypersurface types where the scalar curvature may not be bounded below. In [Buzano et al. 2013] we carried out numerical investigations of steady solitons where the hypersurfaces are the total spaces of Riemannian submersions for which the hypersurface metric involves two functions, one scaling the base and one the fibre of the submersion. We now look numerically at expanding solitons with such hypersurface types, in particular where the hypersurfaces are S^2 or S^3 bundles over quaternionic projective space. We produce numerical evidence of complete expanding gradient Ricci soliton structures in these cases.

Before undertaking our theoretical and numerical investigations, we first prove some general results about expanding solitons of cohomogeneity one type. Some of the results follow from properties of general expanding gradient Ricci solitons. However, the proofs are much simpler and sometimes the statements are sharper, which is helpful in numerical studies. The results include monotonicity and concavity properties for the soliton potential similar to those proved in [Buzano et al.

2013] in the steady case, as well as an upper bound for the mean curvature of the hypersurfaces. To derive this bound, we need to know that complete *non-Einstein* expanding gradient Ricci solitons have infinite volume. We include a proof of this fact here (Proposition 1.22) since we were not able to find an explicit statement in the literature. Finally we derive an asymptotic lower bound for the gradient of the soliton potential, which is in turn used to exhibit a general Lyapunov function for the cohomogeneity one expander equations.

1. Background on cohomogeneity one expanding solitons

We briefly review the formalism [Dancer and Wang 2011] for Ricci solitons of cohomogeneity one. We work on a manifold M with an open dense set foliated by equidistant diffeomorphic hypersurfaces P_t of real dimension n. The dimension of M, the manifold where we construct the soliton, is therefore $n+1$. The metric is then of the form $\bar{g} = dt^2 + g_t$, where g_t is a metric on P_t and t is the arclength coordinate along a geodesic orthogonal to the hypersurfaces. This set-up is more general than the cohomogeneity one ansatz, as it allows us to consider metrics with no symmetry provided that appropriate additional conditions on P_t are satisfied; see the following as well as [Dancer and Wang 2011, Remarks 2.18, 3.18]. We will also suppose that u is a function of t only.

We let r_t denote the Ricci endomorphism of g_t, defined by $\mathrm{Ric}(g_t)(X, Y) = g_t(r_t(X), Y)$ and viewed as an endomorphism via g_t. Also let L_t be the shape operator of the hypersurfaces, defined by the equation $\dot{g}_t = 2g_t L_t$ where g_t is regarded as an endomorphism with respect to a fixed background metric Q. The Levi-Civita connections of $\bar{g}$ and g_t will be denoted by $\overline{\nabla}$ and ∇ respectively. The relative volume $v(t)$ is defined by $d\mu_{g_t} = v(t)\, d\mu_Q$

We assume that the scalar curvature $S_t = \mathrm{tr}(r_t)$ and the mean curvature $\mathrm{tr}(L_t)$ (with respect to the normal $\nu = \partial/\partial t$) are constant on each hypersurface. These assumptions hold, for example, if M is of cohomogeneity one with respect to an isometric Lie group action. They are satisfied also when M is a multiple warped product over an interval.

The gradient Ricci soliton equation now becomes the system

$$-\operatorname{tr}\dot{L} - \mathrm{tr}(L^2) + \ddot{u} + \tfrac{1}{2}\epsilon = 0, \tag{1.1}$$

$$r - (\operatorname{tr} L)L - \dot{L} + \dot{u}L + \tfrac{1}{2}\epsilon\, \mathbb{I} = 0, \tag{1.2}$$

$$d(\operatorname{tr} L) + \delta^{\nabla} L = 0. \tag{1.3}$$

The first two equations represent the components of the equation in the directions normal and tangent to the hypersurfaces P, respectively. The third equation represents the equation in mixed directions — here $\delta^{\nabla} L$ denotes the codifferential for TP-valued 1-forms.

In the warped product case the final equation involving the codifferential automatically holds. This is also true for cohomogeneity one metrics that are *monotypic*, i.e., when there are no repeated real irreducible summands in the isotropy representation of the principal orbits; see [Bérard-Bergery 1982, Proposition 3.18].

There is a conservation law

$$\ddot{u} + (-\dot{u} + \operatorname{tr} L)\,\dot{u} - \epsilon u = C \tag{1.4}$$

for some constant C. Using our equations we may rewrite this as

$$S + \operatorname{tr}(L^2) - (\dot{u} - \operatorname{tr} L)^2 - \epsilon u + \tfrac{1}{2}(n-1)\epsilon = C. \tag{1.5}$$

where $S := \operatorname{tr}(r_t)$ is the scalar curvature S of the principal orbits. If $\bar{R}$ denotes the scalar curvature of the ambient metric $\bar{g}$, then

$$\bar{R} = -2\operatorname{tr}\dot{L} - \operatorname{tr}(L^2) - (\operatorname{tr} L)^2 + S.$$

We can deduce the equality

$$\bar{R} + \dot{u}^2 + \epsilon u = -C - \frac{\epsilon}{2}(n+1). \tag{1.6}$$

We let ξ denote the *dilaton mean curvature*

$$\xi := -\dot{u} + \operatorname{tr} L.$$

This is the mean curvature of the dilaton volume element $e^{-u} d\mu_{\bar{g}}$. It is often useful to define a new independent variable s by

$$\frac{d}{ds} := \frac{1}{\xi}\frac{d}{dt}, \tag{1.7}$$

and use a prime to denote d/ds. We note that (1.1) implies that $\dot{\xi} = -\operatorname{tr}(L^2) + \epsilon/2$.

It is also useful, following [Dancer et al. 2013], to introduce the quantity

$$\mathcal{E} := C + \epsilon u.$$

The conservation law may now be rewritten (for nonzero ϵ) as

$$\ddot{\mathcal{E}} + \xi\dot{\mathcal{E}} - \epsilon\mathcal{E} = 0. \tag{1.8}$$

Note that, for a function $t \mapsto f(t)$, the quantity $\ddot{f} + \xi\dot{f}$ is just the u-Laplacian in the sense of metric measure spaces.

Another useful quantity is the normalised mean curvature

$$\mathcal{H} = \frac{\operatorname{tr} L}{\xi} = 1 + \frac{\dot{u}}{\xi} = 1 + u',$$

which was introduced in [Dancer and Wang 2009a; Dancer et al. 2013].

We now specialise to the case of *expanding solitons*, that is,

$$\epsilon > 0.$$

We shall consider complete noncompact expanding solitons with one special orbit. We may take the interval I over which t ranges to be $[0, \infty)$ with the special orbit placed at $t = 0$. Let k denote the dimension of the collapsing sphere at $t = 0$. We will moreover assume in this section that $u(0) = 0$, since adding a constant to the soliton potential does not affect the equations.

A basic result of B.-L. Chen [2009] together with the strong maximum principle says that for a non-Einstein expanding gradient Ricci soliton $\bar{R} > -\frac{\epsilon}{2}(n+1)$. So we deduce from (1.6) that

$$\mathcal{E} < 0 \quad \text{and} \quad (\dot{u})^2 < -\mathcal{E} := -(C + \epsilon u).$$

Using the first inequality and the smoothness conditions at $t = 0$, we find as in the steady case that $\ddot{u}(0) = C/(k+1) < 0$, so completeness imposes restrictions on our initial conditions.

Integrating the second inequality and using the initial conditions yield

$$0 \le -u(t) < \frac{\epsilon}{4}t^2 + \sqrt{-C}\,t \tag{1.9}$$

and

$$|\dot{u}| < \frac{\epsilon}{2}t + \sqrt{-C}. \tag{1.10}$$

These are just the cohomogeneity one versions of general estimates of the potential due to Z.-H. Zhang [2009].

Proposition 1.11. *For a non-Einstein, complete, expanding gradient Ricci soliton of cohomogeneity one with a special orbit, the soliton potential u is strictly decreasing and strictly concave on* $(0, \infty)$.

Proof. The conservation law (1.8) and the fact that $\mathcal{E}$ is negative and ϵ is positive show that u is strictly concave on a neighbourhood of each critical point t_0. As we noted above, we also have concavity at the special orbit $t = 0$. Now, as in the steady case [Buzano et al. 2013], we see there are no critical points of u in $(0, \infty)$. As $\dot{u}(0) = 0$, we see u is strictly decreasing on $(0, \infty)$.

Now set $y = \dot{u}$ and differentiate (1.4); using (1.1) we obtain

$$\ddot{y} + \xi\,\dot{y} - \Big(\frac{\epsilon}{2} + \operatorname{tr}(L^2)\Big)y = 0.$$

In particular, $\ddot{y} + \xi\,\dot{y} < 0$, since y is negative. Integrating shows $ve^{-u}\dot{y}$ is strictly decreasing, where we recall that v is the relative volume. As t tends to 0, the smoothness conditions imply that $ve^{-u}\dot{y}$ tends to 0, so $\dot{y} = \ddot{u}$ is negative, as required. □

Our next result is inspired by the work of Munteanu and Sesum [2013] for the case of steady solitons.

Proposition 1.12. *For a non-Einstein, complete, expanding gradient Ricci soliton of cohomogeneity one with a special orbit, the volume growth is at least logarithmic.*

Proof. Let $M_t = \pi^{-1}([0, t])$, where π is the projection of M onto the orbit space I. We consider the integral

$$f(t) := \int_{M_t} \Big(\bar{R} + \frac{\epsilon}{2}(n+1)\Big)\, d\mu_{\bar{g}}$$

As we are considering non-Einstein solitons the integrand is positive.

Let $t_0 > 0$ and let $b := f(t_0)$. Using the trace of the soliton equation and also the divergence theorem we have, for $t \geq t_0$,

$$\begin{aligned} 0 < b \leq f(t) &= -\int_{M_t} \bar{\Delta} u \, d\mu_{\bar{g}} \\ &= \int_{\partial M_t} (\bar{\nabla} u) \cdot \Big(-\frac{\partial}{\partial t}\Big)\, d\mu_{\bar{g}}|_{\partial M_t} \\ &= |\dot{u}|\, v(t) \\ &< \Big(\frac{\epsilon}{2} t + \sqrt{-C}\Big) v(t) \end{aligned}$$

where we use (1.10) in the last line. Hence $v(t) > b/\big(\frac{\epsilon}{2}t + \sqrt{-C}\big)$, and integrating yields

$$\mathrm{vol}(M_t) > \mathrm{vol}(M_{t_0}) - \frac{2b}{\epsilon} \log\Big(\frac{\epsilon}{2} t_0 + \sqrt{-C}\Big) + \frac{2b}{\epsilon} \log\Big(\frac{\epsilon}{2} t + \sqrt{-C}\Big). \qquad \square$$

Proposition 1.13. *Let $(M, \bar{g}, u)$ be a non-Einstein, complete, expanding gradient Ricci soliton of cohomogeneity one with a special orbit. Then there exists $t_1 > 0$ such that on (t_1, ∞) we have* $\operatorname{tr} L < \sqrt{n\epsilon/2}$.

Proof. By Cauchy–Schwartz and the concavity result, we have

$$\frac{d}{dt}(\operatorname{tr} L) < \frac{\epsilon}{2} - \operatorname{tr}(L^2) \leq \frac{\epsilon}{2} - \frac{1}{n}(\operatorname{tr} L)^2. \tag{1.14}$$

Note that by the smoothness conditions $\operatorname{tr} L$ is strictly decreasing near $t = 0$, and its limit as t tends to zero from above is $+\infty$.

(i) First let us assume that $d(\operatorname{tr} L)/dt$ is nonnegative at some t_1. The inequality above shows that $|\operatorname{tr} L|^2 < \frac{\epsilon}{2} n$ at $t = t_1$.

Let us consider the solutions of the equation

$$\dot{h} = \frac{\epsilon}{2} - \frac{1}{n} h^2. \tag{1.15}$$

These are the family of increasing functions

$$h(t) = \sqrt{\frac{\epsilon n}{2}}\,\frac{a\exp(t\sqrt{2\epsilon/n}) - 1}{a\exp(t\sqrt{2\epsilon/n}) + 1},$$

where a is a positive constant, as well as the constant functions $\pm\sqrt{\epsilon n/2}$ which form the bounding envelope for this family. Hence $\operatorname{tr} L \leq h^*(t) < \sqrt{\epsilon n/2}$, where $h^*(t)$ is the solution to (1.15) which agrees with $\operatorname{tr} L$ at t_1.

(ii) Next suppose that $d(\operatorname{tr} L)/dt$ is always negative. Now if $\operatorname{tr} L$ is ever zero then it is negative and bounded away from zero on some semi-infinite interval. Recalling that $\operatorname{tr} L = \dot{v}/v$ and integrating, we see that the soliton volume is finite, which contradicts Proposition 1.12. So $\operatorname{tr} L$ is positive on $(0, \infty)$ and, using Proposition 1.11, we see ξ is also positive on this interval. By [Pigola et al. 2011, Theorem 11], ξ tends to infinity as t tends to ∞. But ξ also tends to infinity as t tends to zero, so we have a minimum t_1 where $\dot{\xi}$ vanishes. Now (1.1) shows $\operatorname{tr}(L^2) = \epsilon/2$ at t_1 and Cauchy–Schwartz shows $(\operatorname{tr} L)^2 \leq n\epsilon/2$ at t_1. As $\operatorname{tr} L$ is decreasing, we have the desired result. □

Remark 1.16. This bound on $\operatorname{tr} L$ is best possible, at least if we allow the solitons to be Einstein. Indeed, the negative scalar curvature Einstein metrics of Böhm [1999] give exactly this bound, as $\operatorname{tr} L$ is asymptotic to $n\epsilon/2$.

Next we consider properties of the Lyapunov function $\mathscr{F}_0$, which was introduced by Böhm [1999] for the Einstein case and was subsequently studied in [Dancer et al. 2013; Buzano et al. 2013] for the soliton case. Note that this function was denoted by $\mathscr{F}$ in [Dancer et al. 2013].

Proposition 1.17. *Let $\mathscr{F}_0$ denote the function $v^{2/n}\big(S + \operatorname{tr}((L^{(0)})^2)\big)$ defined on the velocity phase space of the cohomogeneity one expanding gradient Ricci soliton equations, with $L^{(0)}$ representing the trace-free part of L. Then along the trajectory of a complete smooth non-Einstein expanding soliton, $\mathscr{F}_0$ is nonincreasing for sufficiently large t.*

Proof. The formula for $d\mathscr{F}_0/dt$ in [Dancer et al. 2013, Proposition 2.17] shows that the proposition would follow if, for sufficiently large t, one can show that

$$\xi - \frac{1}{n}\operatorname{tr} L = -\dot{u} + \frac{n-1}{n}\operatorname{tr} L \geq 0.$$

We first note that $\operatorname{tr} L$ is eventually bounded below by $-\sqrt{\epsilon n/2}$. Otherwise, at some $t = t_1 > 0$, $\operatorname{tr} L \leq -\sqrt{\epsilon n/2}$ and (1.14) shows that this inequality continues to hold from t_1 onwards. But this would imply that the soliton has finite volume, contradicting Proposition 1.12.

We are now done since part (i) of the next proposition shows that $|\dot{u}| = -\dot{u}$ grows at least linearly for sufficiently large t. In particular, for large enough t, $\mathcal{F}_0$ fails to be strictly decreasing iff the shape operator of the hypersurfaces become diagonal. □

Proposition 1.18. *Let $(M, \bar{g}, u)$ be a complete, non-Einstein, expanding gradient Ricci soliton of cohomogeneity one with a special orbit. Suppose $t_1 > 2\sqrt{5/\epsilon}$ and on $[t_1, +\infty)$ we have an upper bound $\lambda_0 > 0$ for $\operatorname{tr} L$. Set $a := \lambda_0 + \sqrt{-C}$. Then on $[t_1, +\infty)$ we have*

(i) $|\overline{\nabla} u| = -\dot{u}(t) > \dfrac{9}{10}\left(\dfrac{-\dot{u}(t_1)}{\frac{\epsilon}{2}t_1 + a}\right)\left(\dfrac{\epsilon}{2}t + a\right)$,

(ii) $\ddot{u} + \dfrac{\epsilon}{2} = -\operatorname{Ric}_{\bar{g}}\left(\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial t}\right) \leq \dfrac{\epsilon}{2}\left(1 + \dfrac{9}{10}\dfrac{\dot{u}(t_1)}{\frac{\epsilon}{2}t_1 + a}\right)$.

Proof. By assumption and the upper bound (1.10) we have $\xi < \frac{\epsilon}{2}t + a$. Since $\dot{y} = \ddot{u} < 0$ and $y = \dot{u} < 0$ by Proposition 1.11, we see that y satisfies the differential inequality

$$\ddot{y} + \left(\frac{\epsilon}{2}t + a\right)\dot{y} - \frac{\epsilon}{2}y < 0.$$

We will now compare y with solutions of the corresponding equation

$$\ddot{x} + \left(\frac{\epsilon}{2}t + a\right)\dot{x} - \frac{\epsilon}{2}x = 0, \tag{1.19}$$

which can be solved explicitly. This is because, if we differentiate this equation, we obtain

$$\frac{d^3x}{dt^3} + \left(\frac{\epsilon}{2}t + a\right)\ddot{x} = 0,$$

from which we can solve for $\ddot{x}$. Accordingly, upon integration and using (1.19), we obtain

$$x(t) = -\left(\frac{\epsilon}{2}t + a\right)\left(\frac{c_0}{\frac{\epsilon}{2}t_1 + a} - c_1 e^{(\epsilon/4)t_1^2 + at_1}\int_{t_1}^{t}\frac{e^{-(\epsilon/4)\tau^2 - a\tau}}{\left(\frac{\epsilon}{2}\tau + a\right)^2}\,d\tau\right), \tag{1.20}$$

where c_0 and c_1 are arbitrary constants.

In order to apply [Protter and Weinberger 1984, Theorem 13, p. 26], we must choose $x(t_1) \geq y(t_1) = \dot{u}(t_1)$ and $\dot{x}(t_1) \geq \dot{y}(t_1) = \ddot{u}(t_1)$. Since $x(t_1) = -c_0$, we can maximize c_0 by choosing $x(t_1) = \dot{u}(t_1)$. It follows that

$$c_1 = -\ddot{x}(t_1) = -\frac{\epsilon}{2}\,x(t_1) + \left(\frac{\epsilon}{2}t_1 + a\right)\dot{x}(t_1) \geq -\frac{\epsilon}{2}\,\dot{u}(t_1) + \left(\frac{\epsilon}{2}t_1 + a\right)\ddot{u}(t_1).$$

In particular, an admissible choice for c_1 is $c_1 = \frac{\epsilon}{2}c_0 > 0$. With this choice, it remains to find an upper bound for the integral in (1.20).

To do this, we integrate by parts three times and then throw away the resulting term involving integration (this term is negative). Specifically, we have

$$\int_{\lambda_1}^{\lambda} \frac{e^{-\sigma^2/\epsilon}}{\sigma^2}\,d\sigma$$
$$\leq \frac{\epsilon}{2}\left(\frac{e^{-\lambda_1^2/\epsilon}}{\lambda_1^3}\right)\left(1-\frac{3}{2}\frac{\epsilon}{\lambda_1^2}+\frac{15}{\lambda_1^4}\left(\frac{\epsilon}{2}\right)^2-\left(\frac{\lambda_1}{\lambda}\right)^3 e^{-(\lambda^2-\lambda_1^2)/\epsilon}\left(1-\frac{3}{2}\frac{\epsilon}{\lambda^2}+\frac{15}{\lambda^4}\left(\frac{\epsilon}{2}\right)^2\right)\right).$$

Using the change of independent variable $\lambda := \frac{\epsilon}{2}t + a$ and the fact that

$$1-\frac{3\epsilon}{2}x+\frac{15}{4}\epsilon^2x^2 = \left(1-\frac{3\epsilon}{4}x\right)^2+\frac{51}{16}\epsilon^2x^2 \geq \frac{17}{20},$$

we obtain

$$e^{(\epsilon/4)t_1^2+at_1}\int_{t_1}^{t}\frac{e^{-(\epsilon/4)\tau^2-a\tau}}{(\frac{\epsilon}{2}\tau+a)^2}\,d\tau$$
$$\leq \frac{1}{(\frac{\epsilon}{2}t_1+a)^3}\left(1-\frac{\epsilon}{2}\frac{3}{(\frac{\epsilon}{2}t_1+a)^2}+\left(\frac{\epsilon}{2}\right)^2\frac{15}{(\frac{\epsilon}{2}t_1+a)^4}-\frac{17}{20}\left(\frac{\frac{\epsilon}{2}t_1+a}{\frac{\epsilon}{2}t+a}\right)^3\frac{e^{(\epsilon/4)t_1^2+at_1}}{e^{(\epsilon/4)t^2+at}}\right).$$

If we substitute the information above together with the choice $c_1 = \frac{\epsilon}{2}c_0$ in the comparison inequality $\dot{u}(t) \leq x(t)$ (for $t \geq t_1$), we obtain

$$-\dot{u}(t)$$
$$\geq -\frac{\dot{u}(t_1)}{\frac{\epsilon}{2}t_1+a}\left(\frac{\epsilon}{2}t+a\right)\left(1-\frac{\epsilon}{2}\frac{1}{(\frac{\epsilon}{2}t_1+a)^2}\left(1-\frac{\epsilon}{2}\frac{3}{(\frac{\epsilon}{2}t_1+a)^2}+\left(\frac{\epsilon}{2}\right)^2\frac{15}{(\frac{\epsilon}{2}t_1+a)^4}\right)\right)$$
$$\geq -\frac{\dot{u}(t_1)}{\frac{\epsilon}{2}t_1+a}\left(\frac{\epsilon}{2}t+a\right)\left(1-\frac{\epsilon}{2}\frac{1}{(\frac{\epsilon}{2}t_1+a)^2}\right)$$
$$> \frac{9}{10}\left(-\frac{\dot{u}(t_1)}{\frac{\epsilon}{2}t_1+a}\right)\left(\frac{\epsilon}{2}t+a\right)$$

where for the last inequality we used the hypothesis that $t_1 > 2\sqrt{5/\epsilon}$, so that $\frac{\epsilon}{2}t_1 + a > \sqrt{5\epsilon}$. This completes the proof of (i).

The proof of (ii) follows by applying the same estimates to the comparison inequality $\ddot{u}(t) = \dot{y}(t) \leq \dot{x}(t)$ for $t \geq t_1$. Note that by [Dancer and Wang 2000, (2.2)] and (1.2), the quantity $\ddot{u} + \frac{\epsilon}{2}$ is precisely the negative of the Ricci curvature of the soliton metric in the direction $\partial/\partial t$. □

Remark 1.21. In the above proof we can of course take λ_0 to be $\sqrt{\epsilon n/2}$ by Proposition 1.13. Notice, however, that in part (ii) of the proof of Proposition 1.13 one automatically has an upper bound on $\operatorname{tr} L$. So one can apply Proposition 1.18 instead of [Pigola et al. 2011, Theorem 11] to obtain a self-contained proof for Proposition 1.13.

Note also that neither Proposition 1.13 nor 1.18 requires any curvature bounds.

We end this section with a simple generalization of Proposition 1.12 which, as far as we know, has not been explicitly observed in the literature. An analogous result for steady gradient Ricci solitons is [Munteanu and Sesum 2013, Theorem 5.1].

Proposition 1.22. *A complete non-Einstein expanding gradient Ricci soliton has at least logarithmic volume growth.*

We shall give a sketch of the proof only since the basic outline is the same as that for the cohomogeneity one case. One replaces M_t in the proof of Proposition 1.12 by the metric ball $B_p(t)$ of radius t from an arbitrary but fixed point $p \in M$. The integrand in the boundary integral that is left after applying Stokes' theorem can be bounded by $\tilde{c}\,(t+2)\ \mathrm{vol}_{n-1}(\partial B_p(t))$ where $\tilde{c}$ is a positive constant which depends only on n and ϵ; see [Zhang 2009]. One then obtains the inequality

$$\frac{b}{\tilde{c}(t+2)} \le \mathrm{vol}_{n-1}(\partial M_p(t)).$$

Integrating this inequality and applying the coarea formula, one deduces that the volume of $B_p(t)$ grows at least logarithmically in t.

The main technical point in the above is to justify the use of Stokes' theorem as the distance function from p is only Lipschitz continuous. For this one can use the well-known fact that Stokes' theorem holds for Lipschitz domains (see [McLean 2000, Theorem 3.34]), or one can use the approximation arguments of Gaffney [1954] as in [Yau 1976, p. 660] to get a compact exhaustion of the underlying manifold with sufficiently good properties for applying the usual version of Stokes' theorem (see the version of this paper at arXiv:1311.5097).

Remark 1.23. Of course there are noncompact negative Einstein manifolds with finite volume. It is quite probable though that for nontrivial expanders the above volume lower bound is not sharp. Most lower bounds for the volume in the literature involve additional assumptions on the curvature. For example, in [Carrillo and Ni 2009, Proposition 5.1(b)] or [Chen 2012, Theorem 1] a lower bound on the (average) scalar curvature is assumed.

2. Multiple warped product expanders

In this section, we specialise to multiple warped products, that is, metrics of the form

$$\bar{g} = dt^2 + \sum_{i=1}^{r} g_i^2(t)\, h_i \tag{2.1}$$

on $I \times M_1 \times \cdots \times M_r$, where I is an interval in $\mathbb{R}$, $r \ge 2$ and (M_i, h_i) are Einstein manifolds with real dimensions d_i and Einstein constants λ_i. We observe that $n = \sum_i d_i$ is greater than or equal to 3 as long as some M_i is nonflat.

The Ricci endomorphism is now diagonal with components given by blocks $(\lambda_i/g_i^2)\mathbb{I}_{d_i}$, where $i = 1, \ldots, r$ and $\mathbb{I}_m$ denotes the identity matrix of size m. We work with the variables

$$X_i = \frac{\sqrt{d_i}}{\xi} \frac{\dot{g}_i}{g_i}, \tag{2.2}$$

$$Y_i = \frac{\sqrt{d_i}}{\xi} \frac{1}{g_i}, \tag{2.3}$$

$$W = \frac{1}{\xi} := \frac{1}{-\dot{u} + \operatorname{tr} L}, \tag{2.4}$$

for $i = 1, \ldots, r$. The definition of Y_i in [Dancer and Wang 2009a; 2009b] differs from that above by a scale factor of $\sqrt{\lambda_i}$. This choice reflects the fact that we are now allowing one of the λ_i to be zero. As in [Buzano et al. 2013] we have

$$\sum_{j=1}^{r} X_j^2 = \frac{\operatorname{tr}(L^2)}{\xi^2} \quad \text{and} \quad \sum_{j=1}^{r} \lambda_j Y_j^2 = \frac{\operatorname{tr}(r_t)}{\xi^2}.$$

As mentioned earlier, we shall introduce the new independent variable s defined by (1.7) and use a prime to denote differentiation with respect to s.

In these new variables the Ricci soliton system (1.1)–(1.2) becomes

$$X_i' = X_i\left(\sum_{j=1}^{r} X_j^2 - 1\right) + \frac{\lambda_i}{\sqrt{d_i}} Y_i^2 + \frac{\epsilon}{2}(\sqrt{d_i} - X_i)W^2, \tag{2.5}$$

$$Y_i' = Y_i\left(\sum_{j=1}^{r} X_j^2 - \frac{X_i}{\sqrt{d_i}} - \frac{\epsilon}{2} W^2\right), \tag{2.6}$$

$$W' = W\left(\sum_{j=1}^{r} X_j^2 - \frac{\epsilon}{2} W^2\right), \tag{2.7}$$

for $i = 1, \ldots, r$. Note that, in the warped product situation, (1.3) is automatically satisfied.

As in [Buzano et al. 2013] we use $\mathcal{G}$ to denote $\sum_{i=1}^{r} X_i^2$. The quantity $\mathcal{H} = W \operatorname{tr} L$ becomes $\sum_{i=1}^{r} \sqrt{d_i} X_i$ in our new variables. We further have the equation

$$(\mathcal{H} - 1)' = (\mathcal{H} - 1)\left(\mathcal{G} - 1 - \frac{\epsilon}{2} W^2\right) + \mathcal{Q},$$

where

$$\mathcal{Q} = \sum_{i=1}^{r} (X_i^2 + \lambda_i Y_i^2) + \frac{\epsilon(n-1)}{2} W^2 - 1. \tag{2.8}$$

As explained in [Dancer and Wang 2009a], $\mathcal{Q}$ serves as an energy functional in the expanding case, modifying the Lyapunov functional

$$\mathcal{L} := \sum_{i=1}^{r} (X_i^2 + \lambda_i Y_i^2) - 1, \tag{2.9}$$

which plays a key role in the steady case; see [Dancer and Wang 2009b; Buzano et al. 2013]. The general conservation law (1.5) then becomes $\mathcal{Q} = (C + \epsilon u)\, W^2$.

Note that, in our situation, the quantity $\mathcal{Q}$ is no longer a Lyapunov function. However, we do have the equations

$$\begin{aligned}(\mathcal{H}-1)' &= f_1(\mathcal{H}-1) + f_2\mathcal{Q}\\ \mathcal{Q}' &= f_3(\mathcal{H}-1) + f_4\mathcal{Q},\end{aligned}$$

where $f_1 = G - 1 - \frac{\epsilon}{2}W^2$, $f_2 = 1$, $f_3 = \epsilon W^2$, and $f_4 = 2\big(G - \frac{\epsilon}{2}W^2\big)$. The crucial point for us is that in the expanding case both f_2 and f_3 are positive, so the phase plane diagram in the $(\mathcal{H}-1, \mathcal{Q})$-plane shows that the regions $\{\mathcal{H} < 1, \mathcal{Q} < 0\}$ and $\{\mathcal{H} > 1, \mathcal{Q} > 0\}$ are both flow-invariant. Furthermore, the region $\{\mathcal{Q} = 0, \mathcal{H} = 1\}$ of phase space corresponds to Einstein metrics of negative Einstein constant and is of course also flow-invariant.

The above observations are in fact valid for the *general* monotypic cohomogeneity one expanding soliton equations, not just for the warped product case, provided we make the general definition

$$\mathcal{Q} := W^2\mathcal{E} = W^2(C + \epsilon u) \quad \text{and} \quad \mathcal{H} := W \operatorname{tr} L.$$

(The conservation law shows that this is consistent with the earlier formula for $\mathcal{Q}$ that we gave in the warped product case; see [Dancer et al. 2013, (4.6)].) We refer to [Dancer et al. 2013] for a discussion of this topic as well as the qualitatively different situation of shrinking solitons, where ϵ is negative. However, apart from the multiple warped product case, these formulae for $\mathcal{Q}$ involve polynomial or rational expressions in the X_i and Y_i variables which need not be definite, so the estimates obtained are not coercive.

In the warped product case with all λ_i positive, which was the situation examined in [Dancer and Wang 2009a], $\mathcal{Q}$ is, as explained above, a positive definite form (up to an additive constant) in the X_i, Y_i, so we obtained coercive estimates which allowed us to analyse the flow. For the rest of this section, we shall look at the case where the collapsing factor M_1 is S^1, so $d_1 = 1$, $\lambda_1 = 0$, and the remaining Einstein constants λ_i are positive. Then the equation for X_1 becomes

$$X_1' = X_1\left(\sum_{j=1}^{r} X_j^2 - 1\right) + \frac{\epsilon}{2}(1 - X_1)W^2.$$

As $\mathcal{Q}$ now does not include a Y_1 term, the region $\mathcal{Q} < 0$ is no longer precompact. However, we will see by using similar ideas to those in [Buzano et al. 2013] that we can still analyse the flow.

It is clear that we can recover t and g_i from a solution X, Y, W of the above system via the relation $dt = W\,ds$ and the formulae (2.2), (2.3), (2.4). As usual we choose $t = 0$ to correspond to $s = -\infty$. The soliton potential u is recovered by integrating

$$\dot{u} = \operatorname{tr} L - \frac{1}{W} = \frac{\mathcal{H} - 1}{W} = \frac{\sum_{i=1}^{r} \sqrt{d_i} X_i - 1}{W}. \tag{2.10}$$

We next compute the critical points of the soliton system (2.5)–(2.7).

Lemma 2.11. *Let $d_1 = 1$ and $d_i > 1$ for $i > 1$, so that $\lambda_i = 0$ iff $i = 1$. The stationary points of* (2.5), (2.6), (2.7) *in X, Y, W-space consist of*

(i) *the origin*

(ii) *points with $W = 0$, $Y_i = 0$ for all i, and $\sum_{i=1}^{r} X_i^2 = 1$*

(iii) *points given by*

$$W = 0, \quad X_i = \sqrt{d_i}\,\rho_A, \quad Y_i^2 = \frac{d_i}{\lambda_i}\rho_A(1 - \rho_A), \quad i \in A$$

and $X_i = Y_i = 0$ for $i \notin A$, where A is any nonempty subset of $\{2, \dots, r\}$, and $\rho_A = \left(\sum_{j \in A} d_j\right)^{-1}$

(iv) *the line where $W = 0$, $X_i = 0$ for all i, and $Y_i = 0$ for $i > 1$*

(v) *the line where $W = 0$, $X_1 = 1$, and X_i, $Y_i = 0$ for $i > 1$.*

(vi) *the points $E_\pm$ with coordinates*

$$X_i = \frac{\sqrt{d_i}}{n}, \quad Y_i = 0, \quad W = \pm\sqrt{\frac{2}{n\epsilon}}. \qquad \square$$

Note that $\mathcal{L}$ equals -1 in case (i) and (iv), equals 0 in case (ii), (iii) and (v), and equals $(1 - n)/n$ in case(vi). Also $\mathcal{Q}$ is -1 in cases (i) and (iv) and 0 otherwise. Cases (i)–(v) arose in [Buzano et al. 2013] in the steady case. Case (vi) is special to the expanding case and arose in [Dancer and Wang 2009a]. Again the origin is no longer an isolated critical point.

The analysis of the equations is quite similar to that in [Dancer and Wang 2009a], with appropriate changes as in [Buzano et al. 2013] to reflect the fact that one factor M_1 of the product hypersurface is flat. Accordingly, we shall be brief in our discussion.

We look for solutions where the flat factor $M_1 = S^1$ collapses at the end corresponding to $t = 0$ (that is, $s = -\infty$). In our new variables, this translates into considering trajectories in the unstable manifold of the critical point P of (2.5)–(2.7) (of type (v)) given by

$$W = 0, \quad X_1 = 1, \quad Y_1 = 1, \quad X_i = Y_i = 0 \quad (i > 1).$$

Note that at this critical point we have $\mathcal{L} = \mathcal{Q} = 0$ and $\mathcal{G} = \mathcal{H} = 1$.

The linearisation about this critical point is the system

$$\begin{aligned} x_1' &= 2x_1, \\ y_1' &= x_1, \\ x_i' &= 0 \quad (i \geq 2), \\ y_i' &= y_i \quad (i \geq 2), \\ w' &= w \end{aligned}$$

with eigenvalues 2, 1 (r times), and 0 (r times).

The results of [Buzano 2011] now show we have an r-parameter family of trajectories $\gamma(s)$ emanating from P and pointing into the region $\{\mathcal{Q} < 0, \mathcal{H} < 1\}$. Moreover, by the arguments above, such trajectories stay in this region. We can choose the trajectories to have W, Y_i positive for all time, as the loci $\{Y_i = 0\}$ or $\{W = 0\}$ are flow-invariant and the equations are invariant under changing the sign of W and/or of any Y_i.

As mentioned above, as M_1 is flat and Y_1 does not appear in $\mathcal{Q}$, the region $\{\mathcal{Q} < 0\}$ is no longer precompact. However, since the variable Y_1 only enters into the equations through the equation for Y_1', we may follow [Buzano et al. 2013] and consider the subsystem obtained by omitting the $i = 1$ equation in (2.6). The result is a system of equations in W, X_i $(i = 1, \dots, r)$ and Y_i $(i = 2, \dots, r)$, and on this $2r$-dimensional phase space the locus $\{\mathcal{Q} < 0\}$ *is* precompact. Once we have a long-time solution to the subsystem, Y_1 may be recovered via

$$Y_1(s) = Y_1(s_0) \exp\left(\int_{s_0}^{s} \sum_{j=1}^{r} X_j^2 - X_1 - \frac{\epsilon}{2} W^2 \right),$$

where s_0 is a fixed but arbitrary constant.

The critical points of the subsystem are obtained by removing the Y_1-coordinate from those of the full system. In particular, the origin becomes an isolated critical point, and case (v) of Lemma 2.11 gives rise to the special critical point $\hat{P}$ with $W = 0$, $X_1 = 1$, $X_i = 0$ $(i > 1)$, $Y_i = 0$ $(i = 2, \dots, r)$, from which emanates an r-parameter family of local solutions lying in the region

$$\{W > 0, Y_i > 0 \ (i > 1), \mathcal{Q} < 0, \mathcal{H} < 1\}.$$

The r parameters may be thought of as $g_i(0)$, $i > 1$, and the constant C in the conservation law (which has to be negative under the assumption that $u(0) = 0$). Homothetic solutions are eliminated by fixing the value of ϵ.

Precompactness of the region where the subsystem flow lives shows that the variables are bounded, so that the flow exists for all s. Hence the same is true for the original flow also. As in [Dancer and Wang 2009a, Lemma 2.2] we can show that the X_i are positive for all s. It follows that $\mathcal{H} > 0$ and $X_i < 1/\sqrt{d_i}$. Furthermore, we still have the equation

$$\left(\frac{W}{Y_i}\right)' = \frac{X_i}{\sqrt{d_i}}\left(\frac{W}{Y_i}\right),$$

including the possibility $i = 1$. So W/Y_i increases monotonically to a limit $\sigma_i \in (0, \infty]$. (We shall presently show that the σ_i must all be equal to $+\infty$.)

As the trajectories of interest lie in a precompact set, each of them has a nonempty ω-limit set Ω, where we suppressed the dependence on the trajectory. Moreover, each Ω is compact, connected, and invariant under both forward and backward flows.

As in [Dancer and Wang 2009a, p. 1115] we can show that Ω lies in the locus $\{Y_i = 0, 2 \le i \le r\}$. Now, on this locus the flow is just the same as that in [Dancer and Wang 2009a], and the arguments there (see pp. 1116–1120) show as before that Ω contains the origin (in the phase space for the subsystem). The centre manifold argument in [Dancer and Wang 2009a, pp. 1121–1122] then shows the origin is a nonlinear sink, so in fact the trajectory converges to the origin.

Now we can follow the arguments of for Lemma 3.13 in [Dancer and Wang 2009a] to show that

$$\lim_{s\to\infty} \frac{X_i}{W^2} = \Lambda_i := \frac{\lambda_i}{\sigma_i^2\sqrt{d_i}} + \frac{\epsilon}{2}\sqrt{d_i}, \tag{2.12}$$

where $\Lambda_i > 0$. This is valid in particular for $i = 1$, in which case $\Lambda_1 = \frac{\epsilon}{2}$. In fact, the proof of [Dancer and Wang 2009a, Lemma 3.15] shows that σ_i cannot be finite, and so $\Lambda_i/\sqrt{d_i} = \frac{\epsilon}{2}$ for all i. Applying this fact to the relation

$$\frac{\dot{g}_i}{g_i} = \frac{1}{\sqrt{d_i}}\frac{X_i}{W} = \frac{1}{\sqrt{d_i}}\frac{X_i}{W^2}W,$$

it follows that the hypersurfaces have asymptotically decaying principal curvatures.

The limits (2.12) also imply that, for sufficiently large s, there exist $a_1, a_2 > 0$ such that $a_1 W^4 \le \mathcal{G} \le a_2 W^4$, from which we deduce completeness of the soliton metric by using the relation $dt = W\,ds$ and the equation (from (2.7)) $W\,ds = dW/(\mathcal{G} - \frac{\epsilon}{2}W^2)$. We further have $W \sim 1/\sqrt{\epsilon s}$ and $s \sim \frac{1}{4}\epsilon t^2$.

The asymptotics for g_i, $i > 1$, are deduced as in [Dancer and Wang 2009a]. As for g_1, the equation
$$\left(\frac{W}{Y_1}\right)' = \frac{X_i}{\sqrt{d_i}}\left(\frac{W}{Y_1}\right)$$
and $X_1 \sim \frac{\epsilon}{2}W^2 \sim 1/(2s)$ show that $g_1 = W/Y_1$ is also asymptotically linear in t, so we have conical asymptotics for all factors.

Remark 2.13. This contrasts with the steady case, where the asymptotic geometry for $n = 1$, $r = 1$ (the cigar) is different from the paraboloid asymptotics for the Bryant solitons with $n > 1$, $r = 1$. In the steady case with $r > 1$ our work in [Buzano et al. 2013] yielded solitons of *mixed* asymptotic type, where g_1 tends to a positive constant and g_i behaves like $\sqrt{t}$ for $i > 1$.

In the expanding case, both the $n = 1$, $r = 1$ case (due to [Gutperle et al. 2003]) and the $n > 1$, $r = 1$ case (due to R. Bryant) have conical asymptotics, and our solutions here for the $r > 1$ case also exhibit conical behaviour.

We summarise the discussion in this section by the following:

Theorem 2.14. *Let $M_2, \dots, M_r$ be closed Einstein manifolds with positive scalar curvature. There is an r-parameter family of nonhomothetic complete smooth expanding gradient Ricci soliton structures on the trivial rank 2 vector bundle over $M_2 \times \cdots \times M_r$, with conical asymptotics in the sense given above.* □

Remark 2.15. As in [Dancer and Wang 2009a] we can see directly from the equations that the soliton potential u is concave, in accordance with Proposition 1.11. We can similarly deduce directly that $\mathrm{Ric}(\bar{g}) + \frac{\epsilon}{2}\,\bar{g}$ is positive semidefinite, so $-u$ is subharmonic.

Next we note that when $r \geq 2$ the sectional curvatures $\kappa(X \wedge Y)$, for X, Y tangent to different Einstein factors, satisfy $-c_1/t^2 \leq \kappa(X \wedge Y) \leq -c_2/t^2 < 0$ for certain positive constants c_1, c_2. This shows that the hypothesis of $\lim_{t\to\infty} t^2|\mathrm{sect}| = 0$ in many results in [Chen 2012] is not satisfied by our examples. In particular, the simplest hypersurface type in our examples is $S^1 \times S^{n-1}$; see [Chen 2012, Theorem 4].

Furthermore, all sectional curvatures decay faster than $t^{-2+\delta}$ for an arbitarily small $\delta > 0$. Hence the ambient scalar curvature $\bar{R}$ tends to zero. Finally we note that none of the hypotheses (topological or metric) in the recent rigidity theorem of Chodosh [2014] are satisfied by our examples.

3. Complete Einstein metrics with negative scalar curvature

We may also consider the flow of equations (2.5)–(2.7) in the variety $\{\mathcal{Q}=0, \mathcal{H}=1\}$. Such solutions of course correspond to Einstein metrics with negative scalar curvature, the soliton potential now being constant. In the case when $d_i > 1$ for all i, such metrics were constructed earlier in [Böhm 1999] by dynamical systems methods as well. In [Dancer and Wang 2009a, Remark 4.13] we pointed out that a simpler proof of Böhm's result can be obtained using our special variables and the embedding of the Einstein system within the soliton system.

In the present situation, where $d_1 = 1$, the hypersurfaces in the multiple warped product no longer admit a positive Einstein product metric whose hyperbolic cone acts as an attractor for the Einstein system. Nevertheless our setup allows us easily to deduce the following:

Theorem 3.1. *Let $M_2, \ldots, M_r$ be compact Einstein manifolds with positive scalar curvature. There is an $r-1$-parameter family of nonhomothetic complete smooth Einstein metrics on the trivial rank 2 vector bundle over $M_2 \times \cdots \times M_r$.*

To prove the theorem, we consider the $r-1$-parameter family of trajectories emanating from the critical point P and lying in the variety $\{\mathcal{Q} = 0, \mathcal{H} = 1\}$. Note that this variety is smooth.

As in the previous section, we see that the flow is defined for all s by first restricting to the subsystem obtained by omitting the equation for Y_1 and observing that the locus $\{\mathcal{Q} = 0\}$ is compact. As usual we can take Y_i, W positive on our trajectories, and we can show the X_i are positive also. In the following we will work with the subsystem.

The ω-limit set Ω of a fixed trajectory lies within the locus $\{Y_i = 0 : i = 2, \ldots, r\}$ by the same argument as in the soliton case. However, the difference now is that no point in Ω can have W-coordinate equal to 0. Otherwise, $\mathcal{G} = 1$ and such a point is a critical point of type (ii) in Lemma 2.11. The argument in the last part of the proof of [Dancer and Wang 2009a, Proposition 3.6] then leads to a contradiction. This in particular implies that the only critical point of the flow lying in Ω is E_+ (since $W > 0$ along our trajectory).

We next consider the trajectory starting from a noncritical point in Ω.

Recall from [Dancer and Wang 2009a] that on the locus $\{\mathcal{Q} = 0, \mathcal{H} = 1, Y = 0\}$, the quantity $J := G - \frac{\epsilon}{2} W^2$ satisfies $0 \leq J \leq 1$ and the equation

$$J' = 2J(J - 1).$$

Moreover, $J = 1$ exactly when $W = 0$ and $\mathcal{G} = 1$, and $J = 0$ exactly at the critical points $E_\pm$ (of type (vi) in Lemma 2.11). Points with $W > 0$ (resp. $W < 0$) flow to E_+ (resp. E_-) and flow backwards to $W = 0$.

For our trajectory, W is necessarily positive, so we obtain a contradiction since Ω is compact, flow-invariant, and contains no point with zero W-coordinate. We therefore deduce that Ω is $\{E_+\}$. Now it was observed in [Dancer and Wang 2009a, Lemma 3.8] that for the flow on $\{\mathcal{Q} = 0, \mathcal{H} = 1\}$ the point E_+ is a sink, so our (original) trajectory converges to E_+.

As $dt = W\,ds$ and W is converging to a positive constant we deduce the metric is complete. Using (2.2) we see that the metric components g_i^2 grow exponentially fast asymptotically.

We end this section with some consequences of combining our existence theorems with a study of the differential topology of some of our examples.

We will focus on the case where $r = 2$ and M_2 is a homotopy sphere. Recall that Boyer, Galicki and Kollár [Boyer et al. 2005a; 2005b] have constructed Sasakian Einstein metrics with positive scalar curvature on all Kervaire spheres (with dimension $4m+1$) and those homotopy spheres of dimension 7, 11 or 15 which bound parallelizable manifolds. As in [Buzano et al. 2013] we can take these Einstein manifolds or the standard sphere as M_2 in our constructions in Section 2 and Section 3. Since it follows from the independent work of K. Kawakubo [1969] and R. Schultz [1969] that the manifolds $\mathbb{R}^2 \times M_2$ and $\mathbb{R}^2 \times S^q$ are not diffeomorphic if M_2 is an exotic sphere (see [Kwasik and Schultz 2002]), we deduce the following:

Corollary 3.2. *In dimensions* 9, 13, 17 *and all dimensions* $4m+3$ *with* $m \neq 0, 1, 3, 7, 15, 31$, *there exist pairs of homeomorphic but not diffeomorphic manifolds both of which admit non-Einstein, complete, expanding gradient Ricci soliton structures. The same holds for complete Einstein metrics with negative scalar curvature.* □

Note also that our expanding gradient Ricci solitons and negative Einstein manifolds exhibit conical asymptotics. The corresponding cones are differentially of the form $\mathbb{R}_+ \times S^1 \times M_2$, where $\mathbb{R}_+$ is the set of positive real numbers. We are indebted to Ian Hambleton for providing an outline of the proof of the following consequence of the above-mentioned work of Kawakubo and Schultz.

Proposition 3.3. *Let* Σ^q *and* S^q *be, respectively, a nonstandard homotopy sphere and the standard* q*-sphere. Then the open cones* $\mathbb{R}_+ \times S^1 \times \Sigma$ *and* $\mathbb{R}_+ \times S^1 \times S^q$ *are not diffeomorphic.*

Proof (I. Hambleton). Let

$$\phi : \mathbb{R}_+ \times S^1 \times \Sigma^q \to \mathbb{R}_+ \times S^1 \times S^q$$

be an orientation-preserving diffeomorphism. For convenience, let

$$X = S^1 \times \Sigma^q, \quad X_a = \{a\} \times X,$$
$$Y = S^1 \times S^q, \quad Y_b = \{b\} \times Y.$$

By compactness, $\phi(X_1) \subset (a, b) \times Y$ for some $0 < a < b$. Moreover, by Alexander duality (applied to $(a, b) \times Y$ with the ends capped off by attaching $D^2_\pm \times Y$, for example), $\phi(X_1)$ is a two-sided hypersurface that separates $(a, b) \times Y$ into two path-connected open submanifolds of $\mathbb{R}_+ \times Y$.

Let $W_\pm$ denote the closures of these path components. Then, using the diffeomorphism ϕ, which has to preserve the ends of $\mathbb{R}_+ \times X$ and $\mathbb{R}_+ \times Y$, one easily sees that W_- (resp. W_+) is a compact manifold whose boundary consists of Y_a and $\phi(X_1)$ (resp. $\phi(X_1)$ and Y_b). Moreover, by composition with suitable retractions and the restrictions of ϕ or ϕ^{-1} to suitable subsets, one also sees easily that the inclusion of the boundary components into W_- are homotopy equivalences, i.e., W_- is an h-cobordism between its boundary components. Noting that the Whitehead group of $\pi_1(S^1 \times S^q) = \mathbb{Z}$ is trivial and applying the s-cobordism theorem, we get a contradiction to the result of Kawakubo and Schultz that $S^1 \times \Sigma^q$ and $S^1 \times S^q$ are not diffeomorphic. □

Hence we obtain for the dimensions given in Corollary 3.2 pairs of non-Einstein, complete, expanding gradient Ricci solitons (or complete negative Einstein manifolds) whose asymptotic cones are homeomorphic but not diffeomorphic.

4. Numerical examples

We shall now look at some numerical solutions of the equations (1.1)–(1.3). The Ricci soliton equation in the cohomogeneity one setting has an irregular singular point at $t = 0$. We therefore follow the procedure in [Dancer et al. 2013, § 5; Buzano et al. 2013]. That is, we first find a series solution in a neighbourhood of the singular orbit satisfying the appropriate smoothness conditions. We then truncate the series and use the values of the resulting functions at some small $t_0 > 0$ as initial values to generate solutions of the equations for $t > t_0$ via a fourth-order Runge–Kutta scheme. Because the manifolds we are considering are noncompact, we check the numerics obtained against the general asymptotic properties given in Section 1.

The explicit cases that we shall look at are those where the hypersurface is the twistor space of quaternionic projective space and the total space of the corresponding Sp(1) bundle. For these examples, the estimates $\mathcal{Q} < 0$ and $\mathcal{H} < 1$ do not give coercive estimates, and we do not yet have analytical existence proofs. However the numerics give a strong indication that complete expanding solitons exist in these cases.

Let us recall the equations that will be analysed numerically, following [Buzano et al. 2013]. We consider cohomogeneity one manifolds with principal orbits G/K whose isotropy representation consists of two inequivalent $\mathrm{Ad}(K)$-invariant irreducible real summands. We assume that $K \subset H \subset G$, where H, K are closed subgroups of the compact Lie group G such that H/K is a sphere. A G-invariant background metric b is chosen on G/K such that it induces the constant curvature 1 metric on H/K. The cohomogeneity one manifolds are then the vector bundles $G \times_H \mathbb{R}^{d_1+1}$ where $H/K \subset \mathbb{R}^{d_1+1}$ is regarded as the unit sphere.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an $\mathrm{Ad}(K)$-invariant decomposition of the Lie algebra of G, so that $\mathfrak{p}$ is identified with the tangent space of G/K at the base point. We can further decompose $\mathfrak{p}$ into irreducible K-modules, thus $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are respectively the tangent spaces (at the base point) to the sphere H/K and the singular orbit G/H. Their respective dimensions are denoted by d_1 and d_2.

The metrics of cohomogeneity one take the form

$$\bar{g} = dt^2 + g_1(t)^2\, \mathsf{b}|\mathfrak{p}_1 + g_2(t)^2\, \mathsf{b}|\mathfrak{p}_2.$$

Letting $(z_1, \dots, z_6) := (g_1, \dot{g_1}, g_2, \dot{g_2}, u, \dot{u})$, the gradient Ricci soliton equations become

$$\begin{aligned}
\dot{z_1} &= z_2,\\
\dot{z_2} &= -(d_1-1)\frac{z_2^2}{z_1} - d_2\frac{z_2 z_4}{z_3} + z_2 z_6 + \frac{d_1-1}{z_1} + \frac{A_3}{d_1}\frac{z_1^3}{z_3^4} + \frac{\epsilon}{2} z_1,\\
\dot{z_3} &= z_4,\\
\dot{z_4} &= -d_1\frac{z_2 z_4}{z_1} - (d_2-1)\frac{z_4^2}{z_3} + z_4 z_6 + \frac{A_2}{d_2}\frac{1}{z_3} - 2\frac{A_3}{d_2}\frac{z_1^2}{z_3^3} + \frac{\epsilon}{2} z_3,\\
\dot{z_5} &= z_6,\\
\dot{z_6} &= -z_6\left(d_1\frac{z_2}{z_1} + d_2\frac{z_4}{z_3}\right) + z_6^2 + \epsilon z_5 + C,
\end{aligned}$$

where the A_i are positive constants which appear in the scalar curvature function of the principal orbit. Note that, because of the backgound metric chosen, the coefficient A_1/d_1 of the $1/z_1$ term in the second equation becomes $d_1 - 1$, and for expanding solitons we have $\epsilon > 0$.

Recall also the general relation $(d_1+1)\,\ddot{u}(0) = C + \epsilon u$, which follows from the conservation law and the smoothness conditions at $t = 0$. In generating the numerics, we find it convenient to eliminate homothetic solutions by choosing ϵ to be 1. Furthermore, rather than setting $u(0) = 0$, as was done throughout Section 1, we now set the constant C to be zero. It then follows from the necessary condition $\mathcal{E} < 0$ that in the series solution we must arrange for $\ddot{u}(0) = u(0)/(d_1+1) < 0$, with $u(0)$ as an otherwise arbitrary parameter.

Example 1. We set $G = \mathrm{Sp}(m+1)$, $H = \mathrm{Sp}(m) \times \mathrm{Sp}(1)$, and $K = \mathrm{Sp}(m) \times \mathrm{U}(1)$. The principal orbit G/K is diffeomorphic to $\mathbb{CP}^{2m+1}$ and the singular orbit G/H is $\mathbb{HP}^m$. So $d_1 = 2$, $d_2 = 4m$, and $A_2 = 2m(m+2)$, $A_3 = m/2$ (with b chosen to be $-2\,\mathrm{tr}(XY)$). The initial values of $(z_1, \dots, z_6)$ are given by $(0, 1, \bar{h}, 0, \bar{u}, 0)$, where $\bar{h} > 0$ and $\bar{u} < 0$. These give rise to a 2-parameter family of numerical solutions.

In Figure 1 on the next page we plot the functions g_i and u for the cases $m = 1$ and $m = 2$, with parameter values $\bar{h} = 6$ and $\bar{u} = -1$.

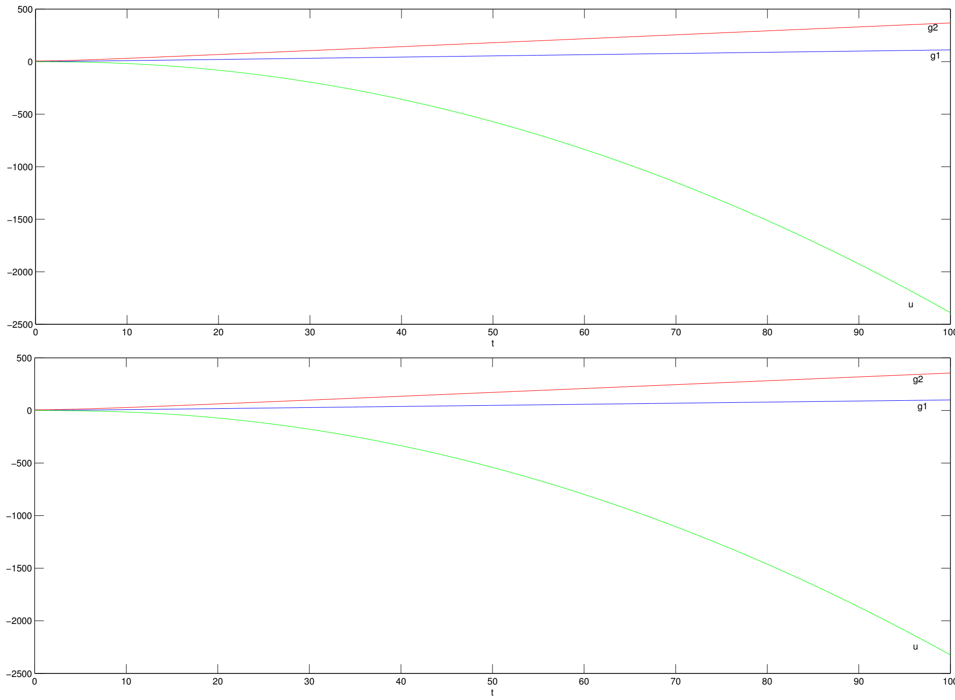

Figure 1. Plots of g_1 (blue), g_2 (red) and u (green) for $m = 1$ (top) and $m = 2$ (bottom).

Note that the soliton potential is concave down and becomes approximately quadratic, in accordance with Proposition 1.11 and Proposition 1.18. The g_i are asymptotically linear.

We have also plotted the quantities

$$\tilde{X}_i = \frac{X_i}{\sqrt{d_i}} \quad \text{and} \quad \tilde{Y}_i = \frac{Y_i}{\sqrt{d_1}}$$

against t in Figures 2 and 3 for the cases $m = 1$ and $m = 2$ respectively. They all converge quickly to 0.

In Figure 4 we plot the ratios $\tilde{X}_1/\tilde{X}_2$ and $\tilde{Y}_1/\tilde{Y}_2$. Note that the second ratio is g_2/g_1, which tends to a positive constant. The first ratio is the ratio of the principal curvatures, $(\dot{g}_2/g_2)\big/(\dot{g}_1/g_1)$, and we see that it quickly approaches 1.

Similar numerical results hold for larger values of m.

Example 2. We next set

$$\begin{aligned} G &= \mathrm{Sp}(m+1) \times \mathrm{Sp}(1), \\ H &= \mathrm{Sp}(m) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1), \\ K &= \mathrm{Sp}(m) \times \Delta\, \mathrm{Sp}(1). \end{aligned}$$

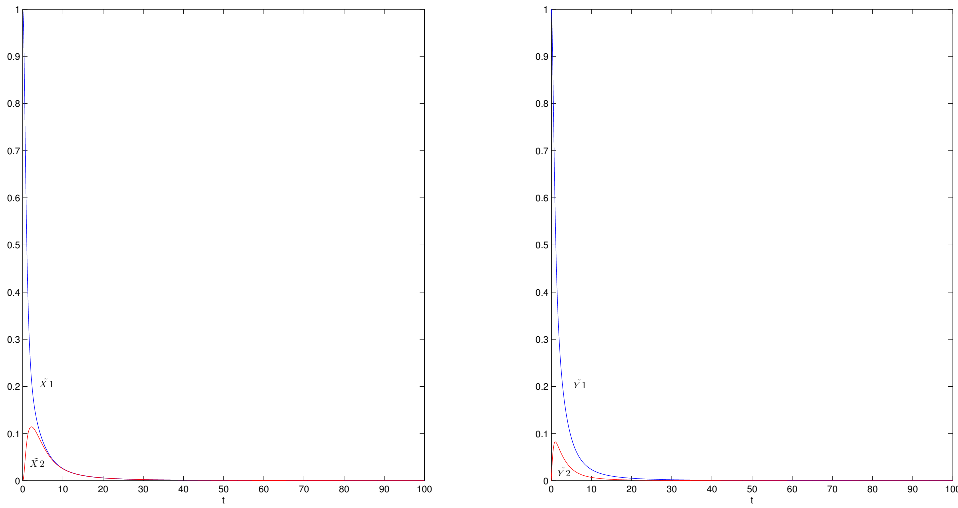

Figure 2. Plots of $\tilde{X}_i$ (left) and $\tilde{Y}_i$ (right) for $i = 1$ (blue) and $i = 2$ (red), in the case $m = 1$.

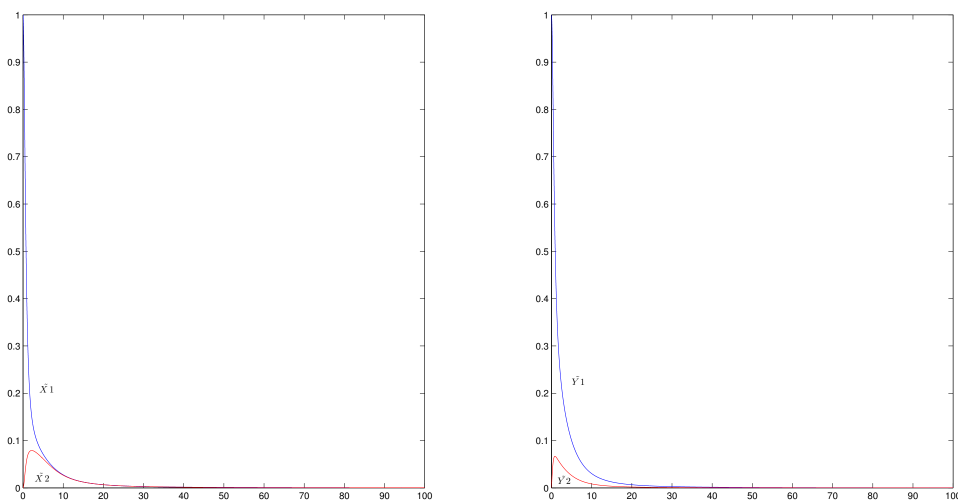

Figure 3. Plots of $\tilde{X}_i$ (left) and $\tilde{Y}_i$ (right) for $i = 1$ (blue) and $i = 2$ (red), in the case $m = 2$.

The principal orbit G/K is diffeomorphic to S^{4m+3} and the singular orbit G/H is again $\mathbb{HP}^m$. So $d_1 = 3$, $d_2 = 4m$, and $A_2 = 4m(m+2)$, $A_3 = 3m/4$ (where b is given by $-2\,\mathrm{tr}(XY)$ on both of the simple factors). The initial values of $(z_1, \ldots, z_6)$ are given by $(0, 1, \bar{h}, 0, \bar{u}, 0)$, where $\bar{h} > 0$ and $\bar{u} < 0$.

For this case we obtain graphs very similar to those in Example 1.

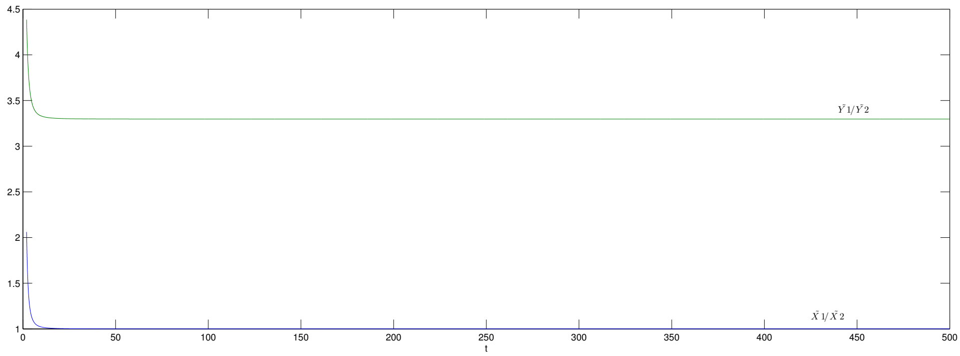

Figure 4. Plots of $\tilde{X}_1/\tilde{X}_2$ (lower curve) and $\tilde{Y}_1/\tilde{Y}_2$ (upper curve).

Based on the last two examples, we would conjecture that on the vector bundles $G \times_H \mathbb{R}^{d_1+1}$, where (G, H, K) are as above, there is a 2-parameter family of nonhomothetic complete expanding gradient Ricci solitons.

References

[Bérard-Bergery 1982] L. Bérard-Bergery, *Sur de nouvelles variétés Riemanniennes d'Einstein*, Institut Élie Cartan **6**, Université Nancy, Nancy, 1982. MR 85b:53048 Zbl 0544.53038

[Böhm 1999] C. Böhm, "Non-compact cohomogeneity one Einstein manifolds", *Bull. Soc. Math. France* **127**:1 (1999), 135–177. MR 2000h:53057 Zbl 0935.53021

[Boyer et al. 2005a] C. P. Boyer, K. Galicki, and J. Kollár, "Einstein metrics on spheres", *Ann. of Math.* (2) **162**:1 (2005), 557–580. MR 2006j:53058 Zbl 1093.53044

[Boyer et al. 2005b] C. P. Boyer, K. Galicki, J. Kollár, and E. Thomas, "Einstein metrics on exotic spheres in dimensions 7, 11, and 15", *Experiment. Math.* **14**:1 (2005), 59–64. MR 2006a:53042 Zbl 1112.53033

[Bryant 2005] R. L. Bryant, "Ricci flow solitons in dimension three with SO(3)-symmetries", preprint, 2005, Available at www.math.duke.edu/~bryant/3DRotSymRicciSolitons.pdf.

[Buzano 2011] M. Buzano, "Initial value problem for cohomogeneity one gradient Ricci solitons", *J. Geom. Phys.* **61**:6 (2011), 1033–1044. MR 2012m:53094 Zbl 1213.53039

[Buzano et al. 2013] M. Buzano, A. S. Dancer, M. Gallaugher, and M. Y. Wang, "A family of steady Ricci solitons and Ricci-flat metrics", preprint, 2013. arXiv 1309.6140

[Carrillo and Ni 2009] J. A. Carrillo and L. Ni, "Sharp logarithmic Sobolev inequalities on gradient solitons and applications", *Comm. Anal. Geom.* **17**:4 (2009), 721–753. MR 3010626 Zbl 1197.53083

[Chen 2009] B.-L. Chen, "Strong uniqueness of the Ricci flow", *J. Differential Geom.* **82**:2 (2009), 363–382. MR 2010h:53095 Zbl 1177.53036

[Chen 2012] C.-W. Chen, "On the asymptotic behavior of expanding gradient Ricci solitons", *Ann. Global Anal. Geom.* **42**:2 (2012), 267–277. MR 2947955 Zbl 1263.53061

[Chodosh 2014] O. Chodosh, "Expanding Ricci solitons asymptotic to cones", *Calc. Var. Partial Differential Equations* **51**:1-2 (2014), 1–15. MR 3247379 arXiv 1303.2983

[Dancer and Wang 2000] A. S. Dancer and M. Y. Wang, "The cohomogeneity one Einstein equations from the Hamiltonian viewpoint", *J. Reine Angew. Math.* **524** (2000), 97–128. MR 2001e:53046 Zbl 0988.53018

[Dancer and Wang 2009a] A. S. Dancer and M. Y. Wang, "Non-Kähler expanding Ricci solitons", *Int. Math. Res. Not.* **2009**:6 (2009), 1107–1133. MR 2009k:53098 Zbl 1183.53060

[Dancer and Wang 2009b] A. S. Dancer and M. Y. Wang, "Some new examples of non-Kähler Ricci solitons", *Math. Res. Lett.* **16**:2 (2009), 349–363. MR 2010e:53108 Zbl 1173.53327

[Dancer and Wang 2011] A. S. Dancer and M. Y. Wang, "On Ricci solitons of cohomogeneity one", *Ann. Global Anal. Geom.* **39**:3 (2011), 259–292. MR 2012a:53124 Zbl 1215.53040

[Dancer et al. 2013] A. S. Dancer, S. J. Hall, and M. Y. Wang, "Cohomogeneity one shrinking Ricci solitons: an analytic and numerical study", *Asian J. Math.* **17**:1 (2013), 33–61. MR 3038724 Zbl 1280.53044

[Gaffney 1954] M. P. Gaffney, "A special Stokes's theorem for complete Riemannian manifolds", *Ann. of Math.* (2) **60** (1954), 140–145. MR 15,986d Zbl 0055.40301

[Gastel and Kronz 2004] A. Gastel and M. Kronz, "A family of expanding Ricci solitons", pp. 81–93 in *Variational problems in Riemannian geometry: bubbles, scans and geometric flows*, edited by P. Baird et al., Progr. Nonlinear Differential Equations Appl. **59**, Birkhäuser, Basel, 2004. MR 2005m:53118 Zbl 1075.53061

[Gutperle et al. 2003] M. Gutperle, M. Headrick, S. Minwalla, and V. Schomerus, "Spacetime energy decreases under world-sheet RG flow", *J. High Energy Phys.* **2003**:1 (2003), Article ID #073. MR 2004a:83072 Zbl 1225.83078

[Jablonski 2013] M. Jablonski, "Homogeneous Ricci solitons", preprint, 2013. To appear in J. Reine Angew. Math. arXiv 1109.6556

[Kawakubo 1969] K. Kawakubo, "Smooth structures on $S^p \times S^q$", *Osaka J. math.* **6** (1969), 165–196. MR 41 #6228 Zbl 0191.54403

[Kwasik and Schultz 2002] S. Kwasik and R. E. Schultz, "Multiplicative stabilization and transformation groups", pp. 147–165 in *Current trends in transformation groups: dedicated to the memory of Professor Katsuo Kawakubo*, edited by A. Bak et al., K-Monogr. Math. **7**, Kluwer Acad. Publ., Dordrecht, 2002. MR 2004c:57055 Zbl 1021.57015

[Lauret 2001] J. Lauret, "Ricci soliton homogeneous nilmanifolds", *Math. Ann.* **319**:4 (2001), 715–733. MR 2002k:53083 Zbl 0987.53019

[Lauret 2009] J. Lauret, "Einstein solvmanifolds and nilsolitons", pp. 1–35 in *New developments in Lie theory and geometry* (Córdoba, 2007), edited by C. S. Gordon et al., Contemp. Math. **491**, Amer. Math. Soc., Providence, RI, 2009. MR 2010g:53081 Zbl 1186.53058

[Lauret 2011] J. Lauret, "Ricci soliton solvmanifolds", *J. Reine Angew. Math.* **650** (2011), 1–21. MR 2012d:53160 Zbl 1210.53051

[McLean 2000] W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, 2000. MR 2001a:35051 Zbl 0948.35001

[Munteanu and Sesum 2013] O. Munteanu and N. Sesum, "On gradient Ricci solitons", *J. Geom. Anal.* **23**:2 (2013), 539–561. MR 3023848 Zbl 1275.53061

[Petersen and Wylie 2009] P. Petersen and W. Wylie, "Rigidity of gradient Ricci solitons", *Pacific J. Math.* **241**:2 (2009), 329–345. MR 2010j:53071 Zbl 1176.53048

[Pigola et al. 2011] S. Pigola, M. Rimoldi, and A. G. Setti, "Remarks on non-compact gradient Ricci solitons", *Math. Z.* **268**:3-4 (2011), 777–790. MR 2012i:53042 Zbl 1223.53034

[Protter and Weinberger 1984] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Springer, New York, 1984. MR 86f:35034 Zbl 0549.35002

[Schultz 1969] R. E. Schultz, "Smooth structures on $S^p \times S^q$", *Ann. of Math.* (2) **90** (1969), 187–198. MR 40 #3560 Zbl 0183.28201

[Schulze and Simon 2013] F. Schulze and M. Simon, “Expanding solitons with non-negative curvature operator coming out of cones”, *Math. Z.* **275**:1-2 (2013), 625–639. MR 3101823 Zbl 1278.53072

[Yau 1976] S. T. Yau, “Some function-theoretic properties of complete Riemannian manifold and their applications to geometry”, *Indiana Univ. Math. J.* **25**:7 (1976), 659–670. MR 54 #5502 Zbl 0335.53041

[Zhang 2009] Z.-H. Zhang, “On the completeness of gradient Ricci solitons”, *Proc. Amer. Math. Soc.* **137**:8 (2009), 2755–2759. MR 2010a:53057 Zbl 1176.53046

Received February 14, 2014.

MARIA BUZANO
DEPARTMENT OF MATHEMATICS AND STATISTICS
MCMASTER UNIVERSITY
1280 MAIN STREET W.
HAMILTON, ON L8S 4K1
CANADA
maria.buzano@gmail.com

ANDREW S. DANCER
JESUS COLLEGE
OXFORD UNIVERSITY
OXFORD OX1 3DW
UNITED KINGDOM
dancer@maths.ox.ac.uk

MICHAEL GALLAUGHER
DEPARTMENT OF MATHEMATICS AND STATISTICS
MCMASTER UNIVERSITY
1280 MAIN STREET W.
HAMILTON, ON L8S 4K1
CANADA
gallaump@mcmaster.ca

MCKENZIE WANG
DEPARTMENT OF MATHEMATICS AND STATISTICS
MCMASTER UNIVERSITY
1280 MAIN STREET W.
HAMILTON, ON L8S 4K1
CANADA
wang@mcmaster.ca

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.395

A NOTE ON L-PACKETS AND ABELIAN VARIETIES OVER LOCAL FIELDS

JEFFREY D. ACHTER AND CLIFTON CUNNINGHAM

A polarized abelian variety (X, λ) of dimension g and good reduction over a local field K determines an admissible representation of $\mathrm{GSpin}_{2g+1}(K)$. We show that the restriction of this representation to $\mathrm{Spin}_{2g+1}(K)$ is reducible if and only if X is isogenous to its twist by the quadratic unramified extension of K. When $g = 1$ and $K = \mathbb{Q}_p$, we recover the well-known fact that the admissible $\mathrm{GL}_2(K)$-representation attached to an elliptic curve E with good reduction is reducible upon restriction to $\mathrm{SL}_2(K)$ if and only if E has supersingular reduction.

Introduction

Consider an elliptic curve $E/\mathbb{Q}_p$ with good reduction. Let π_E be the unramified principal series representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with the same Euler factor as E. Although π_E is irreducible, the restriction of π_E from $\mathrm{GL}_2(\mathbb{Q}_p)$ to its derived group, $\mathrm{SL}_2(\mathbb{Q}_p)$, need not be irreducible. In fact, it is not hard to show that $\pi_E|_{\mathrm{SL}_2(\mathbb{Q}_p)}$ is reducible if and only if the reduction of E is supersingular; see [Anandavardhanan and Prasad 2006, 2.1], for example.

This note generalizes the observation above, as follows. Let K be a non-Archimedean local field with finite residue field and let (X, λ) be a polarized abelian variety over K of dimension g with good reduction. Fix a rational prime ℓ invertible in the residue field of K. Then the associated Galois representation on the rational ℓ-adic Tate module of X takes values in $\mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda) \cong \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$. The eigenvalues of the image of Frobenius under this unramified representation determine an irreducible principal series representation $\pi_{X,\lambda}$ of $\mathrm{GSpin}_{2g+1}(K)$ with the same Euler factor as X. Note that the dual group to GSpin_{2g+1} is GSp_{2g}; note also that $\mathrm{GSpin}_3 \cong \mathrm{GL}_2$ and $\mathrm{GSpin}_5 \cong \mathrm{GSp}_4$, accidentally. In this note we show that the restriction of $\pi_{X,\lambda}$ from $\mathrm{GSpin}_{2g+1}(K)$ to its derived group $\mathrm{Spin}_{2g+1}(K)$

Achter was partially supported by a grant from the Simons Foundation (204164). Cunningham was partially supported by NSERC (DG696158) and PIMS.
MSC2010: primary 22E50, 11G10; secondary 11F70, 14K15, 11S37.
Keywords: abelian varieties, good reduction, local fields, L-packets, admissible representations.

is reducible if and only if X is isogenous to its twist by the quadratic unramified extension of K.

Furthermore, we identify the Langlands parameter $\phi_{X,\lambda}$ for $\pi_{X,\lambda}$ and then show that the corresponding L-packet $\Pi_{X,\lambda}$ contains the equivalence class of $\pi_{X,\lambda}$ only. Then we show that we can detect when X is isogenous to its quadratic unramified twist directly from the local L-packet $\Pi^{\mathrm{der}}_{X,\lambda}$ determined by transferring the Langlands parameter $\phi_{X,\lambda}$ to the derived group $\mathrm{Spin}_{2g+1}(K)$ of $\mathrm{GSpin}_{2g+1}(K)$.

1. Abelian varieties

In this section, we collect some useful facts about abelian varieties, especially over finite fields. Many of the attributes discussed here are isogeny invariants. We write $X \sim Y$ if X and Y are isogenous abelian varieties, and $\mathrm{End}^0(X)$ for the endomorphism algebra $\mathrm{End}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ of X.

1A. ***Base change of abelian varieties.*** Let $X/\mathbb{F}_q$ be an abelian variety of dimension g. Associated to it are the characteristic polynomial $P_{X/\mathbb{F}_q}(T)$ and minimal polynomial $M_{X/\mathbb{F}_q}(T)$ of Frobenius. Then $P_{X/\mathbb{F}_q}(T) \in \mathbb{Z}[T]$ is monic of degree $2g$, and $M_{X/\mathbb{F}_q}(T)$ is the radical of $P_{X/\mathbb{F}_q}(T)$.

The isogeny class of X is completely determined by $P_{X/\mathbb{F}_q}(T)$ [Tate 1966]. It is thus possible to detect from $P_{X/\mathbb{F}_q}(T)$ whether X is simple, but even easier to decide if X is isotypic, which is to say, isogenous to the self-product of a simple abelian variety. Indeed, let $\mathrm{ZEnd}^0(X) \subset \mathrm{End}^0(X)$ be the center of the endomorphism algebra of X. Then

$$\mathrm{ZEnd}^0(X) \cong \mathbb{Q}[T]/(M_{X/\mathbb{F}_q}(T)), \tag{1-1}$$

and X is isotypic if and only if $M_{X/\mathbb{F}_q}(T)$ is irreducible. While it is possible for a simple abelian variety to become reducible after extension of scalars of the base field, isotypicality is preserved by base extension (see [Oort 2008, Claim 10.8], for example).

For a monic polynomial $g(T) = \prod_{1\le j\le N}(T-\tau_j)$ and a natural number r, set $g^{(r)}(T) = \prod_{1\le j\le N}(T-\tau_j^r)$. It is not hard to check that

$$P_{X/\mathbb{F}_{q^r}}(T) = P^{(r)}_{X/\mathbb{F}_q}(T).$$

Lemma 1.1. *Suppose $X/\mathbb{F}_q$ is isotypic, and let $\mathbb{F}_{q^r}/\mathbb{F}_q$ be a finite extension. Let Y be a simple factor of $X_{\mathbb{F}_{q^r}}$. Then there exists some $m \mid r$ such that*

$$M^{(r)}_{X/\mathbb{F}_q}(T) = M_{Y/\mathbb{F}_{q^r}}(T)^m \quad \textit{and} \quad \dim \mathrm{ZEnd}^0(X) = m \dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^r}}).$$

Proof. Write $X_{\mathbb{F}_{q^r}} \sim Y^n$ with Y simple. Then we have two different factorizations of $P_{X/\mathbb{F}_{q^r}}(T)$:

$$P_{X/\mathbb{F}_{q^r}}(T) = (M^{(r)}_{X/\mathbb{F}_q}(T))^d,$$
$$P_{X/\mathbb{F}_{q^r}}(T) = (M_{Y/\mathbb{F}_{q^r}}(T))^e.$$

Since $M_{Y/\mathbb{F}_{q^r}}(T)$ is irreducible (and all polynomials considered here are monic), there exists some integer m such that

$$M^{(r)}_{X/\mathbb{F}_q}(T) = M_{Y/\mathbb{F}_{q^r}}(T)^m.$$

Note that

$$m = \frac{\deg M_{X/\mathbb{F}_q}(T)}{\deg M_{Y/\mathbb{F}_{q^r}}(T)} = [\mathrm{ZEnd}^0(X) : \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^r}})].$$

Let τ be a root of $M_{X/\mathbb{F}_q}(T)$. Then τ^r is a root of $M^{(r)}_{X/\mathbb{F}_q}(T)$, and thus of $M_{Y/\mathbb{F}_q}(T)$; and the inclusion of fields $\mathrm{ZEnd}^0(X_{\mathbb{F}_{q^r}}) \subseteq \mathrm{ZEnd}^0(X)$ is isomorphic to the inclusion of fields $\mathbb{Q}(\tau^r) \subseteq \mathbb{Q}(\tau)$, under (1-1). In particular, $m = [\mathbb{Q}(\tau) : \mathbb{Q}(\tau^r)]$. Since τ satisfies the equation $S^r - \tau^r$ over $\mathbb{Q}(\tau^r)$, its degree over $\mathbb{Q}(\tau^r)$ divides r. □

1B. *Even abelian varieties.* Call an abelian variety $X/\mathbb{F}_q$ *even* if its characteristic polynomial is even:

$$P_{X/\mathbb{F}_q}(T) = P_{X/\mathbb{F}_q}(-T).$$

If X is simple, then it admits a unique nontrivial quadratic twist $X'/\mathbb{F}_q$. For an arbitrary $X/\mathbb{F}_q$, let $X'/\mathbb{F}_q$ be the quadratic twist associated to the cocycle

$$\mathrm{Gal}(\mathbb{F}_q) \to \mathrm{Aut}(X), \quad \mathrm{Fr}_q \mapsto [-1]$$

corresponding to a nontrivial quadratic twist of all simple factors of X.

For future use, we record the following elementary observation:

Lemma 1.2. *Let $X/\mathbb{F}_q$ be an abelian variety. Then X is even if and only if X and X' are isogenous.*

Proof. Use the (canonical, given our construction) isomorphism $X_{\mathbb{F}_{q^2}} \cong X'_{\mathbb{F}_{q^2}}$ to identify $V_\ell X$ and $V_\ell X'$. Then one knows (see [Serre and Tate 1968, p. 506], for example) that $\rho_{X'/\mathbb{F}_q}(\mathrm{Fr}_q) = -\rho_{X/\mathbb{F}_q}(\mathrm{Fr}_q)$, and thus that

$$P_{X'/\mathbb{F}_q}(T) = P_{X/\mathbb{F}_q}(-T).$$

The asserted equivalence now follows from Tate's theorem [1966, Theorem 1]. □

To a large extent, evenness of X is captured by the behavior of the center of $\mathrm{End}^0(X)$ upon quadratic base extension.

Lemma 1.3. *If $X/\mathbb{F}_q$ is even, then*

$$\dim \mathrm{ZEnd}^0(X) = 2 \dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^2}}).$$

Proof. Suppose $X/\mathbb{F}_q$ is even. Then the multiset $\{\tau_1, \dots, \tau_{2g}\}$ of eigenvalues of Frobenius of X is stable under multiplication by -1, and in particular the set of distinct eigenvalues of Frobenius is stable under multiplication by -1. Moreover, this action has no fixed points; and thus $\{\tau_1^2, \dots, \tau_{2g}^2\}$, the set of eigenvalues of $X/\mathbb{F}_{q^2}$, has half as many distinct elements as the original set. The claim now follows from characterization (1-1) of $\mathrm{ZEnd}^0(X)$. □

The converse is almost true.

Proposition 1.4. *Suppose X is isotypic. Then X is even if and only if*

$$\dim \mathrm{ZEnd}^0(X) = 2 \dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^2}}).$$

Proof. Suppose $\dim \mathrm{ZEnd}^0(X) = 2 \dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^2}})$ and let Y be a simple factor of $X_{\mathbb{F}_{q^2}}$. By Lemma 1.1,

$$M^{(2)}_{X/\mathbb{F}_q}(T) = M_{Y/\mathbb{F}_{q^2}}(T)^2. \tag{1-2}$$

Factor the minimal polynomials of X and Y as

$$M_{X/\mathbb{F}_q}(T) = \prod_{1 \le j \le 2h} (T - \tau_j),$$
$$M_{Y/\mathbb{F}_{q^2}}(T) = \prod_{1 \le j \le h} (T - \beta_j).$$

By (1-2), we may order the roots of $M_{X/\mathbb{F}_q}(T)$ so that, for each $1 \le j \le h$, we have

$$\tau_j^2 = \tau_{h+j}^2 = \beta_j,$$

so that $\tau_{h+j} = \pm\tau_j$. In fact, $\tau_{h+j} = -\tau_j$; for otherwise, $M_{X/\mathbb{F}_q}(T)$ would have a repeated root, which contradicts the known semisimplicity of Frobenius. Now, $P_{X/\mathbb{F}_q}(T) = M_{X/\mathbb{F}_q}(T)^d$ for some d. The multiset of eigenvalues of Frobenius of X is thus stable under multiplication by -1, and $X/\mathbb{F}_q$ is even. □

Note that evenness is an assertion about the *multiset* of eigenvalues of Frobenius, while the calculation of $\dim \mathrm{ZEnd}^0(X_{\mathbb{F}_{q^e}})$ only detects the *set* of eigenvalues. Consequently, if one drops the isotypicality assumption in Proposition 1.4, it is easy to write down examples of abelian varieties which are not even but satisfy the criterion on dimensions of centers of endomorphism rings.

Example 1.5. Let $E/\mathbb{F}_q$ be an ordinary elliptic curve; then E is not isogenous to E' over $\mathbb{F}_q$ but $\mathrm{End}^0(E) \cong \mathrm{End}^0(E') \cong L$, a quadratic imaginary field. Set $X = E \times E \times E'$. Then X is not even, since $X' \cong E' \times E' \times E$, but $\mathrm{ZEnd}^0(X) \cong L \times L$ while $\mathrm{ZEnd}^0(X_{\mathbb{F}_{q^2}}) \cong L$. Therefore, $X/\mathbb{F}_q$ satisfies the dimension criterion of Proposition 1.4 but is not even.

Example 1.6. Consider a supersingular elliptic curve $E/\mathbb{F}_q$, where q is an odd power of the prime p. Then $\mathrm{End}^0(E) \cong \mathbb{Q}(\sqrt{-p})$, while $\mathrm{End}^0(E_{\mathbb{F}_{q^2}})$ is the quaternion algebra ramified at p and ∞. In particular, $\mathrm{ZEnd}^0(E)$ is a quadratic imaginary field, while $\mathrm{ZEnd}^0(E_{\mathbb{F}_{q^2}}) \cong \mathbb{Q}$. Therefore, $E/\mathbb{F}_q$ is even.

Example 1.7. In contrast, if $X/\mathbb{F}_q$ is an absolutely simple ordinary abelian variety, then $\mathrm{End}^0(X) = \mathrm{End}^0(X_{\mathbb{F}_{q^2}})$. (This is a consequence of Theorem 7.2 of [Waterhouse 1969], which unfortunately omits the necessary hypothesis of absolute simplicity.)

Example 1.8. Now consider an arbitrary abelian variety $X/\mathbb{F}_q$ and its preferred quadratic twist X'. Then the sum $X \times X'$ is visibly isomorphic to its own quadratic twist, and thus even.

Example 1.9. Let $X/\mathbb{F}_q$ be an abelian variety of dimension g. Suppose there is an integer $N \geq 3$, relatively prime to q, such that $X[N](\mathbb{F}_q) \cong (\mathbb{Z}/N)^{2g}$. Then X is not even. Indeed, if an abelian variety Y over a field k has maximal k-rational N-torsion for $N \geq 3$ and N is invertible in k, then $\mathrm{End}^0(Y) \cong \mathrm{End}^0(Y_{\bar{k}})$ [Silverberg 1992, Theorem 2.4]. By the criterion of Lemma 1.3, if $X/\mathbb{F}_q$ satisfies the hypotheses of the present lemma, then X cannot be even.

1C. *Abelian varieties over local fields.* Now let K be a local field with residue field $\mathbb{F}_q$ and let X/K be an abelian variety with good reduction $X_0/\mathbb{F}_q$. As in 1B, we define a canonical quadratic twist X' of X, associated to the unique nontrivial character

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(K^{\mathrm{unram}}/K) \to \{[\pm 1]\} \subset \mathrm{Aut}(X).$$

Proposition 1.10. *Let X/K be an abelian variety with good reduction $X_0/\mathbb{F}_q$. The following are equivalent*:

(a) *X and X' are isogenous*;

(b) *$X_0/\mathbb{F}_q$ and $X_0'/\mathbb{F}_q$ are isogenous*;

(c) *$X_0/\mathbb{F}_q$ is even.*

Proof. By hypothesis, X spreads out to an abelian scheme $\mathscr{X}/\mathcal{O}_K$ (its Néron model) with special fiber $X_0/\mathbb{F}_q$; the automorphism $[-1] \in \mathrm{End}(X)$ extends to an automorphism of $\mathscr{X}$ and the corresponding twist $\mathscr{X}'$ has generic and special fibers X' and $(X_0)'/\mathbb{F}_q$, respectively. This compatibility explains the equivalence of (a) and (b); the equivalence of (b) and (c) is Lemma 1.2. □

Call X/K *even* if X has good reduction and satisfies any of the equivalent statements in Proposition 1.10.

2. L-packets attached to abelian varieties

2A. *Polarizations.* Let X/k be an abelian variety over an arbitrary field k. Let λ be a polarization on X, i.e., a symmetric isogeny $X \to \hat{X}$ which arises from an ample line bundle on X. Fix a rational prime ℓ invertible in k. The polarization λ on X induces a nondegenerate skew-symmetric pairing $\langle \cdot, \cdot \rangle_\lambda$ on the Tate module $T_\ell X$ and on the rational Tate module $V_\ell X$. Let $\mathrm{GSp}(V_\ell X, \langle \cdot, \cdot \rangle_\lambda)$ be the group of symplectic similitudes of $V_\ell X$ with respect to this pairing; note that $\mathrm{GSp}(V_\ell X, \langle \cdot, \cdot \rangle_\lambda)$ comes with a representation $r_{\lambda,\ell} : \mathrm{GSp}(V_\ell X, \langle \cdot, \cdot \rangle_\lambda) \hookrightarrow \mathrm{GL}(V_\ell X)$. Let $\rho_{X,\ell} : \mathrm{Gal}(k) \to \mathrm{GL}(V_\ell X)$ be the representation on the rational Tate module and let $\rho_{\lambda,\ell} : \mathrm{Gal}(k) \to \mathrm{GSp}(V_\ell X, \langle \cdot, \cdot \rangle_\lambda)$ be the continuous homomorphism such that $\rho_{X,\ell} = r_{\lambda,\ell} \circ \rho_{\lambda,\ell}$.

$$
\begin{tikzcd}
\mathrm{Gal}(\bar{k}/k) \arrow[rr, "\rho_{X,\ell}"] \arrow[dr, "\rho_{X,\lambda,\ell}"'] & & \mathrm{GL}(V_\ell X) \\
& \mathrm{GSp}(V_\ell X, \langle \cdot, \cdot \rangle_\lambda) \arrow[ur, "r_{\lambda,\ell}"'] &
\end{tikzcd}
\tag{2-1}
$$

2B. *Admissible representations attached to abelian varieties with good reduction.* Let K be a local field. Fix a rational prime ℓ invertible in the residue field of K, and thus in K. It will be comforting, though not even remotely necessary, to fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. We will indicate the corresponding complex-valued versions of $\rho_{X,\ell}$, $\rho_{\lambda,\ell}$, and $r_{\lambda,\ell}$ from Section 2A by eliding the subscript ℓ.

In the rest of the paper we will commonly employ the notation $G := \mathrm{GSpin}_{2g+1}$; note that the dual group to G is $\check{G} = \mathrm{GSp}_{2g}$. The derived group $G_{\mathrm{der}} = \mathrm{Spin}_{2g+1}$, which is semisimple and simply connected, will play a role below, as will its dual $\check{G}_{\mathrm{ad}} = \mathrm{PGSp}_{2g}$, which is of adjoint type.

Proposition 2.1. *Let X/K be an abelian variety of dimension g with good reduction and let λ be a polarization on X. There is an irreducible unramified principal series representation $\pi_{X,\lambda}$ of $\mathrm{GSpin}_{2g+1}(K)$, unique up to equivalence, such that*

$$L(z, \rho_X) = L(z, \pi_{X,\lambda}, r_\lambda).$$

Moreover, $|\ |_K^{-1/2} \otimes \pi_{X,\lambda}$ is unitary.

Proof. This is a very small and well-known part of the local Langlands correspondence for $G = \mathrm{GSpin}_{2g+1}$ over K which, in this case, matches unramified principal series representations of $G(K) = \mathrm{GSpin}_{2g+1}(K)$ with unramified Langlands parameters taking values in $\check{G}(\mathbb{C}) = \mathrm{GSp}_{2g}(\mathbb{C})$. For completeness and to introduce notation for later use, we include the details here.

We begin by describing $L(z, \rho_X)$. By [Serre and Tate 1968], the Galois representation $\rho_{X,\ell}$ is unramified and the characteristic polynomial of $\rho_{X,\ell}(\mathrm{Fr}_q)$ has rational coefficients. Accordingly, the Euler factor for $\rho_{X,\ell}$ takes the form

$$L(s,\rho_X)=\frac{(q^s)^{2g}}{P_{X_0/\mathbb{F}_q}(q^s)}.$$

Let $\{\tau_1,\dots,\tau_{2g}\}$ be the (complex) roots of $P_{X_0/\mathbb{F}_q}(T)$. Also by [Serre and Tate 1968], the ℓ-adic realization $\rho_{X,\ell}(\mathrm{Fr}_q)\in \mathrm{GL}(V_\ell X)$ of the Frobenius endomorphism of X is semisimple of weight 1, so each eigenvalue satisfies $|\tau_j|=\sqrt{q}$. Label the roots in such a way that, for each $1\le j\le g$, we have $\tau_{g+j}=q\tau_j^{-1}$; and $\tau_j=\sqrt{q}e^{2\pi i\theta_j}$, where $1>\theta_1\ge\theta_2\ge\cdots\ge\theta_g\ge 0$.

Let T be a K-split maximal torus in G; let $\check{T}$ be the dual torus. Then the Lie algebra of the torus $\check{T}(\mathbb{C})$ may be identified with $X^*(T)\otimes\mathbb{C}$ through the function

$$\exp: X^*(T)\otimes\mathbb{C}\to\check{T}(\mathbb{C})$$

defined by $\check{\alpha}(\exp(x))=e^{2\pi i\langle\check{\alpha},x\rangle}$ for each root $\check{\alpha}$ for $\check{G}$ with respect to $\check{T}$. The Lie algebra of the compact part of $\check{T}(\mathbb{C})$, denoted by $\check{T}(\mathbb{C})^{\mathrm{u}}$ below, is then identified with $X^*(T)\otimes\mathbb{R}$ under exp. We pick a basis $\{e_0,\dots,e_g\}$ for $X^*(T)$ that identifies e_0 with the similitude character for $\check{G}$ and write $\{f_0,\dots,f_g\}$ for the dual basis for $X_*(T)\cong X^*(\check{T})$. Set $\theta_0:=0$ and set $\theta:=\sum_{j=0}^{g}\theta_j e_j$; note that $\theta\in X^*(T)\otimes\mathbb{R}$, so $\exp(\theta)$ lies in $\check{T}(\mathbb{C})^{\mathrm{u}}$. Then $\rho_{X,\lambda}(\mathrm{Fr}_q)=\sqrt{q}\exp(\theta)$.

Let W_K be the Weil group for K. The L-group for T is ${}^LT=\check{T}(\mathbb{C})\times W_K$ since T is K-split. Consider the Langlands parameter

$$\phi: W_K\to {}^LT$$

defined by $\phi(\mathrm{Fr}_q)=\rho_{X,\lambda}(\mathrm{Fr}_q)=\sqrt{q}\exp(\theta)\times\mathrm{Fr}_q$. Let $\chi: T(K)\to\mathbb{C}^\times$ be the quasicharacter of $T(K)$ matching ϕ under the local Langlands correspondence for algebraic tori [Yu 2009]. The character $\chi^{\mathrm{u}}:=|\ |_K^{-1/2}\otimes\chi$ corresponds to the unramified Langlands parameter

$$\phi^{\mathrm{u}}: W_K\to {}^LT$$

defined by $\phi^{\mathrm{u}}(\mathrm{Fr}_q)=\exp(\theta)\times\mathrm{Fr}_q$.

Now pick a Borel subgroup $B\subset G$ over K with reductive quotient T and set

$$\pi_{X,\lambda}:=\mathrm{Ind}_{B(K)}^{G(K)}\chi.$$

Then $\pi_{X,\lambda}$ is an irreducible, unramified principal series representation of $G(K)$. In the same way, the unitary character $\chi^{\mathrm{u}}: K^\times\to\mathbb{C}^\times$ determines the irreducible principal series representation

$$\pi_{X,\lambda}^{\mathrm{u}}:=\mathrm{Ind}_{B(K)}^{G(K)}\chi^{\mathrm{u}}.$$

This admissible representation $\pi^{\mathrm{u}}_{X,\lambda}$ is unitary and enjoys

$$\pi^{\mathrm{u}}_{X,\lambda} = |\ |_K^{-1/2} \otimes \pi_{X,\lambda},$$

as promised.

Having identified the irreducible principal series representation $\pi_{X,\lambda}$ of $G(K)$ attached to (X, λ), we turn to the L function $L(s, \pi_{X,\lambda}, r_\lambda)$. For this it will be helpful to go back and say a few words about the representation $r_{\lambda,\ell} : \mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda) \hookrightarrow \mathrm{GL}(V_\ell X)$.

Let S be a maximal torus in $\mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$ containing $\rho_{X,\ell}(\mathrm{Fr}_q)$ and let S' be a maximal torus in $\mathrm{GL}(V_\ell X)$ containing $r_{\lambda,\ell}(S)$. Let F_ℓ be the splitting extension of S' in $\overline{\mathbb{Q}}_\ell$; observe that this contains the splitting extension of $P_{X_0/\mathbb{F}_q}(T) \in \mathbb{Q}[T]$ in $\overline{\mathbb{Q}}_\ell$. Passing from $\mathbb{Q}_\ell$ to F_ℓ, we may choose bases $\{f_0, f_1, f_2, \ldots, f_g\}$ for $X^*(S)$ and $\{f'_1, f'_2, \ldots, f'_{2g}\}$ for $X^*(S')$ such that the map $X^*(S') \twoheadrightarrow X^*(S)$ induced by the representation $r_{\lambda,\ell}$ is given, for $j = 1, \ldots, g$, by

$$X^*(S') \to X^*(S), \quad f'_j \mapsto f_j, \quad f'_{g+j} \mapsto f_0 - f_{g-j+1}. \tag{2-2}$$

Note that this determines a basis for $V_\ell X \otimes_{\mathbb{Q}_\ell} F_\ell$.

Passing from F_ℓ to $\mathbb{C}$, we have now identified a basis for $V_\ell X \otimes_{\mathbb{Q}_\ell} \mathbb{C}$ which defines

$$\mathrm{GSp}(V_\ell X \otimes_{\mathbb{Q}_\ell} \mathbb{C}, \langle\cdot,\cdot\rangle_\lambda) \cong \mathrm{GSp}_{2g}(\mathbb{C}) = \check{G}(\mathbb{C})$$

inducing $S \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong \check{T}$ and also gives

$$\mathrm{GL}(V_\ell X \otimes_{\mathbb{Q}_\ell} \mathbb{C}) \cong \mathrm{GL}_{2g}(\mathbb{C}).$$

Now (2-1) extends to

$$\begin{tikzcd} \mathrm{Gal}(\overline{K}/K) \arrow[rr, "\rho_X"] \arrow[dr, "\rho_{X,\lambda}"'] & & \mathrm{GL}_{2g}(\mathbb{C}) \\ & \check{G}(\mathbb{C}) \arrow[ur, "r_\lambda"'] & \end{tikzcd} \tag{2-3}$$

It follows immediately that

$$L(s, \pi_{X,\lambda}, r_\lambda) = \prod_{i=1}^{2g} \frac{1}{1 - \tau_i q^{-s}} = \prod_{i=1}^{2g} \frac{q^s}{q^s - \tau_i} = \frac{(q^s)^{2g}}{P_{X_0/\mathbb{F}_q}(q^s)} = L(s, \rho_X),$$

concluding the proof of Proposition 2.1. □

2C. *R-groups.* The irreducible representation $\pi_{X,\lambda}$ of $G(K)$ in Proposition 2.1 is obtained by parabolic induction from an unramified quasicharacter of a split maximal torus $T(K)$. In Section 2E we will use the restriction of this representation to the derived group $G_{\mathrm{der}}(K) = \mathrm{Spin}_{2g+1}(K)$ of $G(K) = \mathrm{GSpin}_{2g+1}(K)$ to study X. While the resulting representation of $G_{\mathrm{der}}(K)$ is again an unramified principal series representation, it need not be irreducible; in fact, we will glean information about X from the components of this representation of $G_{\mathrm{der}}(K)$. With this application in mind, here we review some basic facts about reducible principal series representations of $G_{\mathrm{der}}(K)$.

As in the proof of Proposition 2.1, let B be a Borel subgroup of G with reductive quotient T, a split maximal torus in G. Set $B_{\mathrm{der}} = G_{\mathrm{der}} \cap B$. This is a Borel subgroup of G_{der} with reductive quotient $T_{\mathrm{der}} = T \cap G_{\mathrm{der}}$, a split maximal torus in G_{der}. Let σ be a character of $T_{\mathrm{der}}(K)$. The component structure of the admissible representation $\mathrm{Ind}_{B_{\mathrm{der}}(K)}^{G_{\mathrm{der}}(K)} \sigma$ is governed by the commuting algebra $\mathrm{End}(\mathrm{Ind}_{B_{\mathrm{der}}(K)}^{G_{\mathrm{der}}(K)} \sigma)$, which, in turn, is given by the group algebra $\mathbb{C}[R(\sigma)]$, where $R(\sigma)$ is the Knapp–Stein R-group; see [Keys 1982, Introduction] for a summary and references to original sources, including [Silberger 1979].

The Knapp–Stein R-group $R(\sigma)$ is determined as follows, as explained in [Keys 1982, §3]. Let R be the root system for G with respect to T and let W be the corresponding Weyl group for G. The root system for G_{der} may be identified with R; see Table 1. Set $W_\sigma = \{w \in W \mid {}^w\sigma = \sigma\}$. For each root $\alpha \in R$, let σ_α be the restriction of σ to the rank-1 subtorus $T_\alpha \subseteq T$. Consider the root system $R_\sigma = \{\alpha \in R \mid \sigma_\alpha = 1\}$. Then $R(\sigma) = \{w \in W_\sigma \mid w(R_\sigma) = R_\sigma\}$. The exact sequence

$$1 \to W_\sigma^\circ \to W_\sigma \to R(\sigma) \to 1$$

determines $R(\sigma)$, with $W_\sigma^\circ := \{w_\alpha \mid \alpha \in R_\sigma\}$, the Weyl group of the root system R_σ; see [Keys 1982, §3].

We will need the following alternate characterization of $R(\sigma)$. Let $s \in \check{T}_{\mathrm{ad}}(\mathbb{C})$ be the semisimple element of $\check{G}_{\mathrm{ad}}(\mathbb{C})$ corresponding to the character σ of $T_{\mathrm{der}}(K)$. By Proposition 4 of [Steinberg 1974, §3.5] (see also [Humphreys 1995, §2.2, Theorem]), $Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s)$ is a reductive group with root system $\check{R}_s := \{\check{\alpha} \in \check{R} \mid \check{\alpha}(s) = 1\}$. The bijection between R and $\check{R}$ which comes with the root datum for G restricts to a bijection between R_σ and $\check{R}_s$. Moreover, by that same Proposition 4, the component group of the reductive group $Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s)$ is W_s / W_s°, where W_s° is the Weyl group for the root system $\check{R}_s$ and $W_s = \{w \in W \mid w(s) = s\}$:

$$1 \to W_s^\circ \to W_s \to \pi_0(Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s)) \to 1.$$

Here we have identified the Weyl group W for R with the Weyl group for $\check{R}$. Under that identification, $W_s = W_\sigma$ and $W_\sigma^\circ = W_s^\circ$, so

$$R(\sigma) \cong \pi_0(Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s)),$$

canonically,

2D. ***Component group calculations.*** Now we calculate the group $\pi_0(Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s))$.

Proposition 2.2. *Suppose $t \in \mathrm{GSp}_{2g}(\mathbb{C})$ is semisimple and all eigenvalues have complex modulus 1. Let $s \in \mathrm{PGSp}_{2g}(\mathbb{C})$ be the image of t under $\mathrm{GSp}_{2g}(\mathbb{C}) \to \mathrm{PGSp}_{2g}(\mathbb{C})$. Then, if and only if the characteristic polynomial of $r_\lambda(t)$ is even,*

$$\pi_0(Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s)) \cong \mathbb{Z}/2\mathbb{Z};$$

otherwise, $\pi_0(Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s))$ is trivial.

Proof. Using the notation in the proof of Proposition 2.1, pick $x \in X^*(T) \otimes \mathbb{R}$ with $\exp(x) = t$; of course, x is not uniquely determined by t, as the kernel of $\exp : X^*(T) \otimes \mathbb{R} \to \check{T}(\mathbb{C})$ is the weight lattice for T, which, in this case, is the character lattice $X^*(T)$ itself; see Table 1. Let $v \in X^*(T_{\mathrm{der}}) \otimes \mathbb{R}$ be the image of x under the map $X^*(T) \otimes \mathbb{R} \to X^*(T_{\mathrm{der}}) \otimes \mathbb{R}$ induced from $X^*(T) \to X^*(T_{\mathrm{der}})$; see Table 1. Note that $\exp(v) = s$, where now exp refers to the map $\exp : X^*(T_{\mathrm{der}}) \to \check{T}(\mathbb{C})$ defined as above. Using this map we may identify Lie $\check{T}_{\mathrm{ad}}(\mathbb{C})$ with $X^*(T_{\mathrm{der}}) \otimes \mathbb{C}$; under this identification, the Lie algebra of the compact subtorus of $\check{T}_{\mathrm{ad}}(\mathbb{C})$ may be identified with $X^*(T_{\mathrm{der}}) \otimes \mathbb{R}$, henceforth denoted by V.

Let R_{der} be the root system for G_{der} and let $\langle R_{\mathrm{der}} \rangle$ be the lattice generated by R_{der}. By [Reeder 2010, §2.2],

$$\pi_0(Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s)) \cong \{\gamma \in X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle \mid \gamma(v) = v\}, \tag{2-4}$$

for a canonical action of $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle$ on V, which we will use to calculate $\pi_0(Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s))$. Even before describing this action, however, we remark that (2-4), together with the calculation of $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle$ in Table 1, already gives us good information about $\pi_0(Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s))$: this component group is trivial or $\mathbb{Z}/2\mathbb{Z}$, and in particular, abelian.

In order to describe the action of $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle$ on V and calculate the right-hand side of (2-4), we must introduce yet more notation. Adapting [Bourbaki 1968, VI, §2], let $W_{\mathrm{aff}} := \langle R_{\mathrm{der}} \rangle \rtimes W$ be the affine Weyl group for $\check{G}_{\mathrm{ad}}$ and let $W_{\mathrm{ext}} := X^*(T_{\mathrm{der}}) \rtimes W$ be the extended affine Weyl group for $\check{G}_{\mathrm{ad}}$. (Here we use the coincidence of the weight lattice for G_{der} with the character lattice for G_{der}.) Then W_{ext} is a semidirect product of the Coxeter group W_{aff} by $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle$.

$$1 \to W_{\mathrm{aff}} \to W_{\mathrm{ext}} \to X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle \to 1. \tag{2-5}$$

The quotient $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle$ coincides with the fundamental group $\pi_1(\check{G}_{\mathrm{ad}})$ of $\check{G}_{\mathrm{ad}}$

Semisimple, simply connected		Type: B_g		Adjoint
$G_{\mathrm{der}} = \mathrm{Spin}_{2g+1}$ $T_{\mathrm{der}} = \mathbb{G}_m^g$ $Z(G_{\mathrm{der}}) = \mu_2$	$\rightarrowtail$ $\rightarrowtail$ $\rightarrowtail$	$G = \mathrm{GSpin}_{2g+1}$ $T = \mathbb{G}_m^{g+1}$ $Z(G) = \mathbb{G}_m$	$\twoheadrightarrow$ $\twoheadrightarrow$ $\twoheadrightarrow$	$G_{\mathrm{ad}} = \mathrm{SO}_{2g+1}$ $T_{\mathrm{ad}} = \mathbb{G}_m^g$ $Z(G_{\mathrm{ad}}) = 1$
$X^*(T_{\mathrm{der}}) = \langle e_1, \dots, e_g \rangle$	$0 \mapsfrom e_0$	$X^*(T) = \langle e_0, e_1, \dots, e_g \rangle$	$\leftarrowtail$	$X^*(T_{\mathrm{ad}}) = \langle \alpha_1, \dots, \alpha_g \rangle$
$R_{\mathrm{der}} := R(G_{\mathrm{der}}, T_{\mathrm{der}})$ $= \langle \alpha_1, \dots, \alpha_g \rangle$ $\alpha_1 = e_1 - e_2$ $\alpha_2 = e_2 - e_3$ $\vdots$ $\alpha_{g-1} = e_{g-1} - e_g$ $\alpha_g = e_g$		$R := R(G, T)$ $= \langle \alpha_1, \dots, \alpha_g \rangle$ $\alpha_1 = e_1 - e_2$ $\alpha_2 = e_2 - e_3$ $\vdots$ $\alpha_{g-1} = e_{g-1} - e_g$ $\alpha_g = e_g$		$R_{\mathrm{ad}} := R(G_{\mathrm{ad}}, T_{\mathrm{ad}})$ $= \langle \alpha_1, \dots, \alpha_g \rangle$
$X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle = \mathbb{Z}/2\mathbb{Z}$ weight lattice $= X^*(T_{\mathrm{der}})$		$X^*(T)/\langle R \rangle = \mathbb{Z}$ weight lattice $= X^*(T)$		$X^*(T_{\mathrm{ad}}) = \langle R_{\mathrm{ad}} \rangle$ $\dfrac{X^*(T_{\mathrm{ad}})}{\text{weight lattice}} = \mathbb{Z}/2\mathbb{Z}$
Adjoint		Type: C_g		Semisimple, simply connected
$\check{G}_{\mathrm{ad}} = \mathrm{PGSp}_{2g}$ $\check{T}_{\mathrm{ad}} = \mathbb{G}_m^g$ $Z(\check{G}_{\mathrm{ad}}) = 1$	$\twoheadleftarrow$ $\twoheadleftarrow$ $\twoheadleftarrow$	$\check{G} = \mathrm{GSp}_{2g}$ $\check{T} = \mathbb{G}_m^{g+1}$ $Z(\check{G}) = \mathbb{G}_m$	$\leftarrowtail$ $\leftarrowtail$ $\leftarrowtail$	$\check{G}_{\mathrm{der}} = \mathrm{Sp}_{2g}$ $\check{T}_{\mathrm{der}} = \mathbb{G}_m^g$ $Z(\check{G}_{\mathrm{der}}) = \mu_2$
$X^*(\check{T}_{\mathrm{ad}}) = \langle \check{\alpha}_1, \dots, \check{\alpha}_g \rangle$	$\rightarrowtail$	$X^*(\check{T}) = \langle f_0, f_1, \dots, f_g \rangle$	$f_0 \mapsto 0$	$X^*(\check{T}_{\mathrm{der}}) = \langle f_1, \dots, f_g \rangle$
$\check{R}_{\mathrm{ad}} := R(\check{G}_{\mathrm{ad}}, \check{T}_{\mathrm{ad}})$ $= \langle \check{\alpha}_1, \dots, \check{\alpha}_g \rangle$		$\check{R} := R(\check{G}, \check{T})$ $= \langle \check{\alpha}_1, \dots, \check{\alpha}_g \rangle$ $\check{\alpha}_1 = f_1 - f_2$ $\check{\alpha}_2 = f_2 - f_3$ $\vdots$ $\check{\alpha}_{g-1} = f_{g-1} - f_g$ $\check{\alpha}_g = 2f_g - f_0$		$\check{R}_{\mathrm{der}} := R(\check{G}_{\mathrm{der}}, \check{T}_{\mathrm{der}})$ $= \langle \check{\alpha}'_1, \dots, \check{\alpha}'_g \rangle$ $\check{\alpha}'_1 = f_1 - f_2$ $\check{\alpha}'_2 = f_2 - f_3$ $\vdots$ $\check{\alpha}'_{g-1} = f_{g-1} - f_g$ $\check{\alpha}'_g = 2f_g$
$X^*(\check{T}_{\mathrm{ad}}) = \langle \check{R}_{\mathrm{ad}} \rangle$ $\dfrac{X^*(\check{T}_{\mathrm{ad}})}{\text{weight lattice}} = \mathbb{Z}/2\mathbb{Z}$		$X^*(\check{T})/\langle \check{R} \rangle = \mathbb{Z}$ weight lattice $= \langle \check{R} \rangle$		$X^*(\check{T}_{\mathrm{der}})/\langle \check{R}_{\mathrm{der}} \rangle = \mathbb{Z}/2\mathbb{Z}$ weight lattice $= X^*(\check{T}_{\mathrm{der}})$

Table 1. Based root data for GSpin_{2g+1}, Spin_{2g+1} and SO_{2g+1}.

(see [Steinberg 1968, p. 45] for a table of these finite abelian groups by type). By [Bourbaki 1968, VI, §2.4, Corollary], the minuscule coweights for $\check{G}_{\mathrm{ad}}$ determine a set of representatives for $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}} \rangle$. The basis in Table 1 for the root system

$\check{R}_{\mathrm{ad}}$ determines the alcove

$$C := \{v \in V \mid \langle \check{\alpha}_i, v\rangle > 0, 0 \le i \le n\}$$

in V, where $\check{\alpha}_0$ is the affine root for which $1 - \check{\alpha}_0$ is the longest root with respect to the given basis for $\check{R}_{\mathrm{ad}}$; see [Bourbaki 1968, VI, §2.3]. The closure $\overline{C}$ of C is a fundamental domain for the action of W_{aff} on V. The affine Weyl group W_{aff} acts freely and transitively on the set of alcoves in V. The extended affine Weyl group W_{ext} acts transitively on the set of alcoves, but generally not freely. Since minuscule coweights for $\check{G}_{\mathrm{ad}}$ determine a set of representatives for $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}}\rangle$, and since each such coweight may be identified with a vertex of $\overline{C}$ (not all vertices arise this way), we have

$$\{w \in W_{\mathrm{ext}} \mid w(C) = C\} \cong X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}}\rangle, \tag{2-6}$$

canonically. This describes the action of $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}}\rangle$ on V.

The calculation of $\{\gamma \in X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}}\rangle \mid \gamma(v) = v\}$ now follows easily. Let

$$\{\varpi_1, \dots, \varpi_g\}$$

be the basis of weights for $X^*(T_{\mathrm{der}})$ dual to the basis $\check{R}_{\mathrm{ad}} = \{\check{\alpha}_1, \dots, \check{\alpha}_g\}$ for $X^*(\check{T}_{\mathrm{ad}}) = X_*(T_{\mathrm{der}})$; set $\varpi_0 = 0$. The closure $\overline{C}$ of the alcove C is the convex hull of the vertices $\{v_0, v_1, \dots, v_g\}$ defined by $v_j = (1/b_j)\varpi_j$, where $b_0 = 1$ and the other integers b_j are determined by the longest root in $\check{R}_{\mathrm{ad}}$ according to $\check{\alpha} = \sum_{j=1}^{g} b_j \check{\alpha}_j$. In the case at hand, the longest root is $\check{\alpha} = 2\check{\alpha}_1 + 2\check{\alpha}_2 + \cdots + 2\check{\alpha}_{g-1} + \check{\alpha}_g$, so $b_1 = 2, \dots, b_{g-1} = 2, b_g = 1$. Note that exactly two vertices in $\{v_0, v_1, \dots, v_g\}$ are hyperspecial: v_0 and v_g. Since W_{ext} acts transitively on the alcoves in V and since $\exp : V \to \check{T}_{\mathrm{ad}}(\mathbb{C})$ is W_{ext}-invariant, we may now suppose $v \in \overline{C}$. Express $v \in V$ in the basis of weights for $X^*(T_{\mathrm{der}})$:

$$v = \sum_{j=1}^{g} x_j \varpi_j; \tag{2-7}$$

note that the coefficients in this expansion are precisely the root values $x_j = \check{\alpha}_j(v)$. Then $v \in \overline{C}$ exactly means $x_j \ge 0$. Set $b_0 = 1$ and define $x_0 \ge 0$ so that $\sum_{j=0} b_j x_j = 1$; in other words,

$$v = \sum_{j=0}^{g} x_j \varpi_j, \quad x_0 + 2x_1 + \cdots + 2x_{g-1} + x_g = 1.$$

Under the isomorphism (2-6), the nontrivial element of $X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}}\rangle$ corresponds to $\rho \in W_{\mathrm{ext}}$ defined by $v_j \mapsto v_{g-j}$ for $j = 0, \dots, g$. In terms of the fundamental weights $\{\varpi_0, \varpi_1, \dots, \varpi_g\}$, this affine transformation is defined by $\varpi_j \mapsto \varpi_{g-j}$ for $j = 0, \dots, g$. Thus, $\{\gamma \in X^*(T_{\mathrm{der}})/\langle R_{\mathrm{der}}\rangle \mid \gamma(v) = v\}$ is nontrivial if and only if $\rho(v) = v$, which is to say,

$$x_j \geq 0, \quad j = 1, \dots, g \tag{2-8}$$

and

$$\begin{aligned} x_1 + \cdots + x_{g-1} + x_g &= \tfrac{1}{2}, \\ x_j = x_{g-j}, &\quad j = 1, \dots, g-1. \end{aligned}$$

It only remains to translate the conditions above into conditions on the eigenvalues of $t \in G(\mathbb{C})$. To do that we pass from root values $x_j = \langle \check{\alpha}_j, x\rangle$ to character values $y_j := \langle f_j, x\rangle$. Again using Table 1, we see that these conditions are equivalent to

$$y_1 \geq y_2 \geq \cdots \geq y_g \geq \tfrac{1}{2} y_0 \tag{2-9}$$

and

$$\begin{aligned} y_1 + y_g &= \tfrac{1}{2} + y_0, \\ y_j - y_{j+1} = y_{g-j} - y_{g-j+1}, &\quad j = 1, \dots, g-1. \end{aligned}$$

When combined, these last two conditions take a very simple form:

$$y_0 - y_j = \frac{1}{2} + y_{g-j+1}, \quad j = 1, \dots, g-1. \tag{2-10}$$

Finally, we calculate the characteristic polynomial of $r_\lambda(t)$. Observe that $r_\lambda(t) = r_\lambda(\exp(x)) = \exp(dr_\lambda(x))$, where $dr_\lambda : X^*(T) \to X^*(\mathbb{G}_{\boldsymbol{m}}^{2g})$ is given by (2-2). Set $t_j = e^{2\pi i y_j}$ for $j = 0, \dots, g$. Then constraint (2-10) is equivalent to

$$t_0 t_j^{-1} = -t_{g-j+1}, \quad j = 1, \dots, g-1. \tag{2-11}$$

The characteristic polynomial of $r_\lambda(t)$ is

$$P_{r_\lambda(t)}(T) := \prod_{j=1}^{g} (T - t_j) \prod_{j=1}^{g} (T - t_0 t_j^{-1}). \tag{2-12}$$

When combined with (2-11), it is clear that $P_{r_\lambda(t)}(T)$ is even:

$$\begin{aligned} P_{r_\lambda(t)}(T) &= \prod_{j=1}^{g} (T - t_j) \prod_{j=1}^{g} (T + t_{g-j+1}), \quad \text{(2-11)} \\ &= \prod_{j=1}^{g} (T - t_j) \prod_{i=1}^{g} (T + t_i), \quad j \mapsto g - j + 1 \\ &= \prod_{j=1}^{g} (T^2 - t_j^2). \end{aligned}$$

We have now seen that if $\pi_0(Z_{\mathrm{PGSp}_{2g}(\mathbb{C})}(s))$ is nontrivial, then $P_{r_\lambda(t)}(T)$ is even. To see the converse, suppose $P_{r_\lambda(t)}(T)$ (2-12) is even. Without loss of generality, we may assume the similitude factor t_0 is trivial. Then, after relabeling if necessary, the *symplectic* characteristic polynomial $P_{r_\lambda(t)}(T)$ is even if and only if it takes the form $P_{r_\lambda(t)}(T) = \prod_{j=1}^{g}(T^2 - r_j^2)$, with $r_j^{-1} = -r_{\sigma(j)}$ for some permutation σ of $\{1, \dots, g\}$. Since the roots are the eigenvalues of t, which are unitary by hypothesis, we can order them by angular components, as in (2-9), while replacing σ with the permutation $j \mapsto g - j + 1$, thus bringing us back to (2-10). This concludes the proof of Proposition 2.2. □

2E. *Restriction to the derived group.* In this section we show how to recognize when X/K is even through a simple property of the admissible representation $\pi_{X,\lambda}$ of $G(K)$.

Theorem 2.3. *Let X/K be an abelian variety of dimension g with good reduction and let λ be a polarization on X. The restriction of $\pi_{X,\lambda}$ from* $\mathrm{GSpin}_{2g+1}(K)$ *to* $\mathrm{Spin}_{2g+1}(K)$ *is reducible if and only if X is even.*

Proof. With reference to notation from the proof of Proposition 2.1, set $t = \exp(\theta)$ and let $s \in \check{T}_{\mathrm{ad}}$ be the image of t under $\check{T} \to \check{T}_{\mathrm{ad}}$. The restriction of $\pi_{X,\lambda}$ from $G(K)$ to $G_{\mathrm{der}}(K)$ decomposes into irreducible representations indexed by the component group $\pi_0(Z_{\check{G}_{\mathrm{ad}}(\mathbb{C})}(s))$. Indeed, the irreducible representations of $G_{\mathrm{der}}(K)$ that arise in this way are precisely the irreducible representations appearing in $\mathrm{Ind}_{B_{\mathrm{der}}(K)}^{G_{\mathrm{der}}(K)} \chi_{\mathrm{der}}$, where $B_{\mathrm{der}}(K)$ is a Borel subgroup containing $T_{\mathrm{der}}(K)$ and χ_{der} is the unramified quasicharacter of $T_{\mathrm{der}}(K)$ corresponding to $t_{\mathrm{ad}} \in \check{T}_{\mathrm{ad}}(\mathbb{C})$. The R-group for this unramified principal series representation is $\pi_0(Z_{\check{G}_{\mathrm{ad}}}(s))$. By Proposition 2.2, this group is either trivial or a group of order 2, so either $\pi_{X,\lambda}|_{G_{\mathrm{der}}(K)}$ is irreducible or contains two irreducible admissible representations; also by Proposition 2.2, the latter case occurs if and only if the characteristic polynomial $P_{X_0/\mathbb{F}_q}(T)$ is even, in which case X/K itself is even (Proposition 1.10). □

2F. *L-packet interpretation.* In this section we show how to recognize even abelian varieties over local fields through associated L-packets.

As discussed in Section 2A, every polarized abelian variety (X, λ) over K determines an ℓ-adic Galois representation $\rho_{X,\lambda,\ell} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$. Let W'_K be the Weil–Deligne group for K [Tate 1979, §4.1]. Let $\phi_{X,\lambda,\ell} : W'_K \to \mathrm{Gal}(\overline{K}/K) \to \mathrm{GSp}(V_\ell X, \langle\cdot,\cdot\rangle_\lambda)$ be the Weil–Deligne homomorphism obtained by applying [Deligne 1973, Theorem 8.2] to $\rho_{X,\lambda,\ell}$. We note that ${}^L G = \check{G}(\mathbb{C}) \rtimes W_K = \mathrm{GSp}_{2g}(\mathbb{C}) \times W_K$. Let

$$\phi_{X,\lambda} : W'_K \to \mathrm{Gal}(\overline{K}/K) \to {}^L G$$

be the admissible homomorphism determined by $\phi_{X,\lambda,\ell}$ and the basis for $V_\ell X \otimes_{\mathbb{Q}_\ell} \mathbb{C}$ identified in the proof of Proposition 2.1. The equivalence class of the admissible homomorphism $\phi_{X,\lambda}$ is the Langlands parameter for the polarized abelian variety

(X, λ) over K. We remark that this recipe is valid for all polarized abelian varieties over K, not just those of good reduction. But here we are interested in the case when X has good reduction, in which case $\rho_{X,\lambda}$ is unramified in the strongest sense: the local monodromy operator for the Langlands parameter $\phi_{X,\lambda}$ is trivial ($\phi_{X,\lambda}$ factors through $W'_K \to W_K$) and $\phi_{X,\lambda}$ is trivial on the inertia subgroup I_K of W_K.

Although the full local Langlands correspondence for $G = \mathrm{GSpin}_{2g+1}$ is not yet known, the part which pertains to unramified principal series representations is, allowing us to consider the L-packet $\Pi_{X,\lambda}$ for the Langlands parameter $\phi_{X,\lambda}$. Indeed, we have seen that this L-packet contains the equivalence class of $\pi_{X,\lambda}$, only.

Theorem 2.3 shows that we can detect when X is K-isogenous to its twist over the quadratic unramified extension of K by restricting $\pi_{X,\lambda}$ from $G(K)$ to $G_{\mathrm{der}}(K)$. On the Langlands parameter side, this restriction corresponds to post-composing $\phi_{X,\lambda}$ with ${}^LG \to {}^LG_{\mathrm{ad}}$. Let $\phi^{\mathrm{der}}_{X,\lambda}$ be the Langlands parameter for G_{der}/K defined by the diagram below and let $\Pi^{\mathrm{der}}_{X,\lambda}$ be the corresponding L-packet.

$$\begin{tikzcd} W'_K \arrow[rr, "\phi_{X,\lambda}"] \arrow[dr, dotted, "\phi^{\mathrm{der}}_{X,\lambda}"'] & & {}^LG \arrow[dl] \\ & {}^LG_{\mathrm{ad}} & \end{tikzcd} \tag{2-13}$$

Corollary 2.4. *Let X/K be an abelian variety of dimension g with good reduction and let λ be a polarization on X. The L-packet $\Pi^{\mathrm{der}}_{X,\lambda}$ for* $\mathrm{Spin}_{2g+1}(K)$ *has cardinality* 2 *exactly when X is even*; *otherwise*, *it has cardinality* 1.

Proof. This follows directly from the fact that the R-group for any representation in the restriction of $\pi_{X,\lambda}$ to $G_{\mathrm{der}}(K)$ coincides with the Langlands component group attached to $\phi^{\mathrm{der}}_{X,\lambda}$. (See [Ban and Goldberg 2012] for more instances of this coincidence.) Namely, equivalence classes of representations that live in $\Pi^{\mathrm{der}}_{X,\lambda}$ are parametrized by irreducible representations of the group

$$\mathcal{S}_{\phi^{\mathrm{der}}_{X,\lambda}} := Z_{\check{G}_{\mathrm{ad}}}(\phi^{\mathrm{der}}_{X,\lambda})/Z_{\check{G}_{\mathrm{ad}}}(\phi^{\mathrm{der}}_{X,\lambda})^0\ (Z\check{G}_{\mathrm{ad}})^{W_K}.$$

Since G_{der} is K-split, the action of W_K on $\check{G}_{\mathrm{ad}}$ is trivial, and since $\phi^{\mathrm{der}}_{X,\lambda}$ is unramified, $Z_{\check{G}_{\mathrm{ad}}}(\phi^{\mathrm{der}}_{X,\lambda}) = Z_{\check{G}_{\mathrm{ad}}}(t_{\mathrm{ad}})$, where $t_{\mathrm{ad}} = \phi^{\mathrm{der}}_{X,\lambda}(\mathrm{Fr}_q)$; thus,

$$\mathcal{S}_{\phi^{\mathrm{der}}_{X,\lambda}} = \pi_0(Z_{\check{G}_{\mathrm{ad}}}(t_{\mathrm{ad}})),$$

which is precisely the R-group for $\pi_{X,\lambda}|_{G_{\mathrm{der}}(K)}$ calculated in Theorem 2.3. $\square$

3. Concluding remarks

It is natural to ask how the story above extends to include abelian varieties X over local fields which do not necessarily have good reduction, keeping track of the relation between the ℓ-adic Tate module $T_\ell X$ and the associated Weil–Deligne representations, generalizing [Rohrlich 1994], and the corresponding L-packets. For this it would be helpful to know the full local Langlands correspondence for $\mathrm{GSpin}_{2g+1}(K)$, not just the part which pertains to unramified principal series representations. Since the full local Langlands correspondence for $\mathrm{GSpin}_{2g+1}(K)$ is almost certainly within reach by an adaptation of Arthur's work [2013] on the endoscopic classification of representations, following [Arthur 2004], we have, for the moment, postponed looking into such questions until Arthur's ideas have been adapted to general spin groups.

At the heart of this note we have used a very simple instance of what is, according to a conjecture of Arthur [1989], a very general phenomenon: the coincidence of Knapp–Stein R-groups with the component groups attached to Langlands parameters, sometimes known as Arthur R-groups, as in [Ban and Zhang 2005]. While most known cases of this coincidence appear or are summarized in [Ban and Goldberg 2012], as remarked at the end of the introduction to that paper, there is work remaining for general spin groups.

When some of these missing pieces are available, we intend to use the local results in this note to explore the connection between abelian varieties over number fields and global L-packets of automorphic representations of spin groups and general spin groups, generalizing the results of [Anandavardhanan and Prasad 2006, §2].

Acknowledgements

We thank V. Vatsal, P. Mezo and U. K. Anandavardhanan for helpful conversations, and the referee for a careful reading and useful suggestions.

References

[Anandavardhanan and Prasad 2006] U. K. Anandavardhanan and D. Prasad, "On the SL(2) period integral", *Amer. J. Math.* **128**:6 (2006), 1429–1453. MR 2008b:22014 Zbl 1117.11030

[Arthur 1989] J. G. Arthur, "Unipotent automorphic representations: conjectures", pp. 13–71 in *Orbites unipotentes et représentations, II: Groupes p-adiques et réels*, edited by M. Andler, Astérisque **171-172**, Société Mathématique de France, Paris, 1989. MR 91f:22030 Zbl 0728.22014

[Arthur 2004] J. G. Arthur, "Automorphic representations of GSp(4)", pp. 65–81 in *Contributions to automorphic forms, geometry, and number theory* (Baltimore, 2002), edited by H. Hida et al., Johns Hopkins University Press, Baltimore, 2004. MR 2005d:11074 Zbl 1080.11037

[Arthur 2013] J. G. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications **61**, American Mathematical Society, Providence, RI, 2013. MR 3135650 Zbl 06231010

[Ban and Goldberg 2012] D. Ban and D. Goldberg, "R-groups and parameters", *Pacific J. Math.* **255**:2 (2012), 281–303. MR 2928553 Zbl 1284.22008

[Ban and Zhang 2005] D. Ban and Y. Zhang, "Arthur R-groups, classical R-groups, and Aubert involutions for $\mathrm{SO}(2n+1)$", *Compos. Math.* **141**:2 (2005), 323–343. MR 2006d:22019 Zbl 1131.11033

[Bourbaki 1968] N. Bourbaki, *Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines*, Actualités Scientifiques et Industrielles **1337**, Hermann, Paris, 1968. Translated as *Lie groups and Lie algebras: chapters 4–6*, Springer, Berlin, 2002. MR 39 #1590 Zbl 0186.33001

[Deligne 1973] P. Deligne, "Les constantes des équations fonctionnelles des fonctions L", pp. 501–597 in *Modular functions of one variable, II* (Antwerp, 1972), edited by P. Deligne and W. Kuyk, Lecture Notes in Math. **349**, Springer, Berlin, 1973. MR 50 #2128 Zbl 0271.14011

[Humphreys 1995] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs **43**, American Mathematical Society, Providence, RI, 1995. MR 97i:20057 Zbl 0834.20048

[Keys 1982] C. D. Keys, "On the decomposition of reducible principal series representations of p-adic Chevalley groups", *Pacific J. Math.* **101**:2 (1982), 351–388. MR 84d:22032 Zbl 0438.22010

[Oort 2008] F. Oort, "Abelian varieties over finite fields", pp. 123–188 in *Higher-dimensional geometry over finite fields* (Göttingen, 2007), edited by D. Kaledin and Y. Tschinkel, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur. **16**, IOS, Amsterdam, 2008. MR 2009k:14085 Zbl 1181.14029

[Reeder 2010] M. Reeder, "Torsion automorphisms of simple Lie algebras", *Enseign. Math.* (2) **56**:1-2 (2010), 3–47. MR 2012b:17040 Zbl 1223.17020

[Rohrlich 1994] D. E. Rohrlich, "Elliptic curves and the Weil–Deligne group", pp. 125–157 in *Elliptic curves and related topics*, edited by H. Kisilevsky and M. R. Murty, CRM Proc. Lecture Notes **4**, American Mathematical Society, Providence, RI, 1994. MR 95a:11054 Zbl 0852.14008

[Serre and Tate 1968] J.-P. Serre and J. Tate, "Good reduction of abelian varieties", *Ann. of Math.* (2) **88** (1968), 492–517. MR 38 #4488 Zbl 0172.46101

[Silberger 1979] A. J. Silberger, *Introduction to harmonic analysis on reductive p-adic groups*, Mathematical Notes **23**, Princeton University Press, Tokyo, 1979. MR 81m:22025 Zbl 0458.22006

[Silverberg 1992] A. Silverberg, "Fields of definition for homomorphisms of abelian varieties", *J. Pure Appl. Algebra* **77**:3 (1992), 253–262. MR 93f:14022 Zbl 0808.14037

[Steinberg 1968] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, CT, 1968. MR 57 #6215 Zbl 1196.22001

[Steinberg 1974] R. Steinberg, *Conjugacy classes in algebraic groups*, Lecture Notes in Math. **366**, Springer, Berlin, 1974. MR 50 #4766 Zbl 0281.20037

[Tate 1966] J. Tate, "Endomorphisms of abelian varieties over finite fields", *Invent. Math.* **2** (1966), 134–144. MR 34 #5829 Zbl 0147.20303

[Tate 1979] J. Tate, "Number theoretic background", pp. 3–26 in *Automorphic forms, representations and L-functions* (Corvallis, OR, 1977), vol. 2, edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, American Mathematical Society, Providence, RI, 1979. MR 80m:12009 Zbl 0422.12007

[Waterhouse 1969] W. C. Waterhouse, "Abelian varieties over finite fields", *Ann. Sci. École Norm. Sup.* (4) **2** (1969), 521–560. MR 42 #279 Zbl 0188.53001

[Yu 2009] J.-K. Yu, "On the local Langlands correspondence for tori", pp. 177–183 in *Ottawa lectures on admissible representations of reductive p-adic groups*, edited by C. Cunningham and M. Nevins, Fields Inst. Monogr. **26**, American Mathematical Society, Providence, RI, 2009. MR 2009m:11201 Zbl 1187.11045

Received February 21, 2014. Revised July 25, 2014.

JEFFREY D. ACHTER
COLORADO STATE UNIVERSITY
WEBER BUILDING
FORT COLLINS, CO 80523-1874
UNITED STATES
achter@math.colostate.edu

CLIFTON CUNNINGHAM
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF CALGARY
2500 UNIVERSITY DRIVE NW
CALGARY, AB T2N 1N4
CANADA
cunning@math.ucalgary.ca

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.413

ROTATING DROPS WITH HELICOIDAL SYMMETRY

BENNETT PALMER AND OSCAR M. PERDOMO

It is known that, if we ignore gravitational forces, the shape of an equilibrium drop in $\mathbb{R}^3$ rotating about the z-axis is a surface that satisfies the equation $2H = \Lambda_0 - \frac{1}{2}aR^2$, where H is the mean curvature and R is the distance from a point in the surface to the z-axis. We consider helicoidal immersions in $\mathbb{R}^3$ that satisfy the rotating drop equation. We prove the existence of properly immersed solutions that contain the z-axis. We also show the existence of several families of embedded examples. We describe the set of possible solutions and we show that most of these solutions are not properly immersed and are dense in the region bounded by two concentric cylinders. We show that all properly immersed solutions, besides being invariant under a one-parameter helicoidal group, are invariant under a cyclic group of rotations of the variables x and y.

The second variation of energy for the volume constrained problem with Dirichlet boundary conditions is also studied.

1. Introduction

In this paper we study the equilibrium shape of a rotating liquid drop or liquid film which is invariant under a helicoidal motion of the three dimensional Euclidean space. The subject of rotating drops has been studied by many authors, including Chandrasekhar [1965], Brown and Scriven [1980], Solonnikov [2004]. Our main objective here is to use a new construction, recently developed in [Perdomo 2012], to construct an abundant supply of examples. This construction is closely related to Delaunay's classical construction of the axially symmetric constant mean curvature surfaces whose generating curves are produced by rolling a conic section. A special case of the type of surface which we study here occurs when the rotating drop is a cylinder over a plane curve. We treat that case in detail in [Palmer and Perdomo 2014].

If a rigid object is moved from one position in space to another, this repositioning can be realized via a helicoidal motion of $\mathbb{R}^3$. If this motion is then successively repeated, one arrives at a configuration which is invariant under a helicoidal motion.

MSC2000: 53C43, 53C42, 53C10.
Keywords: rotating drops, mean curvature, helicoidal surfaces.

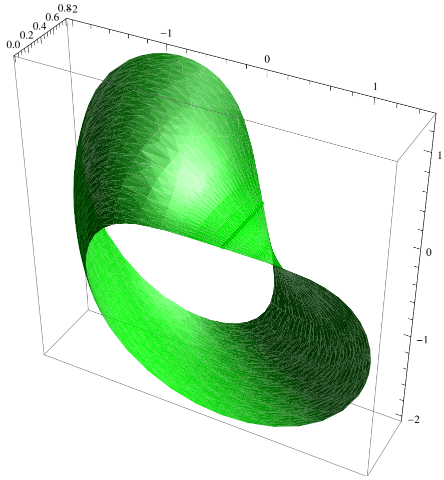

Figure 1. A rotating helicoidal drop.

The simple idea that helicoidal motions give all repeated motions of a rigid object, known in the physical sciences as Pauling's theorem [Cahill 2005; Pauling et al. 1951], is behind many of the occurrences of helicoidal symmetry in nature [Barros and Ferrández 2009], since it allows for extensive growth with a minimal amount of information.

We will consider the equilibrium shape of a liquid drop rotating in a zero gravity environment with a constant angular velocity Ω about a vertical axis. The surface of the drop, which we denote by Σ, is represented as a smooth surface. The bulk of the drop is assumed to be occupied by an incompressible liquid of a constant mass density ρ_1, while the drop is surrounded by a fluid of constant mass density ρ_2. Since the drop is liquid, its free surface energy is proportional to its surface area $\mathcal{A}$, and we take the constant of proportionality to be one. The downward gravitational force is neglected. This is justified if the volume of the drop is sufficiently small compared to the other parameters. The rotation contributes a second energy term of the form $-\Omega^2\Delta\mathcal{I}$, where $\Delta\mathcal{I}$ is difference of moments of inertia about the vertical axis:

$$\Delta\mathcal{I} := (\rho_1-\rho_2)\int_U R^2\,dv.$$

This term represents twice the rotational kinetic energy.

The total energy is thus of the form

$$\mathcal{E} := \mathcal{A} - \frac{\Omega^2}{2}\Delta\mathcal{I} + \Lambda_0\mathcal{V}, \tag{1-1}$$

where $\mathcal{V}$ denotes the volume of the drop and Λ_0 is a Lagrange multiplier. Let $\Delta\rho := \rho_1-\rho_2$; then by introducing a constant $a := (\Delta\rho)\Omega^2$, we can write the functional in the form

$$\mathscr{E}_{a,\Lambda_0} = \mathscr{A} - \frac{a}{2}\int_U R^2\, dV + \Lambda_0 \mathscr{V},$$

where U is the three-dimensional region occupied by the bulk of the drop and $R := \sqrt{x_1^2 + x_2^2}$.

Since we want to consider both embedded and immersed surfaces, we will precisely define the last two terms in the energy in the following way. First, define vector fields on $\mathbb{R}^3$ by

$$W = \nabla' \frac{R^4}{16} = \frac{R^2}{4}(x_1, x_2, 0), \qquad W_0 = \nabla' \frac{R^2}{4} = \tfrac{1}{2}(x_1, x_2, 0).$$

If $\nabla' \cdot$ denotes the divergence operator on $\mathbb{R}^3$, then it is easily checked that $\nabla' \cdot W_0 = 1$ and $\nabla' \cdot W = R^2$ hold. We then define

$$\mathscr{V} := \int_\Sigma W_0 \cdot \nu\, d\Sigma, \qquad \int_U R^2\, dv := \int_\Sigma W \cdot \nu\, d\Sigma.$$

The definitions are valid as long as Σ is immersed and oriented.

We will next derive the first variation of the functional given above. Let

$$X_\epsilon = X + \epsilon(\psi\nu + T) + \cdots$$

be a variation of X, where ψ is a smooth function, ν is the unit normal to the surface and T is a tangent vector field along Σ. The first variation formula for the area gives

$$\delta\mathscr{A} = -\int_\Sigma 2H\psi\, d\Sigma + \oint_{\partial\Sigma} T \cdot n\, ds = -\int_\Sigma 2H\psi\, d\Sigma + \oint_{\partial\Sigma} dX \times \nu \cdot T.$$

We will show in the Appendix that

$$\delta \int_\Omega R^2\, dV = \int_\Sigma \psi R^2\, d\Sigma + \oint_{\partial\Sigma} dX \times W \cdot \delta X, \tag{1-2}$$

where W is a vector field satisfying $\nabla' \cdot W = R^2$ on $\mathbb{R}^3$, and it is well known that the first variation of volume is

$$\delta\mathscr{V} = \int_\Sigma \psi\, d\Sigma + \oint_{\partial\Sigma} dX \times W_0 \cdot \delta X. \tag{1-3}$$

By combining the last three formulas, we arrive at

$$\delta\mathscr{E}_{a,\Lambda_0} = \int_\Sigma \Big(-2H - \frac{a}{2}R^2 + \Lambda_0\Big)\psi\, d\Sigma + \oint_{\partial\Sigma_1} dX \times \Big(\nu - \frac{a}{2}W + \Lambda_0 W_0\Big) \cdot \delta X. \tag{1-4}$$

Regardless of the boundary conditions, a necessary condition for an equilibrium is that the equation

$$2H = -\frac{a}{2}R^2 + \Lambda_0 \tag{1-5}$$

holds in the interior of Σ.

If we assume that the surface has free boundary contained in a supporting surface S having outward unit normal N, then the admissible variations must satisfy the condition $\delta X \cdot N \equiv 0$ on $\partial\Sigma$. In order for the boundary integral in (1-4) to vanish for all admissible variations, $dX \times \nu$ must be parallel to N along the boundary, which means that the surface Σ meets the supporting surface S in a right angle.

We now assume that an equilibrium surface Σ, i.e., a surface satisfying (1-5), is invariant under a helicoidal motion

$$(x_1 + ix_2, x_3) \mapsto (e^{-i\omega t}(x_1 + ix_2), x_3 + t), \tag{1-6}$$

and we will derive a conservation law which characterizes the equilibrium surfaces. We do not assume that the angular velocity ω which determines the pitch of the helicoidal surface is the same as the angular velocity Ω appearing above.

Let Σ_1 denote the compact region in Σ bounded on the sides by two integral curves C_1 and C_2 of the Killing field

$$\mathcal{K}(X) := -\omega E_3 \times X + E_3$$

and bounded below and above by the horizontal planes $x_3 = 0$ and $x_3 = 2\pi/\omega$. Then Σ_1 is a compact surface with oriented boundary $C_1 + \alpha_1 - C_2 - \alpha_2$, where α_1 and α_2 are congruent arcs in the planes $x_3 = 2\pi/\omega$ and $x_3 = 0$, respectively. By the calculations in the Appendix, we have, using (1-5),

$$\delta\mathscr{E}[\Sigma_0] = \oint_{\partial\Sigma_1} dX \times \left(\nu - \frac{a}{2}W + \Lambda_0 W_0\right) \cdot \delta X. \tag{1-7}$$

If we take the variation with $\delta X = E_3$ then, since E_3 generates a translation, the first variation will vanish. Consequently, we obtain

$$0 = \oint_{\partial\Sigma_1} dX \times \left(\nu - \frac{a}{2}\nabla' \frac{R^4}{16} + \Lambda_0 \nabla' \frac{R^2}{4}\right) \cdot E_3. \tag{1-8}$$

Note that the integration over the 1-chain $\alpha_1 - \alpha_2$ yields zero since the two arcs are congruent and are traversed in opposite directions. On C_i, $i = 1, 2$, we have

$\dfrac{dX}{dt} = (-1)^{i+1}\mathcal{K}$. Hence

$$
\begin{aligned}
dX \times \left(\nu - \frac{a}{2}\nabla' \frac{R^4}{16} + \Lambda_0 \nabla' \frac{R^2}{4}\right) \cdot E_3 \\
&= (-1)^{i+1}(-\omega E_3 \times X + E_3) \times \left(\nu - \frac{a}{2}\nabla' \frac{R^4}{16} + \Lambda_0 \nabla' \frac{R^2}{4}\right) \cdot E_3 \, dt \\
&= (-1)^{i}(-\omega E_3 \times X + E_3) \times E_3 \cdot \left(\nu - \frac{a}{2}\nabla' \frac{R^4}{16} + \Lambda_0 \nabla' \frac{R^2}{4}\right) dt \\
&= (-1)^{i+1}\omega(x_1, x_2, 0) \cdot \left(\nu - \frac{a}{8}R^2(x_1, x_2, 0) + \frac{\Lambda_0}{2}(x_1, x_2, 0)\right) dt \\
&= (-1)^{i+1}\omega\left((Q - x_3\nu_3) - \frac{a}{8}R^4 + \frac{\Lambda_0 R^2}{2}\right) dt,
\end{aligned}
$$

where $Q = X \cdot \nu$ is the support function of the surface. Setting $\hat{Q} := Q - x_3\nu_3$, we can conclude from this that the integral

$$\int_{C_i} \left(\hat{Q} - \frac{a}{8}R^4 + \frac{\Lambda_0 R^2}{2}\right) dt$$

is independent of i. Also, it is easily checked that the integrand is, in fact, constant on each helix C_i, and we obtain the result that

$$2\hat{Q} + \Lambda_0 R^2 - a\frac{R^4}{4} \equiv \text{constant}. \tag{1-9}$$

Proposition 1.1. *Let Σ be a helicoidal surface. A necessary and sufficient condition that Σ is a critical point for the functional $\mathscr{E}_{a,\Lambda_0}$ is that* (1-9) *holds.*

Proof. The necessity was shown above, so we now show that the condition is sufficient. We can assume that the helicoidal symmetry group of the surface fixes the vertical axis.

Any helicoidal surface arises as the orbit of a planar "generating curve" α under a helicoidal motion. We let s be the arc-length coordinate of α and we let t denote a coordinate for the helices which are the orbits of points in α. Local calculations which can be found in [Perdomo 2012] show that the mean curvature H and the third component of the normal ν_3 are functions of s alone. Also, it is clear that the function R^2 only depends on s.

It is easy to see that if ν_3 vanishes on any arc of α, then this arc is necessarily circular. It is clear that the orbit of a circular arc satisfying (1-9) is an equilibrium surface for the functional $\mathscr{E}_{a,\Lambda_0}$. Now consider a connected arc $\eta \subset \alpha$ on which (say) $\nu_3 > 0$ holds almost everywhere. If (1-5) does not hold on α, we can assume, by replacing α with a subarc if necessary, that $-2H - aR^2/2 + \Lambda_0 > 0$ also holds almost everywhere on α. Let Σ_1 denote the compact domain consisting of the orbit of the arc α for $0 \le t \le 2\pi/\omega$. The boundary of Σ_1 consists of two helices C_1, C_2 together with two arcs α_1, α_2, both congruent to α.

We take the first variation of $\mathscr{E}_{a,\Lambda_0}$ with the variation field being the constant vector E_3. Since E_3 is the generator of a one-parameter family of isometries, this first variation vanishes. We express the first variation as in (1-4). Taking into account (1-9), the contribution to the boundary integral is zero since it is given by the right-hand side of (1-8), which vanishes. Also, the integrals over α_1 and α_2 cancel each other since these arcs are congruent and are traversed in opposite directions. We then obtain from the calculations given above that

$$0 = \int_{\Sigma_1} \bigl(-2H - \tfrac{1}{2}aR^2 + \Lambda_0\bigr)\nu_3 \, d\Sigma,$$

which is a contradiction since the integrand is positive almost everywhere on Σ_1. $\square$

This result can easily be modified for axially symmetric surfaces. In that case, the Killing field used is simply $E_3 \times X$, and the helices are replaced by circles and (1-9) still holds.

2. Treadmill sled coordinates analysis

We will be considering immersions of the form

$$\phi(s,t) = \bigl(x(s)\cos\omega t + y(s)\sin\omega t,\ -x(s)\sin\omega t + y(s)\cos\omega t,\ t\bigr),$$

with the curve $\alpha(s) = (x(s), y(s))$ parametrized by arc length. We will refer to the curve α as the profile curve of the surface. The surface given as the image of ϕ is the orbit of α under the helicoidal motion (1-6). For $\theta(s)$ defined by

$$x'(s) = \cos\theta(s) \quad \text{and} \quad y'(s) = \sin\theta(s),$$

following [Perdomo 2012] we define the *treadmill sled coordinates* $\xi_1(s)$, $\xi_2(s)$ by

(2-1) $\xi_1(s) = x(s)\cos\theta(s) + y(s)\sin\theta(s), \qquad \xi_2(s) = x(s)\sin\theta(s) - y(s)\cos\theta(s).$

The Gauss map of the immersion ϕ is given by

$$\nu = \frac{1}{\sqrt{1+w^2\xi_1^2}}(\sin(\theta-\omega t), -\cos(\theta-\omega t), -\omega\xi_1),$$

and so, by a direct calculation, we obtain $\hat{Q} = \xi_2/\sqrt{1+\omega^2\xi_1^2}$. Finally, using that $R^2 = x^2 + y^2 = \xi_1^2 + \xi_2^2$, we see from (1-9) that the immersion ϕ represents a rotating helicoidal drop if and only if there holds

$$G(\xi_1,\xi_2) := \frac{2\xi_2}{\sqrt{1+\omega^2\xi_1^2}} + \Lambda_0(\xi_1^2+\xi_2^2) - \frac{a}{4}(\xi_1^2+\xi_2^2)^2 \equiv \text{constant} =: C. \tag{2-2}$$

A direct computation shows that the equation $2H = \Lambda_0 - \frac{1}{2}aR^2$ reduces to

$$\theta'(s) = \frac{2w^2\xi_2 - 2\Lambda_0(1+w^2\xi_1^2)^{3/2} + a(\xi_1^2+\xi_2^2)(1+w^2\xi_1^2)^{3/2}}{1+w^2(\xi_1^2+\xi_2^2)}. \tag{2-3}$$

From the definitions of ξ_1, ξ_2 and θ, we get that $\xi_1' = 1 - \xi_2\theta'$ and $\xi_2' = \xi_1\theta'$. Using (2-3), we conclude that ξ_1 and ξ_2 must satisfy

(2-4)
$$\xi_1' = f_1(\xi_1,\xi_2) = \frac{(1+w^2\xi_1^2)\big(2+\sqrt{1+w^2\xi_1^2}\,\xi_2(2\Lambda_0 - a(\xi_1^2+\xi_2^2))\big)}{2(1+w^2(\xi_1^2+\xi_2^2))},$$

(2-5)
$$\xi_2' = f_2(\xi_1,\xi_2) = \frac{\xi_1\left(2w^2\xi_2 - 2\Lambda_0(1+w^2\xi_1^2)^{3/2} + a(\xi_1^2+\xi_2^2)(1+w^2\xi_1^2)^{3/2}\right)}{2(1+w^2(\xi_1^2+\xi_2^2))}.$$

This system of ordinary differential equations for ξ_1 and ξ_2 provides a different proof of the fact that the $G(\xi_1(s), \xi_2(s))$ must be constant, because we can check directly that

$$\frac{\partial G}{\partial \xi_1} = -\frac{2+2w^2(\xi_1^2+\xi_2^2)}{(1+w^2\xi_1^2)^{3/2}} f_2 \quad \text{and} \quad \frac{\partial G}{\partial \xi_2} = \frac{2+2w^2(\xi_1^2+\xi_2^2)}{(1+w^2\xi_1^2)^{3/2}} f_1.$$

Remark 2.1. The level sets of G are symmetric with respect to the ξ_2-axis. Therefore, in order to understand the level set of G, it is enough to understand those points in the level set with $\xi_1 \geq 0$.

In order to study the level sets of the function G we replace the variables ξ_1 and ξ_2 with the variables r and ξ_2, where

$$r = \xi_1^2 + \xi_2^2.$$

Making this change, we obtain that the equation $G = C$ reduces to

$$\frac{2\xi_2}{\sqrt{1+\omega^2 r - \omega^2\xi_2^2}} + \Lambda_0 r - \frac{a}{4}r^2 = C.$$

In fact, this equation is exactly the one appearing in (1-9).

We have

$$\xi_2 = \frac{(4C + r(-4\Lambda_0 + ar))\sqrt{1+r\omega^2}}{\sqrt{64 + (4C + r(-4\Lambda_0 + ar))^2\,\omega^2}}.$$

By Remark 2.1, it is enough to consider those points with $\xi_1 \geq 0$. Since $\xi_1 = \sqrt{r - \xi_2^2}$, we get that

$$\xi_1 = \frac{\sqrt{p(r,a,\Lambda,C)}}{\sqrt{64 + (4C + r(-4\Lambda_0 + ar))^2\,\omega^2}},$$

where

$$p(r,a,\Lambda,C) = -16C^2+64r+32C\Lambda_0 r-16\Lambda_0^2r^2-8aCr^2+8a\Lambda_0 r^3-a^2r^4. \tag{2-6}$$

Remark 2.2. Since $p(r,a,\Lambda,C)$ is a polynomial in r of degree 4 with negative leading coefficient when $a\neq 0$ and p is a polynomial of degree two when $a=0$, we get that the values of r for which $p(r,a,\Lambda,C)$ is positive are bounded. Since $r=\xi_1^2+\xi_2^2=x^2+y^2$, we conclude that the profile curve of any helicoidal rotating drop is bounded.

Definition 2.3. Let r_1 and r_2 be nonnegative values such that $p(r_1)=p(r_2)=0$ and $p(r)>0$ for all $r\in(r_1,r_2)$. We define $\rho:[r_1,r_2]\to\mathbb{R}^2$ by

$$\rho(r)=\left(\frac{\sqrt{p(r,a,\Lambda,C)}}{\sqrt{64+(4C+r(-4\Lambda_0+ar))^2\,\omega^2}},\ \frac{(4C+r(-4\Lambda_0+ar))\sqrt{1+r\omega^2}}{\sqrt{64+(4C+r(-4\Lambda_0+ar))^2\,\omega^2}}\right).$$

Remark 2.4. As pointed out before, all the level sets of the function G are bounded. The map ρ parametrizes half of the level set $G=C$.

Definition 2.5. In the case that the level set $G=C$ is a regular closed curve or a union of regular closed curves, we define a fundamental piece of the profile curve as a simple connected part of the profile curve such that the parametrized curve (ξ_1,ξ_2), given by (2-1), corresponds to exactly one closed curve in the level set of $G=C$.

Remark 2.6. From the definition of treadmill sled given in [Perdomo 2012], we obtain that the profile curve of the solutions of the helicoidal rotational drop equation are characterized by the property that their treadmill sleds are the level sets of G. In other words, using the notation of [ibid.], we have $TS(\alpha)=\beta$, where β is a parametrization of a connected component of the level set of $G=C$ and α is the profile curve of the helicoidal rotating drop. We will see that, for a few exceptional examples, the profile curve is a bounded complete curve having a circle as a limit cycle. For the nonexceptional examples we can define an initial and final point of the fundamental piece, and the whole profile curve is the union of rotated fundamental pieces. If $R_1=\min\{|m|:m\in TS(\alpha)\}$ and $R_2=\max\{|m|:m\in TS(\alpha)\}$ and $\Delta\tilde\theta$ is the variation of the angle between $\overrightarrow{0p_1}$ and $\overrightarrow{0p_2}$, where p_1 and p_2 are the initial and final points of a fundamental piece, then, the profile curve is properly immersed if $\Delta\tilde\theta/\pi$ is a rational number, otherwise the profile curve is dense in the set $\{(x,y)\in\mathbb{R}^2 : R_1\le|(x,y)|\le R_2\}$.

We compute the variation $\Delta\tilde\theta$ in terms of the parameter r. We assume that $\alpha(s)$ is the profile curve of a helicoidal rotational drop. Recall that we are assuming that s is the arc-length parameter for the curve α. If $\beta(s)=(\xi_1(s),\xi_2(s))$, then

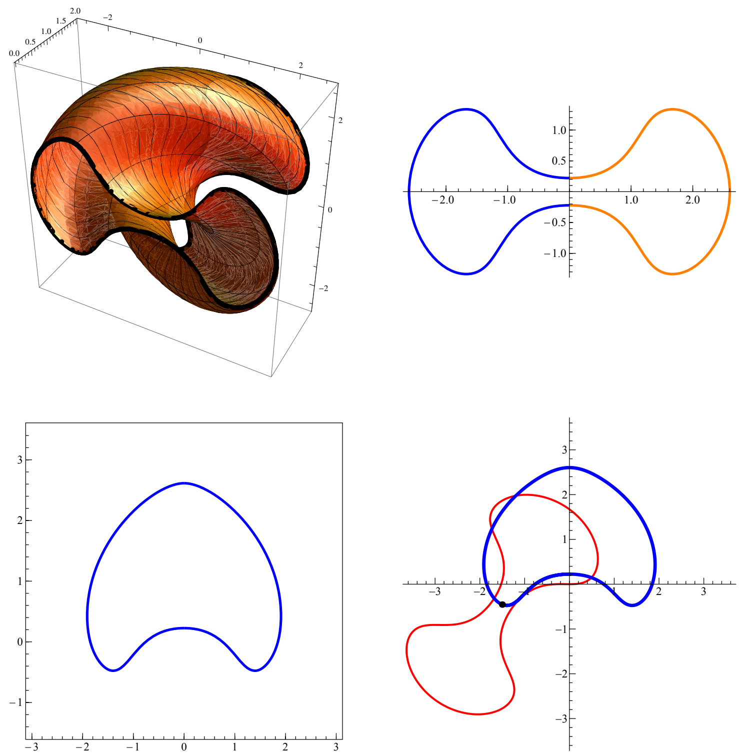

Figure 2. Top left: a helicoidal rotational drop. Top right: the corresponding profile curve, emphasizing the fundamental piece. Here $c = 0.5$, $a = 1$, $\Lambda_0 = 1$ and $\omega = 1.1805$. Bottom left: the (algebraic) level set $G = C$, i.e., the solution set to the equation

$$\frac{2\xi_2}{\sqrt{1+1.1805\xi_1^2}} + (\xi_1^2+\xi_2^2) - \tfrac{1}{4}(\xi_1^2+\xi_2^2)^2 = \tfrac{1}{2}.$$

Bottom right: how treadmill sled of the profile curve produces the level set $G = C$. In this example the treadmill sled of the profile curve will go over the level set $G = C$ two times.

$\beta(\sigma(r)) = \rho(r)$ for some function $s = \sigma(r)$. By the chain rule, we have

$$\frac{ds}{dr}Y = \frac{d\sigma}{dr} = \frac{|\rho'(s)|}{|\beta'(s)|} = \sqrt{\frac{|\rho'(r)|^2}{f_1^2(s)+f_2^2(s)}} \tag{2-7}$$
$$= \frac{1}{2}\sqrt{\frac{64+(4C+r(-4\Lambda_0+ar))^2\omega^2}{p(r,a,C)}}.$$

If $\tilde\theta$ denotes the polar angle of the profile curve, that is, if $\tilde\theta(s)$ satisfies the equation

$\alpha(s) = (x(s), y(s)) = R(s)(\cos\tilde{\theta}(s), \sin\tilde{\theta}(s))$, then $\tilde{\theta}'(s) = \xi_2(s)/r$, and

$$\frac{d\tilde{\theta}}{dr} = \frac{d\tilde{\theta}}{ds}\frac{ds}{dr} = \frac{1}{2}\frac{(4C + ar^2 - 4\Lambda_0 r)\sqrt{1+r\omega^2}}{r\sqrt{p(r,a,C)}}. \tag{2-8}$$

Since the map $\rho : [r_1, r_2] \to \mathbb{R}^2$ parametrizes half of the treadmill sled of the fundamental piece of the profile curve, we obtain the following expression for function $\Delta\tilde{\theta}$ defined in Remark 2.6:

$$\Delta\tilde{\theta} = \Delta\tilde{\theta}(C, a, \omega, r_1, r_2) = \int_{r_1}^{r_2} \frac{(4C + ar^2 - 4\Lambda_0 r)\sqrt{1+r\omega^2}}{r\sqrt{p(r,a,C)}}\,dr. \tag{2-9}$$

Remark 2.7. If we have a helicoidal rotational drop Σ and we multiply every point by a positive fixed number λ, that is, if we consider the surface $\lambda\Sigma$, then this new surface satisfies the equation of the rotating drop for some other values of Λ_0 and a. If we change the orientation of the profile curve of a surface Σ that satisfies the equation of the rotating drop with values Λ_0, a and H, then the reparametrized surface satisfies the equation with values $-\Lambda_0$, $-a$ and $-H$. With these two observations in mind, in order to consider all the helicoidal rotational drops, up to parametrizations, rigid motions and dilations, it is enough to consider two cases: (I) $\Lambda_0 = 0$ and $a = -1$, and (II) $\Lambda_0 = 1$ and a is any real number.

Case I: $\boldsymbol{\Lambda_0 = 0}$ and $\boldsymbol{a = -1}$. In this case, the polynomial $p(r, a, \Lambda, C)$ reduces to

$$q = q(r, C) = -16C^2 + 64r + 8Cr^2 - r^4.$$

Recall that we are interested in finding two consecutive positive roots of the polynomial q. Notice that when C is a negative large number then the polynomial q has no roots, and when C is a positive large number then the polynomial q has more than one root. In every case, $q(0) = -16C^2 \le 0$, and the limit when $r \to \infty$ of $q(r)$ is negative infinity. The following lemma was proven in [Palmer and Perdomo 2014] and provides the number of possible roots of $q(r, C)$ in terms of the values of C. For completeness reasons we will present the proof in this paper as well.

Lemma 2.8. *For any $C > C_0 = -3/2^{2/3}$, the polynomial $q(r, C)$ has exactly two nonnegative real roots. When $C = C_0$, $\sqrt[3]{4}$ is the only real root of $q(r, C)$, and when $C < C_0$, $q(r, C)$ has no real roots.*

Proof. We have $q'(r) = 64 + 16Cr - 4r^3$. A direct computation shows that the only real solution of the system

$$q(r, C) = 0 \quad \text{and} \quad q'(r, C) = 0$$

is $C = -3/2^{2/3}$ and $r = 2^{2/3}$. This also follows from the fact that a Gröbner basis of the polynomials $\{q, q'\}$ is the set $\{27+4C^3, -4C^2+9r\}$. Since

$$q''(2^{2/3}) = -48\sqrt[3]{2} < 0,$$

the polynomial q has either 0, 1 or 2 real roots for values of (C, r) near $(-3/\sqrt[3]{4}, \sqrt[3]{4})$. A direct computation shows that the only roots of q when $C = 0$ are $r = 0$ and $r = 4$, and that q has no real roots when $C = -2$. By continuity, we conclude that the lemma holds. Recall that $q(0) = -16C^2$ and $q(r) \to -\infty$ as $r \to \infty$. Notice that, if for some value of C the polynomial $q(r, C)$ has more than 2 roots, there should exist another solution of the equations $\{q(r, C) = 0,\ q'(r, C) = 0\}$, which is impossible. □

Now we will compute the limit of $\Delta\tilde{\theta}$ when C goes to C_0. We will use the following lemma from [Perdomo 2010].

Lemma 2.9. *Let $f(c, r)$ and $g(r, c)$ be smooth functions such that*

$$g(r_0, C_0) = \frac{\partial g}{\partial r}(r_0, C_0) = 0 \quad \textit{and} \quad \frac{\partial^2 g}{\partial r^2}(C_0, r_0) = -2A,$$

where $A > 0$. If $\{C_n\}$, $\{u_n\}$ and $\{v_n\}$ are sequences such that

$$C_n \to C_0, \quad u_n, v_n \to r_0,$$

with

$$u_n < r_0 < v_n, \quad g(u_n, c_n) = g(v_n, c_n) = 0 \quad \textit{and} \quad g(r, c_n) > 0 \textit{ for all } r \in (u_n, v_n),$$

then

$$\int_{u_n}^{v_n} \frac{f(c, r)\, dr}{\sqrt{g(c, r)}} \longrightarrow f(C_0, r_0)\frac{\pi}{\sqrt{A}} \quad \textit{as} \quad n \longrightarrow \infty.$$

Notice that helicoidal rotating drops are defined when C takes values from $C_0 = -3/\sqrt[3]{4}$ to ∞. When $C = C_0$, the only root of the polynomial q is $r_0 = \sqrt[3]{4}$. If we apply Lemma 2.9 with

$$f(r, c) = \frac{(4C - r^2)\sqrt{1 + r\omega^2}}{r} \quad \text{and} \quad g(r, c) = q(r, C)$$

to the integral given in (2-9), we obtain that

$$\lim_{C \to C_0^+} \Delta\tilde{\theta} = B(\omega) = -\frac{2\pi}{\sqrt{3}}\sqrt{1 + \sqrt[3]{4}\omega^2}. \tag{2-10}$$

Remark 2.10. Recall that whenever $\Delta\tilde{\theta} = n2\pi/m$ for some pair of integers m and n, then the entire profile curve is properly immersed and it is invariant under the group $\mathbb{Z}_m$.

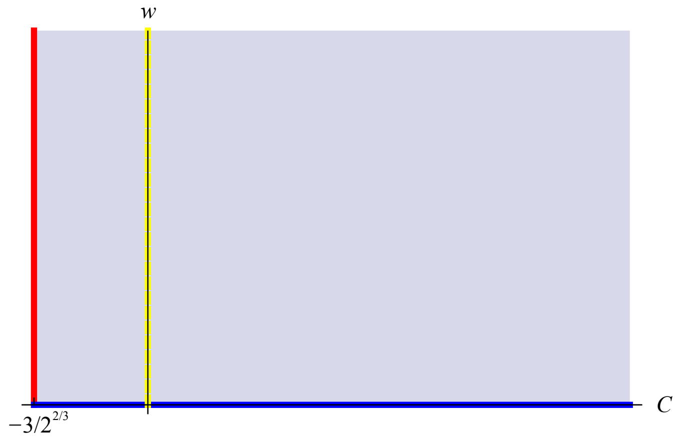

Figure 3. Moduli space of the helicoidal drops with $a = -1$ and $\Lambda = 0$. All points on the vertical red line correspond to a single surface, a round cylinder. Points on the vertical yellow line correspond to helicoidal rotational drops that contain the axis of symmetry. Points on the horizontal blue line correspond to cylindrical rotational drops.

Remark 2.11. Up to dilations and rigid motions, the moduli space for all helicoidal rotating drops with $\Lambda_0 = 0$ is the plane region

$$\{(C, \omega) : C \geq C_0 = -3/\sqrt[3]{4},\ \omega > 0\}.$$

Moreover, for any $\omega > 0$, the surface associated with the point $(C, \omega) = (C_0, \omega)$ is a round cylinder of radius $\sqrt[3]{2}$, because it can be easily checked that when $C = C_0$, then, for any ω, the level set $G = C$ reduces to the point $\{(0, -\sqrt{2})\}$.

Case II: $\Lambda_0 = 1$. First note that the case $a = 0$ corresponds to a helicoidal surface with constant mean curvature. These surfaces were studied using similar techniques in [Perdomo 2012], and for this reason we will assume here that $a \neq 0$. In this case, the polynomial $p(r, a, \Lambda, C)$ reduces to

$$q = q(r, C, a) = -16C^2 + 64r + 32Cr - 16r^2 - 8aCr^2 + 8ar^3 - a^2r^4. \tag{2-11}$$

Recall that we are interested in finding two consecutive positive roots of the polynomial q. The roots of the polynomial q given in (2-11) were analyzed in [Palmer and Perdomo 2014]. In order to describe the roots of q we need to define the following functions.

Definition 2.12. Let $h(R) = 2(R-1)/R^3$, and define the functions $R_1 : \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $R_2 : \left(0, \frac{8}{27}\right] \to \mathbb{R}$ and $R_3 : \left(0, \frac{8}{27}\right] \to \mathbb{R}$ by

$$R_i(a) = R \quad \text{such that } h(R) = a, \text{ with } \begin{cases} R < 1 & \text{if } i = 1, \\ 1 < R \le \frac{3}{2} & \text{if } i = 2, \\ \frac{3}{2} \le R < \infty & \text{if } i = 3. \end{cases}$$

We also define the functions $r_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $r_2 : \left(0, \frac{8}{27}\right] \to \mathbb{R}$ and $r_3 : \left(0, \frac{8}{27}\right] \to \mathbb{R}$ by

$$r_1(a) = R_1^2(a), \quad r_2(a) = R_2^2(a), \quad r_3(a) = R_3^2(a).$$

The following lemma was proven in [Palmer and Perdomo 2014] and provides the number of roots of q depending on the values a and C.

Lemma 2.13. *Let r_1, r_2 and r_3 be as in Definition 2.12 and for $i = 1, 2, 3$ define*

$$C_i = \frac{16 - 8r_i + 6a{r_i}^2 - a^2{r_i}^3}{4(-2 + ar_i)}$$

and

$$q = q(r, a, C) = -16C^2 + 64r + 32Cr - 16r^2 - 8aCr^2 + 8ar^3 - a^2r^4.$$

Recall that the domain of C_i is the same domain of r_i. That is, the domain of $C_1(a)$ is $\mathbb{R} \setminus \{0\}$ and the domain of $C_2(a)$ and $C_3(a)$ is the interval $\left(0, \frac{8}{27}\right]$. For any $a \neq 0$ and any C, the polynomial q has nonnegative real roots whose multiplicities are as follows. Let $\mathcal{N}(q)$ denote the number of distinct real roots of q. There are four cases to consider:

Case 1: *If $a < 0$, then $C_1(a) < 0$, and*

$$\mathcal{N}(q) = \begin{cases} 0 & \text{if } C < C_1(a), \\ 1 & \text{if } C = C_1(a), \\ 2 & \text{if } C > C_1(a). \end{cases}$$

Case 2: *If $0 < a < \frac{8}{27}$, then*

$$C_2(a) < 0, \quad C_1(a) > 0, \quad C_2(a) < C_3(a) < C_1(a),$$

and

$$\mathcal{N}(q) = \begin{cases} 0 & \text{if } C > C_1(a), \\ 1 & \text{if } C = C_1(a), \\ 2 & \text{if } C_3(a) < C < C_1(a), \\ 3 & \text{if } C = C_3(a), \\ 4 & \text{if } C_2(a) < C < C_3(a), \\ 3 & \text{if } C = C_2(a), \\ 3 & \text{if } C < C_2(a). \end{cases}$$

When $C = C_3(a)$, the second root has multiplicity two. When $C = C_2(a)$, the first root has multiplicity two.

Case 3: *If* $a = \frac{8}{27}$, *then*

$$C_1(a) = 9, \quad C_2(a) = C_3(a) = -\tfrac{9}{8},$$

and

$$\mathcal{N}(q) = \begin{cases} 2 & \textit{if } C < -\frac{9}{8}, \\ 2 & \textit{if } C = -\frac{9}{8}, \\ 2 & \textit{if } -\frac{9}{8} < C < 9, \\ 1 & \textit{if } C = 9, \\ 0 & \textit{if } C > 9. \end{cases}$$

When $C = -\frac{9}{8}$, *the roots are* $\frac{9}{4}$ *with multiplicity three and* $\frac{81}{4}$ *with multiplicity one. When* $C = 9$, *the only real root is* 9, *with multiplicity two.*

Case 4: *If* $a > \frac{8}{27}$, *then* $C_1(a) > 0$, *and*

$$\mathcal{N}(q) = \begin{cases} 0 & \textit{if } C > C_1(a), \\ 1 & \textit{if } C = C_1(a), \\ 2 & \textit{if } C < C_1(a). \end{cases}$$

Now that we have discussed the roots of the polynomial q we can describe the moduli space of all helicoidal rotating drops with $\Lambda = 1$.

Theorem 2.14. *Let* $\Lambda_0 = 1$ *and let* $\Delta\tilde{\theta}$ *be the function defined in* (2-9). *Let*

$$\begin{aligned} \Omega_1 &= \{(a, C, \omega) : C > C_1(a),\ a < 0,\ w > 0\}, \\ \Omega_2 &= \left\{(a, C, \omega) : C_2(a) < C < C_3(a),\ 0 < a < \tfrac{8}{27},\ \omega > 0\right\}, \\ \Omega_3 &= \{(a, C, \omega) : C < C_1(a),\ a > 0,\ \omega > 0\}, \\ \Omega &= \Omega_1 \cup \Omega_3 \setminus \Omega_2 \\ \beta_1 &= \{(a, C, \omega) : C = C_1(a),\ a \neq 0,\ \omega > 0\}, \\ \beta_2 &= \left\{(a, C, \omega) : C = C_2(a),\ 0 < a < \tfrac{8}{27},\ \omega > 0\right\}, \\ \beta_3 &= \left\{(a, C, \omega) : C = C_3(a),\ 0 < a < \tfrac{8}{27},\ \omega > 0\right\}. \end{aligned}$$

Under the convention that a point (a, C, ω) *represents a helicoidal rotating drop if the treadmill sled of its profile curve is contained in the level set* $G = C$, *we have*:

(i) *Every point* (a, C, ω) *in the interior of* Ω *represents a helicoidal rotating drop with its fundamental piece having finite length. The treadmill sleds of the profile curves of these surfaces are parametrized by* ρ *defined for values of* r *between the only two roots of the polynomial* $q(r, a, C)$.

(ii) *Every point in* Ω_2 *represents two helicoidal rotating drops, both having fundamental pieces of finite length. The treadmill sleds of the profile curves of these surfaces are parametrized by* ρ *defined for those values of* r *that lie between* *the first and second root of the polynomial $q(r, a, C)$ and the third and fourth root of the polynomial $q(r, a, C)$ respectively.*

(iii) *Every point (a, C, ω) in the set β_1 represents a circular helicoidal rotating drop. This cylinder is the same for all values of ω.*

(iv) *Every point (a, C, ω) in the set β_2 represents two helicoidal rotating drops: a circular cylinder and a noncircular cylinder with bounded length of its fundamental piece. The circular cylinder is the same for all values of w.*

(v) *Every point in the set β_3 represents three helicoidal rotating drops. One is a circular cylinder, which is the same for all values of w. The second one has a treadmill sled parametrized by ρ defined for those values of r that lie between the first and second root of the polynomial $q(r, a, C)$. Recall that the second root has multiplicity two. The third surface has a treadmill sled parametrized by ρ defined for those values of r that lie between the second and third root of the polynomial $q(r, a, C)$. The second and third surfaces are not properly immersed and their profile curves have a circle as a limit cycle and they have infinite winding number with respect to a point interior to this circle. Solutions whose profile curves possess a circle as a limit cycle will be called* helicoidal drops of exceptional type.

(vi) *Points of the form $(a, C, \omega) = \left(\frac{8}{27}, -\frac{9}{8}, \omega\right)$ represent two helicoidal rotating drops: a circular cylinder, which is the same for all values of ω, and one helicoidal drop of exceptional type.*

(vii) *Up to a rigid motion, every helicoidal drop falls into one of the cases above.*

(viii) *Every helicoidal drop that is not exceptional is either properly immersed (when $\Delta\tilde{\theta}(a, C, \omega)/\pi$ is a rational number) or it is dense in the region bounded by two round cylinders (when $\Delta\tilde{\theta}(a, C, \omega)/\pi$ is an irrational number).*

Proof. We already know that the treadmill sled of the profile curve of any helicoidal rotating drop satisfies the equation

$$G(\xi_1, \xi_2) = \frac{2\xi_2}{\sqrt{1+\omega^2\xi_1^2}} + \Lambda_0(\xi_1^2 + \xi_2^2) - \frac{a}{4}(\xi_1^2 + \xi_2^2)^2 = C.$$

We also know that, up to rigid motions, the treadmill sled of a curve determines the curve; see [Perdomo 2012]. Since any level set of G can be parametrized using the map ρ given in Definition 2.3, and every parametrization of a level set of G is defined for values of r where the polynomial q is positive, it then follows from Lemma 2.13 that every helicoidal rotating drop can be represented as one of the cases (i), (ii), (iii), (iv), (v) and (vi). It is worth recalling (see Remark 2.4) that the parametrization ρ only covers half of the level set of the map G. Each one of these

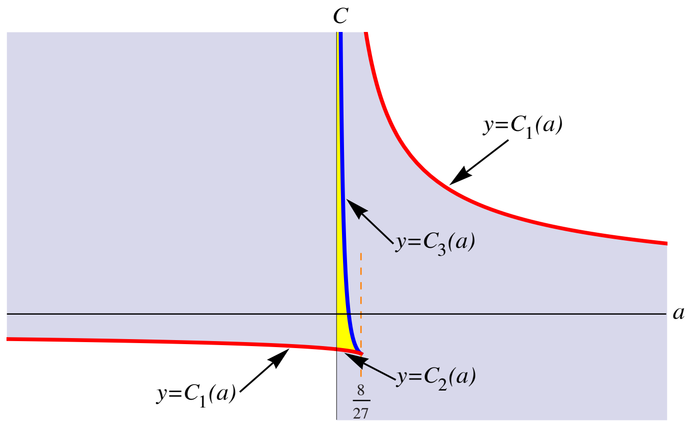

Figure 4. Graphs of the functions C_1, C_2 and C_3; these functions are used in Theorem 2.14 to describe the moduli space of all helicoidal rotational drops with $\Lambda_0 = 1$.

level sets is symmetric with respect to the ξ_2-axis, and the parametrization ρ covers the half on the right.

Notice that when the profile curve is a circle, the level set $G = C$ reduces to a point. In this case we will take the parametrization ρ to be defined just in a point, a root with multiplicity two of the polynomial q.

When case (i) occurs, q has only two simple roots, x_1 and x_2, with $x_1 < x_2$. We can check that the derivative of q at x_1 is positive while the derivative of q at x_2 is negative, so the length of the fundamental piece, according to (2-7), reduces to

$$\int_{x_1}^{x_2} \sqrt{\frac{64 + (4C + r(-4\Lambda_0 + ar))^2\omega^2}{q(r, a, C)}}\, dr,$$

which converges. Therefore the length of the fundamental piece is finite.

For values of C, ω and a that fall into case (ii), the polynomial q has four roots $x_1 < x_2 < x_3 < x_4$, and q is positive from x_1 to x_2 and from x_3 to x_4. Also, the level set of G has two connected components. Half of each connected component of $G = C$ can be parametrized using the map ρ. One half of the connected component of $G = C$ uses the domain (x_1, x_2) for ρ and the half of the other connected component of $G = C$ uses the domain (x_3, x_4) for ρ. The proof that the length of the fundamental piece of each surface is finite follows as in the proof in case (i).

For values of (a, C, ω) that satisfy the case (iii), the polynomial q has only one root $x_1 = r_1$ with multiplicity two. We take $R = \sqrt{x_1}$. A direct calculation shows that if $a > 0$, then $R_1(a) = -R$ and if we consider the profile curve

$$\alpha(s) = \left(R \sin \frac{s}{R}, -R \cos \frac{s}{R}\right),$$

then

$$\xi_1 = 0, \quad \xi_2 = R \quad \text{and} \quad G(\xi_1, \xi_2) = 2R + \Lambda_0 R^2 - \tfrac{1}{4} a R^4.$$

Using the definition of C_1 and the fact that $R_1(a) = -R$, we can check that the expression $G(\xi_1, \xi_2)$ reduces to $C = C_1(a)$, which was our goal in order to show that the point (a, C, ω) represents a round cylinder. Similarly, a direct verification shows that if $a < 0$, then $R_1(a) = R$ and if we consider the profile curve

$$\alpha(s) = \left(R \sin \frac{s}{R}, R \cos \frac{s}{R}\right),$$

then $\xi_1 = 0$, $\xi_2 = -R$ and $G(\xi_1, \xi_2) = -2R + \Lambda_0 R^2 - \frac{1}{4} a R^4$. Using the definition of C_1 and the fact that $R_1(a) = R$, we can check that the expression $G(\xi_1, \xi_2)$ reduces to $C = C_1(a)$. Since r_1 is independent of w these cylinders are independent of the value of w. This finish the proof of part (iii).

For values of (a, C, ω) that fall into case (iv), the polynomial q has three roots $x_1 < x_2 < x_3$, where $x_1 = r_1$ has multiplicity two and x_2 and x_3 are simple. The polynomial q is positive for $r \in (x_2, x_3)$. In this case the level set $G = C$ is the union of the point $(0, -\sqrt{x_1})$ and a closed curve. If we consider the cylinder of radius $\sqrt{x_1}$ oriented by the inward-pointing normal, then a direct computation shows that its mean curvature is $2H = 1 - \frac{1}{2} a r_1$. Therefore this circular cylinder is a helicoidal rotating drop for the given parameters. Note that this cylinder is independent of w. The treadmill sled of the profile curve of the other rotating drop is the level closed curve component of $G = C$; half of this part can be parametrized by the map ρ with domain (x_2, x_3).

For values of (a, C, ω) in case (v), the polynomial q has only three roots $x_1 < x_2 < x_3$, where $x_2 = r_2$ has multiplicity two and x_1 and x_3 are simple. The polynomial q is positive for $r \in (x_1, x_2) \cup (x_2, x_3)$. In this case the level set $G = C$ is connected but it self-intersects at the point $(0, -\sqrt{x_2})$. Any part of a curve that crosses the ξ_2-axis nonhorizontally cannot be the treadmill sled of a regular curve (see Proposition 2.11 in [Perdomo 2013]). Therefore the correct way to view the level set $G = C$ in this case is not as a connected closed curve that self-intersects but as the union of two curves and a point. Figure 5 shows one of these level sets.

One can check that the circular cylinder with radius $\sqrt{r_2}$, oriented by the inward-pointing normal, satisfies the equation $2H = 1 - \frac{1}{2} a r_2$; therefore this circular cylinder is a helicoidal rotating drop for the given parameters. The treadmill sled associated with the profile curve of this round cylinder reduces to the point $(0, -\sqrt{x_2})$. The set $G = C \setminus \{(0, -\sqrt{x_2})\}$ has two connected components. One of these connected components can be parametrized using the map ρ with $r \in (x_1, x_2)$ and the other using the map ρ with $r \in (x_2, x_3)$. Each of these connected components is the treadmill sled of the fundamental curve for a rotating helicoidal drop whose length

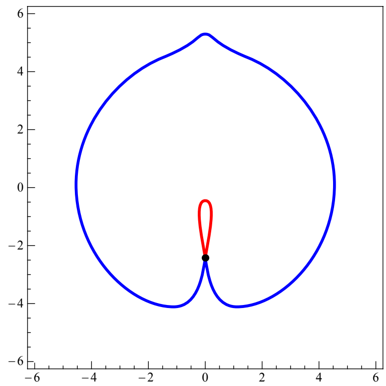

Figure 5. The level set $G = C$ when $a = 0.2$, $\omega = 3$, $\Lambda_0 = 1$ and $C = C_3(0.2)$. For helicoidal drops in case (v) of Theorem 2.14, the level set of G should be regarded as the union of two curves and a point. Each curve is the treadmill sled of an exceptional helicoidal rotating drop and the point is the treadmill sled of a circular cylinder.

is unbounded. Specifically, their lengths are given by the divergent integrals

$$\int_{x_1}^{x_2} \sqrt{\frac{64 + (4C + r(-4\Lambda_0 + ar))^2\omega^2}{q(r, a, C)}}\, dr$$

and

$$\int_{x_2}^{x_3} \sqrt{\frac{64 + (4C + r(-4\Lambda_0 + ar))^2\omega^2}{q(r, a, C)}}\, dr,$$

respectively. Moreover, using the definition of treadmill sled, we notice that the function giving the distance to the origin of the profile curve, $|(x(s), y(s)|$, agrees with the function giving the distance to the origin of the level set $G = C$ given by $|(\xi_1(s), \xi_2(s))| = |\rho(\sigma^{-1}(s))|$. Therefore, as r approaches r_2, $s = \sigma(r)$ goes to $-\infty$ and the function $|(x(s), y(x)|$ approaches $\sqrt{r_2}$. Since polar angle of the profile curve can be calculated by integrating the expression in (2-8), we conclude that $\tilde{\theta}(r)$ also goes to $-\infty$ as r approaches r_2. We conclude that the profile curve has a circle of radius $\sqrt{r_2}$ as a limit cycle and it has infinite winding number with respect to a point interior to this circle.

For values of (a, C, w) that satisfy the case (vi), the polynomial q has only two roots $x_1 < x_2$, where $x_1 = \frac{9}{4}$ has multiplicity three and $x_2 = \frac{81}{4}$ is simple. The polynomial q is positive for $r \in (x_1, x_2)$. In this case the level set $G = C$ is connected but it has a singularity at the point $\left(0, -\frac{3}{2}\right)$. We can check that the circular cylinder with radius $\frac{3}{2}$ oriented by the inward-pointing normal satisfies the equation $2H = 1 - \frac{1}{2}ar_1$; therefore this circular cylinder is a helicoidal rotating drop.

The treadmill sled associated to the profile curve of this round cylinder reduces to the point $\left(0, -\frac{3}{2}\right)$. The set $G = C$ minus the point $\left(0, -\frac{3}{2}\right)$ is connected and half of it can be parametrized using the map ρ with $r \in \left(\frac{9}{4}, \frac{81}{4}\right)$ This part of the set $G = C$ is the treadmill sled of the fundamental curve of a rotating helicoidal drop whose length is not bounded.

Since we know that the profile curve of every rotating helicoidal drop satisfies the integral equation $G = C$ and cases (i)–(vi) cover all the possibilities for the level sets of G, then every rotating helicoidal drop falls into one of the first six cases of this proposition. This proves (vii).

In order to prove (viii) we notice that when a helicoidal rotating drop is not exceptional, it has a fundamental piece with finite length whose treadmill sled is a closed regular curve (a connected component of the set $G = C$). By the properties of the treadmill sled operator (in particular, the one that states that the treadmill sled inverse is unique up to rotations about the origin), we have that the whole profile curve is a union of rotations of the fundamental piece. The angle of rotation is given by $\Delta\tilde{\theta} = \Delta\tilde{\theta}(C, a, \omega, x_1, x_2)$. The profile curve is invariant under the group $\mathbb{G}$ of rotations of the form

$$(y_1, y_2) \mapsto (\cos(n\Delta\tilde{\theta})y_1 + \sin(n\Delta\tilde{\theta})y_2, -\sin(n\Delta\tilde{\theta})y_1 + \cos(n\Delta\theta)y_2), \tag{2-12}$$

where $n \in \mathbb{Z}$. It is clear that if $\Delta\tilde{\theta}/\pi$ is a rational number then the group $\mathbb{G}$ is finite and the helicoidal surface is properly immersed. Moreover, if $\Delta\tilde{\theta}/\pi$ is not a rational number, then the group $\mathbb{G}$ is not finite and the helicoidal drop is dense in the region bounded by the two cylinders of radius $\sqrt{r_1}$ and $\sqrt{r_2}$. A more detailed explanation of this last statement can be found in [Perdomo 2012]. $\square$

Embedded and properly embedded examples. In this subsection we will find some embedded examples and we will show their profile curves. As pointed out before, when the helicoidal drop is not exceptional, its profile curve is a union of rotations of fundamental pieces that ends up being invariant under the group $\mathbb{G}$ of rotations defined by (2-12). It is not difficult to see that a necessary condition for the helicoidal drop to be embedded is that $\Delta\tilde{\theta} = 2\pi/m$ for some integer m. We will show that this condition is not sufficient. In order to catch the potentially embedded examples we need to understand the function $\Delta\tilde{\theta}(C, a, \omega, x_1, x_2)$. As a very elementary technique to solve the equation $\Delta\tilde{\theta} = 2\pi/m$, we will use the intermediate value theorem. We know that for any a, there is a first (or last) value of C, $C_0(a, x_1, x_2)$, for which the function $\Delta\tilde{\theta}$ is defined. We will compute the limit of $\Delta\tilde{\theta}$ when C goes to C_0 using Lemma 2.9. The graphs shown in this paper were generated using the software Mathematica 8.

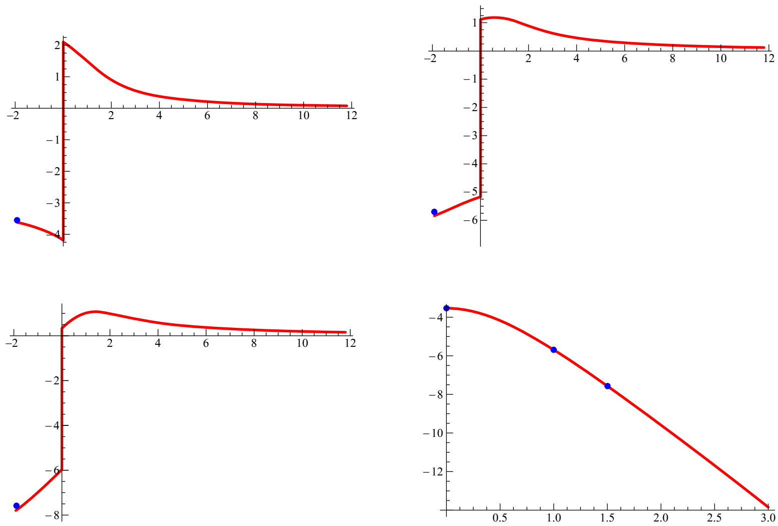

Figure 6. Top left, top right, bottom left: graphs of $\Delta\tilde{\theta}$ in terms of C when $a = -1$, $\Lambda_0 = 1$; $\omega = 0$, $\omega = 1$, and $\omega = 1.5$, respectively. Bottom right: graph of B in terms of ω when $a = -1$, $\Lambda_0 = 0$. This shows how the beginnings of the other three graphs change when ω changes. The highlighted points in this graph ($\omega = 0$, 1 and 1.5) correspond to the highlighted points in the other three graphs. For $\omega = 0$ there is no solution of the equation $\Delta\tilde{\theta} = -2\pi/m$ with negative values of C. We see that, for some values of ω, the equation $\Delta\tilde{\theta} = -2\pi$ has a solution with C negative, which is responsible for the existence of embedded examples with $\Lambda_0 = 0$ and $a = -1$.

Embedded examples with $\boldsymbol{\Lambda_0 = 0}$ and $\boldsymbol{a = -1}$. From Lemma 2.8 we know that the polynomial p has two nonnegative roots if and only if $C > C_0 = -3/\sqrt[3]{4}$. A direct application of Lemma 2.9 shows:

Proposition 2.15. *If $\Lambda_0 = 0$, $a = -1$ and for any $C > C_0$, x_1 and x_2 denote the two roots of the polynomial $q(r, C) = -16C^2 + 64r + 8Cr^2 - r^4$, then*

$$\lim_{C \to C_0^+} \Delta\theta(C, \omega, x_1, x_2) = \int_{x_1}^{x_2} \frac{(4C - r^2)\sqrt{1 + r\omega^2}}{r\sqrt{q(r, C)}}\, dr$$
$$= B(\omega) = -\frac{2\pi\sqrt{1 + \sqrt[3]{4}\omega^2}}{\sqrt{3}}.$$

Using the intermediate value theorem, we can numerically solve the equation $\Delta\tilde{\theta} = -2\pi$ for values of w and C. The images in Figures 7 and 8 show some of

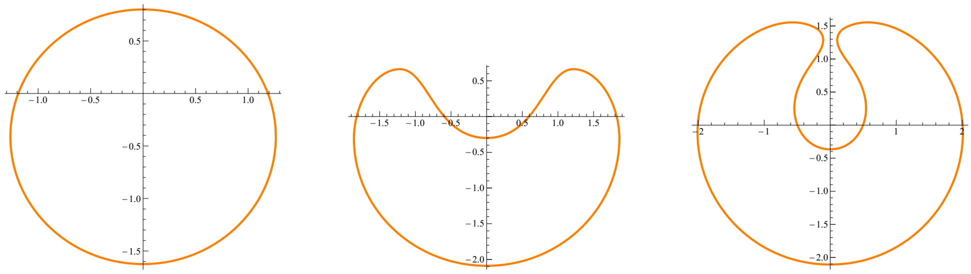

Figure 7. Profile curves of some embedded helicoidal rotating drops when $\Lambda_0 = 0$ and $a = -1$. Left: $C = -1.5$, $\omega = 1.17684$. Middle: $C = 0.6$, $\omega = 3.04646$. Right: $C = 0.735$, $\omega = 4.65615$. When C is close to the critical value C_0, the embedded examples are close to a round cylinder. As C increases, the shape develops a self-intersection.

the resulting profile curves and the corresponding surfaces. They also show the values of C and w that solve the equation $\Delta\tilde{\theta} = -2\pi$.

Embedded examples with $\Lambda_0 = 1$ and $a \neq 0$. We now show some embedded examples in this case. Again, the intermediate value theorem is used to numerically solve the equation $\Delta\tilde{\theta} = 2\pi/m$. A direct application of Lemma 2.9 shows:

Proposition 2.16. *Let r_1, r_2 and r_3 be as in Definition 2.12 and let C_1, C_2 and C_3 be as in Lemma 2.13. Let us define the two bounds*

$$b_i(a, w) = \frac{\pi(4C_i + ar_i^2 - 4r_i)\sqrt{1 + r_i w^2}}{r_i\sqrt{16 + 8a(C - 3r_i) + 6a^2 r_i^2}}, \qquad i = 1, 2.$$

(a) *If $a < 0$ then $\lim_{C\to C_1(a)^+} \Delta\tilde{\theta}(C, a, x_1, x_2) = b_1(a)$, where x_1 and x_2 are the first two roots of the polynomial $q(r, C, a)$*

(b) *If $a > 0$ then $\lim_{C\to C_1(a)^-} \Delta\tilde{\theta}(C, a, x_1, x_2) = b_1(a)$, where x_1 and x_2 are the first two roots of the polynomial $q(r, C, a)$*

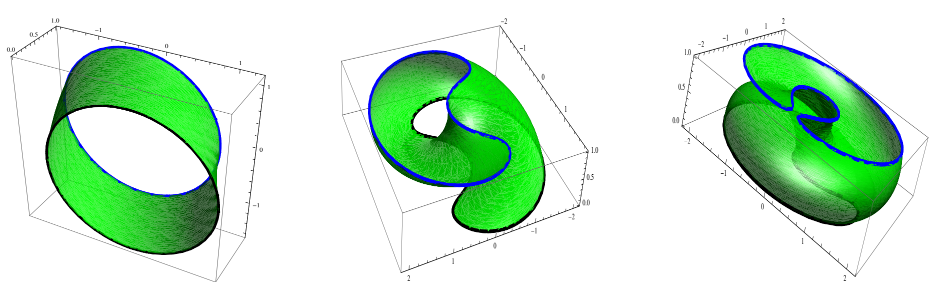

Figure 8. Surfaces associated with the profile curve in Figure 7.

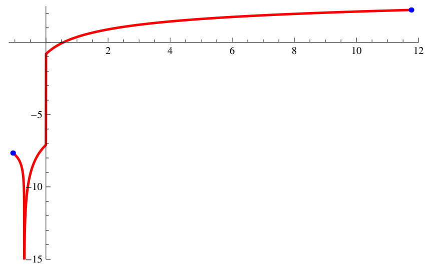

Figure 9. The graph of the function $\Delta\tilde{\theta}$ when $a = 0.2$ and $\omega = 0.15$. In this case $C_2 \approx -1.065$, $b_2 \approx -7.66$, $C_3 \approx -0.698$, $C_1 \approx 11.76$ and $b_1 \approx 2.23$. The points (C_2, b_2) and (C_1, b_1) have been highlighted.

(c) *If* $a > 0$ *then* $\lim_{C \to C_2(a)^+} \Delta\tilde{\theta}(C, a, x_1, x_2) = b_2(a)$, *where* x_1 *and* x_2 *are the first two roots of the polynomial* $q(r, C, a)$.

Proof. Since

$$\Delta\tilde{\theta} = \int_{x_1}^{x_2} \frac{(4C + ar^2 - 4r)\sqrt{1 + r\omega^2}}{r\sqrt{q}}\, dr,$$

in every case, when C approaches the limit value, the two roots approach r_i $(i = 1, 2)$, which is a root of q with multiplicity two. Therefore Lemma 2.9 applies and the proposition follows. Notice that the value A in Lemma 2.9 is given by

$$A = -\tfrac{1}{2} q''(r_i) = 16 + 8a(C - 3r_i) + 6a^2 r_i^2. \qquad \square$$

Remark 2.17. When $0 < a < \frac{8}{27}$, the domain of $\Delta\tilde{\theta}$ is $(C_2(a), C_1(a))$, and

$$\lim_{C \to C_2(a)^+} \Delta\tilde{\theta}(C) = b_2(a, w), \qquad \lim_{C \to C_1(a)^-} \Delta\tilde{\theta}(C) = b_1(a, w).$$

There is a vertical asymptote at $C = C_3(a)$ and a jump discontinuity at $C = 0$.

Taking a look at Figure 9, we notice that, when $\omega = 0.15$ and $a = 0.2$, and for any integer $m > 2$, the equation $\Delta\tilde{\theta} = 2\pi/m$ has a solution. We have numerically solved this equation for $m = 4$ and $m = 8$. Figures 10 and 11 provides a picture of the profile curves of the properly immersed examples.

If we decrease the value of a while keeping the value of ω constant, we can again solve the equation $\Delta\tilde{\theta} = \pi/4$, but this time the helicoidal rotational drop is embedded. See Figure 12.

Finally, we would like to show that if we increase ω then it is possible to solve the equation $\Delta\tilde{\theta} = 2\pi$ (see Figure 13).

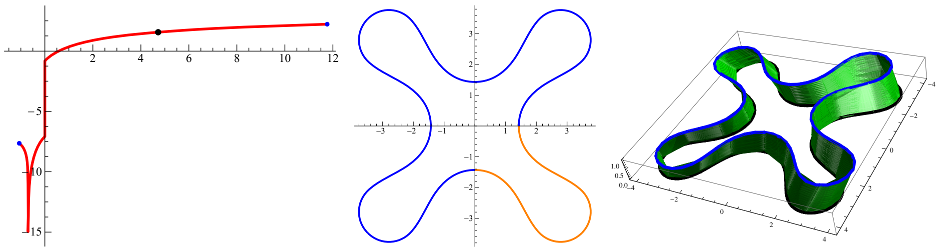

Figure 10. The embedded helicoidal rotational drop obtained by solving the equation $\Delta\tilde{\theta} = \frac{\pi}{2}$ when $a = 0.2$ and $\omega = 0.15$. In the middle graph, $\Lambda_0 = 1$ and $C = 4.72283$.

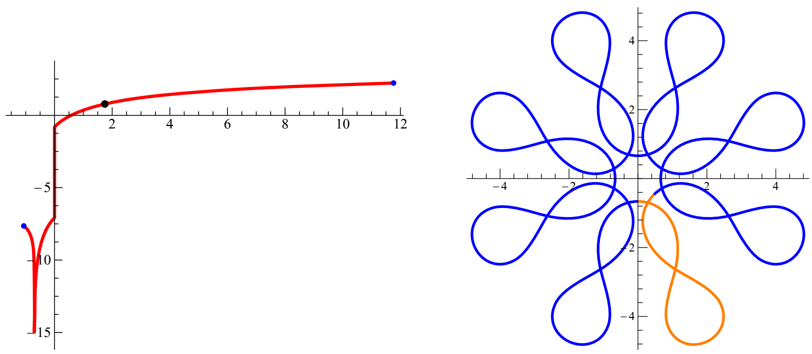

Figure 11. The properly immersed helicoidal rotational drop obtained by solving the equation $\Delta\tilde{\theta} = \frac{\pi}{4}$ when $a = 0.2$ and $\omega = 0.15$. In the graph on the right, $\Lambda_0 = 1$ and $C = 1.7453$.

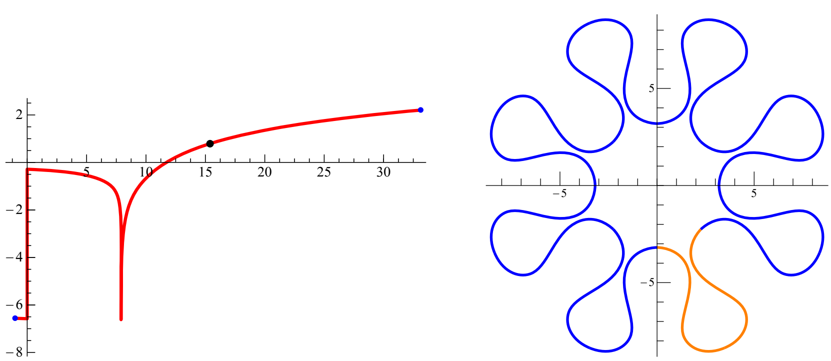

Figure 12. The embedded helicoidal rotational drop obtained by solving the equation $\Delta\tilde{\theta} = \frac{\pi}{4}$ when $a = 0.05$ and $\omega = 0.15$. In the graph on the right, $\Lambda_0 = 1$ and $C = 15.3877$.

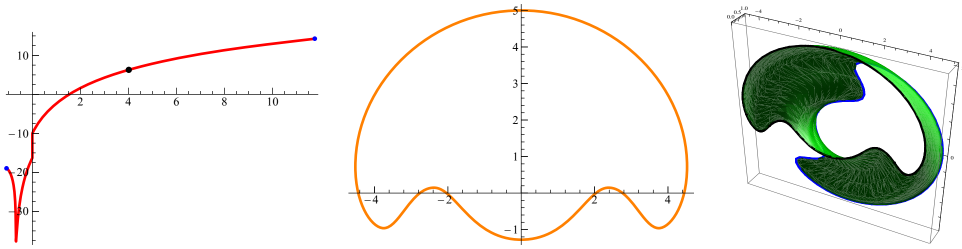

Figure 13. The embedded helicoidal rotational drop obtained by solving the equation $\Delta\tilde{\theta} = 2\pi$ when $a = 0.2$ and $\omega = 2$. In the middle graph, $\Lambda_0 = 1$ and $C = 4.0134$.

3. Second variation

For any sufficiently smooth surface, we define an invariant $\ell = 2H + \frac{1}{2}aR^2$. The first variation formula (1-4) restricted to compactly supported variations can then be expressed as

$$\delta\mathscr{E}_{a,\Lambda_0} = -\int_\Sigma (\ell - \Lambda_0)\psi \, d\Sigma,$$

where $\psi := \delta X \cdot \nu$. We assume that the surface is in equilibrium, so that $\ell - \Lambda_0 \equiv 0$ holds. The second variation is thus

$$\delta^2\mathscr{E}_{a,\Lambda_0} = -\int_\Sigma \psi(\delta\ell)\, d\Sigma.$$

A well-known formula for the pointwise variation of the mean curvature is

$$2\delta H = \hat{L}[\psi] + 2\nabla H \cdot \delta X, \tag{3-1}$$

where $\hat{L} = \Delta + |d\nu|^2$. Also

$$\delta R^2 = 2\sum_{i=1,2} x_i \delta X \cdot E_i = 2\sum_{i=1,2} x_i\big(\psi\nu_i + (\delta X)^T \cdot E_i\big) = 2\psi\hat{Q} + 2\nabla' R^2 \cdot (\delta X)^T,$$

where $\hat{Q} = x_1\nu_1 + x_2\nu_2$. Combining this with (3-1), we have

$$\delta\ell = L[\psi] + \nabla\ell \cdot T, \tag{3-2}$$

where $L[\psi] = \Delta\psi + (|d\nu|^2 + a\hat{Q})\psi$. Since we are assuming $\ell \equiv \Lambda_0 = \text{constant}$, the second term above vanishes and the second variation formula for variations vanishing on $\partial\Sigma$ then reads

$$\begin{aligned} \delta\mathscr{E}_{a,\Lambda_0} &= -\int_\Sigma \psi L[\psi]\, d\Sigma = -\int_\Sigma \psi\big(\Delta\psi + (|d\nu|^2 + a\hat{Q}\psi)\big)\, d\Sigma \\ &= \int_\Sigma |\nabla\psi|^2 - (|d\nu|^2 + a\hat{Q})\psi^2\, d\Sigma. \end{aligned} \tag{3-3}$$

This formula can be found in [López 2010]. As usual, an equilibrium surface will be called stable if the second variation is nonnegative for all compactly supported variations satisfying the additional condition

$$\int_\Sigma \psi\, d\Sigma = 0. \tag{3-4}$$

This is just the first-order condition which is necessary and sufficient for the variation to be volume-preserving.

For a part of the surface of the form $\alpha \times [-h/2, h/2]$, this can be written

$$\begin{aligned}\delta^2\mathscr{E}_{a,\Lambda_0} = \int_\alpha \int_{-h/2}^{h/2} \biggl(\frac{1}{\sqrt{1+\omega^2\xi_1^2}} \bigl([1+\omega^2 R^2]\psi_s^2 - 2\omega\xi_2\psi_s\psi_t + \psi_t^2 \bigr) \\ - (4H^2 - 2K + a\xi_2)\sqrt{1+\omega^2\xi_1^2}\,\psi^2 \biggr)\, dt\, ds,\end{aligned} \tag{3-5}$$

where K denotes the Gaussian curvature. Choosing $\psi = \sin(2\pi t/h)$ gives

$$\int_\Sigma |\nabla\psi|^2\, d\Sigma = \frac{2\pi^2}{h} \int_\alpha \frac{1}{\sqrt{1+\omega^2\xi_1^2}}\, ds.$$

In addition, for this choice of ψ, we have $\psi \equiv 0$ on the boundary and the mean value of ψ on $\alpha \times [-h/2, h/2]$ is zero.

Lemma 3.1. *We have*

$$\int_\alpha K\sqrt{1+\omega^2\xi_1^2}\, ds = 0,$$

and hence

$$\int_{\alpha\times[-h/2,h/2]} K\, d\Sigma = 0.$$

Proof. From calculations found in [Perdomo 2012], one finds

$$K = \frac{-\omega^2(1+\kappa\xi_2)}{(1+\omega^2\xi_1^2)^2} = \frac{-\omega^2(\xi_1)_s}{(1+\omega^2\xi_1^2)^2},$$

so

$$\int_\alpha K\sqrt{1+\omega^2\xi_1^2}\, ds = \int_\alpha \frac{-\omega^2(\xi_1)_s}{(1+\omega^2\xi_1^2)^{3/2}}\, ds = 0,$$

since the last integrand is the s-derivative of a function of ξ_1. □

Proposition 3.2. *A necessary condition for the stability of $\alpha \times [-h/2, h/2]$ for the fixed boundary problem is that*

$$\frac{2\pi^2}{h^2} \int_\alpha \frac{1}{\sqrt{1+\omega^2\xi_1^2}}\, ds \geq \int_\alpha 4H^2\sqrt{1+\omega^2\xi_1^2}\, ds + a\mathscr{A} \tag{3-6}$$

holds. Equivalently, this can be expressed as

$$\text{(3-7)} \quad \frac{2\pi^2}{h^2}\int_\alpha \frac{1}{\sqrt{1+\omega^2\xi_1^2}}\,ds \ge \int_{\alpha\times[-h/2,h/2]} 4H^2\,d\Sigma + a\mathcal{V}(\alpha\times[-h/2,h/2]).$$

Proof. We choose $\psi = \sin(2\pi t/h)$ in the second variation formula. For this choice of ψ, we have $\psi \equiv 0$ on the boundary and the mean value of ψ on $\alpha\times[-h/2,h/2]$ is zero. The result then follows directly from (3-5) and the previous lemma. □

The bound (3-6) gives a condition on the maximum height of a stable helicoidal surface in terms of the geometry of the generating curve.

There is no possible way to obtain a positive lower bound for the right-hand side of (3-6). For a round cylinder of radius R, the equation

$$\frac{2\xi_2}{\sqrt{1+\omega^2\xi_1^2}} + \Lambda_0 R^2 - \frac{aR^4}{4} = c$$

becomes

$$2R + \Lambda_0 R^2 - \frac{aR^4}{4} = c,$$

so for arbitrary a, we can simply define c by this equation, and hence the cylinder will be an equilibrium surface. For a cylinder, the potential in the second variation formula is

$$4H^2 - 2K + a\hat{Q} = \frac{1}{R^2} + aR,$$

so for $a \ll 0$ the potential is nonpositive and the cylinder is stable for arbitrary heights.

We will now give an upper bound for the height of a stable helicoidal equilibrium surface which is valid for any such surface which is not a cylinder over a planar curve. This upper bound will only depend on the generating curve. In [Palmer and Perdomo 2014], this estimate is modified so that it applies to noncircular cylindrical equilibrium surfaces as well.

Theorem 3.3. *For a helicoidal surface which is not a round cylinder, a necessary condition for the stability of the part of the surface between horizontal planes separated by a distance h is that*

$$\text{(3-8)} \qquad \frac{4\pi^2 e^4}{h^2} \ge \frac{\omega^2 \oint_\alpha \dfrac{(1+\omega^2R^2)(1+\kappa\xi_2)^2}{(1+\omega^2\xi_1^2)^{7/2}}\,ds}{\oint_\alpha \dfrac{1}{1+\omega^2\xi_1^2}\,ds} (\ge \omega^2).$$

The result also holds true if $a = 0$, i.e., if the surface has constant mean curvature.

Remark. In [Palmer and Perdomo 2014] a similar estimate is given for cylindrical surfaces which are not round cylinders.

Proof. To begin, note that the third component of the normal ν_3 satisfies $L[\nu_3]=0$ since vertical translation is a symmetry of the normal. Also, this function will vanish identically if and only if the surface is a cylinder.

The function ν_3 can be written as [Perdomo 2012]

$$\nu_3 = \frac{\omega\xi_1}{\sqrt{1+\omega^2\xi_1^2}},$$

so ν_3 is a function of s only. Using local coordinate expressions found in [Perdomo 2012], we can write

$$\begin{aligned} 0 = L[\nu_3] &= \frac{1}{\sqrt{g}}(\sqrt{g}\, g^{11}(\nu_3)_s)_s + (|d\nu|^2 + a\xi_2)\nu_3 \\ &= \frac{1}{\sqrt{1+\omega^2\xi_1^2}}\left[\frac{1+\omega^2R^2}{\sqrt{1+\omega^2\xi_1^2}}(\nu_3)_s\right]_s + (|d\nu|^2 + a\xi_2)\nu_3 =: \mathscr{L}[\nu_3]. \end{aligned}$$

Note that $(|d\nu|^2 + a\xi_2)$ only depends on s. For any smooth function $u = u(s)$, there holds

$$\mathscr{L}[e^u] = e^u(\mathscr{L}[u] + g^{11}u_s^2) = e^u(\mathscr{L}[u] + (1+\omega R^2)u_s^2).$$

If we now take $\psi = e^{\nu_3(s)}\sin(2\pi t/h)$, then (3-4) holds, and from (3-5) we get

$$\delta^2\mathscr{E}_{a,\Lambda_0} = \frac{2\pi^2}{h}\oint_\alpha \frac{e^{2\nu_3}}{1+\omega^2\xi_1^2}\,ds - \frac{h}{2}\oint_\alpha \frac{e^{2\nu_3}(1+\omega^2R^2)}{\sqrt{1+\omega^2\xi_1^2}}((\nu_3)_s)^2\,ds.$$

Using $-1 \le \nu_3 \le 1$ and using

$$(\nu_3)_s = (\omega\xi_1(1+\omega^2\xi_1^2)^{-1/2})_s = (\xi_1)_s\omega(1+\omega^2\xi_1^2)^{-3/2} = \omega(1+\kappa\xi_2)(1+\omega^2\xi_1^2)^{-3/2}$$

yields the result. □

Appendix

We assume that Σ is contained in a three-dimensional region Ω and that $\partial\Sigma$ is contained in a supporting surface S which is part of $\partial\Omega$. We assume that there is a (not necessarily connected) domain $S_1 \subset S$ such that $\Sigma \cup S_1$ bounds a subregion $\Omega_1 \subset \Omega$. The volume of Ω_1 will be denoted by $\mathscr{V}$.

Let ϕ be a solution of $\Delta'\phi = 1$ in Ω with $\nabla'\phi \cdot N = 0$ on S, where N is the outward-pointing normal to S. This boundary value problem is underdetermined and is solvable provided S is not closed.

We subject the surface to a variation that keeps $\partial\Sigma$ on S. We write $\delta X =: T + \psi\nu \perp N$ along $\partial\Sigma$, and

$$\mathscr{V} = \int_\Sigma \nabla'\phi \cdot \nu\, d\Sigma.$$

We have

$$\begin{aligned}\delta\mathcal{V} &= \int_\Sigma \nabla'_{T+\psi\nu}\nabla'\phi\cdot\nu + \nabla'\phi\cdot\delta\nu\,d\Sigma + \int_\Sigma \nabla'\phi\cdot\nu(\nabla\cdot T - 2H\psi)\,d\Sigma \\ &= \int_\Sigma \psi\nabla'_\nu\nabla'\phi\cdot\nu - 2H\psi\nabla'\phi\cdot\nu - \nabla\phi\cdot\nabla\psi\,d\Sigma + \oint_{\partial\Sigma}(\nabla'\phi\cdot\nu)T\cdot n\,ds \\ &= \int_\Sigma \psi\nabla'_\nu\nabla'\phi\cdot\nu - 2H\psi\nabla'\phi\cdot\nu + \psi\Delta\phi\,d\Sigma + \oint_{\partial\Sigma}((\nabla'\phi\cdot\nu)T - \psi\nabla\phi)\cdot n\,ds.\end{aligned}$$

A well-known formula relating the Laplacian on a submanifold to the Laplacian on the ambient space gives $\Delta'\phi = \nabla'_\nu\nabla'\phi\cdot\nu - 2H\nabla'\phi + \Delta\phi$. Therefore we obtain

$$\delta\mathcal{V} = \int_\Sigma \psi\,d\Sigma + \oint dX\times\nabla'\phi\cdot\delta X.$$

However, all of dX, $\nabla'\phi$ and δX are perpendicular to N on $\partial\Sigma$, so the line integral above vanishes.

To obtain (1-2), we let W be a vector field on Ω satisfying $\nabla'\cdot W = R^2$ and $W\cdot N = 0$ along S. This boundary value problem is underdetermined and is solvable provided S is not closed. Then, by the divergence theorem,

$$\int_{\Omega_1} R^2\,d^3x = \int_\Sigma W\cdot\nu\,d\Sigma,$$

so that

$$\begin{aligned}\delta\int_{\Omega_1} R^2\,d^3x &= \int_\Sigma \nabla'_{T+\psi\nu}W\cdot\nu + W\cdot\delta\nu\,d\Sigma + \int_\Sigma W\cdot\nu(\nabla\cdot T - 2H\psi)\,d\Sigma \\ &= \int_\Sigma \psi\nabla'_\nu W\cdot\nu - W\cdot\nabla\psi - 2H\psi W\cdot\nu\,d\Sigma + \oint_{\partial\Sigma}(W\cdot\nu)T\cdot n\,ds \\ &= \int_\Sigma \psi\nabla'\cdot W\,d\Sigma + \oint_{\partial\Sigma}((W\cdot\nu)T - \psi W)\cdot n\,ds \\ &= \int_\Sigma \psi R^2\,d\Sigma + \oint_{\partial\Sigma} dX\times W\cdot\delta X.\end{aligned}$$

Again, all of dX, W and δX are perpendicular to N along $\partial\Sigma$ so the line integral will vanish.

If the pair (ϕ, W) used above are replaced by another pair $(\underline{\phi}, \underline{W})$ satisfying the same equations ($\Delta'\underline{\phi} = 1$ and $\nabla'\underline{W} = R^2$), the divergence theorem yields

$$\int_\Sigma \nabla'\underline{\phi}\cdot\nu\,d\Sigma = \mathcal{V} + c_1 \quad\text{and}\quad \int_\Sigma W\cdot\nu\,d\Sigma = \int_{\Omega_1} R^2\,d^3x + c_2$$

for constants c_1 and c_2. Thus, these replacements will not affect the variational formulas for these integrals.

References

[Barros and Ferrández 2009] M. Barros and A. Ferrández, "A conformal variational approach for helices in nature", *J. Math. Phys.* **50**:10 (2009), 103529. MR 2010k:53004 Zbl 1283.53003

[Brown and Scriven 1980] R. A. Brown and L. E. Scriven, "The shape and stability of rotating liquid drops", *Proc. Roy. Soc. London Ser. A* **371**:1746 (1980), 331–357. MR 82m:76027 Zbl 0435.76073

[Cahill 2005] K. Cahill, "Helices in biomolecules", *Phys. Rev. E* **72**:6 (2005), 062901.

[Chandrasekhar 1965] S. Chandrasekhar, "The stability of a rotating liquid drop", *Proc. Roy. Soc. Ser. A* **286**:1404 (1965), 1–26. MR 32 #708 Zbl 0137.23004

[López 2010] R. López, "Stationary rotating surfaces in Euclidean space", *Calc. Var. Partial Differential Equations* **39**:3-4 (2010), 333–359. MR 2011m:53012 Zbl 1203.53008

[Palmer and Perdomo 2014] B. Palmer and O. Perdomo, "Equilibrium shapes of cylindrical rotating liquid drops", preprint, 2014. arXiv 1404.2976

[Pauling et al. 1951] L. Pauling, R. B. Corey, and H. R. Branson, "The structure of proteins: Two hydrogen-bonded helical configurations of the polypeptide chains", *Proc. Nat. Aca. Sci. U. S. A.* **37**:4 (1951), 205–211.

[Perdomo 2010] O. Perdomo, "Embedded constant mean curvature hypersurfaces on spheres", *Asian J. Math.* **14**:1 (2010), 73–108. MR 2012a:53113 Zbl 1213.53080

[Perdomo 2012] O. Perdomo, "A dynamical interpretation of the profile curve of CMC twizzler surfaces", *Pacific J. Math.* **258**:2 (2012), 459–485. MR 2981962 Zbl 1252.53007

[Perdomo 2013] O. Perdomo, "Helicoidal minimal surfaces in $\mathbb{R}^3$", *Illinois J. Math.* **57**:1 (2013), 87–104. MR 3224562 Zbl 06322984

[Solonnikov 2004] V. A. Solonnikov, "On instability of axially symmetric equilibrium figures of rotating viscous incompressible liquid", *Zap. Nauchn. Sem.* (*POMI*) **318** (2004), 277–297. Translated in *J. Math. Sci.* **136**:2 (2006), 3812–3825. MR 2006a:35257 Zbl 1075.35042

Received March 4, 2014. Revised May 21, 2014.

BENNETT PALMER
DEPARTMENT OF MATHEMATICS
IDAHO STATE UNIVERSITY
POCATELLO, ID 83209
UNITED STATES
palmbenn@isu.edu

OSCAR M. PERDOMO
DEPARTMENT OF MATHEMATICAL SCIENCES
CENTRAL CONNECTICUT STATE UNIVERSITY
1615 STANLEY STREET
MARCUS WHITE 111
NEW BRITAIN, CT 06050
UNITED STATES
perdomoosm@ccsu.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.443

THE BIDUAL OF A RADICAL OPERATOR ALGEBRA CAN BE SEMISIMPLE

CHARLES JOHN READ

The paper of Sidney (Denny) L. Gulick ("Commutativity and ideals in the biduals of topological algebras", *Pacific J. Math.* 18, 1966) contains some good mathematics, but it also contains an error. It claims that for a Banach algebra A, the intersection of the Jacobson radical of A^{**} with A is precisely the radical of A (this is claimed for either of the Arens products on A^{**}). In this paper we begin with a simple counterexample to that claim, in which A is a radical operator algebra, but not every element of A lies in the radical of A^{**}. We then develop a more complicated example $\mathscr{A}$, which, once again, is a radical operator algebra, but $\mathscr{A}^{**}$ is semisimple. So rad $\mathscr{A}^{**} \cap \mathscr{A}$ is zero, but rad $\mathscr{A} = \mathscr{A}$. We conclude by examining the uses Gulick's paper has been put to since 1966 (at least 8 subsequent papers refer to it), and we find that most authors have used the correct material from that paper, and avoided using the wrong result. We reckon, then, that we are not the first to suspect that the result rad $A^{**} \cap A =$ rad A was wrong; but we believe we are the first to provide "neat" counterexamples as described.

1. Introduction

The theorem in which Gulick [1966] makes the claim $\operatorname{rad} A^{**} \cap A = \operatorname{rad} A$ is Theorem 4.6. We believe that the place where his proof breaks down is nearby, in the proof of Lemma 4.5, the seventh line: "note that M_E is once again a maximal regular left ideal in E". We could not see why this should be so, and Theorem 4.6 is definitely false; this introductory section contains a counterexample.

We shall always be working with operator algebras (norm-closed subalgebras of the algebra $B(H)$ of all operators on a Hilbert space H), so the question of which Arens product is involved need never be addressed, for as is well known, every operator algebra is Arens regular — the two products coincide.

Let us conclude this introduction with the simpler counterexample mentioned in the abstract.

Read is grateful for support from UK Research Council grant EP/K019546/1, and for helpful suggestions from David Blecher.
MSC2010: 46H05, 47L50.
Keywords: radical Banach algebra, bidual operator algebra.

Let H be a Hilbert space with orthonormal basis $(e_i)_{i\in\mathbb{N}}$. Let $T_0 : H \to H$ be the operator with

$$T_0 e_i = \begin{cases} e_{i+1} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even.} \end{cases} \tag{1}$$

For $n \in \mathbb{N}$, let $T_n : H \to H$ be the rank-1 operator with

$$T_n e_i = \begin{cases} e_{i+1} & \text{if } i = 2n, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Let A denote the operator algebra (the norm-closed subalgebra of $B(H)$) generated by $\{T_n : n \in \mathbb{N}_0\}$.

Lemma 1.1. *A is radical.*

Proof. First, $T_0^2 = 0$ and each T_n $(n \geq 1)$ has rank 1, so everything in A is of form $\lambda T_0 + K$, where $\lambda \in \mathbb{C}$ and K is a compact operator. Second, the subspaces $E_k = \overline{\mathrm{lin}}\{e_i : i > k\} \subset H$ are invariant for every T_n (and hence for every $T \in A$); indeed, every $T \in A$ maps E_k into E_{k+1} $(k \in \mathbb{N}_0)$. So, let $T = \lambda T_0 + K \in A$, with $\lambda \in \mathbb{C}$ and $K \in K(H)$. It is enough to show that T is quasinilpotent. Since K is compact, the norms $\varepsilon_n = \left\| K|_{E_n} \right\|$ tend to zero as $n \to \infty$. Furthermore, since $T_0^2 = 0$, we have

$$\left\| T^2|_{E_n} \right\| = \left\| \lambda T_0 K + \lambda K T_0 + K^2|_{E_n} \right\| \leq 2|\lambda|\varepsilon_n + \varepsilon_n^2 = \delta_n, \tag{3}$$

with $\delta_n \to 0$ as $n \to \infty$. Now $T^{2k} = T^2|_{E_{2k-2}} T^2|_{E_{2k-4}} \ldots T^2|_{E_2} T^2|_{E_0}$; hence

$$\|T^{2k}\| \leq \prod_{j=0}^{k-1} \delta_{2j},$$

so $\|T^{2k}\|^{1/k} \to 0$. Plainly T^2, and hence T itself, is quasinilpotent. $\square$

Theorem 1.2. *$T_0 \notin \operatorname{rad} A^{**}$, so $A = \operatorname{rad} A \subsetneq A \cap \operatorname{rad} A^{**}$.*

Proof. Now $A \subset B(H)$, and $B(H)$ is of course a dual Banach algebra, so there is a natural projection from $B(H)^{**}$ (the third dual of the Banach space of trace class operators on H) onto $B(H)$. This projection is an algebra homomorphism, so when we restrict it to $A^{**} \subset B(H)^{**}$, we get a representation of A^{**} acting on H, such that the canonical image $A \subset A^{**}$ acts on H in its usual way, and the representation of A^{**} consists of the weak-* closure of A in $B(H)$.

Among the operators in this weak-* closure is the weak-* convergent sum $T = \sum_{n=1}^{\infty} T_n$, with

$$T e_i = \begin{cases} e_{i+1} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \tag{4}$$

The product TT_0 has $TT_0e_i = e_{i+2}$ (if i is odd) or $TT_0e_i = 0$ (if i is even); so $\|(TT_0)^k\| = 1$ for all k, and indeed 1 is in the spectrum of TT_0. If $\tau \in A^{**}$ is any element represented as T by this representation, then $1 \in \mathrm{Sp}(\tau T_0)$. So T_0 does not lie in the Jacobson radical of A^{**}, by a well-known characterization of that radical. □

Note that the proof given above does not depend on the faithfulness (injectivity) of the natural representation of A^{**} in $B(H)$. However, when we give the more complicated counterexample — when we make the claim that the bidual of our radical algebra $\mathscr{A}$ is semisimple — we will have to show that the analogous representation for the bidual of that algebra is indeed faithful.

2. The main construction

We now seek to develop the example given in the introduction into an example $\mathscr{A}$ where $\mathscr{A}$ is radical but $\mathscr{A}^{**}$ is semisimple.

Definition 2.1. Let S denote the free unital semigroup on two generators g, h. If $s \in S$ with $s = \gamma_n\gamma_{n-1}\dots\gamma_2\gamma_1 = \prod_{j=0}^{n-1}\gamma_{n-j}$, and each $\gamma_i \in \{g, h\}$, we define the length $l(s) = n$ and the depth $\rho(s) = \#\{i : 1 \le i \le n, \gamma_i = h\}$; and if $n > 0$ (that is, if $s \ne 1$, the unit), we define the predecessor $p(s) = \prod_{j=1}^{n-1}\gamma_{n-j}$. We define $S^- = S \setminus \{1\}$.

We define the Cayley graph G of S to be an abstract directed graph with vertex set S, and a directed edge $p(s) \to s$ for each $s \in S^-$.

Note that G is an infinite tree with root vertex 1, such that every vertex $s \in S$ has two outward edges leaving it (the edges $s \to gs$ and $s \to hs$) and every vertex $s \in S^-$ has a single edge entering it (the edge $p(s) \to s$). If $l(s) = k$, the unique directed path from 1 to s consists of $k+1$ vertices $1 \to p^{k-1}(s) \to p^{k-2}(s) \to \dots \to p(s) \to s$.

Definition 2.2. For $s \in S$ we define the weight $w(s) = 2^{-\rho(s)}$, and if $l(s) = l$ we define

$$W(s) = \prod_{j=0}^{l-1} w(p^j s). \tag{5}$$

We define a Hilbert space $\mathscr{H} = l^2(S, W)$ to be the collection of all formal sums $\boldsymbol{x} = \sum_{s\in S} x_s \cdot s$ with $x_s \in \mathbb{C}$ and

$$\|\boldsymbol{x}\|^2 = \sum_{s\in S} W(s)^2 |x_s|^2 < \infty. \tag{6}$$

We define a particular subset $\mathscr{C} \subset S^-$, the *colour set*

$$\mathscr{C} = \{g^k : k \in \mathbb{N}\} \cup \{g^k hs : k \in \mathbb{N}_0, s \in S, 1 + l(s) \mid k\} \tag{7}$$

(here and elsewhere we use "$1+l(s) \mid k$" for "$1+l(s)$ divides k").

We define a *colour map* $\mu : S^- \to \mathscr{C}$ recursively as follows:

(8)
$$\mu(s) = \begin{cases} s & \text{if } s \in \mathscr{C}, \\ \mu(p^{n-k'}y) & \text{if } s = g^k hy,\ y \in S,\ l(y) = n,\ 1 \le k' \le n,\ k \equiv k' \bmod n+1. \end{cases}$$

Note that (8) really "works" as a recursive definition, because if $s \notin \mathscr{C}$, we necessarily have $s = g^k hy$ for some $k \in \mathbb{N}$ such that $1+l(y) \nmid k$; so writing $n = l(y)$, there is a unique $k' \in [1, n]$ such that $k' \equiv k \pmod{n+1}$. The iterated predecessor $p^{n-k'}y$ will not be equal to 1 because $k' > 0$ and $l(y) = n$, so $\mu(p^{n-k'}y)$ will be (recursively) defined. Note that for $s \in S$, the colour $\mu(hs)$ is always equal to hs, while the colour $\mu(gs)$ is either gs itself, or one of the iterated predecessors of gs. So we never have $\mu(gs) = \mu(hs)$ for any $s \in S$.

Definition 2.3. For each colour $c \in \mathscr{C}$, we define a linear map $T_c \in B(\mathscr{H})$ by its action on the basis S: for each $s \in S$, we define

$$T_c(s) = \begin{cases} gs & \text{if } \mu(gs) = c, \\ hs & \text{if } \mu(hs) = c, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Each T_c is a weighted shift operator (for S is an orthogonal, though not an orthonormal, basis of H). Writing $e_s = W(s)^{-1} \cdot s$ ($s \in S$) for the corresponding orthonormal basis, and giving due regard to the fact that $W(s)/W(p(s)) = w(s)$ for each $s \in S^-$, we have

$$T_c(e_s) = \begin{cases} w(gs)e_{gs} & \text{if } \mu(gs) = c, \\ w(hs)e_{hs} & \text{if } \mu(hs) = c, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

This implies that for each $c \in \mathscr{C}$,

$$\|T_c\| = \max\{w(x) : \mu(x) = c\} = w(c) = 2^{-\rho(c)}. \tag{11}$$

Definition 2.4. We define two families of *coordinatewise* orthogonal projections on $\mathscr{H}$. For $n \in \mathbb{N}_0$, P_n is the orthogonal projection onto $\overline{\mathrm{lin}}\{s \in S : \rho(s) = n\}$, and $\overline{P}_n = \sum_{i=0}^n P_i$; while π_n is the orthogonal projection onto $\mathrm{lin}\{s \in S : l(s) = n\}$, and $\overline{\pi}_n = \sum_{i=0}^n \pi_n$.

We also define, for $n \in \mathbb{N}_0$, a subgraph $G^{(n)}$ of G, obtained from G by deleting some of the edges. Specifically, $G^{(n)}$ is a graph with vertex set S and a directed edge $p(s) \to s$ for every $s \in S$ such that the *colour depth* $\rho\mu(s)$ is no greater than n. (Equivalently, we obtain $G^{(n)}$ by deleting from G every edge $p(s) \to s$ such that the colour depth $\rho\mu(s)$ is greater than n). If $K \subset G^{(n)}$ is a connected component, we define the coordinatewise projection $Q_{n,K}$ by

$$Q_{n,K}(s) = \begin{cases} s & \text{if } s \in K, \\ 0 & \text{otherwise} \end{cases} \qquad (s \in S). \tag{12}$$

We define $H_{n,K} = Q_{n,K}(\mathcal{H})$.

Note that while π_n has finite rank 2^n, the projection P_n always has infinite rank (even when $n = 0$, when it is the orthogonal projection onto $\overline{\text{lin}}\{g^k : k \geq 0\}$).

Definition 2.5. We define an algebra $\mathcal{A}_0 \subset B(\mathcal{H})$. $\mathcal{A}_0$ is the nonunital subalgebra of $B(\mathcal{H})$ generated by the operators T_c $(c \in \mathcal{C})$. We define the operator algebra $\mathcal{A} = \overline{A_0}$, the norm closure of A_0 in $B(\mathcal{H})$. We define $\mathcal{A}^{(n)} \subset \mathcal{A}_0$ to be the linear span of products $T = T_{c_k} T_{c_{k-1}} \dots T_{c_2} T_{c_1} = \prod_{i=0}^{k-1} T_{c_{k-i}}$ such that $c_i \in \mathcal{C}$ and $\max\{\rho(c_i) : 1 \leq i \leq k\} = n$. We define $\overline{\mathcal{A}}^{(n)} = \sum_{r=0}^{n} \mathcal{A}^{(r)}$, the subalgebra of $\mathcal{A}_0$ generated by maps T_c $(c \in \mathcal{C})$ with $\rho(c) \leq n$.

For $n, r \geq 0$, let $S_{n,r} = \{s \in S :$ the path from 1 to s in G contains exactly r edges $p(u) \to u$ with colour depth $\rho\mu(u) > n\}$. Let $P_{n,r}$ be the orthogonal projection onto $\overline{\text{lin}}(S_{n,r})$, and let $\overline{P}_{n,r} = \sum_{t=0}^{r} P_{n,t}$.

Note that $S_{n,0} = \{s \in S : \rho(s) \leq n\}$, so $P_{n,0} = \overline{P}_n$ for each $n \in \mathbb{N}_0$.

Lemma 2.6. (*a*) *For each $n \in \mathbb{N}_0$, the subspaces* $\ker \overline{P}_n$, $\ker \overline{\pi}_n \subset \mathcal{H}$ *are invariant for $\mathcal{A}$. Further, $\mathcal{A}$ maps* $\ker \overline{\pi}_n$ *into* $\ker \overline{\pi}_{n+1}$ *for each n.*

(*b*) *For each component K of $G^{(n)}$, the subspace $H_{n,K}$ is invariant for $\overline{\mathcal{A}}^{(n)}$ and also for the hermitian conjugate $(\overline{\mathcal{A}}^{(n)})^*$. The component of $G^{(n)}$ containing* 1 *is $S_{n,0}$, and the associated projection is $\overline{P}_n$.*

(*c*) *Every map T_c with $\rho(c) > n$ maps $\mathcal{H}$ into* $\ker \overline{P}_n$.

(*d*) *For $T \in \mathcal{A}_0$, the decomposition $T = \sum_{n=1}^{\infty} T^{(n)}$, with $T^{(n)} \in \mathcal{A}^{(n)}$, is unique and continuous*; *writing $\overline{T}^{(n)} = \sum_{i=0}^{n} T^{(i)}$, we have $\|\overline{T}^{(n)}\| \leq \|T\|$ for every n and T*; *in fact $\overline{T}^{(n)} = \sum_{r=0}^{\infty} P_{n,r} T P_{n,r}$ in the strong operator topology, while $T - \overline{T}^{(n)} = \sum_{r=0}^{\infty} (1 - \overline{P}_{n,r}) T P_{n,r}$.*

(*e*) *For all $s \in S$ we have $\rho\mu(s) \leq \rho(s)$, with equality if $s \in hS$.*

Proof. (a) is obvious because the generating maps T_c all map an element $s \in S$ to gs, hs, or zero; and we have $\rho(gs) \geq \rho(s)$, $\rho(hs) \geq \rho(s)$ and $l(gs) = l(s) + 1$, $l(hs) = l(s) + 1$ for all $s \in S$.

For $c \in \mathcal{C}$, we have $\langle T_c s, t \rangle \neq 0$ $(s, t \in S)$ only when there is an edge $s \to t$ in G, and $\mu(t) = c$. So if T is in $\overline{\mathcal{A}}^{(n)}$, the algebra generated by maps T_c with $\rho(c) \leq n$, and if $\langle Ts, t \rangle \neq 0$, then there is a path from s to t in G, and each edge $p(u) \to u$ in that path has $\rho\mu(u) \leq n$, so the edge $p(u) \to u$ is present in the graph $G^{(n)}$. Thus s, t belong to the same component of $G^{(n)}$. So for a connected component

$K \subset G^{(n)}$, the associated subspace $H_{n,K}$ is invariant for both $\mathcal{A}^{(n)}$ and $(\mathcal{A}^{(n)})^*$, establishing the first part of (b).

The component of $G^{(n)}$ containing 1 is the set of $s \in S$ such that the path from 1 to s in G contains only edges $p(u) \to u$ with $\rho\mu(u) \leq n$. Now for any $u \in S$, $\mu(u)$ is either u itself or one of the iterated predecessors $p^i(u)$; taking predecessors cannot increase the depth $\rho(u)$, so $\rho\mu(u) \leq \rho(u)$ for all u. If $s \in S$ with $\rho(s) \leq n$, then every edge $p(u) \to u$ in the path from 1 to s has colour depth $\rho\mu(u) \leq n$ also, so s lies in the component of $G^{(n)}$ containing 1. Conversely, if $\rho(s) > n$ then we have $s = g^k ht$ for some $t \in S$ and $k \in \mathbb{N}_0$; the edge $t \to ht$ is part of the path from 1 to s, and $ht \in \mathcal{C}$ by (7), so the colour depth satisfies $\rho\mu(ht) = \rho(ht) = \rho(s) > n$, and therefore s is not in the connected component of $G^{(n)}$ containing 1. Therefore that component is precisely $\{s : \rho(s) \leq n\}$, and the associated coordinatewise projection is $\overline{P}_n$. Thus we have established the second part of (b), and also part (e).

For part (c), note that T_c maps $\mathcal{H}$ into $\overline{\text{lin}}\{x \in S : \mu(x) = c\}$; if $\rho(c) > n$ then this subspace is contained in $\overline{\text{lin}}\{x \in S : \rho\mu(x) > n\} \subset \overline{\text{lin}}\{x \in S : \rho(x) > n\} \subset \ker \overline{P}_n$, where the first $\subset$ depends on part (e).

To prove part (d), we note that the edges of $G^{(n)}$ include the edge $p(u) \to u$ only if $\rho\mu(u) \leq n$; hence the set $S_{n,r}$ is a union of some of the components K of $G^{(n)}$. So by part (b) of this lemma, each image $P_{n,r}\mathcal{H}$ is $\mathcal{A}^{(n)}$-invariant; but for $c \in \mathcal{C}$ with $\rho(c) > n$, T_c maps $P_{n,r}\mathcal{H}$ into $P_{n,r+1}\mathcal{H}$ because $\langle T_c s, t\rangle \neq 0$ $(s, t \in S)$ only when $s = p(t)$ and the colour depth $\rho\mu(t)$ is greater than n. Now take any $T \in \mathcal{A}_0$ and write $T = \sum_i T^{(i)}$ with each $T^{(i)} \in \mathcal{A}^{(i)}$. We have $\overline{T}^{(n)} = \sum_{i=0}^n T^{(i)} \in \overline{\mathcal{A}}^{(n)}$, so each $P_{n,r}\mathcal{H}$ is a $\overline{T}^{(n)}$-invariant subspace; but $T - \overline{T}^{(n)}$ maps $P_{n,r}\mathcal{H}$ into $\bigoplus_{i=r+1}^\infty P_{n,i}\mathcal{H}$. Therefore we have

$$\overline{T}^{(n)} = \sum_{r=0}^\infty \overline{T}^{(n)} P_{n,r} = \sum_{r=0}^\infty P_{n,r} T^{(n)} P_{n,r} = \sum_{r=0}^\infty P_{n,r} T P_{n,r}, \tag{13}$$

while $T - \overline{T}^{(n)} = \sum_{r=1}^\infty (1 - \overline{P}_{n,r}) T P_{n,r}$ as required by the lemma. This shows that the decomposition $T = \sum_{i=0}^\infty T^{(i)}$ is indeed unique, and furthermore the compression $\overline{T}^{(n)}$ as given by (13) plainly satisfies $\|\overline{T}^{(n)}\| \leq \|T\|$. Thus the lemma is proved. □

Definition 2.7. Let us write $\mathcal{B}^{(n)}$ ($\overline{\mathcal{B}}^{(n)}$) for the norm closure of $\mathcal{A}^{(n)}$ ($\overline{\mathcal{A}}^{(n)}$) in $B(\mathcal{H})$. Let us write Δ_n for the map $\mathcal{A}_0 \to \mathcal{A}^{(n)}$ with $\Delta_n(T)$ the unique element $T^{(n)} \in \mathcal{A}^{(n)}$ such that $T = \sum_{n=0}^\infty T^{(n)}$; and let $\overline{\Delta}_n : \mathcal{A}_0 \to \overline{\mathcal{A}}^{(n)}$ be the map $\sum_{i=0}^n \Delta_i$.

The maps Δ_n, $\overline{\Delta}_n$ are uniformly norm-bounded by part (d) of the previous lemma; so they extend continuously to maps $\Delta_n : \mathcal{A} \to \mathcal{B}^{(n)}$ and $\overline{\Delta}_n : \mathcal{A} \to \overline{\mathcal{B}}^{(n)}$; and because of the uniform bound on $\|\overline{\Delta}_n\|$ (each $\overline{\Delta}_n$ is contractive), we have $T = \sum_{n=0}^\infty \Delta_n T = \sum_{n=0}^\infty T^{(n)}$, with $T^{(n)} \in \mathcal{B}^{(n)}$, for all $T \in \mathcal{A}$. The formulae $\overline{\Delta}_n T = \overline{T}^{(n)} = \sum_{r=0}^\infty P_{n,r} T P_{n,r}$ and $T - \overline{T}^{(n)} = \sum_{r=0}^\infty (1 - \overline{P}_{n,r}) T P_{n,r}$ remain true in the strong operator topology.

3. $\mathscr{A}$ is radical

In order to prove that our algebra $\mathscr{A}$ is radical, the main theorem we need is the following:

Theorem 3.1. *Every $T \in \mathscr{A}^{(n)}$, or the norm closure thereof, satisfies*

$$(1 - \overline{\pi}_k)\overline{P}_n T \to 0 \quad \text{as} \quad k \to \infty. \tag{14}$$

Indeed, $\overline{P}_n T$ is a compact operator. Furthermore, every $T \in \overline{\mathscr{A}}^{(n)}$ satisfies

$$\|T\| = \|\overline{P}_n T \overline{P}_n\|. \tag{15}$$

Proof of Theorem 3.1, first part. From Definition 2.2, we find that if $s \neq c$ but $\mu(s) = c$, then we must have $s = g^k hyc$ for some $k \in \mathbb{N}_0$ and $y \in S$. In particular, $\rho(s) > \rho(c)$. So if $\rho(c) = n$, then the map $\overline{P}_n T_c$ in fact has rank 1; it maps $p(c)$ to c, and all other $s \in S$ to zero. Any product $T = \prod_{i=0}^{k-1} T_{c_{k-i}}$ with $c_i \in \mathscr{C}$ and $\max\{\rho(c_i) : 1 \leq i \leq k\} = n$ accordingly satisfies $\overline{P}_n T = \prod_{i=0}^{k-1} \overline{P}_n T_{c_{k-i}}$ (because $\ker \overline{P}_n$ is an invariant subspace for each T_{c_j}) so the rank of $\overline{P}_n T$ is at most 1. $\mathscr{A}^{(n)}$ is the linear span of such maps, so any $T \in \mathscr{A}^{(n)}$, or its norm closure, will have $\overline{P}_n T$ a compact operator; hence $\|(1 - \overline{\pi}_k)\overline{P}_n T\| \to 0$ as $k \to \infty$. □

To prove the second part of the theorem, we need certain preliminaries, which we give in the following two lemmas, the first of which is rather elementary:

Lemma 3.2. *Let $M \in M_{m+1}(\mathbb{C})$ be a strictly lower triangular matrix, and let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on $\mathbb{C}^{m+1}$, with $\|\lambda_0, \lambda_1, \ldots, \lambda_m\| = \big(\sum_{i=0}^m \omega_i^2 |\lambda_i|^2\big)^{1/2}$ and $\|\lambda_0, \lambda_1, \ldots, \lambda_m\|' = \big(\sum_{i=0}^m (\omega_i')^2 |\lambda_i|^2\big)^{1/2}$ for positive constants ω_i, ω_i' (with $i = 1, \ldots, m$). Suppose we have*

$$\frac{\omega_{i+1}'}{\omega_i'} \leq \frac{1}{2} \cdot \frac{\omega_{i+1}}{\omega_i} \tag{16}$$

for each $i = 0, \ldots, m-1$. Then

$$\|M\|' \leq \|M\|. \tag{17}$$

Proof. Let $(e_i)_{i=0}^m$ be the unit vectors of $\mathbb{C}^{m+1}$, and write $Me_i = \sum_{j>i} M_{j,i} e_j$. We may assume $\|M\| = 1$, in which case $|M_{j,i}| \leq \|e_i\|/\|e_j\| = \omega_i/\omega_j$ for all i and j. For $k \in [1, m]$, the weighted shift matrix $M^{(k)}$ with

$$M^{(k)} e_i = \begin{cases} M_{i+k,i} e_{i+k} & \text{if } i + k \leq m, \\ 0 & \text{if } i + k > m \end{cases} \tag{18}$$

satisfies

$$\|M^{(k)}\|' = \max_{i \in [0, m-k]} \big|M_{i+k,i}\big| \frac{\omega_{i+k}'}{\omega_i'} \leq \max_{i \in [0, m-k]} \left(\frac{\omega_i}{\omega_{i+k}}\right) \cdot \frac{\omega_{i+k}'}{\omega_i'} \leq 2^{-k},$$

by (16). Then $M = \sum_{k=1}^{m} M^{(k)}$, so $\|M\|' \le \sum_{k=1}^{m} 2^{-k} < 1$. □

Lemma 3.3. (*a*) *Let K be a connected component of $G^{(n)}$. Then either K equals $S_{n,0}$, the component which contains* 1, *or K consists of a path*

$$y \to gy \to g^2 y \to \cdots \to g^m y$$

for some $y \in S$ and $m \in \mathbb{N}$ such that the colour depths satisfy $\rho\mu(y) > n$ and $\rho\mu(g^{m+1}y) > n$, but $\rho\mu(g^i y) \le n$ for $i \in [1, m]$. Furthermore, there is a path $s_0 \to s_1 \to \cdots \to s_m$ in the component $S_{n,0}$ such that the colours satisfy $\mu(s_i) = \mu(g^i y)$ for each $i \in [1, m]$.

(*b*) *For every $T \in \mathcal{A}$, we have $\|(1 - \overline{P}_n)T\| \to 0$ as $n \to \infty$.*

Proof. (a) Suppose $K \ne S_{n,0}$. Since K cannot meet $S_{n,0}$, every vertex $x \in K$ must have $\rho(x) > n$. But if $x \to x'$ is an edge in K, we must have $\rho\mu(x') \le n$, and therefore $\mu(x') \ne x'$, so $x' \notin \mathcal{C}$, so $x' = g^k hz$ for some $z \in S$ and $k > 0$ with $\rho(hz) = \rho(x') > n$. Indeed, we must have $1 + l(z) \nmid k$. Every edge of K must be of form $x \to gx$ rather than $x \to hx$, so K does indeed consist of a path (finite or infinite) of form $g^r hz \to g^{r+1}hz \to g^{r+2}hz \to \cdots$, for some $r \ge 0$. But we have the condition $1 + l(z) \nmid k$ for any k such that $k > r$ and $g^k hz$ is in the path; so the path is finite. Its last vertex must be $g^t hz$ for some t with $t - r \le 1 + l(z)$. Writing $m = t - r$ and $y = g^r hz$, we see that $K = \{g^i y : i = 0, \ldots, m\}$.

If $r > 0$, we must have $\rho\mu(y) = \rho\mu(g^r hz) > n$ or we could continue the path in K backwards to include the vertex $g^{r-1}hy$. If $r = 0$, we have $\mu(y) = \mu(hz) = hz$, so $\rho\mu(y) > n$ anyway. Also, we must have $\rho\mu(g^{t+1}hy) > n$ or we could include the vertex $g^{t+1}hy$ in our component K. For $i \in (r, t]$ we have $\rho\mu(g^i hy) \le n$ because the edge $g^{i-1}hy \to g^i hy$ lies in K. Thus the component K is as described in part (a) of this lemma.

To complete the proof of part (a), we claim that there is a sequence $s_0 \to s_1 \to \cdots \to s_m \in S_{n,0}$ such that $\mu(s_i) = \mu(g^i y)$ for each $i \in [1, m]$. This is proved by induction on $l(y) = \min\{l(u) : u \in K\}$. If $l(y) \le n$, there is nothing to prove because the component is $S_{n,0}$ after all. If the component K is not $S_{n,0}$, write $K = \{g^i hz : r \le i \le r + m\}$. We return to (8) to compute the colours $\mu(g^i hz)$ for $i \in (r, r + m]$. Writing $l = l(z)$ and $z = \prod_{i=0}^{l-1} z_{l-i}$ ($z_j \in \{g, h\}$), we find that if $i' \in [1, l]$ is the unique integer with $i' \equiv i \pmod{1 + l}$, then $\mu(g^i hz) = \mu\bigl(\prod_{j=0}^{i'-1} z_{i'-j}\bigr) = \mu(p^{l-i'} z)$. If $r_0 \in [0, l]$ satisfies $r_0 \equiv r \pmod{l + 1}$, then the sequence $\mu(g^i y)$ ($i = 1, \ldots, m$) is the sequence $\mu(p^{l-r_0-i} z)$ ($i = 1, \ldots, m$). The vertices $(p^{l-r_0-i} z)_{i=0}^m$ form a path in G which, since it involves the same colours for $i > 0$, is also a path in $G^{(n)}$. So this path is part of a component K' of $G^{(n)}$. If $K' = S_{n,0}$ we are done; if not, we note that the minimum length of an element of K' is strictly less than $l(y)$, so the result follows by induction hypothesis.

(b) Let $c \in \mathscr{C}$. From (10), for $s \in S$ we have

$$(1-\overline{P}_n)T_c e_s = \begin{cases} w(gs)e_{gs} & \text{if } \mu(gs)=c \text{ and } \rho(gs)>n, \\ w(hs)e_{hs} & \text{if } \mu(hs)=c \text{ and } \rho(hs)>n, \\ 0 & \text{otherwise.} \end{cases}$$

But $w(x)=2^{-\rho(x)}$, so $\|(1-\overline{P}_n)T_c\| \le 2^{-n-1}$. We will also have $\|(1-\overline{P}_n)T\| \to 0$ for any operator T in the norm-closed right ideal generated by the operators T_c. But this right ideal is the entire algebra $\mathscr{A}$. □

Proof of Theorem 3.1, second part. By Lemma 2.6(b), when $T \in \overline{\mathscr{A}}^{(n)}$ we have $T=\sum_K Q_{n,K}TQ_{n,K}$, where the sum is taken over the connected components K of $G^{(n)}$. So

$$\|T\| = \sup_K \|Q_{n,K}TQ_{n,K}\|. \tag{19}$$

If K equals $S_{n,0}$, the component containing 1, then the norm $\|Q_{n,K}TQ_{n,K}\|$ equals $\|\overline{P}_nT\overline{P}_n\|$. If K is any other component, we claim that the norm is at most $\|\overline{P}_nT\overline{P}_n\|$. By Lemma 3.3(a), we can write $K=\{g^iy : 0 \le i \le m\}$ for suitable $y \in S$ and m; writing γ_i for the colour $\mu(g^iy)$, there is also a set $\kappa=\{s_i : 0 \le i \le m\} \subset S_{n,0}$ such that the colour $\mu(s_i)$ equals γ_i for $i \in [1,m]$. Let q denote the orthogonal (coordinatewise) projection onto $\operatorname{lin}(\kappa)$. If $c_1, c_2, \dots, c_r \in \mathscr{C}$, then the compression $\tau_1 = Q_{n,K}T_{c_r}T_{c_{r-1}}\dots T_{c_1}Q_{n,K}$ sends g^iy to $g^{i+r}y$, if $i+r \le m$ and $c_i=\gamma_{r+i}$ for each $i=1,\dots,r$; otherwise, we have $\tau_1g^iy=0$. Similarly, the compression $\tau_2 = qT_{c_r}T_{c_{r-1}}\dots T_{c_1}q$ sends s_i to s_{i+r} if $i+r \le m$ and $c_i=\gamma_{r+i}$ for each $i=1,\dots,r$; otherwise, we have $\tau_2s_i=0$. So the compressions τ_1 and τ_2 are intertwined by the map η sending g^iy to s_i for each i. Indeed, if $T \in \overline{\mathscr{A}}^{(n)}$, the compressions $\tau=Q_{n,K}TQ_{n,K}$ and $\tau'=qTq$ are intertwined, with $\eta\tau=\tau'\eta$. So τ has the same $(m+1)\times(m+1)$ matrix M with respect to the basis $(g^iy)_{i=0}^m$ of $Q_{n,K}\mathscr{H}$, as τ' has with respect to the basis $(s_i)_{i=0}^m$ of $q\mathscr{H}$. M is strictly lower triangular, because all such compressions qTq map s_i into $\operatorname{lin}\{s_j : j>i\}$ for each i. The norm on $q\mathscr{H}$ is given by $\left\|\sum_{i=0}^m \lambda_i s_i\right\| = \left(\sum_{i=0}^m \omega_i^2|\lambda_i|^2\right)^{1/2}$, where $\omega_i=W(s_i)$. The norm on $Q_{n,K}\mathscr{H}$ is likewise given by $\left\|\sum_{i=0}^m \lambda_i g^iy\right\| = \left(\sum_{i=0}^m (\omega_i')^2|\lambda_i|^2\right)^{1/2}$, where $\omega_i'=W(g^iy)$. For $0 \le i < m$, the ratio ω_{i+1}/ω_i equals $W(s_{i+1})/W(s_i)=w(s_{i+1})$ because there is an edge $s_i \to s_{i+1}$ in G; and $w(s_{i+1}) \ge 2^{-n}$ because $s_{i+1} \in S_{n,0}$ so $\rho(s_{i+1}) \le n$. On the other hand, the ratio ω_{i+1}'/ω_i' equals $W(g^{i+1}y)/W(g^iy)=w(g^{i+1}y) \le 2^{-n-1}$, because $g^{i+1}y \notin S_{n,0}$ so $\rho(g^{i+1}y) \ge n+1$. We deduce that $\omega_{i+1}'/\omega_i' \le \frac{1}{2}\cdot\omega_{i+1}/\omega_i$. By Lemma 3.2, we have $\|Q_{n,K}TQ_{n,K}\|=\|\tau\| \le \|\tau'\|$, and of course $\|\tau'\| \le \|\overline{P}_nT\overline{P}_n\|$ because the orthogonal projection satisfies $q \le \overline{P}_n$. By (19), the norm of T is the supremum of $\|\overline{P}_nT\overline{P}_n\|$ and the norms $\|Q_{n,K}TQ_{n,K}\|$ for all other connected components $K \subset G^{(n)}$; so $\|T\|=\|\overline{P}_nT\overline{P}_n\|$ as claimed by the theorem. □

We can now prove the main theorem of this section:

Theorem 3.4. *$\mathcal{A}$ is radical.*

Proof. If not, let $T \in \mathcal{A}$ have spectral radius at least 1. By Lemma 3.3, there is an $n \in \mathbb{N}$ such that $\|(1-\overline{P}_n)T\| \le \frac{1}{2}$. We claim that the spectral radius of the compression $\overline{P}_n T \overline{P}_n$ is at least 1. For by Lemma 2.6(a), for each $k \in \mathbb{N}$ we have $T^k = \overline{P}_n T^k \overline{P}_n + (1-\bar{P}_n)T^k$ (any $k \in \mathbb{N}$) because $\ker \overline{P}_n$ is an invariant subspace for $\mathcal{A}$; indeed, $T^k = (\overline{P}_n T \overline{P}_n)^k + (1-\overline{P}_n)T^k$, because the compression map $T \to \overline{P}_n T \overline{P}_n$ is an algebra homomorphism on $\mathcal{A}$. So for all $k > 0$, $T^k = (\overline{P}_n T \overline{P}_n)^k + (1-\overline{P}_n)T \cdot T^{k-1}$, and hence

$$
\begin{aligned}
\|T^k\| &\le \|(\overline{P}_n T \overline{P}_n)^k\| + \tfrac{1}{2} \cdot \|T^{k-1}\| \\
&\le \|(\overline{P}_n T \overline{P}_n)^k\| + \tfrac{1}{2} \cdot \|(\overline{P}_n T \bar{P}_n)^{k-1}\| + \tfrac{1}{4} \cdot \|T^{k-2}\| \\
&\le \cdots \le 2^{-k} + \sum_{j=0}^{k-1} 2^{-j} \|(\overline{P}_n T \overline{P}_n)^{k-j}\|.
\end{aligned}
$$

If the spectral radius of $\overline{P}_n T \overline{P}_n$ is less than 1, we can find $r < 1$ and $C > 0$ such that $\|(\overline{P}_n T \overline{P}_n)^j\| \le C r^j$ for all $j \in \mathbb{N}$, so we have $1 \le \|T^k\| \le 2^{-k} + \sum_{j=0}^{k-1} C \cdot 2^{-j} \cdot r^{k-j}$ $\le 2^{-k} + kC \max\bigl(\frac{1}{2}, r\bigr)^k$ for all $k \in \mathbb{N}$. This is a contradiction for large k, so the spectral radius of the compression $\overline{P}_n T \overline{P}_n$ must be at least 1.

It is thus sufficient to show that for each $T \in \mathcal{A}$ and $n \in \mathbb{N}$, the compression $\overline{P}_n T \overline{P}_n$ is quasinilpotent. Let us prove this by induction on n, beginning with the not-quite-trivial case $n = 0$.

By Lemma 2.6(d) (and its generalisation to $T \in \mathcal{A}$ rather than $T \in \mathcal{A}_0$ as discussed after Definition 2.7), we have $\overline{P}_0 T \overline{P}_0 = \overline{P}_0 \overline{T}^{(0)} \overline{P}_0 = \overline{P}_0 T^{(0)} \overline{P}_0$ for any $T \in \mathcal{A}$; and $T^{(0)} \in \mathcal{B}^{(0)}$. By Theorem 3.1, we have $(1-\overline{\pi}_k)\overline{P}_0 T^{(0)} \to 0$, and by Lemma 2.6(a), $T^{(0)}$ maps $\ker \overline{\pi}_k$ into $\ker \overline{\pi}_{k+1}$ for every k. Writing $\varepsilon_k = \|(1-\overline{\pi}_k)\overline{P}_0 T^{(0)}\|$, we have $\varepsilon_k \to 0$, and

$$
\begin{aligned}
(\overline{P}_0 T \overline{P}_0)^k &= (\overline{P}_0 T^{(0)} \overline{P}_0)^k \\
&= (1-\overline{\pi}_{k-1})\overline{P}_0 T^{(0)} (1-\overline{\pi}_{k-2})\overline{P}_0 T^{(0)} (1-\overline{\pi}_{k-3}) \\
&\qquad \ldots \overline{P}_0 T^{(0)} (1-\overline{\pi}_0)\overline{P}_0 T^{(0)} \overline{P}_0,
\end{aligned}
$$

so $\|(\overline{P}_0 T \overline{P}_0)^k\| \le \prod_{j=0}^{k-1} \varepsilon_j$, and hence $\overline{P}_0 T \overline{P}_0$ is indeed quasinilpotent.

Proceeding to the case of a general $n \in \mathbb{N}$, we note that for $T \in \mathcal{A}$, $\overline{P}_n T \overline{P}_n = \overline{P}_n \overline{T}^{(n)} \overline{P}_n = \overline{P}_n (T^{(n)} + \overline{T}^{(n-1)}) \overline{P}_n$, where $T^{(n)} \in \mathcal{B}^{(n)}$ and $\overline{T}^{(n-1)} \in \overline{\mathcal{B}}^{(n-1)}$.

Writing $\tau = \overline{T}^{(n-1)}$, we have $\tau^k \in \overline{\mathcal{B}}^{(n-1)}$ for all k, so by Theorem 3.1, $\|\tau^k\| = \|\overline{P}_{n-1} \tau^k \overline{P}_{n-1}\|$ for all k. But $\ker \overline{P}_{n-1}$ is an invariant subspace for $\mathcal{A}$, so

$$
\overline{P}_{n-1} \tau^k \overline{P}_{n-1} = (\overline{P}_{n-1} \tau \overline{P}_{n-1})^k;
$$

and our induction hypothesis tell us that $\overline{P}_{n-1}\tau\overline{P}_{n-1}$ is quasinilpotent. Thus $\|\tau^k\|^{1/k} \to 0$ as $k \to \infty$, and also $\|(\overline{P}_n\tau\overline{P}_n)^k\|^{1/k} = \|\overline{P}_n\tau^k\overline{P}_n\|^{1/k} \to 0$ as $k \to \infty$. So $\overline{P}_n\overline{T}^{(n-1)}$ is quasinilpotent.

Meanwhile $\sigma = \overline{P}_nT^{(n)}$ is a compact operator by Theorem 3.1, satisfying $\varepsilon_k = \|(1-\overline{\pi}_k)\sigma\| \to 0$ as $k \to \infty$; and both σ and τ map $\ker\overline{\pi}_k$ into $\ker\overline{\pi}_{k+1}$ for each k.

Let us pick an arbitrary $\delta > 0$ and choose $C > 0$ such that $\|(\overline{P}_n\overline{T}^{(n-1)})^k\| \le C\cdot\delta^k$ for all $k \in \mathbb{N}_0$. Then for any $k \in \mathbb{N}$, we have

$$\begin{aligned}(\overline{P}_nT\overline{P}_n)^k &= (\overline{P}_nT^{(n-1)} + \sigma)^k\overline{P}_n \\ &= \sum_{r=0}^{k}\sum_{\substack{i_0+i_1+\cdots+i_r=k-r\\ i_j\in\mathbb{N}_0}}(\overline{P}_nT^{(n-1)})^{i_0}\cdot\prod_{j=1}^{r}\sigma\cdot(\overline{P}_nT^{(n-1)})^{i_j}\cdot\overline{P}_n,\end{aligned}$$

and writing $u_j = \sum_{t=j}^{r}(1+i_t)-1$, the product from $j=1$ to r is equal to $\prod_{j=1}^{r}(1-\overline{\pi}_{u_j})\sigma(\overline{P}_nT^{(n-1)})^{i_j}$; so

$$\|(\overline{P}_nT\overline{P}_n)^k\| \le \sum_{r=0}^{k}\sum_{\substack{i_0+i_1+\cdots+i_r=k-r\\ i_j\in\mathbb{N}_0}} C^{r+1}\delta^{k-r}\cdot\prod_{j=1}^{r}\varepsilon_{u_j}.$$

Now $u_j \ge j-1$ in all cases, so writing $\eta_j = \prod_{j=1}^{r}\varepsilon_{j-1}$, we have

$$\|(\overline{P}_nT\overline{P}_n)^k\| \le \sum_{r=0}^{k}\sum_{\substack{i_0+i_1+\cdots+i_r=k-r\\ i_j\in\mathbb{N}_0}} C^{r+1}\delta^{k-r}\eta_r = \sum_{r=0}^{k}\binom{k}{r}C^{r+1}\delta^{k-r}\eta_r.$$

But $\eta_r^{1/r} \to 0$, so we can choose $D > 0$ such that $\eta_r \le D\cdot(\delta/C)^r$ for all r; substituting this in the previous equation, we find that $\|(\overline{P}_nT\overline{P}_n)^k\| \le \sum_{r=0}^{k}\binom{k}{r}CD\delta^k = CD\cdot(2\delta)^k$. So the spectral radius of $\overline{P}_nT\overline{P}_n$ is at most 2δ; but $\delta > 0$ was arbitrary, so $\overline{P}_nT\overline{P}_n$ is quasinilpotent. Therefore every $T \in \mathcal{A}$ is quasinilpotent; $\mathcal{A}$ is a radical Banach algebra. □

4. $\overline{\mathcal{A}}^{w*}$ is semisimple

We wish to prove the second half of our main result, namely that the bidual $\mathcal{A}^{**}$ is semisimple. We shall do this by showing that the weak-* closure $\overline{\mathcal{A}}^{w*}$ of $\mathcal{A}$ in $B(\mathcal{H})$ is semisimple, and then show that the natural representation $\theta : \mathcal{A}^{**} \to B(\mathcal{H})$, whose image is $\overline{\mathcal{A}}^{w*}$, is faithful, so that $\mathcal{A}^{**}$ itself is semisimple. (Our "natural representation" is the restriction to $\mathcal{A}^{**}$ of the natural projection $\mathcal{T}^{***} \to \mathcal{T}^*$, where $\mathcal{T}$ are the trace-class operators on $\mathcal{H}$, and $\mathcal{T}^* = B(\mathcal{H})$, $\mathcal{T}^{***} = B(\mathcal{H})^{**}$).

In this section, we show that $\overline{\mathcal{A}}^{w*}$, very unlike $\mathcal{A}$ itself, is semisimple.

Definition 4.1. Let $\mathscr{C}^{<\infty}$ be the collection of all finite sequences $(c_1, c_2, \ldots, c_m)$ of colours $c_i \in \mathscr{C}$, for $m \in \mathbb{N}$ (we exclude $m = 0$). For $\boldsymbol{c} = (c_1, c_2, \ldots, c_m) \in \mathscr{C}^{<\infty}$, let $T_{\boldsymbol{c}}$ denote the operator $\prod_{i=1}^{m} T_{c_i} \in \mathscr{A}_0$. Let $S_{\mathscr{A}} \subset \mathscr{C}^{<\infty}$ be the set of $\boldsymbol{c} \in \mathscr{C}$ such that $T_{\boldsymbol{c}} \neq 0$.

We think of $S_{\mathscr{A}}$ as the "support" of $\mathscr{A}$, because clearly every $T \in \mathscr{A}_0$ is equal to a sum

$$T = \sum_{\boldsymbol{c} \in S_{\mathscr{A}}} \lambda_{\boldsymbol{c}} \cdot T_{\boldsymbol{c}}, \tag{20}$$

the coefficients $\lambda_{\boldsymbol{c}} \in \mathbb{C}$ being finitely nonzero.

Lemma 4.2. *Given $T \in \mathscr{A}_0$, the coefficients $\lambda_{\boldsymbol{c}}(T)$ such that $T = \sum_{\boldsymbol{c} \in S_{\mathscr{A}}} \lambda_{\boldsymbol{c}}(T) \cdot T_{\boldsymbol{c}}$ are unique, and they are weak-* continuous linear functionals of T.*

Proof. For $c \in \mathscr{C}$, (10) tells us that $\langle T_c e_s, e_t \rangle \neq 0$ if and only if $s = p(t)$ and the colour $\mu(t) = c$, in which case it is equal to $w(t)$. Any easy induction then tells us that for $\boldsymbol{c} = (c_1, c_2, \ldots, c_m) \in S_{\mathscr{A}}$, $\langle T_{\boldsymbol{c}} e_s, e_t \rangle \neq 0$ if and only if $s = p^m(t)$ and, for each $i = 1, \ldots, m$, the colour $\mu(p^{i-1}t)$ equals c_i. In that case, $\langle T_{\boldsymbol{c}} e_s, e_t \rangle = \prod_{i=0}^{m-1} w(p^i t) = W(t)/W(s)$. So for fixed s, t, the colour sequence $\boldsymbol{c} \in S_{\mathscr{A}}$ such that $\langle T_{\boldsymbol{c}} e_s, e_t \rangle \neq 0$ is unique if it exists; and since $T_{\boldsymbol{c}} \neq 0$ for $\boldsymbol{c} \in S_{\mathscr{A}}$, for fixed $\boldsymbol{c} \in S_{\mathscr{A}}$ there is at least one pair $s, t \in S$ such that $\langle T_{\boldsymbol{c}} e_s, e_t \rangle \neq 0$.

Given $T \in \mathscr{A}_0$, $T = \sum_{\boldsymbol{c} \in S_{\mathscr{A}}} \lambda_{\boldsymbol{c}} \cdot T_{\boldsymbol{c}}$, we therefore have

$$\lambda_{\boldsymbol{c}} = \lambda_{\boldsymbol{c}}(T) = \frac{W(s)}{W(t)} \langle T e_s, e_t \rangle, \tag{21}$$

where s, t is any pair such that $\langle T_{\boldsymbol{c}} e_s, e_t \rangle \neq 0$. Now $\lambda_{\boldsymbol{c}}$ is indeed uniquely determined by T, and it is indeed a weak-* continuous function of T; (21) even equates $\lambda_{\boldsymbol{c}} \in B(H)_*$ with an element of $\mathscr{T}$ of rank 1. □

Given two elements $\boldsymbol{c} = (c_1, \ldots, c_m)$, $\boldsymbol{d} = (d_1, \ldots d_n)$ in $\mathscr{C}$, we can define the product $\boldsymbol{c} \cdot \boldsymbol{d}$ to be the sequence $(c_1, \ldots, c_m, d_1, \ldots, d_n)$. From (20), we see that for $T, T' \in \mathscr{A}_0$, we have

$$\lambda_{\boldsymbol{c}}(TT') = \sum_{\substack{\boldsymbol{d}, \boldsymbol{e} \in S_{\mathscr{A}}, \\ \boldsymbol{d} \odot \boldsymbol{e} = \boldsymbol{c}}} \lambda_{\boldsymbol{d}}(T) \cdot \lambda_{\boldsymbol{e}}(T'), \tag{22}$$

where the product $\boldsymbol{d} \odot \boldsymbol{e}$ denotes concatenation of sequences. The sum is always finite (it has $m - 1$ terms when $\boldsymbol{c} = (c_1, \ldots, c_m)$), so (22) remains true even when we extend $\lambda_{\boldsymbol{c}}$ to the weak-* closure $\overline{\mathscr{A}}^{w*}$ of $\mathscr{A}_0$.

Now for each $c \in \mathscr{C}$, (10) tells us that the left support projection $l(T_c)$ for the operator T_c is the orthogonal projection onto $\overline{\text{lin}}\{e_t : t \in S^-, \mu(t) = c\}$. We also have $\|T_c\| = w(c) = 2^{-\rho(c)} \leq 1$. These left support projections are mutually orthogonal for different colours c. The corresponding right support projection $r(T_c)$ is the projection onto $\overline{\operatorname{lin}}\{e_s : s \in S, s = p(t), \mu(t) = c\}$. These right support projections are not mutually orthogonal, but nevertheless, for each $s \in S$ there are only two $t \in S^-$ such that $s = p(t)$, so the norm of any sum $\sum_{c\in S_{\mathscr{A}}} \lambda_c r(T_c)$ is at most $2 \cdot \sup\{|\lambda_c| : c \in S_{\mathscr{A}}\}$. Hence for any sequence $\boldsymbol{x} \in l^\infty(S_{\mathscr{A}})$, the formal sum

$$T = \sum_{c\in S_{\mathscr{A}}} \frac{x_c}{w_c} \cdot T_c \tag{23}$$

satisfies

$$T^*T = \sum_{c,d\in S_{\mathscr{A}}} \frac{x_c^* x_d}{w_c w_d} T_c^* l(T_c) l(T_d) T_d = \sum_{c\in S_{\mathscr{A}}} \frac{|x_c|^2}{w_c^2} T_c^* T_c \le \sum_{c\in S_{\mathscr{A}}} |x_c|^2 r(T_c),$$

in particular $\|T^*T\| \le 2 \cdot \|\boldsymbol{x}\|_\infty^2$. So the sum T in fact converges in the weak-* topology to an element of $\overline{\mathscr{A}}^{w*}$ of norm at most $\sqrt{2} \cdot \|\boldsymbol{x}\|_\infty$.

Theorem 4.3. *$\overline{\mathscr{A}}^{w*}$ is semisimple.*

Proof. Let $T \in \overline{\mathscr{A}}^{w*}$, $T \neq 0$. We claim that $T \notin \operatorname{rad} \overline{\mathscr{A}}^{w*}$. Let us choose $s, t \in S$ such that $\langle Te_s, e_t\rangle \neq 0$.

Suppose first that $s \neq 1$. Let $l_0 = l(s) > 0$, and for $i = 1, \dots, l_0$, write $d_i = \mu(p^{i-1}s) = \mu(p^{i+m-1}t)$. Writing $\boldsymbol{d} = (d_1, \dots, d_{l_0}) \in \mathscr{C}^{<\infty}$, we will have $T_{\boldsymbol{d}}(1) = s$, so $\boldsymbol{d} \in S_{\mathscr{A}}$ and the product $T' = T \cdot T_{\boldsymbol{d}}$ satisfies $\langle T'e_1, e_t\rangle \neq 0$. Furthermore, in order to show $T \notin \operatorname{rad} \overline{\mathscr{A}}^{w*}$ it is enough to show that $T' \notin \operatorname{rad} \overline{\mathscr{A}}^{w*}$, because the radical is an ideal. So, we can replace T with T' if necessary, and assume that $\langle Te_1, e_t\rangle \neq 0$.

Then $\lambda_{\boldsymbol{c}}(T) \neq 0$, where $\boldsymbol{c} = (c_1, c_2, \dots, c_l) \in S_{\mathscr{A}}$ is the unique sequence such that $l = l(t)$ (so $1 = p^l t$), and the colours $\mu(p^{i-1}t)$ $(i = 1, \dots, l)$ are c_i. Write $\xi_m = g^{(m-1)(l+1)}ht$, and let $E \subset \mathscr{C}$ be the collection $\{\xi_m : m \in \mathbb{N}_0\}$ (noting from (7) that these elements are truly elements of the colour set $\mathscr{C}$). Let us also note that the weight $w_{\xi_m} = 2^{-\rho(\xi_m)} = 2^{-(1+\rho(t))}$ is independent of m. So $U = \sum_{c\in E} T_c \in \overline{\mathscr{A}}^{w*}$ (for U is a weak-* convergent sum like T in (23)). We claim that the product $U \cdot T \in \overline{\mathscr{A}}^{w*}$ is not quasinilpotent, so UT and T itself are not in the radical of $\overline{\mathscr{A}}^{w*}$. To prove this, we compute the inner product $\langle (UT)^m e_1, e_{\xi_m}\rangle$ for every $m \in \mathbb{N}$. Obviously $\lambda_{\boldsymbol{d}}(U) = 1$ (if $\boldsymbol{d} \in E$) or zero otherwise.

Now the length L equals $l(\xi_m) = m(1 + l)$, and the colour sequence $\mu(p^{i-1}\xi_m)$ (with $i = 1, \dots, L$) is obtained from (8) as follows: if $1+l \mid i-1$, we have $p^{i-1}\xi_m = g^{(m-1-r)(l+1)}ht$ (with $r = (i-1)/(1+l) \in [0, m)$), and $\mu(p^{i-1}\xi_m) = p^{i-1}\xi_m = \xi_{m-r} \in E$. But if $1+l \nmid i-1$, then writing $i-1 = r(l+1)+j$ (with $r \in [0, m)$, $j \in [1, l]$), if $r = m-1$ we have $p^{i-1}\xi_m = p^j ht = p^{j-1}t$, so $\mu(p^{j-1}\xi_m) = c_j$; but if $r < m-1$ we have $p^{i-1}\xi_m = g^{(m-2-r)(l+1)+l+1-j}ht$ and the recursive definition in (8) tells us that $\mu(p^{i-1}\xi_m) = \mu(p^{l-(l+1-j)}t) = \mu(p^{j-1}t) = c_j$ also. So for all i,

$1 \le i \le L$, we have

$$\mu(p^{i-1}\xi_m) = \begin{cases} \xi_{m-r} \in E & \text{if } i-1 = r(1+l), \\ c_j \notin E & \text{if } i-1 \equiv j \pmod{l+1},\ 1 \le j \le l. \end{cases} \tag{24}$$

The full sequence $(\mu(p^{i-1}\xi_m))_{i=1}^{L} \in S_{\mathscr{A}}$ is the concatenation $\bigodot_{r=0}^{m-1}(\xi_{m-r} \odot \boldsymbol{c})$, where we slightly abuse notation by writing ξ_{m-r} for the sequence of length 1 in $\mathscr{C}^{<\infty}$. Now from (21), we have the inner product

$$\langle (UT)^m e_1, e_{\xi_m} \rangle = W(\xi_m) \cdot \lambda_{\xi_m}((UT)^m); \tag{25}$$

and using (22) $2m$ times, we have

$$\lambda_{\xi_m}((UT)^m) = \sum_{\substack{\boldsymbol{d}^{(1)},\boldsymbol{c}^{(1)},\dots,\boldsymbol{d}^{(m)},\boldsymbol{c}^{(m)} \in S_{\mathscr{A}}, \\ \bigodot_{i=1}^{m}(\boldsymbol{d}^{(i)} \odot \boldsymbol{c}^{(i)}) = \bigodot_{r=0}^{m-1}(\xi_{m-r} \odot \boldsymbol{c})}} \prod_{i=1}^{m} \lambda_{\boldsymbol{d}^{(i)}}(U)\lambda_{\boldsymbol{c}^{(i)}}(T).$$

But the coefficient $\lambda_{\boldsymbol{d}}(U)$ can only be nonzero if the sequence $\boldsymbol{d}$ has length 1 and consists of one of the colours $\xi_j \in E$ (in which case the coefficient is equal to 1). There are only m such colours in the sequence $\bigodot_{r=1}^{m}(\xi_{m-r} \odot \boldsymbol{c})$, and the rest of the sequence consists precisely of m copies of $\boldsymbol{c}$, so in fact

$$\lambda_{\xi_m}((UT)^m) = \lambda_{\boldsymbol{c}}(T)^m. \tag{26}$$

Equation (26) makes the rest of the proof rather straightforward. Substituting it in (25), we have $\|(UT)^m\| \ge |\langle (UT)^m e_1, e_{\xi_m} \rangle| = |\lambda_{\boldsymbol{c}}(T)|^m \cdot W(\xi_m)$; where writing $L = m(1+l)$ as usual, we have $W(\xi_m) = \prod_{j=1}^{L} w(p^{j-1}\xi_m) = 2^{-\sum_{j=1}^{L}\rho(p^{j-1}\xi_m)}$, from Definition 2.2. But $\xi_m = g^{(m-1)(l+1)}ht$, so $\rho(\xi_m) = 1 + \rho(t)$. And $\rho(p^i\xi_m) \le \rho(\xi_m)$ for all $i \ge 0$, so for all $m \in \mathbb{N}$,

$$\|(UT)^m\| \ge |\lambda_{\boldsymbol{c}}(T)|^m \cdot 2^{-L(1+\rho(t))} = |\lambda_{\boldsymbol{c}}(T)|^m \cdot 2^{-m(1+l)(1+\rho(t))}.$$

Accordingly $UT \in \overline{\mathscr{A}}^{w*}$ is not a quasinilpotent operator, and $T \notin \operatorname{rad} \overline{\mathscr{A}}^{w*}$. $\square$

5. $\mathscr{A}^{**}$ is semisimple

Let $\theta_0 : \mathscr{T}^{***} \to \mathscr{T}^* = B(\mathscr{H})$ be the natural projection, which is an algebra homomorphism, and let $\theta = \theta_0|_{\mathscr{A}^{**}}$ be the restriction, which is a representation of $\mathscr{A}^{**}$. If $\tau \in \mathscr{A}^{**}$ is a weak-* limit of operators T_α in $\mathscr{A}$, then for each $\eta, \zeta \in \mathscr{H}$, we have $\langle \theta(\tau)\eta, \zeta \rangle = \lim_\alpha \langle T_\alpha \eta, \zeta \rangle$, so $\theta(\tau)$ is the $\sigma(B(\mathscr{H}), \mathscr{T})$-limit of the operators T_α, and the image $\theta(\mathscr{A}^{**})$ is contained in the weak-* closure $\overline{\mathscr{A}}^{w*}$ of $\mathscr{A}$ in $B(\mathscr{H})$. Conversely, the image of the unit ball of $\mathscr{A}^{**}$, being the weak-* continuous image of a weak-* compact set, is weak-* compact, and therefore contains the weak-* closure $\overline{B}^{w*}$ of the unit ball of $\mathscr{A}$. It is a consequence of the Hahn–Banach theorem that $\overline{\mathscr{A}}^{w*}$ is equal to the union $\bigcup_{n=1}^{\infty} n \cdot \overline{B}^{w*}$, so we have $\theta(\mathscr{A}^{**}) = \overline{\mathscr{A}}^{w*}$, which by Theorem 4.3

is semisimple. To deduce that $\mathscr{A}^{**}$ is semisimple, we need only prove that θ is a faithful (injective) representation.

Theorem 5.1. *The representation $\theta : \mathscr{A}^{**} \to B(H)$ is faithful.*

Proof. Let $\tau \in \mathscr{A}^{**}$ with $\|\tau\| = 1$. We claim that $\theta(\tau) \neq 0$. To establish this, we first prove the following lemma:

Lemma 5.2. *If $\tau \in \mathscr{A}^{**}$ with $\|\tau\| = 1$, then for every $\varepsilon > 0$, there are $n \in \mathbb{N}$ and $\phi \in B(H)^*$ with $\|\phi\| = 1$, such that the compression $\phi_n = \overline{P}_n \cdot \phi \cdot \overline{P}_n$ satisfies $|\langle \tau, \phi_n \rangle| > 1 - \varepsilon$.*

Proof. If $a, b \in B(H)$ and Q is an orthogonal projection, then simple calculations yield the inequalities

$$\|aQ + b(1-Q)\| \leq \sqrt{\|aQ\|^2 + \|b(1-Q)\|^2},$$
$$\|Qa + (1-Q)b\| \leq \sqrt{\|Qa\|^2 + \|(1-Q)b\|^2}.$$

When these are dualized, the directions of the inequalities are reversed: if $\phi, \psi \in B(H)^*$, then

$$\|\phi \cdot Q + \psi \cdot (1-Q)\| \geq \sqrt{\|\phi \cdot Q\|^2 + \|\psi \cdot (1-Q)\|^2},$$
$$\|Q \cdot \phi + (1-Q) \cdot \psi\| \geq \sqrt{\|Q \cdot \phi\|^2 + \|(1-Q) \cdot \psi\|^2}. \tag{27}$$

For every $\eta > 0$ there is a $\phi \in \mathscr{A}^*$ such that $\|\phi\| = 1$ and $\langle \tau, \phi \rangle > 1 - \eta$. There is also a witness $T \in \mathscr{A}$ such that $\|T\| = 1$ and $\langle \phi, T \rangle > 1 - \eta$. By Lemma 3.3(b) there is an $n \in \mathbb{N}$ such that $\|(1 - \overline{P}_n)T\| < \eta$. Hence, $|\langle \phi - \phi_n, T \rangle| \leq \|(1 - \overline{P}_n)T\| + \|\overline{P}_n T(1 - \overline{P}_n)\| = \|(1 - \overline{P}_n)T\| < \eta$ also (because $\ker \overline{P}_n$ is an invariant subspace for $\mathscr{A}$), and so $\|\phi_n\| \geq |\langle \phi_n, T \rangle| > 1 - 2\eta$. By (27) we therefore have $\|(1 - \overline{P}_n) \cdot \phi\|, \|\phi \cdot (1 - \overline{P}_n)\| < \sqrt{1 - (1 - 2\eta)^2} < 2\sqrt{\eta}$, and hence $\|\phi - \phi_n\| < 4\sqrt{\eta}$. Since $\langle \tau, \phi \rangle > 1 - \eta$, we have $|\langle \tau, \phi_n \rangle| > 1 - \eta - \|\phi - \phi_n\| \geq 1 - \eta - 4\sqrt{\eta}$. Appropriate choice of $\eta > 0$ yields $|\langle \tau, \phi_n \rangle| > 1 - \varepsilon$ as required. □

We now prove Theorem 5.1. Let $\tau \in \mathscr{A}^{**}$ with $\|\tau\| = 1$, and assume towards a contradiction that $\theta(\tau) = 0$. Write $\gamma_n = \sup\{|\langle \overline{P}_n \cdot \phi \cdot \overline{P}_n, \tau \rangle| : \phi \in B(H)^*, \|\phi\| = 1\}$. The sequence γ_n is nondecreasing, and by Lemma 5.2 we have $\gamma_n \to 1$. Pick then $N \in \mathbb{N}$ such that $\gamma_N > 0$, and let $n \leq N$ be the least natural number such that $\gamma_n = \gamma_N$. For each $\varepsilon > 0$ we can find $\phi \in B(H)^*$, $\|\phi\| = 1$ such that $\langle \overline{P}_n \cdot \phi \cdot \overline{P}_n, \tau \rangle \geq \gamma_N - \varepsilon$.

Given such an $\varepsilon > 0$ and ϕ, we write ϕ_1 for a weak-* accumulation point of the functionals $\overline{\pi}_k \cdot \phi$; but actually, we claim that ϕ_1 is the norm-convergent limit of $\overline{\pi}_k \cdot \phi$. For the norms $\|\overline{\pi}_k \cdot \phi\|$ are a nondecreasing sequence tending to a limit l; (27), with $Q = \overline{\pi}_k$ and $\psi = \overline{\pi}_m \cdot \phi$, tells us that for $m > k$ we have $\|\overline{\pi}_k \cdot \phi\|^2 + \|(\overline{\pi}_m - \overline{\pi}_k) \cdot \phi\|^2 \geq \|\overline{\pi}_m \cdot \phi\|^2$, so $\|(\overline{\pi}_m - \overline{\pi}_k) \cdot \phi\| \to 0$ as $k, m \to \infty$;

so the sequence $(\overline{\pi}_k \cdot \phi)_{k\in\mathbb{N}}$ satisfies the Cauchy criterion and is norm-convergent. Each projection $\overline{\pi}_k$ is of finite rank, so $\overline{\pi}_k \cdot \phi$ belongs to the trace-class operators $\mathcal{T}$. Therefore, $\phi_1 \in \mathcal{T}$. But the difference $\phi - \phi_1 = \lim_k(1-\overline{\pi}_k)\cdot\phi$ will annihilate any compact operator.

We therefore claim that $n > 1$. For by Theorem 3.1, whenever $T \in \mathcal{A}$ the operator $P_1 T = P_1 T^{(1)}$ is a compact operator, so $\langle P_1 T P_1, \phi\rangle = \langle P_1 T P_1, \phi_1\rangle$. We may write τ as a weak-* convergent limit $\tau = \lim_{w*} T_\alpha$ for $T_\alpha \in \mathcal{A}$ with $\|T_\alpha\| = 1$. Then $\gamma_N - \varepsilon \le \langle P_1\cdot\phi\cdot P_1, \tau\rangle = \lim_\alpha\langle T_\alpha, P_1\cdot\phi\cdot P_1\rangle = \lim_\alpha\langle P_1 T_\alpha P_1, \phi\rangle = \lim_\alpha\langle P_1 T_\alpha P_1, \phi_1\rangle = \lim_\alpha\langle T_\alpha, P_1\cdot\phi_1\cdot P_1\rangle = \langle\tau, P_1\cdot\phi_1\cdot P_1\rangle$. For small ε this implies $\langle\tau, P_1\cdot\phi_1\cdot P_1\rangle \neq 0$. But $P_1\cdot\phi_1\cdot P_1 \in \mathcal{T} = B(H)_*$, so $\theta(\tau)$ is not the zero operator in $B(H)$, a contradiction. Therefore we have $n > 1$.

Given $n > 1$, we again pick $\varepsilon > 0$ and find $\phi \in B(H)^*$, $\|\phi\| = 1$ such that

$$\langle \overline{P}_n\cdot\phi\cdot\overline{P}_n, \tau\rangle \ge \gamma_N - \varepsilon > 0. \tag{28}$$

The norm limit $\phi_1 = \lim_k \overline{\pi}_k\cdot\phi$ is again in $\mathcal{T}$. However, the difference $\phi - \phi_1$ will not necessarily annihilate $\overline{P}_n T\overline{P}_n$ for $T \in \mathcal{A}$, because though $\phi - \phi_1$ annihilates $K(H)$, the operator $\overline{P}_n T\overline{P}_n$ need not be compact. Rather, for $T \in \mathcal{A}$ we have $\overline{P}_n T\overline{P}_n = \overline{P}_n\overline{T}^{(n)}\overline{P}_n$, where $\overline{T}^{(n)} = \overline{\Delta}_n(T)$ as in Definition 2.7; and $\overline{T}^{(n)} = \overline{T}^{(n-1)} + T^{(n)}$, where the operator $\overline{P}_n T^{(n)}$ is compact by Theorem 3.1. So $\langle\overline{P}_n T^{(n)}\bar{P}_n, \phi - \phi_1\rangle = 0$ for all $T \in \mathcal{A}$. Writing $\tau = \lim_\alpha T_\alpha$ for a suitable net (T_α) in $\mathcal{A}$, we have $\langle T_\alpha^{(n)}, \overline{P}_n(\phi-\phi_1)\overline{P}_n\rangle = 0$ for all α. Because $\phi_1 \in \mathcal{T}$ and $\theta(\tau) = 0$ by hypothesis, if we write $\beta = \lim_\alpha\langle\overline{T}_\alpha^{(n-1)}, \overline{P}_n(\phi-\phi_1)\overline{P}_n\rangle$, we will have $0 = \langle\overline{P}_n\phi_1\overline{P}_n, \tau\rangle = \langle\overline{P}_n\phi\bar{P}_n, \tau\rangle - \lim_\alpha\langle\overline{P}_n(\phi-\phi_1)\overline{P}_n, T_\alpha\rangle = \langle\overline{P}_n\phi\overline{P}_n, \tau\rangle - \beta$. By (28), we have $|\beta| \ge \gamma_N - \varepsilon$.

For each $T \in \mathcal{A}$ and $n > 1$, the norms of $\overline{T}^{(n-1)}$ and $\overline{P}_{n-1}\overline{T}^{(n-1)}\overline{P}_{n-1} = \overline{P}_{n-1}T\overline{P}_{n-1}$ are the same by (15). Thus there is a unique map $\eta : \overline{P}_{n-1}\cdot\mathcal{A}\cdot\bar{P}_{n-1} \to \overline{\mathcal{A}}^{(n-1)}$ which is a right inverse to the compression $p : \mathcal{A} \to \overline{P}_{n-1}\cdot\mathcal{A}\cdot\overline{P}_{n-1}$ with $p(T) = \overline{P}_{n-1}T\overline{P}_{n-1}$ ($T \in \mathcal{A}$); and $\|\eta\| = 1$. We will have $\eta\cdot p = \overline{\Delta}_{n-1}$. Let us write $\psi = (\overline{P}_n(\phi-\phi_1)\overline{P}_n)\circ\eta$. Then $\psi \in (\overline{P}_{n-1}\cdot\mathcal{A}\cdot\overline{P}_{n-1})^*$ with $\|\psi\| \le 1$.

By the Hahn–Banach theorem, we can extend ψ to $\overline{P}_{n-1}\cdot B(H)\cdot\overline{P}_{n-1}$ with the same norm; and then extend to all of $B(H)$ so that $\psi = \psi\circ p$ (where we abuse notation slightly by writing p for the compression $B(H) \to \overline{P}_{n-1}\cdot B(H)\cdot\overline{P}_{n-1}$ also).

Then for $T \in \mathcal{A}$ we have $\psi(T) = \psi\circ\eta p(T) = \psi(\overline{\Delta}_{n-1}(T))$; so

$$|\langle\psi, \tau\rangle| = \lim_\alpha|\langle\psi, \bar{\Delta}_{n-1}T_\alpha\rangle| = \lim_\alpha|\langle\overline{P}_n(\phi-\phi_1)\overline{P}_n, \bar{T}_\alpha^{(n-1)}\rangle| = |\beta| \ge \gamma_N - \varepsilon.$$

Since $\psi = \psi\circ p = \overline{P}_{n-1}\cdot\psi\cdot\overline{P}_{n-1}$, we find that

$$\gamma_{n-1} = \sup\{|\langle\overline{P}_{n-1}\cdot\phi\cdot\overline{P}_{n-1}, \tau\rangle| : \phi \in B(H)^*, \|\phi\| = 1\}$$

is at least $\gamma_N - \varepsilon$. But $\varepsilon > 0$ is arbitrary, so $\gamma_{n-1} = \gamma_N$, and n was not the minimal integer with $\gamma_n = \gamma_N$, contrary to hypothesis. This contradiction proves the theorem. □

6. References to Gulick's paper

Having established that the result rad $A^{**} \cap A = \operatorname{rad} A$ of Gulick is wrong, let us look at papers which have referenced [Gulick 1966] and try to establish that no further damage has been done.

The lengthy paper of Dales and Lau [2005] refers to [Gulick 1966], but does not use the false Theorem 4.6; private communication with my colleague Garth Dales reveals a history of previous suspicion about that result, but no actual counterexamples as presented here. The paper of Daws, Haydon, Schlumprecht and White [Daws et al. 2012] refers to (the proof of) Theorem 3.3 of [Gulick 1966], which we believe to be completely correct. Likewise the paper of Bouziad and Filali [2011] quotes the proof, given by Gulick [1966, Lemma 5.2], that the radical of $L^\infty(G)^*$ is nonseparable for any nondiscrete locally compact group G. This proof also is perfectly valid. The earlier paper of Granirer [1973] makes reference to that same, correct, lemma. Tomiuk [1981] likewise refers to Gulick's untainted Theorem 5.5. A. Ülger [1987] solves one of the problems posed by Gulick [1966]. Finally, Tomiuk and Wong [1970] make a passing reference to [Gulick 1966] in their paper on Arens products.

We have not found a case in which another author has used the false Theorem 4.6 from Gulick's paper, or anything tainted by it. This chimes with our reckoning that more than one author apart from ourselves has suspected that that theorem is false. So, the general literature on Banach algebras is not seriously harmed; but it was nonetheless high time that these counterexamples were made known so that such errors will not occur in the future.

References

[Bouziad and Filali 2011] A. Bouziad and M. Filali, "On the size of quotients of function spaces on a topological group", *Studia Math.* **202**:3 (2011), 243–259. MR 2012c:43004 Zbl 1234.43001

[Dales and Lau 2005] H. G. Dales and A. T.-M. Lau, *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. **836**, Amer. Math. Soc., Providence, RI, 2005. MR 2006k:43002 Zbl 1075.43003

[Daws et al. 2012] M. Daws, R. Haydon, T. Schlumprecht, and S. White, "Shift invariant preduals of $\ell_1(\mathbb{Z})$", *Israel J. Math.* **192**:2 (2012), 541–585. MR 3009735 Zbl 1269.46008

[Granirer 1973] E. E. Granirer, "The radical of $L^\infty(G)^*$", *Proc. Amer. Math. Soc.* **41** (1973), 321–324. MR 48 #4646 Zbl 0278.43009

[Gulick 1966] S. L. Gulick, "Commutativity and ideals in the biduals of topological algebras", *Pacific J. Math.* **18** (1966), 121–137. MR 33 #3118 Zbl 0198.17901

[Tomiuk 1981] B. J. Tomiuk, "Arens regularity and the algebra of double multipliers", *Proc. Amer. Math. Soc.* **81**:2 (1981), 293–298. MR 81m:46065 Zbl 0489.46040

[Tomiuk and Wong 1970] B. J. Tomiuk and P.-k. Wong, "The Arens product and duality in B^*-algebras", *Proc. Amer. Math. Soc.* **25** (1970), 529–535. MR 41 #4256 Zbl 0198.17902

[Ülger 1987] A. Ülger, "Weakly compact bilinear forms and Arens regularity", *Proc. Amer. Math. Soc.* **101**:4 (1987), 697–704. MR 88k:46056 Zbl 0635.46044

Received March 5, 2014.

CHARLES JOHN READ
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF LEEDS
LEEDS LS2 9JT
UNITED KINGDOM
read@maths.leeds.ac.uk

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.461

DIMENSION JUMPS IN BOTT–CHERN AND AEPPLI COHOMOLOGY GROUPS

JIEZHU LIN AND XUANMING YE

Let X be a compact complex manifold, and let $\pi : \mathscr{X} \to B$ be a small deformation of X, the dimensions of the Bott–Chern cohomology groups $H^{p,q}_{\mathrm{BC}}(X(t))$ and Aeppli cohomology groups $H^{p,q}_{\mathrm{A}}(X(t))$ may vary under this deformation. In this paper, we will study the deformation obstructions of a (p,q) class in the central fiber X. In particular, we obtain an explicit formula for the obstructions and apply this formula to the study of small deformations of the Iwasawa manifold.

1. Introduction

Let X be a compact complex manifold and $\pi : \mathscr{X} \to B$ be a family of complex manifolds such that $\pi^{-1}(0) = X$, where $\mathscr{X}$ is a complex manifold and B is a neighborhood of the origin. Let $X_t = \pi^{-1}(t)$ denote the fiber of π over the point $t \in B$. In [Ye 2008], the author studied the jumping phenomenon of Hodge numbers $h^{p,q}$ of X by studying the deformation obstructions of a (p,q) class in the central fiber X. In particular, the author obtained an explicit formula for the obstructions and applied it to the study of small deformations of the Iwasawa manifold. Besides the Hodge numbers, the dimensions of Bott–Chern cohomology groups and the dimensions of Aeppli cohomology groups are also important invariants of complex structures. D. Angella [2013] studied the small deformations of the Iwasawa manifold and found that the dimensions of Bott–Chern and Aeppli cohomology groups are not deformation invariants.

In this paper, we will study the Bott–Chern and Aeppli cohomologies by studying the hypercohomology of the complex $\mathscr{B}^{\bullet}_{p,q}$ constructed in [Schweitzer 2007]. M. Schweitzer [2007] proved that

$$H^{p,q}_{\mathrm{BC}}(X) \cong \mathbb{H}^{p+q}(X, \mathscr{B}^{\bullet}_{p,q}) \quad \text{and} \quad H^{p,q}_{\mathrm{A}}(X) \cong \mathbb{H}^{p+q+1}(X, \mathscr{B}^{\bullet}_{p+1,q+1}).$$

This work was supported by National Natural Science Foundation of China (Grant No. 11201491 and 11201090), Doctoral Fund of Ministry of Education of China (Grant No. 20124410120001 and 20120171120009) and the Foundation of Research Funds for Young Teachers Training Project (Grant No. 34000-31610248).

MSC2010: 14B12, 14C30, 14F05, 14F40, 18G40.

Keywords: Bott–Chern cohomology, Aeppli cohomology, deformation, obstruction, Kodaira–Spencer classes.

As in [Ye 2008], we will study the jumping phenomenons from the viewpoint of obstruction theory. More precisely, for a certain small deformation $\mathscr{X}$ of X parameterized by a base B and a certain class $[\theta]$ of the hypercohomology group $\mathbb{H}^l(X, \mathscr{B}^\bullet_{p,q})$, we will try to find out the obstruction to extend it to an element of the relative hypercohomology group $\mathbb{H}^l(\mathscr{X}, \mathscr{B}^\bullet_{p,q;\mathscr{X}/B})$. We will call those elements which have nontrivial obstruction the obstructed elements. And then we will see that these elements will play an important role when we study the jumping phenomenon, because we will see that the existence of obstructed elements is a sufficient condition for the variation of the dimensions of Bott–Chern and Aeppli cohomologies.

In Section 2 we will summarize the results of M. Schweitzer about Bott–Chern and Aeppli cohomologies, from which we can define the relative Bott–Chern and Aeppli cohomologies on X_n, where X_n is the n-th order deformation of $\pi : \mathscr{X} \to B$. We will also introduce some important maps which will be used in the calculation of the obstructions in Section 4. In Section 3 we will try to explain why we need to consider the obstructed elements. The relation between the jumping phenomenon of the dimensions of Bott–Chern and Aeppli cohomologies and the obstructed elements is the following.

Theorem 3.1. *Let $\pi : \mathscr{X} \to B$ be a small deformation of the central fiber compact complex manifold X. Now we consider* $\dim \mathbb{H}^l(X(t), \mathscr{B}^\bullet_{p,q;t})$ *as a function of $t \in B$. It jumps at $t = 0$ if there exists an element $[\theta]$ either in $\mathbb{H}^l(X, \mathscr{B}^\bullet_{p,q})$ or in $\mathbb{H}^{l-1}(X, \mathscr{B}^\bullet_{p,q})$ and a minimal natural number $n \geq 1$ such that the n-th order obstruction is nonzero*:

$$\mathrm{o}_n([\theta]) \neq 0.$$

In Section 4 we will get a formula for the obstruction to the extension we mentioned above.

Theorem 4.4. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X. For arbitrary $[\theta]$ belongs to $\mathbb{H}^l(X, \mathscr{B}^\bullet_{p,q})$, suppose we can extend $[\theta]$ to order $n-1$ in $\mathbb{H}^l(X_{n-1}, \mathscr{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. Denote such element by $[\theta_{n-1}]$. The obstruction of the extension of $[\theta]$ to n-th order is given by*

$$\mathrm{o}_n([\theta]) = -\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ \partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]) - \bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ \bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]),$$

where κ_n is the n-th order Kodaira–Spencer class and $\bar\kappa_n$ is the n-th order Kodaira–Spencer class of the deformation $\bar\pi : \bar{\mathscr{X}} \to \bar B$. The maps $\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}$ and $\bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}$ are defined in Section 2.

In Section 5 we will use this formula to study carefully the example given by Iku Nakamura and D. Angella, that is, the small deformation of the Iwasawa manifold and discuss some phenomena.

2. The relative Bott–Chern and Aeppli cohomologies of X_n and the representation of their cohomology classes

2A. ***The Bott–Chern and Aeppli cohomologies and hypercohomologies.*** All the details of this subsection can be found in [Schweitzer 2007]. Let X be a compact complex manifold. The Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(X)$, and more generally the terms $E^{p,q}_r(X)$ in the Frölicher spectral sequence [Frölicher 1955], are well-known finite dimensional invariants of the complex manifold X. On the other hand, the Bott–Chern and Aeppli cohomologies define additional complex invariants of X given, respectively, by [Bott and Chern 1965; Aeppli 1965]

$$H^{p,q}_{\mathrm{BC}}(X)=\frac{\ker\{d\colon \mathscr{A}^{p,q}(X)\to\mathscr{A}^{p+q+1}(X)\}}{\operatorname{im}\{\partial\bar\partial\colon \mathscr{A}^{p-1,q-1}(X)\to\mathscr{A}^{p,q}(X)\}},$$

and

$$H^{p,q}_{\mathrm{A}}(X)=\frac{\ker\{\partial\bar\partial\colon \mathscr{A}^{p,q}(X)\to\mathscr{A}^{p+1,q+1}(X)\}}{\operatorname{im}\{\partial\colon \mathscr{A}^{p-1,q}(X)\to\mathscr{A}^{p,q}(X)\}+\operatorname{im}\{\bar\partial\colon \mathscr{A}^{p,q-1}(X)\to\mathscr{A}^{p,q}(X)\}}.$$

By the Hodge theory developed in [Schweitzer 2007], all these complex invariants are also finite dimensional, and $H^{p,q}_{\mathrm{A}}(X)\cong H^{n-q,n-p}_{\mathrm{BC}}(X)$. Notice that $H^{q,p}_{\mathrm{BC}}(X)$ is isomorphic to $H^{p,q}_{\mathrm{BC}}(X)$ by complex conjugation. For any $r\ge 1$ and for any p,q, there are natural maps

$$H^{p,q}_{\mathrm{BC}}(X)\to E^{p,q}_r(X)\quad\text{and}\quad E^{p,q}_r(X)\to H^{p,q}_{\mathrm{A}}(X).$$

Recall that $E^{p,q}_1(X)$ is isomorphic to $H^{p,q}_{\bar\partial}(X)$ and that the terms for $r=\infty$ provide a decomposition of the de Rham cohomology of X: $H^k_{\mathrm{dR}}(X,\mathbb{C})\cong\oplus_{p+q=k}E^{p,q}_\infty(X)$. From now on we shall denote by $h^{p,q}_{\mathrm{BC}}(X)$ the dimension of the cohomology group $H^{p,q}_{\mathrm{BC}}(X)$. The Hodge numbers will be denoted simply by $h^{p,q}(X)$ and the Betti numbers by $b_k(X)$. For any given $p\ge 1$, $q\ge 1$, we define the complex of sheaves $\mathscr{L}^\bullet_{p,q}$ by

$$\mathscr{L}^k_{p,q}=\bigoplus_{\substack{r+s=k\\ r<p,\,s<q}}\mathscr{A}^{r,s}\quad\text{if } k\le p+q-2,\qquad \mathscr{L}^{k-1}_{p-1,q-1}=\bigoplus_{\substack{r+s=k\\ r\ge p,\,s\ge q}}\mathscr{A}^{r,s}\quad\text{if } k\ge p+q,$$

and the differential

$$\mathscr{L}^0_{p,q}\xrightarrow{\mathrm{pr}_{\mathscr{L}^1_{p,q}}\circ d}\mathscr{L}^1_{p,q}\xrightarrow{\mathrm{pr}_{\mathscr{L}^2_{p,q}}\circ d}\cdots\longrightarrow\mathscr{L}^{k-2}_{p,q}\xrightarrow{\partial\bar\partial}\mathscr{L}^{k-1}_{p,q}\xrightarrow{d}\mathscr{L}^k_{p,q}\xrightarrow{d}\cdots,$$

where $\mathscr{A}^{r,s}$ are the sheaves of smooth (r,s)-forms and pr is the projection operator.

Then by the above construction, we have the following isomorphisms:

$$H^{p,q}_{\mathrm{BC}}(X)=H^{p+q-1}(\mathscr{L}^\bullet_{p,q}(X))\cong\mathbb{H}^{p+q-1}(X,\mathscr{L}^\bullet_{p,q}),$$

$$H^{p,q}_{\mathrm{A}}(X)=H^{p+q}(\mathscr{L}^\bullet_{p+1,q+1}(X))\cong\mathbb{H}^{p+q}(X,\mathscr{L}^\bullet_{p+1,q+1}),$$

because $\mathscr{L}^k_{p,q}$ are soft.

We define a sub complex $\mathscr{S}^\bullet_{p,q}$ of $\mathscr{L}^\bullet_{p,q}$ by

$$(\mathscr{S}'^\bullet_p, \partial) : \mathbb{O} \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0, \quad (\mathscr{S}''^\bullet_q, \bar\partial) : \bar{\mathbb{O}} \to \bar\Omega^1 \to \cdots \to \bar\Omega^{q-1} \to 0,$$

$$\mathscr{S}^\bullet_{p,q} = \mathscr{S}'^\bullet_p + \mathscr{S}''^\bullet_q : \mathbb{O} + \bar{\mathbb{O}} \to \Omega^1 \oplus \bar\Omega^1 \to \cdots \to \Omega^{p-1} \oplus \bar\Omega^{p-1} \to \bar\Omega^p \to \cdots \to \bar\Omega^{q-1} \to 0.$$

Note that the inclusion $\mathscr{S}^\bullet \subset \mathscr{L}^\bullet$ is a quasiisomorphism [Schweitzer 2007]. There is another complex $\mathscr{B}^\bullet_{p,q}$ used in [Schweitzer 2007], defined by

$$\mathscr{B}^\bullet_{p,q} : \mathbb{C} \xrightarrow{(+,-)} \mathbb{O} \oplus \bar{\mathbb{O}} \to \Omega^1 \oplus \bar\Omega^1 \to \cdots \to \Omega^{p-1} \oplus \bar\Omega^{p-1} \to \bar\Omega^p \to \cdots \to \bar\Omega^{q-1} \to 0.$$

and the following morphism from $\mathscr{B}^\bullet_{p,q}$ to $\mathscr{S}^\bullet_{p,q}[1]$ is a quasiisomorphism [Schweitzer 2007]:

$$\begin{array}{ccccccc}
\mathbb{C} & \xrightarrow{(+,-)} & \mathbb{O} \oplus \bar{\mathbb{O}} & \to & \Omega^1 \oplus \bar\Omega^1 & \to & \cdots \\
\downarrow & & \downarrow + & & \downarrow & & \\
0 & \longrightarrow & \mathbb{O} + \bar{\mathbb{O}} & \to & \Omega^1 \oplus \bar\Omega^1 & \to & \cdots .
\end{array}$$

Therefore we have

$$H^{p,q}_{\mathrm{BC}}(X) \cong \mathbb{H}^{p+q}(X, \mathscr{L}^\bullet_{p,q}[1]) \cong \mathbb{H}^{p+q}(X, \mathscr{S}^\bullet_{p,q}[1]) \cong \mathbb{H}^{p+q}(X, \mathscr{B}^\bullet_{p,q}),$$

and

$$H^{p,q}_{\mathrm{A}}(X) \cong \mathbb{H}^{p+q}(X, \mathscr{L}^\bullet_{p+1,q+1}) \cong \mathbb{H}^{p+q}(X, \mathscr{S}^\bullet_{p+1,q+1}) \cong \mathbb{H}^{p+q+1}(X, \mathscr{B}^\bullet_{p+1,q+1}).$$

2B. ***The relative Bott–Chern and Aeppli cohomologies of X_n.*** Here we make some definitions in order to construct the relative Bott–Chern and Aeppli cohomologies of X_n. Suppose X is a compact complex manifold.

- Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$.
- For every integer $n \geq 0$, set $B_n = \operatorname{Spec} \mathbb{O}_{B,0}/m_0^{n+1}$ — the n-th order infinitesimal neighborhood of the closed point 0 of the base B.
- Let $X_n \subset \mathscr{X}$ be the complex space over B_n.
- Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X, and denote $\pi^*(m_0)$ by $\mathscr{M}_0$.
- Complex conjugation gives another complex structure of the differential manifold of $\mathscr{X}$; we denote this manifold by $\bar{\mathscr{X}}$, and π induces a deformation $\bar\pi : \bar{\mathscr{X}} \to \bar{B}$ of $\bar{X}$. Then we have $\bar{X}_n$ and $\bar\pi_n : \bar{X}_n \to \bar{B}_n$.
- Let $\mathscr{C}^\omega_B$ be the sheaf of $\mathbb{C}$-valued real analytic functions on B.
- Set $\mathbb{O}^\omega_{\mathscr{X}} = \pi^*(\mathscr{C}^\omega_B)$, $\bar{\mathbb{O}}^\omega_{\mathscr{X}} = \bar\pi^*(\mathscr{C}^\omega_B)$; let m^ω_0 be the maximal ideal of $\mathscr{C}^\omega_{B,0}$ and let $\mathscr{M}^\omega_0 = \pi^*(m^\omega_0)$, $\bar{\mathscr{M}}^\omega_0 = \bar\pi^*(m^\omega_0)$.
- For any sheaf of $\mathbb{O}_{\mathscr{X}}$- (resp. $\bar{\mathbb{O}}_{\mathscr{X}}$-) modules $\mathscr{F}$, set $\mathscr{F}^\omega = \mathscr{F} \otimes_{\mathbb{O}_{\mathscr{X}}} \mathbb{O}^\omega_{\mathscr{X}}$ (resp. $\bar{\mathscr{F}}^\omega = \mathscr{F} \otimes_{\bar{\mathbb{O}}_{\mathscr{X}}} \bar{\mathbb{O}}^\omega_{\mathscr{X}}$).
- Let $\mathbb{O}^\omega_{X_n} = \mathbb{O}^\omega_{\mathscr{X},0}/(\mathscr{M}^\omega_0)^{n+1}$ and $\mathbb{O}^\omega_{\bar{X}_n} = \bar{\mathbb{O}}^\omega_{\mathscr{X},0}/(\bar{\mathscr{M}}^\omega_0)^{n+1}$.

- For any sheaf of $\mathcal{O}_{X_n}$ (resp. $\mathcal{O}_{\overline{X}_n}$) modules $\mathcal{F}$, set $\mathcal{F}^\omega = \mathcal{F} \otimes_{\mathcal{O}_{X_n}} \mathcal{O}^\omega_{X_n}$ (resp. $\overline{\mathcal{F}}^\omega = \mathcal{F} \otimes_{\mathcal{O}_{\overline{X}_n}} \mathcal{O}^\omega_{\overline{X}_n}$).
- For any given $p \geq 1$, $q \geq 1$, we define the complex $\mathcal{S}^\bullet_{X_n/B_n} = \mathcal{S}^\bullet_{p,q;X_n/B_n}$ by

$$(\mathcal{S}'^\bullet_{p;X_n/B_n}, \partial_{X_n/B_n}) : \mathcal{O}^\omega_{X_n} \to \Omega^{1;\omega}_{X_n/B_n} \to \cdots \to \Omega^{p-1;\omega}_{X_n/B_n} \to 0,$$

$$(\mathcal{S}''^\bullet_{q;X_n/B_n}, \bar\partial_{X_n/B_n}) : \mathcal{O}^\omega_{\overline{X}_n} \to \overline{\Omega}^{1;\omega}_{X_n/B_n} \to \cdots \to \overline{\Omega}^{q-1;\omega}_{X_n/B_n} \to 0,$$

$$\begin{aligned}\mathcal{S}^\bullet_{p,q;X_n/B_n} = \mathcal{S}'^\bullet_{p;X_n/B_n} + \mathcal{S}''^\bullet_{q;X_n/B_n} : \mathcal{O}^\omega_{X_n} + \mathcal{O}^\omega_{\overline{X}_n} \to \Omega^{1;\omega}_{X_n/B_n} \oplus \overline{\Omega}^{1;\omega}_{X_n/B_n} \to \cdots \\ \to \Omega^{p-1;\omega}_{X_n/B_n} \oplus \overline{\Omega}^{p-1;\omega}_{X_n/B_n} \to \overline{\Omega}^{p;\omega}_{X_n/B_n} \to \cdots \to \overline{\Omega}^{q-1;\omega}_{X_n/B_n} \to 0.\end{aligned}$$

- Finally, define $\mathcal{B}^\bullet_{p,q;X_n/B_n}$ by

$$\begin{aligned}\mathcal{B}^\bullet_{p,q;X_n/B_n} : \mathbb{C}^\omega_{B_n} \xrightarrow{(+,-)} \mathcal{O}^\omega_{X_n} \oplus \mathcal{O}^\omega_{\overline{X}_n} \to \Omega^{1;\omega}_{X_n/B_n} \oplus \overline{\Omega}^{1;\omega}_{X_n/B_n} \to \cdots \\ \to \Omega^{p-1;\omega}_{X_n/B_n} \oplus \overline{\Omega}^{p-1;\omega}_{X_n/B_n} \to \overline{\Omega}^{p;\omega}_{X_n/B_n} \to \cdots \to \overline{\Omega}^{q-1;\omega}_{X_n/B_n} \to 0,\end{aligned}$$

  where $\mathbb{C}^\omega_{B_n} = \pi^{-1}(\mathcal{C}^\omega_{B,0}/(m^\omega_0)^{n+1})$.

Now we are ready to define the relative Bott–Chern and Aeppli cohomologies of X_n:

$$H^{p,q}_{\mathrm{BC}}(X_n/B_n) \cong \mathbb{H}^{p+q}(X, \mathcal{S}^\bullet_{p,q;X_n/B_n}[1]) \cong \mathbb{H}^{p+q}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n}),$$

and

$$H^{p,q}_{\mathrm{A}}(X_n/B_n) \cong \mathbb{H}^{p+q}(X, \mathcal{S}^\bullet_{p+1,q+1;X_n/B_n}) \cong \mathbb{H}^{p+q+1}(X_n, \mathcal{B}^\bullet_{p+1,q+1;X_n/B_n}).$$

2C. ***Representation of the relative Bott–Chern and Aeppli cohomology classes.*** In this subsection we will follow [Schweitzer 2007] to construct a hypercocycle in $\check{Z}^{p+q}(X, \mathcal{B}^\bullet_{p,q})$ to represent the relative Bott–Chern cohomology classes. Let $[\theta]$ be an element of $H^{p,q}_{\mathrm{BC}}(X)$, represented by a closed (p,q)-form θ. It is defined in $\mathbb{H}^{p+q}(X, \mathcal{L}_{p,q}[1]^\bullet)$ by a hypercocycle, still denoted by θ and defined by $\theta^{p,q} = \theta|_{U_j}$ and $\theta^{r,s} = 0$ otherwise. For given $p \geq 1$ and $q \geq 1$, there exists a hypercocycle $w = (c; u^{r,0}; v^{0,s}) \in \check{Z}^{p+q}(X, \mathcal{B}^\bullet_{p,q})$ and an hypercochain $\alpha = (\alpha^{r,s}) \in \check{C}^{p+q-1}(X, \mathcal{L}_{p,q}[1]^\bullet)$ such that $\theta = \check{\delta}\alpha + w$. We represent the data in the following table:

$$\theta \longleftrightarrow \left[\begin{array}{c|ccc} \theta_v^{0,q-1} & & & \\ \vdots & & \alpha^{r,s} & \\ \theta_v^{0,0} & & & \\ \hline \theta_c & \theta_u^{0,0} & \cdots & \theta_u^{p-1,0} \end{array}\right].$$

The equality $\theta = \check{\delta}\alpha + w$ corresponds to the relations

$$\begin{aligned}\theta^{p,q} &= \partial\bar{\partial}\alpha^{p-1,q-1}\\ (-1)^{r+s}\check{\delta}\alpha^{r,s} &= \bar{\partial}\alpha^{r,s-1} + \partial\alpha^{r-1,s}\\ (-1)^{s}\check{\delta}\alpha^{0,s} &= \bar{\partial}\alpha^{0,s-1} + \theta_v^{0,s}\\ (-1)^{r}\check{\delta}\alpha^{r,0} &= \theta_u^{r,0} + \partial\alpha^{r-1,0}\\ \check{\delta}\alpha^{0,0} &= \theta_u^{0,0} + \theta_v^{0,0}\\ \check{\delta}\theta_u^{0,0} &= \theta_c,\end{aligned}$$

where $1 \le r \le p-1$ and $1 \le s \le q-1$. Note that these relations involve relations of the hypercocycles for θ_u and θ_v:

$$(-1)^r\check{\delta}\theta_u^{r,0} = \partial\theta_u^{r-1,0}, \quad (-1)^s\check{\delta}\theta_v^{0,s} = \bar{\partial}\theta_v^{0,s-1},$$

with the same conditions on r and s. If $q = 0$, we simply have

$$\theta \longleftrightarrow \big(\theta_c, \theta_u^{0,0}, \dots, \theta_u^{p-1,0}\big)$$

with the relations

$$\theta^{p,0} = \partial\theta_u^{p-1,0}, \quad (-1)^r\check{\delta}\theta_u^{r,0} = \partial\theta_u^{r-1,0}, \quad \check{\delta}\theta_u^{0,0} = \theta_c,$$

for $1 \le r \le p-1$. Similarly, if $p = 0$, we have

$$\theta \longleftrightarrow \big(\theta_c, \theta_v^{0,0}, \dots, \theta_v^{0,q-1}\big)$$

with the relations (where $1 \le s \le q-1$)

$$\theta^{0,q} = -\bar{\partial}\theta_v^{0,q-1}, \quad (-1)^s\check{\delta}\theta_v^{0,s} = \bar{\partial}\theta_v^{0,s-1}, \quad -\check{\delta}\theta_v^{0,0} = \theta_c.$$

Similarly, let $[\theta]$ be an element of $H_{\mathrm{BC}}^{p,q}(X_n/B_n)$, then it can be represented by a Čech hypercocycle θ_u, θ_v and θ_c of $\check{Z}^{p+q}(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ with the relations

$$\begin{aligned}(-1)^r\check{\delta}\theta_u^{r,0} &= \partial\theta_u^{r-1,0} & (-1)^s\check{\delta}\theta_v^{0,s} &= \bar{\partial}\theta_v^{0,s-1}\\ \check{\delta}\theta_u^{0,0} &= \theta_c, & -\check{\delta}\theta_v^{0,0} &= \theta_c,\end{aligned}$$

where $1 \le r \le p-1$ and $1 \le s \le q-1$; while for an element $[\theta]$ of $H_{\mathrm{A}}^{p,q}(X_n/B_n)$, it can be represented by a Čech hypercocycle θ_u and θ_v of $\check{Z}^{p+q+1}(X, \mathcal{B}^\bullet_{p+1,q+1;X_n/B_n})$ with the relations

$$\begin{aligned}(-1)^r\check{\delta}\theta_u^{r,0} &= \partial\theta_u^{r-1,0} & (-1)^s\check{\delta}\theta_v^{0,s} &= \bar{\partial}\theta_v^{0,s-1}\\ \check{\delta}\theta_u^{0,0} &= \theta_c, & -\check{\delta}\theta_v^{0,0} &= \theta_c,\end{aligned}$$

where $1 \le r \le p$ and $1 \le s \le q$.

Before the end of this section, we will introduce some important maps which will be used in the computation in Section 4.

Define

$$\partial^{\bar\partial,\mathscr{B}}_{X_n/B_n} : H^\bullet(X_n, \Omega^{p-1;\omega}_{X_n/B_n}) \to \mathbb{H}^{\bullet+p}(X_n, \mathscr{B}^\bullet_{p,q;X_n/B_n})$$

as follows. Let $[\theta]$ be an element of $H^\bullet(X_n, \Omega^{p-1;\omega}_{X_n/B_n})$ then θ can be represented by a cocycle of $\check{Z}^\bullet(X, \Omega^{p-1;\omega}_{X_n/B_n})$, we define $\partial^{\bar\partial,\mathscr{B}}_{X_n/B_n}([\theta])$ to be the cohomology class associated to the hypercocycle in $\check{Z}^{p+\bullet}(X, \mathscr{B}^\bullet_{p,q;X_n/B_n})$ given by

$$\theta^{p-1,0}_u = \theta, \quad \theta^{r,0}_u = 0 \text{ for } 0 \le r \le p-2 \quad \theta^{0,s}_v = 0 \text{ for } 0 \le s \le q-1, \quad \text{and} \quad \theta_c = 0.$$

When $\bullet < 0$, $\partial^{\bar\partial,\mathscr{B}}_{X_n/B_n}$ is defined to be 0.

Lemma 2.1. *The map $\partial^{\bar\partial,\mathscr{B}}_{X_n/B_n}$ is well defined.*

Proof. It is easy to check that the hypercochain given by θ_u, θ_v and θ_c is a hypercocycle. On the other hand, suppose there exists a cochain α' in $\check{C}^{\bullet-1}(X, \Omega^{p-1;\omega}_{X_n/B_n})$ such that $\check\delta\alpha' = \theta$. Then if we take a hypercochain α in $\check{C}^{p+\bullet-1}(X, \mathscr{B}^\bullet_{p,q;X_n/B_n})$ given by $\alpha^{p-1,0}_u = (-1)^{p-1}\alpha'$, $\alpha^{r,0}_u = 0$ for $0 \le r \le p-2$, $\alpha^{0,s}_v = 0$ for $0 \le s \le q-1$, and $\alpha_c = 0$, we have $\check{\boldsymbol{\delta}}\alpha = \partial^{\bar\partial,\mathscr{B}}_{X_n/B_n}([\theta])$. Therefore $\partial^{\bar\partial,\mathscr{B}}_{X_n/B_n}([\theta]) = 0$. □

Similarly, we can define

$$\bar\partial^{\partial,\mathscr{B}}_{X_n/B_n} : H^\bullet(\overline{X}_n, \overline{\Omega}^{q-1;\omega}_{X_n/B_n}) \to \mathbb{H}^{\bullet+q}(X_n, \mathscr{B}^\bullet_{p,q;X_n/B_n})$$

as follows. Let $[\theta]$ be an element of $H^\bullet(\overline{X}_n, \overline{\Omega}^{q-1;\omega}_{X_n/B_n})$. Then θ can be represented by a cocycle of $\check{Z}^\bullet(\overline{X}, \overline{\Omega}^{q-1;\omega}_{X_n/B_n})$; we define $\bar\partial^{\partial,\mathscr{B}}_{X_n/B_n}([\theta])$ to be the cohomology class associated to the hypercocycle in $\check{Z}^{q+\bullet}(X, \mathscr{B}^\bullet_{p,q;X_n/B_n})$ given by $\theta^{0,q-1}_v = \theta$, $\theta^{0,r}_v = 0$ for all $0 \le r \le q-2$, $\theta^{r,0}_u = 0$ for all $0 \le r \le p-1$, and $\theta_c = 0$ (when $\bullet < 0$, this map is defined to be 0). This map is also well defined and the proof is just as Lemma 2.1.

Define

$$\partial^{\mathscr{B},\bar\partial}_{X_n/B_n} : \mathbb{H}^{\bullet+p}(X_n, \mathscr{B}^\bullet_{p,q;X_n/B_n}) \to H^\bullet(X_n, \Omega^{p;\omega}_{X_n/B_n})$$

as follows. Let $[\theta]$ be an element of $\mathbb{H}^{\bullet+p}(X_n, \mathscr{B}^\bullet_{p,q;X_n/B_n})$. Then θ can be represented by a hypercocycle of $\check{Z}^{p+\bullet}(X, \mathscr{B}^\bullet_{p,q;X_n/B_n})$, and we define $\partial^{\mathscr{B},\bar\partial}_{X_n/B_n}([\theta])$ to be the cohomology class associated to the cocycle in $\check{Z}^\bullet(X, \Omega^{p;\omega}_{X_n/B_n})$ given by $\partial_{X_n/B_n}\theta^{p-1,0}_u$(when $\bullet < 0$, this map is defined to be 0).

Lemma 2.2. *The map $\partial^{\mathscr{B},\bar\partial}_{X_n/B_n}$ is well defined.*

Proof. First we check that the cochain given by $\partial_{X_n/B_n}\theta^{p-1,0}_u$ is a cocycle. In fact, since θ is a hypercocycle in $\check{Z}^{p+\bullet}(X, \mathscr{B}^\bullet_{p,q;X_n/B_n})$, we have

$$(-1)^{p-1}\check\delta\theta^{p-1,0}_u = \partial_{X_n/B_n}\theta^{p-2,0}_u,$$

therefore

$$\check\delta\partial_{X_n/B_n}\theta^{p-1,0}_u = (-1)^p\partial_{X_n/B_n} \circ \partial_{X_n/B_n}\theta^{p-2,0}_u = 0.$$

On the other hand, suppose there exists a hypercochain $\alpha \in \check{C}^{p+\bullet-1}(X, \mathscr{B}^{\bullet}_{p,q;X_n/B_n})$ such that $\check{\delta}\alpha = \theta$. Then if we take a cochain $\alpha' \in \check{C}^{\bullet-1}(X, \Omega^{p;\omega}_{X_n/B_n})$ given by $\alpha' = (-1)^p \partial_{X_n/B_n} \alpha_u^{p-1,0}$, we have

$$\begin{aligned}\check{\delta}\alpha' = (-1)^p \check{\delta}\partial_{X_n/B_n}\alpha_u^{p-1,0} &= (-1)^{p+1}\partial_{X_n/B_n}\check{\delta}\alpha_u^{p-1,0} \\ &= (-1)^{p+1+p-1}\partial_{X_n/B_n}\theta_u^{p-1,0} = \partial^{\mathscr{B},\bar{\partial},}_{X_n/B_n}([\theta]).\end{aligned}$$

Therefore $\partial^{\bar{\partial},\mathscr{B}}_{X_n/B_n}([\theta]) = 0$. □

Similarly, we can define

$$\bar{\partial}^{\mathscr{B},\partial}_{X_n/B_n} : \mathbb{H}^{\bullet+q}(X_n, \mathscr{B}^{\bullet}_{p,q;X_n/B_n}) \to H^{\bullet}(\overline{X}_n, \Omega^{q;\omega}_{X_n/B_n}).$$

Let $[\theta]$ be an element of $\mathbb{H}^{\bullet+q}(X_n, \mathscr{B}^{\bullet}_{p,q;X_n/B_n})$ then θ can be represented by a hypercocycle of $\check{Z}^{q+\bullet}(X, \mathscr{B}^{\bullet}_{p,q;X_n/B_n})$, we define $\bar{\partial}^{\mathscr{B},\partial}_{X_n/B_n}([\theta])$ to be the cohomology class associated to the cocycle in $\check{Z}^{\bullet}(X, \overline{\Omega}^{q;\omega}_{X_n/B_n})$ given by $\bar{\partial}_{X_n/B_n}\theta_u^{0,q-1}$(when $\bullet < 0$, this map is defined to be 0). This map is also well defined and the proof is just as Lemma 2.2.

Remark 2.3. The natural maps from $H^{p,q}_{\mathrm{BC}}(X_n/B_n)$ to $H^q(X_n, \Omega^{p;\omega}_{X_n/B_n})$ and from $H^q(X_n, \Omega^{p;\omega}_{X_n/B_n})$ to $H^{p,q}_{\mathrm{A}}(X_n/B_n)$ mentioned in Section 2A respectively are exactly the map

$$\begin{aligned}&\partial^{\mathscr{B},\bar{\partial}}_{X_n/B_n} : \mathbb{H}^{q+p}(X_n, \mathscr{B}^{\bullet}_{p,q;X_n/B_n})(\cong H^{p,q}_{\mathrm{BC}}(X_n/B_n)) \to H^q(X_n, \Omega^{p;\omega}_{X_n/B_n}),\\ &\partial^{\bar{\partial},\mathscr{B}}_{X_n/B_n} : H^q(X_n, \Omega^{p;\omega}_{X_n/B_n}) \to \mathbb{H}^{q+p+1}(X_n, \mathscr{B}^{\bullet}_{p+1,q+1;X_n/B_n})(\cong H^{p,q}_{\mathrm{A}}(X_n/B_n)),\end{aligned}$$

and we denote these maps by $r_{BC,\bar{\partial}}$ and $r_{\bar{\partial},A}$.

We also denote the maps

$$\begin{aligned}&\partial^{\bar{\partial},\mathscr{B}}_{X_n/B_n} : H^q(X_n, \Omega^{p-1;\omega}_{X_n/B_n}) \to \mathbb{H}^{q+p}(X_n, \mathscr{B}^{\bullet}_{p,q;X_n/B_n})(\cong H^{p,q}_{\mathrm{BC}}(X_n/B_n)),\\ &\partial^{\mathscr{B},\bar{\partial}}_{X_n/B_n} : \mathbb{H}^{q+p+1}(X_n, \mathscr{B}^{\bullet}_{p+1,q+1;X_n/B_n})(\cong H^{p,q}_{\mathrm{A}}(X_n/B_n)) \to H^q(X_n, \Omega^{p+1;\omega}_{X_n/B_n})\end{aligned}$$

by $\partial^{\bar{\partial},BC}_{X_n/B_n}$ and $\partial^{A,\bar{\partial}}_{X_n/B_n}$.

The following lemma is an important observation which will be used for the computation in Section 4.

Lemma 2.4. *Let $[\theta]$ be an element of $\mathbb{H}^l(X_n, \mathscr{B}^{\bullet}_{p,q;X_n/B_n})$ which is represented by an element $\theta \in \check{Z}^l(X, \mathscr{B}^{\bullet}_{p,q;X_n/B_n})$ given by θ_u, θ_v and θ_c. Then $\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})$ is a hypercoboundary.*

Proof. The hypercochain $\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})$ is given by $(\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0}))_u^{r,0} = \partial_{X_n/B_n}\theta_u^{r-1,0}$ for $0 < r \le p-1$, $(\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0}))_u^{0,0} = 0$, $(\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0}))_v^{0,s} = 0$ for $0 \le s \le q-1$, and $(\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0}))_c = 0$. Let α be the hypercochain in

$\check{C}^l(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ given by $\alpha_u^{r,0} = -r\theta_u^{r,0}$ for $0 \le r \le p-1$, $\alpha_v^{0,s} = 0$ for $0 \le s \le q-1$, and $\alpha_c = 0$. It is easy to see that

$$\begin{aligned}
(\check{\boldsymbol{\delta}}\alpha)_u^{r,0} &= (-1)^r \check{\delta}\alpha_u^{r,0} + \partial_{X_n/B_n}\alpha_u^{r-1,0} \\
&= (-1)^r \check{\delta}(-r)\theta_u^{r,0} - \partial_{X_n/B_n}(r-1)\theta_u^{r-1,0} \\
&= \partial_{X_n/B_n}\theta_u^{r-1,0} \\
&= \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})_u^{r,0}, \quad \text{for } 0 < r \le p-1, \\
(\check{\boldsymbol{\delta}}\alpha)_u^{0,0} &= 0 = \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})_u^{0,0}, \\
(\check{\boldsymbol{\delta}}\alpha)_v^{0,s} &= 0 = \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})_v^{0,s}, \quad \text{for } 0 \le s \le q-1 \quad \text{and} \\
(\check{\boldsymbol{\delta}}\alpha)_c &= 0 = \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})_c.
\end{aligned}$$

Therefore $\check{\boldsymbol{\delta}}\alpha = \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})$, and $\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})$ is a hypercoboundary. □

The following lemma can be proven similarly.

Lemma 2.5. *Let $[\theta]$ be an element of $\mathbb{H}^l(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ which is represented by an element θ in $\check{Z}^l(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ given by θ_u, θ_v and θ_c, then $\bar\partial_{X_n/B_n}(\theta - \theta_v^{0,q-1})$ is a hypercoboundary.*

3. The jumping phenomenon and obstructions

There is a Hodge theory also for Bott–Chern and Aeppli cohomologies, see [Schweitzer 2007]. More precisely, for a fixed Hermitian metric on X,

$$H^{\bullet,\bullet}_{\mathrm{BC}}(X) \simeq \ker \tilde\Delta_{\mathrm{BC}} \quad \text{and} \quad H^{\bullet,\bullet}_{\mathrm{A}}(X) \simeq \ker \tilde\Delta_{\mathrm{A}},$$

where

$$\begin{aligned}
\tilde\Delta_{\mathrm{BC}} &:= (\partial\bar\partial)(\partial\bar\partial)^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\bar\partial^*\partial)(\bar\partial^*\partial)^* + (\bar\partial^*\partial)^*(\bar\partial^*\partial) + \bar\partial^*\bar\partial + \partial^*\partial, \\
\tilde\Delta_{\mathrm{A}} &:= \partial\partial^* + \bar\partial\bar\partial^* + (\partial\bar\partial)^*(\partial\bar\partial) + (\partial\bar\partial)(\partial\bar\partial)^* + (\bar\partial\partial^*)^*(\bar\partial\partial^*) + (\bar\partial\partial^*)(\bar\partial\partial^*)^*
\end{aligned}$$

are 4-th order elliptic self-adjoint differential operators. In particular,

$$\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\sharp}(X) < +\infty \ \text{ for } \sharp \in \{\bar\partial, \partial, BC, A\}.$$

Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold and B is a neighborhood of the origin in $\mathbb{C}$. Note that $h^{p,q}_{\mathrm{BC}}(X(t))$ and $h^{p,q}_{\mathrm{A}}(X(t))$ are semicontinuous functions of $t \in B$ where $X(t) = \pi^{-1}(t)$ [Schweitzer 2007]. Denote the $\tilde\Delta_{\mathrm{BC}}$ operator and the $\tilde\Delta_{\mathrm{A}}$ on $X(t)$ by $\tilde\Delta_{BC,t}$ and $\tilde\Delta_{A,t}$. From the proof of the semicontinuity of $h^{p,q}_{\mathrm{BC}}(X(t))$ (resp. $h^{p,q}_{\mathrm{A}}(X(t))$) in [Schweitzer 2007], we can see that $h^{p,q}_{\mathrm{BC}}(X(t))$ (resp. $h^{p,q}_{\mathrm{A}}(X(t))$) does not jump at the point $t = 0$ if and only if all the $\tilde\Delta_{BC,0}$- (resp. $\tilde\Delta_{A,0}$)-harmonic forms on X can be extended to relative $\tilde\Delta_{BC,t}$- (resp. $\tilde\Delta_{A,t}$)-harmonic forms on a neighborhood of $0 \in B$ which are real analytic in the direction of B, since the $\tilde\Delta_{BC,t}$ (resp. $\tilde\Delta_{A,t}$)

varies real analytically on B. The above condition is equivalent to the following: all the cohomology classes $[\theta]$ in $H^{p,q}_{\mathrm{BC}}(X)$ (resp. $H^{p,q}_{\mathrm{A}}(X)$) can be extended to a relative d_t − closed (resp. $\partial_t\bar{\partial}_t$ − closed) forms $\theta(t)$ such that $[\theta(t)] \neq 0$ on a neighborhood of $0 \in B$ which are real analytic on the direction of B. Therefore in order to study the jumping phenomenon, we need to study the extension obstructions. So we need to study the obstructions of the extension of the cohomology classes in $\mathbb{H}^\bullet(X, \mathcal{B}^\bullet_{p,q})$ to the relative cohomology classes in $\mathbb{H}^\bullet(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$. Set

$$\begin{aligned}
&\mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}} \\
&\quad = \pi^{-1}(m_0^\omega/(m_0^\omega)^{n+1}) \overset{(+,-)}{\to} \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{O}^\omega_{X_n} \oplus \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \mathcal{O}^\omega_{\bar{X}_n} \\
&\qquad \to \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \Omega^{1;\omega}_{X_n/B_n} \oplus \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \bar{\Omega}^{1;\omega}_{X_n/B_n} \to \cdots \\
&\qquad\quad \to \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \Omega^{p-1;\omega}_{X_n/B_n} \oplus \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \bar{\Omega}^{p-1;\omega}_{X_n/B_n} \\
&\qquad\qquad \to \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \bar{\Omega}^{p;\omega}_{X_n/B_n} \to \cdots \to \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \bar{\Omega}^{q-1;\omega}_{X_n/B_n} \to 0.
\end{aligned}$$

Now we consider the exact sequence

$$0 \to \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}} \to \mathcal{B}^\bullet_{p,q;X_n/B_n} \to \mathcal{B}^\bullet_{p,q;X_0/B_0} \to 0,$$

which induces a long exact sequence

$$\begin{aligned}
0 &\to \mathbb{H}^0(X_n, \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}) \to \mathbb{H}^0(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n}) \\
&\quad \to \mathbb{H}^0(X, \mathcal{B}^\bullet_{p,q;X_0/B_0}) \to \mathbb{H}^1(X_n, \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}) \to \cdots .
\end{aligned}$$

Let $[\theta]$ be a cohomology class in $\mathbb{H}^l(X, \mathcal{B}^\bullet_{p,q;X_0/B_0})$. The obstruction for the extension of $[\theta]$ to a relative cohomology classes in $\mathbb{H}^l(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ comes from the nontrivial image of the connecting homomorphism $\delta^* : \mathbb{H}^l(X, \mathcal{B}^\bullet_{p,q;X_0/B_0}) \to \mathbb{H}^{l+1}(X_n, \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. We denote this obstruction by $\mathrm{o}_n([\theta])$. On the other hand, for a given real direction $\partial/\partial x$ on B, if there exits $n \in \mathbb{N}$, such that $\mathrm{o}_i([\theta]) = 0$ for all $i \leq n$ and $\mathrm{o}_n([\theta]) \neq 0$, then let θ_{n-1} be a $n-1$-st order extension of θ to a relative cohomology class in $\mathbb{H}^l(X_{n-1}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. Then $\check{\boldsymbol{\delta}}\theta_{n-1} = 0$ up to order $n-1$. Now, it is easy to check that $\check{\boldsymbol{\delta}}\theta_{n-1}/x^n$ is an extension of a nontrivial cohomology class $[\check{\boldsymbol{\delta}}\theta_{n-1}/x^n(0)]$ in $\mathbb{H}^{l+1}(X, \mathcal{B}_{p,q})$, while $[\check{\boldsymbol{\delta}}\theta_{n-1}/x^n(x_0)]$ is trivial in $X(x_0)$ as a cohomology classes in $\mathbb{H}^{l+1}(X(x_0), \mathcal{B}_{p,q;x_0})$ if $x_0 \neq 0$. The above discussion leads to the following theorem.

Theorem 3.1. *Let $\pi : \mathcal{X} \to B$ be a small deformation of the central fiber compact complex manifold X. Now we consider* $\dim \mathbb{H}^l(X(t), \mathcal{B}^\bullet_{p,q;t})$ *as a function of $t \in B$. This function jumps at $t = 0$ if there exists an element $[\theta]$ either in $\mathbb{H}^l(X, \mathcal{B}^\bullet_{p,q})$ or in $\mathbb{H}^{l-1}(X, \mathcal{B}^\bullet_{p,q})$ and a minimal natural number $n \geq 1$ such that the n-th order obstruction satisfies*

$$\mathrm{o}_n([\theta]) \neq 0.$$

4. The formula for the obstructions

Since the obstructions we discussed in the previous section are so important when we consider the problem of jumping phenomenon of Bott–Chern cohomology and Aeppli cohomology, in this section we try to find an explicit calculation for such obstructions. As in [Ye 2008, §3], we need some preparation. Cover X by open sets U_i such that, for arbitrary i, U_i is small enough. More precisely, U_i is Stein and the following exact sequence splits:

$$0 \to \pi_n^*(\Omega_{B_n})^{\omega}(U_i) \to \Omega^{\omega}_{X_n}(U_i) \to \Omega^{\omega}_{X_n/B_n}(U_i) \to 0,$$
$$0 \to \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega}(U_i) \to \Omega^{\omega}_{\bar{X}_n}(U_i) \to \bar{\Omega}^{\omega}_{X_n/B_n}(U_i) \to 0.$$

So we have a map $\varphi_i : \Omega^{\omega}_{X_n/B_n}(U_i) \oplus \bar{\Omega}^{\omega}_{X_n/B_n}(U_i) \to \Omega^{\omega}_{X_n}(U_i) \oplus \Omega^{\omega}_{\bar{X}_n}(U_i)$ such that

$$\varphi_i|_{\Omega^{\omega}_{X_n/B_n}(U_i)}(\Omega^{\omega}_{X_n/B_n}(U_i)) \oplus \pi_n^*(\Omega_{B_n})^{\omega}(U_i) \cong \Omega^{\omega}_{X_n}(U_i),$$
$$\varphi_i|_{\bar{\Omega}^{\omega}_{X_n/B_n}(U_i)}(\bar{\Omega}^{\omega}_{X_n/B_n}(U_i)) \oplus \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega}(U_i) \cong \Omega^{\omega}_{\bar{X}_n}(U_i).$$

This decomposition determines a local decomposition of the exterior derivative ∂_{X_n} (resp. $\bar{\partial}_{X_n}$) in $\Omega^{\bullet;\omega}_{X_n}$ (resp. $\bar{\Omega}^{\bullet;\omega}_{X_n}$) on each U_i:

$$\partial_{X_n} = \partial^i_{B_n} + \partial^i_{X_n/B_n} \quad (\text{resp. } \partial_{\bar{X}_n} = \bar{\partial}^i_{B_n} + \bar{\partial}^i_{X_n/B_n}).$$

By definition, ∂_{X_n/B_n} and $\bar{\partial}_{X_n/B_n}$ are given by $\varphi_i^{-1} \circ \partial^i_{X_n/B_n} \circ \varphi_i$ and $\varphi_i^{-1} \circ \bar{\partial}^i_{X_n/B_n} \circ \varphi_i$.

Denote the set of alternating q-cochains β with values in $\mathscr{F}$ by $\check{C}^q(\boldsymbol{U}, \mathscr{F})$, that is, to each $q+1$-tuple, $i_0 < i_1 \cdots < i_q$, β assigns a section $\beta(i_0, i_1, \dots, i_q)$ of $\mathscr{F}$ over $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q}$. Let us still use φ_i to denote the map

$$\begin{aligned}\varphi_i : \pi_n^*(\Omega_{B_n})^{\omega} \wedge \Omega^{p;\omega}_{X_n/B_n}(U_i) \oplus \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega} \wedge \bar{\Omega}^{p;\omega}_{X_n/B_n}(U_i) \\ \to \Omega^{p+1;\omega}_{X_n}(U_i) \oplus \Omega^{p+1;\omega}_{\bar{X}_n}(U_i) \\ \omega_1 \wedge \beta_{i_1} \wedge \cdots \wedge \beta_{i_p} + \omega_2 \wedge \beta'_{j_1} \wedge \cdots \wedge \beta'_{j_p} \mapsto \omega_1 \wedge \varphi_i(\beta_{i_1}) \wedge \cdots \wedge \varphi_i(\beta_{i_p}) \\ +\omega_2 \wedge \varphi_i(\beta'_{j_1}) \wedge \cdots \wedge \varphi_i(\beta'_{j_p}).\end{aligned}$$

Define

$$\varphi : \check{C}^q\big(\boldsymbol{U}, \pi_n^*(\Omega_{B_n})^{\omega} \wedge \Omega^{p;\omega}_{X_n/B_n} \oplus \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega} \wedge \bar{\Omega}^{p;\omega}_{X_n/B_n}\big) \to \check{C}^q\big(\boldsymbol{U}, \Omega^{p+1;\omega}_{X_n} \oplus \Omega^{p+1;\omega}_{\bar{X}_n}\big)$$
$$\text{by} \quad \varphi(\beta)(i_0, i_1, \dots, i_q) = \varphi_{i_0}(\beta(i_0, i_1, \dots, i_q))$$

for all $\beta \in \check{C}^q(\boldsymbol{U}, \pi_n^*(\Omega_{B_n})^{\omega} \wedge \Omega^{p;\omega}_{X_n/B_n} \oplus \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega} \wedge \bar{\Omega}^{p;\omega}_{X_n/B_n})$, where $i_0 < i_1 < \cdots < i_q$. Define the total Lie derivative with respect to B_n:

$$L_{B_n} : \check{C}^q\big(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\bar{X}_n}\big) \to \check{C}^q\big(\boldsymbol{U}, \Omega^{p+1;\omega}_{X_n} \oplus \Omega^{\omega}_{X_n} \wedge \Omega^{p;\omega}_{\bar{X}_n}\big)$$
$$\text{by} \quad L_{B_n}(\beta)(i_0, i_1, \dots, i_q) = \partial^{i_0}_{B_n}(\beta(i_0, i_1, \dots, i_q))$$

for $\beta \in \check{C}^q(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\bar{X}_n})$, where $i_0 < i_1 < \cdots < i_q$ (see [Katz and Oda 1968]).

Define, for each U_i, the total interior product with respect to B_n

$$I^i : \Omega^{p;\omega}_{X_n}(U_i) \oplus \Omega^{p;\omega}_{\overline{X}_n}(U_i) \to \Omega^{p;\omega}_{X_n}(U_i) \oplus \Omega^{p;\omega}_{\overline{X}_n}(U_i)$$

by

$$\begin{aligned} I^i(\mu_1 \partial_{X_n} g_1 \wedge \partial_{X_n} g_2 \wedge \cdots \wedge \partial_{X_n} g_p + \mu_2 \partial_{\overline{X}_n} g'_1 \wedge \partial_{\overline{X}_n} g'_2 \wedge \cdots \wedge \partial_{\overline{X}_n} g'_p) \\ = \mu_1 \sum_{j=1}^{p} \partial_{X_n} g_1 \wedge \cdots \wedge \partial_{X_n} g_{j-1} \wedge \partial^i_{B_n}(g_j) \wedge \partial_{X_n} g_{j+1} \wedge \cdots \wedge \partial_{X_n} g_p \\ + \mu_2 \sum_{j=1}^{p} \partial_{\overline{X}_n} g'_1 \wedge \cdots \wedge \partial_{\overline{X}_n} g'_{j-1} \wedge \partial^i_{\overline{B}_n}(g'_j) \wedge \partial_{\overline{X}_n} g'_{j+1} \wedge \cdots \wedge \partial_{\overline{X}_n} g'_p. \end{aligned}$$

When $p = 0$, we put $I^i = 0$ (see [Katz and Oda 1968]). Finally, define

$$\lambda : \check{C}^q\big(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\overline{X}_n}\big) \to \check{C}^{q+1}\big(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\overline{X}_n}\big)$$
$$\text{by} \quad (\lambda\beta)(i_0, \cdots, i_{q+1}) = (I^{i_0} - I^{i_1})\beta(i_1, \cdots, i_{q+1})$$

for all $\beta \in \check{C}^q(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\overline{X}_n})$.

This gives the following lemma, proved identically to [Ye 2008, Lemma 3.1].

Lemma 4.1. $$\lambda \circ \varphi \equiv \delta \circ \varphi - \varphi \circ \delta.$$

With the above preparation, we are ready to study the jumping phenomenon of the dimensions of Bott–Chern or Aeppli cohomology groups. Suppose we can extend an arbitrary $[\theta] \in \mathbb{H}^l(X, \mathcal{B}_{p,q})$ to order $n-1$ in $\mathbb{H}^l(X_{n-1}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. Denote such an element by $[\theta_{n-1}]$. In what follows, we try to find the obstruction of the extension of $[\theta_{n-1}]$ to n-th order. Consider the exact sequence

$$0 \to (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0} \to \mathcal{B}^\bullet_{p,q;X_n/B_n} \to \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}} \to 0$$

which induces a long exact sequence

$$\begin{aligned} 0 \to \mathbb{H}^0\big(X_n, (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big) \to \mathbb{H}^0\big(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n}\big) \\ \to \mathbb{H}^0\big(X_{n-1}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}\big) \to \mathbb{H}^1\big(X_n, (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big) \to \cdots. \end{aligned}$$

Let $[\theta]$ be a cohomology class in $\mathbb{H}^l(X, \mathcal{B}^\bullet_{p,q;X_0/B_0})$.

The obstruction for $[\theta_{n-1}]$ comes from the nontrivial image of the connecting homomorphism

$$\delta^* : \mathbb{H}^l\big(X_{n-1}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}\big) \to \mathbb{H}^{l+1}\big(X_n, (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big).$$

Now we are ready to calculate the formula for the obstructions. Let $\tilde{\theta}$ be an element of $\check{C}^l(\boldsymbol{U}, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ such that its quotient image in $\check{C}^l(\boldsymbol{U}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$ is θ_{n-1}. Then $\delta^*([\theta_{n-1}])= [\check{\boldsymbol{\delta}}(\tilde{\theta})]$, which is an element of

$$\mathbb{H}^{l+1}\big(X_n, (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big) \cong (\mathrm{m}^\omega_0)^n/(\mathrm{m}^\omega_0)^{n+1} \otimes \mathbb{H}^{l+1}\big(X, \mathcal{B}^\bullet_{p,q;X_0/B_0}\big).$$

Let r_{X_n} be the restriction to the space X_n^ω (the topological space X with structure sheaf $\mathcal{O}_{X_n}^\omega$) and set

$$\begin{aligned}
&\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}},\\
&\qquad = \pi^{-1}(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}})\\
&\qquad\qquad \xrightarrow{(-,+)} \pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \otimes \mathcal{O}^\omega_{X_{n-1}}\\
&\qquad\qquad\qquad\qquad \oplus \bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \otimes \mathcal{O}^\omega_{\bar{X}_{n-1}}\\
&\qquad\qquad \to \pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \Omega^{1;\omega}_{X_{n-1}/B_{n-1}}\\
&\qquad\qquad\qquad\qquad \oplus \bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \bar{\Omega}^{1;\omega}_{X_{n-1}/B_{n-1}} \to \cdots\\
&\qquad\qquad \to \pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \Omega^{p-1;\omega}_{X_{n-1}/B_{n-1}}\\
&\qquad\qquad\qquad\qquad \oplus \bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \bar{\Omega}^{p-1;\omega}_{X_{n-1}/B_{n-1}}\\
&\qquad\qquad \to \bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \bar{\Omega}^{p;\omega}_{X_{n-1}/B_{n-1}} \to \cdots\\
&\qquad\qquad \to \bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \bar{\Omega}^{q-1;\omega}_{X_{n-1}/B_{n-1}} \to 0,
\end{aligned}$$

where

$$\begin{aligned}
\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) &= \pi_{n-1}^{-1}(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \otimes_{\pi_{n-1}^{-1}(\mathscr{C}_B^\omega)} \mathcal{O}^\omega_{X_{n-1}},\\
\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) &= \pi_{n-1}^{-1}(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \otimes_{\pi_{n-1}^{-1}(\mathscr{C}_B^\omega)} \bar{\mathcal{O}}^\omega_{X_{n-1}}.
\end{aligned}$$

In order to give the obstructions an explicit calculation, we need to consider the map

$$\begin{aligned}
\rho : \mathbb{H}^l\big(X_n, (\mathcal{M}_0^\omega)^n/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big)&\\
\to \mathbb{H}^l\big(X_{n-1}, \pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}\big)&
\end{aligned}$$

which is defined by $\rho[\sigma] = [\varphi^{-1} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{\bar{B}_n}) \circ \varphi(\sigma)]$, where φ^{-1} is the quotient map

$$\begin{aligned}
&\check{C}^\bullet\big(\boldsymbol{U}, \pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \Omega^{p;\omega}_{X_n|X_{n-1}} \oplus \bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \Omega^{p;\omega}_{\bar{X}_n|\bar{X}_{n-1}}\big)\\
&\quad \to \check{C}^\bullet\big(\boldsymbol{U}, \pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \Omega^{p;\omega}_{X_{n-1}/B_{n-1}}\\
&\qquad\qquad\qquad \oplus \bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}} + \bar{\Omega}^\omega_{B_n|B_{n-1}}) \wedge \bar{\Omega}^{p;\omega}_{X_{n-1}/B_{n-1}}\big).
\end{aligned}$$

And we have the following lemmas; the proofs are identical to those of [Ye 2008, Lemma 3.2 and Lemma 3.3].

Lemma 4.2. *The map ρ is well defined.*

Lemma 4.3. *Furthermore, $\rho([\check{\boldsymbol{\delta}}(\tilde{\theta})])$ is exactly $\mathrm{o}_n([\theta])$ in Section 3.*

Now consider the following exact sequence. The connecting homomorphism of the associated long exact sequence gives the Kodaira–Spencer class of order n [Voisin 1996, §1.3.2];

$$0 \to \pi_{n-1}^{*}(\Omega_{B_n|B_{n-1}})^{\omega} \to \Omega^{\omega}_{X_n|X_{n-1}} \to \Omega^{\omega}_{X_{n-1}/B_{n-1}} \to 0.$$

If we wedge the above exact sequence with $\Omega^{p-1;\omega}_{X_{n-1}/B_{n-1}}$, we get a new exact sequence. The connecting homomorphism of such an exact sequence gives us a map from $H^q(X_{n-1}, \Omega^{p;\omega}_{X_{n-1}/B_{n-1}})$ to $H^{q+1}(X_{n-1}, \pi^*(\Omega_{B_n|B_{n-1}})^{\omega} \wedge \Omega^{p-1;\omega}_{X_{n-1}/B_{n-1}})$.

Denote such a map by $\kappa_n \llcorner$, for it is simply the inner product with the Kodaira–Spencer class of order n. With the above preparation, we are ready to prove the main theorem of this section.

Theorem 4.4. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X. Suppose we can extend an arbitrary $[\theta] \in \mathbb{H}^l(X, \mathscr{B}^{\bullet}_{p,q})$ to order $n-1$ in $\mathbb{H}^l(X_{n-1}, \mathscr{B}^{\bullet}_{p,q;X_{n-1}/B_{n-1}})$. Denote such an element by $[\theta_{n-1}]$. The obstruction of the extension of $[\theta]$ to n-th order is given by*

$$\mathrm{o}_n([\theta]) = -\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ \partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]) - \bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ \bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]),$$

where κ_n is the n-th order Kodaira–Spencer class and $\bar\kappa_n$ is the n-th order Kodaira–Spencer class of the deformation $\bar\pi : \bar{\mathscr{X}} \to \bar B$. The maps $\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}$ and $\bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}$ are those defined in Section 2.

Proof. Note that $\mathrm{o}_n([\theta]) = \rho \circ \delta(\tilde\theta) = [\varphi^{-1} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ \varphi \circ \delta(\tilde\theta)]$. Because $(L_{B_n} + L_{\bar B_n} + \partial_{X_n/B_n} + \bar\partial_{X_n/B_n}) \circ \check{\boldsymbol\delta} = -\check{\boldsymbol\delta} \circ (L_{B_n} + L_{\bar B_n} + \partial_{X_n/B_n} + \bar\partial_{X_n/B_n})$,

$$\begin{aligned}
r_{X_{n-1}} &\circ (L_{B_n} + L_{\bar B_n}) \circ \varphi \circ \check{\boldsymbol\delta}(\tilde\theta) \\
&\equiv r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ (\check{\boldsymbol\delta} \circ \varphi - \lambda \circ \varphi)(\tilde\theta) \\
&\equiv r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ \check{\boldsymbol\delta} \circ \varphi(\tilde\theta) \\
&\equiv -r_{X_{n-1}} \circ \big(\partial^{\bullet}_{X_n/B_n} \circ \check{\boldsymbol\delta} + \check{\boldsymbol\delta} \circ \partial^{\bullet}_{X_n/B_n} + \bar\partial^{\bullet}_{X_n/B_n} \circ \check{\boldsymbol\delta} + \check{\boldsymbol\delta} \circ \bar\partial^{\bullet}_{X_n/B_n} \\
&\qquad\qquad + \check{\boldsymbol\delta} \circ (L_{B_n} + L_{\bar B_n})\big) \circ \varphi(\tilde\theta) \\
&\equiv -r_{X_{n-1}} \circ \big(\partial^{\bullet}_{X_n/B_n} \circ \check{\boldsymbol\delta} + \check{\boldsymbol\delta} \circ \partial^{\bullet}_{X_n/B_n} + \bar\partial^{\bullet}_{X_n/B_n} \circ \check{\boldsymbol\delta} + \check{\boldsymbol\delta} \circ \bar\partial^{\bullet}_{X_n/B_n}\big) \circ \varphi(\tilde\theta) \\
&\qquad\qquad - \check{\boldsymbol\delta} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ \varphi(\tilde\theta).
\end{aligned}$$

Therefore

$$
\begin{aligned}
&[\varphi^{-1}\circ r_{X_{n-1}}\circ(L_{B_n}+L_{\bar{B}_n})\circ\varphi\circ\delta(\tilde{\theta})]\\
&\quad=[-\varphi^{-1}\circ r_{X_{n-1}}\circ(\partial^{\bullet}_{X_n/B_n}\circ\check{\boldsymbol{\delta}}+\check{\boldsymbol{\delta}}\circ\partial^{\bullet}_{X_n/B_n}+\bar{\partial}^{\bullet}_{X_n/B_n}\circ\check{\boldsymbol{\delta}}+\check{\boldsymbol{\delta}}\circ\bar{\partial}^{\bullet}_{X_n/B_n})\circ\varphi(\tilde{\theta})]\\
&\quad=-[\partial_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})+\bar{\partial}_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})\\
&\qquad+\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\widetilde{\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1})})+\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}(\theta_{n-1})}).
\end{aligned}
$$

Since, for $0\le r\le p-1$,

$$
\begin{aligned}
(\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}))^{p-1,0}_{u}&=(\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}))^{0,q-1}_{v}\\
&=\check{\delta}(\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}))^{r,0}_{u}=0,
\end{aligned}
$$

and $\check{\delta}(\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}))^{0,s}_{v}=0$ for $0\le s\le q-1$, we know that

$$
\begin{aligned}
\partial_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})+\bar{\partial}_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})&\\
=\check{\boldsymbol{\delta}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}).&
\end{aligned}
$$

Therefore, $[\partial_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})+\bar{\partial}_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})]=0$. And from Lemma 2.4 and Lemma 2.5,

$$
\begin{aligned}
&[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\widetilde{\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1})})]\\
&\qquad=\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1}-\theta^{p-1,0}_{n-1;u})}+\partial_{X_{n-1}/B_{n-1}}\theta^{p-1,0}_{n-1;u}\big)\big]\\
&\qquad=\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\partial_{X_{n-1}/B_{n-1}}\theta^{p-1,0}_{n-1;u}}\big)\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}(\theta_{n-1})})]\\
&\qquad=\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}(\theta_{n-1}-\theta^{0,q-1}_{n-1;v})}+\bar{\partial}_{X_{n-1}/B_{n-1}}\theta^{0,q-1}_{n-1;v}\big)\big]\\
&\qquad=\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}\theta^{0,q-1}_{n-1;v}}\big)\big]
\end{aligned}
$$

By the definition of the maps $\partial^{\bar{\partial},\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\partial^{\mathscr{B},\bar{\partial}}_{X_{n-1}/B_{n-1}}$ and [Ye 2008, Lemma 3.4],

$$
\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\partial_{X_{n-1}/B_{n-1}}\theta^{p-1,0}_{n-1;u}}\big)\big]=\partial^{\bar{\partial},\mathscr{B}}_{X_{n-1}/B_{n-1}}\circ\kappa_n\llcorner\circ\partial^{\mathscr{B},\bar{\partial}}_{X_{n-1}/B_{n-1}}([\theta_{n-1}])
$$

and similarly, we have

$$
\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}\theta^{0,q-1}_{n-1;v}}\big)\big]=\bar{\partial}^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}\circ\bar{\kappa}_n\llcorner\circ\bar{\partial}^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]).
$$

Finally,

$$
\begin{aligned}
&[\varphi^{-1}\circ r_{X_{n-1}}\circ(L_{B_n}+L_{\bar{B}_n})\circ\varphi\circ\delta(\tilde{\theta})]\\
&\quad=-\partial^{\bar{\partial},\mathscr{B}}_{X_{n-1}/B_{n-1}}\circ\kappa_n\llcorner\circ\partial^{\mathscr{B},\bar{\partial}}_{X_{n-1}/B_{n-1}}([\theta_{n-1}])-\bar{\partial}^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}\circ\bar{\kappa}_n\llcorner\circ\bar{\partial}^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]).\quad\square
\end{aligned}
$$

We apply the above theorem and Theorem 3.1 in order to study the jumping phenomenon of the dimensions of Bott–Chern(Aeppli) cohomology groups, and obtain the following theorems.

Theorem 4.5. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X. If there exists an element $[\theta^1]$ in $H^{p,q}_{\mathrm{BC}}(X)$ or an element $[\theta^2]$ in $H^{p-1,q-1}_{\mathrm{A}}(X)$ and a minimal natural number $n \geq 1$ such that the n-th order obstruction $o_n([\theta^1]) \neq 0$ or $o_n([\theta^2]) \neq 0$, then the $h^{\mathrm{BC}}_{p,q}(X(t))$ will jump at the point $t = 0$. The formulas for the obstructions are given by*

$$o_n([\theta^1]) = -\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ r_{BC,\bar\partial}([\theta^1_{n-1}]) - \bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ r_{BC,\partial}([\theta^1_{n-1}]);$$

$$o_n([\theta^2]) = -\partial^{\bar\partial,BC}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ \partial^{A,\bar\partial}_{X_{n-1}/B_{n-1}}([\theta^2_{n-1}]) - \bar\partial^{\partial,BC}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ \bar\partial^{A,\partial}_{X_{n-1}/B_{n-1}}([\theta^2_{n-1}]).$$

Theorem 4.6. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X. If there exists an element $[\theta^1]$ in $H^{p,q}_{\mathrm{A}}(X)$ or an element $[\theta^2]$ in $\mathbb{H}^{p+q}(X, \mathscr{B}^\bullet_{p+1,q+1})$ and a minimal natural number $n \geq 1$ such that the n-th order obstruction $o_n([\theta^1]) \neq 0$ or $o_n([\theta^2]) \neq 0$, then the $h^{\mathrm{A}}_{p,q}(X(t))$ will jump at the point $t = 0$. The formulas for the obstructions are given by*

$$o_n([\theta^1]) = -\partial^{\bar\partial,BC}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ \partial^{A,\bar\partial}_{X_{n-1}/B_{n-1}}([\theta^1_{n-1}]) - \bar\partial^{\partial,BC}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ \bar\partial^{A,\partial}_{X_{n-1}/B_{n-1}}([\theta^1_{n-1}]).$$

$$o_n([\theta^2]) = -r_{\bar\partial,A} \circ \kappa_n \llcorner \circ \partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}([\theta^2_{n-1}]) - r_{\partial,A} \circ \bar\kappa_n \llcorner \circ \bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta^2_{n-1}]).$$

By these theorems, we can deduce the following corollaries immediately.

Corollary 4.7. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Suppose that up to order n, the maps $r_{BC,\bar\partial} : H^{p,q}_{\mathrm{BC}}(X_n/B_n) \to H^q(X_n, \Omega^{p;\omega}_{X_n/B_n})$ and $r_{BC,\partial} : H^{p,q}_{\mathrm{BC}}(X_n/B_n) \to H^p(\overline{X}_n, \overline{\Omega}^{q;\omega}_{X_n/B_n})$ are 0. Any element $[\theta] \in H^{p,q}_{\mathrm{BC}}(X)$ can be extended to order $n+1$ in $H^{p,q}_{\mathrm{BC}}(X_{n+1}/B_{n+1})$.*

Proof. This result can be shown by induction on k.

Suppose that the corollary is proved for $k-1$, then we can extend $[\theta]$ to and element $[\theta_{k-1}]$ in $H^{p,q}_{\mathrm{BC}}(X_{k-1}/B_{k-1})$. By Theorem 4.5 , the obstruction for the extension of $[\theta]$ to k-th order comes from:

$$o_k([\theta]) = -\partial^{\bar\partial,\mathscr{B}}_{X_{k-1}/B_{k-1}} \circ \kappa_k \llcorner \circ r_{BC,\bar\partial}([\theta_{k-1}]) - \bar\partial^{\partial,\mathscr{B}}_{X_{k-1}/B_{k-1}} \circ \bar\kappa_k \llcorner \circ r_{BC,\partial}([\theta_{k-1}]).$$

By the assumption, $r_{BC,\bar\partial} : H^{p,q}_{\mathrm{BC}}(X_{k-1}/B_{k-1}) \to H^q(X_{k-1}, \Omega^{p;\omega}_{X_{k-1}/B_{k-1}})$ and $r_{BC,\partial} : H^{p,q}_{\mathrm{BC}}(X_{k-1}/B_{k-1}) \to H^p(\overline{X}_{k-1}, \overline{\Omega}^{q;\omega}_{X_{k-1}/B_{k-1}})$ are 0, where $k \leq n+1$. So we have $r_{BC,\partial}([\theta_{k-1}]) = 0$ and $r_{BC,\bar\partial}([\theta_{k-1}]) = 0$. So the obstruction $o_k([\theta])$ is trivial which means $[\theta]$ can be extended to k-th order. $\square$

Since

$$\partial^{A,\bar\partial}_{X_n/B_n} : H^{p,q}_{\mathrm{A}}(X_n/B_n) \to H^q(X_n, \Omega^{p+1;\omega}_{X_n/B_n})$$

is the composition of

$$\begin{aligned}\partial^{A,BC}_{X_n/B_n} &: H^{p,q}_{\mathrm{A}}(X_n/B_n) \to H^{p+1,q}_{\mathrm{BC}}(X_n/B_n) \quad \text{and}\\ r_{BC,\bar\partial} &: H^{p+1,q}_{\mathrm{BC}}(X_n/B_n) \to H^q(X_n, \Omega^{p+1;\omega}_{X_n/B_n}).\end{aligned}$$

With the same proof of the above corollary, we have the following result and we omit the proof.

Corollary 4.8. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Suppose, up to order n, the maps $r_{BC,\bar\partial} : H^{p+1,q}_{\mathrm{BC}}(X_n/B_n) \to H^q(X_n, \Omega^{p+1;\omega}_{X_n/B_n})$ and $r_{BC,\partial} : H^{p,q+1}_{\mathrm{BC}}(X_n/B_n) \to H^p(\bar X_n, \bar\Omega^{q+1;\omega}_{X_n/B_n})$ is 0. Any $[\theta]$ that belongs to $H^{p,q}_{\mathrm{A}}(X)$ can be extended to order $n+1$ in $H^{p,q}_{\mathrm{A}}(X_{n+1}/B_{n+1})$.*

5. An example

In this section, we will use the formulas in Theorems 4.5 and 4.6 to study the jumping phenomenon of the dimensions of Bott–Chern and Aeppli cohomology groups $h^{p,q}_{\mathrm{BC}}$ and $h^{p,q}_{\mathrm{A}}$, respectively, of small deformations of Iwasawa manifold. It was Kodaira who first calculated small deformations of Iwasawa manifold [Nakamura 1975]. In the first part of this section, let us recall his result.

Set

$$G = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\} \cong \mathbb{C}^3,$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} : \omega_i \in \mathbb{Z} + \mathbb{Z}\sqrt{-1} \right\}.$$

The multiplication is defined by

$$\begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2+\omega_2 & z_3+\omega_1 z_2+\omega_3 \\ 0 & 1 & z_1+\omega_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$X = G/\Gamma$ is called the Iwasawa manifold. We may consider $X = \mathbb{C}^3/\Gamma$. The element $g \in \Gamma$ operates on $\mathbb{C}^3$ as follows:

$$z_1' = z_1 + \omega_1, \quad z_2' = z_2 + \omega_2, \quad z_3' = z_3 + \omega_1 z_2 + \omega_3,$$

where $g = (\omega_1, \omega_2, \omega_3)$ and $z' = z \cdot g$.

There exist holomorphic 1-forms $\varphi_1, \varphi_2, \varphi_3$ which are linearly independent at every point on X and are given by

$$\varphi_1 = dz_1, \quad \varphi_2 = dz_2, \quad \varphi_3 = dz_3 - z_1 dz_2,$$

so that

$$d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2.$$

On the other hand we have holomorphic vector fields $\theta_1, \theta_2, \theta_3$ on X given by

$$\theta_1 = \frac{\partial}{\partial z_1}, \quad \theta_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, \quad \theta_3 = \frac{\partial}{\partial z_3}.$$

It is easily seen that

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \quad [\theta_1, \theta_3] = [\theta_2, \theta_3] = 0.$$

In view of [Nakamura 1975, Theorem 3], $H^1(X, \mathbb{O}_X)$ is spanned by $\bar{\varphi}_1, \bar{\varphi}_2$. Since Θ is isomorphic to $\mathbb{O}^3$, $H^1(X, T_X)$ is spanned by $\theta_i \bar{\varphi}_\lambda$, $i = 1, 2, 3$, $\lambda = 1, 2$.

Consider the small deformation of X given by

$$\psi(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \bar{\varphi}_\lambda t - (t_{11}t_{22} - t_{21}t_{12}) \theta_3 \bar{\varphi}_3 t^2.$$

We summarize the numerical characters of deformations. The deformations are divided into the following three classes, which are characterized by the following values of the parameters (all the details can be found in [Angella 2013]):

class (i): $t_{11} = t_{12} = t_{21} = t_{22} = 0$;

class (ii): $D(\boldsymbol{t}) = 0$ and $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$;

- class (ii.a): $D(\boldsymbol{t}) = 0$ and rank $S = 1$;
- class (ii.b): $D(\boldsymbol{t}) = 0$ and rank $S = 2$;

class (iii): $D(\boldsymbol{t}) \neq 0$;

- class (iii.a): $D(\boldsymbol{t}) \neq 0$ and rank $S = 1$;
- class (iii.b): $D(\boldsymbol{t}) \neq 0$ and rank $S = 2$;

where the matrix S is defined by

$$S := \begin{pmatrix} \bar{\sigma}_{1\bar{1}} & \bar{\sigma}_{2\bar{2}} & \bar{\sigma}_{1\bar{2}} & \bar{\sigma}_{2\bar{1}} \\ \sigma_{1\bar{1}} & \sigma_{2\bar{2}} & \sigma_{2\bar{1}} & \sigma_{1\bar{2}} \end{pmatrix}$$

where $\sigma_{1\bar{1}}, \sigma_{1\bar{2}}, \sigma_{2\bar{1}}, \sigma_{2\bar{2}} \in \mathbb{C}$ and $\sigma_{12} \in \mathbb{C}$ are complex numbers depending only on $\boldsymbol{t}$ such that

$$d\varphi_t^3 =: \sigma_{12}\varphi_t^1 \wedge \varphi_t^2 + \sigma_{1\bar{1}}\varphi_t^1 \wedge \bar{\varphi}_t^1 + \sigma_{1\bar{2}}\varphi_t^1 \wedge \bar{\varphi}_t^2 + \sigma_{2\bar{1}}\varphi_t^2 \wedge \bar{\varphi}_t^1 + \sigma_{2\bar{2}}\varphi_t^2 \wedge \bar{\varphi}_t^2.$$

$\mathbf{H}^{\bullet}_{\mathrm{dR}}$	$\mathbf{b_1}$	$\mathbf{b_2}$	$\mathbf{b_3}$	$\mathbf{b_4}$	$\mathbf{b_5}$
$\mathbb{I}_3$ and (i), (ii), (iii)	4	8	10	8	4

$\mathbf{H}^{\bullet\bullet}_{\bar\partial}$	$\mathbf{h}^{1,0}_{\bar\partial}$	$\mathbf{h}^{0,1}_{\bar\partial}$	$\mathbf{h}^{2,0}_{\bar\partial}$	$\mathbf{h}^{1,1}_{\bar\partial}$	$\mathbf{h}^{0,2}_{\bar\partial}$	$\mathbf{h}^{3,0}_{\bar\partial}$	$\mathbf{h}^{2,1}_{\bar\partial}$	$\mathbf{h}^{1,2}_{\bar\partial}$	$\mathbf{h}^{0,3}_{\bar\partial}$	$\mathbf{h}^{3,1}_{\bar\partial}$	$\mathbf{h}^{2,2}_{\bar\partial}$	$\mathbf{h}^{1,3}_{\bar\partial}$	$\mathbf{h}^{3,2}_{\bar\partial}$	$\mathbf{h}^{2,3}_{\bar\partial}$
$\mathbb{I}_3$ and (i)	3	2	3	6	2	1	6	6	1	2	6	3	2	3
(ii)	2	2	2	5	2	1	5	5	1	2	5	2	2	2
(iii)	2	2	1	5	2	1	4	4	1	2	5	1	2	2
$\mathbf{H}^{\bullet\bullet}_{\mathrm{BC}}$	$\mathbf{h}^{1,0}_{\mathrm{BC}}$	$\mathbf{h}^{0,1}_{\mathrm{BC}}$	$\mathbf{h}^{2,0}_{\mathrm{BC}}$	$\mathbf{h}^{1,1}_{\mathrm{BC}}$	$\mathbf{h}^{0,2}_{\mathrm{BC}}$	$\mathbf{h}^{3,0}_{\mathrm{BC}}$	$\mathbf{h}^{2,1}_{\mathrm{BC}}$	$\mathbf{h}^{1,2}_{\mathrm{BC}}$	$\mathbf{h}^{0,3}_{\mathrm{BC}}$	$\mathbf{h}^{3,1}_{\mathrm{BC}}$	$\mathbf{h}^{2,2}_{\mathrm{BC}}$	$\mathbf{h}^{1,3}_{\mathrm{BC}}$	$\mathbf{h}^{3,2}_{\mathrm{BC}}$	$\mathbf{h}^{2,3}_{\mathrm{BC}}$
$\mathbb{I}_3$ and (i)	2	2	3	4	3	1	6	6	1	2	8	2	3	3
(ii.a)	2	2	2	4	2	1	6	6	1	2	7	2	3	3
(ii.b)	2	2	2	4	2	1	6	6	1	2	6	2	3	3
(iii.a)	2	2	1	4	1	1	6	6	1	2	7	2	3	3
(iii.b)	2	2	1	4	1	1	6	6	1	2	6	2	3	3
$\mathbf{H}^{\bullet\bullet}_{\mathrm{A}}$	$\mathbf{h}^{1,0}_{\mathrm{A}}$	$\mathbf{h}^{0,1}_{\mathrm{A}}$	$\mathbf{h}^{2,0}_{\mathrm{A}}$	$\mathbf{h}^{1,1}_{\mathrm{A}}$	$\mathbf{h}^{0,2}_{\mathrm{A}}$	$\mathbf{h}^{3,0}_{\mathrm{A}}$	$\mathbf{h}^{2,1}_{\mathrm{A}}$	$\mathbf{h}^{1,2}_{\mathrm{A}}$	$\mathbf{h}^{0,3}_{\mathrm{A}}$	$\mathbf{h}^{3,1}_{\mathrm{A}}$	$\mathbf{h}^{2,2}_{\mathrm{A}}$	$\mathbf{h}^{1,3}_{\mathrm{A}}$	$\mathbf{h}^{3,2}_{\mathrm{A}}$	$\mathbf{h}^{2,3}_{\mathrm{A}}$
$\mathbb{I}_3$ and (i)	3	3	2	8	2	1	6	6	1	3	4	3	2	2
(ii.a)	3	3	2	7	2	1	6	6	1	2	4	2	2	2
(ii.b)	3	3	2	6	2	1	6	6	1	2	4	2	2	2
(iii.a)	3	3	2	7	2	1	6	6	1	1	4	1	2	2
(iii.b)	3	3	2	6	2	1	6	6	1	1	4	1	2	2

Table 1. Dimensions of the cohomologies of the Iwasawa manifold and of its small deformations [Angella 2013].

The first order asymptotic behavior of $\sigma_{12},\ \sigma_{1\bar{1}},\ \sigma_{1\bar{2}},\ \sigma_{2\bar{1}},\ \sigma_{2\bar{2}}$ for $\boldsymbol{t}$ near 0 in classes (i), (ii) or (iii) is

$$\sigma_{12} = -1 + \mathrm{o}(|t|)\boldsymbol{t} \quad \sigma_{1\bar{1}} = t_{21} + \mathrm{o}(|t|)\boldsymbol{t} \quad \sigma_{1\bar{2}} = t_{22} + \mathrm{o}(|t|)\boldsymbol{t}$$
$$\sigma_{2\bar{1}} = -t_{11} + \mathrm{o}(|t|)\boldsymbol{t} \quad \sigma_{2\bar{2}} = -t_{12} + \mathrm{o}(|t|)\boldsymbol{t}.$$

From Table 1, we know that the jumping phenomenon happens in $h^{2,0}_{\mathrm{BC}}$, $h^{0,2}_{\mathrm{BC}}$ and $h^{2,2}_{\mathrm{BC}}$ of the Bott–Chern cohomology and symmetrically happens in $h^{3,1}_{\mathrm{A}}$, $h^{1,3}_{\mathrm{A}}$ and $h^{1,1}_{\mathrm{A}}$ of the Aeppli cohomology. Now let us explain the jumping phenomenon of the dimensions of Bott–Chern and Aeppli cohomologies by using the obstruction formula. From [Angella 2013, §4], it follows that the Bott–Chern cohomology groups in bidegree (2, 0), (0, 2), (2, 2) are

$$
\begin{aligned}
H^{2,0}_{\mathrm{BC}}(X) &= \mathrm{Span}_{\mathbb{C}}\{[\varphi_1\wedge\varphi_2], [\varphi_2\wedge\varphi_3], [\varphi_3\wedge\varphi_1]\},\\
H^{0,2}_{\mathrm{BC}}(X) &= \mathrm{Span}_{\mathbb{C}}\{[\bar\varphi_1\wedge\bar\varphi_2], [\bar\varphi_2\wedge\bar\varphi_3], [\bar\varphi_3\wedge\bar\varphi_1]\},\\
H^{2,2}_{\mathrm{BC}}(X) &= \mathrm{Span}_{\mathbb{C}}\big\{[\varphi_2\wedge\varphi_3\wedge\bar\varphi_1\wedge\bar\varphi_2], [\varphi_3\wedge\varphi_1\wedge\bar\varphi_1\wedge\bar\varphi_2],\\
&\qquad [\varphi_1\wedge\varphi_2\wedge\bar\varphi_2\wedge\bar\varphi_3], [\varphi_2\wedge\varphi_3\wedge\bar\varphi_2\wedge\bar\varphi_3], [\varphi_3\wedge\varphi_1\wedge\bar\varphi_2\wedge\bar\varphi_3],\\
&\qquad [\varphi_1\wedge\varphi_2\wedge\bar\varphi_3\wedge\bar\varphi_1], [\varphi_2\wedge\varphi_3\wedge\bar\varphi_3\wedge\bar\varphi_1], [\varphi_3\wedge\varphi_1\wedge\bar\varphi_3\wedge\bar\varphi_1]\big\},
\end{aligned}
$$

and the Aeppli cohomology groups in bidegree (3, 1), (1, 3), (1, 1) are

$$H_{\mathrm{A}}^{3,1}(X) = \mathrm{Span}_{\mathbb{C}}\{[\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \bar{\varphi}_1], [\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \bar{\varphi}_2], [\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \bar{\varphi}_3]\},$$
$$H_{\mathrm{A}}^{1,3}(X) = \mathrm{Span}_{\mathbb{C}}\{[\varphi_1 \wedge \bar{\varphi}_1 \wedge \bar{\varphi}_2 \wedge \bar{\varphi}_3], [\varphi_2 \wedge \bar{\varphi}_1 \wedge \bar{\varphi}_2 \wedge \bar{\varphi}_3], [\varphi_3 \wedge \bar{\varphi}_1 \wedge \bar{\varphi}_2 \wedge \bar{\varphi}_3]\},$$
$$H_{\mathrm{A}}^{1,1}(X) = \mathrm{Span}_{\mathbb{C}}\big\{[\varphi_1 \wedge \bar{\varphi}_1], [\varphi_1 \wedge \bar{\varphi}_2], [\varphi_1 \wedge \bar{\varphi}_3], [\varphi_2 \wedge \bar{\varphi}_1], [\varphi_2 \wedge \bar{\varphi}_2], [\varphi_2 \wedge \bar{\varphi}_3], [\varphi_3 \wedge \bar{\varphi}_1], [\varphi_3 \wedge \bar{\varphi}_2]\big\}.$$

For example, let us first consider $h_{\mathrm{BC}}^{2,0}$ under a class (ii) deformation. The Kodaira–Spencer class of the this deformation is $\psi_1(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \bar{\varphi}_\lambda$, and $\overline{\psi}_1(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} \bar{t}_{i\lambda} \bar{\theta}_i \varphi_\lambda$, with $t_{11}t_{22} - t_{21}t_{12} = 0$. It is easy to check that

$$\mathrm{o}_1(\varphi_1 \wedge \varphi_2) = -\partial(\mathrm{int}(\psi_1(t))(\varphi_1 \wedge \varphi_2)) - \bar{\partial}(\mathrm{int}(\overline{\psi}_1(t))(\varphi_1 \wedge \varphi_2)) = 0,$$
$$\mathrm{o}_1(t_{11}\varphi_2 \wedge \varphi_3 - t_{21}\varphi_1 \wedge \varphi_3) = -\partial((t_{11}t_{22} - t_{21}t_{12})\varphi_3 \wedge \bar{\varphi}_2) = 0,$$
$$\mathrm{o}_1(\varphi_2 \wedge \varphi_3) = t_{21}\varphi_1 \wedge \varphi_2 \wedge \bar{\varphi}_1 + t_{22}\varphi_1 \wedge \varphi_2 \wedge \bar{\varphi}_2,$$
$$\mathrm{o}_1(\varphi_1 \wedge \varphi_3) = t_{11}\varphi_1 \wedge \varphi_2 \wedge \bar{\varphi}_1 + t_{12}\varphi_1 \wedge \varphi_2 \wedge \bar{\varphi}_2.$$

Therefore, for an element of the subspace $\mathrm{Span}_{\mathbb{C}}\{[\varphi_1 \wedge \varphi_2], [t_{11}\varphi_2 \wedge \varphi_3 - t_{21}\varphi_1 \wedge \varphi_3]\}$, the first order obstruction is trivial, while, since $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$, at least one of the obstructions $\mathrm{o}_1(\varphi_2 \wedge \varphi_3)$, $\mathrm{o}_1(\varphi_1 \wedge \varphi_3)$ is nontrivial. This partly explains why the Hodge number $h_{\mathrm{BC}}^{2,0}$ jumps from 3 to 2. For another example, let us consider $h_{\mathrm{A}}^{1,1}$ under a class (ii) deformation. It is easy to check that all the first order obstructions of the cohomology classes are trivial. However, if we want to study the jumping phenomenon, we also need to consider the obstructions that come from $\mathbb{H}^2(X, \mathcal{B}^{\bullet}_{2,2})$. It is easy to check that

$$\mathbb{H}^2(X, \mathcal{B}^{\bullet}_{2,2}) = \mathrm{Span}_{\mathbb{C}}\{[\varphi_3], [\bar{\varphi}_3]\},$$
$$\mathrm{o}_1(\varphi_3) = t_{11}\varphi_2 \wedge \bar{\varphi}_1 + t_{12}\varphi_2 \wedge \bar{\varphi}_2 - t_{21}\varphi_1 \wedge \bar{\varphi}_1 - t_{22}\varphi_1 \wedge \bar{\varphi}_2,$$
$$\mathrm{o}_1(\bar{\varphi}_3) = \bar{t}_{11}\bar{\varphi}_2 \wedge \varphi_1 + \bar{t}_{12}\bar{\varphi}_2 \wedge \varphi_2 - \bar{t}_{21}\bar{\varphi}_1 \wedge \varphi_1 - \bar{t}_{22}\bar{\varphi}_1 \wedge \varphi_2.$$

Note that the first order of S is

$$\begin{pmatrix} \bar{t}_{21} & -\bar{t}_{12} & \bar{t}_{22} & -\bar{t}_{11} \\ t_{21} & -t_{12} & -t_{11} & t_{22} \end{pmatrix}.$$

If the rank of the first order of S is 1, then there exist c_1, c_2 such that

$$\mathrm{o}_1(c_1\varphi_3 + c_2\bar{\varphi}_3) \neq 0.$$

If the rank of the first order of S is 2, then for all c_1, c_2

$$\mathrm{o}_1(c_1\varphi_3 + c_2\bar{\varphi}_3) \neq 0,$$

and exactly these obstructions make $h_{\mathrm{A}}^{1,1}$ jump from 8 to 7 in class (ii.a) and from 8 to 6 in class (ii.b).

To end the section, we give the following observation as an application of the formulas in Theorems 4.5 and 4.6.

Proposition 5.1. *Let X be a non-Kähler nilpotent complex parallelisable manifold whose dimension is more than* 2, *and $\pi : \mathscr{X} \to B$ be the versal deformation family of X. Then the number $h_{\mathrm{A}}^{1,1}$ will jump in any neighborhood of* $0 \in B$.

Proof. Let φ_i, for $i = 1, \dots, m = \dim_{\mathbb{C}} X$, be the linearly independent holomorphic 1-forms of X. It is easy to check that $\bar{\varphi}_i$ are $\partial\bar{\partial}$-closed and therefore each $\bar{\varphi}_i$ represents an element of $H_{\mathrm{A}}^{0,1}(X)$. On the other hand, by [Macrì 2013, Theorem 3], we know that the dimension of $H_{\mathrm{A}}^{0,1}(X)$ is less than or equal to m. Therefore $\bar{\varphi}_i$, $i = 1, \dots, n$, give us a base of $H_{\mathrm{A}}^{0,1}(X)$. So $\partial : H_{\mathrm{A}}^{0,1}(X) \to H^1(X, \Omega_X)$ is trivial. Then we know that $r_{\bar{\partial},A} : H^1(X, \Omega_X) \to H_{\mathrm{A}}^{1,1}(X)$ is injective. From the proof of [Ye 2008, Proposition 4.2], we know there exists an element $[\theta]$ in $H^0(X, \Omega_X)$ whose $\mathrm{o}_1([\theta]) \neq 0$. Since $\bar{\partial}\theta = 0$, θ also represents an element in $\mathbb{H}^2(X, \mathscr{B}^{\bullet}_{2,2})$; let us denote it by $[\theta]_{\mathscr{B}}$. By Theorem 4.6 one can see that $\mathrm{o}_1([\theta]_{\mathscr{B}}) = -r_{\bar{\partial},A}(\mathrm{o}_1([\theta]))$ in this case. From the injectivity of $r_{\bar{\partial},A}$, we know that $\mathrm{o}_1([\theta]_{\mathscr{B}}) \neq 0$. Therefore the number $h_{\mathrm{A}}^{1,1}$ will jump in any neighborhood of $0 \in B$. □

References

[Aeppli 1965] A. Aeppli, "On the cohomology structure of Stein manifolds", pp. 58–70 in *Proceedings of the Conference on Complex Analysis* (Minneapolis, MN, 1964), edited by A. Aeppli et al., Springer, Berlin, 1965. MR 36 #4588 Zbl 0166.33902

[Angella 2013] D. Angella, "The cohomologies of the Iwasawa manifold and of its small deformations", *J. Geom. Anal.* **23**:3 (2013), 1355–1378. MR 3078358 Zbl 1278.32013

[Bott and Chern 1965] R. Bott and S.-S. Chern, "Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections", *Acta Math.* **114** (1965), 71–112. MR 32 #3070 Zbl 0148.31906

[Frölicher 1955] A. Frölicher, "Relations between the cohomology groups of Dolbeault and topological invariants", *Proc. Nat. Acad. Sci. USA* **41** (1955), 641–644. MR 17,409a Zbl 0065.16502

[Katz and Oda 1968] N. M. Katz and T. Oda, "On the differentiation of de Rham cohomology classes with respect to parameters", *J. Math. Kyoto Univ.* **8** (1968), 199–213. MR 38 #5792 Zbl 0165.54802

[Macrì 2013] M. Macrì, "Cohomological properties of unimodular six dimensional solvable Lie algebras", *Differential Geom. Appl.* **31**:1 (2013), 112–129. MR 3010082 Zbl 1263.53045 arXiv 1111.5958v2

[Nakamura 1975] I. Nakamura, "Complex parallelisable manifolds and their small deformations", *J. Differential Geometry* **10** (1975), 85–112. MR 52 #14389 Zbl 0297.32019

[Schweitzer 2007] M. Schweitzer, "Autour de la cohomologie de Bott–Chern", preprint, 2007. arXiv 0709.3528v1

[Voisin 1996] C. Voisin, *Symétrie miroir*, Panoramas et Synthèses **2**, Société Mathématique de France, Paris, 1996. MR 97i:32026 Zbl 0849.14001

[Ye 2008] X. Ye, "The jumping phenomenon of Hodge numbers", *Pacific J. Math.* **235**:2 (2008), 379–398. MR 2009a:32016 Zbl 1194.32007

Received March 6, 2014. Revised July 27, 2014.

JIEZHU LIN
SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE
GUANGZHOU UNIVERSITY
KEY LABORATORY OF MATHEMATICS AND INTERDISCIPLINARY SCIENCES OF GUANGDONG
HIGHER EDUCATION INSTITUTE
NO 230, WAIHUAN ROAD WEST
GUANGZHOU, 510006
CHINA
jlin@gzhu.edu.cn

XUANMING YE
SCHOOL OF MATHEMATICS AND COMPUTATIONAL SCIENCE
SUN YAT-SEN UNIVERSITY
GUANGZHOU, 510275
CHINA
yexm3@mail.sysu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.483

FIXED-POINT RESULTS AND THE HYERS–ULAM STABILITY OF LINEAR EQUATIONS OF HIGHER ORDERS

BING XU, JANUSZ BRZDĘK AND WEINIAN ZHANG

We present a general method for investigation of the Hyers–Ulam stability of linear equations (differential, difference, functional, integral) of higher orders. It is shown that in many cases, that kind of stability for such equations is a consequence of a similar property of the corresponding first-order equations. Some particular examples of applications for differential, integral, difference and functional equations are described. The method is based on some fixed-point results that are proved in this paper.

1. Introduction

Sometimes we have to deal with functions that satisfy some equations only approximately. One of the possible ways to treat them is just to replace such functions by suitably corresponding exact solutions to those equations. Therefore it seems to be important to know when, why and to what extent we can do this, and what errors we thus commit. Some tools for evaluation of that issue are offered by the theory of Ulam-type stability.

Some information on that theory and further references concerning it are given in Section 3. The following definition somehow describes the main ideas of that kind of stability ($\mathbb{N}$ stands for the set of positive integers, $\mathbb{R}_+ := [0, \infty)$, and C^D denotes the family of all functions mapping a set $D \neq \varnothing$ into a set $C \neq \varnothing$):

Definition 1.1. Let $n \in \mathbb{N}$, A be a nonempty set, (X, d) be a metric space, $\mathscr{E} \subset \mathscr{C} \subset {\mathbb{R}_+}^{A^n}$ be nonempty, $\mathscr{T}$ be an operator (not necessarily linear) mapping $\mathscr{C}$ into ${\mathbb{R}_+}^{A}$, and $\mathscr{F}_1$, $\mathscr{F}_2$ be operators (not necessarily linear) mapping a nonempty $\mathscr{D} \subset X^A$ into X^{A^n}. We say that the operator equation

$$(1) \qquad \mathscr{F}_1\varphi(x_1, \ldots, x_n) = \mathscr{F}_2\varphi(x_1, \ldots, x_n)$$

is $(\mathscr{E}, \mathscr{T})$-stable, provided for every $\varepsilon \in \mathscr{E}$ and $\varphi_0 \in \mathscr{D}$ with

$$(2) \qquad d(\mathscr{F}_1\varphi_0(x_1, \ldots, x_n), \mathscr{F}_2\varphi_0(x_1, \ldots, x_n)) \leq \varepsilon(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in A,$$

Bing Xu has been supported by NSFC (China) Grant No. 11101295. Weinian Zhang has been supported by NSFC Grant No. 11231001, SRFDP 20120181110062 and PCSIRT IRT1273.
MSC2010: primary 39B82, 47A63, 47J99; secondary 34K20, 39B12, 47H10.
Keywords: fixed point, Hyers–Ulam stability, linear equation, linear operator, operator equation.

there exists a solution $\varphi \in \mathcal{D}$ of (1) such that

$$d(\varphi(x), \varphi_0(x)) \leq \mathcal{T}\varepsilon(x), \quad x \in A. \tag{3}$$

Roughly speaking, $(\mathscr{E}, \mathcal{T})$-stability of (1) means that every approximate (in the sense of (2)) solution of (1) is always close (in the sense of (3)) to an exact solution to (1).

In the particular case when $\mathscr{E}$ contains only all constant functions, $(\mathscr{E}, \mathcal{T})$-stability is called Hyers–Ulam stability. In this paper we describe (in the terms of fixed points) a general method for investigation of the Hyers–Ulam stability of various higher-order linear (differential, integral, difference or functional) equations in a single variable, that is, for $n = 1$. In this way we show how to generalize and easily extend numerous results given in, e.g., [Takahasi et al. 2002; Miura et al. 2003a; 2003b; 2004; 2012; Jung 2004; 2005; 2006; Popa 2005b; Trif 2006; Wang et al. 2008; Brzdęk et al. 2008; 2010; 2011b; Li and Shen 2009; Brzdęk and Jung 2010].

In what follows, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also, X is a Banach space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $m \in \mathbb{N}$ is fixed and in general we assume that $m > 1$ (unless explicitly stated otherwise), S is a nonempty set, and $a_0, \ldots, a_{m-1} \in \mathbb{K}$. Additionally, $\mathcal{U}$ is a linear subspace of X^S (the linear space over $\mathbb{K}$ of all the functions mapping S into X), $F \in \mathcal{U}$ is fixed, $\mathscr{L} : \mathcal{U} \to X^S$ is a linear operator, $P_m : \mathbb{C} \to \mathbb{C}$ is a polynomial given by $P_m(z) := z^m + \sum_{j=0}^{m-1} a_j z^j$ and $r_1, \ldots, r_m \in \mathbb{C}$ are the roots of the equation

$$P_m(z) = 1. \tag{4}$$

Moreover, we write $\mathcal{U}_m := \{f \in \mathcal{U} : \mathscr{L}^i f \in \mathcal{U} \text{ for } i = 1, \ldots, m-1\}$ and define a linear operator $P_m(\mathscr{L}) : \mathcal{U}_m \to X^S$ by $P_m(\mathscr{L}) := \mathscr{L}^m + \sum_{j=0}^{m-1} a_j \mathscr{L}^j$, where $\mathscr{L}^0 := \mathcal{I}$ is the identity operator (i.e., $\mathcal{I} f = f$ for $f \in X^S$) and $\mathscr{L}^k := \mathscr{L} \circ \mathscr{L}^{k-1}$ for any $k \in \mathbb{N}$. In the next section we present some fixed-point results for the operator

$$\mathcal{P}_m^F := P_m(\mathscr{L}) + F$$

(i.e., $\mathcal{P}_m^F(\varphi) = P_m(\mathscr{L})(\varphi) + F$ for $\varphi \in \mathcal{U}_m$).

2. Fixed-point results

For the sake of simplicity we use the notion $\|f\| := \sup_{x \in S} \|f(x)\|$ for $f \in X^S$, which can be considered as an extension (because it admits an infinite value) of the usual supremum norm $\|\cdot\|_\infty$ defined on the linear space (over $\mathbb{K}$) of all bounded functions from X^S. In this section, we write

$$\mathscr{L}_i^v := \mathscr{L} + (1 - r_i)\mathcal{I} - v, \quad v \in X^S, \ i \in \{1, \ldots, m\} \tag{5}$$

(i.e., $\mathcal{L}_i^v(\varphi) := \mathcal{L}(\varphi) + (1 - r_i)\varphi - v$ for $\varphi \in \mathcal{U}$), provided $r_1, \dots, r_m \in \mathbb{K}$. The next two fixed-point theorems are the main tools in this paper; we present their proofs at the end of the paper.

Theorem 2.1. *Let $r_1, \dots, r_m \in \mathbb{K}$ and $\xi_i : \mathbb{R}_+ \to \mathbb{R}_+$ for $i = 1, \dots, m$. Suppose that*

$$\delta := \|\mathcal{P}_m^F \varphi_s - \varphi_s\| < \infty \tag{6}$$

for some $\varphi_s \in \mathcal{U}_m$ and that the following fixed-point property holds for $i = 1, \dots, m$:

($\mathfrak{L}_i$) *For every $\psi, v \in \mathcal{U}$ such that $\delta := \|\mathcal{L}_i^v \psi - \psi\| < \infty$, there is a fixed point $\phi \in \mathcal{U}$ of $\mathcal{L}_i^v$ such that $\|\psi - \phi\| \le \xi_i(\delta)$.*

Then there exists a fixed point $\varphi \in \mathcal{U}_m$ of $\mathcal{P}_m^F$ such that

$$\|\varphi_s - \varphi\| \le \xi_m \circ \dots \circ \xi_1(\delta). \tag{7}$$

Moreover, if $\mathcal{L}(\mathcal{U}) \subset \mathcal{U}$ and there is an $L \in \mathbb{R}_+$ with $|r_i| > L$ for $i = 1, \dots, m$ and $\|\mathcal{L} f\| \le L\|f\|$ for $f \in \mathcal{U}$, then there is exactly one fixed point $\varphi \in \mathcal{U}$ of $\mathcal{P}_m^F$ with

$$\|\varphi_s - \varphi\| < \infty. \tag{8}$$

Remark 2.2. From the proof of Theorem 2.1 (see Section 5), φ is equal to ϕ_m, with ϕ_m obtained, step by step, by the following procedure.

Write $\phi_0 = -F$, $\psi_m = \varphi_s$, and $\psi_j(z) = \mathcal{L}\psi_{j+1} - r_{j+1}\psi_{j+1}$ for $j = 1, \dots, m-1$. Then, for $i = 1, \dots, m$, $\phi_i \in \mathcal{U}$ is a fixed point of the operator $\mathcal{L} + (1 - r_i)\mathcal{I} - \phi_{i-1}$ with $\|\psi_i - \phi_i\| \le \xi_i \circ \dots \circ \xi_1(\delta)$. By ($\mathfrak{L}_i$), such a $\phi_i \in \mathcal{U}$ exists for each $i \in \{1, \dots, m\}$. In many cases such a ϕ_i can be described quite precisely (see, e.g., Remark 2.4).

Concerning operators satisfying ($\mathfrak{L}_i$), some recent results can be found in, e.g., [Brzdęk and Jung 2011, Theorem 5.1; Badora and Brzdęk 2012, Theorem 2.1].

In what follows, we say that $\mathcal{U}$ is *closed in the norm* $\|\cdot\|_\infty$ if $\mathcal{U}$ contains every function $f \in X^S$ for which there is a sequence of functions (f_n) in $\mathcal{U}$ that is uniformly convergent to f (i.e., $\lim_{n\to\infty}\|f - f_n\|_\infty = 0$).

Theorem 2.3. *Let $\mathcal{L}(\mathcal{U}) \subset \mathcal{U}$ and let $\mathcal{U}$ be closed in the norm $\|\cdot\|_\infty$. Suppose that there are $\kappa \in \mathbb{R}_+$ and $\varphi_s \in \mathcal{U}$ such that* (6) *holds, that*

$$\|\mathcal{L} f\| \le \kappa\|f\|, \quad f \in \mathcal{U}, \tag{9}$$

and that one of the following two conditions is valid:

(α) *$r_i \in \mathbb{K}$ and $|r_i| > \kappa$ for $i = 1, \dots, m$;*

(β) *$|r_i| > 2\kappa$ for $i = 1, \dots, m$.*

Then there is a unique fixed point $\varphi \in \mathcal{U}$ of $\mathcal{P}_m^F$ such that $\|\varphi_s - \varphi\| < \infty$; moreover,

$$\|\varphi_s - \varphi\| \le \frac{\delta}{(|r_1| - \rho\kappa) \cdots (|r_m| - \rho\kappa)}, \tag{10}$$

where

$$\rho := \begin{cases} 1 & \textit{if } (\alpha) \textit{ holds}, \\ 2 & \textit{if } (\beta) \textit{ holds}. \end{cases}$$

Remark 2.4. From the proof of Theorem 2.3 (see Section 6) and Remark 2.2, we can deduce that the function φ can be described analogously to Remark 2.2, with the functions ϕ_i (denoted in $(\mathfrak{L}_i)$ by ϕ) given by $\phi_i(x) := \lim_{n\to\infty} \mathcal{T}_i{}^n \psi_i(x)$ for $i = 1, \dots, m$, $x \in S$, where $\mathcal{T}_i$ is defined by (32).

3. Hyers–Ulam stability

Let $b_0, \dots, b_m \in \mathbb{K}$, with $b_m \neq 0$, let $Q_m : \mathbb{C} \to \mathbb{C}$ be given by $Q_m(z) := \sum_{j=0}^{m} b_j z^j$, and let $q_1, \dots, q_m \in \mathbb{C}$ be the roots of the equation

$$Q_m(z) = 0. \tag{11}$$

We define a linear operator $Q_m(\mathcal{L}) : \mathcal{U}_m \to X^S$ by $Q_m(\mathcal{L}) := \sum_{j=0}^{m} b_j \mathcal{L}^j$. In this section, we describe some direct consequences of Theorems 2.1 and 2.3 concerning the Hyers–Ulam stability of the operator equation

$$Q_m(\mathcal{L})\varphi = G \tag{12}$$

(for functions $\varphi \in \mathcal{U}_m$ and with a fixed $G \in X^S$), under the assumption

$(\mathcal{G})$ $G \in \mathcal{U}$ or (12) has a solution $\hat{\varphi} \in \mathcal{U}_m$.

Let us mention that Hyers–Ulam stability is related to the notions of shadowing and controlled chaos (see, e.g., [Pilyugin 1999; Palmer 2000; Hayes and Jackson 2005; Stević 2008]) as well as the theories of perturbation (see, e.g., [Chang and Howes 1984; Lin and Zhou 1995]) and optimization. At the moment it is a very popular subject of investigation (for more details, references and examples of some recent results, see, e.g., [Hyers 1941; Ulam 1964; Forti 1995; 2007; Hyers et al. 1998; Jung 2001; 2011; Agarwal et al. 2003; Popa 2005a; Jabłoński and Reich 2006; Bahyrycz 2007; Jung and Rassias 2007; 2008; Moszner 2009; Paneah 2009; Ciepliński 2010; 2011; 2012b; Sikorska 2010; Forti and Sikorska 2011; Piszczek 2013a; 2013b]).

Under suitable assumptions, we have the following natural examples of (12):

- The linear differential equation

$$b_m\varphi^{(m)}(z) + b_{m-1}\varphi^{(m-1)}(z) + \cdots + b_1\varphi'(z) + b_0\varphi(z) = G(z). \tag{13}$$

- The linear recurrence (or difference) equation

$$b_m\varphi(n+m) + b_{m-1}\varphi(n+m-1) + \cdots + b_1\varphi(n+1) + b_0\varphi(n) = G(n). \tag{14}$$

- The well-known linear functional equation

$$b_m\varphi(f^m(z)) + b_{m-1}\varphi(f^{m-1}(z)) + \cdots + b_1\varphi(f(z)) + b_0\varphi(z) = G(z). \tag{15}$$

For results on the Hyers–Ulam stability of (13) see [Miura et al. 2003b] (with $G(z) \equiv 0$). Equation (14) is a discrete case of (15); its Hyers–Ulam stability was discussed in [Popa 2005a; 2005b; Brzdęk et al. 2006; 2010]. Equation (15) is one of the most important functional equations, and many results on its solutions can be found in [Kuczma 1968; Kuczma et al. 1990] (see also the references therein); its Hyers–Ulam stability was discussed, e.g., in [Kim 2000; Trif 2002] for $m = 1$ and in [Trif 2006; Brzdęk et al. 2008; 2011b; Brzdęk and Jung 2010] for $m > 1$.

The fixed-point approach has been already applied in the investigation of the Hyers–Ulam stability [Baker 1991; Jung and Chang 2005; Jung and Kim 2006; Mirzavaziri and Moslehian 2006; Jung 2007; Brzdęk et al. 2011a; Brzdęk and Ciepliński 2011; Ciepliński 2012a]. In this section we continue this direction and present two corollaries on such stability, obtained from Theorems 2.1 and 2.3. The first one corresponds to the results in [Brzdęk et al. 2008]. Namely, it states that, in some cases, the Hyers–Ulam stability of (12) can be derived from the analogous properties of the corresponding first-order operator equations, which we express in the form of the following hypothesis:

($\mathcal{H}_i$) $\rho_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that, for every $\varphi_s, \eta \in \mathcal{U}$ and $\delta \in \mathbb{R}_+$ with $\|\mathcal{L}\varphi_s - q_i\varphi_s - \eta\| \le \delta$, there is $\varphi \in \mathcal{U}$ such that $\|\varphi_s - \varphi\| \le \rho_i(\delta)$ and

$$\mathcal{L}\varphi = q_i\varphi + \eta. \tag{16}$$

For examples of operators satisfying ($\mathcal{H}_i$) see [Brzdęk et al. 2010; 2011b] and Section 4.

Corollary 3.1. *Suppose that $G \in X^S$, ($\mathcal{G}$) and ($\mathcal{H}_i$) hold for $i = 1, \dots, m$, $\delta \in \mathbb{R}_+$, and $\varphi_s \in \mathcal{U}_m$ satisfies*

$$\|Q_m(\mathcal{L})\varphi_s - G\| \le \delta. \tag{17}$$

Then there exists a solution $\varphi \in \mathcal{U}_m$ of (12) *such that*

$$\|\varphi_s - \varphi\| \le \rho_m \circ \cdots \circ \rho_1\left(\frac{\delta}{|b_m|}\right). \tag{18}$$

Moreover, if $\mathcal{L}(\mathcal{U}) \subset \mathcal{U}$, and there is $L \in \mathbb{R}_+$ with $\|\mathcal{L}f - \mathcal{L}g\| \le L\|f - g\|$ for $f, g \in \mathcal{U}$ and $|q_i| > L$ for $i = 1, \dots, m$, then there is exactly one solution $\varphi \in \mathcal{U}$ of (12) *with*

$$\|\varphi_s - \varphi\| < \infty. \tag{19}$$

Proof. Assume first that $G \in \mathcal{U}$. Let

$$F = -\frac{1}{b_m}G, \quad a_0 = \frac{b_0}{b_m} + 1, \quad \text{and} \quad a_i = \frac{b_i}{b_m} \quad \text{for } i = 1, \dots, m-1.$$

Then (17) implies that $\|P_m^F(\mathscr{L})\varphi_s - \varphi_s\| \le \delta/|b_m|$. Further, it is easily seen that $q_1, \dots, q_m$ are the roots of (4) and $(\mathscr{H}_i)$ yields $(\mathfrak{L}_i)$ for $i = 1, \dots, m$ with $\xi_i = \rho_i$. So, by Theorem 2.1, there is a fixed point $\varphi \in \mathscr{U}_m$ of $P_m^F(\mathscr{L})$ such that (18) holds. Clearly φ is a solution to (12).

Now consider the case where (12) has a solution $\hat{\varphi} \in \mathscr{U}_m$. Let $\zeta_s := \varphi_s - \hat{\varphi}$. Then $\zeta_s \in \mathscr{U}_m$ and $\|Q_m(\mathscr{L})\zeta_s - G_0\| = \|Q_m(\mathscr{L})\zeta_s\| \le \delta$, where $G_0 \in X^S$ and $G_0(x) \equiv 0$. Clearly $G_0 \in \mathscr{U}$. Hence, by the first part of the proof, there exists a solution $\zeta \in \mathscr{U}_m$ of (12) (with $G = G_0$) such that

$$\|\zeta_s - \zeta\| \le \rho_m \circ \cdots \circ \rho_1\left(\frac{\delta}{|b_m|}\right).$$

Now, it is easily seen that $\varphi := \zeta + \hat{\varphi}$ is a solution of (12), and (18) holds.

It remains to show the statement concerning the uniqueness of φ. So, suppose that there is an $L \in \mathbb{R}_+$ such that $\|\mathscr{L}f\| \le L\|f\|$ for $f \in \mathscr{U}$ and $|q_i| > L$ for $i = 1, \dots, m$. Then such a φ is the unique fixed point of $\mathscr{P}_m^F$ satisfying (19), and therefore it is also the unique solution of (12) such that (19) is valid. □

It is easily seen that [Brzdęk et al. 2008, Theorem 1] is a particular case of our Corollary 3.1.

Remark 3.2. In the case where $|q_i| < L$ for some $i \in \{1, \dots, m\}$, it follows from [Brzdęk et al. 2010, Theorem 3(c)] that in the general situation we may not have uniqueness of φ in Corollary 3.1.

Corollary 3.3. *Let* $\mathscr{L}(\mathscr{U}) \subset \mathscr{U}$, $G \in X^S$, $(\mathscr{G})$ *be valid*, $\mathscr{U}$ *be closed in the supremum norm* $\|\cdot\|_\infty$, $\delta, \kappa \in \mathbb{R}_+$, $\varphi_s \in \mathscr{U}$, (17) *hold, and*

$$\|\mathscr{L}f - \mathscr{L}g\| \le \kappa\|f - g\|, \quad f, g \in \mathscr{U}. \tag{20}$$

Assume that one of the following two conditions is valid:

(α) $q_i \in \mathbb{K}$ *and* $|q_i| > \kappa$ *for* $i = 1, \dots, m$;

(β) $|q_i| > 2\kappa$ *for* $i = 1, \dots, m$.

Then there is a unique solution $\varphi \in \mathscr{U}$ *of* (12) *with* $\|\varphi_s - \varphi\| < \infty$; *moreover*,

$$\|\varphi_s - \varphi\| \le \frac{\delta}{|b_m|(|q_1| - \rho\kappa)\cdots(|q_m| - \rho\kappa)}, \tag{21}$$

where

$$\rho := \begin{cases} 1 & \textit{if } (\alpha) \textit{ holds}, \\ 2 & \textit{if } (\beta) \textit{ holds}. \end{cases}$$

Proof. Arguing analogously as in the proof of Corollary 3.1, we deduce the statement from Theorem 2.3. □

If $\sigma : S \to \mathbb{K}$ is bounded, $f : S \to S$, and $\mathscr{L}g := \sigma g \circ f$ for $g \in \mathscr{U}$, then (20) holds with $\kappa := \sup_{t\in S} |\sigma(t)|$. So, Corollary 3.3 yields the following result, which complements (and generalizes to a certain extent) some results in [Trif 2006; Brzdęk et al. 2008; 2011b]:

Corollary 3.4. *Let one of the conditions* (α), (β) *of* Corollary 3.3 *hold, and let* $\sigma : S \to \mathbb{K}$, $G \in X^S$, $f : S \to S$, $\delta \in \mathbb{R}_+$, $\varphi_s : S \to X$, $\kappa := \sup_{t\in S} |\sigma(t)| < |q_j|$ *for* $j = 1, \dots, m$, *and*

$$\sup_{t\in S}\left\| \sum_{j=0}^{m} b_j \sigma_j(t)\varphi_s(f^j(t)) - G(t)\right\| \le \delta, \tag{22}$$

where $\sigma_0(t) = 1$ *and* $\sigma_j(t) = \sigma_{j-1}(t)\sigma(f^{j-1}(t))$ *for* $t \in S$, $j = 1, \dots, m$. *Then there is a unique solution* $\varphi : S \to X$ *of the functional equation*

$$\sum_{j=0}^{m} b_j \sigma_j(t)\varphi(f^j(t)) = G(t) \tag{23}$$

such that (21) *holds. Moreover, if S is endowed with a topology and* $\sigma_1, \dots, \sigma_m$ *and f are continuous, then the following two statements are valid*:

(i) *If φ_s and G are continuous, then φ is continuous.*

(ii) *If $X = \mathbb{K}$ and φ_s and G are Borel measurable, then φ is Borel measurable.*

Proof. It is enough to take $\mathscr{L}\xi = \sigma\, \xi \circ f$ for $\xi \in X^S$ in Corollary 3.3. Moreover, if S is endowed with a topology and $\sigma_1, \dots, \sigma_m$ and f are continuous, then taking $\mathscr{U} := \{\xi \in X^S : \xi \text{ is continuous}\}$ or $\mathscr{U} := \{\xi \in X^S : \xi \text{ is Borel measurable}\}$ we obtain that φ is continuous or Borel measurable, respectively. □

Remark 3.5. The form of σ_j seems to be a bit complicated for greater m, but for instance with $m = 2$, (23) has the simple and quite general form

$$b_2\sigma(t)\sigma(f(t))\varphi(f^2(t)) + b_1\sigma(t)\varphi(f(t)) + b_0\varphi(t) = G(t).$$

4. Some further consequences

Let I be an open real interval, let $C^1(I, X)$ denote the space of strongly differentiable functions mapping I into X, and let $\mathscr{U} = C^1(I, X)$ and $\mathscr{L} = d/dt$. In the next remark we show that, in view of [Miura et al. 2004, Remark 1, Corollaries 2, 3] (see also [Takahasi et al. 2002]), hypothesis ($\mathscr{H}_i$) holds for each $i \in \{1, \dots, m\}$ such that $\Re q_i \neq 0$, where $\Re z$ denotes the real part of the complex number z for $\mathbb{K} = \mathbb{C}$ and $\Re z = z$ for $\mathbb{K} = \mathbb{R}$, with

$$\rho_i(\delta) := \frac{\delta}{|\Re q_i|}, \quad \delta \in \mathbb{R}_+. \tag{24}$$

Moreover, in the case where I is finite and $\Re q_i = 0$, $(\mathscr{H}_i)$ holds for each $i \in \{1, \dots, m\}$, with

$$\rho_i(\delta) := d(I)\delta, \quad \delta \in \mathbb{R}_+, \tag{25}$$

where $d(I)$ denotes the diameter of I; see [Miura et al. 2004, Remark 1, Corollary 4].

Remark 4.1. For $i \in \{1, \dots, m\}$ and $\delta \in \mathbb{R}_+$, we write

$$\rho_i(\delta) = \begin{cases} d(I)\delta & \text{if } \Re q_i = 0 \text{ and } d(I) < \infty, \\ \delta/|\Re q_i| & \text{if } \Re q_i \neq 0. \end{cases} \tag{26}$$

Let $\mathscr{L} = d/dt$ and $\eta \in \mathscr{U}$. Take $\varphi_s \in \mathscr{U}$, $i \in \{1, \dots, m\}$, and $\delta \in \mathbb{R}_+$ with $\|\mathscr{L}\varphi_s - q_i\varphi_s - \eta\| \leq \delta$. There is a solution $\varphi_0 \in \mathscr{U}$ of the equation $\mathscr{L}\varphi_0 = q_i\varphi_0 + \eta$. Write $\varphi_1 = \varphi_s - \varphi_0$. Then $\|\mathscr{L}\varphi_1 - q_i\varphi_1\| = \|\mathscr{L}\varphi_s - q_i\varphi_s - \eta\| \leq \delta$. Hence, according to the results in [Miura et al. 2004], there is $\hat{\varphi} \in \mathscr{U}$ such that $\|\varphi_1 - \hat{\varphi}\| \leq \rho_i(\delta)$ and $\mathscr{L}\hat{\varphi} = q_i\hat{\varphi}$. Now, it is easily seen that $\varphi := \hat{\varphi} + \varphi_0$ satisfies (16) and that $\|\varphi_s - \varphi\| = \|\varphi_1 - \hat{\varphi}\| \leq \rho_i(\delta)$.

If $d(I) = \infty$, then $(\mathscr{H}_i)$ may not be valid for $i \in \{1, \dots, m\}$ with $\Re q_i = 0$ (see, e.g., [Takahasi et al. 2002, Theorem 2.1(iii)]).

In view of Remark 4.1, from Theorem 2.1 we can deduce the following corollary, which generalizes the main result obtained in [Miura et al. 2003b], although it was proved in that paper by a different method.

Corollary 4.2. *Let I be an open real interval, and let $G \in C^0(I, X)$, $q_i \in \mathbb{K}$ for $i = 1, \dots, m$, $\delta \in \mathbb{R}_+$ and $\varphi_s \in C^m(I, X)$ satisfy*

$$\|b_m\varphi_s^{(m)} + \cdots + b_1\varphi_s' + b_0\varphi_s - G\| \leq \delta. \tag{27}$$

Suppose that $d(I) < \infty$ or $\Re q_i \neq 0$ for $i = 1, \dots, m$. Then there exists a solution $\varphi \in C^m(I, X)$ of (13) *such that*

$$\|\varphi_s - \varphi\| \leq \frac{\delta}{|b_m|} \prod_{i=1}^{m} D_i, \tag{28}$$

where

$$D_i = \begin{cases} d(I) & \text{if } \Re q_i = 0 \text{ and } d(I) < \infty, \\ 1/|\Re q_i| & \text{if } \Re q_i \neq 0. \end{cases} \tag{29}$$

Proof. There is a function $\varphi_0 \in C^m(I, X)$ satisfying the equation

$$b_m\varphi_0^{(m)}(x) + b_{m-1}\varphi_0^{(m-1)}(x) + \cdots + b_1\varphi_0'(x) + b_0\varphi_0(x) = G(x).$$

Let $\zeta_s := \varphi_s - \varphi_0$. Then, by (27),

$$\|b_m\zeta_s^{(m)} + b_{m-1}\zeta_s^{(m-1)} + \cdots + b_1\zeta_s' + b_0\zeta_s\| \le \delta,$$

i.e., (17) is valid with $G(x) \equiv 0$. As we have already observed in Remark 4.1, hypothesis $(\mathcal{H}_i)$ holds for $i = 1, \ldots, m$ with $\mathcal{U} = C^1(I, X)$, $\mathcal{L} = d/dt$ and ρ_i given by (26). So, by Corollary 3.1, there is a solution $\zeta \in C^m(I, X)$ of the equation

$$b_m\zeta^{(m)}(x) + b_{m-1}\zeta^{(m-1)}(x) + \cdots + b_1\zeta'(x) + b_0\zeta(x) = 0$$

such that

$$\|\zeta_s - \zeta\| \le \frac{\delta}{|b_m|}\prod_{i=1}^{m} D_i,$$

where D_i is given by (29). Now, it is easily seen that $\varphi := \zeta + \varphi_0$ is a solution of (13), and (28) holds. $\square$

Similar results for integral equations can be derived from [Miura et al. 2012] in analogous ways. It seems that so far that no paper has been published containing stability results (of the type discussed in this paper) for linear integral equations of higher orders.

5. Proof of Theorem 2.1

This proof proceeds via induction with respect to m. The case $m = 1$ is a consequence of $(\mathfrak{L}_1)$ with $v = -F$. Assume that the theorem is true for $m = k$. Let $\varphi_s \in \mathcal{U}_m$ satisfy (6) with $m = k + 1$, which in view of the Viète formulas can be written in the form

$$\begin{aligned}\delta = \|\mathcal{P}^F_{k+1}\varphi_s - \varphi_s\| &= \left\|\mathcal{L}^{k+1}\varphi_s + \sum_{j=0}^{k} a_j\mathcal{L}^j\varphi_s + F - \varphi_s\right\| \\ &= \left\|\mathcal{L}^{k+1}\varphi_s + (-1)\left(\sum_{j=1}^{k+1} r_j\right)\mathcal{L}^k\varphi_s + \cdots + (-1)^{k+1}r_1\cdot\cdots\cdot r_{k+1}\varphi_s + F\right\|.\end{aligned}$$

Let $\psi_s := \mathcal{L}\varphi_s - r_{k+1}\varphi_s$. Since $\mathcal{L} : \mathcal{U} \to X^S$ is a linear operator, we have

$$\mathcal{L}^p\psi_s = \mathcal{L}^{p+1}\varphi_s - r_{k+1}\mathcal{L}^p\varphi_s$$

for $p = 1, \dots, k$, whence

$$\begin{aligned}
&\left\| \mathscr{L}^k \psi_s + (-1)\left(\sum_{j=1}^{k} r_j\right)\mathscr{L}^{k-1}\psi_s + \cdots + [(-1)^k r_1 \cdots r_k + 1]\psi_s + F - \psi_s \right\| \\
&\qquad = \left\| \mathscr{L}^k \psi_s + (-1)\left(\sum_{j=1}^{k} r_j\right)\mathscr{L}^{k-1}\psi_s + \cdots + (-1)^k r_1 \cdots r_k \psi_s + F \right\| \\
&\qquad = \Bigg\| \mathscr{L}^{k+1}\varphi_s - r_{k+1}\mathscr{L}^k\varphi_s + (-1)\left(\sum_{j=1}^{k} r_j\right)(\mathscr{L}^k\varphi_s - r_{k+1}\mathscr{L}^{k-1}\varphi_s) \\
&\qquad\qquad\qquad + \cdots + (-1)^k r_1 \cdots r_k(\mathscr{L}\varphi_s - r_{k+1}\varphi_s) + F \Bigg\| \\
&\qquad = \left\| \mathscr{L}^{k+1}\varphi_s + (-1)\left(\sum_{j=1}^{k+1} r_j\right)\mathscr{L}^k\varphi_s + \cdots + (-1)^{k+1} r_1 \cdots r_{k+1}\varphi_s + F \right\| \le \delta.
\end{aligned}$$

Since, by the Viète formulas, $r_1, \dots, r_k$ are the roots of the equation

$$\begin{aligned}
1 &= (z - r_1)(z - r_2) \cdots (z - r_k) + 1 \\
&= z^k + (-1)\left(\sum_{j=1}^{k} r_j\right)z^{k-1} + \cdots + [(-1)^k r_1 \cdots r_k + 1]z^0,
\end{aligned}$$

by the inductive assumption, there is $\psi \in \mathscr{U}_k$ such that

$$\psi = \mathscr{L}^k\psi + (-1)\left(\sum_{j=1}^{k} r_j\right)\mathscr{L}^{k-1}\psi + \cdots + [(-1)^k r_1 \cdots r_k + 1]\psi + F \tag{30}$$

and $\|\mathscr{L}^{\psi}_{k+1}\varphi_s - \varphi_s\| = \|\mathscr{L}\varphi_s - r_{k+1}\varphi_s - \psi\| = \|\psi_s - \psi\| \le \xi_k \circ \cdots \circ \xi_1(\delta)$. Hence, in view of $(\mathfrak{L}_{k+1})$, there is a fixed point $\varphi \in \mathscr{U}$ of $\mathscr{L}^{\psi}_{k+1}$ with

$$\|\varphi_s - \varphi\| \le \xi_{k+1}(\xi_k \circ \cdots \circ \xi_1(\delta)).$$

Note that $\mathscr{L}\varphi = \psi + r_{k+1}\varphi$, whence $\mathscr{L}\varphi \in \mathscr{U}$, which means that $\varphi \in \mathscr{U}_2$. Analogously, step by step, finally we get $\varphi \in \mathscr{U}_{k+1}$. Consequently, (30) yields

$$\begin{aligned}
0 &= \mathscr{L}^{k+1}\varphi - r_{k+1}\mathscr{L}^k\varphi + (-1)\left(\sum_{j=1}^{k} r_j\right)(\mathscr{L}^k\varphi - r_{k+1}\mathscr{L}^{k-1}\varphi) \\
&\qquad\qquad + \cdots + (-1)^k r_1 \cdots r_k(\mathscr{L}\varphi - r_{k+1}\varphi) + F \\
&= \mathscr{L}^{k+1}\varphi + (-1)\left(\sum_{j=1}^{k+1} r_j\right)\mathscr{L}^k\varphi + \cdots + (-1)^{k+1} r_1 \cdots r_{k+1}\varphi + F \\
&= P_m(\mathscr{L})\varphi + F - \varphi.
\end{aligned}$$

It remains to prove the statement of the uniqueness of φ. Notice that if $\varphi_1, \varphi_2 \in \mathcal{U}$ are both fixed points of $\mathcal{P}_m^F$ with $\|\varphi_s - \varphi_i\| < \infty$ for $i = 1, 2$, then $\|\varphi_1 - \varphi_2\| < \infty$. So, it suffices to prove that $\varphi_1 = \varphi_2$ if $\varphi_1, \varphi_2 \in \mathcal{U}$ are fixed points of $\mathcal{P}_m^F$ such that

$$M := \|\varphi_1 - \varphi_2\| < \infty. \tag{31}$$

The proof of uniqueness proceeds via induction with respect to m. For $m = 1$ we have $r_1 = 1 - a_0$ and, therefore, for arbitrary fixed points $\varphi_1, \varphi_2 \in \mathcal{U}$ of $\mathcal{P}_m^F$ satisfying (31), we have $|r_1|^n \|\varphi_1 - \varphi_2\| = \|\mathcal{L}^n \varphi_1 - \mathcal{L}^n \varphi_2\| \leq L^n M$ for $n \in \mathbb{N}$, whence $\varphi_1 = \varphi_2$, because $|r_1| > L$. We further assume that the fact is true for $m = k$. Consider fixed points $\varphi_1, \varphi_2 \in \mathcal{U}$ of $\mathcal{P}_{k+1}^F$ satisfying (31), and write $\phi_i := \mathcal{L}\varphi_i - r_{k+1}\varphi_i$ for $i = 1, 2$. Then, arguing analogously as before for ψ_s, we see that $\phi_1, \phi_2 \in \mathcal{U}$ are fixed points of $\mathcal{P}_m^F$ with $m = k$ and appropriate (possibly different) $a_0, \dots, a_{k-1}$. Moreover, $\|\phi_1 - \phi_2\| \leq (L + |r_{k+1}|)M$. Hence, according to the inductive assumption, $\phi_1 = \phi_2$ and, analogously to the case $m = 1$, finally we obtain that $\varphi_1 = \varphi_2$. □

6. Proof of Theorem 2.3

First, consider the case of (α). In view of Theorem 2.1, it is enough to show that $(\mathfrak{L}_i)$ holds for $i = 1, \dots, m$. Fix $i \in \{1, \dots, m\}$, $v \in \mathcal{U}$ and $\psi \in \mathcal{U}$ and assume that $\delta_0 := \|\mathcal{L}_i^v \psi - \psi\| < \infty$. Write

$$\mathcal{T}_i := \frac{1}{r_i}(\mathcal{L} - v). \tag{32}$$

In view of (5), $\|\mathcal{T}_i \psi - \psi\| \leq \delta_0 / |r_i|$, and, for every $f, g \in \mathcal{U}$,

$$\|\mathcal{T}_i f - \mathcal{T}_i g\| = \left\| \frac{1}{r_i}\mathcal{L}f - \frac{1}{r_i}\mathcal{L}g \right\| \leq \frac{\kappa}{|r_i|}\|f - g\|.$$

Define a generalized metric (i.e., admitting an infinite value) d in X^S by $d(f, g) = \|f - g\|$ for $f, g \in X^S$ (see [Luxemburg 1958]). Applying the fixed-point alternative of J. B. Diaz and B. Margolis [1968, p. 306–307], we see that (for the generalized metric d) the limit $\phi := \lim_{n\to\infty} \mathcal{T}_i{}^n \psi$ exists in X^S and ϕ is the unique fixed point of $\mathcal{T}_i$ with

$$\|\psi - \phi\| \leq \frac{\delta_0}{|r_i|} \, \frac{1}{1 - \kappa/|r_i|} = \frac{\delta_0}{|r_i| - \kappa}.$$

Since the sequence $(\mathcal{T}_i{}^n \psi)$ converges to ϕ uniformly and $\mathcal{U}$ is closed in the norm $\|\cdot\|_\infty$, ϕ belongs to $\mathcal{U}$. Next, $\mathcal{L}_i^v \phi = \mathcal{L}\phi + (1 - r_i)\phi - v = r_i \mathcal{T}_i \phi + v + (1 - r_i)\phi - v = \phi$, implying that $(\mathfrak{L}_i)$ is valid, which completes the proof in the case of (α).

Now, consider the case when (β) is valid and $\mathbb{K} = \mathbb{R}$. As is well-known (see, e.g., [Fabian et al. 2001, p. 39; Ferrera and Muñoz 2003] or [Kadison and Ringrose 1997, p. 66, Exercise 1.9.6]), X^2 is a complex Banach space with linear structure defined by the operations $(x, y) + (z, w) := (x + z, y + w)$ and

$(\alpha + i\beta)(x, y) := (\alpha x - \beta y, \beta x + \alpha y)$ and the Taylor norm $\|\cdot\|_T$ given by $\|(x, y)\|_T := \sup_{0\le\theta\le 2\pi} \|(\cos\theta)x + (\sin\theta)y\|$ for $x, y, z, w \in X$, $\alpha, \beta \in \mathbb{R}$. Clearly,

$$\max\{\|x\|, \|y\|\} \le \|(x, y)\|_T \le \|x\| + \|y\|, \quad x, y \in X. \tag{33}$$

We write $p_i(w_1, w_2) := w_i$ for $w_1, w_2 \in X$, $i = 1, 2$ and $\|\mu\|_T := \sup_{x\in S} \|\mu(x)\|_T$ for $\mu \in (X^2)^S$. Let $\mathcal{U}_0 := \{\mu : S \to X^2 : p_i \circ \mu \in \mathcal{U}, i \in \{1, 2\}\}$ and $\mathcal{L}_0\mu(x) := \big(\mathcal{L}(p_1 \circ \mu)(x), \mathcal{L}(p_2 \circ \mu)(x)\big)$ for $\mu \in \mathcal{U}_0$, $x \in S$. Since $\mathcal{L}$ is linear and $\mathcal{U}$ is a linear subspace of X over $\mathbb{K} = \mathbb{R}$, we see that $\mathcal{U}_0$ is a linear subspace of X^2 over $\mathbb{C}$ and $\mathcal{L}_0$ is a linear operator (also over $\mathbb{C}$) such that $\mathcal{L}_0(\mathcal{U}_0) \subset \mathcal{U}_0$.

Choose $\mu \in (X^2)^S$ and consider a sequence (μ_n) in $\mathcal{U}_0$ which is uniformly convergent to μ (in the Taylor norm). Then, by (33),

$$\max\{\|p_1 \circ \mu_n - p_1 \circ \mu\|, \|p_2 \circ \mu_n - p_2 \circ \mu\|\} \le \|\mu_n - \mu\|_T, \quad n \in \mathbb{N},$$

which means that $p_i \circ \mu_n$ is uniformly convergent to $p_i \circ \mu$ for $i = 1, 2$. Consequently, $p_1 \circ \mu$, $p_2 \circ \mu \in \mathcal{U}$, whence $\mu \in \mathcal{U}_0$. Thus, $\mathcal{U}_0$ is closed in the supremum norm connected with the norm $\|\cdot\|_T$. Further, according to (9) and (33), we have

$$\begin{aligned}\|\mathcal{L}_0\mu\|_T &= \|(\mathcal{L}(p_1 \circ \mu), \mathcal{L}(p_2 \circ \mu))\|_T \le \|\mathcal{L}(p_1 \circ \mu)\| + \|\mathcal{L}(p_2 \circ \mu)\| \\ &\le \kappa\|p_1 \circ \mu\| + \kappa\|p_2 \circ \mu\| \le 2\kappa \max\{\|p_1 \circ \mu\|, \|p_2 \circ \mu\|\} \le 2\kappa\|\mu\|_T\end{aligned}$$

for every $\mu \in \mathcal{U}_0$. We write $\chi := (\varphi_s, 0)$ and $v_0 = (v, 0)$ for $v \in \mathcal{U}$. Then we have $\|\mathcal{L}_0\chi + (1 - r_i)\chi - v_0 - \chi\|_T = \|\mathcal{L}\varphi_s + (1 - r_i)\varphi_s - v - \varphi_s\| = \delta < \infty$, because $p_2 \circ \chi(x) = 0$ for $x \in S$. So, we have again the case of (α), where $\mathcal{L}, \kappa, \varphi_s, F, \mathcal{U}, \mathcal{P}_m^F$ are replaced with $\mathcal{L}_0, 2\kappa, \chi, F_0 := (F, 0), \mathcal{U}_0$ and $\widehat{\mathcal{P}_m^F} := P_m(\mathcal{L}_0) + F_0$, respectively. So, by the first part of the proof, there is a fixed point $H \in \mathcal{U}_0$ of $\widehat{\mathcal{P}_m^F}$ with

$$\|\chi - H\|_T \le \frac{\delta}{(|r_1| - 2\kappa)\cdots(|r_m| - 2\kappa)}. \tag{34}$$

Observe that $\varphi := p_1 \circ H$ is a fixed point of $\mathcal{P}_m^F$. Moreover, by (34), (10) holds with $\rho = 2$.

It remains to prove the statement of the uniqueness of φ. Let $\varphi_0 \in \mathcal{U}$ be a fixed point of $\mathcal{P}_m^F$ such that $\|\varphi_s - \varphi_0\| < \infty$. Write $H_0(x) := (\varphi_0(x), 0)$ for $x \in S$. Note that $H_0 \in \mathcal{U}_0$ is a fixed point of $\widehat{\mathcal{P}_m^F}$. Moreover, $\|\chi - H_0\|_T = \|\varphi_s - \varphi\| < \infty$. By Theorem 2.1 (with $L = 2\kappa$), we deduce that $H_0 = H$, whence $\varphi_0 = p_1 \circ H_0 = p_1 \circ H = \varphi$. □

References

[Agarwal et al. 2003] R. P. Agarwal, B. Xu, and W. Zhang, "Stability of functional equations in single variable", *J. Math. Anal. Appl.* **288**:2 (2003), 852–869. MR 2004k:39072 Zbl 1053.39042

[Badora and Brzdęk 2012] R. Badora and J. Brzdęk, "A note on a fixed point theorem and the Hyers–Ulam stability", *J. Difference Equ. Appl.* **18**:6 (2012), 1115–1119. MR 2929641 Zbl 1254.39012

[Bahyrycz 2007] A. Bahyrycz, "Forti's example of an unstable homomorphism equation", *Aequationes Math.* **74**:3 (2007), 310–313. MR 2009d:39043 Zbl 1165.39021

[Baker 1991] J. A. Baker, "The stability of certain functional equations", *Proc. Amer. Math. Soc.* **112**:3 (1991), 729–732. MR 91j:39003 Zbl 0735.39004

[Brzdęk and Ciepliński 2011] J. Brzdęk and K. Ciepliński, "A fixed point approach to the stability of functional equations in non-Archimedean metric spaces", *Nonlinear Anal.* **74**:18 (2011), 6861–6867. MR 2012g:39035 Zbl 1237.39022

[Brzdęk and Jung 2010] J. Brzdęk and S.-M. Jung, "A note on stability of a linear functional equation of second order connected with the Fibonacci numbers and Lucas sequences", *J. Inequal. Appl.* **2010** (2010), Art. ID #793947. MR 2678910 Zbl 1198.39036

[Brzdęk and Jung 2011] J. Brzdęk and S.-M. Jung, "A note on stability of an operator linear equation of the second order", *Abstr. Appl. Anal.* **2011** (2011), Art. ID #602713. MR 2819774 Zbl 1273.39023

[Brzdęk et al. 2006] J. Brzdęk, D. Popa, and B. Xu, "Note on nonstability of the linear recurrence", *Abh. Math. Sem. Univ. Hamburg* **76** (2006), 183–189. MR 2007k:39051 Zbl 1126.39019

[Brzdęk et al. 2008] J. Brzdęk, D. Popa, and B. Xu, "Hyers–Ulam stability for linear equations of higher orders", *Acta Math. Hungar.* **120**:1-2 (2008), 1–8. MR 2009e:39027 Zbl 1174.39012

[Brzdęk et al. 2010] J. Brzdęk, D. Popa, and B. Xu, "Remarks on stability of linear recurrence of higher order", *Appl. Math. Lett.* **23**:12 (2010), 1459–1463. MR 2718530 Zbl 1206.39019

[Brzdęk et al. 2011a] J. Brzdęk, J. Chudziak, and Z. Páles, "A fixed point approach to stability of functional equations", *Nonlinear Anal.* **74**:17 (2011), 6728–6732. MR 2012f:39060 Zbl 1236.39022

[Brzdęk et al. 2011b] J. Brzdęk, D. Popa, and B. Xu, "On approximate solutions of the linear functional equation of higher order", *J. Math. Anal. Appl.* **373**:2 (2011), 680–689. MR 2011m:39042 Zbl 1204.65147

[Chang and Howes 1984] K. W. Chang and F. A. Howes, *Nonlinear singular perturbation phenomena: theory and applications*, Applied Mathematical Sciences **56**, Springer, New York, 1984. MR 86e:34090 Zbl 0559.34013

[Ciepliński 2010] K. Ciepliński, "Generalized stability of multi-additive mappings", *Appl. Math. Lett.* **23**:10 (2010), 1291–1294. MR 2011e:39031 Zbl 1204.39026

[Ciepliński 2011] K. Ciepliński, "Stability of multi-additive mappings in non-Archimedean normed spaces", *J. Math. Anal. Appl.* **373**:2 (2011), 376–383. MR 2720687 Zbl 1204.39027

[Ciepliński 2012a] K. Ciepliński, "Applications of fixed point theorems to the Hyers–Ulam stability of functional equations: a survey", *Ann. Funct. Anal.* **3**:1 (2012), 151–164. MR 2903276 Zbl 1252.39032

[Ciepliński 2012b] K. Ciepliński, "Stability of multi-additive mappings in β-Banach spaces", *Nonlinear Anal.* **75**:11 (2012), 4205–4212. MR 2921983 Zbl 1243.39022

[Diaz and Margolis 1968] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space", *Bull. Amer. Math. Soc.* **74** (1968), 305–309. MR 36 #3333 Zbl 0157.29904

[Fabian et al. 2001] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics **8**, Springer, New York, 2001. MR 2002f:46001 Zbl 0981.46001

[Ferrera and Muñoz 2003] J. Ferrera and G. A. Muñoz, "A characterization of real Hilbert spaces using the Bochnak complexification norm", *Arch. Math.* (*Basel*) **80**:4 (2003), 384–392. MR 2004b:46026 Zbl 1035.46016

[Forti 1995] G. L. Forti, "Hyers–Ulam stability of functional equations in several variables", *Aequationes Math.* **50**:1-2 (1995), 143–190. MR 96i:39033 Zbl 0836.39007

[Forti 2007] G.-L. Forti, "Elementary remarks on Ulam–Hyers stability of linear functional equations", *J. Math. Anal. Appl.* **328**:1 (2007), 109–118. MR 2007i:39046 Zbl 1111.39026

[Forti and Sikorska 2011] G.-L. Forti and J. Sikorska, "Variations on the Drygas equation and its stability", *Nonlinear Anal.* **74**:2 (2011), 343–350. MR 2011j:39048 Zbl 1201.39023

[Hayes and Jackson 2005] W. Hayes and K. R. Jackson, "A survey of shadowing methods for numerical solutions of ordinary differential equations", *Appl. Numer. Math.* **53**:2-4 (2005), 299–321. MR 2005m:65127 Zbl 1069.65075

[Hyers 1941] D. H. Hyers, "On the stability of the linear functional equation", *Proc. Nat. Acad. Sci. USA* **27** (1941), 222–224. MR 2,315a Zbl 0061.26403

[Hyers et al. 1998] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications **34**, Birkhäuser, Boston, 1998. MR 99i:39035 Zbl 0907.39025

[Jabłoński and Reich 2006] W. Jabłoński and L. Reich, "Stability of the translation equation in rings of formal power series and partial extensibility of one-parameter groups of truncated formal power series", *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber.* (*II*) **215** (2006), 127–137. MR 2008j:39023 Zbl 1159.39316

[Jung 2001] S.-M. Jung, *Hyers–Ulam–Rassias stability of functional equations in mathematical analysis*, Hadronic Press, Palm Harbor, FL, 2001. MR 2003b:39035 Zbl 0980.39024

[Jung 2004] S.-M. Jung, "Hyers–Ulam stability of linear differential equations of first order", *Appl. Math. Lett.* **17**:10 (2004), 1135–1140. MR 2005g:34117 Zbl 1061.34039

[Jung 2005] S.-M. Jung, "Hyers–Ulam stability of linear differential equations of first order, III", *J. Math. Anal. Appl.* **311**:1 (2005), 139–146. MR 2006i:39044 Zbl 1087.34534

[Jung 2006] S.-M. Jung, "Hyers–Ulam stability of linear differential equations of first order, II", *Appl. Math. Lett.* **19**:9 (2006), 854–858. MR 2007a:39037 Zbl 1125.34328

[Jung 2007] S.-M. Jung, "A fixed point approach to the stability of isometries", *J. Math. Anal. Appl.* **329**:2 (2007), 879–890. MR 2007k:39054 Zbl 1153.39309

[Jung 2011] S.-M. Jung, *Hyers–Ulam–Rassias stability of functional equations in nonlinear analysis*, Springer Optimization and Its Applications **48**, Springer, New York, 2011. MR 2012c:39001 Zbl 1221.39038

[Jung and Chang 2005] Y.-S. Jung and I.-S. Chang, "The stability of a cubic type functional equation with the fixed point alternative", *J. Math. Anal. Appl.* **306**:2 (2005), 752–760. MR 2005m:39055 Zbl 1077.39026

[Jung and Kim 2006] S.-M. Jung and T.-S. Kim, "A fixed point approach to the stability of the cubic functional equation", *Bol. Soc. Mat. Mexicana* (3) **12**:1 (2006), 51–57. MR 2008d:39038 Zbl 1133.39028

[Jung and Rassias 2007] S.-M. Jung and T. M. Rassias, "Ulam's problem for approximate homomorphisms in connection with Bernoulli's differential equation", *Appl. Math. Comput.* **187**:1 (2007), 223–227. MR 2323574 Zbl 1118.39014

[Jung and Rassias 2008] S.-M. Jung and T. M. Rassias, "Generalized Hyers–Ulam stability of Riccati differential equation", *Math. Inequal. Appl.* **11**:4 (2008), 777–782. MR 2458171 Zbl 1200.39010

[Kadison and Ringrose 1997] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras, I: Elementary theory*, Graduate Studies in Mathematics **15**, American Mathematical Society, Providence, RI, 1997. MR 98f:46001a Zbl 0888.46039

[Kim 2000] G. H. Kim, "On the stability of generalized gamma functional equation", *Int. J. Math. Math. Sci.* **23**:8 (2000), 513–520. MR 2001g:39050 Zbl 0955.39010

[Kuczma 1968] M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne **46**, Państwowe Wydawnictwo Naukowe, Warsaw, 1968. MR 37 #4441 Zbl 0196.16403

[Kuczma et al. 1990] M. Kuczma, B. Choczewski, and R. Ger, *Iterative functional equations*, Encyclopedia Math. Appl. **32**, Cambridge University Press, 1990. MR 92f:39002 Zbl 0703.39005

[Li and Shen 2009] Y. Li and Y. Shen, "Hyers–Ulam stability of nonhomogeneous linear differential equations of second order", *Int. J. Math. Math. Sci.* **2009** (2009), Art. ID #576852. MR 2011d:34012 Zbl 1181.34014

[Lin and Zhou 1995] Z.-C. Lin and M.-R. Zhou, *Perturbation methods in applied mathematics*, Jiangsu Education Press, Nanjing, 1995. In Chinese.

[Luxemburg 1958] W. A. J. Luxemburg, "On the convergence of successive approximations in the theory of ordinary differential equations, II", *Indag. Math.* **20** (1958), 540–546. MR 23 #A1866 Zbl 0084.07703

[Mirzavaziri and Moslehian 2006] M. Mirzavaziri and M. S. Moslehian, "A fixed point approach to stability of a quadratic equation", *Bull. Braz. Math. Soc.* (*N.S.*) **37**:3 (2006), 361–376. MR 2007i:39040 Zbl 1118.39015

[Miura et al. 2003a] T. Miura, S. Miyajima, and S.-E. Takahasi, "A characterization of Hyers–Ulam stability of first order linear differential operators", *J. Math. Anal. Appl.* **286**:1 (2003), 136–146. MR 2004h:34012 Zbl 1045.47037

[Miura et al. 2003b] T. Miura, S. Miyajima, and S.-E. Takahasi, "Hyers–Ulam stability of linear differential operator with constant coefficients", *Math. Nachr.* **258** (2003), 90–96. MR 2004d:34113 Zbl 1039.34054

[Miura et al. 2004] T. Miura, S.-M. Jung, and S.-E. Takahasi, "Hyers–Ulam–Rassias stability of the Banach space valued linear differential equations $y' = \lambda y$", *J. Korean Math. Soc.* **41**:6 (2004), 995–1005. MR 2005f:34139 Zbl 1069.34079

[Miura et al. 2012] T. Miura, G. Hirasawa, S.-E. Takahasi, and T. Hayata, "A note on the stability of an integral equation", pp. 207–222 in *Functional equations in mathematical analysis*, edited by T. M. Rassias and J. Brzdęk, Springer Optimization and Its Applications **52**, Springer, New York, 2012. MR 3059690 Zbl 1248.45008

[Moszner 2009] Z. Moszner, "On the stability of functional equations", *Aequationes Math.* **77**:1-2 (2009), 33–88. MR 2010b:39040 Zbl 1207.39044

[Palmer 2000] K. Palmer, *Shadowing in dynamical systems: theory and applications*, Mathematics and its Applications **501**, Kluwer, Dordrecht, 2000. MR 2002k:37034 Zbl 0997.37001

[Paneah 2009] B. Paneah, "A new approach to the stability of linear functional operators", *Aequationes Math.* **78**:1-2 (2009), 45–61. MR 2010k:39038 Zbl 1207.39046

[Pilyugin 1999] S. Y. Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Math. **1706**, Springer, Berlin, 1999. MR 2001b:37030 Zbl 0954.37014

[Piszczek 2013a] M. Piszczek, "On selections of set-valued inclusions in a single variable with applications to several variables", *Results Math.* **64**:1-2 (2013), 1–12. MR 3095123 Zbl 1277.39032

[Piszczek 2013b] M. Piszczek, "The properties of functional inclusions and Hyers–Ulam stability", *Aequationes Math.* **85**:1-2 (2013), 111–118. MR 3028204 Zbl 1271.39031

[Popa 2005a] D. Popa, "Hyers–Ulam–Rassias stability of a linear recurrence", *J. Math. Anal. Appl.* **309**:2 (2005), 591–597. MR 2006a:39045 Zbl 1079.39027

[Popa 2005b] D. Popa, "Hyers–Ulam stability of the linear recurrence with constant coefficients", *Adv. Difference Equ.* **2005**:2 (2005), 101–107. MR 2006k:39009 Zbl 1095.39024

[Sikorska 2010] J. Sikorska, "On a direct method for proving the Hyers–Ulam stability of functional equations", *J. Math. Anal. Appl.* **372**:1 (2010), 99–109. MR 2011f:39042 Zbl 1198.39039

[Stević 2008] S. Stević, "Bounded solutions of a class of difference equations in Banach spaces producing controlled chaos", *Chaos Solitons Fractals* **35**:2 (2008), 238–245. MR 2008h:39010 Zbl 1142.39013

[Takahasi et al. 2002] S.-E. Takahasi, T. Miura, and S. Miyajima, "On the Hyers–Ulam stability of the Banach space-valued differential equation $y' = \lambda y$", *Bull. Korean Math. Soc.* **39**:2 (2002), 309–315. MR 2003c:34090 Zbl 1011.34046

[Trif 2002] T. Trif, "On the stability of a general gamma-type functional equation", *Publ. Math. Debrecen* **60**:1-2 (2002), 47–61. MR 2003e:39052 Zbl 1004.39023

[Trif 2006] T. Trif, "Hyers–Ulam–Rassias stability of a linear functional equation with constant coefficients", *Nonlinear Funct. Anal. Appl.* **11**:5 (2006), 881–889. MR 2008c:39045 Zbl 1115.39028

[Ulam 1964] S. M. Ulam, *Problems in modern mathematics*, Wiley, New York, 1964. MR 43 #6031 Zbl 0137.24201

[Wang et al. 2008] G. Wang, M. Zhou, and L. Sun, "Hyers–Ulam stability of linear differential equations of first order", *Appl. Math. Lett.* **21**:10 (2008), 1024–1028. MR 2450633 Zbl 1159.34041

Received March 28, 2014.

BING XU
DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU, SICHUAN, 610064
CHINA
xb4106@sina.com

JANUSZ BRZDĘK
DEPARTMENT OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
PODCHORĄŻYCH 2
30-084 KRAKÓW
POLAND
jbrzdek@up.krakow.pl

WEINIAN ZHANG
DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU, SICHUAN, 610064
CHINA
matzwn@126.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.499

COMPLETE CURVATURE HOMOGENEOUS METRICS ON $\mathrm{SL}_2(\mathbb{R})$

BENJAMIN SCHMIDT AND JON WOLFSON

A construction is described that associates to each positive smooth function $F : S^1 \to \mathbb{R}$ a smooth Riemannian metric g_F on $\mathrm{SL}_2(\mathbb{R}) \cong \mathbb{R}^2 \times S^1$ that is complete and curvature homogeneous. The construction respects moduli: positive smooth functions F and G lie in the same $\mathrm{Diff}(S^1)$ orbit if and only if the associated metrics g_F and g_G lie in the same $\mathrm{Diff}(\mathrm{SL}_2(\mathbb{R}))$ orbit.

The constructed metrics all have curvature tensor modeled on the same algebraic curvature tensor. Moreover, the following are shown to be equivalent: F is constant, g_F is left-invariant, and $(\mathrm{SL}_2(\mathbb{R}), g_F)$ Riemannian covers a finite volume manifold. Applications of the construction are discussed.

1. Introduction

Let (M, g) be a connected Riemannian manifold, ∇ its Levi-Civita connection, and R its curvature tensor. Then (M, g) is said to be curvature homogeneous of order k if for every $p, q \in M$ there exists a linear isometry $I : T_pM \to T_qM$ such that

$$I^*(\nabla^i R)_q = (\nabla^i R)_p$$

for each $i = 0, 1, \ldots, k$. When M is curvature homogeneous of order 0, M is simply said to be *curvature homogeneous*. Locally homogeneous (M, g) are clearly curvature homogeneous of all orders. I. M. Singer proved the converse in a seminal paper:

Theorem 1.1 [Singer 1960]. *A connected and complete d-dimensional Riemannian manifold (M, g) that is curvature homogeneous of order at least $d(d-1)/2 - 1$ is locally homogeneous. If, in addition, M is simply connected, then (M, g) is homogeneous.*

While Singer's theorem ensures that completeness and curvature homogeneity of sufficiently large order implies local homogeneity, there exist examples of complete

---

Schmidt was partially supported by NSF grant DMS-1207655.
MSC2010: primary 53C21; secondary 22F30.
Keywords: curvature homogeneous space, homogeneous space, constant vector curvature.

and curvature homogeneous Riemannian manifolds that are not locally homogeneous. We refer the reader to [Boeckx et al. 1996] for an extensive collection of examples and additional references. In this note we prove:

Theorem 1.2. *There is a construction that associates to each positive smooth function $F : S^1 \to \mathbb{R}$ a complete and curvature homogeneous Riemannian metric g_F on $\mathrm{SL}_2(\mathbb{R})$. In this construction, the following are equivalent*:

(1) *F is constant.*

(2) *The metric g_F is left-invariant.*

(3) *$(\mathrm{SL}_2(\mathbb{R}), g_F)$ Riemannian covers a finite volume manifold.*

Theorem 1.2 is related to a conjecture attributed to Gromov by Berger [1988] that we now describe. Let T denote a fixed algebraic curvature tensor on Euclidean space $\mathbb{E}^n$ and let M denote a connected, smooth n-manifold. A Riemannian metric h on M with curvature tensor R is said to be modeled on T if for each $x \in M$ there is a linear isometry $I : T_x M \to \mathbb{E}^n$ such that $I^*(T) = R_x$. It is clear that such a Riemannian metric h is curvature homogeneous and that $\mathrm{Diff}(M)$ acts on the space of such metrics by pullback. Let $\mathcal{M}(M, T)$ denote the space of $\mathrm{Diff}(M)$ orbits of *complete* Riemannian metrics on M with curvature tensor modeled on T.

Conjecture 1.3 (Gromov). *If M is compact, then the moduli space $\mathcal{M}(M, T)$ is finite-dimensional.*

It is known that the assumption of compactness in Gromov's conjecture cannot in general be replaced by an assumption of completeness on the metrics under consideration. For example, infinite-dimensional moduli spaces of complete metrics with curvature tensors modeled on certain *reducible* symmetric spaces are constructed in [Tricerri and Vanhecke 1989; Kowalski et al. 1992] (see also [Boeckx et al. 1996, Propositions 4.15–4.16]).

Question [Tricerri and Vanhecke 1989, Problem 2]. Do the isometry classes of the germs of Riemannian metrics which have the curvature tensor of a given "irreducible" homogeneous Riemannian manifold depend on a finite number of parameters?

As explained in Section 3, the Riemannian metrics constructed in Theorem 1.2 all have curvature tensors modeled on a fixed algebraic curvature tensor that we will call T throughout. The algebraic curvature tensor T is modeled on the curvature tensor of an irreducible left-invariant metric on $\mathrm{SL}_2(\mathbb{R})$. Our next theorem describes the moduli space of these metrics.

Theorem 1.4. *Let F and G be two positive smooth functions on the circle. Then there exists a diffeomorphism $\Phi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ such that $\Phi^*(g_G) = g_F$ if and only if there exists a diffeomorphism $\phi : S^1 \to S^1$ such that $F = \phi^*(G)$.*

The space of $\mathrm{Diff}(S^1)$ orbits of positive smooth functions on S^1 is easily seen to be infinite-dimensional. Hence, Theorems 1.2 and 1.4 yield the following negative answer to Tricerri and Vanhecke's problem:

Corollary 1.5. *There is an algebraic curvature tensor T modeled on an irreducible left-invariant metric on* $\mathrm{SL}_2(\mathbb{R})$ *such that the moduli space* $\mathcal{M}(\mathrm{SL}_2(\mathbb{R}), T)$ *is infinite-dimensional.*

Our construction also has an application to the problem of finding isocurved deformations of homogeneous Riemannian spaces. Let (M, g) be a homogeneous Riemannian manifold. Kowalski [1999] defines an *isocurved deformation* of g to be a family of smooth Riemannian metrics $\{g_t \mid t \in [0, 1]\}$ on M satisfying:

(1) Each (M, g_t) is a curvature homogeneous space with curvature tensor modeled on (M, g).

(2) The metrics g_t depend smoothly on t and $g_0 = g$.

(3) (M, g_{t_1}) is not locally isometric to (M, g_{t_2}) when $t_1 \neq t_2$.

If, in addition, the metrics g_t with $t \in (0, 1)$ are not locally homogeneous, then the isocurved deformation is said to be *proper*.

A proper isocurved deformation of an irreducible homogeneous metric g_0 on the three-dimensional Lie group $E(1, 1)$ is constructed in [Kowalski 1999]. However, the metric g_1 in the deformation is not complete, and the completeness of the intermediate metrics is not determined. Problem 1 in [Kowalski 1999] asks to find a proper isocurved deformation of an irreducible homogenous Riemannian manifold through *complete* Riemannian metrics.

Corollary 1.6. *Let $F : S^1 \to \mathbb{R}$ be a nonconstant smooth positive function with a critical value not equal to one, and let $F_t = (1-t) + tF$. Then the family of metrics*

$$\{g_t = g_{F_t} \mid t \in [0, 1]\}$$

is a proper isocurved deformation of the irreducible homogeneous Riemannian manifold $(\mathrm{SL}_2(\mathbb{R}), g_1)$ *through complete Riemannian metrics.*

Proof. As remarked above, each of the metrics g_t is modeled on a fixed algebraic curvature tensor T; their smoothness in the parameter t will be evident from the construction. The metric g_0 is homogeneous, each of the metrics g_t is complete, and each of the metrics g_t with $t > 0$ is not locally homogeneous by Theorem 1.2; the irreducibility of the metric g_0 is clear. It remains to check that the metrics g_t are pairwise nonisometric. This follows from Theorem 1.4 after checking that the functions F_t pairwise lie in different $\mathrm{Diff}(S^1)$ orbits. This is an immediate consequence of the fact that the number of critical points and the associated critical values of smooth functions on S^1 are $\mathrm{Diff}(S^1)$-invariants. □

Theorem 1.2 is also related to a classification result for *constant vector curvature* three-manifolds contained in [Schmidt and Wolfson 2013] that will be used in Section 3. A Riemannian manifold (M, g) has constant vector curvature ε if each tangent vector $v \in TM$ lies in a tangent plane of sectional curvature ε. This curvature condition was introduced as a pointwise analogue of the higher rank condition for Riemannian manifolds. Motivated by a number of results on rank-rigidity such as [Ballmann 1985; Burns and Spatzier 1987; Connell 2002; Constantine 2008; Hamenstädt 1991; Shankar et al. 2005], the present authors proved the following rigidity result for constant vector curvature -1 three-manifolds:

Theorem 1.7 [Schmidt and Wolfson 2013, Theorem 1.1]. *Suppose that M is a finite volume three-manifold with constant vector curvature -1. If $\sec \leq -1$, then M is real hyperbolic. If $\sec \geq -1$ and M is not real hyperbolic, then its universal covering is isometric to a left-invariant metric on one of the Lie groups $E(1,1)$ or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ with sectional curvatures having range $[-1, 1]$.*

As will be explained in Section 3, the metrics constructed in Theorem 1.2 all have constant vector curvature -1 and sectional curvatures having range $[-1, 1]$. Therefore, it is not possible to remove the finite volume hypothesis in Theorem 1.7 in the case when $\sec \geq -1$.

2. $\mathrm{SL}_2(\mathbb{R})$

Let $\mathrm{SL}_2(\mathbb{R})$ denote the Lie group consisting of 2×2 real matrices of determinant one and let $e \in \mathrm{SL}_2(\mathbb{R})$ denote the identity element. Its Lie algebra $\mathrm{sl}_2(\mathbb{R}) \cong T_e\,\mathrm{SL}_2(\mathbb{R})$ consists of 2×2 real matrices with trace equal to zero. Consider the following three one-parameter subgroups of $\mathrm{SL}_2(\mathbb{R})$:

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\},$$

$$A = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

The multiplication map $K \times N \times A \to \mathrm{SL}_2(\mathbb{R})$, $(k, n, a) \mapsto kna$ is a diffeomorphism, yielding the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = KNA$.

Define trace zero matrices $E_1, E_2, E_3 \in \mathrm{sl}_2(\mathbb{R})$ by

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad E_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Then $\{E_1, E_2, E_3\}$ is a basis for the Lie algebra $\mathrm{sl}_2(\mathbb{R})$. Moreover, E_1, E_2, and E_3 are the infinitesimal generators of the one-parameter subgroups K, N, and A, respectively. This Lie algebra basis satisfies the bracket relations

$$[E_1, E_2] = 2E_3, \quad [E_2, E_3] = -E_2, \quad [E_1, E_3] = E_1 - 2E_2. \tag{2-1}$$

The vectors E_i have unique extensions to left-invariant vector fields on $\mathrm{SL}_2(\mathbb{R})$ that we also denote by E_i. Declaring the left-invariant framing $\{E_1, E_2, E_3\}$ of $\mathrm{SL}_2(\mathbb{R})$ to be orthonormal determines a left-invariant Riemannian metric on $\mathrm{SL}_2(\mathbb{R})$. Throughout the remainder of this paper, we let g_1 denote this left-invariant metric. The pullback of its curvature tensor via a linear isometry from Euclidean space $\mathbb{E}^3$ to $T_e\,\mathrm{SL}_2(\mathbb{R})$ defines an algebraic curvature tensor that we denote by T in the remainder of the paper. In the next section, we give the construction of Theorem 1.2. The metrics constructed will all have curvature tensors modeled on the algebraic curvature tensor T.

3. The construction

Note that the subgroup K of $\mathrm{SL}_2(\mathbb{R})$ is diffeomorphic to S^1. Throughout what follows, we assume that a diffeomorphism between K and S^1 has been fixed, identifying positive smooth functions on K with those on S^1. A positive smooth function $F : K \to \mathbb{R}$ determines a positive smooth function $\bar{F} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ as follows. Given $g \in \mathrm{SL}_2(\mathbb{R})$, there is a unique expression $g = kna$ with $k \in K$, $n \in N$, and $a \in A$ by the Iwasawa decomposition. Define $\bar{F}(g) = \bar{F}(kna) = F(k)$.

Alternatively, the bracket relations (2-1) show that the left-invariant vector fields E_2 and E_3 span an involutive plane distribution; the foliation of $\mathrm{SL}_2(\mathbb{R})$ by integral surfaces of this distribution coincides with the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left-cosets of the subgroup NA. As NA is a closed subgroup of $\mathrm{SL}_2(\mathbb{R})$, the natural projection map to the space of left-cosets

$$\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})\,/NA$$

is smooth. Note that the space of cosets $\mathrm{SL}_2(\mathbb{R})\,/NA$ is diffeomorphic to K. Then $\bar{F} = F \circ \pi$ is constant on the leaves of the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left-cosets of NA. We summarize this in the following lemma.

Lemma 3.1. *Smooth functions $F : K \to \mathbb{R}$ lift to smooth functions $\bar{F} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ satisfying $E_2(\bar{F}) = E_3(\bar{F}) = 0$.*

Let $F : K \to \mathbb{R}$ be a smooth and positive function and $\bar{F} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ its associated lift. Define a framing $\{e_1, e_2, e_3\}$ of $\mathrm{SL}_2(\mathbb{R})$ by

$$e_1 = \bar{F} E_1, \quad e_2 = E_2, \quad e_3 = E_3. \tag{3-1}$$

We call such a framing an F-framing. The bracket relations for an F-framing are easy to deduce from (2-1) and the fact that $E_2(\bar{F}) = E_3(\bar{F}) = 0$. They are given by

$$[e_1, e_2] = 2\bar{F}e_3, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = e_1 - 2\bar{F}e_2. \tag{3-2}$$

Definition 3.2. Given a smooth positive function $F : K \to \mathbb{R}$, the F-metric on $\mathrm{SL}_2(\mathbb{R})$ is the Riemannian metric denoted by g_F which is defined by declaring the associated F-framing to be g_F orthonormal.

Note that for the function F which is identically one on K, the associated F-metric is the left-invariant metric g_1 described in Section 2. We remark that the space of F-metrics is path connected. Indeed, given two positive functions F_0 and F_1 on K, the metrics $g_{(1-t)F_0+tF_1}$ with $t \in [0, 1]$ define the path joining g_{F_0} to g_{F_1}. As we shall show, all F-metrics have curvature tensors modeled on the algebraic curvature tensor T.

In order to calculate the curvatures of an F-metric, we first calculate the Christoffel symbols. As an F-framing is by definition orthonormal for the metric g_F, Koszul's formula reads

$$g_F(\nabla_{e_i} e_j, e_k) = \tfrac{1}{2}\bigl\{g_F([e_i, e_j], e_k) - g_F([e_j, e_k], e_i) + g_F([e_k, e_i], e_j)\bigr\}. \tag{3-3}$$

Combining (3-2) and (3-3) yields

$$\begin{aligned}
&\nabla_{e_1} e_3 = e_1 - 2\bar{F}e_2, && \nabla_{e_2} e_3 = -e_2,\\
&\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0,\\
&\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = e_3,\\
&\nabla_{e_1} e_2 = 2\bar{F}e_3, && \nabla_{e_1} e_1 = -e_3,\\
&\nabla_{e_3} e_3 = 0.
\end{aligned} \tag{3-4}$$

We let R_{ijkl} denote the component of the curvature tensor

$$R(e_i, e_j, e_k, e_l) = g_F(\nabla_{e_i}\nabla_{e_j} e_k - \nabla_{e_j}\nabla_{e_i} e_k - \nabla_{[e_i,e_j]} e_k, e_l).$$

Tedious but straightforward calculations using (3-2), (3-4), and the fact that $e_2(\bar{F}) = e_3(\bar{F}) = 0$ show that

$$R_{1221} = 1, \quad R_{1331} = -1 = R_{2332}, \quad R_{ijkl} = 0 \text{ if three indices are distinct.} \tag{3-5}$$

The symmetries of the curvature tensor determine its remaining components.

Corollary 3.3. *An F-metric g_F is curvature homogeneous and has curvature tensor modeled on the algebraic curvature tensor T. An F-framing diagonalizes the Ricci tensor. If σ is a two-plane and $v = \sum_{i=1}^{3} c_i e_i$ is a unit vector orthogonal to σ, then*

$$\sec(\sigma) = c_3^2 - c_1^2 - c_2^2.$$

Consequently, g_F has constant vector curvature -1, e_3 lies in the intersection of all curvature -1 planes, and the range of sectional curvatures for an F-metric is $[-1, 1]$.

Proof. To prove the first claim, note that by (3-5), the curvatures of an F-metric with respect to an F-framing do not depend on the function $F : K \to \mathbb{R}$. Therefore, they all have curvature tensors modeled on the curvature tensor of the F-metric corresponding to $F \equiv 1$ which is the left-invariant metric g_1 constructed at the end of the previous section.

The fact that an F-framing diagonalizes the Ricci tensor is immediate from (3-5). This fact and [Schmidt and Wolfson 2013, Lemma 2.2] yield the curvature formula. The curvature formula implies the last statement. □

Lemma 3.4. *An F-metric g_F is complete.*

Proof. Let $F : K \to \mathbb{R}$ be a positive smooth function and g_F the associated F-metric. As K is compact, there exists $M > 1$ such that $1/M < F < M$. Consider the Riemannian metrics $M^{-2}g_1$ and $M^2 g_1$ obtained by scaling the left-invariant metric g_1. The induced norms satisfy

$$M^{-1}\|v\|_{g_1} = \|v\|_{M^{-2}g_1} < \|v\|_{g_F} < \|v\|_{M^2 g_1} = M\|v\|_{g_1}$$

for each tangent vector $v \in T\,\mathrm{SL}_2(\mathbb{R})$. Consequently, the induced path metrics satisfy

$$M^{-1}d_{g_1}(p,q) \le d_{g_F}(p,q) \le M d_{g_1}(p,q)$$

for any pair of points $p, q \in \mathrm{SL}_2(\mathbb{R})$. As d_{g_1} Cauchy sequences converge, the same is true of d_{g_F} Cauchy sequences. □

The following lemma may be of interest to some readers. It is not used in the proof of our main results and may be skipped.

Lemma 3.5. *For any F-metric g_F, the foliation of* $\mathrm{SL}_2(\mathbb{R})$ *by left-cosets of NA is a foliation by totally geodesic hyperbolic planes.*

Proof. Let $F : K \to \mathbb{R}$ be a smooth positive function, g_F the associated F-metric, and $\{e_1, e_2, e_3\}$ the associated F-framing. The leaves of the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left cosets of NA are precisely the integral surfaces of the involutive plane distribution $e_2 \wedge e_3$. These leaves are totally geodesic since by (3-4), $\nabla_{e_2} e_1 = \nabla_{e_3} e_1 = 0$. By (3-5), $R_{2332} = -1$, so that the leaves are hyperbolic. As NA is diffeomorphic to $\mathbb{R}^2$, the leaves are hyperbolic planes. □

To complete the proof of Theorem 1.2 from the introduction, it remains to establish the following proposition.

Proposition 3.6. *For a positive smooth function $F : K \to \mathbb{R}$, the following are equivalent*:

(1) *F is constant.*

(2) *The metric g_F is left-invariant.*

(3) $(\mathrm{SL}_2(\mathbb{R}), g_F)$ *Riemannian covers a finite volume manifold.*

Proof. Let $F : K \to \mathbb{R}$ be a positive smooth function and g_F the associated F-metric on $\mathrm{SL}_2(\mathbb{R})$.

(1) $\implies$ (2): Because F is constant, so is its lift $\bar{F}$. The associated F-framing $\{e_1 = \bar{F} E_1, e_2 = E_2, e_3 = E_3\}$ is easily seen to be left-invariant since the framing $\{E_1, E_2, E_3\}$ is left-invariant. Therefore g_F is a left-invariant metric.

(2) $\implies$ (3): This is an easy consequence of the fact that $\mathrm{SL}_2(\mathbb{R})$ admits lattice subgroups.

(3) $\implies$ (1): Let M denote the finite volume manifold Riemannian covered by $(\mathrm{SL}_2(\mathbb{R}), g_F)$. We first claim that the metric g_F is locally homogeneous. Indeed, by Corollary 3.3, M has constant vector curvature -1 and sectional curvatures with range $[-1, 1]$. By Theorem 1.7, the universal covering $(\widetilde{\mathrm{SL}_2(\mathbb{R})}, \tilde{g}_F)$ is left-invariant (and homogeneous), whence g_F is locally homogeneous.

Let $\bar{F}$ denote the lift of F to $\mathrm{SL}_2(\mathbb{R})$ and let $\{e_1, e_2, e_3\}$ be the associated F-framing. Let $p, q \in \mathrm{SL}_2(\mathbb{R})$ be two points. As g_F is locally homogeneous, there is an $r > 0$ and an isometry I between the balls of radius r centered at p and q with $I(p) = q$:

$$I : B(p, r) \to B(q, r).$$

The derivative map $dI : TB(p, r) \to TB(q, r)$ preserves the line field spanned by e_3 and the perpendicular plane field $e_1 \wedge e_2$ by the curvature formula in Corollary 3.3. Therefore, there exists a smooth map $\theta : B(q, r) \to \mathbb{R}$ such that $dI(e_3) = \pm e_3$ and such that the restriction of dI to the plane field $e_1 \wedge e_2$ has matrix representation given by either

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

with respect to the $\{e_1, e_2\}$ framing.

By (3-2),

$$dI_p([e_1, e_2]_p) = dI_p(2\bar{F}(p)e_3) = \pm 2\bar{F}(p)e_3 \in T_q \, \mathrm{SL}_2(\mathbb{R}),$$

where the sign is $+$ if dI preserves the orientation of e_3 and is $-$ if the orientation is reversed. A simple calculation yields

$$g_F\big([dI_p(e_1), dI_p(e_2)]_q, e_3\big)_q = \pm [e_1, e_2]_q = \pm 2\bar{F}(q),$$

where the sign is $+$ if dI preserves the orientation of the plane field $e_1 \wedge e_2$ and is $-$ if the orientation is reversed.

Since $dI_p([e_1, e_2]_p) = [dI_p(e_1), dI_p(e_2)]_q$, we have $\overline{F}(p) = \pm\overline{F}(q)$. As $\overline{F}$ is everywhere positive, it must be the case that $\overline{F}(p) = \overline{F}(q)$. Therefore F is constant, concluding the proof. $\square$

We conclude the paper with a proof of Theorem 1.4, restated for the reader's convenience, followed by a conjecture.

Theorem 3.7. *Let F and G be two positive smooth functions on the circle. Then there exists a diffeomorphism $\Phi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ such that $\Phi^*(g_G) = g_F$ if and only if there exists a diffeomorphism $\phi : S^1 \to S^1$ such that $F = \phi^*(G)$.*

Proof. Recall that a diffeomorphism between S^1 and K has been fixed, identifying positive smooth functions on these two spaces.

First, assume that there is a diffeomorphism $\phi : K \to K$ such that $\phi^*(G) = F$. Define a diffeomorphism $\Phi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ as follows. By the Iwasawa decomposition, each $g \in \mathrm{SL}_2(\mathbb{R})$ has a unique expression $g = kna$; define $\Phi(g) = \Phi(kna) = \phi(k)na$. It is routine to check that $\Phi^*(g_G) = g_F$.

Assume that $\Phi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ is a diffeomorphism satisfying $\Phi^*(g_G) = g_F$. Let $\overline{F}$ and $\overline{G}$ denote the lifts of F and G to $\mathrm{SL}_2(\mathbb{R})$ and let $\{e_1, e_2, e_3\}$ and $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ denote the associated F-framing and G-framing of $T\,\mathrm{SL}_2(\mathbb{R})$, respectively. Since $e_2 = \tilde{e}_2$, $e_3 = \tilde{e}_3$, and e_1 and $\tilde{e}_1$ are positively parallel, these framings induce the same orientation of $\mathrm{SL}_2(\mathbb{R})$.

As $\Phi : (\mathrm{SL}_2(\mathbb{R}), g_F) \to (\mathrm{SL}_2(\mathbb{R}), g_G)$ is an isometry, it preserves the sectional curvatures of planes. By Corollary 3.3, it follows that the derivative map

$$d\Phi : T\,\mathrm{SL}_2(\mathbb{R}) \to T\,\mathrm{SL}_2(\mathbb{R})$$

satisfies $d\Phi(e_3) = \pm\tilde{e}_3$ and maps the plane field $e_1 \wedge e_2$ isometrically to the plane field $\tilde{e}_1 \wedge \tilde{e}_2$. Therefore, there exists a smooth map

$$\theta : (\mathrm{SL}_2(\mathbb{R}), g_G) \to \mathbb{R}$$

such that the matrix representation of

$$d\Phi|_{e_1 \wedge e_2} : e_1 \wedge e_2 \to \tilde{e}_1 \wedge \tilde{e}_2$$

with respect to the ordered framings $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

depending on whether $d\Phi|_{e_1 \wedge e_2}$ preserves or reverses orientation.

By (3-2),

$$d\Phi([e_1, e_2]) = d\Phi(2\overline{F} e_3) = \pm 2\overline{F}\tilde{e}_3.$$

A simple calculation shows that

$$[d\Phi(e_1), d\Phi(e_2)] = \pm(-\tilde{e}_1(\theta)\tilde{e}_1 - \tilde{e}_2(\theta)\tilde{e}_2 + 2\overline{G}\tilde{e}_3),$$

where the sign $\pm$ is $+$ if and only if $d\Phi|_{e_1\wedge e_2}$ is orientation-preserving.

Since $d\Phi([e_1, e_2]) = [d\Phi(e_1), d\Phi(e_2)]$, comparing $\tilde{e}_3$ components, we have $\overline{F} = \pm\Phi^*(\overline{G})$. As both $\overline{F}$ and $\overline{G}$ are positive, we have

$$\overline{F} = \Phi^*(\overline{G}). \tag{3-6}$$

Consequently, $d\Phi(e_3) = \tilde{e}_3$ if and only if $d\Phi|_{e_1\wedge e_2}$ is orientation-preserving. In particular, Φ is orientation-preserving.

Comparing $\tilde{e}_1$ and $\tilde{e}_2$ components yields

$$\tilde{e}_1(\theta) = \tilde{e}_2(\theta) = 0. \tag{3-7}$$

By (3-2) and (3-7),

$$2\overline{G}\tilde{e}_3(\theta) = [\tilde{e}_1, \tilde{e}_2](\theta) = (\tilde{e}_1\tilde{e}_2 - \tilde{e}_2\tilde{e}_1)(\theta) = 0.$$

As $\overline{G}$ is nonzero, it follows that $\tilde{e}_3(\theta) = 0$, whence $\theta : (\mathrm{SL}_2(\mathbb{R}), g_G) \to \mathbb{R}$ is globally constant. In what follows, we will consider the two cases $d\Phi(e_3) = \tilde{e}_3$ and $d\Phi(e_3) = -\tilde{e}_3$ separately.

Case I: $d\Phi(e_3) = \tilde{e}_3$. As Φ is orientation-preserving, we have that $d\Phi|_{e_1\wedge e_2}$ is orientation-preserving. Using (3-2) twice, we obtain successively

$$g_G(d\Phi([e_2, e_3]), \tilde{e}_1) = \sin\theta \quad \text{and} \quad g_G([d\Phi(e_2), d\Phi(e_3)], \tilde{e}_1) = -\sin\theta.$$

As $d\Phi([e_2, e_3]) = [d\Phi(e_2), d\Phi(e_3)]$, it follows that $\sin\theta = 0$ and that θ is an integral multiple of π.

As θ is an integral multiple of π, the derivative map $d\Phi$ preserves the plane distribution $e_2 \wedge e_3$. Consequently, the diffeomorphism Φ preserves the foliation of $\mathrm{SL}_2(\mathbb{R})$ by left-cosets of NA and descends to a diffeomorphism ϕ of K. By (3-6), $F = \phi^*(G)$, concluding the proof in this case.

Case II: $d\Phi(e_3) = -\tilde{e}_3$. As Φ is orientation-preserving, we have that $d\Phi|_{e_1\wedge e_2}$ is orientation-reversing. Using (3-2) twice, we obtain successively

$$g_G(d\Phi([e_2, e_3]), \tilde{e}_2) = \cos\theta \quad \text{and} \quad g_G([d\Phi(e_2), d\Phi(e_3)], \tilde{e}_2) = 2\overline{G}\sin\theta - \cos\theta.$$

As $d\Phi([e_2, e_3]) = [d\Phi(e_2), d\Phi(e_3)]$, it follows that $\cos\theta = \overline{G}\sin\theta$. As θ is constant, so is $\overline{G}$. By (3-6), $\overline{F} = \overline{G}$ are equal constants. Hence, $F = G$ are equal constants, concluding the proof. □

Conjecture 3.8. *The metrics g_F constructed in this paper describe all of the complete Riemannian metrics on* $\mathrm{SL}_2(\mathbb{R})$ (*up to isometry*) *that are modeled on the curvature tensor T.*

References

[Ballmann 1985] W. Ballmann, "Nonpositively curved manifolds of higher rank", *Ann. of Math.* (2) **122**:3 (1985), 597–609. MR 87e:53059 Zbl 0585.53031

[Berger 1988] M. Berger, "L'œuvre d'André Lichnerowicz en géométrie riemannienne", pp. 11–24 in *Physique quantique et géométrie* (Paris, 1986), edited by D. Bernard and Y. Choquet-Bruhat, Travaux en Cours **32**, Hermann, Paris, 1988. MR 89i:01060 Zbl 0642.53002

[Boeckx et al. 1996] E. Boeckx, O. Kowalski, and L. Vanhecke, *Riemannian manifolds of conullity two*, World Scientific, River Edge, NJ, 1996. MR 98h:53075 Zbl 0904.53006

[Burns and Spatzier 1987] K. Burns and R. Spatzier, "Manifolds of nonpositive curvature and their buildings", *Inst. Hautes Études Sci. Publ. Math.* 65 (1987), 35–59. MR 88g:53050 Zbl 0643.53037

[Connell 2002] C. Connell, "A characterization of homogeneous spaces with positive hyperbolic rank", *Geom. Dedicata* **93** (2002), 205–233. MR 2003h:53059 Zbl 1032.53042

[Constantine 2008] D. Constantine, "2-frame flow dynamics and hyperbolic rank-rigidity in nonpositive curvature", *J. Mod. Dyn.* **2**:4 (2008), 719–740. MR 2010e:53076 Zbl 1157.53335

[Hamenstädt 1991] U. Hamenstädt, "A geometric characterization of negatively curved locally symmetric spaces", *J. Differential Geom.* **34**:1 (1991), 193–221. MR 92i:53046 Zbl 0733.53018

[Kowalski 1999] O. Kowalski, "On isocurved deformations of a homogeneous Riemannian space", pp. 163–171 in *New developments in differential geometry* (Budapest, 1996), edited by J. Szenthe, Kluwer Acad. Publ., Dordrecht, 1999. MR 2000a:53089 Zbl 0946.53024

[Kowalski et al. 1992] O. Kowalski, F. Tricerri, and L. Vanhecke, "Curvature homogeneous Riemannian manifolds", *J. Math. Pures Appl.* (9) **71**:6 (1992), 471–501. MR 94c:53064 Zbl 0836.53029

[Schmidt and Wolfson 2013] B. Schmidt and J. Wolfson, "Three-manifolds with constant vector curvature", preprint, 2013. To appear in *Indiana Univ. Math. J.* arXiv 1110.4619

[Shankar et al. 2005] K. Shankar, R. Spatzier, and B. Wilking, "Spherical rank rigidity and Blaschke manifolds", *Duke Math. J.* **128**:1 (2005), 65–81. MR 2006b:53056 Zbl 1082.53051

[Singer 1960] I. M. Singer, "Infinitesimally homogeneous spaces", *Comm. Pure Appl. Math.* **13** (1960), 685–697. MR 24 #A1100 Zbl 0171.42503

[Tricerri and Vanhecke 1989] F. Tricerri and L. Vanhecke, "Curvature homogeneous Riemannian manifolds", *Ann. Sci. École Norm. Sup.* (4) **22**:4 (1989), 535–554. MR 91f:53044 Zbl 0698.53033

Received March 31, 2014.

BENJAMIN SCHMIDT
DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824
UNITED STATES
schmidt@math.msu.edu

JON WOLFSON
DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824
UNITED STATES
wolfson@math.msu.edu

CONTENTS

Volume 273, no. 1 and no. 2

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 273 No. 2 February 2015

0030-8730(201502)273:2;1-4