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We consider complete multiple warped product type Riemannian metrics on manifolds of the form $\mathbb{R}^2 \times M_2 \times \cdots \times M_r$, where $r \geq 2$ and M_i are arbitrary closed Einstein spaces with positive scalar curvature. We construct on these spaces a family of non-Kähler, non-Einstein, expanding gradient Ricci solitons with conical asymptotics as well as a family of Einstein metrics with negative scalar curvature. The 2-dimensional Euclidean space factor allows us to obtain homeomorphic but not diffeomorphic examples which have analogous cone structure behaviour at infinity. We also produce numerical evidence for complete expanding solitons on the vector bundles whose sphere bundles are the twistor or Sp(1) bundles over quaternionic projective space.

0. Introduction

In [Buzano et al. 2013] we constructed complete steady gradient Ricci soliton structures (including Ricci-flat metrics) on manifolds of the form $\mathbb{R}^2 \times M_2 \times \cdots \times M_r$, where M_i, $2 \leq i \leq r$, are arbitrary closed Einstein manifolds with positive scalar curvature. We also produced numerical solutions of the steady gradient Ricci soliton equation on certain nontrivial $\mathbb{R}^3$ and $\mathbb{R}^4$ bundles over quaternionic projective spaces. In the current paper we will present the analogous results for the case of *expanding* solitons on the same underlying manifolds.

Recall that a gradient Ricci soliton is a manifold M together with a smooth Riemannian metric g and a smooth function u, called the soliton potential, which give a solution to the equation

$$\mathrm{Ric}(g) + \mathrm{Hess}(u) + \frac{\epsilon}{2}\, g = 0 \tag{0.1}$$

for some constant ϵ. The soliton is then called expanding, steady, or shrinking according to whether ϵ is greater than, equal to, or less than zero.

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A gradient Ricci soliton is called *complete* if the metric g is complete. The completeness of the vector field ∇u follows from that of the metric; see [Zhang 2009]. If the metric of a gradient Ricci soliton is Einstein, then either $\operatorname{Hess} u = 0$ (i.e., ∇u is parallel) or we are in the case of the Gaussian soliton; see [Petersen and Wylie 2009; Pigola et al. 2011].

At present most examples of non-Kählerian expanding solitons arise from left-invariant metrics on nilpotent and solvable Lie groups (resp. *nilsolitons*, *solvsolitons*), as a result of work by J. Lauret [2001; 2011], M. Jablonski [2013], and many others (see the survey [Lauret 2009]). These expanders are however not of gradient type, i.e., they satisfy the more general equation

$$\operatorname{Ric}(g) + \frac{1}{2}\,\mathsf{L}_X g + \frac{\epsilon}{2}\, g = 0, \tag{0.2}$$

where X is a vector field on M and L denotes Lie differentiation.

A large class of complete, non-Einstein, non-Kählerian expanders of gradient type (with dimension ≥ 3) consists of an r-parameter family of solutions to (0.1) on $\mathbb{R}^{k+1} \times M_2 \times \cdots \times M_r$ where $k > 1$ and M_i are positive Einstein manifolds. The special case $r = 1$ (i.e., no M_i) is due to R. Bryant [2005] and the solitons have positive sectional curvature. The $r = 2$ case is due to Gastel and Kronz [2004], who adapted Böhm's construction of complete Einstein metrics with negative scalar curvature to the soliton case. The case of arbitrary r was treated in [Dancer and Wang 2009a] via a generalization of the dynamical system studied by Bryant. The soliton metrics in this family are all of multiple warped product type. In other words, the manifold is thought of as being foliated by hypersurfaces of the form $S^k \times M_2 \times \cdots \times M_r$ each equipped with a product metric depending smoothly on a real parameter t. As $k \geq 2$ in these works, the hypersurfaces and the asymptotic cones have finite fundamental group.

More recently, Schulze and Simon [2013] constructed expanding gradient Ricci solitons with nonnegative curvature operator in arbitrary dimensions by studying the scaling limits of the Ricci flow on complete open Riemannian manifolds with nonnegative bounded curvature operator and positive asymptotic volume ratio.

As pointed out in [Buzano et al. 2013], the situation of multiple warped products on nonnegative Einstein manifolds is rather special because of the automatic lower bound on the scalar curvature of the hypersurfaces. This leads, in the case where all factors have positive scalar curvature, $k > 1$, to definiteness of certain energy functionals occurring in the analysis of the dynamical system arising from (0.1), and hence to coercive estimates on the flow. In the present case, where one factor is a circle, i.e., $k = 1$, we can pass, as in [Buzano et al. 2013], to a subsystem where coercivity holds, and this is enough for the analysis to proceed. The new solitons obtained, like those of [Dancer and Wang 2009a], have conical asymptotics and are not of Kähler type (Theorem 2.14). We note that the lowest-dimensional solitons we obtain form a 2-parameter family on $\mathbb{R}^2 \times S^2$. The special case $r = 1$ was analysed earlier by the physicists Gutperle, Headrick, Minwalla and Schomerus [Gutperle et al. 2003].

As in [Buzano et al. 2013], we also obtain a family of solutions to our soliton equations that yield complete Einstein metrics of negative scalar curvature (Theorem 3.1). These are analogous to the metrics discovered by Böhm [1999]. Recall that for Böhm's construction the fact that the hyperbolic cone over the product Einstein metric on the hypersurface acts as an attractor plays an important role in the convergence proof for the Einstein trajectories. When $k = 1$, however, no product metric on the hypersurface can be Einstein with positive scalar curvature, so the hyperbolic cone construction cannot be exploited directly. It turns out that the analysis of the soliton case already contains most of the analysis required for the Einstein case. The new Einstein metrics we obtain have exponential volume growth.

The fact that $k = 1$ (rather than $k > 1$) allows for some new phenomena displayed by the asymptotic cones of some of our expander and Einstein examples. This is a consequence of the striking observation of Kwasik and Schultz [2002] that for an exotic sphere Σ and the standard sphere S of the same dimension, $\mathbb{R}^2 \times \Sigma$ is not diffeomorphic to $\mathbb{R}^2 \times \mathsf{S}$, but if we replace $\mathbb{R}^2$ by $\mathbb{R}^3$ in the products the resulting spaces do become diffeomorphic. In fact, the open cones $\mathbb{R}_+ \times S^1 \times \Sigma$ and $\mathbb{R}_+ \times S^1 \times \mathsf{S}$ are also homeomorphic but not diffeomorphic. As a result, we obtain examples of pairs of expanders and negative Einstein manifolds whose asymptotic cones are also homeomorphic but not diffeomorphic. These results are described in greater detail at the end of Section 3; see Corollary 3.2 and Proposition 3.3.

To make further progress in the search for expanders, we need to consider more complicated hypersurface types where the scalar curvature may not be bounded below. In [Buzano et al. 2013] we carried out numerical investigations of steady solitons where the hypersurfaces are the total spaces of Riemannian submersions for which the hypersurface metric involves two functions, one scaling the base and one the fibre of the submersion. We now look numerically at expanding solitons with such hypersurface types, in particular where the hypersurfaces are S^2 or S^3 bundles over quaternionic projective space. We produce numerical evidence of complete expanding gradient Ricci soliton structures in these cases.

Before undertaking our theoretical and numerical investigations, we first prove some general results about expanding solitons of cohomogeneity one type. Some of the results follow from properties of general expanding gradient Ricci solitons. However, the proofs are much simpler and sometimes the statements are sharper, which is helpful in numerical studies. The results include monotonicity and concavity properties for the soliton potential similar to those proved in [Buzano et al.

2013] in the steady case, as well as an upper bound for the mean curvature of the hypersurfaces. To derive this bound, we need to know that complete *non-Einstein* expanding gradient Ricci solitons have infinite volume. We include a proof of this fact here (Proposition 1.22) since we were not able to find an explicit statement in the literature. Finally we derive an asymptotic lower bound for the gradient of the soliton potential, which is in turn used to exhibit a general Lyapunov function for the cohomogeneity one expander equations.

1. Background on cohomogeneity one expanding solitons

We briefly review the formalism [Dancer and Wang 2011] for Ricci solitons of cohomogeneity one. We work on a manifold M with an open dense set foliated by equidistant diffeomorphic hypersurfaces P_t of real dimension n. The dimension of M, the manifold where we construct the soliton, is therefore $n+1$. The metric is then of the form $\bar{g} = dt^2 + g_t$, where g_t is a metric on P_t and t is the arclength coordinate along a geodesic orthogonal to the hypersurfaces. This set-up is more general than the cohomogeneity one ansatz, as it allows us to consider metrics with no symmetry provided that appropriate additional conditions on P_t are satisfied; see the following as well as [Dancer and Wang 2011, Remarks 2.18, 3.18]. We will also suppose that u is a function of t only.

We let r_t denote the Ricci endomorphism of g_t, defined by $\mathrm{Ric}(g_t)(X, Y) = g_t(r_t(X), Y)$ and viewed as an endomorphism via g_t. Also let L_t be the shape operator of the hypersurfaces, defined by the equation $\dot{g}_t = 2g_t L_t$ where g_t is regarded as an endomorphism with respect to a fixed background metric Q. The Levi-Civita connections of $\bar{g}$ and g_t will be denoted by $\overline{\nabla}$ and ∇ respectively. The relative volume $v(t)$ is defined by $d\mu_{g_t} = v(t)\, d\mu_Q$

We assume that the scalar curvature $S_t = \mathrm{tr}(r_t)$ and the mean curvature $\mathrm{tr}(L_t)$ (with respect to the normal $\nu = \partial/\partial t$) are constant on each hypersurface. These assumptions hold, for example, if M is of cohomogeneity one with respect to an isometric Lie group action. They are satisfied also when M is a multiple warped product over an interval.

The gradient Ricci soliton equation now becomes the system

$$-\,\mathrm{tr}\,\dot{L} - \mathrm{tr}(L^2) + \ddot{u} + \tfrac{1}{2}\epsilon = 0, \tag{1.1}$$

$$r - (\mathrm{tr}\,L)L - \dot{L} + \dot{u}L + \tfrac{1}{2}\epsilon\,\mathbb{I} = 0, \tag{1.2}$$

$$d(\mathrm{tr}\,L) + \delta^{\nabla} L = 0. \tag{1.3}$$

The first two equations represent the components of the equation in the directions normal and tangent to the hypersurfaces P, respectively. The third equation represents the equation in mixed directions — here $\delta^{\nabla} L$ denotes the codifferential for TP-valued 1-forms.

In the warped product case the final equation involving the codifferential automatically holds. This is also true for cohomogeneity one metrics that are *monotypic*, i.e., when there are no repeated real irreducible summands in the isotropy representation of the principal orbits; see [Bérard-Bergery 1982, Proposition 3.18].

There is a conservation law

$$\ddot{u} + (-\dot{u} + \operatorname{tr} L)\,\dot{u} - \epsilon u = C \tag{1.4}$$

for some constant C. Using our equations we may rewrite this as

$$S + \operatorname{tr}(L^2) - (\dot{u} - \operatorname{tr} L)^2 - \epsilon u + \tfrac{1}{2}(n-1)\epsilon = C. \tag{1.5}$$

where $S := \operatorname{tr}(r_t)$ is the scalar curvature S of the principal orbits. If $\bar{R}$ denotes the scalar curvature of the ambient metric $\bar{g}$, then

$$\bar{R} = -2\operatorname{tr}\dot{L} - \operatorname{tr}(L^2) - (\operatorname{tr} L)^2 + S.$$

We can deduce the equality

$$\bar{R} + \dot{u}^2 + \epsilon u = -C - \frac{\epsilon}{2}(n+1). \tag{1.6}$$

We let ξ denote the *dilaton mean curvature*

$$\xi := -\dot{u} + \operatorname{tr} L.$$

This is the mean curvature of the dilaton volume element $e^{-u}d\mu_{\bar{g}}$. It is often useful to define a new independent variable s by

$$\frac{d}{ds} := \frac{1}{\xi}\frac{d}{dt}, \tag{1.7}$$

and use a prime to denote d/ds. We note that (1.1) implies that $\dot{\xi} = -\operatorname{tr}(L^2) + \epsilon/2$.

It is also useful, following [Dancer et al. 2013], to introduce the quantity

$$\mathcal{E} := C + \epsilon u.$$

The conservation law may now be rewritten (for nonzero ϵ) as

$$\ddot{\mathcal{E}} + \xi\dot{\mathcal{E}} - \epsilon\mathcal{E} = 0. \tag{1.8}$$

Note that, for a function $t \mapsto f(t)$, the quantity $\ddot{f} + \xi\dot{f}$ is just the u-Laplacian in the sense of metric measure spaces.

Another useful quantity is the normalised mean curvature

$$\mathcal{H} = \frac{\operatorname{tr} L}{\xi} = 1 + \frac{\dot{u}}{\xi} = 1 + u',$$

which was introduced in [Dancer and Wang 2009a; Dancer et al. 2013].

We now specialise to the case of *expanding solitons*, that is,

$$\epsilon > 0.$$

We shall consider complete noncompact expanding solitons with one special orbit. We may take the interval I over which t ranges to be $[0, \infty)$ with the special orbit placed at $t = 0$. Let k denote the dimension of the collapsing sphere at $t = 0$. We will moreover assume in this section that $u(0) = 0$, since adding a constant to the soliton potential does not affect the equations.

A basic result of B.-L. Chen [2009] together with the strong maximum principle says that for a non-Einstein expanding gradient Ricci soliton $\bar{R} > -\frac{\epsilon}{2}(n+1)$. So we deduce from (1.6) that

$$\mathcal{E} < 0 \quad \text{and} \quad (\dot{u})^2 < -\mathcal{E} := -(C + \epsilon u).$$

Using the first inequality and the smoothness conditions at $t = 0$, we find as in the steady case that $\ddot{u}(0) = C/(k+1) < 0$, so completeness imposes restrictions on our initial conditions.

Integrating the second inequality and using the initial conditions yield

$$0 \le -u(t) < \frac{\epsilon}{4}t^2 + \sqrt{-C}\,t \tag{1.9}$$

and

$$|\dot{u}| < \frac{\epsilon}{2}t + \sqrt{-C}. \tag{1.10}$$

These are just the cohomogeneity one versions of general estimates of the potential due to Z.-H. Zhang [2009].

Proposition 1.11. *For a non-Einstein, complete, expanding gradient Ricci soliton of cohomogeneity one with a special orbit, the soliton potential u is strictly decreasing and strictly concave on* $(0, \infty)$.

Proof. The conservation law (1.8) and the fact that $\mathcal{E}$ is negative and ϵ is positive show that u is strictly concave on a neighbourhood of each critical point t_0. As we noted above, we also have concavity at the special orbit $t = 0$. Now, as in the steady case [Buzano et al. 2013], we see there are no critical points of u in $(0, \infty)$. As $\dot{u}(0) = 0$, we see u is strictly decreasing on $(0, \infty)$.

Now set $y = \dot{u}$ and differentiate (1.4); using (1.1) we obtain

$$\ddot{y} + \xi\dot{y} - \left(\frac{\epsilon}{2} + \operatorname{tr}(L^2)\right)y = 0.$$

In particular, $\ddot{y} + \xi\dot{y} < 0$, since y is negative. Integrating shows $ve^{-u}\dot{y}$ is strictly decreasing, where we recall that v is the relative volume. As t tends to 0, the smoothness conditions imply that $ve^{-u}\dot{y}$ tends to 0, so $\dot{y} = \ddot{u}$ is negative, as required. $\square$

Our next result is inspired by the work of Munteanu and Sesum [2013] for the case of steady solitons.

Proposition 1.12. *For a non-Einstein, complete, expanding gradient Ricci soliton of cohomogeneity one with a special orbit, the volume growth is at least logarithmic.*

Proof. Let $M_t = \pi^{-1}([0, t])$, where π is the projection of M onto the orbit space I. We consider the integral

$$f(t) := \int_{M_t} \Big(\bar{R} + \frac{\epsilon}{2}(n+1) \Big)\, d\mu_{\bar{g}}$$

As we are considering non-Einstein solitons the integrand is positive.

Let $t_0 > 0$ and let $b := f(t_0)$. Using the trace of the soliton equation and also the divergence theorem we have, for $t \geq t_0$,

$$\begin{aligned} 0 < b \leq f(t) &= -\int_{M_t} \overline{\Delta} u \, d\mu_{\bar{g}} \\ &= \int_{\partial M_t} (\overline{\nabla} u) \cdot \Big(-\frac{\partial}{\partial t} \Big)\, d\mu_{\bar{g}}\big|_{\partial M_t} \\ &= |\dot{u}|\, v(t) \\ &< \Big(\frac{\epsilon}{2} t + \sqrt{-C} \Big) v(t) \end{aligned}$$

where we use (1.10) in the last line. Hence $v(t) > b/\big(\frac{\epsilon}{2}t + \sqrt{-C}\big)$, and integrating yields

$$\operatorname{vol}(M_t) > \operatorname{vol}(M_{t_0}) - \frac{2b}{\epsilon} \log\Big(\frac{\epsilon}{2} t_0 + \sqrt{-C} \Big) + \frac{2b}{\epsilon} \log\Big(\frac{\epsilon}{2} t + \sqrt{-C} \Big). \qquad \square$$

Proposition 1.13. *Let $(M, \bar{g}, u)$ be a non-Einstein, complete, expanding gradient Ricci soliton of cohomogeneity one with a special orbit. Then there exists $t_1 > 0$ such that on (t_1, ∞) we have* $\operatorname{tr} L < \sqrt{n\epsilon/2}$.

Proof. By Cauchy–Schwartz and the concavity result, we have

$$\frac{d}{dt}(\operatorname{tr} L) < \frac{\epsilon}{2} - \operatorname{tr}(L^2) \leq \frac{\epsilon}{2} - \frac{1}{n}(\operatorname{tr} L)^2. \tag{1.14}$$

Note that by the smoothness conditions $\operatorname{tr} L$ is strictly decreasing near $t = 0$, and its limit as t tends to zero from above is $+\infty$.

(i) First let us assume that $d(\operatorname{tr} L)/dt$ is nonnegative at some t_1. The inequality above shows that $|\operatorname{tr} L|^2 < \frac{\epsilon}{2} n$ at $t = t_1$.

Let us consider the solutions of the equation

$$\dot{h} = \frac{\epsilon}{2} - \frac{1}{n} h^2. \tag{1.15}$$

These are the family of increasing functions

$$h(t) = \sqrt{\frac{\epsilon n}{2}}\, \frac{a \exp(t\sqrt{2\epsilon/n}) - 1}{a \exp(t\sqrt{2\epsilon/n}) + 1},$$

where a is a positive constant, as well as the constant functions $\pm\sqrt{\epsilon n/2}$ which form the bounding envelope for this family. Hence $\operatorname{tr} L \leq h^*(t) < \sqrt{\epsilon n/2}$, where $h^*(t)$ is the solution to (1.15) which agrees with $\operatorname{tr} L$ at t_1.

(ii) Next suppose that $d(\operatorname{tr} L)/dt$ is always negative. Now if $\operatorname{tr} L$ is ever zero then it is negative and bounded away from zero on some semi-infinite interval. Recalling that $\operatorname{tr} L = \dot{v}/v$ and integrating, we see that the soliton volume is finite, which contradicts Proposition 1.12. So $\operatorname{tr} L$ is positive on $(0, \infty)$ and, using Proposition 1.11, we see ξ is also positive on this interval. By [Pigola et al. 2011, Theorem 11], ξ tends to infinity as t tends to ∞. But ξ also tends to infinity as t tends to zero, so we have a minimum t_1 where $\dot{\xi}$ vanishes. Now (1.1) shows $\operatorname{tr}(L^2) = \epsilon/2$ at t_1 and Cauchy–Schwartz shows $(\operatorname{tr} L)^2 \leq n\epsilon/2$ at t_1. As $\operatorname{tr} L$ is decreasing, we have the desired result. □

Remark 1.16. This bound on $\operatorname{tr} L$ is best possible, at least if we allow the solitons to be Einstein. Indeed, the negative scalar curvature Einstein metrics of Böhm [1999] give exactly this bound, as $\operatorname{tr} L$ is asymptotic to $n\epsilon/2$.

Next we consider properties of the Lyapunov function $\mathscr{F}_0$, which was introduced by Böhm [1999] for the Einstein case and was subsequently studied in [Dancer et al. 2013; Buzano et al. 2013] for the soliton case. Note that this function was denoted by $\mathscr{F}$ in [Dancer et al. 2013].

Proposition 1.17. *Let $\mathscr{F}_0$ denote the function $v^{2/n}\big(S + \operatorname{tr}((L^{(0)})^2)\big)$ defined on the velocity phase space of the cohomogeneity one expanding gradient Ricci soliton equations, with $L^{(0)}$ representing the trace-free part of L. Then along the trajectory of a complete smooth non-Einstein expanding soliton, $\mathscr{F}_0$ is nonincreasing for sufficiently large t.*

Proof. The formula for $d\mathscr{F}_0/dt$ in [Dancer et al. 2013, Proposition 2.17] shows that the proposition would follow if, for sufficiently large t, one can show that

$$\xi - \frac{1}{n}\operatorname{tr} L = -\dot{u} + \frac{n-1}{n}\operatorname{tr} L \geq 0.$$

We first note that $\operatorname{tr} L$ is eventually bounded below by $-\sqrt{\epsilon n/2}$. Otherwise, at some $t = t_1 > 0$, $\operatorname{tr} L \leq -\sqrt{\epsilon n/2}$ and (1.14) shows that this inequality continues to hold from t_1 onwards. But this would imply that the soliton has finite volume, contradicting Proposition 1.12.

We are now done since part (i) of the next proposition shows that $|\dot{u}| = -\dot{u}$ grows at least linearly for sufficiently large t. In particular, for large enough t, $\mathcal{F}_0$ fails to be strictly decreasing iff the shape operator of the hypersurfaces become diagonal. □

Proposition 1.18. *Let $(M, \bar{g}, u)$ be a complete, non-Einstein, expanding gradient Ricci soliton of cohomogeneity one with a special orbit. Suppose $t_1 > 2\sqrt{5/\epsilon}$ and on $[t_1, +\infty)$ we have an upper bound $\lambda_0 > 0$ for* $\operatorname{tr} L$. *Set $a := \lambda_0 + \sqrt{-C}$. Then on $[t_1, +\infty)$ we have*

(i) $|\overline{\nabla} u| = -\dot{u}(t) > \dfrac{9}{10}\left(\dfrac{-\dot{u}(t_1)}{\frac{\epsilon}{2}t_1 + a}\right)\left(\dfrac{\epsilon}{2}t + a\right),$

(ii) $\ddot{u} + \dfrac{\epsilon}{2} = -\operatorname{Ric}_{\bar{g}}\left(\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial t}\right) \le \dfrac{\epsilon}{2}\left(1 + \dfrac{9}{10}\dfrac{\dot{u}(t_1)}{\frac{\epsilon}{2}t_1 + a}\right).$

Proof. By assumption and the upper bound (1.10) we have $\xi < \frac{\epsilon}{2}t + a$. Since $\dot{y} = \ddot{u} < 0$ and $y = \dot{u} < 0$ by Proposition 1.11, we see that y satisfies the differential inequality

$$\ddot{y} + \left(\frac{\epsilon}{2}t + a\right)\dot{y} - \frac{\epsilon}{2}y < 0.$$

We will now compare y with solutions of the corresponding equation

$$\ddot{x} + \left(\frac{\epsilon}{2}t + a\right)\dot{x} - \frac{\epsilon}{2}x = 0, \tag{1.19}$$

which can be solved explicitly. This is because, if we differentiate this equation, we obtain

$$\frac{d^3x}{dt^3} + \left(\frac{\epsilon}{2}t + a\right)\ddot{x} = 0,$$

from which we can solve for $\ddot{x}$. Accordingly, upon integration and using (1.19), we obtain

$$x(t) = -\left(\frac{\epsilon}{2}t + a\right)\left(\frac{c_0}{\frac{\epsilon}{2}t_1 + a} - c_1 e^{(\epsilon/4)t_1^2 + at_1}\int_{t_1}^{t} \frac{e^{-(\epsilon/4)\tau^2 - a\tau}}{\left(\frac{\epsilon}{2}\tau + a\right)^2}\,d\tau\right), \tag{1.20}$$

where c_0 and c_1 are arbitrary constants.

In order to apply [Protter and Weinberger 1984, Theorem 13, p. 26], we must choose $x(t_1) \ge y(t_1) = \dot{u}(t_1)$ and $\dot{x}(t_1) \ge \dot{y}(t_1) = \ddot{u}(t_1)$. Since $x(t_1) = -c_0$, we can maximize c_0 by choosing $x(t_1) = \dot{u}(t_1)$. It follows that

$$c_1 = -\ddot{x}(t_1) = -\frac{\epsilon}{2}\,x(t_1) + \left(\frac{\epsilon}{2}t_1 + a\right)\dot{x}(t_1) \ge -\frac{\epsilon}{2}\,\dot{u}(t_1) + \left(\frac{\epsilon}{2}t_1 + a\right)\ddot{u}(t_1).$$

In particular, an admissible choice for c_1 is $c_1 = \frac{\epsilon}{2}c_0 > 0$. With this choice, it remains to find an upper bound for the integral in (1.20).

To do this, we integrate by parts three times and then throw away the resulting term involving integration (this term is negative). Specifically, we have

$$\int_{\lambda_1}^{\lambda} \frac{e^{-\sigma^2/\epsilon}}{\sigma^2}\, d\sigma$$
$$\leq \frac{\epsilon}{2}\left(\frac{e^{-\lambda_1^2/\epsilon}}{\lambda_1^3}\right)\left(1-\frac{3}{2}\frac{\epsilon}{\lambda_1^2}+\frac{15}{\lambda_1^4}\left(\frac{\epsilon}{2}\right)^2-\left(\frac{\lambda_1}{\lambda}\right)^3 e^{-(\lambda^2-\lambda_1^2)/\epsilon}\left(1-\frac{3}{2}\frac{\epsilon}{\lambda^2}+\frac{15}{\lambda^4}\left(\frac{\epsilon}{2}\right)^2\right)\right).$$

Using the change of independent variable $\lambda := \frac{\epsilon}{2}t + a$ and the fact that

$$1-\frac{3\epsilon}{2}x+\frac{15}{4}\epsilon^2x^2 = \left(1-\frac{3\epsilon}{4}x\right)^2+\frac{51}{16}\epsilon^2x^2 \geq \frac{17}{20},$$

we obtain

$$e^{(\epsilon/4)t_1^2+at_1}\int_{t_1}^{t} \frac{e^{-(\epsilon/4)\tau^2-a\tau}}{\left(\frac{\epsilon}{2}\tau+a\right)^2}\, d\tau$$
$$\leq \frac{1}{\left(\frac{\epsilon}{2}t_1+a\right)^3}\left(1-\frac{\epsilon}{2}\frac{3}{\left(\frac{\epsilon}{2}t_1+a\right)^2}+\left(\frac{\epsilon}{2}\right)^2\frac{15}{\left(\frac{\epsilon}{2}t_1+a\right)^4}-\frac{17}{20}\left(\frac{\frac{\epsilon}{2}t_1+a}{\frac{\epsilon}{2}t+a}\right)^3\frac{e^{(\epsilon/4)t_1^2+at_1}}{e^{(\epsilon/4)t^2+at}}\right).$$

If we substitute the information above together with the choice $c_1 = \frac{\epsilon}{2}c_0$ in the comparison inequality $\dot{u}(t) \leq x(t)$ (for $t \geq t_1$), we obtain

$$-\dot{u}(t)$$
$$\geq -\frac{\dot{u}(t_1)}{\frac{\epsilon}{2}t_1+a}\left(\frac{\epsilon}{2}t+a\right)\left(1-\frac{\epsilon}{2}\frac{1}{\left(\frac{\epsilon}{2}t_1+a\right)^2}\left(1-\frac{\epsilon}{2}\frac{3}{\left(\frac{\epsilon}{2}t_1+a\right)^2}+\left(\frac{\epsilon}{2}\right)^2\frac{15}{\left(\frac{\epsilon}{2}t_1+a\right)^4}\right)\right)$$
$$\geq -\frac{\dot{u}(t_1)}{\frac{\epsilon}{2}t_1+a}\left(\frac{\epsilon}{2}t+a\right)\left(1-\frac{\epsilon}{2}\frac{1}{\left(\frac{\epsilon}{2}t_1+a\right)^2}\right)$$
$$> \frac{9}{10}\left(-\frac{\dot{u}(t_1)}{\frac{\epsilon}{2}t_1+a}\right)\left(\frac{\epsilon}{2}t+a\right)$$

where for the last inequality we used the hypothesis that $t_1 > 2\sqrt{5/\epsilon}$, so that $\frac{\epsilon}{2}t_1 + a > \sqrt{5\epsilon}$. This completes the proof of (i).

The proof of (ii) follows by applying the same estimates to the comparison inequality $\ddot{u}(t) = \dot{y}(t) \leq \dot{x}(t)$ for $t \geq t_1$. Note that by [Dancer and Wang 2000, (2.2)] and (1.2), the quantity $\ddot{u} + \frac{\epsilon}{2}$ is precisely the negative of the Ricci curvature of the soliton metric in the direction $\partial/\partial t$. □

Remark 1.21. In the above proof we can of course take λ_0 to be $\sqrt{\epsilon n/2}$ by Proposition 1.13. Notice, however, that in part (ii) of the proof of Proposition 1.13 one automatically has an upper bound on $\operatorname{tr} L$. So one can apply Proposition 1.18 instead of [Pigola et al. 2011, Theorem 11] to obtain a self-contained proof for Proposition 1.13.

Note also that neither Proposition 1.13 nor 1.18 requires any curvature bounds.

We end this section with a simple generalization of Proposition 1.12 which, as far as we know, has not been explicitly observed in the literature. An analogous result for steady gradient Ricci solitons is [Munteanu and Sesum 2013, Theorem 5.1].

Proposition 1.22. *A complete non-Einstein expanding gradient Ricci soliton has at least logarithmic volume growth.*

We shall give a sketch of the proof only since the basic outline is the same as that for the cohomogeneity one case. One replaces M_t in the proof of Proposition 1.12 by the metric ball $B_p(t)$ of radius t from an arbitrary but fixed point $p \in M$. The integrand in the boundary integral that is left after applying Stokes' theorem can be bounded by $\tilde{c}\,(t+2)\ \mathrm{vol}_{n-1}(\partial B_p(t))$ where $\tilde{c}$ is a positive constant which depends only on n and ϵ; see [Zhang 2009]. One then obtains the inequality

$$\frac{b}{\tilde{c}(t+2)} \le \mathrm{vol}_{n-1}(\partial M_p(t)).$$

Integrating this inequality and applying the coarea formula, one deduces that the volume of $B_p(t)$ grows at least logarithmically in t.

The main technical point in the above is to justify the use of Stokes' theorem as the distance function from p is only Lipschitz continuous. For this one can use the well-known fact that Stokes' theorem holds for Lipschitz domains (see [McLean 2000, Theorem 3.34]), or one can use the approximation arguments of Gaffney [1954] as in [Yau 1976, p. 660] to get a compact exhaustion of the underlying manifold with sufficiently good properties for applying the usual version of Stokes' theorem (see the version of this paper at arXiv:1311.5097).

Remark 1.23. Of course there are noncompact negative Einstein manifolds with finite volume. It is quite probable though that for nontrivial expanders the above volume lower bound is not sharp. Most lower bounds for the volume in the literature involve additional assumptions on the curvature. For example, in [Carrillo and Ni 2009, Proposition 5.1(b)] or [Chen 2012, Theorem 1] a lower bound on the (average) scalar curvature is assumed.

2. Multiple warped product expanders

In this section, we specialise to multiple warped products, that is, metrics of the form

$$\bar{g} = dt^2 + \sum_{i=1}^{r} g_i^2(t)\, h_i \tag{2.1}$$

on $I \times M_1 \times \cdots \times M_r$, where I is an interval in $\mathbb{R}$, $r \ge 2$ and (M_i, h_i) are Einstein manifolds with real dimensions d_i and Einstein constants λ_i. We observe that $n = \sum_i d_i$ is greater than or equal to 3 as long as some M_i is nonflat.

The Ricci endomorphism is now diagonal with components given by blocks $(\lambda_i/g_i^2)\mathbb{I}_{d_i}$, where $i = 1, \ldots, r$ and $\mathbb{I}_m$ denotes the identity matrix of size m. We work with the variables

$$X_i = \frac{\sqrt{d_i}}{\xi}\frac{\dot{g}_i}{g_i}, \tag{2.2}$$

$$Y_i = \frac{\sqrt{d_i}}{\xi}\frac{1}{g_i}, \tag{2.3}$$

$$W = \frac{1}{\xi} := \frac{1}{-\dot{u} + \operatorname{tr} L}, \tag{2.4}$$

for $i = 1, \ldots, r$. The definition of Y_i in [Dancer and Wang 2009a; 2009b] differs from that above by a scale factor of $\sqrt{\lambda_i}$. This choice reflects the fact that we are now allowing one of the λ_i to be zero. As in [Buzano et al. 2013] we have

$$\sum_{j=1}^{r} X_j^2 = \frac{\operatorname{tr}(L^2)}{\xi^2} \quad \text{and} \quad \sum_{j=1}^{r} \lambda_j Y_j^2 = \frac{\operatorname{tr}(r_t)}{\xi^2}.$$

As mentioned earlier, we shall introduce the new independent variable s defined by (1.7) and use a prime to denote differentiation with respect to s.

In these new variables the Ricci soliton system (1.1)–(1.2) becomes

$$X_i' = X_i\left(\sum_{j=1}^{r} X_j^2 - 1\right) + \frac{\lambda_i}{\sqrt{d_i}}Y_i^2 + \frac{\epsilon}{2}(\sqrt{d_i} - X_i)W^2, \tag{2.5}$$

$$Y_i' = Y_i\left(\sum_{j=1}^{r} X_j^2 - \frac{X_i}{\sqrt{d_i}} - \frac{\epsilon}{2}W^2\right), \tag{2.6}$$

$$W' = W\left(\sum_{j=1}^{r} X_j^2 - \frac{\epsilon}{2}W^2\right), \tag{2.7}$$

for $i = 1, \ldots, r$. Note that, in the warped product situation, (1.3) is automatically satisfied.

As in [Buzano et al. 2013] we use $\mathcal{G}$ to denote $\sum_{i=1}^{r} X_i^2$. The quantity $\mathcal{H} = W \operatorname{tr} L$ becomes $\sum_{i=1}^{r} \sqrt{d_i} X_i$ in our new variables. We further have the equation

$$(\mathcal{H} - 1)' = (\mathcal{H} - 1)\left(\mathcal{G} - 1 - \frac{\epsilon}{2}W^2\right) + \mathcal{Q},$$

where

$$\mathcal{Q} = \sum_{i=1}^{r}(X_i^2 + \lambda_i Y_i^2) + \frac{\epsilon(n-1)}{2}W^2 - 1. \tag{2.8}$$

As explained in [Dancer and Wang 2009a], $\mathcal{Q}$ serves as an energy functional in the expanding case, modifying the Lyapunov functional

$$\mathcal{L} := \sum_{i=1}^{r} (X_i^2 + \lambda_i Y_i^2) - 1, \tag{2.9}$$

which plays a key role in the steady case; see [Dancer and Wang 2009b; Buzano et al. 2013]. The general conservation law (1.5) then becomes $\mathcal{Q} = (C + \epsilon u)\, W^2$.

Note that, in our situation, the quantity $\mathcal{Q}$ is no longer a Lyapunov function. However, we do have the equations

$$\begin{aligned}(\mathcal{H} - 1)' &= f_1(\mathcal{H} - 1) + f_2 \mathcal{Q}\\ \mathcal{Q}' &= f_3(\mathcal{H} - 1) + f_4 \mathcal{Q},\end{aligned}$$

where $f_1 = G - 1 - \frac{\epsilon}{2} W^2$, $f_2 = 1$, $f_3 = \epsilon W^2$, and $f_4 = 2\big(G - \frac{\epsilon}{2} W^2\big)$. The crucial point for us is that in the expanding case both f_2 and f_3 are positive, so the phase plane diagram in the $(\mathcal{H} - 1, \mathcal{Q})$-plane shows that the regions $\{\mathcal{H} < 1, \mathcal{Q} < 0\}$ and $\{\mathcal{H} > 1, \mathcal{Q} > 0\}$ are both flow-invariant. Furthermore, the region $\{\mathcal{Q} = 0, \mathcal{H} = 1\}$ of phase space corresponds to Einstein metrics of negative Einstein constant and is of course also flow-invariant.

The above observations are in fact valid for the *general* monotypic cohomogeneity one expanding soliton equations, not just for the warped product case, provided we make the general definition

$$\mathcal{Q} := W^2 \mathcal{E} = W^2 (C + \epsilon u) \quad \text{and} \quad \mathcal{H} := W \operatorname{tr} L.$$

(The conservation law shows that this is consistent with the earlier formula for $\mathcal{Q}$ that we gave in the warped product case; see [Dancer et al. 2013, (4.6)].) We refer to [Dancer et al. 2013] for a discussion of this topic as well as the qualitatively different situation of shrinking solitons, where ϵ is negative. However, apart from the multiple warped product case, these formulae for $\mathcal{Q}$ involve polynomial or rational expressions in the X_i and Y_i variables which need not be definite, so the estimates obtained are not coercive.

In the warped product case with all λ_i positive, which was the situation examined in [Dancer and Wang 2009a], $\mathcal{Q}$ is, as explained above, a positive definite form (up to an additive constant) in the X_i, Y_i, so we obtained coercive estimates which allowed us to analyse the flow. For the rest of this section, we shall look at the case where the collapsing factor M_1 is S^1, so $d_1 = 1$, $\lambda_1 = 0$, and the remaining Einstein constants λ_i are positive. Then the equation for X_1 becomes

$$X_1' = X_1 \bigg(\sum_{j=1}^{r} X_j^2 - 1 \bigg) + \frac{\epsilon}{2} (1 - X_1) W^2.$$

As $\mathcal{Q}$ now does not include a Y_1 term, the region $\mathcal{Q} < 0$ is no longer precompact. However, we will see by using similar ideas to those in [Buzano et al. 2013] that we can still analyse the flow.

It is clear that we can recover t and g_i from a solution X, Y, W of the above system via the relation $dt = W\,ds$ and the formulae (2.2), (2.3), (2.4). As usual we choose $t = 0$ to correspond to $s = -\infty$. The soliton potential u is recovered by integrating

$$\dot{u} = \operatorname{tr} L - \frac{1}{W} = \frac{\mathcal{H} - 1}{W} = \frac{\sum_{i=1}^{r} \sqrt{d_i} X_i - 1}{W}. \tag{2.10}$$

We next compute the critical points of the soliton system (2.5)–(2.7).

Lemma 2.11. *Let $d_1 = 1$ and $d_i > 1$ for $i > 1$, so that $\lambda_i = 0$ iff $i = 1$. The stationary points of* (2.5), (2.6), (2.7) *in X, Y, W-space consist of*

(i) *the origin*

(ii) *points with $W = 0$, $Y_i = 0$ for all i, and $\sum_{i=1}^{r} X_i^2 = 1$*

(iii) *points given by*

$$W = 0, \quad X_i = \sqrt{d_i}\,\rho_A, \quad Y_i^2 = \frac{d_i}{\lambda_i}\rho_A(1 - \rho_A), \quad i \in A$$

and $X_i = Y_i = 0$ for $i \notin A$, where A is any nonempty subset of $\{2, \dots, r\}$, *and $\rho_A = \left(\sum_{j \in A} d_j\right)^{-1}$*

(iv) *the line where $W = 0$, $X_i = 0$ for all i, and $Y_i = 0$ for $i > 1$*

(v) *the line where $W = 0$, $X_1 = 1$, and X_i, $Y_i = 0$ for $i > 1$.*

(vi) *the points $E_\pm$ with coordinates*

$$X_i = \frac{\sqrt{d_i}}{n}, \quad Y_i = 0, \quad W = \pm\sqrt{\frac{2}{n\epsilon}}. \qquad \square$$

Note that $\mathcal{L}$ equals -1 in case (i) and (iv), equals 0 in case (ii), (iii) and (v), and equals $(1 - n)/n$ in case(vi). Also $\mathcal{Q}$ is -1 in cases (i) and (iv) and 0 otherwise. Cases (i)–(v) arose in [Buzano et al. 2013] in the steady case. Case (vi) is special to the expanding case and arose in [Dancer and Wang 2009a]. Again the origin is no longer an isolated critical point.

The analysis of the equations is quite similar to that in [Dancer and Wang 2009a], with appropriate changes as in [Buzano et al. 2013] to reflect the fact that one factor M_1 of the product hypersurface is flat. Accordingly, we shall be brief in our discussion.

We look for solutions where the flat factor $M_1 = S^1$ collapses at the end corresponding to $t = 0$ (that is, $s = -\infty$). In our new variables, this translates into considering trajectories in the unstable manifold of the critical point P of (2.5)–(2.7) (of type (v)) given by

$$W = 0, \quad X_1 = 1, \quad Y_1 = 1, \quad X_i = Y_i = 0 \quad (i > 1).$$

Note that at this critical point we have $\mathcal{L} = \mathcal{Q} = 0$ and $\mathcal{G} = \mathcal{H} = 1$.

The linearisation about this critical point is the system

$$\begin{aligned} x_1' &= 2x_1, \\ y_1' &= x_1, \\ x_i' &= 0 \quad\; (i \geq 2), \\ y_i' &= y_i \quad (i \geq 2), \\ w' &= w \end{aligned}$$

with eigenvalues 2, 1 (r times), and 0 (r times).

The results of [Buzano 2011] now show we have an r-parameter family of trajectories $\gamma(s)$ emanating from P and pointing into the region $\{\mathcal{Q} < 0, \mathcal{H} < 1\}$. Moreover, by the arguments above, such trajectories stay in this region. We can choose the trajectories to have W, Y_i positive for all time, as the loci $\{Y_i = 0\}$ or $\{W = 0\}$ are flow-invariant and the equations are invariant under changing the sign of W and/or of any Y_i.

As mentioned above, as M_1 is flat and Y_1 does not appear in $\mathcal{Q}$, the region $\{\mathcal{Q} < 0\}$ is no longer precompact. However, since the variable Y_1 only enters into the equations through the equation for Y_1', we may follow [Buzano et al. 2013] and consider the subsystem obtained by omitting the $i = 1$ equation in (2.6). The result is a system of equations in W, X_i ($i = 1, \ldots, r$) and Y_i ($i = 2, \ldots, r$), and on this $2r$-dimensional phase space the locus $\{\mathcal{Q} < 0\}$ *is* precompact. Once we have a long-time solution to the subsystem, Y_1 may be recovered via

$$Y_1(s) = Y_1(s_0) \exp\left(\int_{s_0}^{s} \sum_{j=1}^{r} X_j^2 - X_1 - \frac{\epsilon}{2} W^2 \right),$$

where s_0 is a fixed but arbitrary constant.

The critical points of the subsystem are obtained by removing the Y_1-coordinate from those of the full system. In particular, the origin becomes an isolated critical point, and case (v) of Lemma 2.11 gives rise to the special critical point $\hat{P}$ with $W = 0$, $X_1 = 1$, $X_i = 0$ ($i > 1$), $Y_i = 0$ ($i = 2, \ldots, r$), from which emanates an r-parameter family of local solutions lying in the region

$$\{W > 0, Y_i > 0 \ (i > 1), \mathcal{Q} < 0, \mathcal{H} < 1\}.$$

The r parameters may be thought of as $g_i(0)$, $i > 1$, and the constant C in the conservation law (which has to be negative under the assumption that $u(0) = 0$). Homothetic solutions are eliminated by fixing the value of ϵ.

Precompactness of the region where the subsystem flow lives shows that the variables are bounded, so that the flow exists for all s. Hence the same is true for the original flow also. As in [Dancer and Wang 2009a, Lemma 2.2] we can show that the X_i are positive for all s. It follows that $\mathcal{H} > 0$ and $X_i < 1/\sqrt{d_i}$. Furthermore, we still have the equation

$$\left(\frac{W}{Y_i}\right)' = \frac{X_i}{\sqrt{d_i}}\left(\frac{W}{Y_i}\right),$$

including the possibility $i = 1$. So W/Y_i increases monotonically to a limit $\sigma_i \in (0, \infty]$. (We shall presently show that the σ_i must all be equal to $+\infty$.)

As the trajectories of interest lie in a precompact set, each of them has a nonempty ω-limit set Ω, where we suppressed the dependence on the trajectory. Moreover, each Ω is compact, connected, and invariant under both forward and backward flows.

As in [Dancer and Wang 2009a, p. 1115] we can show that Ω lies in the locus $\{Y_i = 0, 2 \leq i \leq r\}$. Now, on this locus the flow is just the same as that in [Dancer and Wang 2009a], and the arguments there (see pp. 1116–1120) show as before that Ω contains the origin (in the phase space for the subsystem). The centre manifold argument in [Dancer and Wang 2009a, pp. 1121–1122] then shows the origin is a nonlinear sink, so in fact the trajectory converges to the origin.

Now we can follow the arguments of for Lemma 3.13 in [Dancer and Wang 2009a] to show that

$$\lim_{s\to\infty} \frac{X_i}{W^2} = \Lambda_i := \frac{\lambda_i}{\sigma_i^2\sqrt{d_i}} + \frac{\epsilon}{2}\sqrt{d_i}, \tag{2.12}$$

where $\Lambda_i > 0$. This is valid in particular for $i = 1$, in which case $\Lambda_1 = \frac{\epsilon}{2}$. In fact, the proof of [Dancer and Wang 2009a, Lemma 3.15] shows that σ_i cannot be finite, and so $\Lambda_i/\sqrt{d_i} = \frac{\epsilon}{2}$ for all i. Applying this fact to the relation

$$\frac{\dot{g}_i}{g_i} = \frac{1}{\sqrt{d_i}}\frac{X_i}{W} = \frac{1}{\sqrt{d_i}}\frac{X_i}{W^2}W,$$

it follows that the hypersurfaces have asymptotically decaying principal curvatures.

The limits (2.12) also imply that, for sufficiently large s, there exist $a_1, a_2 > 0$ such that $a_1 W^4 \leq \mathcal{G} \leq a_2 W^4$, from which we deduce completeness of the soliton metric by using the relation $dt = W\,ds$ and the equation (from (2.7)) $W\,ds = dW/(\mathcal{G} - \frac{\epsilon}{2}W^2)$. We further have $W \sim 1/\sqrt{\epsilon s}$ and $s \sim \frac{1}{4}\epsilon t^2$.

The asymptotics for g_i, $i > 1$, are deduced as in [Dancer and Wang 2009a]. As for g_1, the equation

$$\left(\frac{W}{Y_1}\right)' = \frac{X_i}{\sqrt{d_i}}\left(\frac{W}{Y_1}\right)$$

and $X_1 \sim \frac{\epsilon}{2}W^2 \sim 1/(2s)$ show that $g_1 = W/Y_1$ is also asymptotically linear in t, so we have conical asymptotics for all factors.

Remark 2.13. This contrasts with the steady case, where the asymptotic geometry for $n = 1$, $r = 1$ (the cigar) is different from the paraboloid asymptotics for the Bryant solitons with $n > 1$, $r = 1$. In the steady case with $r > 1$ our work in [Buzano et al. 2013] yielded solitons of *mixed* asymptotic type, where g_1 tends to a positive constant and g_i behaves like $\sqrt{t}$ for $i > 1$.

In the expanding case, both the $n = 1$, $r = 1$ case (due to [Gutperle et al. 2003]) and the $n > 1$, $r = 1$ case (due to R. Bryant) have conical asymptotics, and our solutions here for the $r > 1$ case also exhibit conical behaviour.

We summarise the discussion in this section by the following:

Theorem 2.14. *Let $M_2, \ldots, M_r$ be closed Einstein manifolds with positive scalar curvature. There is an r-parameter family of nonhomothetic complete smooth expanding gradient Ricci soliton structures on the trivial rank 2 vector bundle over $M_2 \times \cdots \times M_r$, with conical asymptotics in the sense given above.* □

Remark 2.15. As in [Dancer and Wang 2009a] we can see directly from the equations that the soliton potential u is concave, in accordance with Proposition 1.11. We can similarly deduce directly that $\mathrm{Ric}(\bar{g}) + \frac{\epsilon}{2}\,\bar{g}$ is positive semidefinite, so $-u$ is subharmonic.

Next we note that when $r \geq 2$ the sectional curvatures $\kappa(X \wedge Y)$, for X, Y tangent to different Einstein factors, satisfy $-c_1/t^2 \leq \kappa(X \wedge Y) \leq -c_2/t^2 < 0$ for certain positive constants c_1, c_2. This shows that the hypothesis of $\lim_{t\to\infty} t^2|\mathrm{sect}| = 0$ in many results in [Chen 2012] is not satisfied by our examples. In particular, the simplest hypersurface type in our examples is $S^1 \times S^{n-1}$; see [Chen 2012, Theorem 4].

Furthermore, all sectional curvatures decay faster than $t^{-2+\delta}$ for an arbitarily small $\delta > 0$. Hence the ambient scalar curvature $\bar{R}$ tends to zero. Finally we note that none of the hypotheses (topological or metric) in the recent rigidity theorem of Chodosh [2014] are satisfied by our examples.

3. Complete Einstein metrics with negative scalar curvature

We may also consider the flow of equations (2.5)–(2.7) in the variety $\{\mathcal{Q} = 0, \mathcal{H} = 1\}$. Such solutions of course correspond to Einstein metrics with negative scalar curvature, the soliton potential now being constant. In the case when $d_i > 1$ for all i, such metrics were constructed earlier in [Böhm 1999] by dynamical systems methods as well. In [Dancer and Wang 2009a, Remark 4.13] we pointed out that a simpler proof of Böhm's result can be obtained using our special variables and the embedding of the Einstein system within the soliton system.

In the present situation, where $d_1 = 1$, the hypersurfaces in the multiple warped product no longer admit a positive Einstein product metric whose hyperbolic cone acts as an attractor for the Einstein system. Nevertheless our setup allows us easily to deduce the following:

Theorem 3.1. *Let $M_2, \dots, M_r$ be compact Einstein manifolds with positive scalar curvature. There is an $r-1$-parameter family of nonhomothetic complete smooth Einstein metrics on the trivial rank 2 vector bundle over $M_2 \times \cdots \times M_r$.*

To prove the theorem, we consider the $r-1$-parameter family of trajectories emanating from the critical point P and lying in the variety $\{\mathcal{Q} = 0, \mathcal{H} = 1\}$. Note that this variety is smooth.

As in the previous section, we see that the flow is defined for all s by first restricting to the subsystem obtained by omitting the equation for Y_1 and observing that the locus $\{\mathcal{Q} = 0\}$ is compact. As usual we can take Y_i, W positive on our trajectories, and we can show the X_i are positive also. In the following we will work with the subsystem.

The ω-limit set Ω of a fixed trajectory lies within the locus $\{Y_i = 0 : i = 2, \dots, r\}$ by the same argument as in the soliton case. However, the difference now is that no point in Ω can have W-coordinate equal to 0. Otherwise, $\mathcal{G} = 1$ and such a point is a critical point of type (ii) in Lemma 2.11. The argument in the last part of the proof of [Dancer and Wang 2009a, Proposition 3.6] then leads to a contradiction. This in particular implies that the only critical point of the flow lying in Ω is E_+ (since $W > 0$ along our trajectory).

We next consider the trajectory starting from a noncritical point in Ω.

Recall from [Dancer and Wang 2009a] that on the locus $\{\mathcal{Q} = 0, \mathcal{H} = 1, Y = 0\}$, the quantity $J := G - \frac{\epsilon}{2} W^2$ satisfies $0 \leq J \leq 1$ and the equation

$$J' = 2J(J - 1).$$

Moreover, $J = 1$ exactly when $W = 0$ and $\mathcal{G} = 1$, and $J = 0$ exactly at the critical points $E_\pm$ (of type (vi) in Lemma 2.11). Points with $W > 0$ (resp. $W < 0$) flow to E_+ (resp. E_-) and flow backwards to $W = 0$.

For our trajectory, W is necessarily positive, so we obtain a contradiction since Ω is compact, flow-invariant, and contains no point with zero W-coordinate. We therefore deduce that Ω is $\{E_+\}$. Now it was observed in [Dancer and Wang 2009a, Lemma 3.8] that for the flow on $\{\mathcal{Q} = 0, \mathcal{H} = 1\}$ the point E_+ is a sink, so our (original) trajectory converges to E_+.

As $dt = W\,ds$ and W is converging to a positive constant we deduce the metric is complete. Using (2.2) we see that the metric components g_i^2 grow exponentially fast asymptotically.

We end this section with some consequences of combining our existence theorems with a study of the differential topology of some of our examples.

We will focus on the case where $r = 2$ and M_2 is a homotopy sphere. Recall that Boyer, Galicki and Kollár [Boyer et al. 2005a; 2005b] have constructed Sasakian Einstein metrics with positive scalar curvature on all Kervaire spheres (with dimension $4m + 1$) and those homotopy spheres of dimension 7, 11 or 15 which bound parallelizable manifolds. As in [Buzano et al. 2013] we can take these Einstein manifolds or the standard sphere as M_2 in our constructions in Section 2 and Section 3. Since it follows from the independent work of K. Kawakubo [1969] and R. Schultz [1969] that the manifolds $\mathbb{R}^2 \times M_2$ and $\mathbb{R}^2 \times S^q$ are not diffeomorphic if M_2 is an exotic sphere (see [Kwasik and Schultz 2002]), we deduce the following:

Corollary 3.2. *In dimensions* 9, 13, 17 *and all dimensions* $4m+3$ *with* $m \neq 0$, 1, 3, 7, 15, 31, *there exist pairs of homeomorphic but not diffeomorphic manifolds both of which admit non-Einstein, complete, expanding gradient Ricci soliton structures. The same holds for complete Einstein metrics with negative scalar curvature.* □

Note also that our expanding gradient Ricci solitons and negative Einstein manifolds exhibit conical asymptotics. The corresponding cones are differentially of the form $\mathbb{R}_+ \times S^1 \times M_2$, where $\mathbb{R}_+$ is the set of positive real numbers. We are indebted to Ian Hambleton for providing an outline of the proof of the following consequence of the above-mentioned work of Kawakubo and Schultz.

Proposition 3.3. *Let* Σ^q *and* S^q *be, respectively, a nonstandard homotopy sphere and the standard* q*-sphere. Then the open cones* $\mathbb{R}_+ \times S^1 \times \Sigma$ *and* $\mathbb{R}_+ \times S^1 \times S^q$ *are not diffeomorphic.*

Proof (I. Hambleton). Let

$$\phi : \mathbb{R}_+ \times S^1 \times \Sigma^q \to \mathbb{R}_+ \times S^1 \times S^q$$

be an orientation-preserving diffeomorphism. For convenience, let

$$X = S^1 \times \Sigma^q, \quad X_a = \{a\} \times X,$$
$$Y = S^1 \times S^q, \quad Y_b = \{b\} \times Y.$$

By compactness, $\phi(X_1) \subset (a, b) \times Y$ for some $0 < a < b$. Moreover, by Alexander duality (applied to $(a, b) \times Y$ with the ends capped off by attaching $D^2_\pm \times Y$, for example), $\phi(X_1)$ is a two-sided hypersurface that separates $(a, b) \times Y$ into two path-connected open submanifolds of $\mathbb{R}_+ \times Y$.

Let $W_\pm$ denote the closures of these path components. Then, using the diffeomorphism ϕ, which has to preserve the ends of $\mathbb{R}_+ \times X$ and $\mathbb{R}_+ \times Y$, one easily sees that W_- (resp. W_+) is a compact manifold whose boundary consists of Y_a and $\phi(X_1)$ (resp. $\phi(X_1)$ and Y_b). Moreover, by composition with suitable retractions and the restrictions of ϕ or ϕ^{-1} to suitable subsets, one also sees easily that the inclusion of the boundary components into W_- are homotopy equivalences, i.e., W_- is an h-cobordism between its boundary components. Noting that the Whitehead group of $\pi_1(S^1 \times S^q) = \mathbb{Z}$ is trivial and applying the s-cobordism theorem, we get a contradiction to the result of Kawakubo and Schultz that $S^1 \times \Sigma^q$ and $S^1 \times S^q$ are not diffeomorphic. $\square$

Hence we obtain for the dimensions given in Corollary 3.2 pairs of non-Einstein, complete, expanding gradient Ricci solitons (or complete negative Einstein manifolds) whose asymptotic cones are homeomorphic but not diffeomorphic.

4. Numerical examples

We shall now look at some numerical solutions of the equations (1.1)–(1.3). The Ricci soliton equation in the cohomogeneity one setting has an irregular singular point at $t = 0$. We therefore follow the procedure in [Dancer et al. 2013, § 5; Buzano et al. 2013]. That is, we first find a series solution in a neighbourhood of the singular orbit satisfying the appropriate smoothness conditions. We then truncate the series and use the values of the resulting functions at some small $t_0 > 0$ as initial values to generate solutions of the equations for $t > t_0$ via a fourth-order Runge–Kutta scheme. Because the manifolds we are considering are noncompact, we check the numerics obtained against the general asymptotic properties given in Section 1.

The explicit cases that we shall look at are those where the hypersurface is the twistor space of quaternionic projective space and the total space of the corresponding Sp(1) bundle. For these examples, the estimates $\mathcal{Q} < 0$ and $\mathcal{H} < 1$ do not give coercive estimates, and we do not yet have analytical existence proofs. However the numerics give a strong indication that complete expanding solitons exist in these cases.

Let us recall the equations that will be analysed numerically, following [Buzano et al. 2013]. We consider cohomogeneity one manifolds with principal orbits G/K whose isotropy representation consists of two inequivalent $\mathrm{Ad}(K)$-invariant irreducible real summands. We assume that $K \subset H \subset G$, where H, K are closed subgroups of the compact Lie group G such that H/K is a sphere. A G-invariant background metric b is chosen on G/K such that it induces the constant curvature 1 metric on H/K. The cohomogeneity one manifolds are then the vector bundles $G \times_H \mathbb{R}^{d_1+1}$ where $H/K \subset \mathbb{R}^{d_1+1}$ is regarded as the unit sphere.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an $\mathrm{Ad}(K)$-invariant decomposition of the Lie algebra of G, so that $\mathfrak{p}$ is identified with the tangent space of G/K at the base point. We can further decompose $\mathfrak{p}$ into irreducible K-modules, thus $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are respectively the tangent spaces (at the base point) to the sphere H/K and the singular orbit G/H. Their respective dimensions are denoted by d_1 and d_2.

The metrics of cohomogeneity one take the form

$$\bar{g} = dt^2 + g_1(t)^2 \, \mathsf{b}|\mathfrak{p}_1 + g_2(t)^2 \, \mathsf{b}|\mathfrak{p}_2.$$

Letting $(z_1, \ldots, z_6) := (g_1, \dot{g_1}, g_2, \dot{g_2}, u, \dot{u})$, the gradient Ricci soliton equations become

$$\begin{aligned}
\dot{z_1} &= z_2, \\
\dot{z_2} &= -(d_1 - 1)\frac{z_2^2}{z_1} - d_2\frac{z_2 z_4}{z_3} + z_2 z_6 + \frac{d_1 - 1}{z_1} + \frac{A_3}{d_1}\frac{z_1^3}{z_3^4} + \frac{\epsilon}{2} z_1, \\
\dot{z_3} &= z_4, \\
\dot{z_4} &= -d_1\frac{z_2 z_4}{z_1} - (d_2 - 1)\frac{z_4^2}{z_3} + z_4 z_6 + \frac{A_2}{d_2}\frac{1}{z_3} - 2\frac{A_3}{d_2}\frac{z_1^2}{z_3^3} + \frac{\epsilon}{2} z_3, \\
\dot{z_5} &= z_6, \\
\dot{z_6} &= -z_6\left(d_1\frac{z_2}{z_1} + d_2\frac{z_4}{z_3}\right) + z_6^2 + \epsilon z_5 + C,
\end{aligned}$$

where the A_i are positive constants which appear in the scalar curvature function of the principal orbit. Note that, because of the backgound metric chosen, the coefficient A_1/d_1 of the $1/z_1$ term in the second equation becomes $d_1 - 1$, and for expanding solitons we have $\epsilon > 0$.

Recall also the general relation $(d_1 + 1)\,\ddot{u}(0) = C + \epsilon u$, which follows from the conservation law and the smoothness conditions at $t = 0$. In generating the numerics, we find it convenient to eliminate homothetic solutions by choosing ϵ to be 1. Furthermore, rather than setting $u(0) = 0$, as was done throughout Section 1, we now set the constant C to be zero. It then follows from the necessary condition $\mathcal{E} < 0$ that in the series solution we must arrange for $\ddot{u}(0) = u(0)/(d_1 + 1) < 0$, with $u(0)$ as an otherwise arbitrary parameter.

Example 1. We set $G = \mathrm{Sp}(m+1)$, $H = \mathrm{Sp}(m) \times \mathrm{Sp}(1)$, and $K = \mathrm{Sp}(m) \times \mathrm{U}(1)$. The principal orbit G/K is diffeomorphic to $\mathbb{CP}^{2m+1}$ and the singular orbit G/H is $\mathbb{HP}^m$. So $d_1 = 2$, $d_2 = 4m$, and $A_2 = 2m(m+2)$, $A_3 = m/2$ (with b chosen to be $-2\,\mathrm{tr}(XY)$). The initial values of $(z_1, \ldots, z_6)$ are given by $(0, 1, \bar{h}, 0, \bar{u}, 0)$, where $\bar{h} > 0$ and $\bar{u} < 0$. These give rise to a 2-parameter family of numerical solutions.

In Figure 1 on the next page we plot the functions g_i and u for the cases $m = 1$ and $m = 2$, with parameter values $\bar{h} = 6$ and $\bar{u} = -1$.

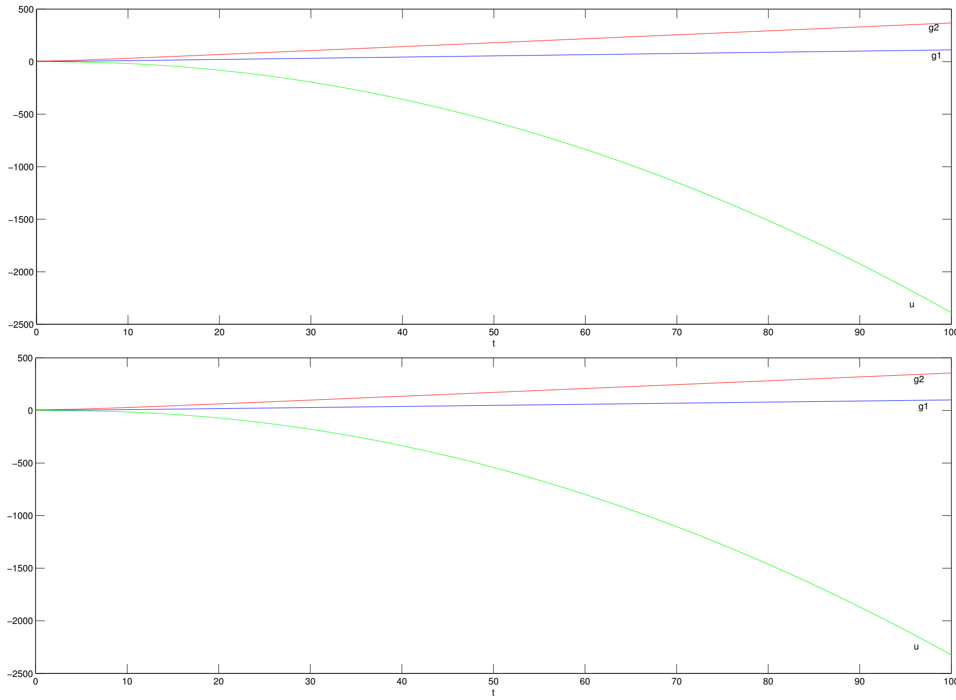

Figure 1. Plots of g_1 (blue), g_2 (red) and u (green) for $m = 1$ (top) and $m = 2$ (bottom).

Note that the soliton potential is concave down and becomes approximately quadratic, in accordance with Proposition 1.11 and Proposition 1.18. The g_i are asymptotically linear.

We have also plotted the quantities

$$\tilde{X}_i = \frac{X_i}{\sqrt{d_i}} \quad \text{and} \quad \tilde{Y}_i = \frac{Y_i}{\sqrt{d_1}}$$

against t in Figures 2 and 3 for the cases $m = 1$ and $m = 2$ respectively. They all converge quickly to 0.

In Figure 4 we plot the ratios $\tilde{X}_1/\tilde{X}_2$ and $\tilde{Y}_1/\tilde{Y}_2$. Note that the second ratio is g_2/g_1, which tends to a positive constant. The first ratio is the ratio of the principal curvatures, $(\dot{g}_2/g_2)\big/(\dot{g}_1/g_1)$, and we see that it quickly approaches 1.

Similar numerical results hold for larger values of m.

Example 2. We next set

$$\begin{aligned} G &= \mathrm{Sp}(m+1) \times \mathrm{Sp}(1), \\ H &= \mathrm{Sp}(m) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1), \\ K &= \mathrm{Sp}(m) \times \Delta\, \mathrm{Sp}(1). \end{aligned}$$

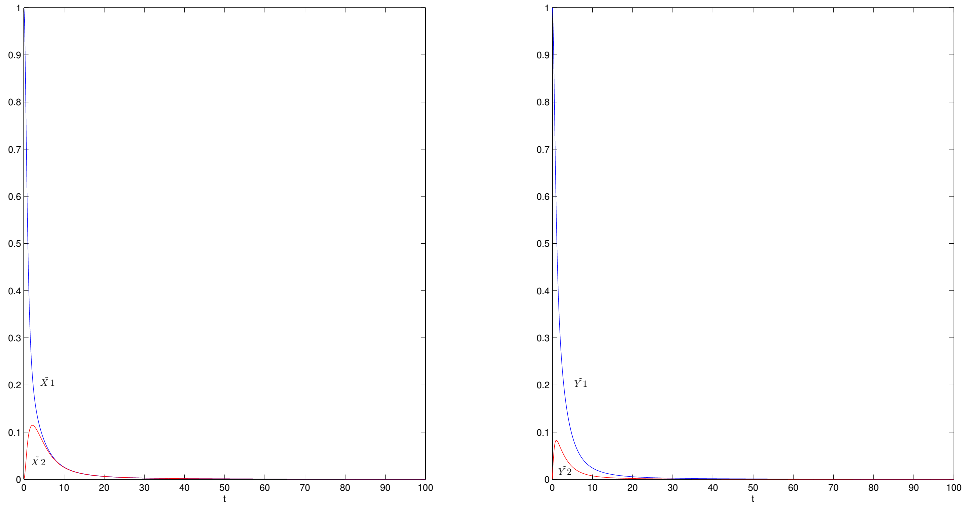

Figure 2. Plots of $\tilde{X}_i$ (left) and $\tilde{Y}_i$ (right) for $i = 1$ (blue) and $i = 2$ (red), in the case $m = 1$.

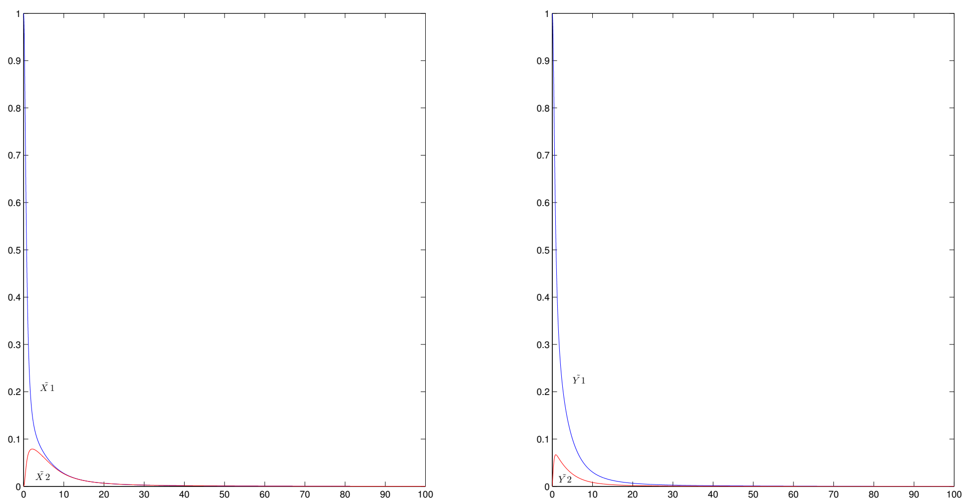

Figure 3. Plots of $\tilde{X}_i$ (left) and $\tilde{Y}_i$ (right) for $i = 1$ (blue) and $i = 2$ (red), in the case $m = 2$.

The principal orbit G/K is diffeomorphic to S^{4m+3} and the singular orbit G/H is again $\mathbb{HP}^m$. So $d_1 = 3$, $d_2 = 4m$, and $A_2 = 4m(m+2)$, $A_3 = 3m/4$ (where b is given by $-2\,\mathrm{tr}(XY)$ on both of the simple factors). The initial values of $(z_1, \ldots, z_6)$ are given by $(0, 1, \bar{h}, 0, \bar{u}, 0)$, where $\bar{h} > 0$ and $\bar{u} < 0$.

For this case we obtain graphs very similar to those in Example 1.

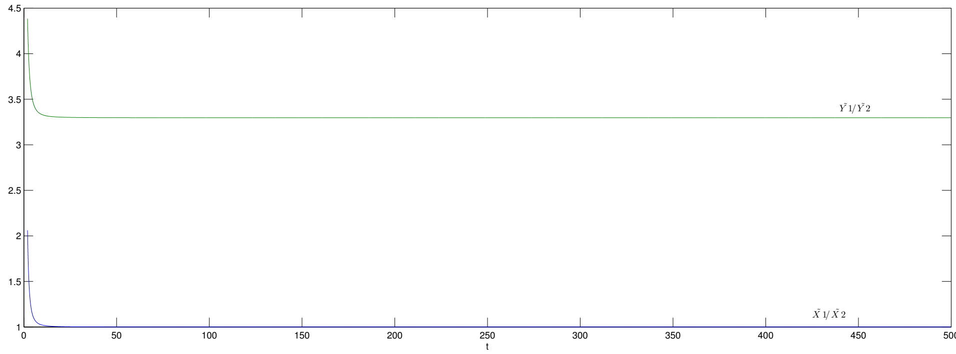

Figure 4. Plots of $\tilde{X}_1/\tilde{X}_2$ (lower curve) and $\tilde{Y}_1/\tilde{Y}_2$ (upper curve).

Based on the last two examples, we would conjecture that on the vector bundles $G \times_H \mathbb{R}^{d_1+1}$, where (G, H, K) are as above, there is a 2-parameter family of nonhomothetic complete expanding gradient Ricci solitons.

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