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PRESCRIBING THE BOUNDARY GEODESIC CURVATURE ON A COMPACT SCALAR-FLAT RIEMANN SURFACE VIA A FLOW METHOD

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We study the problem of prescribing the boundary geodesic curvature on a compact scalar-flat Riemann surface. We use the negative gradient flow method. We prove the global existence and the convergence of the flow as time goes to infinity under sufficient conditions on the prescribed function.

1. Introduction

Let (M, g_0) be a compact Riemann surface with boundary equipped with a scalar-flat metric g_0. Given a function f on ∂M, does there exist a scalar-flat metric g which is pointwise conformal to g_0, i.e., a $g = e^{2u} g_0$ such that f is the geodesic curvature of ∂M under the metric g? This problem is equivalent to solving the boundary value problem

$$\begin{cases} \Delta_{g_0} u = 0 & \text{in } M, \\ \partial_n u + k_0 = f e^u & \text{on } \partial M, \end{cases} \tag{1-1}$$

where ∂_n is the outward-pointing normal derivative operator with respect to g_0 and k_0 is the geodesic curvature of ∂M under the metric g_0. We may assume without loss of generality that k_0 is a constant since there always exists such a metric in the conformal class of g_0. Let us first derive necessary conditions for (1-1) to have a solution. By integrating (1-1), we obtain

$$\int_{\partial M} f e^u \, ds_{g_0} = k_0 \mathscr{L}(\partial M), \tag{1-2}$$

where $\mathscr{L}(\partial M) = \mathscr{L}(\partial M, g_0)$ is the arc length of ∂M, and

$$\int_{\partial M} f \, ds_{g_0} = -\int_M |\nabla_{g_0} u|^2 e^{-u} \, dA_{g_0} + k_0 \int_{\partial M} e^{-u} \, ds_{g_0}. \tag{1-3}$$

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Depending on the sign of k_0, with the help of (1-2) and (1-3), we conclude that the geodesic curvature candidate f should satisfy the conditions

$$\begin{cases} \text{(i)}\ \max_{x\in\partial M} f(x) > 0 & \text{when } k_0 > 0,\\ \text{(ii)}\ \max_{x\in\partial M} f(x) > 0 \ \text{ and } \displaystyle\int_{\partial M} f\, ds_{g_0} < 0 & \text{when } k_0 = 0,\\ \text{(iii)}\ \displaystyle\int_{\partial M} f(x)\, ds_{g_0} < 0 & \text{when } k_0 < 0. \end{cases} \tag{1-4}$$

Remark 1.1. Notice that for $k_0 = 0$, it is not hard to see that if (1-1) has a solution, then either $f \equiv 0$ or $\max_{x\in\partial M} f(x) > 0$ and $\int_{\partial M} f\, ds_{g_0} < 0$. However, we do not include the case $f \equiv 0$ in (ii). This is because (1-1) becomes trivial in that case. Hence, we only consider the case $f \not\equiv 0$ in this paper.

To the author's knowledge, there are very few papers concerned with sufficient conditions on f for the existence of a solution to problem (1-1). Cherrier [1984] studied the regularity issue for (1-1), and for $k_0 = 0$ he showed that if condition (ii) in (1-4) holds then the equation has a nontrivial solution. This implies that condition (ii) is necessary and sufficient for (1-1) to have a nontrivial solution. Kazdan and Warner [1975] found the similar condition in the prescribed Gaussian curvature problem on Riemann surfaces without boundary. For $k_0 < 0$, Ho [2011] proved that (1-1) has a solution provided the prescribed function f is strictly negative by using a flow method. He considered the evolution problem

$$\begin{cases} \dfrac{\partial g}{\partial t} = (\alpha(t) f - k) g & \text{in } \partial M,\\ K = 0 & \text{on } M, \end{cases} \tag{1-5}$$

where $\alpha(t) = 2\pi\chi(M)/\big(\int_{\partial M} f\, ds_g\big)$, and k and K are the geodesic curvature and Gaussian curvature of the time metric $g(t)$. Such a flow has been used in many works; see for instance [Brendle 2002a; 2003; Struwe 2005; Malchiodi and Struwe 2006; Chen and Xu 2012] and the literature therein. When $k_0 > 0$ and $M = D$ (the unit disc in the plane), Liu and Huang [2005] showed that there exists a solution of (1-1) if f possesses some kind of symmetries, while for a more general smooth function, Chang and Liu [1996] obtained an existence result through the Morse theory method.

In this paper, we will use the negative gradient flow introduced in [Baird et al. 2004; 2006] to investigate the problem of prescribing the geodesic curvature when the candidate curvature function f is not necessarily of constant sign. This gradient flow will be different from (1-5). To be precise, it is introduced in the following way. Motivated by [Chang and Liu 1996], we consider the functional

$$J(u) = \int_{\partial M} \frac{1}{2}\partial_n u \cdot u + k_0 u\, ds_{g_0}$$

on the Sobolev space $H := \{u \in H^1(M) : \Delta_{g_0} u = 0 \text{ in } M\}$ under the constraint

$$u \in X := \left\{ u \in H : L(u) := \int_{\partial M} e^u f \, ds_{g_0} = k_0 \mathscr{L}(\partial M) \right\}.$$

Note that the set X is not empty, thanks to the conditions in (1-4). From the Moser–Trudinger inequality with boundary [Li and Liu 2005, Theorem A]

$$\int_{\partial M} e^u \, ds_{g_0} \le C \exp\left\{ \frac{1}{4\pi} \int_M |\nabla u|^2 \, dA_{g_0} + \frac{1}{\mathscr{L}(\partial M)} \int_{\partial M} u \, ds_{g_0} \right\}, \tag{1-6}$$

where the constant C depends on M and g_0, it follows that L is well-defined on H. Since H is restricted to the set of harmonic functions, we may assume that H is equipped with the scalar product

$$\langle u, v \rangle = \int_{\partial M} \partial_n u \cdot v + u \cdot v \, ds_{g_0},$$

for $u, v \in H$. Hence the associated norm on H is given by

$$\|u\|^2 = \int_{\partial M} \partial_n u \cdot u + u^2 \, ds_{g_0}.$$

The functionals J and L are analytic, and their gradients are given by

$$\langle \nabla J(u), \phi \rangle = \int_{\partial M} (\partial_n u + k_0) \phi \, ds_{g_0} \quad \text{for all } \phi \in H, \tag{1-7}$$

which implies that

$$\nabla J(u) = (\partial_n + I)^{-1} (\partial_n u + k_0),$$

and

$$\langle \nabla L(u), \phi \rangle = \int_{\partial M} e^u f \phi \, ds_{g_0} \quad \text{for all } \phi \in H, \tag{1-8}$$

which implies that

$$\nabla L(u) = (\partial_n + I)^{-1} (e^u f),$$

where I is the identity transformation.

Since $\nabla L(u) \ne 0$ for all $u \in X$ by the hypothesis (1-4), the set X is a regular hypersurface of H. A unit normal field at a point u in X is given by

$$N(u) = \frac{\nabla L(u)}{\|\nabla L(u)\|}. \tag{1-9}$$

The gradient of the functional J with respect to the hypersurface X is thus defined by

$$\nabla^X J(u) = \nabla J(u) - \langle \nabla J(u), N(u) \rangle N(u). \tag{1-10}$$

Then the negative gradient flow of J with respect to the hypersurface X is

$$\begin{cases} \partial_t u = -\nabla^X J(u), \\ u(0) = u_0 \in X. \end{cases} \tag{1-11}$$

If the flow (1-11) exists for all time and converges at infinity, then the limit function u_∞ produces a solution of (1-1) and so defines a metric of geodesic curvature f. In this paper, we will show the long-time existence of a solution of (1-11) and its convergence as $t \to \infty$ under sufficient conditions on the prescribed function f. We also describe the asymptotic behavior of the flow at infinity.

2. Statement of the results

We will first show the long-time existence of the solution to (1-11).

Theorem 2.1. *Let (M, g_0) be a compact scalar-flat Riemann surface with boundary and let $f \in C^0(\partial M)$ satisfy the appropriate condition in (1-4). Then for any $u_0 \in X$, there exists a unique global solution $u \in C^\infty([0, \infty[, H)$ of (1-11). In addition, the energy identity*

$$\int_0^t \|\partial_\tau u(\tau)\|^2 \, d\tau + J(u(t)) = J(u_0), \tag{2-1}$$

holds for all $t \geq 0$.

We will study the convergence of the global solution depending on the sign of k_0. When $k_0 > 0$, we only consider the case $M = D$, the unit disc. Let $u_0 \in X$ and $u : [0, \infty[\to X$ be the solution of (1-11) obtained in Theorem 2.1.

Theorem 2.2. *Suppose that $k_0 = 0$. Let $f \in C^0(\partial M)$ satisfy the conditions*

$$\max_{x \in \partial M} f(x) > 0 \quad \text{and} \quad \int_{\partial M} f(x) \, ds_{g_0} < 0;$$

then u converges in H as $t \to \infty$ to a function $u_\infty \in H \cap C^\alpha(\partial M)$ with the property that the function $v_\infty = u_\infty + \lambda$ is a solution of

$$\begin{cases} \Delta_{g_0} v_\infty = 0 & \text{in } M, \\ \partial_n v_\infty + k_0 = f e^{v_\infty} & \text{on } \partial M, \end{cases}$$

for some constant λ. Moreover, there exist two constants $\beta, \delta > 0$ such that

$$\|u(t) - u_\infty\| \leq \beta (1 + t)^{-\delta}$$

for all $t \geq 0$.

Corollary 2.3. *Suppose that $k_0 = 0$. Let $f \in C^0(\partial M)$ satisfy the conditions $\max_{x \in \partial M} f(x) > 0$ and $\int_{\partial M} f(x) \, ds_{g_0} < 0$; then there exists a metric conformal to g_0 with associated geodesic curvature f.*

For the negative case, we have:

Theorem 2.4. *Suppose that* $k_0 < 0$. *Let* $f \in C^0(\partial M)$ *satisfy the condition* $\int_{\partial M} f(x)\, ds_{g_0} < 0$; *then there exists a positive constant* $\bar{C}$ *depending only on the function* $f^-(x) = \max(-f(x), 0)$, g_0 *and* M, *such that if* u_0 *satisfies*

$$e^{\xi \|u_0\|^2} \max_{x\in\partial M} f(x) \le \bar{C}, \tag{2-2}$$

where $\xi > 1$ *is a constant depending only on* g_0 *and* M, *then* u *converges in* H *as* $t \to \infty$ *to a solution* $u_\infty \in H \cap C^\alpha(\partial M)$ *of* (1-1). *Moreover, there exist two constants* $\beta, \delta > 0$ *such that*

$$\|u(t) - u_\infty\| \le \beta(1+t)^{-\delta}$$

for all $t \ge 0$. *In particular, if* $f \le 0$, *then* u *converges in* H *as* $t \to \infty$ *to a solution* $u_\infty \in H \cap C^\alpha(\partial M)$ *of* (1-1) *and* $\|u(t) - u_\infty\| \le \beta(1+t)^{-\delta}$ *for all* $t \ge 0$.

Corollary 2.5. *Suppose that* $k_0 < 0$. *Let* $f \in C^0(\partial M)$ *satisfy the condition* $\int_{\partial M} f(x)\, ds_{g_0} < 0$. *There exists a positive constant* $\bar{C}$ *depending only on the function* f^-, g_0 *and* M, *such that if* f *satisfies*

$$\max_{x\in\partial M} f(x) \le \bar{C},$$

then (1-1) *admits a solution* $u \in H \cap C^\alpha(\partial M)$. *In particular, if* $f \le 0$, *then* (1-1) *admits a solution* $u \in H \cap C^\alpha(\partial M)$.

We now consider the positive case. In this case, we assume that $M = D$, the unit disc. Suppose that the function f is invariant under a group G of isometries of $\partial D = S^1$ (f is a G-invariant function). Then we can establish the convergence.

Recall that a function on S^1 is said to be G-invariant if it satisfies

$$f(\sigma x) = f(x) \quad \text{for all } x \in S^1 \text{ and } \sigma \in G.$$

Let Σ denote the set of fixed points of G, that is,

$$\Sigma = \{x \in S^1 : \sigma x = x \text{ for all } \sigma \in G\};$$

we have the following result:

Theorem 2.6. *Let* $f \in C^0(\partial M)$ *be a function invariant under a group* G *of isometries of* S^1 *with* $\max_{x\in S^1} f(x) > 0$, *and let* $u_0 \in X$ *also be invariant under* G. *If either*

(i) $\Sigma = \varnothing$, *or*

(ii) $\max_{p\in\Sigma} f(p) \le e^{-J(u_0)/2\pi}$,

then u *converges in* H *as* $t \to \infty$ *to a* G*-invariant solution* $u_\infty \in H \cap C^\alpha(S^1)$ *of* (1-1). *Moreover, there exist two constants* $\beta, \delta > 0$ *such that*

$$\|u(t) - u_\infty\| \le \beta(1+t)^{-\delta}$$

for all $t \ge 0$.

Let $a \in D$ and denote by Φ_a the Möbius transformation given by

$$\Phi_a = \frac{z+a}{\bar{a}z+1}.$$

For a suitable choice of the initial data u_0, we have the following:

Corollary 2.7. *Let $f \in C^0(S^1)$ be a function with $\max_{x\in S^1} f(x) > 0$ which is invariant under a group of isometries of S^1. If either*

(i) $\Sigma = \varnothing$, *or*

(ii) *there exists $a_0 \in \Sigma$ such that*

$$\max_{p\in\Sigma} f(p) \le \max\left(0, \fint_{S^1} f \circ \Phi_{a_0}\, ds_{g_0}\right), \tag{2-3}$$

then (1-6) admits a G-invariant solution $u \in H \cap C^\alpha(S^1)$. In particular, if

$$\max_{p\in\Sigma} f(p) \le \max\left(0, \fint_{S^1} f\, ds_{g_0}\right), \tag{2-4}$$

then (1-6) admits a G-invariant solution $u \in H \cap C^\alpha(S^1)$.

3. Long-time existence

In this section, we first show that the solution of the flow (1-11) is well-defined on $[0, \infty[$. Then we show the convergence of the flow under the assumption of uniform boundedness of the conformal factor u. To do so, we will first prove:

Lemma 3.1. *The linear mapping $(\partial_n + I)^{-1} : L^2(\partial M) \to H$ is compact.*

Proof. Let S be a bounded set in $L^2(\partial M)$. Then there exists a sequence $(\phi_i)_i \subset S$ that weakly converges to a function ϕ_∞ in $L^2(\partial M)$. Define $u_i = (\partial_n + I)^{-1}\phi_i$ and $u_\infty = (\partial_n + I)^{-1}\phi_\infty$. We then have

$$(\partial_n + I)(u_i - u_\infty) = \phi_i - \phi_\infty.$$

Hence, u_i weakly converges to u_∞ in H. By the compact embedding $H \hookrightarrow L^2(\partial M)$, u_i strongly converges to u_∞ in $L^2(\partial M)$. Now, a simple calculation, Hölder's inequality and boundedness of ϕ_i and ϕ_∞ in $L^2(\partial M)$ yield

$$\begin{aligned}\|u_i - u_\infty\|^2 &= \int_{\partial M} (\partial_n + I)(u_i - u_\infty) \cdot (u_i - u_\infty)\, ds_{g_0} \\ &= \int_{\partial M} (\phi_i - \phi_\infty) \cdot (u_i - u_\infty)\, ds_{g_0} \le C\|u_i - u_\infty\|_{L^2(\partial M)} \to 0.\end{aligned}$$

Hence, $(\partial_n + I)^{-1}\phi_i$ strongly converges to $(\partial_n + I)^{-1}\phi_\infty$ in H. This implies that $(\partial_n + I)^{-1}(S)$ is relatively compact in H. Therefore, the linear mapping $(\partial_n + I)^{-1}$ is compact. □

Proof of Theorem 2.1. Since the functionals J and L are C^∞ on H and $\nabla L(u) \neq 0$ for all $u \in H$, it follows that $\nabla^X J$ is C^∞ on H, and the short-time existence follows from the classical Cauchy–Lipschitz theorem. We now extend this short-time solution to $[0, \infty[$.

Since $\nabla J(u) = -(\partial_n + I)^{-1}(u - k_0) + u$ and $(\partial_n + I)^{-1}$ is a bounded linear map by Lemma 3.1, it follows that

$$\|\partial_t u\| = \|\nabla^X J(u)\| \leq \|\nabla J(u)\| \leq C_0\|u\| + C_0.$$

From the inequality above, we deduce that, for all $t < T$,

$$\|u(t)\| \leq (\|u_0\| + 1)e^{C_0 T},$$

which ensures that the solution u is globally defined on $[0, \infty[$.

Now, using (1-9)–(1-11) we can obtain

$$\begin{aligned} \frac{dJ(u)}{dt} &= \langle \nabla J(u), \partial_t u\rangle = \langle \nabla J(u), -\nabla^X J(u)\rangle \\ &= -\|\nabla^X J(u)\|^2 = -\|\partial_t u\|^2. \end{aligned} \tag{3-1}$$

Integrating the equality above from 0 to t yields the energy identity (2-1), which completes the proof. □

Next, we wish to establish convergence at infinity under the assumption of uniform boundedness of the global solution u in H. For this we will prove:

Lemma 3.2. *Let $u : [0, \infty[\to H$ be the solution of* (1-11). *If u satisfies*

$$\|u(t)\| \leq C \tag{3-2}$$

for all $t > 0$, where C is a positive constant independent of t, then $u(t)$ converges in H as $t \to \infty$ to a function $u_\infty \in H \cap C^\alpha(\partial M)$ $(0 < \alpha < 1)$. If $k_0 \neq 0$, then u_∞ is a solution of (1-1). *If $k_0 = 0$, then the function $v_\infty = u_\infty + \lambda$ is a solution of*

$$e^{-v_\infty}(\partial_n v_\infty + k_0) = f,$$

for some constant λ. Moreover, there exist two constants $\beta, \delta > 0$ such that

$$\|u(t) - u_\infty\| \leq \beta(1+t)^{-\delta}$$

for all $t \geq 0$.

Proof. The energy identity (2-1) and (3-2) imply that

$$\int_0^\infty \|\partial_t u\|^2\, dt \leq J(u_0) + c \sup_t \|u(t)\| \leq J(u_0) + C_1,$$

where $C_1 > 0$ is a constant depending on M, g_0 and the constant C in (3-2). Thus, there exists a sequence $t_k \to \infty$ such that

$$\|\partial_t u(t_k)\| = \|\nabla^X J(u(t_k))\| \to 0. \tag{3-3}$$

From (3-2), we have $\|u(t_k)\| \leq C$; hence there exist a function $u_\infty \in H$ and a subsequence of t_k (again denoted by t_k), such that

$$\begin{cases} u(t_k) \to u_\infty & \text{weakly in } H, \\ u(t_k) \to u_\infty & \text{strongly in } L^2(\partial M). \end{cases} \tag{3-4}$$

It follows from (3-2) and (1-6) that for all $p \in \mathrm{R}$, there exists a positive constant $C(p)$ such that

$$\int_{\partial M} e^{pu(t_k)}\, ds_{g_0} \leq C(p). \tag{3-5}$$

A straightforward computation from (3-4) and (3-5) shows that for all $p \geq 1$,

$$\lim_{k\to\infty} \|fe^{u(t_k)} - fe^{u_\infty}\|_{L^p(\partial M)} = 0. \tag{3-6}$$

Since $u(t_k) \in X$, which means that

$$\int_{\partial M} fe^{u(t_k)}\, ds_{g_0} = k_0 \mathscr{L}(\partial M),$$

we conclude from (3-6) that $u_\infty \in X$.

Next, we show that $\nabla^X J(u_\infty) = 0$. Recall that

$$\nabla^X J(u(t)) = \nabla J(u(t)) - \big\langle \nabla J(u(t)), \nabla L(u(t))\big\rangle \frac{\nabla L(u(t))}{\|\nabla L(u(t))\|^2} \tag{3-7}$$

with

$$\nabla L(u(t)) = (\partial_n + I)^{-1}(fe^{u(t)}) \tag{3-8}$$

and

$$\nabla J(u(t)) = -[(\partial_n + I)^{-1} - I]u + (\partial_n + I)^{-1} k_0 \tag{3-9}$$

Since $(\partial_n + I)^{-1} : L^2(\partial M) \to H$ is compact by Lemma 3.1, we deduce from (3-4) and (3-6), as well as from (3-7)–(3-9) above, that $\nabla^X J(u(t_k))$ weakly converges in H to $\nabla^X J(u_\infty)$. It follows from (3-3) that $\nabla^X J(u_\infty) = 0$. Therefore,

$$(\partial_n + I)^{-1}(\partial_n u_\infty + k_0) = \eta(u_\infty)(\partial_n + I)^{-1}(fe^{u_\infty}),$$

where $\eta(u_\infty)$ is a constant. Hence,

$$\partial_n u_\infty + k_0 = \eta(u_\infty) fe^{u_\infty}. \tag{3-10}$$

From (3-6) , (3-10) and [Brendle 2002b, Lemma 3.2], it follows that $\|\nabla u_\infty\|_{L^p(\partial M)} \le C$. Since $u_\infty \in H$, we have, by the Sobolev embedding theorem, $u_\infty \in L^p$ for all $2 \le p < \infty$. Hence $u_\infty \in W^{1,p}(\partial M)$ with $2 \le p < \infty$. By the Sobolev embedding theorem again, we obtain $u_\infty \in H \cap C^\alpha(\partial M)$ for all $0 < \alpha < 1$.

Suppose that $k_0 \neq 0$; then since $u_\infty \in X$, by integrating (3-10), we deduce that $\eta(u_\infty) = 1$ and u_∞ is a solution of (1-1). On the other hand, for $k_0 = 0$, if $\eta(u_\infty) = 0$, then $\partial_n u_\infty = 0$ and hence u_∞ is a constant, contradicting (1-4) and the fact proved above that $u_\infty \in X$; if $\eta(u_\infty) < 0$, then $v_\infty = u_\infty + \log(-\eta(u_\infty))$ is a solution of $e^{-v_\infty}\partial_n v_\infty = -f$. However, by integrating this equation, one has $\int_M f\, ds_{g_0} = \int_M e^{-v_\infty}|\nabla v_\infty|^2\, ds_{g_0} > 0$, contradicting (1-4). Hence, the only possibility is $\eta(u_\infty) > 0$, and for this case, one can see that $v_\infty = u_\infty + \log \eta(u_\infty)$ is a solution of $e^{-v_\infty}(\partial_n v_\infty + k_0) = f$.

In order to prove the asymptotic behavior of the flow, we need to show that

$$\lim_{k\to\infty} \|u(t_k) - u_\infty\| = 0. \tag{3-11}$$

Since $\nabla^X J(u_\infty) = 0$, it follows from (3-7) and (3-9) that

$$\|u(t_k) - u_\infty\| \le \|\nabla^X J(u(t_k))\| + \|(\partial_n + I)^{-1}(u(t_k) - u_\infty)\| \\ + C(\|(\partial_n + I)^{-1}(f e^{u_k} - f e^{u_\infty})\| + |\eta(u(t_k)) - \eta(u_\infty)|).$$

At this point, (3-11) follows from (3-3), Lemma 3.1 and (3-6).

Finally, we will end the proof of the lemma by showing that there exist two constants $\beta, \delta > 0$ such that for all $t \ge 0$,

$$\|u(t) - u_\infty\| \le \beta(1+t)^{-\delta}. \tag{3-12}$$

Before doing this, we will cite a version of the Łojasiewicz–Simon inequality:

Lemma 3.3 [Baird et al. 2004]. *Let X be an analytic manifold modeled on a Hilbert space $\mathcal{H}$ and suppose that $\mathcal{J} : X \to R$ is an analytic function on a neighborhood of a point $\tilde u \in X$ satisfying*:

(i) $\nabla\mathcal{J}(\tilde u) = 0$.

(ii) $\nabla^2\mathcal{J}(\tilde u) : T_{\tilde u}X \to T_{\tilde u}X$ *is a Fredholm operator.*

Here, $\nabla\mathcal{J}$ denotes the gradient in X of $\mathcal{J}$ and we consider the second derivative $\nabla^2\mathcal{J}(\tilde u)$ as a linear map $\nabla^2\mathcal{J}(\tilde u) : T_{\tilde u}X \to T_{\tilde u}X$ by using the inner product on $T_{\tilde u}X$. Then there exist constants $\mu > 0$ and $0 < \theta < \frac{1}{2}$ such that if $u \in B(\tilde u, \mu)$ (the geodesic ball of radius μ centered on $\tilde u$), we have

$$\|\nabla\mathcal{J}(u)\| \ge |\mathcal{J}(u) - \mathcal{J}(\tilde u)|^{1-\theta}.$$

Now we apply Lemma 3.3 to the functional J in a neighborhood of the point u_∞.

Since L is an analytic function on H, X is an analytic manifold. Moreover, $J : X \subset H \to R$ is analytic and $\nabla^X J(u_\infty) = 0$. Let $\Pi_{u_\infty} : H \to T_{u_\infty}X$ be the projection onto $T_{u_\infty}X$. From (3-7)–(3-9), it follows that for all $v \in T_{u_\infty}X$,

$$\nabla^2 J(u_\infty)(v) = (I + \Pi_{u_\infty} A)(v),$$

where $A : H \to H$ is defined by

$$A(v) = -(\partial_n + I)^{-1}(v) - \left\langle \nabla J(u_\infty), \frac{\nabla L(u_\infty)}{\|\nabla L(u_\infty)\|^2} \right\rangle (\partial_n + I)^{-1}(f e^{u_\infty} v) + \left\langle \nabla J(u_\infty), \frac{\nabla L(u_\infty)}{\|\nabla L(u_\infty)\|^2} \right\rangle \left\langle v, \frac{\nabla \|\nabla L(u_\infty)\|}{\|\nabla L(u_\infty)\|} \right\rangle \nabla L(u_\infty).$$

It is not difficult to check that A is a compact operator since $(\partial_n + I)^{-1}$ is a compact operator. Since Π_{u_∞} is a continuous map, it follows that $\Pi_{u_\infty} A$ is also compact. Hence, we conclude that $\nabla^2 J(u_\infty)$ is a Fredholm operator. It follows from Lemma 3.3 that there exist constants $\mu > 0$ and $0 < \theta < \frac{1}{2}$ such that if $\|u(t) - u_\infty\| < \mu$, then

$$\|\nabla^X J(u(t))\| \geq (J(u(t)) - J(u_\infty))^{1-\theta}. \tag{3-13}$$

We may assume that $J(u(t) - J(u_\infty)) > 0$ for all $t \geq 0$. Otherwise, if there exists $\tilde{t} \geq 0$ such that $J(u(\tilde{t})) = J(u_\infty)$, then since J is nonincreasing and the solution of (1-1) is unique, it follows that $u(t) \equiv u_\infty$ for all $t \geq \tilde{t}$. Therefore the solution is stationary and the estimate (3-12) is trivial. In view of (3-1), we have

$$-\frac{dJ(u(t))}{dt} = \|\nabla^X J(u(t))\| \|\partial_t u(t)\|. \tag{3-14}$$

From (3-11), we deduce that for all $\epsilon > 0$, there exists $N > 0$ such that

$$\|u(t_n) - u_\infty\| \leq \frac{\epsilon}{2} \quad \text{and} \quad \frac{1}{\theta}(J(u(t_n)) - J(u_\infty))^\theta \leq \frac{\epsilon}{2} \qquad \text{for all } n \geq N.$$

Let $\epsilon = \frac{1}{2}\mu$ and $t^* = \sup\{t \geq t_N : \|u(\tau) - u_\infty\| < \mu \text{ for all } \tau \in [t_N, t]\}$. Suppose that $t^* < \infty$. It follows from (3-13) and (3-14) that

$$-\frac{d}{dt}[(J(u(t)) - J(u_\infty))^\theta] \geq \theta \|\partial_t u(t)\| \tag{3-15}$$

for all $t \in [t_N, t^*]$. Integrating (3-15) and using the monotonicity of J yields

$$\|u(t^*) - u(t_N)\| \leq \int_{t_N}^{t^*} \|\partial_\tau u(\tau)\| \, d\tau \leq \frac{1}{\theta}(J(u(t_N)) - J(u_\infty))^\theta < \frac{\epsilon}{2}.$$

Recalling that $\epsilon = \frac{1}{2}\mu$, we have

$$\|u(t^*) - u_\infty\| \leq \|u(t^*) - u(t_N)\| + \|u(t_N) - u_\infty\| < \frac{\mu}{2},$$

which contradicts the definition of t^*. Hence $t^* = \infty$. This implies that estimate (3-13) holds for all $t \geq t_N$. Now, set $h(t) = J(u(t)) - J(u_\infty)$. Then from (3-13) and (3-14) again, it follows that

$$-\frac{dh(t)}{dt} = \|\nabla^X J(u(t))\|^2 \geq h^{2(1-\theta)}(t),$$

or, equivalently,

$$\frac{d}{dt} h^{2\theta-1}(t) \geq (1-2\theta),$$

for all $t \geq t_N$. Since $0 < \theta < \frac{1}{2}$, we can deduce that

$$h(t) \leq (h^{2\theta-1}(t_N) + (1-2\theta)(t-t_N))^{1/(2\theta-1)} \leq Ct^{-\delta'}, \tag{3-16}$$

where $\delta' = 1/(1-2\theta)$ and C are positive constants. We fix $t > t_N$ and integrate (3-15) from t to t_n (with n sufficiently large) to obtain, by estimate (3-16),

$$\|u(t) - u(t_n)\| \leq \int_t^{t_n} \|\partial_\tau u(\tau)\| \, d\tau \leq \frac{1}{\theta}(J(u(t)) - J(u_\infty))^\theta = \frac{1}{\theta} h^\theta(t) \leq \frac{1}{\theta} C t^{-\theta\delta'}.$$

By letting $n \to \infty$, we obtain

$$\|u(t) - u_\infty\| \leq \frac{1}{\theta} C t^{-\theta\delta'}$$

for all $t > t_N$. However, for $t \leq t_N$, $\|u(t) - u_\infty\|$ is bounded, so there exist two positive constants $\delta = \theta\delta'$ and β with

$$\|u(t) - u_\infty\| \leq \beta(1+t)^{-\delta}$$

for all $t \geq 0$. □

4. Convergence

In this section, we will apply Lemma 3.2 to obtain convergence. We thus only need to prove uniform boundedness in H of the global solution $u : [0, \infty) \to H$ in Theorem 2.1.

Proof of Theorem 2.2. Suppose that $k_0 = 0$. Writing $\mathbf{1}$ for the constant function, in view of (1-7) and (1-8) we have for $u \in X$ that

$$\langle \nabla J(u), \mathbf{1} \rangle = \int_{\partial M} \partial_n u \cdot \mathbf{1} \, ds_{g_0} = 0 \quad \text{and} \quad \langle \nabla L(u), \mathbf{1} \rangle = \int_{\partial M} e^u f \cdot \mathbf{1} \, ds_{g_0} = 0.$$

From this it follows that

$$0 = \langle \partial_t u, \mathbf{1} \rangle = \int_{\partial M} (\partial_n \partial_t u + \partial_t u) \mathbf{1} \, ds_{g_0},$$

which implies that

$$\partial_t \int_{\partial M} u(t)\, ds_{g_0} = 0.$$

Hence,

$$\int_{\partial M} u(t,\cdot)\, ds_{g_0} = \int_{\partial M} u_0\, ds_{g_0}. \tag{4-1}$$

From now on, we set $\bar{u} = (1/\mathscr{L}(\partial M)) \int_{\partial M} u\, ds_{g_0}$. Since $k_0 = 0$, the energy identity (2-1) yields

$$J(u) = \frac{1}{2}\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} \le J(u_0). \tag{4-2}$$

In order to show that $\|u(t)\| \le C$, it remains to bound $\int_{\partial M} u^2\, ds_{g_0}$. By Poincaré's inequality, we have

$$\|u-\bar{u}\|_{L^2}^2 \le \lambda_1^{-1} \int_{\partial M} \partial_n u \cdot u\, ds_{g_0}, \tag{4-3}$$

where λ_1 is the first nonzero Steklov eigenvalue. From (4-3) and (4-1), it follows that

$$\begin{aligned}\int_{\partial M} u^2\, ds_{g_0} &\le \lambda_1^{-1} \int_{\partial M} \partial_n u \cdot u\, ds_{g_0} + \mathscr{L}(\partial M)\bar{u}^2\\ &= \lambda_1^{-1} \int_{\partial M} \partial_n u \cdot u\, ds_{g_0} + \mathscr{L}(\partial M)\overline{u_0}^2.\end{aligned}$$

Hence, we deduce from (4-2) that $\int_{\partial M} u^2\, ds_{g_0}$ is bounded. □

Proof of Theorem 2.4. Suppose that $k_0 < 0$; without loss of generality we assume that $k_0 = -1$. We first prove that the solution u satisfies a nonconcentration lemma:

Lemma 4.1. *Let K be a measurable subset of ∂M with $\mathscr{L}(K) > 0$. Then there exist a constant $\alpha > 1$ depending on M and g_0 and a constant $C_K > 1$ depending on M, g_0 and $\mathscr{L}(K)$ such that*

$$\int_{\partial M} e^u\, ds_{g_0} \le C_K e^{\alpha \|u_0\|^2} \max\left(\left(\int_K e^u\, ds_{g_0}\right)^{\alpha}, 1\right).$$

Proof. Step 1. We claim that there exists a positive constant C depending on M and g_0 such that, for any measurable subset K of M with $\mathscr{L}(K) > 0$, we have

$$\int_{\partial M} u\, ds_{g_0} \le |J(u_0)| + \frac{C}{\mathscr{L}(K)} + \frac{2\sqrt{2}\mathscr{L}(\partial M)}{\mathscr{L}(K)} \max\left(\int_K u\, ds_{g_0}, 0\right). \tag{4-4}$$

Fix $t > 0$. Suppose that $\int_{\partial M} u\, ds_{g_0} > 0$, otherwise estimate (4-4) is trivial. By the energy identity (2-1), we have

$$\frac{1}{2}\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} \le J(u_0) + \int_{\partial M} u\, ds_{g_0}. \tag{4-5}$$

It follows from (4-5) and (4-3) that

$$\int_{\partial M} u^2\,ds_{g_0} \le \frac{2}{\lambda_1} J(u_0) + \frac{2}{\lambda_1}\int_{\partial M} u\,ds_{g_0} + \frac{1}{\mathcal{L}(\partial M)}\Big(\int_{\partial M} u\,ds_{g_0}\Big)^2. \tag{4-6}$$

Now, we consider the following two cases:

Case (i): $\int_K u(t)\,ds_{g_0} \le 0$. Then

$$\Big(\int_{\partial M} u\,ds_{g_0}\Big)^2 \le \Big(\int_{K^c} u\,ds_{g_0}\Big)^2 \le \mathcal{L}(K^c)\int_{\partial M} u^2\,ds_{g_0},$$

where K^c denotes the compliment of K in ∂M. Plugging this inequality into (4-6) yields

$$\frac{\mathcal{L}(K)}{\mathcal{L}(\partial M)}\int_{\partial M} u^2\,ds_{g_0} \le \frac{2}{\lambda_1} J(u_0) + \frac{2}{\lambda_1}\int_{\partial M} u\,ds_{g_0}. \tag{4-7}$$

On the other hand, by Young's inequality, we have

$$\Big|\int_{\partial M} u\,ds_{g_0}\Big| \le \epsilon\int_{\partial M} u^2\,ds_{g_0} + (4\epsilon)^{-1}\mathcal{L}(\partial M).$$

Taking $\epsilon = \lambda_1\mathcal{L}(K)/(4\mathcal{L}(\partial M))$ and substituting into (4-7) gives

$$\int_{\partial M} u^2\,ds_{g_0} \le \frac{4\mathcal{L}(\partial M)}{\lambda_1\mathcal{L}(K)} J(u_0) + \frac{2\mathcal{L}^3(\partial M)}{\lambda_1^2\mathcal{L}^2(K)}. \tag{4-8}$$

Since

$$\Big(\int_{\partial M} u\,ds_{g_0}\Big)^2 \le \mathcal{L}(\partial M)\int_{\partial M} u^2\,ds_{g_0},$$

it follows from (4-8) that

$$\begin{aligned}\Big(\int_{\partial M} u\,ds_{g_0}\Big)^2 &\le \frac{4\mathcal{L}^2(\partial M)}{\lambda_1\mathcal{L}(K)} J(u_0) + \frac{2\mathcal{L}^4(\partial M)}{\lambda_1^2\mathcal{L}^2(K)}\\ &\le |J(u_0)|^2 + \frac{4\mathcal{L}^4(\partial M)}{\lambda_1^2\mathcal{L}^2(K)} + \frac{2\mathcal{L}^4(\partial M)}{\lambda_1^2\mathcal{L}^2(K)}.\end{aligned}$$

Therefore,

$$\int_{\partial M} u\,ds_{g_0} \le |J(u_0)| + \frac{C}{\mathcal{L}(K)},$$

where C is a constant depending on $\mathcal{L}(\partial M)$ and g_0. This establishes case (i).

Case (ii): $\int_K u\,ds_{g_0} > 0$. Rewrite (4-6) as

$$\begin{aligned}\int_{\partial M} u^2\,ds_{g_0} \le \frac{2}{\lambda_1} J(u_0) &+ \frac{2}{\lambda_1}\int_{\partial M} u\,ds_{g_0}\\ &+ \frac{1}{\mathcal{L}(\partial M)}\Big\{\Big(\int_K u\,ds_{g_0}\Big)^2 + \Big(\int_{K^c} u\,ds_{g_0}\Big)^2 + 2\Big(\int_K u\,ds_{g_0}\Big)\Big(\int_{K^c} u\,ds_{g_0}\Big)\Big\}.\end{aligned}$$

By Young's inequality and the fact that

$$\left(\int_{K^c} u\,ds_{g_0}\right)^2 \le \mathscr{L}(K^c)\int_{\partial M} u^2\,ds_{g_0},$$

we have

$$2\left(\int_K u\,ds_{g_0}\right)\left(\int_{K^c} u\,ds_{g_0}\right) \le \frac{2\mathscr{L}(K^c)}{\mathscr{L}(K)}\left(\int_K u\,ds_{g_0}\right)^2 + \frac{\mathscr{L}(K)}{2}\int_{\partial M} u^2\,ds_{g_0}.$$

Hence we arrive at

$$\frac{\mathscr{L}(K)}{2\mathscr{L}(\partial M)}\int_{\partial M} u^2\,ds_{g_0} \le \frac{2}{\lambda_1}J(u_0) + \frac{2}{\lambda_1}\int_{\partial M} u\,ds_{g_0} + \frac{2}{\mathscr{L}(K)}\left(\int_K u\,ds_{g_0}\right)^2.$$

By Young's inequality again, we obtain

$$\left|\int_{\partial M} u\,ds_{g_0}\right| \le \frac{\lambda_1\mathscr{L}(K)}{8\mathscr{L}(\partial M)}\int_{\partial M} u^2\,ds_{g_0} + \frac{2\mathscr{L}^2(\partial M)}{\lambda_1\mathscr{L}(K)}.$$

Therefore,

$$\int_{\partial M} u^2\,ds_{g_0} \le \frac{8\mathscr{L}(\partial M)}{\lambda_1\mathscr{L}(K)}J(u_0) + \frac{16\mathscr{L}^3(\partial M)}{\lambda_1^2\mathscr{L}^2(K)} + \frac{8\mathscr{L}(\partial M)}{\mathscr{L}^2(K)}\left(\int_K u\,ds_{g_0}\right)^2.$$

Since

$$\left(\int_{\partial M} u\,ds_{g_0}\right)^2 \le \mathscr{L}(\partial M)\left(\int_{\partial M} u^2\,ds_{g_0}\right),$$

it follows that

$$\left(\int_{\partial M} u\,ds_{g_0}\right)^2 \le \frac{8\mathscr{L}^2(\partial M)}{\lambda_1\mathscr{L}(K)}|J(u_0)| + \frac{16\mathscr{L}^4(\partial M)}{\lambda_1^2\mathscr{L}^2(K)} + \frac{8\mathscr{L}^2(\partial M)}{\mathscr{L}^2(K)}\left(\int_K u\,ds_{g_0}\right)^2,$$

which implies that

$$\int_{\partial M} u\,ds_{g_0} \le |J(u_0)| + \frac{C}{\mathscr{L}(K)} + \frac{2\sqrt{2}\mathscr{L}(\partial M)}{\mathscr{L}(K)}\int_K u\,ds_{g_0},$$

for a constant C depending on M and g_0. This establishes (4-4).

Step 2. We are in position to establish the lemma using the result in Step 1.

The energy identity (2-1) yields

$$\begin{aligned}\frac{1}{2}\int_{\partial M} \partial_n u\cdot u\,ds_{g_0} &\le J(u_0) + \int_{\partial M} u\,ds_{g_0}\\ &= J(u_0) + \mathscr{L}(\partial M)\bar{u} + \int_{\partial M}(u-\bar{u})\,ds_{g_0}.\end{aligned}$$

From the Young and Poincaré inequalities, it follows that

$$\begin{aligned}\frac{1}{2}\int_{\partial M}\partial_n u\cdot u\,ds_{g_0} &\le J(u_0)+\mathcal{L}(\partial M)\bar{u}+\frac{1}{4\epsilon}\mathcal{L}(\partial M)+\epsilon\int_{\partial M}(u-\bar{u})^2\,ds_{g_0}\\ &\le J(u_0)+\mathcal{L}(\partial M)\bar{u}+\frac{1}{4\epsilon}\mathcal{L}(\partial M)+\frac{\epsilon}{\lambda_1}\int_{\partial M}\partial_n u\cdot u\,ds_{g_0}.\end{aligned}$$

Taking $\epsilon=\frac{1}{4}\lambda_1$ gives

$$\int_{\partial M}\partial_n u\cdot u\,ds_{g_0}\le 4J(u_0)+4\mathcal{L}(\partial M)\bar{u}+\frac{4}{\lambda_1}\mathcal{L}(\partial M).$$

Using the inequality (1-6), we deduce that

$$\int_{\partial M}e^u\,ds_{g_0}\le C\exp\left\{\frac{J(u_0)}{\pi}+\frac{\mathcal{L}(\partial M)}{\pi\lambda_1}+\left(\frac{1}{\mathcal{L}(\partial M)}+\frac{1}{\pi}\right)\int_{\partial M}u\,ds_{g_0}\right\}. \tag{4-9}$$

Notice that we have

$$\begin{aligned}J(u_0)&=\int_{\partial M}\left(\frac{1}{2}\partial_n u_0-1\right)u_0\,ds_{g_0}\\ &\le\int_{\partial M}\partial_n u_0\cdot u_0\,ds_{g_0}+\int_{\partial M}u_0^2\,ds_{g_0}+\mathcal{L}(\partial M)\\ &=\|u_0\|^2+\mathcal{L}(\partial M).\end{aligned}$$

Plugging this inequality into (4-9) yields

$$\int_{\partial M}e^u\,ds_{g_0}\le C\exp\left\{\frac{\|u_0\|^2}{\pi}+B\int_{\partial M}u\,ds_{g_0}\right\},$$

where B and C are positive constants depending on M and g_0.

It follows from (4-4) that

$$\int_{\partial M}e^u\,ds_{g_0}\le C'_K\exp\left\{A_1\|u_0\|^2+\frac{B_1}{\mathcal{L}(K)}\max\left(\int_K u\,ds_{g_0},0\right)\right\},$$

where A_1, B_1 depend on M and g_0 and C'_K is a positive constant depending on M, $\mathcal{L}(K)$ and g_0. Moreover, we set $\alpha=\max(A_1,B_1)+1$. Then we have

$$\int_{\partial M}e^u\,ds_{g_0}\le C'_K\exp\left\{\alpha\|u_0\|^2+\frac{\alpha}{\mathcal{L}(K)}\max\left(\int_K u\,ds_{g_0},0\right)\right\}. \tag{4-10}$$

By Jensen's inequality, we have

$$\exp\left\{\frac{1}{\mathcal{L}(K)}\int_K u\,ds_{g_0}\right\}\le\frac{1}{\mathcal{L}(K)}\int_K e^u\,ds_{g_0}.$$

Hence

$$\exp\left\{\frac{\alpha}{\mathcal{L}(K)}\int_K u\,ds_{g_0}\right\}\le\left(\frac{1}{\mathcal{L}(K)}\int_K e^u\,ds_{g_0}\right)^{\alpha}.$$

But

$$\exp\left\{\frac{\alpha}{\mathcal{L}(K)}\max\left(\int_K u\,ds_{g_0},0\right)\right\}=\max\left(\exp\left\{\frac{\alpha}{\mathcal{L}(K)}\int_K u\,ds_{g_0}\right\},1\right)$$
$$\leq \max\left(\left(\frac{1}{\mathcal{L}(K)}\int_K e^u\,ds_{g_0}\right)^{\alpha},1\right).$$

Inequality (4-10) thus implies that

$$\int_{\partial M} e^u\,ds_{g_0}\leq C'_K e^{\alpha\|u_0\|^2}\max\left(\left(\frac{1}{\mathcal{L}(K)}\right)^{\alpha},1\right)\times\max\left(\left(\int_K e^u\,ds_{g_0}\right)^{\alpha},1\right)$$
$$\leq C_K e^{\alpha\|u_0\|^2}\max\left(\left(\int_K e^u\,ds_{g_0}\right)^{\alpha},1\right),$$

where C_K is a constant depending on $\mathcal{L}(K)$, M and g_0, which we can suppose to be greater than 1. This completes the proof. □

The estimate of Lemma 4.1 will allow us to uniformly bound $\int_{\partial M} e^u\,ds_{g_0}$. Let

$$f^+=\max(f,0)\quad\text{and}\quad K=\left\{x\in\partial M : f(x)\leq\tfrac{1}{2}\min_{x\in\partial M} f(x)\right\}.$$

Notice that we have $u_0\in X$. Then

$$\mathcal{L}(\partial M)=\int_{\partial M} -fe^{u_0}\,ds_{g_0}=\int_{\partial M} f^- e^{u_0}\,ds_{g_0}-\int_{\partial M} f^+ e^{u_0}\,ds_{g_0}$$

implies that

$$\frac{\mathcal{L}(\partial M)}{-\min_{x\in\partial M} f(x)}\leq\int_{\partial M} e^{u_0}\,ds_{g_0}. \tag{4-11}$$

However,

$$\int_{\partial M} u_0\,ds_{g_0}\leq\int_{\partial M} u_0^2\,ds_{g_0}+\mathcal{L}(\partial M),$$

which, together with inequality (1-6), implies that

$$\int_{\partial M} e^{u_0}\,ds_{g_0}\leq C_1\exp\left\{C_1\left(\int_{\partial M}\partial_n u_0\cdot u_0\,ds_{g_0}+\int_{\partial M} u_0^2\,ds_{g_0}\right)\right\} \tag{4-12}$$
$$=C_1 e^{C_1\|u_0\|^2},$$

where C_1, which we may assume to be greater than 1, is a constant depending on M and g_0. Hence,

$$\frac{\mathcal{L}(\partial M)}{-\min_{x\in\partial M} f(x)}\leq C_1 e^{C_1\|u_0\|^2}. \tag{4-13}$$

Now set $\gamma = C_K(8C_1)^\alpha e^{(C_1+1)\alpha\|u_0\|^2}$ ($C_K > 1$ and $\alpha > 1$ are constants in Lemma 4.1). Suppose that condition (2-2) of Theorem 2.4,

$$e^{\xi\|u_0\|^2}\max_{x\in\partial M} f(x) \le \overline{C},$$

holds, with $\overline{C} = -\min_{x\in\partial M} f(x)/(8^\alpha C_K C_1^{\alpha-1})$ and $\xi = \alpha(C_1+1) - C_1$. We wish to show that

$$\int_{\partial M} e^{u(t)}\,ds_{g_0} \le 2\gamma \quad \text{for all } t \ge 0. \tag{4-14}$$

Let

$$I = \left\{ t \ge 0 : \int_{\partial M} e^{u(\tau)}\,ds_{g_0} \le 2\gamma \text{ for all } \tau \in [0, t] \right\}.$$

From (4-12), it follows that $0 \in I$. Let $T = \sup I$. Suppose $T < \infty$. Then by continuity of the map $t \to \int_{\partial M} e^{u(t)}\,ds_{g_0}$, we have

$$\int_{\partial M} e^{u(T)}\,ds_{g_0} = 2\gamma. \tag{4-15}$$

We consider two cases:

Case (i): $$\int_{\partial M} f^+ e^{u(T)}\,ds_{g_0} \le \frac{1}{2}\int_{\partial M} f^- e^{u(T)}\,ds_{g_0}.$$

Using the fact that $u(T) \in X$, we get

$$\int_{\partial M} f^- e^{u(T)}\,ds_{g_0} \le -2\int_{\partial M} f e^{u(T)}\,ds_{g_0} = 2\mathscr{L}(\partial M). \tag{4-16}$$

Since $f^-(x) \ge \frac{1}{2}(-\min_{x\in\partial M} f(x))$ for all $x \in K$, it follows from (4-16) and (4-11) that

$$\int_K e^{u(T)}\,ds_{g_0} \le \frac{4\mathscr{L}(\partial M)}{-\min_{x\in\partial M} f(x)} \le 4C_1 e^{C_1\|u_0\|^2}.$$

We thus deduce from Lemma 4.1 that

$$\begin{aligned}
\int_{\partial M} e^{u(T)}\,ds_{g_0} &\le C_K e^{\alpha\|u_0\|^2}\max\left(\left(\int_K e^{u(T)}\,ds_{g_0}\right)^\alpha, 1\right)\\
&\le C_K e^{\alpha\|u_0\|^2}\max((2C_1)^\alpha e^{\alpha C_1\|u_0\|^2}, 1)\\
&= C_K(2C_1)^\alpha e^{(C_1+1)\alpha\|u_0\|^2} < \gamma,
\end{aligned}$$

which contradicts (4-15).

Case (ii): $$\int_{\partial M} f^+ e^{u(T)}\, ds_{g_0} > \frac{1}{2}\int_{\partial M} f^- e^{u(T)}\, ds_{g_0}.$$

Since $f^-(x) \geq \frac{1}{2}(-\min_{x\in\partial M} f(x))$ for all $x \in K$, it follows from (4-15) that

$$\begin{aligned} -\frac{\min_{x\in\partial M} f(x)}{2}\int_K e^{u(T)}\, ds_{g_0} &\leq \int_{\partial M} f^- e^{u(T)}\, ds_{g_0} \\ &\leq 2\int_{\partial M} f^+ e^{u(T)}\, ds_{g_0} \leq 4\gamma \max_{x\in\partial M} f(x). \end{aligned}$$

Then condition (2-2) of Theorem 2.4 implies that

$$\int_K e^{u(T)}\, ds_{g_0} \leq 8\frac{\gamma \max_{x\in\partial M} f(x)}{-\min_{x\in\partial M} f(x)} \leq 8C_1 e^{C_1\|u_0\|^2}.$$

As before, by Lemma 4.1, we have

$$\int_{\partial M} e^{u(T)}\, ds_{g_0} \leq \gamma,$$

which contradicts (4-15) again. We thus conclude that (4-14) holds.

Now from Jensen's inequality, (4-14) implies that

$$\bar{u} = \frac{1}{\mathcal{L}(\partial M)}\int_{\partial M} u(t)\, ds_{g_0} \leq C, \tag{4-17}$$

where C is a constant depending on M, g_0, f and u_0. The energy identity gives

$$\frac{1}{2}\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} - \int_{\partial M} (u-\bar{u})\, ds_{g_0} - \mathcal{L}(\partial M)\bar{u} \leq J(u_0). \tag{4-18}$$

On the other hand, Young's inequality gives

$$\left|\int_{\partial M} (u-\bar{u})\, ds_{g_0}\right| \leq \epsilon\|u-\bar{u}\|_{L^2}^2 + \frac{1}{4\epsilon}\mathcal{L}(\partial M). \tag{4-19}$$

Setting $\epsilon = \frac{1}{4}\lambda_1$ and using Poincaré's inequality, we deduce from (4-18) and (4-19) that

$$\frac{1}{4}\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} - \mathcal{L}(\partial M)\bar{u} \leq J(u_0) + \frac{1}{\lambda_1}\mathcal{L}(\partial M). \tag{4-20}$$

Using (4-17), we have

$$\int_{\partial M} \partial_n u \cdot u\, ds_{g_0} \leq C.$$

To show that $\|u(t)\| \leq C$, it remains to bound $\int_{\partial M} u^2\, ds_{g_0}$. From (4-20), it follows that

$$-\mathcal{L}(\partial M)\bar{u} \leq C;$$

hence, we deduce that $\bar{u} \geq C$. Combining this with (4-17) yields

$$|\bar{u}| \leq C.$$

Now Poincaré's inequality implies that

$$\|u-\bar{u}\|_{L^2}^2 \leq \frac{1}{\lambda_1}\int_{\partial M} \partial_n u \cdot u \, ds_{g_0} \leq C.$$

Thus, $\int_{\partial M} u^2 \, ds_{g_0} \leq C$. □

Proofs of Theorem 2.6 and Corollary 2.7. Suppose that $k_0 > 0$. In this case, we consider $M = D$, the unit disc. Then $k_0 = 1$. Hence, if $u \in X$, we have $\int_{\partial M} f e^u \, ds_{g_0} = 2\pi$.

Proof of Theorem 2.6. Let $v_a = u \circ \Phi_a + \log|\Phi_a'|$, where Φ_a is the Möbius transformation. From [Chang and Liu 1996, Theorem 2.1] and (2-1), it follows that

$$J(v_a) = J(u) \leq J(u_0), \tag{4-21}$$

and since $u \in X$, we have

$$\int_{S^1} f \circ \Phi_a e^{v_a} \, ds_{g_0} = \int_{S^1} f e^u \, ds_{g_0} = 2\pi. \tag{4-22}$$

From (4-22), we deduce that

$$\int_{S^1} e^{v_a} \, ds_{g_0} \geq \frac{2\pi}{\max_{x\in S^1} f(x)}. \tag{4-23}$$

It is well-known that for all $t > 0$, there exists $a(t) \in D$ such that

$$\int_{S^1} x_i e^{v_{a(t)}} \, ds_{g_0} = 0 \quad \text{for } i = 1, 2. \tag{4-24}$$

Set $v(t) = v_{a(t)}$ and $\Phi(t) = \Phi_{a(t)}$. From now on, we assume that C is a constant only depending on u_0 and $\sup_{x\in S^1} f(x)$. In view of (4-23) and (4-24), it follows from the Osgood–Phillips–Sarnak inequality (see [Osgood et al. 1988]) that

$$C \leq \fint_{S^1} e^{v(t)} \, ds_{g_0} \leq \exp\left\{\fint_{S^1} \left(\frac{1}{4}\partial_n v(t) + 1\right) v(t) \, ds_{g_0}\right\}. \tag{4-25}$$

It follows from (4-21) and (4-25) that

$$\frac{1}{2}\int_{S^1} \partial_n v(t) \cdot v(t) \, ds_{g_0} + \int_{S^1} v(t) \, ds_{g_0} \leq C, \tag{4-26}$$

and

$$\frac{1}{4}\int_{S^1} \partial_n v(t) \cdot v(t) \, ds_{g_0} + \int_{S^1} v(t) \, ds_{g_0} \geq C. \tag{4-27}$$

By taking the difference between (4-26) and (4-27), we obtain

$$\int_{S^1} \partial_n v(t) \cdot v(t)\, ds_{g_0} \le C. \tag{4-28}$$

Now, combining (4-26) and (4-27) yields

$$\left| \int_{S^1} v(t)\, ds_{g_0} \right| \le C. \tag{4-29}$$

Therefore, by the Lebedev–Milin inequality (see [Chang and Liu 1996, (1.12)]), we deduce from (4-28) and (4-29) that for all $p > 1$,

$$\int_{S^1} e^{|pv(t)|}\, ds_{g_0} \le C(p). \tag{4-30}$$

It follows from (4-30) that

$$\int_{S^1} v^2(t)\, ds_{g_0} \le C,$$

which, together with (4-28), implies that

$$\|v(t)\| \le C. \tag{4-31}$$

Next, we wish to prove that u is uniformly bounded in H. To do so, we first establish the following lemma.

Lemma 4.2. *Either*:

(i) *there exists a constant C such that $\|u(t)\| \le C$; or*,

(ii) *there exists a sequence $t_n \to \infty$ and a point $a_\infty \in S^1$ such that for all $r > 0$*,

$$\lim_{n\to\infty} \int_{\mathcal{S}(a_\infty, r)} f e^{u(t_n)}\, ds_{g_0} = 2\pi, \tag{4-32}$$

where $\mathcal{S}(a_\infty, r)$ is an arc in S^1 centered at a_∞ and with radius r. Moreover, for all $q \in S^1 \backslash \{a_\infty\}$ and all $0 < r < \operatorname{dist}(q, a_\infty)$, we have

$$\lim_{n\to\infty} \int_{\mathcal{S}(q,r)} f e^{u(t_n)}\, ds_{g_0} = 0.$$

Proof. There are two possibilities:

Case (i): $\limsup_{t\to\infty} |a(t)| < 1$. Then we have for all $t \ge 0$ that $0 < C_1 \le |\Phi'| \le C_2$. Hence, it follows from (4-31) that

$$\int_{S^1} |u(t)|\, ds_{g_0} \le C. \tag{4-33}$$

Combining (4-33) with the energy identity (2-1) yields

$$\int_{S^1} \partial_n u(t) \cdot u(t)\, ds_{g_0} \le C. \tag{4-34}$$

Hence, using Poincaré's inequality, we deduce from (4-33) and (4-34) that

$$\|u(t)\| \le C.$$

Case (ii): there exist a sequence $t_n \to \infty$ and $a_\infty \in S^1$ such that $a(t_n) \to a_\infty$. From the estimate (4-31), it follows that there exist a subsequence of t_n, still denoted by t_n, and a function $v_\infty \in H$, such that

$$\begin{cases} v(t_n) \to v_\infty & \text{weakly in } H, \\ v(t_n) \to v_\infty & \text{strongly in } L^2. \end{cases}$$

Let $r > 0$ and set $K_n = (\Phi(t_n))^{-1}(\mathcal{S}(a_\infty, r))$. Then we have

$$\left| \int_{S^1} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} - \int_{K_n} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} \right| \le \max_{x \in S^1} f(x) \left(\mathcal{L}(K_n^c) \int_{S^1} e^{|2v(t_n)|}\, ds_{g_0} \right)^{\frac{1}{2}}.$$

Since $\lim_{t\to\infty} \Phi(t_n)(x) = a_\infty$ a.e., it follows that $\lim_{n\to\infty} \mathcal{L}(K_n) = 2\pi$. Thus, we deduce from (4-30) that

$$\begin{aligned} \int_{\mathcal{S}(a_\infty, r)} f e^{u(t_n)}\, ds_{g_0} &= \int_{K_n} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} \\ &= \int_{S^1} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} + \epsilon_n, \end{aligned} \tag{4-35}$$

with $\lim\limits_{n\to\infty} \epsilon_n = 0$. In view of (4-22), we have

$$\int_{S^1} f \circ \Phi(t_n) e^{v(t_n)}\, ds_{g_0} = 2\pi.$$

Therefore, it follows from (4-35) that (4-32) holds. □

Now, we suppose that $u(t) \ne u_0$ for all $t > 0$ (otherwise the solution is stationary and the convergence is obvious). Since u_0 is G-invariant, by using the uniqueness of the solution u, it is not hard to conclude that u is also G-invariant. Again from the uniqueness of u, we can see from the energy identity (2-1) that

$$J(u(t)) < J(u(t')) \quad \text{for } t > t'. \tag{4-36}$$

Case (i): $\Sigma=\varnothing$. Suppose that u is not uniformly bounded in H. So from Lemma 4.2, there exists a point $a_\infty \in S^1$ satisfying (4-32) for all $r>0$. Since $\Sigma=\varnothing$, there exists $\sigma \in G$ such that $\sigma(a_\infty) \neq a_\infty$. Now, for all $r>0$, we have

$$\lim_{n\to\infty} \int_{\mathcal{S}(\sigma(a_\infty),r)} f e^{u(t_n)}\, ds_{g_0} = \lim_{n\to\infty} \int_{\mathcal{S}(a_\infty,r)} f e^{u(t_n)}\, ds_{g_0} = 2\pi,$$

which contradicts Lemma 4.2(ii).

Case (ii): $\Sigma \neq \varnothing$. Suppose that u is not uniformly bounded in H. So from Lemma 4.2, there exists a point $a_\infty \in S^1$ satisfying (4-32) for all $r>0$. If $a_\infty \notin \Sigma$, then in the same way as in case (i) above, we arrive at a contradiction. Otherwise, we have for all $r>0$ that

$$\begin{aligned}\int_{\mathcal{S}(a_\infty,r)} f e^{u(t_n)}\, ds_{g_0} &\le \max_{x\in\mathcal{S}(a_\infty,r)} f(x) \int_{\mathcal{S}(a_\infty,r)} e^{u(t_n)}\, ds_{g_0} \\ &\le \max\Bigl(\max_{x\in\mathcal{S}(a_\infty,r)} f(x),0\Bigr) \int_{\mathcal{S}(a_\infty,r)} e^{u(t_n)}\, ds_{g_0} \\ &\le \max\Bigl(\max_{x\in\mathcal{S}(a_\infty,r)} f(x),0\Bigr) \int_{S^1} e^{u(t_n)}\, ds_{g_0}.\end{aligned} \tag{4-37}$$

Now, we may write the Lebedev–Milin inequality as

$$⨍_{S^1} e^{u(t_n)}\, ds_{g_0} \le e^{J(u(t_n))/2\pi},$$

which, together with (4-36), yields

$$⨍_{S^1} e^{u(t_n)}\, ds_{g_0} \le e^{J(u(t_n))/2\pi} \le e^{J(u_0)/2\pi}. \tag{4-38}$$

Plugging (4-38) into (4-37) and letting $n\to\infty$, we obtain

$$2\pi \le 2\pi \max\Bigl(\max_{x\in\mathcal{S}(a_\infty,r)} f(x),0\Bigr) e^{J(u_0)/2\pi}. \tag{4-39}$$

Estimate (4-39) implies that $f(a_\infty)>0$ so that

$$1 \le f(a_\infty) e^{J(u_0)/2\pi}.$$

Hence

$$f(a_\infty) > e^{-J(u_0)/2\pi},$$

which contradicts assumption (ii) of the theorem. This establishes Theorem 2.6. □

Proof of Corollary 2.7. If $\Sigma = \varnothing$, then the result of Corollary 2.7 is a direct consequence of Theorem 2.6. Suppose now that $\Sigma \neq \varnothing$ and let f satisfy inequality (2-3): if $\fint_{S^1} f \circ \Phi_{a_0}\, ds_{g_0} \le 0$, then $\sup_{p\in\Sigma} f(p) \le 0$; so condition (ii) of Theorem 2.6 is satisfied. Otherwise, if $\fint_{S^1} f \circ \Phi_{a_0}\, ds_{g_0} > 0$, we let $u^* = \log|\Phi'_{a_0}|$. Then we have $J(u^*) = 0$ (see [Chang and Liu 1996]). Now set $u_0 = u^* + C$, where C is a constant satisfying

$$e^C \int_{S^1} f \circ \Phi_{a_0}\, ds_{g_0} = 2\pi.$$

This implies that $u_0 \in X$. Since $a_0 \in \Sigma$, it is not difficult to see that u_0 is G-invariant. Hence we conclude that condition (ii) of Theorem 2.6 is equivalent to

$$\max_{p\in\Sigma} f(p) \le \fint_{S^1} f \circ \Phi_{a_0}\, ds_{g_0}.$$

This completes the proof of Corollary 2.7. □

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References

[Baird et al. 2004] P. Baird, A. Fardoun, and R. Regbaoui, "The evolution of the scalar curvature of a surface to a prescribed function", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **3**:1 (2004), 17–38. MR 2005e:53100 Zbl 1170.58306

[Baird et al. 2006] P. Baird, A. Fardoun, and R. Regbaoui, "Q-curvature flow on 4-manifolds", *Calc. Var. Partial Differential Equations* **27**:1 (2006), 75–104. MR 2007d:53107 Zbl 1094.53024

[Brendle 2002a] S. Brendle, "Curvature flows on surfaces with boundary", *Math. Ann.* **324**:3 (2002), 491–519. MR 2003j:53103 Zbl 1024.53045

[Brendle 2002b] S. Brendle, "A generalization of the Yamabe flow for manifolds with boundary", *Asian J. Math.* **6**:4 (2002), 625–644. MR 2003m:53052 Zbl 1039.53035

[Brendle 2003] S. Brendle, "Global existence and convergence for a higher order flow in conformal geometry", *Ann. of Math.* (2) **158**:1 (2003), 323–343. MR 2004e:53098 Zbl 1042.53016

[Chang and Liu 1996] K. C. Chang and J. Q. Liu, "A prescribing geodesic curvature problem", *Math. Z.* **223**:2 (1996), 343–365. MR 97h:53043 Zbl 0865.53033

[Chen and Xu 2012] X. Chen and X. Xu, "The scalar curvature flow on S^n—perturbation theorem revisited", *Invent. Math.* **187**:2 (2012), 395–506. MR 2885623 Zbl 1275.53056

[Cherrier 1984] P. Cherrier, "Problèmes de Neumann non linéaires sur les variétés Riemanniennes", *J. Funct. Anal.* **57**:2 (1984), 154–206. MR 86c:58154 Zbl 0552.58032

[Ho 2011] P. T. Ho, "Prescribed curvature flow on surfaces", *Indiana Univ. Math. J.* **60**:5 (2011), 1517–1542. MR 2996999 Zbl 06122677

[Kazdan and Warner 1975] J. L. Kazdan and F. W. Warner, "Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures", *Ann. of Math.* (2) **101** (1975), 317–331. MR 51 #11349 Zbl 0297.53020

[Li and Liu 2005] Y. Li and P. Liu, "A Moser–Trudinger inequality on the boundary of a compact Riemann surface", *Math. Z.* **250**:2 (2005), 363–386. MR 2007b:58036 Zbl 1070.26017

[Liu and Huang 2005] P. Liu and W. Huang, "On prescribing geodesic curvature on D^2", *Nonlinear Anal.* **60**:3 (2005), 465–473. MR 2005h:53057 Zbl 1086.53054

[Malchiodi and Struwe 2006] A. Malchiodi and M. Struwe, "Q-curvature flow on S^4", *J. Differential Geom.* **73**:1 (2006), 1–44. MR 2007c:53087 Zbl 1099.53034

[Osgood et al. 1988] B. Osgood, R. Phillips, and P. Sarnak, "Extremals of determinants of Laplacians", *J. Funct. Anal.* **80**:1 (1988), 148–211. MR 90d:58159 Zbl 0653.53022

[Struwe 2005] M. Struwe, "A flow approach to Nirenberg's problem", *Duke Math. J.* **128**:1 (2005), 19–64. MR 2005m:53125 Zbl 1087.53034

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