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SOME RESULTS ON ARCHIMEDEAN RANKIN–SELBERG INTEGRALS

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We use a notion of derivatives of smooth representations of moderate growth of GL(n, $\mathbb{R}$) and exceptional poles to study local Rankin–Selberg integrals. We obtain various results which are archimedean analogs of p-adic results obtained by Cogdell and Piatetski-Shapiro.

1. Introduction

Let F be a p-adic field, π a smooth admissible representation of $\mathrm{GL}(n, F)$. J. Bernstein and A. Zelevinsky [1977] defined the notion of derivatives for π, denoted by $\pi^{(k)}$, $n \geq k \geq 0$, which is a useful tool to study properties of π.

If π' is another smooth admissible representation of $\mathrm{GL}(n, F)$, when both π and π' are generic with associated Whittaker models $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \psi^{-1})$, where ψ is a fixed nontrivial additive character of F, we have the following local Rankin–Selberg integrals:

$$I(s, W, W', \Phi) = \int_{N_n \backslash \mathrm{GL}_n} W(g) W'(g) \Phi(\epsilon_n g) |\det g|^s \, dg$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \bar{\psi})$, $\Phi \in \mathcal{S}(F^n)$ a Schwartz function, s a complex number, and $\epsilon_n = (0, 0, \ldots, 1) \in F^n$.

By the work of H. Jacquet, J. Shalika and Piatetski-Shapiro [1983], these integrals converge in some right half-plane of s, and have a meromorphic continuation to the whole plane. Suppose s_0 is a pole with the expansion

$$I(s, W, W', \Phi) = \frac{B_{s_0}(W, W', \Phi)}{(q^s - q^{s_0})^d} + \cdots .$$

Note that the Schwartz function space $\mathcal{S}(F^n)$ has a filtration

$$0 \subset \mathcal{S}^0 \subset \mathcal{S}(F^n),$$

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where $\mathcal{S}^0 = \{\Phi \in \mathcal{S}(F^n) : \Phi(0) = 0\}$. Cogdell and Piatetski-Shapiro [≥ 2015] defined s_0 to be an exceptional pole if the leading coefficient $B_{s_0}(W, W', \Phi)$ vanishes identically on $\mathcal{S}^0$, and used it together with derivatives to analyze the poles of local Rankin–Selberg integrals. As a consequence, they can compute the local L-factor for a pair of generic representations on general linear groups in terms of L-functions of the inducing datum.

It is interesting to see if there is an analogous theory for $\mathrm{GL}(n, \mathbb{R})$, and there is in fact some work in this direction; for example, [Chang and Cogdell 1999]. In this paper, we will take one more step towards such an archimedean theory, based on results in that reference. There are a couple of difficulties in the archimedean case. First of all, we need an appropriate theory of "derivatives". In a recent preprint, A. Aizenbud, D. Gourevitch and S. Sahi [Aizenbud et al. 2012] defined the derivatives for smooth representations of moderate growth on $\mathrm{GL}(n, \mathbb{R})$ as the inverse limit of certain coinvariants. But this seems complicated for our applications to local Rankin–Selberg integrals.

Here we simply take the naive analog of p-adic derivatives as our archimedean derivatives. It is a component in the $\mathfrak{n}$-homology, where $\mathfrak{n}$ is the nilradical of some parabolic subalgebra. The advantages of this definition are that it is relatively easier to deal with, and compatible with Rankin–Selberg integrals. But it is also interesting to see if one can relate the derivatives defined in [ibid.] to integrals $I(s, W, W', \Phi)$ in some way.

For the exceptional poles, the situation again is a little more complicated. The leading coefficients in the expansion of $I(s, W, W', \Phi)$ at a pole will involve a finite-dimensional representation of $\mathrm{GL}(n, \mathbb{R})$, due to the nature of the differences between Schwartz functions on $\mathbb{R}$ and the p-adic field F. To be more precise, the Schwartz function space $\mathcal{S}_n = \mathcal{S}_n(\mathbb{R}^n)$ has a natural filtration. Let

$$\mathcal{S}_n^m = \{f \in \mathcal{S} : f \text{ vanishes to order at least } m \text{ at zero}\}.$$

Then each $\mathcal{S}_n^m$ is a closed subspace, and we have a filtration

$$\mathcal{S}_n = \mathcal{S}_n^0 \supset \mathcal{S}_n^1 \supset \cdots \supset \mathcal{S}_n^m \supset \cdots,$$

where $\mathcal{S}_n^m / \mathcal{S}_n^{m+1}$ is isomorphic to the space of homogeneous polynomials on $\mathbb{R}^n$ of degree m, denoted as E_n^m — a finite-dimensional representation of $\mathrm{GL}(n, \mathbb{R})$.

At a pole s_0, $I(s, W, W', \Phi)$ has an expansion

$$I(s, W, W', \Phi) = \frac{B_{s_0}(W, W', \Phi)}{(q^s - q^{s_0})^d} + \cdots,$$

and we say s_0 is an exceptional pole of type 1 and level m if B_{s_0} vanishes identically on $\mathcal{S}^{m+1}$, but not on $\mathcal{S}^m$.

In general, we say s_0 is an exceptional pole of type 2 and level m, for π and π', if there exists a continuous trilinear form

$$l : V \times V' \times E_n^m \to \mathbb{C}$$

such that, for $g \in \mathrm{GL}(n, \mathbb{R})$,

$$l(g \cdot W, g \cdot W', g \cdot \Phi_n) = |\det g|^{-s_0} l(W, W', \Phi_n).$$

It follows that an exceptional pole of type 1 is also of type 2.

We can now state our main results. We say π is in general position as in [Chang and Cogdell 1999] (or see Section 2 for more details). We refer to page 294 for a definition of depth of exceptional poles of type 1.

Theorem. *Let π and π' be irreducible generic Casselman–Wallach representations of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_m(\mathbb{R})$ in general position.*

Case $m = n$: *Any pole of the Rankin–Selberg integrals for π and π' is an exceptional pole of type 2 for a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$, $0 \le k \le n-1$. On the other hand, any exceptional pole of type 1 of depth 0 for a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$, $0 \le k \le n-1$, is a pole of the Rankin–Selberg integrals of π and π'.*

Case $m < n$: *Any pole of the Rankin–Selberg integrals for π and π' is an exceptional pole of type 2 for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $1 \le k \le m$. On the other hand, any exceptional pole of type 1 of depth 0 for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $1 \le k \le m$, is a pole of the Rankin–Selberg integrals of π and π'.* □

The first remark is that these are not the exact archimedean analog we are seeking. We expect that the poles of Rankin–Selberg integrals are exactly exceptional poles of type 1 for pairs of components of derivatives of π and π'. A missing point here is that we haven't obtained the asymptotic results analogous to those in [Cogdell and Piatetski-Shapiro ≥ 2015, Section 1.4]; this will be addressed in the future.

We also remark here that the same ideas and techniques of this paper can also be applied to local exterior square L-integrals in [Jacquet and Shalika 1990]; this will appear in a forthcoming paper.

The paper is organized as follows. In Section 2 we review some preliminaries. In Section 3 we define the derivatives and obtain some basic properties. Section 4 is devoted to the study of exceptional poles. We obtain the main results in Section 5 for $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$, and in Section 6 we discuss the case $\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_m(\mathbb{R})$, $m < n$.

2. Notations and preliminaries

In this section, we introduce some notations and results needed in this paper.

Let $G_n = \mathrm{GL}_n(\mathbb{R})$ be the general linear group of invertible $n \times n$ matrices over $\mathbb{R}$, and $K = K_n = O(n)$ be the orthogonal subgroup of G_n, which is a maximal compact subgroup of G_n. We use $\mathfrak{g} = \mathfrak{g}_n$, $\mathfrak{k} = \mathfrak{k}_n$ to denote the complexified Lie algebras of G_n and K_n respectively. Let N_n be the upper triangular unipotent subgroup of G_n. Fix ψ as the additive character of $\mathbb{R}$ given by $\psi(x) = \exp(2\pi\sqrt{-1}x)$, and define a character on N_n, still denoted as ψ, by

$$\psi(u) = \psi\Big(\sum_i u_{i,i+1}\Big),$$

where $u = (u_{ij}) \in N_n$. Let μ be the differential of ψ; then μ is a linear form on $\mathfrak{n}_n$, the Lie algebra of N_n, vanishing on $[\mathfrak{n}_n, \mathfrak{n}_n]$.

A smooth representation (π, V) is called generic if it admits a nontrivial Whittaker functional. A Whittaker functional Λ with respect to μ on (π, V) is a continuous linear functional on V satisfying

$$\Lambda(\pi(X)v) = \mu(X)\Lambda(v)$$

for all $X \in \mathfrak{n}_n$, $v \in V$.

If π is generic, let Λ be the Whittaker functional on π, and for any $v \in V$ define a function $W_v : G_n \to \mathbb{C}$ by $W_v(g) = \Lambda(\pi(g)v)$. Then W_v is called the Whittaker function on G_n corresponding to v, and the space $\mathcal{W}(\pi, \psi) = \{W_v : v \in V\}$ is called the Whittaker model of π.

Throughout the paper, we will work with smooth representations of moderate growth. Suppose V is a Fréchet space. A smooth representation (π, V) is called a representation of moderate growth if, for every seminorm ρ on V, there exists a positive integer N and a seminorm ν such that for every $g \in G_n$, $v \in V$, we have

$$|\pi(g)v|_\rho \le \|g\|^N |v|_\nu,$$

where $\|g\| = \mathrm{Tr}(g\,{}^tg) + \mathrm{Tr}(g^{-1}g^\iota)$ and $g^\iota = {}^tg^{-1}$. If in addition every irreducible representation of K has finite multiplicity in π, we will say π is admissible.

We have the following important result of Casselman and Wallach.

Theorem 2.1. *For any finitely generated admissible* $(\mathfrak{g}, K)$*-module* W, *there exists exactly one smooth representation of moderate growth on a Fréchet space* V, *up to canonical topological isomorphism, such that the underlying* $(\mathfrak{g}, K)$*-module* V_K *is isomorphic to* W. *Moreover, the assignment* $W \to V$ *is an exact functor from the category of finitely generated admissible modules to the category of smooth admissible finitely generated Fréchet representations of moderate growth.*

Proof. See, for example, [Wallach 1992, Chapter 12]. $\square$

Remark. We refer to V in this theorem as the completion or globalization of W, and we refer to smooth admissible finitely generated Fréchet representations of moderate growth (π, V) as Casselman–Wallach representations.

For irreducible Casselman–Wallach representations, by results of J. Shalika [1974], there exists at most one Whittaker functional with respect to a given nontrivial ψ, unique up to a scalar.

For a given smooth representation V of G_n, and a nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, we use $H_0(\mathfrak{n}, V)$ to denote the quotient of V by the closure of the subspace spanned by $\{X \cdot v : X \in \mathfrak{n}, v \in V\}$. When W is a $(\mathfrak{g}, K)$-module, use $H_0(\mathfrak{n}, W)$ to denote $W/\mathfrak{n}W$. Similarly, if N is a unipotent subgroup of G_n, denote by $H_0(N, V)$ the quotient of V by the closure of the subspace spanned by vectors $\{\pi(u)v - v : u \in N, v \in V\}$.

If (π, V) is an irreducible Casselman–Wallach representation of G_n, V_K denotes its K-finite vectors. If P is a standard parabolic subgroup of G_n, denote its Levi decomposition by $P = MN$, where N is the unipotent subgroup of P and M is the Levi component. Let $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{n}$ be their complexified Lie algebras, respectively. It is a result of B. Casselman that $H_0(\mathfrak{n}, V_K)$ is nonzero. By results of Stafford and N. Wallach it is an admissible $(\mathfrak{m}, K \cap M)$ module. Moreover, it is finitely generated over $U(\mathfrak{m})$; here $U(\mathfrak{m})$ denotes the universal enveloping algebra of $\mathfrak{m}$. See, for example, [Borel and Wallach 2000] for more details.

For $H_0(\mathfrak{n}, V)$, M acts naturally on this quotient, which is also a Fréchet space. This gives a smooth representation of M, which is also of moderate growth.

Naturally $H_0(\mathfrak{n}, V_K)$ embeds into $H_0(\mathfrak{n}, V)$, sending $v + \mathfrak{n}V_K$ to $v + \overline{\mathfrak{n}V}$ for any $v \in V_K$. Moreover, we have the following.

Proposition 2.2. *$H_0(\mathfrak{n}, V)$ is a Casselman–Wallach representation of M, and its $K \cap M$-finite vectors are exactly $H_0(\mathfrak{n}, V_K)$; so it is the completion of $H_0(\mathfrak{n}, V_K)$.*

Proof. The image of the embedding $H_0(\mathfrak{n}, V_K) \to H_0(\mathfrak{n}, V)$ is a $(\mathfrak{m}, K \cap M)$-module, and is dense in $H_0(\mathfrak{n}, V)$. Hence $H_0(\mathfrak{n}, V_K)$ can be identified with the underlying $(\mathfrak{m}, K \cap M)$-module of $H_0(\mathfrak{n}, V)$. As $H_0(\mathfrak{n}, V_K)$ is nonzero, finitely generated and admissible, so is $H_0(\mathfrak{n}, V)$. Hence $H_0(\mathfrak{n}, V)$ is the completion of $H_0(\mathfrak{n}, V_K)$. □

Remark. According to an unpublished result of B. Casselman, $H_*(\mathfrak{n}, V)$ is the completion of $H_*(\mathfrak{n}, V_K)$; see [Bunke and Olbrich 1997, Theorem 1.5].

For any two smooth representations of moderate growth (π, V) and (ρ, W) of G_n and G_m, respectively, denote by $(\pi \widehat{\otimes} \rho, V \widehat{\otimes} W)$ the complete projective tensor product. It is also a smooth representation of moderate growth on $G_n \times G_m$.

Now if π (or π') is an irreducible admissible representation of G_n (or G_m), by the local Langlands correspondence π (or π') corresponds to an n- (or m-) dimensional semisimple representation of the Weil group $W_{\mathbb{R}}$, denoted as ρ (or ρ'). Now consider the tensor product $\rho \otimes \rho'$, which defines a semisimple representation of $W_{\mathbb{R}}$ with dimension mn. Then one can associate a local L-factor, denoted by $L(s, \pi \times \pi')$, to $\rho \otimes \rho'$, which is a product of gamma functions. For more details, see, for example, [Knapp 1994].

For any $W_v \in \mathcal{W}(\pi, \psi)$, define $\widetilde{W}_v(g) = W_v(\omega_n g^\iota)$, where

$$\omega_n = \begin{pmatrix} 0 & \cdots & 1 \\ & \iddots & \\ 1 & \cdots & 0 \end{pmatrix}$$

and $g^\iota = {}^t g^{-1}$. Then by [Jacquet and Shalika 1981], it is known that $\{\widetilde{W}_v : v \in V\}$ is a Whittaker model for $\tilde{\pi}$ with respect to $\bar{\psi}$, the contragredient of π.

To introduce the local Rankin–Selberg integrals, assume (π, V) and (π', V') are generic irreducible Casselman–Wallach representations of G_n and G_m, respectively, with Whittaker models $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \bar{\psi})$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions on $\mathbb{R}^n$.

If $m = n$, set

$$I(s, W, W', \Phi) = \int_{N_n \backslash G_n} W(g) W'(g) \Phi(\epsilon_n g) |\det g|^s \, dg$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \bar{\psi})$, $\Phi \in \mathcal{S}(\mathbb{R}^n)$, and $\epsilon_n = (0, 0, \dots, 1) \in \mathbb{R}^n$.

If $n > m$, set

$$I(s, W, W') = \int_{N_m \backslash G_m} W \begin{pmatrix} g & 0 \\ 0 & I_{n-m} \end{pmatrix} W'(g) |\det g|^{s - \frac{n-m}{2}} \, dg.$$

In general, for $0 \le j \le n - m - 1$, set

$$I_j(s, W, W') = \int_{M(m \times j, \mathbb{R})} \int_{N_m \backslash G_m} W \begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{n-m-j} \end{pmatrix} W'(g) |\det g|^{s - \frac{n-m}{2}} \, dg \, dX.$$

The following theorem is due to Jacquet and Shalika; see, for example, [Jacquet 2009].

Theorem 2.3. (1) *These integrals converge for* $\mathrm{Re}(s) \gg 0$.

(2) *Each integral has a meromorphic continuation to all* $s \in \mathbb{C}$, *which is a holomorphic multiple of* $L(s, \pi \times \pi')$.

(3) *The following functional equations are satisfied*:

$$I_j(1 - s, \widetilde{W}, \widetilde{W}') = \omega'(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) I_{n-m-1-j}(s, W, W')$$

and

$$I(1 - s, \widetilde{W}, \widetilde{W}', \widehat{\Phi}) = \omega'(-1)^{n-1} I(s, W, W', \Phi),$$

where $\widehat{\Phi}$ is the Fourier transform of Φ, given by

$$\widehat{\Phi}(X) = \int \Phi(Y)\psi(-\mathrm{Tr}({}^tXY))\,dY. \qquad \square$$

Now we recall some results from [Cogdell and Piatetski-Shapiro 2004] which are essential in Section 4, which studies exceptional poles.

The first result is an extension of the Dixmier–Malliavin theorem. Let (π, V) be a continuous representation of G_n on a Fréchet space V. Still use π to denote the smooth representation of G_n induced from π on the smooth vectors V^∞ of V.

Proposition 2.4 [Cogdell and Piatetski-Shapiro 2004, Proposition 1.1]. *Let $v_k \to v_0$ be a convergent sequence in V^∞. Then there exists a finite set of functions $f_j \in \mathscr{C}_c^\infty(G_n)$ and a collection of vectors $v_{k,j} \in V^\infty$ such that $v_k = \sum_j \pi(f_j)v_{k,j}$ for all $k \geq 0$, and such that $v_{k,j} \to v_{0,j}$ as $k \to \infty$ for each j.* $\square$

The second result is about the continuity of archimedean Rankin–Selberg integrals. Let (π, V) and (π', V') be generic irreducible Casselman–Wallach representations of G_n and G_m, respectively. For $W \in \mathscr{W}(\pi, \psi)$, $W' \in \mathscr{W}(\pi', \psi^{-1})$, $\Phi \in \mathscr{S}(\mathbb{R}^n)$, we have:

Theorem 2.5. *The linear functionals*

$$\Lambda_s = \frac{I(s, W, W')}{L(\pi \times \pi')}, \qquad n > m,$$

$$\Lambda_s = \frac{I(s, W, W', \Phi)}{L(\pi \times \pi')}, \quad n = m,$$

are uniformly continuous in s on compact sets with respect to the topologies involved.

Proof. See [Cogdell and Piatetski-Shapiro 2004, Theorem 1.1]. $\square$

Remark. As noted in [Cogdell and Piatetski-Shapiro 2004], here we claim the result is true for all s.

To end this section, let's explain irreducible representations in general position, following [Chang and Cogdell 1999]. Let $P = MN$ be a parabolic subgroup of G_n, with $M = G_1^p \times G_2^q$ and $p + 2q = n$. Write C_2 for the cyclic group $\{\pm 1\}$, $G_1 \simeq \mathbb{R}_{>0} \times C_2$ and $G_2 \simeq \mathbb{R}_{>0} \times \mathrm{SL}_2^{\pm}$, where $\mathrm{SL}_2^{\pm}$ stands for the subgroup of G_2 consisting of matrices with determinant ± 1. So $M = (\mathbb{R}_{>0})^{p+q} \times C_2^p \times (\mathrm{SL}_2^{\pm})^q$.

Let T_m be the discrete series of $\mathrm{SL}_2^{\pm}$ with parameter $m \in \mathbb{Z}_{>0}$. We will use notation $(s_1, \ldots, s_p)$ to denote the character on $(\mathbb{R}_{>0})^p$ sending $(x_1, \ldots, x_p)$ to $\prod_{i=1}^p x_i^{s_i}$. And let ϵ be a character on C_2. Then form the tensor product

$$\sigma = (s_1, \ldots, s_p, 2t_1, \ldots, 2t_q) \otimes (\epsilon_1 \otimes \cdots \otimes \epsilon_p \otimes T_{m_1} \otimes \cdots \otimes T_{m_q}).$$

This is a representation on M, and then we get the normalized parabolic induced representation $\mathrm{Ind}(\sigma)$. We say $\pi = \mathrm{Ind}(\sigma)$ is a representation in general position if

$$s_i, t_j, s_i - s_j \notin \mathbb{Z} \text{ for } i \neq j, \quad t_i - t_j \notin \tfrac{1}{2}\mathbb{Z} \text{ for } i \neq j, \quad s_i - t_j \notin \tfrac{1}{2}\mathbb{Z}.$$

It is known that these induced representations are irreducible and generic, see [Chang and Cogdell 1999] for more information.

3. Archimedean derivatives

In this section we introduce archimedean derivatives. First we need more notation. For any $1 \leq l \leq n$, let U_{n-l+1} be the unipotent radical of the standard parabolic subgroup associated to the partition $(n-l, 1, \ldots, 1)$, that is, the subgroup of N_n consisting of matrices having the form

$$\begin{pmatrix} I_{n-l} & x \\ 0 & u \end{pmatrix},$$

where x is a $(n-l) \times l$ matrix and $u \in N_l$ is an upper triangular matrix with 1 on the diagonal. Note that $U_1 = N_n$. Denote by $\mathfrak{u}_{n-l+1}$ the corresponding Lie algebras. Define a linear form μ_{n-l+1} on each $\mathfrak{u}_{n-l+1}$ by

$$\mu_{n-l+1}(X) = \mu(X_{n-l+1,n-l+2} + \cdots + X_{n-1,n}).$$

Now let (π, V) be a Casselman–Wallach representation of G_n, V_K its underlying $(\mathfrak{g}, K)$-module. For $1 \leq l \leq n$, let V_l be the closure of the subspace spanned by $\{X \cdot v - \mu_{n-l+1}(X)v : v \in V, X \in \mathfrak{u}_{n-l+1}\}$.

Definition. For each integer $0 \leq l \leq n$, we define the l-th derivative of π, denoted by $(\pi^{(l)}, V^{(l)})$, as follows:

(1) If $l = 0$, put $(\pi^{(0)}, V^{(0)}) = (\pi, V)$.

(2) If $1 \leq l \leq n$, put $V^{(l)} = V/V_l$, and define the action $\pi^{(l)}$ by

$$\pi^{(l)}(g) \cdot (v + V_l) = |\det g|^{-l/2} \pi(g)v + V_l \quad \text{for any } g \in G_{n-l}.$$

To continue, we need more notation. Use $P_{n-l,l}$ to denote the standard parabolic subgroup of G_n associated to the partition $(n-l, l)$ of n. It has Levi decomposition $P_{n-l,l} = M_{n-l,l} N_{n-l,l}$, with Levi component $M_{n-l,l}$ isomorphic to $G_{n-l} \times G_l$ and unipotent part $N_{n-l,l}$. Let $\mathfrak{p}_{n-l,l}$, $\mathfrak{m}_{n-l,l}$ and $\mathfrak{n}_{n-l,l}$ be their complexified Lie algebras, respectively.

Note that we have the decomposition

$$\begin{pmatrix} I_{n-l} & x \\ 0 & v \end{pmatrix} = \begin{pmatrix} I_{n-l} & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} I_{n-l} & x \\ 0 & I_l \end{pmatrix},$$

so we can write $U_{n-l+1} = V_{n-l+1} N_{n-l,l}$, where V_{n-l+1} is the standard unipotent subgroup N_l of G_l embedded in G_n in the right lower corner.

Let $\mathfrak{v}_{n-l+1}$ be the complexified Lie algebra of V_{n-l+1}. Note that the character μ_{n-l+1} is trivial on $\mathfrak{n}_{n-l,l}$. Let Y_l be the closure of the space spanned by

$$\{X \cdot \bar{v} - \mu_{n-l+1}(X)\bar{v} : X \in \mathfrak{v}_{n-l+1}, \bar{v} \in H_0(\mathfrak{n}_{n-l,l}, V)\}.$$

Since $V^{(l)} = V/V_l$ and

$$V_l = \overline{\{X \cdot v - \mu_{n-l+1}(X)v : v \in V, X \in \mathfrak{u}_{n-l+1}\}},$$

note that $\mathfrak{u}_{n-l+1} = \mathfrak{v}_{n-l+1} + \mathfrak{n}_{n-l,l}$. Then

$$H_0(\mathfrak{n}_{n-l,l}, V)/Y_l$$
$$= (V/\overline{\mathfrak{n}_{n-l,l} V}) \big/ \big(\overline{\{X \cdot v - \mu_{n-l+1}(X)v : X \in \mathfrak{u}_{n-l+1}, v \in V\}}/\overline{\mathfrak{n}_{n-l,l} V}\big) = V/V_l.$$

Thus, we have verified the following proposition.

Proposition 3.1. $$V^{(l)} = H_0(\mathfrak{n}_{n-l,l}, V)/Y_l.$$ □

The following result states that the derivatives $\pi^{(l)}$ belong to a nice class of representations.

Proposition 3.2. *For each l, $\pi^{(l)}$ is a Casselman–Wallach representation of G_{n-l}.*

Proof. This follows from the fact that the $\mathfrak{n}$-homology $H_0(\mathfrak{n}, V)$ is admissible. □

Now assume (π, V) is an irreducible smooth admissible generic representation of moderate growth on G_n in general position as in Section 2. Denote by V_K its K-finite vectors, which is an irreducible admissible $(\mathfrak{g}, K)$-module. For the rest of this section, unless otherwise stated, we will drop the subscript for the standard upper triangular parabolic subgroup $P = MN$ associated with the partition $(n-k, k)$ of n, to simplify notation.

By [Chang and Cogdell 1999, Theorem 4.2] the $\mathfrak{n}$-homology $V_K/\mathfrak{n}V_K$ is nonzero and is a semisimple $(\mathfrak{m}, K \cap M)$-module. By Proposition 2.2, $V/\overline{\mathfrak{n}V}$ is the smooth completion of $V_K/\mathfrak{n}V_K$. It follows that $V/\overline{\mathfrak{n}V}$ is also semisimple, so we can write

$$V/\overline{\mathfrak{n}V} = \bigoplus_{i=1}^{r} A_i,$$

where each A_i is an irreducible smooth admissible representation of moderate growth on M and hence, by results of D. Gourevitch and A. Kemarsky [2013], isomorphic to $\rho_i \widehat{\otimes} \sigma_i$, where each ρ_i and σ_i are irreducible smooth representations of moderate growth on G_{n-k} and G_k, respectively. Note that it is possible to have $A_i \cong A_j$ for $i \neq j$. We use $\rho_{i,K}$ and $\sigma_{i,K}$ to denote the representations on the underlying K-finite modules. Let p_i be the natural projection from $V_K/\mathfrak{n}V_K$ onto

$\rho_{i,K}\otimes\sigma_{i,K}$, and also be the projection from $V/\overline{\mathfrak{n}V}$ onto $\rho_i\widehat{\otimes}\sigma_i$. We will also use p to denote the projections $V\to V/\overline{\mathfrak{n}V}$.

Lemma 3.3. *For each i, ρ_i and σ_i are generic representations.*

Proof. This follows from [Chang and Cogdell 1999, Theorem 4.2]. See Remarks 4.3 there. □

Denote by $\mathcal{W}(\rho_i,\psi)$ the Whittaker model for ρ_i.

Proposition 3.4. *For every $W_i\in\mathcal{W}(\rho_i,\psi)$ and every $\Phi\in\mathcal{S}(\mathbb{R}^{n-k})$, there is a Whittaker function $W_v\in\mathcal{W}(\pi,\psi)$ such that*

$$W_v\begin{pmatrix} g & 0\\ 0 & I_k\end{pmatrix}=W_i(g)\Phi(\epsilon_{n-k}g).$$

Proof. The projection p_i from $V_K/\mathfrak{n}V_K$ onto $\rho_{i,K}\otimes\sigma_{i,K}$ induces an injective intertwining map $V_K\to\mathrm{Ind}(|\det|^{-k/2}\rho_{i,K}\otimes|\det|^{(n-k)/2}\sigma_{i,K})$. This extends to an injective map

$$V\to\mathrm{Ind}(|\det|^{-k/2}\rho_i\widehat{\otimes}|\det|^{(n-k)/2}\sigma_i).$$

Denote by Q its quotient; we have a short exact sequence of smooth representations of moderate growth

$$0\longrightarrow V\longrightarrow\mathrm{Ind}(|\det|^{-k/2}\rho_i\widehat{\otimes}|\det|^{(n-k)/2}\sigma_i)\longrightarrow Q\longrightarrow 0. \tag{1}$$

The underlying $(\mathfrak{g},K)$-modules also form a short exact sequence

$$0\longrightarrow V_K\longrightarrow\mathrm{Ind}(|\det|^{-k/2}\rho_{i,K}\otimes|\det|^{(n-k)/2}\sigma_{i,K})\longrightarrow Q_K\longrightarrow 0. \tag{2}$$

By taking the dual (contragredient representation) of the short exact sequence (2), we have

$$0\longrightarrow Q_K^*\longrightarrow\big(\mathrm{Ind}(|\det|^{-k/2}\rho_{i,K}\otimes|\det|^{(n-k)/2}\sigma_{i,K})\big)^*\longrightarrow V_K^*\longrightarrow 0.$$

By [Wallach 1988, Lemma 4.5.2], we have

$$0\longrightarrow Q_K^*\longrightarrow\mathrm{Ind}\big((|\det|^{-k/2}\rho_{i,K})^*\otimes(|\det|^{(n-k)/2}\sigma_{i,K})^*\big)\longrightarrow V_K^*\longrightarrow 0,$$

which induces a short exact sequence for their smooth completions:

$$0\longrightarrow Q^*\longrightarrow\mathrm{Ind}\big((|\det|^{-k/2}\rho_i)^*\widehat{\otimes}(|\det|^{(n-k)/2}\sigma_i)^*\big)\longrightarrow V^*\longrightarrow 0. \tag{3}$$

Now for any representation (τ,U), define representation (τ^s,U) by $\tau^s(g)\cdot u=\tau({}^tg^{-1})\cdot u$ for any $g\in G_n$, $u\in U$; then τ^s is isomorphic to τ^* when τ is irreducible, by [Aizenbud et al. 2008, Theorem 2.4.2]. Note that we are working in the same space, but simply changing the action. So if we have a short exact sequence

$$0\longrightarrow(\tau_1,U_1)\longrightarrow(\tau_2,U_2)\longrightarrow(\tau_3,U_3)\longrightarrow 0,$$

applying the operation 's', we then have a new exact sequence

$$0 \longrightarrow (\tau_1^s, U_1) \longrightarrow (\tau_2^s, U_2) \longrightarrow (\tau_3^s, U_3) \longrightarrow 0.$$

Now apply operation 's' to the sequence (3); then we have

$$(4)\quad 0 \longrightarrow (Q^*)^s \longrightarrow \big(\mathrm{Ind}((|\det|^{-k/2}\rho_i)^* \hat{\otimes} (|\det|^{(n-k)/2}\sigma_i)^*)\big)^s \longrightarrow (V^*)^s \longrightarrow 0.$$

It follows that the sequence (4) becomes

$$0 \longrightarrow (Q^*)^s \longrightarrow \mathrm{Ind}_{P'}^{G_n}\big((|\det|^{-k/2}\rho_i)^{*s} \hat{\otimes} (|\det|^{(n-k)/2}\sigma_i)^{*s}\big) \longrightarrow (V^*)^s \longrightarrow 0.$$

Since π, ρ_i and σ_i are irreducible, the above is

$$0 \longrightarrow (Q^*)^s \longrightarrow \mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \hat{\otimes} |\det|^{(n-k)/2}\sigma_i) \longrightarrow V \longrightarrow 0.$$

Let Λ be the unique (up to a constant) continuous Whittaker functional on V. Composed with the projection

$$\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \hat{\otimes} |\det|^{(n-k)/2}\sigma_i) \longrightarrow V,$$

we get a nontrivial continuous Whittaker functional Λ' on

$$\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \hat{\otimes} |\det|^{(n-k)/2}\sigma_i).$$

By the last conclusion of [Wallach 1992, Theorem 15.4.1], there is a linear bijection between the space of Whittaker functionals on

$$\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \hat{\otimes} |\det|^{(n-k)/2}\sigma_i)$$

and the space of Whittaker functionals on

$$|\det|^{-k/2}\rho_i \hat{\otimes} |\det|^{(n-k)/2}\sigma_i.$$

By Lemma 3.3, the latter space has dimension 1, thus there is a unique (up to a constant) continuous Whittaker functional on $\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \hat{\otimes} |\det|^{(n-k)/2}\sigma_i)$, and it must be Λ'. Then we can conclude that the space of Whittaker functions $\mathcal{W}(\pi, V)$ for π and that for $\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \hat{\otimes} |\det|^{(n-k)/2}\sigma_i)$, are the same.

So in order to prove the existence of W_v in $\mathcal{W}(\pi, V)$ as in the proposition, it suffices to find some Whittaker function for $\mathrm{Ind}_{P'}^{G_n}(|\det|^{-k/2}\rho_i \hat{\otimes} |\det|^{(n-k)/2}\sigma_i)$ with the required property. Now this follows from [Jacquet 2009, Proposition 14.1], which finishes the proof. ☐

Corollary 3.5. *For every Whittaker function W_i in any irreducible component of $\pi^{(k)}$, and any Schwartz function Φ on $\mathbb{R}^{n-k}$, we can always find some $W_v \in \mathcal{W}(\pi, \psi)$ such that*

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = W_i(g)\Phi(\epsilon_{n-k} g)|\det g|^{k/2}.$$

Proof. This follows from the fact that $\pi^{(k)}$ is isomorphic to $|\det|^{-k/2}\bigoplus_i \rho_i$. □

4. Exceptional poles

In this section, we will introduce two types of exceptional poles and discuss their basic properties. Set

$$\mathcal{S}_n^m = \{f \in \mathcal{S} : f \text{ vanishes to order at least } m \text{ at zero}\};$$

then we have a filtration of closed subspaces for the Schwartz function space $\mathcal{S}_n = \mathcal{S}_n(\mathbb{R}^n)$:

$$\mathcal{S}_n = \mathcal{S}_n^0 \supset \mathcal{S}_n^1 \supset \cdots \supset \mathcal{S}_n^m \supset \cdots .$$

$\mathcal{S}_n^m/\mathcal{S}_n^{m+1}$ is isomorphic to the space of homogeneous polynomials on $\mathbb{R}^n$ of degree m, denoted by E_n^m. The group G_n acts on $\mathcal{S}_n$ from the right, which preserves this filtration, and therefore induces an action on E_n^m.

Let π and π' be irreducible generic Casselman–Wallach representations on G_n. The Rankin–Selberg integrals for π and π', are given by

$$I(s, W, W', \Phi) = \int_{N_n\backslash G_n} W(g)W'(g)\Phi(\epsilon_n g)|\det g|^s \, dg$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, $\Phi \in \mathcal{S}$, where $\epsilon_n = (0, 0, \ldots, 1) \in \mathbb{R}^n$, $s \in \mathbb{C}$. By Theorem 2.3, these integrals converge when s is in some right half-plane, and have a meromorphic continuation to the whole complex plane.

For any integer $1 \le k \le n$, for $v \in \pi$, $v' \in \pi'$ and $\Phi \in \mathcal{S}_k$, we define the following family of integrals:

$$I_k(s, W_v, W_{v'}, \Phi) = \int_{N_k\backslash G_k} W_v\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} W_{v'}\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix}\Phi(\epsilon_k g)|\det g|^{s-n+k} \, dg.$$

Lemma 4.1. *The integrals I_k belong to the space of Rankin–Selberg integrals for π and π'.*

Proof. This follows from [Jacquet 2009, Proposition 6.1 and Lemma 14.1]. □

Thus it follows that I_k converges when $\mathrm{Re}(s)$ is large and has a meromorphic continuation to the whole complex plane. Suppose s_0 is a pole of order d for the integral $I_k(s, W, W', \Phi)$, with Laurent expansion

$$I_k(s, W, W', \Phi) = \frac{B_{s_0,k}(W, W', \Phi)}{(s - s_0)^d} + \cdots ,$$

where $B_{s_0,k}(W, W', \Phi)$ is a trilinear form on $V \times V' \times \mathcal{S}_k$ satisfying the following invariance property:

$$B_{s_0,k}(g \cdot W, g \cdot W', g \cdot \Phi) = |\det g|^{-s_0+n-k} B_{s_0,k}(W, W', \Phi)$$

for any $g \in G_k$, $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, $\Phi \in \mathcal{S}_k$.

Proposition 4.2. *The trilinear form $B_{s_0,k}$ is continuous with respect to the topologies involved.*

Proof. When $k = n$, the continuity of $B_{s_0,n}$ follows from Theorem 2.5. When $k < n$, we will use the constructions in the proof of [Jacquet 2009, Lemma 14.1] to prove the continuity.

Now suppose $v_l \to v$, $v'_l \to v'$ and $\Phi_l \to \Phi$; then write

$$I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \frac{B_{s_0,k}(v_l, v'_l, \Phi_l)}{(s-s_0)^d} + \cdots$$

and

$$I_k(s, W_v, W_{v'}, \Phi) = \frac{B_{s_0,k}(v, v', \Phi)}{(s-s_0)^d} + \cdots .$$

Then we want to show that $B_{s_0,k}(v_l, v'_l, \Phi_l) \to B_{s_0,k}(v, v', \Phi)$ as $l \to \infty$.

Let Ψ_l and Ψ be Schwartz functions on $\mathbb{R}^k$ whose Fourier transforms are given by $\widehat{\Psi}_l = \Phi_l$, $\widehat{\Psi} = \Phi$. Since Fourier transform is a topological isomorphism on Schwartz function space, it follows that $\Psi_l \to \Psi$. Now we set

$$u_l = \int \pi \begin{pmatrix} I_k & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} v_l \Psi_l(x)\, dx$$

and

$$u = \int \pi \begin{pmatrix} I_k & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} v \Psi(x)\, dx.$$

Claim 1. *If f is a Schwartz function on $\mathbb{R}^k$, the map $(f, v) \mapsto \pi(f)v$ is a continuous map from $V \times \mathcal{S}_k$ to V, where*

$$\pi(f)v = \int_{\mathbb{R}^k} f(x)\pi(x)v\, dx.$$

Proof of Claim 1. Suppose $f_l \to f$ in $\mathcal{S}_k$, $v_l \to v$ in V. We want to show that $\pi(f_l)v_l \to \pi(f)v$.

Because (π, V) is of moderate growth, for any seminorm $|\cdot|_1$ on V there exists a seminorm $|\cdot|_2$ on V, a positive integer N_0, and a positive number C, such that for any $v \in V$ and $x \in \mathbb{R}^k$, we have $|\pi(x)v|_1 \le C(1 + \|x\|^2)^{N_0}|v|_2$. Here we identify x with

$$\begin{pmatrix} I_k & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} \in G_n,$$

and $\|x\|$ denotes the usual Euclidean norm of x. Then we have

$$\begin{aligned}|\pi(f_l)v_l-\pi(f)v|_1 &\le |\pi(f_l)v_l-\pi(f)v_l|_1+|\pi(f)v_l-\pi(f)v|_1\\ &\le \int |f_l(x)-f(x)\|\pi(x)v_l|_1\,dx+\int|f(x)\|\pi(x)(v_l-v)|_1\,dx\\ &\le C|v_l|_2\int|f_l(x)-f(x)|(1+\|x\|^2)^{N_0}\,dx\\ &\qquad\qquad +C|v_l-v|_2\int|f(x)|(1+\|x\|^2)^{N_0}\,dx.\end{aligned}$$

Since $v_l\to v$, $|v_l|_2$ is bounded for any l, and $|v_l-v|_2\to 0$ as $l\to\infty$. Because $f_l\to f$ in $\mathcal{S}_k$,

$$\int|f_l(x)-f(x)|(1+\|x\|^2)^{N_0}\,dx\to 0\quad\text{as}\quad l\to\infty.$$

Hence $\pi(f_l)v_l\to\pi(f)v$ as $l\to\infty$, which proves the claim. □

So, by Claim 1, $u_l\to u$. And by the first conclusion of [Jacquet 2009, Proposition 6.1], we have

$$W_{u_l}\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}=W_{v_l}\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}\Phi_l(\epsilon_k g)$$

and

$$W_{u}\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}=W_{v}\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}\Phi(\epsilon_k g).$$

Thus

$$I_k(s,W_{v_l},W_{v_l'},\Phi_l)=\int W_{u_l}\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}W_{v_l'}\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}|\det g|^{s-n+k}\,dg$$

and

$$I_k(s,W_{v},W_{v'},\Phi)=\int W_{u}\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}W_{v'}\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}|\det g|^{s-n+k}\,dg.$$

We will view $w_l=u_l\otimes v_l'$ as an element in $\sigma=\pi\widehat{\otimes}\pi'$; consequently $W_{w_l}(g)=W_{u_l}(g)W_{v_l'}(g)\in\mathcal{W}(\pi\widehat{\otimes}\pi',\psi\otimes\psi^{-1})$, and we have

$$I_k(s,W_{v_l},W_{v_l'},\Phi_l)=\int W_{w_l}\left(\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix},\begin{pmatrix}g&0\\0&I_{n-k}\end{pmatrix}\right)|\det g|^{s-n+k}\,dg. \tag{5}$$

Similarly, write $w = u \otimes v' \in \sigma = \pi \widehat{\otimes} \pi'$; then we have $w_l \to w$ and

$$(6) \qquad I_k(s, W_v, W_{v'}, \Phi) = \int W_w\left(\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix}\right) |\det g|^{s-n+k}\, dg.$$

Now by Proposition 2.4[1] applied to the group $\mathbb{R}^k \times \mathbb{R}^\times$, there exists a finite set of functions $f_j(x,h) \in \mathscr{C}_c^\infty(\mathbb{R}^k \times \mathbb{R}^\times)$ and vectors $w_{l,j} \in \pi \widehat{\otimes} \pi'$ with $l \geq 0$ such that

$$w_l = \sum_j \sigma(f_j) w_{l,j} \quad \text{for all } l \geq 1, \quad w = \sum_j \sigma(f_j) w_{0,j},$$

and $w_{l,j} \to w_{0,j}$ for each j.

More precisely, we can write

$$w_l = \sum_j \int \sigma\left(\begin{pmatrix} a^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} a^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}\right) w_{l,j}\, f_j(x,h)\, dx\, d^\times h$$

and

$$w = \sum_j \int \sigma\left(\begin{pmatrix} a^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} a^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}\right) w_{0,j}\, f_j(x,h)\, dx\, d^\times h,$$

where $a(h) = \operatorname{diag}(h, 1, \ldots, 1)$.

Then the integrals (5) and (6) now become

$$\begin{aligned} &I_k(s, W_{v_l}, W_{v_l'}, \Phi_l) \\ &= \sum_j \int W_{w_{l,j}}\left(\begin{pmatrix} ga^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} ga^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}\right) \\ &\qquad\qquad \times f_j(x,h) |\det g|^{s-n+k}\, dg\, dx\, d^\times h, \end{aligned}$$

and

$$\begin{aligned} &I_k(s, W_v, W_{v'}, \Phi) \\ &= \sum_j \int W_{w_{0,j}}\left(\begin{pmatrix} ga^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}, \begin{pmatrix} ga^{-1}(h) & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix}\right) \\ &\qquad\qquad \times f_j(x,h) |\det g|^{s-n+k}\, dg\, dx\, d^\times h. \end{aligned}$$

[1]There is a change of topology for the convergence in Proposition 2.4 in general, but in our special case considered here, the topologies involved are the same.

Make the change of variable $ga^{-1}(h) \to g$; we have the integrals

$$(7)\quad I_k(s, W_{v_l}, W_{v_l'}, \Phi_l) = \sum_j \int W_{w_{l,j}}\left(\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}\right) \times f_j(x,h)|\det g|^{s-n+k}|h|^{s+1-n+k}\, dg\, dx\, d^\times h$$

and

$$(8)\quad I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int W_{w_{0,j}}\left(\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}\right) \times f_j(x,h)|\det g|^{s-n+k}|h|^{s+1-n+k}\, dg\, dx\, d^\times h.$$

Now we will view $f_j(x,h)$ as Schwartz functions on $\mathbb{R}^{k+1}$ which vanish on $\mathbb{R}^k \times \{0\}$. Then let

$$e_{l,j} = \int \sigma\left(\begin{pmatrix} I_{k+1} & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-2}\end{pmatrix}, \begin{pmatrix} I_{k+1} & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-2}\end{pmatrix}\right) w_{l,j} f_j(y)\, dy$$

and

$$e_{0,j} = \int \sigma\left(\begin{pmatrix} I_{k+1} & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-2}\end{pmatrix}, \begin{pmatrix} I_{k+1} & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-k-2}\end{pmatrix}\right) w_{0,j} f_j(y)\, dy,$$

where $y = (x,h) \in \mathbb{R}^{k+1}$.

Thus it follows that, $e_{l,j} \to e_{0,j}$ for each j, and we have

$$(9)\quad I_k(s, W_{v_l}, W_{v_l'}, \Phi_l) = \sum_j \int W_{e_{l,j}}\left(\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}\right) \times |\det g|^{s-n+k}|h|^{s+1-n+k}\, dg\, dx\, d^\times h$$

and

$$(10)\quad I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int W_{e_{0,j}}\left(\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1}\end{pmatrix}\right) \times |\det g|^{s-n+k}|h|^{s+1-n+k}\, dg\, dx\, d^\times h.$$

As in [Jacquet 2009, Lemma 14.1],

$$f \to \int f\begin{pmatrix} g & 0 & 0 \\ x & h & 0 \\ 0 & 0 & I_{n-k-1} \end{pmatrix} dx|\det g|^{-1}\, d^{\times}h$$

gives an invariant measure on $N_{k+1}\backslash G_{k+1}$. Thus, we can rewrite these integrals as

$$(11)\quad I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \sum_j \int_{N_{k+1}\backslash G_{k+1}} W_{e_{l,j}}\left(\begin{pmatrix} g & 0 \\ 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & I_{n-k-1}\end{pmatrix}\right) \times |\det g|^{s+1-n+k}\, dg$$

and

$$(12)\quad I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int_{N_{k+1}\backslash G_{k+1}} W_{e_{0,j}}\left(\begin{pmatrix} g & 0 \\ 0 & I_{n-k-1}\end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & I_{n-k-1}\end{pmatrix}\right) \times |\det g|^{s+1-n+k}\, dg,$$

which are the same type integrals as (5) and (6) belonging to I_{k+1}.

So by induction, we may assume $k = n-1$ in the integrals (5) and (6); then integrals (7) and (8) now become

$$(13)\quad I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \sum_j \int W_{w_{l,j}}\left(\begin{pmatrix} g & 0 \\ x & h\end{pmatrix}, \begin{pmatrix} g & 0 \\ x & h\end{pmatrix}\right) f_j(x,h)|\det g|^{s-1}|h|^s\, dg\, dx\, d^{\times}h$$

and

$$(14)\quad I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int W_{w_{0,j}}\left(\begin{pmatrix} g & 0 \\ x & h\end{pmatrix}, \begin{pmatrix} g & 0 \\ x & h\end{pmatrix}\right) f_j(x,h)|\det g|^{s-1}|h|^s\, dg\, dx\, d^{\times}h.$$

Write

$$g' = \begin{pmatrix} g & 0 \\ x & h\end{pmatrix} \in G_n,$$

and view $f_j(x,h)$ as Schwartz functions on $\mathbb{R}^n$ which vanish on $\mathbb{R}^{n-1}\times\{0\}$; then the above integrals now become

$$I_k(s, W_{v_l}, W_{v'_l}, \Phi_l) = \sum_j \int W_{w_{l,j}}(g', g') f_j(\epsilon_n g')|\det g'|^s |\det g'|^{-1}\, dg'\, dx\, d^{\times}h$$

and

$$I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int W_{w_{0,j}}(g', g') f_j(\epsilon_n g') |\det g'|^s |\det g'|^{-1} \, dg' \, dx \, d^\times h.$$

Again, as in [Jacquet 2009, Lemma 14.1],

$$f \mapsto \int f(g') \, dx |\det g'|^{-1} \, d^\times h$$

gives an invariant measure on $N_n \setminus G_n$. We can rewrite the above integrals as

(15) $$I_k(s, W_{v_l}, W_{v_l'}, \Phi_l) = \sum_j \int_{G_n} W_{w_{l,j}}(g', g') f_j(\epsilon_n g') |\det g'|^s \, dg'$$

and

(16) $$I_k(s, W_v, W_{v'}, \Phi) = \sum_j \int_{G_n} W_{w_{0,j}}(g', g') f_j(\epsilon_n g') |\det g'|^s \, dg'.$$

It follows that

$$B_{s_0,k}(v_l, v_l', \Phi_l) = \sum_j B_{s_0,n}(w_{l,j}, f_j),$$

and similarly

$$B_{s_0,k}(v, v', \Phi) = \sum_j B_{s_0,n}(w_{0,j}, f_j).$$

Since $w_{l,j} \to w_{0,j}$ for each j, and the form $B_{s_0,n}$ is continuous, we conclude that $B_{s_0,k}$ is continuous. This completes the proof. □

Definition. We say a pole s_0 is an exceptional pole of type 1, with level m and depth $n-k$, if the corresponding $B_{s_0,k}$ is zero on $\mathcal{S}_k^{m+1}$, but not identically zero on $\mathcal{S}_k^m$. In this case, we also say s_0 is an exceptional pole for the integrals $I_k(s, W_v, W_{v'}, \Phi)$.

Remark. If s_0 is an exceptional pole of order m, then B_{s_0} defines a continuous linear form on $V \times V' \times E_k^m$ such that, for any $g \in G_k$,

$$B_{s_0,k}(g \cdot W, g \cdot W', g \cdot \Phi) = |\det g|^{-s_0+n-k} B_{s_0,k}(W, W', \Phi).$$

Definition. We say a complex number s_0 is an exceptional pole of type 2, with level m, for π and π', if there exists a continuous trilinear form

$$l : V \times V' \times E_n^m \to \mathbb{C}$$

such that for $g \in G_n$,

$$l(g \cdot W, g \cdot W', g \cdot \Phi_n) = |\det g|^{-s_0} l(W, W', \Phi_n).$$

Remark. It follows that an exceptional pole of type 1 with level m and depth 0 is also of type 2 with level m.

Next we want to relate the exceptional poles for the integrals I_k to the exceptional poles of type 2 for the components of $\pi^{(n-k)}$ and $\pi'^{(n-k)}$.

Lemma 4.3. *If $X = (X_{ij}) \in \mathfrak{n}_{k,n-k}$, then there exists a linear form P_X on $\mathbb{R}^k$ such that for any $v \in V$ we have*

$$W_{\pi(X)\cdot v}\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} = P_X(\epsilon_k g) W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix}.$$

Proof. First, it is easy to see that

$$\begin{aligned} W_{\pi(X)\cdot v}\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} &= \frac{d}{dt}\Big|_{t=0} W_v\left(\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix}\begin{pmatrix} I_k & tX \\ 0 & I_{n-k} \end{pmatrix}\right) \\ &= 2\pi\sqrt{-1}\sum_{j=1}^{k} g_{kj} X_{j,k+1} W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix}. \end{aligned}$$

So define a linear form $P_X(a_1, \ldots, a_k) = 2\pi\sqrt{-1}\sum_{j=1}^{k} X_{j,k+1} a_j$ on $\mathbb{R}^k$; then $P_X(\epsilon_k g) = 2\pi\sqrt{-1}\sum_{j=1}^{k} g_{kj} X_{j,k+1}$, which proves the lemma. □

Proposition 4.4. *Let s_0 be an exceptional pole of level m for the integrals I_k; then the continuous trilinear form $B_{s_0,k}$ defines a continuous trilinear form on $V/\overline{\mathfrak{n}V} \times V'/\overline{\mathfrak{n}V'} \times E_k^m$.*

Proof. It suffices to show that the form $B_{s_0,k}$ vanishes on $\overline{\mathfrak{n}V}$ and $\overline{\mathfrak{n}V'}$ when restricted to $\mathcal{S}_k^m$.

For any $W_{\pi(X)\cdot v}$, $X \in \mathfrak{n}$, any $W_{v'}$ and any $\Phi \in \mathcal{S}_k^m$, by Lemma 4.3 we have

$$W_{\pi(X)\cdot v}\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} = P_X(\epsilon_k g) W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

for some linear form P_X on $\mathbb{R}^k$.

It follows that

$$\begin{aligned} I_k(s, W_{\pi(X)\cdot v}, W_{v'}, \Phi) \\ &= \int W_{\pi(X)\cdot v}\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} W_{v'}\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} \Phi(\epsilon_k g)|\det g|^{s-k+n}\, dg \\ &= \int_{N_k\backslash G_k} W_v\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} W_{v'}\begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} \Psi(\epsilon_k g)|\det g|^{s-k+n}\, dg, \end{aligned}$$

where $\Psi_k(\epsilon_k g) = P_X(\epsilon_k g)\Phi(\epsilon_k g)$.

Since $\Phi \in \mathcal{S}_k^m$, thus $\Psi = P_X\Phi \in \mathcal{S}_k^{m+1}$. Note that s_0 is an exceptional pole with level m, so

$$B_{s_0,k}(W_{\pi(X)\cdot v}, W_{v'}, \Phi) = B_{s_0,k}(W_{\cdot v}, W_{v'}, \Psi_k) = 0.$$

Similarly, $B_{s_0,k}$ vanishes when $v' \in \overline{\mathfrak{n}V'}$. Thus the proposition follows. □

Theorem 4.5. *If s_0 is an exceptional pole of type 1 with level m and depth $n-k$, then s_0 is an exceptional pole of type 2 with level m for some components of $\pi^{(n-k)}$ and $\pi'^{(n-k)}$.*

Proof. Note that we have the decompositions

$$V/\overline{\mathfrak{n}V} = \bigoplus_i (\rho_i, A_i)\widehat{\otimes}(\sigma_i, B_i)$$

and

$$V'/\overline{\mathfrak{n}V'} = \bigoplus_i (\rho_i', A_i')\widehat{\otimes}(\sigma_i', B_i').$$

By Proposition 4.4, if s_0 is an exceptional pole of level m for I_k, $B_{s_0,k}$ defines a nontrivial continuous trilinear form on $V/\overline{\mathfrak{n}V} \times V'/\overline{\mathfrak{n}V'} \times E_k^m$. Thus it has to be nontrivial on some components

$$B_{s_0,k} : (\rho_i, A_i)\widehat{\otimes}(\sigma_i, B_i) \times (\rho_j', A_j')\widehat{\otimes}(\sigma_j', B_j') \times E_k^m \to \mathbb{C},$$

which implies it is also nontrivial on the subspace $A_i \otimes B_i \times A_i' \otimes B_i' \times E_k^m$.

Now fix $v_2 \in B_i$, $v_2' \in B_i'$, so that $B_{s_0,k}$ is nontrivial on $A_i \otimes v_2 \times A_i' \otimes v_2' \times E_k^m$. Then the restriction of $B_{s_0,k}$ to this subspace induces a nontrivial continuous trilinear form, still denoted as $B_{s_0,k}$, on $A_i \times A_i' \times E_k^m$, with

$$B_{s_0,k}(g \cdot v_1, g \cdot v_1', g.\Phi) = |\det g|^{-s_0+n-k} B_{s_0,k}(v_1, v_1', \Phi)$$

for any $v_1 \in A_i$, $v_1' \in A_i'$, $\Phi_i \in E_k^m$ and $g \in G_k$. Note that $|\det|^{(n-k)/2}\rho_i$ is a component for $\pi^{(n-k)}$, thus we have proved the theorem. □

5. Rankin–Selberg integrals: $G_n \times G_n$

Suppose a pole s_0 is not exceptional for the integrals I_n, and that we have the Laurent expansion

$$I_n(s, W, W', \Phi) = \frac{B_{s_0}(W, W', \Phi)}{(s-s_0)^d} + \cdots,$$

and B_{s_0} is continuous on $V \times V' \times E_n^m$ with the invariance property

$$B_{s_0,n}(g \cdot W, g \cdot W', g \cdot \Phi) = |\det g|^{-s_0} B_{s_0,n}(W, W', \Phi).$$

Since s_0 is not exceptional, for any integer m, we can find some $\Phi \in \mathcal{S}^m$ such that the form $B_{s_0,n}(W, W', \Phi)$ is nonzero for some choices of W and W'. Because of the continuity of $B_{s_0,n}$, we may further assume W and W' are both K_n-finite. By Iwasawa decomposition, we have

$$I_n(s, W, W', \Phi) = \int_{K_n}\int_{N_n\backslash P_n} W(pk)W'(pk)|\det p|^{s-1}\int_{\mathbb{R}^\times}\omega(a)\omega'(a)|a|^{ns}\Phi(\epsilon_n ak)\,d^\times a\,dp\,dk.$$

Take $\{W_i\}$ to be some base vectors in the K-span subspace of W, and we write $W(gk)=\sum_i f_i(k)W_i(g)$, where f_i are continuous functions on K. Similarly, write $W'(gk)=\sum_i f_i'(k)W_i'(g)$, where $\{W_j'\}$ are some base vectors of the K-span subspace of W', and f_i' are continuous functions on K. Now $I(s, W, W', \Phi)$ equals

$$\sum_{i,j}\int_{N_n\backslash P_n} W_i(p)W_j(p)|\det p|^{s-1}\int_{\mathbb{R}^\times}\omega(a)\omega'(a)|a|^{ns} \times\int_K f_i(k)f_j'(k)\Phi(\epsilon_n ak)\,dk\,d^\times a\,dp.$$

Lemma 5.1. *For any continuous function $f(k)$ on K, the function*

$$\Psi(a)=\int_K f(k)\Phi(\epsilon_n ak)\,dk$$

belongs to $\mathcal{S}^m(\mathbb{R})$ if Φ is in $\mathcal{S}^m(\mathbb{R}^n)$.

Proof. We will only check that $\Psi(a)$ vanishes at least to order m around 0; other verifications are routine and will be omitted. Since Φ vanishes at 0 at least to order m, by [Trèves 1967, Theorem 38.1] there exists a homogeneous polynomial $P(x_1,\dots,x_n)$ of degree m such that the Taylor expansion of Φ at 0 has the form

$$\Phi(x_1,\dots,x_n)=P(x_1,\dots,x_n)+\cdots.$$

Then

$$\Psi(a)=\int_K f(k)\Phi(\epsilon_n ak)dk=\int_K f(k)P(\epsilon_n ak)\,dk+\cdots = a^m\int_K f(k)P(\epsilon_n k)\,dk+\cdots.$$

This shows that $\Psi(a)$ vanishes at least to order m at 0, which finishes the proof. □

Lemma 5.2. *If $\Phi\in\mathcal{S}^m(\mathbb{R})$ for some $m>0$, then as a function of $s\in\mathbb{C}$, the function*

$$\int_0^\infty a^s\Phi(a)\,d^\times a$$

is holomorphic in the half-plane $\mathrm{Re}(s)>-m$.

Proof. Since Φ is a Schwartz function, the integral

$$\int_\epsilon^\infty a^s\Phi(a)\,d^\times a$$

is holomorphic in s, when ϵ is away from 0.

In a neighborhood of 0, when $\mathrm{Re}(s) > -m$ and $\Phi \in \mathcal{S}^m(\mathbb{R})$, the function $a^s\Phi(a)$ is continuous. Thus

$$\int_0^{\epsilon} a^s \Phi(a)\, d^{\times}a$$

is also holomorphic in s. □

By Lemma 5.1, as a function of a, the integral

$$\int_K f_i(k) f_j'(k) \Phi(\epsilon a k)\, dk$$

belongs to $\mathcal{S}_n^m(\mathbb{R})$, and by Lemma 5.2, when we choose m large enough, the function

$$\int_{\mathbb{R}^{\times}} \omega(a)\omega'(a)|a|^{ns} \int_K f_i(k) f_j'(k)\Phi(\epsilon a k)\, d^{\times}a\, dk$$

is holomorphic in the half-plane containing s_0. Hence the pole s_0 has to occur in the sum

$$\sum_{i,j} \int_{N_n \backslash P_n} W_i(p) W_j'(p) |\det p|^{s-1}\, dp,$$

and we may assume one of the terms

$$\int_{N_n \backslash P_n} W_i(p) W_j'(p) |\det p|^{s-1}\, dp$$

contains the pole s_0. But this integral descends to the integral

$$\int_{N_{n-1} \backslash G_{n-1}} W_i \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W_j' \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} |\det g|^{s-1}\, dg$$

on $N_{n-1} \setminus G_{n-1}$.

Each $W_v\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ can be written as a finite sum

$$\sum_i W_i \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \Phi_i(\epsilon_{n-1} g)$$

for some functions $W_i \in \mathcal{W}(\pi, \psi)$ and Schwartz functions Φ_i on $\mathbb{R}^{n-1}$. Thus the above integral becomes

$$\sum_i \int_{N_{n-1} \backslash G_{n-1}} W_i \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W' \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \Phi_i(\epsilon_{n-1} g) |\det g|^{s-1}\, dg,$$

which are integrals belonging to I_{n-1}. So we have the following corollary.

Proposition 5.3. *If a pole s_0 of I_n of order d is not exceptional of type* 1, *then it occurs as a pole of order d for the integrals I_{n-1}.*

In general, we have the following reduction result.

Proposition 5.4. *If a pole s_0 of I_k is not an exceptional pole for the integrals I_k, then it is a pole for I_{k-1}.*

Proof. By [Jacquet and Shalika 1990, Proposition 2], there exists a finite set of functions $\{\xi\}$ on $(\mathbb{R}^\times)^k$, which have the form $\xi(z_1, \dots, z_k) = \prod_{j=1}^k \chi_j(z_j)(\log|z_j|)^{n_j}$, where χ_j is a character on $\mathbb{R}^\times$, and Schwartz functions ϕ_ξ on $\mathbb{R}^k \times O(n)$, such that

$$W_v(\alpha x) = \sum_\xi \xi(a_1, \dots, a_k)\phi_\xi(a_1, \dots, a_k, x),$$

where $x \in O(n)$ and

$$\alpha = \mathrm{diag}(a_1 \cdots a_k, a_2 \cdots a_k, \dots, a_{k-1}a_k, a_k),$$

which will be viewed as

$$\mathrm{diag}(a_1 \cdots a_k, a_2 \cdots a_k, \dots, a_{k-1}a_k, a_k, 1, \dots, 1) \in G_n.$$

Since ϕ_ξ is a Schwartz function, for each x, it has a Taylor expansion around 0,

$$\phi_\xi(a_1, \dots, a_k, x) = f(x)P_\xi(a_1, \dots, a_k) + \cdots,$$

where $f(x)$ is some continuous function of x, and P_ξ denotes the sum of leading coefficients in the Taylor expansion, which is a polynomial in $a_1, \dots, a_k$.

It follows that, around 0, we can write

$$W_v(\alpha x) = \sum_\xi \{f(x)\xi(a_1, \dots, a_k)P_\xi(a_1, \dots, a_k) + \cdots\}. \tag{17}$$

Similarly, around 0, we have

$$W_{v'}(\alpha x) = \sum_{\xi'} \{f'(x)\xi'(a_1, \dots, a_k)P_{\xi'}(a_1, \dots, a_k) + \cdots\}. \tag{18}$$

By Iwasawa decomposition, we have

$$I_k = \int W_v\begin{pmatrix} pax & 0 \\ 0 & I_{n-k}\end{pmatrix} W_{v'}\begin{pmatrix} pax & 0 \\ 0 & I_{n-k}\end{pmatrix} \times \Phi(\epsilon_k ax)|\det p|^{s-n+k-1}|a|^{k(s-n+k)}\,dp\,dx\,d^\times a,$$

with $p \in N_k \setminus P_k$, where P_k is the mirabolic subgroup in G_k, $x \in O(k)$, $a \in \mathbb{R}^\times$.

Note that $N_k \setminus P_k = N_{k-1} \setminus G_{k-1}$, so we can write $pax = n_{k-1}\alpha yx$ for some $n_{k-1} \in N_{k-1}$,

$$\alpha = \mathrm{diag}(a_1 \cdots a_{k-1}a, \dots, a_{k-1}a, a, 1, \dots, 1),$$

and $y \in O(k-1)$.

Thus by (17), around 0 we have

$$W_v(pax) = \psi(n_k) \sum_{\xi} \{ f(yx)\xi(a_1, \dots, a_{k-1}, a) P_\xi(a_1, \dots, a_{k-1}, a) + \cdots \}$$

and

$$W_{v'}(pax) = \psi^{-1}(n_k) \sum_{\xi'} \{ f'(yx)\xi'(a_1, \dots, a_{k-1}, a) P_{\xi'}(a_1, \dots, a_{k-1}, a) + \cdots \}.$$

Note that the poles of I_k are caused by the integration around 0, and in a neighborhood of 0, the integral is

$$\sum_{\xi,\xi'} \int f(yx) f'(yx)\, dy\, dx \int (\xi P_\xi \xi' P_{\xi'})(a_1, \dots, a_{k-1}, a) \\ \times \Phi(\epsilon_k ax) |a|^{k(s-n+k)} |a_1|^{c_1} \cdots |a_{k-1}|^{c_k}\, d^\times a \cdots d^\times a_{k-1} + \cdots ,$$

where $c_1, \dots, c_{k-1}$ are some complex numbers depending on s.

First, since s_0 is a pole for this integral and $O(k)$, $O(k-1)$ are compact, it follows that this pole occurs as a pole for the integral with respect to the variables $a_1, \dots, a_{k_1}, a$, and the integrals with respect to x, y are nonzero.

Since s_0 is not an exceptional pole, we can choose the Schwartz function Φ so that the integral on a in the above expression is holomorphic in a region containing s_0. Thus the pole is caused by the integration with respect to the variables $a_1, \dots, a_{k-1}$. This implies that the integral

$$\int W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k+1} \end{pmatrix} W_v' \begin{pmatrix} g & 0 \\ 0 & I_{n-k+1} \end{pmatrix} |\det g|^{s-n+k-1}\, dg$$

has the pole s_0. This integral belongs to the integrals I_{k-1}, and the proposition follows. □

Corollary 5.5. *Any pole of the Rankin–Selberg integrals I_n for π and π' is an exceptional pole of type 2 for some components of $\pi^{(k)}$ and $\pi'^{(k)}$, $0 \le k < n$.*

For the other direction, suppose σ and σ' are a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$ respectively.

Proposition 5.6. *Any Rankin–Selberg integral of σ and σ' can be written as a sum of Rankin–Selberg integrals of π and π'.*

Proof. For any $W_{v_1} \in \mathcal{W}(\sigma, \psi)$, $W_{v_1'} \in \mathcal{W}(\sigma', \psi^{-1})$, and $\Phi \in \mathcal{S}_{n-k}$, we have the Rankin–Selberg integral for σ and σ':

$$I(s, W_{v_1}, W_{v_1'}, \Phi) = \int_{N_{n-k}\backslash G_{n-k}} W_{v_1}(g) W_{v_1'}(g) \Phi(\epsilon_{n-k} g) |\det g|^s\, dg.$$

By Corollary 3.5, there exists some $W_v \in \mathcal{W}(\pi, \psi)$ such that

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = W_{v_1}(g)\Phi(\epsilon_{n-k} g)|\det g|^{k/2}.$$

Thus, the above integral is

$$I(s, W_{v_1}, W_{v_1'}, \Phi) = \int_{N_{n-k}\backslash G_{n-k}} W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W_{v_1'}(g)|\det g|^{s-k/2}\, dg,$$

which can be written as

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = \sum_j W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} \Phi_j(\epsilon_{n-k} g)$$

with $W_j \in \mathcal{W}(\pi, \psi)$, and Schwartz functions Φ_j on $\mathbb{R}^{n-k}$. So we have

$$I(s, W_{v_1}, W_{v_1'}, \Phi) = \sum_j \int W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W_{v_1'}(g)\Phi_j(\epsilon_{n-k} g)|\det g|^{s-k/2}\, dg.$$

Using Corollary 3.5 again, we have

$$I(s, W_{v_1}, W_{v_1'}, \Phi) = \sum_j \int W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W_j' \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} |\det g|^{s-k}\, dg$$

for some $W_j' \in \mathcal{W}(\pi', \psi^{-1})$. Then by [Jacquet 2009, Lemma 14.1], each integral on the right side can be written as a Rankin–Selberg integral for π and π'. Thus the proposition follows. □

Corollary 5.7. *Any exceptional pole of type* 1 *of depth* 0 *for Rankin–Selberg integrals of σ and σ' is a pole of the Rankin–Selberg integrals I_n for π and π'.*

Summarizing the above, we obtain the main result of this section.

Theorem 5.8. *Let π and π' be irreducible generic Casselman–Wallach representations of G_n in general position. Then any pole of the Rankin–Selberg integrals for π and π' is an exceptional pole of type* 2 *for a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$,* $0 \le k \le n-1$. *On the other hand, any exceptional pole of type* 1 *of depth* 0 *for a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$,* $0 \le k \le n-1$, *is a pole of the Rankin–Selberg integrals of π and π'.*

6. Case $G_n \times G_m$, $m < n$

This section is devoted to the case $G_n \times G_m$, $m < n$, using the same ideas and techniques as in the previous section. We will indicate the necessary changes and omit details.

Now suppose π and π' are generic irreducible Casselman–Wallach representations of G_n and G_m in general position, respectively. Let $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\pi',\psi^{-1})$ be their Whittaker models. The family of integrals is given by

$$I(s,W,W') = \int_{N_m\backslash G_m} W\begin{pmatrix} g & 0 \\ 0 & I_{n-m}\end{pmatrix} W'(g)|\det g|^{s-\frac{n-m}{2}}\,dg,$$

and for $1 \le j \le n-m-1$

$$I^j(s,W,W') \\ = \int_{M(m\times j,\mathbb{R})}\int_{N_m\backslash G_m} W\begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{n-m-j}\end{pmatrix} W'(g)|\det g|^{s-\frac{n-m}{2}}\,dg\,dX,$$

with $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\pi',\psi^{-1})$. We will only consider the integrals $I(s,W,W')$ since they have the same poles with the same multiplicities as $I^j(s,W,W')$ for each j.

For each $1 \le k \le m$, let Φ be a Schwartz function on $\mathbb{R}^k$, and introduce

$$I_k(s,W,W',\Phi) \\ = \int_{N_k\backslash G_k} W\begin{pmatrix} g & 0 \\ 0 & I_{n-k}\end{pmatrix} W'\begin{pmatrix} g & 0 \\ 0 & I_{m-k}\end{pmatrix} \Phi(\epsilon_k g)|\det g|^{s-\frac{n+m}{2}+k}\,dg.$$

By [Jacquet 2009, Lemma 14.1], the integrals I_k belong to the family I_m, which implies that they are convergent when $\mathrm{Re}(s)$ is large, and have meromorphic continuations to the whole plane.

At a pole s_0 for $I_k(s,W,W',\Phi)$, we have an expansion

$$I_k(s,W,W',\Phi) = \frac{B_{s_0,k}(W,W',\Phi)}{(s-s_0)^d} + \cdots,$$

where $B_{s_0,k}(W,W',\Phi)$ is a trilinear form on $V \times V' \times \mathcal{S}_k$ satisfying the following invariance property: for any $g \in G_k$,

$$B_{s_0,k}(g\cdot W, g\cdot W', g\cdot\Phi) = |\det g|^{-s_0+\frac{n+m}{2}-k} B_{s_0,k}(W,W',\Phi).$$

Similar to Proposition 4.2, we can show $B_{s_0,k}$ is continuous.

Definition. We say a pole s_0 is an exceptional pole of type 1, with level l and depth $m-k$, if the corresponding $B_{s_0,k}$ is zero on $\mathcal{S}_k^{l+1}$, but not identically zero on $\mathcal{S}_k^{l}$. In this case, we also say s_0 is an exceptional pole for the integrals $I_k(s,W_v,W_{v'},\Phi)$.

Definition. We say a complex number s_0 is an exceptional pole of type 2, with level l, for π and π', if there exists a continuous trilinear form

$$l : V \times V' \times E_k^l \to \mathbb{C}$$

such that

$$l(g \cdot W, g \cdot W', g \cdot \Phi) = |\det g|^{-s_0 + \frac{n-m}{2}} l(W, W', \Phi).$$

Remark. If s_0 is an exceptional pole of type 1 with level l and depth 0, then s_0 is also an exceptional pole of type 2 with level l for π and π'.

Along the same lines, we have the following theorem.

Theorem 6.1. *If s_0 is an exceptional pole of type 1 with level l and depth $m-k$, then s_0 is an exceptional pole of type 2 with level l for some components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$.*

The main reduction step is the following analog to Proposition 5.4, with essentially the same proof.

Proposition 6.2. *If a pole s_0 of I_k is not an exceptional pole for these integrals, then it is a pole of I_{k-1}.*

As a corollary, we have:

Corollary 6.3. *Any pole of the Rankin–Selberg integrals I_m for π and π' is an exceptional pole of type 2 for some components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $0 \le k < m$.*

A converse statement is also true.

Proposition 6.4. *Any exceptional pole of type 1 of depth 0 for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$ is a pole of the Rankin–Selberg integrals I_n for π and π'.*

The main result of this section is the following.

Theorem 6.5. *Let π and π' be irreducible generic Casselman–Wallach representations of G_n and G_m in general position. Then any pole of the Rankin–Selberg integrals for π and π' is an exceptional pole of type 2 for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $1 \le k \le m$. On the other hand, any exceptional pole of type 1 of depth 0 for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $1 \le k \le m$, is a pole of the Rankin–Selberg integrals of π and π'.*

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