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It is known that, if we ignore gravitational forces, the shape of an equilibrium drop in \mathbb{R}^3 rotating about the z -axis is a surface that satisfies the equation $2H = \Lambda_0 - \frac{1}{2}aR^2$, where H is the mean curvature and R is the distance from a point in the surface to the z -axis. We consider helicoidal immersions in \mathbb{R}^3 that satisfy the rotating drop equation. We prove the existence of properly immersed solutions that contain the z -axis. We also show the existence of several families of embedded examples. We describe the set of possible solutions and we show that most of these solutions are not properly immersed and are dense in the region bounded by two concentric cylinders. We show that all properly immersed solutions, besides being invariant under a one-parameter helicoidal group, are invariant under a cyclic group of rotations of the variables x and y .

The second variation of energy for the volume constrained problem with Dirichlet boundary conditions is also studied.

1. Introduction

In this paper we study the equilibrium shape of a rotating liquid drop or liquid film which is invariant under a helicoidal motion of the three dimensional Euclidean space. The subject of rotating drops has been studied by many authors, including Chandrasekhar [1965], Brown and Scriven [1980], Solonnikov [2004]. Our main objective here is to use a new construction, recently developed in [Perdomo 2012], to construct an abundant supply of examples. This construction is closely related to Delaunay's classical construction of the axially symmetric constant mean curvature surfaces whose generating curves are produced by rolling a conic section. A special case of the type of surface which we study here occurs when the rotating drop is a cylinder over a plane curve. We treat that case in detail in [Palmer and Perdomo 2014].

If a rigid object is moved from one position in space to another, this repositioning can be realized via a helicoidal motion of \mathbb{R}^3 . If this motion is then successively repeated, one arrives at a configuration which is invariant under a helicoidal motion.

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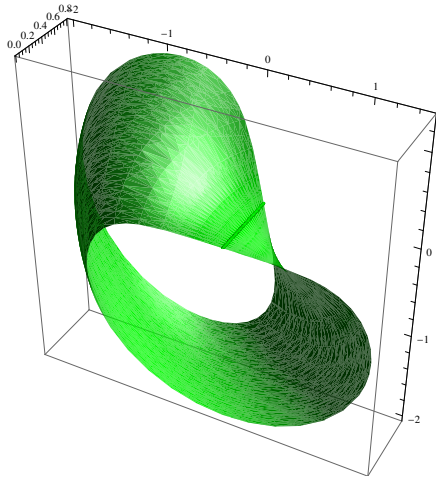


Figure 1. A rotating helicoidal drop.

The simple idea that helicoidal motions give all repeated motions of a rigid object, known in the physical sciences as Pauling’s theorem [Cahill 2005; Pauling et al. 1951], is behind many of the occurrences of helicoidal symmetry in nature [Barros and Ferrández 2009], since it allows for extensive growth with a minimal amount of information.

We will consider the equilibrium shape of a liquid drop rotating in a zero gravity environment with a constant angular velocity Ω about a vertical axis. The surface of the drop, which we denote by Σ , is represented as a smooth surface. The bulk of the drop is assumed to be occupied by an incompressible liquid of a constant mass density ρ_1 , while the drop is surrounded by a fluid of constant mass density ρ_2 . Since the drop is liquid, its free surface energy is proportional to its surface area \mathcal{A} , and we take the constant of proportionality to be one. The downward gravitational force is neglected. This is justified if the volume of the drop is sufficiently small compared to the other parameters. The rotation contributes a second energy term of the form $-\Omega^2 \Delta \mathcal{F}$, where $\Delta \mathcal{F}$ is difference of moments of inertia about the vertical axis:

$$\Delta \mathcal{F} := (\rho_1 - \rho_2) \int_U R^2 dv.$$

This term represents twice the rotational kinetic energy.

The total energy is thus of the form

$$(1-1) \quad \mathcal{E} := \mathcal{A} - \frac{\Omega^2}{2} \Delta \mathcal{F} + \Lambda_0 \mathcal{V},$$

where \mathcal{V} denotes the volume of the drop and Λ_0 is a Lagrange multiplier. Let $\Delta \rho := \rho_1 - \rho_2$; then by introducing a constant $a := (\Delta \rho) \Omega^2$, we can write the

functional in the form

$$\mathcal{E}_{a,\Lambda_0} = \mathcal{A} - \frac{a}{2} \int_U R^2 dV + \Lambda_0 \mathcal{V},$$

where U is the three-dimensional region occupied by the bulk of the drop and $R := \sqrt{x_1^2 + x_2^2}$.

Since we want to consider both embedded and immersed surfaces, we will precisely define the last two terms in the energy in the following way. First, define vector fields on \mathbb{R}^3 by

$$W = \nabla' \frac{R^4}{16} = \frac{R^2}{4} (x_1, x_2, 0), \quad W_0 = \nabla' \frac{R^2}{4} = \frac{1}{2} (x_1, x_2, 0).$$

If $\nabla' \cdot$ denotes the divergence operator on \mathbb{R}^3 , then it is easily checked that $\nabla' \cdot W_0 = 1$ and $\nabla' \cdot W = R^2$ hold. We then define

$$\mathcal{V} := \int_{\Sigma} W_0 \cdot \nu d\Sigma, \quad \int_U R^2 dv := \int_{\Sigma} W \cdot \nu d\Sigma.$$

The definitions are valid as long as Σ is immersed and oriented.

We will next derive the first variation of the functional given above. Let

$$X_{\epsilon} = X + \epsilon(\psi \nu + T) + \dots$$

be a variation of X , where ψ is a smooth function, ν is the unit normal to the surface and T is a tangent vector field along Σ . The first variation formula for the area gives

$$\delta \mathcal{A} = - \int_{\Sigma} 2H\psi d\Sigma + \oint_{\partial\Sigma} T \cdot n ds = - \int_{\Sigma} 2H\psi d\Sigma + \oint_{\partial\Sigma} dX \times \nu \cdot T.$$

We will show in the [Appendix](#) that

$$(1-2) \quad \delta \int_{\Omega} R^2 dV = \int_{\Sigma} \psi R^2 d\Sigma + \oint_{\partial\Sigma} dX \times W \cdot \delta X,$$

where W is a vector field satisfying $\nabla' \cdot W = R^2$ on \mathbb{R}^3 , and it is well known that the first variation of volume is

$$(1-3) \quad \delta \mathcal{V} = \int_{\Sigma} \psi d\Sigma + \oint_{\partial\Sigma} dX \times W_0 \cdot \delta X.$$

By combining the last three formulas, we arrive at

$$(1-4) \quad \delta \mathcal{E}_{a,\Lambda_0} = \int_{\Sigma} \left(-2H - \frac{a}{2} R^2 + \Lambda_0 \right) \psi d\Sigma + \oint_{\partial\Sigma_1} dX \times \left(\nu - \frac{a}{2} W + \Lambda_0 W_0 \right) \cdot \delta X.$$

Regardless of the boundary conditions, a necessary condition for an equilibrium is that the equation

$$(1-5) \quad 2H = -\frac{a}{2}R^2 + \Lambda_0$$

holds in the interior of Σ .

If we assume that the surface has free boundary contained in a supporting surface S having outward unit normal N , then the admissible variations must satisfy the condition $\delta X \cdot N \equiv 0$ on $\partial\Sigma$. In order for the boundary integral in (1-4) to vanish for all admissible variations, $dX \times \nu$ must be parallel to N along the boundary, which means that the surface Σ meets the supporting surface S in a right angle.

We now assume that an equilibrium surface Σ , i.e., a surface satisfying (1-5), is invariant under a helicoidal motion

$$(1-6) \quad (x_1 + ix_2, x_3) \mapsto (e^{-i\omega t}(x_1 + ix_2), x_3 + t),$$

and we will derive a conservation law which characterizes the equilibrium surfaces. We do not assume that the angular velocity ω which determines the pitch of the helicoidal surface is the same as the angular velocity Ω appearing above.

Let Σ_1 denote the compact region in Σ bounded on the sides by two integral curves C_1 and C_2 of the Killing field

$$\mathcal{H}(X) := -\omega E_3 \times X + E_3$$

and bounded below and above by the horizontal planes $x_3 = 0$ and $x_3 = 2\pi/\omega$. Then Σ_1 is a compact surface with oriented boundary $C_1 + \alpha_1 - C_2 - \alpha_2$, where α_1 and α_2 are congruent arcs in the planes $x_3 = 2\pi/\omega$ and $x_3 = 0$, respectively. By the calculations in the [Appendix](#), we have, using (1-5),

$$(1-7) \quad \delta\mathcal{E}[\Sigma_0] = \oint_{\partial\Sigma_1} dX \times \left(\nu - \frac{a}{2}W + \Lambda_0 W_0 \right) \cdot \delta X.$$

If we take the variation with $\delta X = E_3$ then, since E_3 generates a translation, the first variation will vanish. Consequently, we obtain

$$(1-8) \quad 0 = \oint_{\partial\Sigma_1} dX \times \left(\nu - \frac{a}{2}\nabla' \frac{R^4}{16} + \Lambda_0 \nabla' \frac{R^2}{4} \right) \cdot E_3.$$

Note that the integration over the 1-chain $\alpha_1 - \alpha_2$ yields zero since the two arcs are congruent and are traversed in opposite directions. On C_i , $i = 1, 2$, we have

$\frac{dX}{dt} = (-1)^{i+1}\mathcal{J}t$. Hence

$$\begin{aligned} dX &\times \left(\nu - \frac{a}{2}\nabla' \frac{R^4}{16} + \Lambda_0 \nabla' \frac{R^2}{4} \right) \cdot E_3 \\ &= (-1)^{i+1}(-\omega E_3 \times X + E_3) \times \left(\nu - \frac{a}{2}\nabla' \frac{R^4}{16} + \Lambda_0 \nabla' \frac{R^2}{4} \right) \cdot E_3 dt \\ &= (-1)^i(-\omega E_3 \times X + E_3) \times E_3 \cdot \left(\nu - \frac{a}{2}\nabla' \frac{R^4}{16} + \Lambda_0 \nabla' \frac{R^2}{4} \right) dt \\ &= (-1)^{i+1}\omega(x_1, x_2, 0) \cdot \left(\nu - \frac{a}{8}R^2(x_1, x_2, 0) + \frac{\Lambda_0}{2}(x_1, x_2, 0) \right) dt \\ &= (-1)^{i+1}\omega \left((Q - x_3\nu_3) - \frac{a}{8}R^4 + \frac{\Lambda_0 R^2}{2} \right) dt, \end{aligned}$$

where $Q = X \cdot \nu$ is the support function of the surface. Setting $\hat{Q} := Q - x_3\nu_3$, we can conclude from this that the integral

$$\int_{C_i} \left(\hat{Q} - \frac{a}{8}R^4 + \frac{\Lambda_0 R^2}{2} \right) dt$$

is independent of i . Also, it is easily checked that the integrand is, in fact, constant on each helix C_i , and we obtain the result that

$$(1-9) \quad 2\hat{Q} + \Lambda_0 R^2 - a \frac{R^4}{4} \equiv \text{constant}.$$

Proposition 1.1. *Let Σ be a helicoidal surface. A necessary and sufficient condition that Σ is a critical point for the functional $\mathcal{E}_{a,\Lambda_0}$ is that (1-9) holds.*

Proof. The necessity was shown above, so we now show that the condition is sufficient. We can assume that the helicoidal symmetry group of the surface fixes the vertical axis.

Any helicoidal surface arises as the orbit of a planar ‘‘generating curve’’ α under a helicoidal motion. We let s be the arc-length coordinate of α and we let t denote a coordinate for the helices which are the orbits of points in α . Local calculations which can be found in [Perdomo 2012] show that the mean curvature H and the third component of the normal ν_3 are functions of s alone. Also, it is clear that the function R^2 only depends on s .

It is easy to see that if ν_3 vanishes on any arc of α , then this arc is necessarily circular. It is clear that the orbit of a circular arc satisfying (1-9) is an equilibrium surface for the functional $\mathcal{E}_{a,\Lambda_0}$. Now consider a connected arc $\eta \subset \alpha$ on which (say) $\nu_3 > 0$ holds almost everywhere. If (1-5) does not hold on α , we can assume, by replacing α with a subarc if necessary, that $-2H - aR^2/2 + \Lambda_0 > 0$ also holds almost everywhere on α . Let Σ_1 denote the compact domain consisting of the orbit

of the arc α for $0 \leq t \leq 2\pi/\omega$. The boundary of Σ_1 consists of two helices C_1, C_2 together with two arcs α_1, α_2 , both congruent to α .

We take the first variation of $\mathcal{E}_{a,\Lambda_0}$ with the variation field being the constant vector E_3 . Since E_3 is the generator of a one-parameter family of isometries, this first variation vanishes. We express the first variation as in (1-4). Taking into account (1-9), the contribution to the boundary integral is zero since it is given by the right-hand side of (1-8), which vanishes. Also, the integrals over α_1 and α_2 cancel each other since these arcs are congruent and are traversed in opposite directions. We then obtain from the calculations given above that

$$0 = \int_{\Sigma_1} \left(-2H - \frac{1}{2}aR^2 + \Lambda_0\right) \nu_3 \, d\Sigma,$$

which is a contradiction since the integrand is positive almost everywhere on Σ_1 . \square

This result can easily be modified for axially symmetric surfaces. In that case, the Killing field used is simply $E_3 \times X$, and the helices are replaced by circles and (1-9) still holds.

2. Treadmill sled coordinates analysis

We will be considering immersions of the form

$$\phi(s, t) = (x(s) \cos \omega t + y(s) \sin \omega t, -x(s) \sin \omega t + y(s) \cos \omega t, t),$$

with the curve $\alpha(s) = (x(s), y(s))$ parametrized by arc length. We will refer to the curve α as the profile curve of the surface. The surface given as the image of ϕ is the orbit of α under the helicoidal motion (1-6). For $\theta(s)$ defined by

$$x'(s) = \cos \theta(s) \quad \text{and} \quad y'(s) = \sin \theta(s),$$

following [Perdomo 2012] we define the *treadmill sled coordinates* $\xi_1(s), \xi_2(s)$ by

$$(2-1) \quad \xi_1(s) = x(s) \cos \theta(s) + y(s) \sin \theta(s), \quad \xi_2(s) = x(s) \sin \theta(s) - y(s) \cos \theta(s).$$

The Gauss map of the immersion ϕ is given by

$$\nu = \frac{1}{\sqrt{1 + \omega^2 \xi_1^2}} (\sin(\theta - \omega t), -\cos(\theta - \omega t), -\omega \xi_1),$$

and so, by a direct calculation, we obtain $\hat{Q} = \xi_2/\sqrt{1 + \omega^2 \xi_1^2}$. Finally, using that $R^2 = x^2 + y^2 = \xi_1^2 + \xi_2^2$, we see from (1-9) that the immersion ϕ represents a rotating helicoidal drop if and only if there holds

$$(2-2) \quad G(\xi_1, \xi_2) := \frac{2\xi_2}{\sqrt{1 + \omega^2 \xi_1^2}} + \Lambda_0(\xi_1^2 + \xi_2^2) - \frac{a}{4}(\xi_1^2 + \xi_2^2)^2 \equiv \text{constant} =: C.$$

A direct computation shows that the equation $2H = \Lambda_0 - \frac{1}{2}aR^2$ reduces to

$$(2-3) \quad \theta'(s) = \frac{2w^2\xi_2 - 2\Lambda_0(1 + w^2\xi_1^2)^{3/2} + a(\xi_1^2 + \xi_2^2)(1 + w^2\xi_1^2)^{3/2}}{1 + w^2(\xi_1^2 + \xi_2^2)}.$$

From the definitions of ξ_1 , ξ_2 and θ , we get that $\xi_1' = 1 - \xi_2\theta'$ and $\xi_2' = \xi_1\theta'$. Using (2-3), we conclude that ξ_1 and ξ_2 must satisfy

(2-4)

$$\xi_1' = f_1(\xi_1, \xi_2) = \frac{(1 + w^2\xi_1^2)(2 + \sqrt{1 + w^2\xi_1^2}\xi_2(2\Lambda_0 - a(\xi_1^2 + \xi_2^2)))}{2(1 + w^2(\xi_1^2 + \xi_2^2))},$$

(2-5)

$$\xi_2' = f_2(\xi_1, \xi_2) = \frac{\xi_1(2w^2\xi_2 - 2\Lambda_0(1 + w^2\xi_1^2)^{3/2} + a(\xi_1^2 + \xi_2^2)(1 + w^2\xi_1^2)^{3/2})}{2(1 + w^2(\xi_1^2 + \xi_2^2))}.$$

This system of ordinary differential equations for ξ_1 and ξ_2 provides a different proof of the fact that the $G(\xi_1(s), \xi_2(s))$ must be constant, because we can check directly that

$$\frac{\partial G}{\partial \xi_1} = -\frac{2 + 2w^2(\xi_1^2 + \xi_2^2)}{(1 + w^2\xi_1^2)^{3/2}}f_2 \quad \text{and} \quad \frac{\partial G}{\partial \xi_2} = \frac{2 + 2w^2(\xi_1^2 + \xi_2^2)}{(1 + w^2\xi_1^2)^{3/2}}f_1.$$

Remark 2.1. The level sets of G are symmetric with respect to the ξ_2 -axis. Therefore, in order to understand the level set of G , it is enough to understand those points in the level set with $\xi_1 \geq 0$.

In order to study the level sets of the function G we replace the variables ξ_1 and ξ_2 with the variables r and ξ_2 , where

$$r = \xi_1^2 + \xi_2^2.$$

Making this change, we obtain that the equation $G = C$ reduces to

$$\frac{2\xi_2}{\sqrt{1 + \omega^2r - \omega^2\xi_2^2}} + \Lambda_0r - \frac{a}{4}r^2 = C.$$

In fact, this equation is exactly the one appearing in (1-9).

We have

$$\xi_2 = \frac{(4C + r(-4\Lambda_0 + ar))\sqrt{1 + r\omega^2}}{\sqrt{64 + (4C + r(-4\Lambda_0 + ar))^2\omega^2}}.$$

By Remark 2.1, it is enough to consider those points with $\xi_1 \geq 0$. Since $\xi_1 = \sqrt{r - \xi_2^2}$, we get that

$$\xi_1 = \frac{\sqrt{p(r, a, \Lambda, C)}}{\sqrt{64 + (4C + r(-4\Lambda_0 + ar))^2\omega^2}},$$

where

$$(2-6) \quad p(r, a, \Lambda, C) = -16C^2 + 64r + 32C\Lambda_0r - 16\Lambda_0^2r^2 - 8aCr^2 + 8a\Lambda_0r^3 - a^2r^4.$$

Remark 2.2. Since $p(r, a, \Lambda, C)$ is a polynomial in r of degree 4 with negative leading coefficient when $a \neq 0$ and p is a polynomial of degree two when $a = 0$, we get that the values of r for which $p(r, a, \Lambda, C)$ is positive are bounded. Since $r = \xi_1^2 + \xi_2^2 = x^2 + y^2$, we conclude that the profile curve of any helicoidal rotating drop is bounded.

Definition 2.3. Let r_1 and r_2 be nonnegative values such that $p(r_1) = p(r_2) = 0$ and $p(r) > 0$ for all $r \in (r_1, r_2)$. We define $\rho : [r_1, r_2] \rightarrow \mathbb{R}^2$ by

$$\rho(r) = \left(\frac{\sqrt{p(r, a, \Lambda, C)}}{\sqrt{64 + (4C + r(-4\Lambda_0 + ar))^2 \omega^2}}, \frac{(4C + r(-4\Lambda_0 + ar))\sqrt{1 + r\omega^2}}{\sqrt{64 + (4C + r(-4\Lambda_0 + ar))^2 \omega^2}} \right).$$

Remark 2.4. As pointed out before, all the level sets of the function G are bounded. The map ρ parametrizes half of the level set $G = C$.

Definition 2.5. In the case that the level set $G = C$ is a regular closed curve or a union of regular closed curves, we define a fundamental piece of the profile curve as a simple connected part of the profile curve such that the parametrized curve (ξ_1, ξ_2) , given by (2-1), corresponds to exactly one closed curve in the level set of $G = C$.

Remark 2.6. From the definition of treadmill sled given in [Perdomo 2012], we obtain that the profile curve of the solutions of the helicoidal rotational drop equation are characterized by the property that their treadmill sleds are the level sets of G . In other words, using the notation of [ibid.], we have $TS(\alpha) = \beta$, where β is a parametrization of a connected component of the level set of $G = C$ and α is the profile curve of the helicoidal rotating drop. We will see that, for a few exceptional examples, the profile curve is a bounded complete curve having a circle as a limit cycle. For the nonexceptional examples we can define an initial and final point of the fundamental piece, and the whole profile curve is the union of rotated fundamental pieces. If $R_1 = \min\{|m| : m \in TS(\alpha)\}$ and $R_2 = \max\{|m| : m \in TS(\alpha)\}$ and $\Delta\tilde{\theta}$ is the variation of the angle between $\vec{0p_1}$ and $\vec{0p_2}$, where p_1 and p_2 are the initial and final points of a fundamental piece, then, the profile curve is properly immersed if $\Delta\tilde{\theta}/\pi$ is a rational number, otherwise the profile curve is dense in the set $\{(x, y) \in \mathbb{R}^2 : R_1 \leq |(x, y)| \leq R_2\}$.

We compute the variation $\Delta\tilde{\theta}$ in terms of the parameter r . We assume that $\alpha(s)$ is the profile curve of a helicoidal rotational drop. Recall that we are assuming that s is the arc-length parameter for the curve α . If $\beta(s) = (\xi_1(s), \xi_2(s))$, then

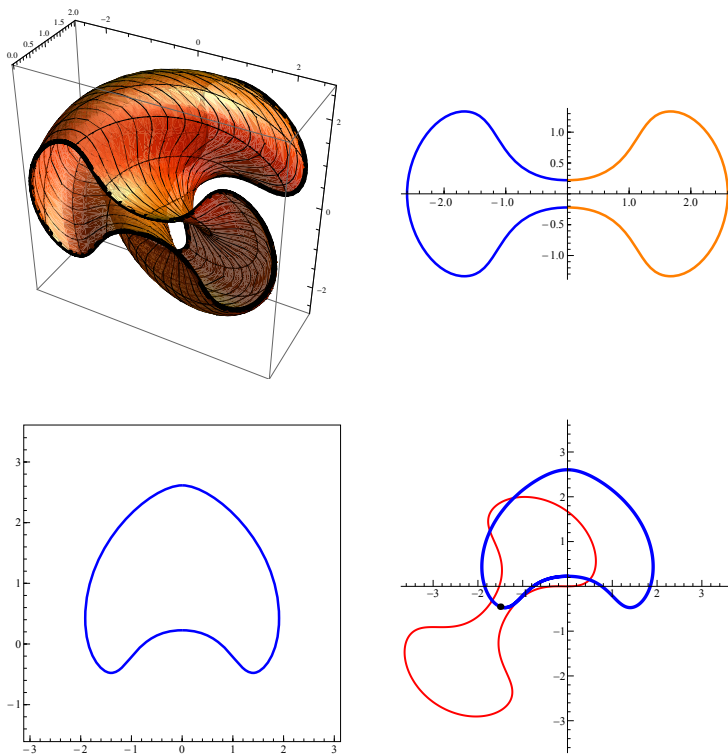


Figure 2. Top left: a helicoidal rotational drop. Top right: the corresponding profile curve, emphasizing the fundamental piece. Here $c = 0.5$, $a = 1$, $\Lambda_0 = 1$ and $\omega = 1.1805$. Bottom left: the (algebraic) level set $G = C$, i.e., the solution set to the equation

$$\frac{2\xi_2}{\sqrt{1 + 1.1805\xi_1^2}} + (\xi_1^2 + \xi_2^2) - \frac{1}{4}(\xi_1^2 + \xi_2^2)^2 = \frac{1}{2}.$$

Bottom right: how treadmill sled of the profile curve produces the level set $G = C$. In this example the treadmill sled of the profile curve will go over the level set $G = C$ two times.

$\beta(\sigma(r)) = \rho(r)$ for some function $s = \sigma(r)$. By the chain rule, we have

$$\begin{aligned} (2-7) \quad \frac{ds}{dr} Y &= \frac{d\sigma}{dr} = \frac{|\rho'(s)|}{|\beta'(s)|} = \sqrt{\frac{|\rho'(r)|^2}{f_1^2(s) + f_2^2(s)}} \\ &= \frac{1}{2} \sqrt{\frac{64 + (4C + r(-4\Lambda_0 + ar))^2 \omega^2}{p(r, a, C)}}. \end{aligned}$$

If $\tilde{\theta}$ denotes the polar angle of the profile curve, that is, if $\tilde{\theta}(s)$ satisfies the equation

$\alpha(s) = (x(s), y(s)) = R(s)(\cos \tilde{\theta}(s), \sin \tilde{\theta}(s))$, then $\tilde{\theta}'(s) = \xi_2(s)/r$, and

$$(2-8) \quad \frac{d\tilde{\theta}}{dr} = \frac{d\tilde{\theta}}{ds} \frac{ds}{dr} = \frac{1}{2} \frac{(4C + ar^2 - 4\Lambda_0 r)\sqrt{1 + r\omega^2}}{r\sqrt{p(r, a, C)}}.$$

Since the map $\rho : [r_1, r_2] \rightarrow \mathbb{R}^2$ parametrizes half of the treadmill sled of the fundamental piece of the profile curve, we obtain the following expression for function $\Delta\tilde{\theta}$ defined in [Remark 2.6](#):

$$(2-9) \quad \Delta\tilde{\theta} = \Delta\tilde{\theta}(C, a, \omega, r_1, r_2) = \int_{r_1}^{r_2} \frac{(4C + ar^2 - 4\Lambda_0 r)\sqrt{1 + r\omega^2}}{r\sqrt{p(r, a, C)}} dr.$$

Remark 2.7. If we have a helicoidal rotational drop Σ and we multiply every point by a positive fixed number λ , that is, if we consider the surface $\lambda\Sigma$, then this new surface satisfies the equation of the rotating drop for some other values of Λ_0 and a . If we change the orientation of the profile curve of a surface Σ that satisfies the equation of the rotating drop with values Λ_0, a and H , then the reparametrized surface satisfies the equation with values $-\Lambda_0, -a$ and $-H$. With these two observations in mind, in order to consider all the helicoidal rotational drops, up to parametrizations, rigid motions and dilations, it is enough to consider two cases: (I) $\Lambda_0 = 0$ and $a = -1$, and (II) $\Lambda_0 = 1$ and a is any real number.

Case I: $\Lambda_0 = 0$ and $a = -1$. In this case, the polynomial $p(r, a, \Lambda, C)$ reduces to

$$q = q(r, C) = -16C^2 + 64r + 8Cr^2 - r^4.$$

Recall that we are interested in finding two consecutive positive roots of the polynomial q . Notice that when C is a negative large number then the polynomial q has no roots, and when C is a positive large number then the polynomial q has more than one root. In every case, $q(0) = -16C^2 \leq 0$, and the limit when $r \rightarrow \infty$ of $q(r)$ is negative infinity. The following lemma was proven in [\[Palmer and Perdomo 2014\]](#) and provides the number of possible roots of $q(r, C)$ in terms of the values of C . For completeness reasons we will present the proof in this paper as well.

Lemma 2.8. *For any $C > C_0 = -3/2^{2/3}$, the polynomial $q(r, C)$ has exactly two nonnegative real roots. When $C = C_0, \sqrt[3]{4}$ is the only real root of $q(r, C)$, and when $C < C_0, q(r, C)$ has no real roots.*

Proof. We have $q'(r) = 64 + 16Cr - 4r^3$. A direct computation shows that the only real solution of the system

$$q(r, C) = 0 \quad \text{and} \quad q'(r, C) = 0$$

is $C = -3/2^{2/3}$ and $r = 2^{2/3}$. This also follows from the fact that a Gröbner basis

of the polynomials $\{q, q'\}$ is the set $\{27 + 4C^3, -4C^2 + 9r\}$. Since

$$q''(2^{2/3}) = -48\sqrt[3]{2} < 0,$$

the polynomial q has either 0, 1 or 2 real roots for values of (C, r) near $(-3/\sqrt[3]{4}, \sqrt[3]{4})$. A direct computation shows that the only roots of q when $C = 0$ are $r = 0$ and $r = 4$, and that q has no real roots when $C = -2$. By continuity, we conclude that the lemma holds. Recall that $q(0) = -16C^2$ and $q(r) \rightarrow -\infty$ as $r \rightarrow \infty$. Notice that, if for some value of C the polynomial $q(r, C)$ has more than 2 roots, there should exist another solution of the equations $\{q(r, C) = 0, q'(r, C) = 0\}$, which is impossible. \square

Now we will compute the limit of $\Delta\tilde{\theta}$ when C goes to C_0 . We will use the following lemma from [Perdomo 2010].

Lemma 2.9. *Let $f(c, r)$ and $g(r, c)$ be smooth functions such that*

$$g(r_0, C_0) = \frac{\partial g}{\partial r}(r_0, C_0) = 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial r^2}(C_0, r_0) = -2A,$$

where $A > 0$. If $\{C_n\}$, $\{u_n\}$ and $\{v_n\}$ are sequences such that

$$C_n \rightarrow C_0, \quad u_n, v_n \rightarrow r_0,$$

with

$$u_n < r_0 < v_n, \quad g(u_n, c_n) = g(v_n, c_n) = 0 \quad \text{and} \quad g(r, c_n) > 0 \text{ for all } r \in (u_n, v_n),$$

then

$$\int_{u_n}^{v_n} \frac{f(c, r) dr}{\sqrt{g(c, r)}} \longrightarrow f(C_0, r_0) \frac{\pi}{\sqrt{A}} \quad \text{as } n \longrightarrow \infty.$$

Notice that helicoidal rotating drops are defined when C takes values from $C_0 = -3/\sqrt[3]{4}$ to ∞ . When $C = C_0$, the only root of the polynomial q is $r_0 = \sqrt[3]{4}$. If we apply Lemma 2.9 with

$$f(r, c) = \frac{(4C - r^2)\sqrt{1 + r\omega^2}}{r} \quad \text{and} \quad g(r, c) = q(r, C)$$

to the integral given in (2-9), we obtain that

$$(2-10) \quad \lim_{C \rightarrow C_0^+} \Delta\tilde{\theta} = B(\omega) = -\frac{2\pi}{\sqrt{3}}\sqrt{1 + \sqrt[3]{4}\omega^2}.$$

Remark 2.10. Recall that whenever $\Delta\tilde{\theta} = n2\pi/m$ for some pair of integers m and n , then the entire profile curve is properly immersed and it is invariant under the group \mathbb{Z}_m .

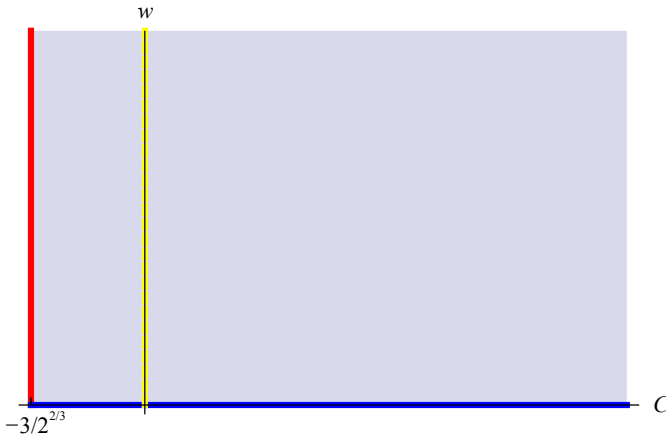


Figure 3. Moduli space of the helicoidal drops with $a = -1$ and $\Lambda = 0$. All points on the vertical red line correspond to a single surface, a round cylinder. Points on the vertical yellow line correspond to helicoidal rotational drops that contain the axis of symmetry. Points on the horizontal blue line correspond to cylindrical rotational drops.

Remark 2.11. Up to dilations and rigid motions, the moduli space for all helicoidal rotating drops with $\Lambda_0 = 0$ is the plane region

$$\{(C, \omega) : C \geq C_0 = -3/\sqrt[3]{4}, \omega > 0\}.$$

Moreover, for any $\omega > 0$, the surface associated with the point $(C, \omega) = (C_0, \omega)$ is a round cylinder of radius $\sqrt[3]{2}$, because it can be easily checked that when $C = C_0$, then, for any ω , the level set $G = C$ reduces to the point $\{(0, -\sqrt{2})\}$.

Case II: $\Lambda_0 = 1$. First note that the case $a = 0$ corresponds to a helicoidal surface with constant mean curvature. These surfaces were studied using similar techniques in [Perdomo 2012], and for this reason we will assume here that $a \neq 0$. In this case, the polynomial $p(r, a, \Lambda, C)$ reduces to

$$(2-11) \quad q = q(r, C, a) = -16C^2 + 64r + 32Cr - 16r^2 - 8aCr^2 + 8ar^3 - a^2r^4.$$

Recall that we are interested in finding two consecutive positive roots of the polynomial q . The roots of the polynomial q given in (2-11) were analyzed in [Palmer and Perdomo 2014]. In order to describe the roots of q we need to define the following functions.

Definition 2.12. Let $h(R) = 2(R - 1)/R^3$, and define the functions $R_1 : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$, $R_2 : (0, \frac{8}{27}] \rightarrow \mathbb{R}$ and $R_3 : (0, \frac{8}{27}] \rightarrow \mathbb{R}$ by

$$R_i(a) = R \quad \text{such that } h(R) = a, \quad \text{with } \begin{cases} R < 1 & \text{if } i = 1, \\ 1 < R \leq \frac{3}{2} & \text{if } i = 2, \\ \frac{3}{2} \leq R < \infty & \text{if } i = 3. \end{cases}$$

We also define the functions $r_1 : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$, $r_2 : (0, \frac{8}{27}] \rightarrow \mathbb{R}$ and $r_3 : (0, \frac{8}{27}] \rightarrow \mathbb{R}$ by

$$r_1(a) = R_1^2(a), \quad r_2(a) = R_2^2(a), \quad r_3(a) = R_3^2(a).$$

The following lemma was proven in [Palmer and Perdomo 2014] and provides the number of roots of q depending on the values a and C .

Lemma 2.13. *Let r_1, r_2 and r_3 be as in Definition 2.12 and for $i = 1, 2, 3$ define*

$$C_i = \frac{16 - 8r_i + 6ar_i^2 - a^2r_i^3}{4(-2 + ar_i)}$$

and

$$q = q(r, a, C) = -16C^2 + 64r + 32Cr - 16r^2 - 8aCr^2 + 8ar^3 - a^2r^4.$$

Recall that the domain of C_i is the same domain of r_i . That is, the domain of $C_1(a)$ is $\mathbb{R} \setminus \{0\}$ and the domain of $C_2(a)$ and $C_3(a)$ is the interval $(0, \frac{8}{27}]$. For any $a \neq 0$ and any C , the polynomial q has nonnegative real roots whose multiplicities are as follows. Let $\mathcal{N}(q)$ denote the number of distinct real roots of q . There are four cases to consider:

Case 1: *If $a < 0$, then $C_1(a) < 0$, and*

$$\mathcal{N}(q) = \begin{cases} 0 & \text{if } C < C_1(a), \\ 1 & \text{if } C = C_1(a), \\ 2 & \text{if } C > C_1(a). \end{cases}$$

Case 2: *If $0 < a < \frac{8}{27}$, then*

$$C_2(a) < 0, \quad C_1(a) > 0, \quad C_2(a) < C_3(a) < C_1(a),$$

and

$$\mathcal{N}(q) = \begin{cases} 0 & \text{if } C > C_1(a), \\ 1 & \text{if } C = C_1(a), \\ 2 & \text{if } C_3(a) < C < C_1(a), \\ 3 & \text{if } C = C_3(a), \\ 4 & \text{if } C_2(a) < C < C_3(a), \\ 3 & \text{if } C = C_2(a), \\ 3 & \text{if } C < C_2(a). \end{cases}$$

When $C = C_3(a)$, the second root has multiplicity two. When $C = C_2(a)$, the first root has multiplicity two.

Case 3: If $a = \frac{8}{27}$, then

$$C_1(a) = 9, \quad C_2(a) = C_3(a) = -\frac{9}{8},$$

and

$$\mathcal{N}(q) = \begin{cases} 2 & \text{if } C < -\frac{9}{8}, \\ 2 & \text{if } C = -\frac{9}{8}, \\ 2 & \text{if } -\frac{9}{8} < C < 9, \\ 1 & \text{if } C = 9, \\ 0 & \text{if } C > 9. \end{cases}$$

When $C = -\frac{9}{8}$, the roots are $\frac{9}{4}$ with multiplicity three and $\frac{81}{4}$ with multiplicity one. When $C = 9$, the only real root is 9, with multiplicity two.

Case 4: If $a > \frac{8}{27}$, then $C_1(a) > 0$, and

$$\mathcal{N}(q) = \begin{cases} 0 & \text{if } C > C_1(a), \\ 1 & \text{if } C = C_1(a), \\ 2 & \text{if } C < C_1(a). \end{cases}$$

Now that we have discussed the roots of the polynomial q we can describe the moduli space of all helicoidal rotating drops with $\Lambda = 1$.

Theorem 2.14. *Let $\Lambda_0 = 1$ and let $\Delta\tilde{\theta}$ be the function defined in (2-9). Let*

$$\begin{aligned} \Omega_1 &= \{(a, C, \omega) : C > C_1(a), a < 0, \omega > 0\}, \\ \Omega_2 &= \{(a, C, \omega) : C_2(a) < C < C_3(a), 0 < a < \frac{8}{27}, \omega > 0\}, \\ \Omega_3 &= \{(a, C, \omega) : C < C_1(a), a > 0, \omega > 0\}, \\ \Omega &= \Omega_1 \cup \Omega_3 \setminus \Omega_2 \\ \beta_1 &= \{(a, C, \omega) : C = C_1(a), a \neq 0, \omega > 0\}, \\ \beta_2 &= \{(a, C, \omega) : C = C_2(a), 0 < a < \frac{8}{27}, \omega > 0\}, \\ \beta_3 &= \{(a, C, \omega) : C = C_3(a), 0 < a < \frac{8}{27}, \omega > 0\}. \end{aligned}$$

Under the convention that a point (a, C, ω) represents a helicoidal rotating drop if the treadmill sled of its profile curve is contained in the level set $G = C$, we have:

- (i) Every point (a, C, ω) in the interior of Ω represents a helicoidal rotating drop with its fundamental piece having finite length. The treadmill sleds of the profile curves of these surfaces are parametrized by ρ defined for values of r between the only two roots of the polynomial $q(r, a, C)$.
- (ii) Every point in Ω_2 represents two helicoidal rotating drops, both having fundamental pieces of finite length. The treadmill sleds of the profile curves of these surfaces are parametrized by ρ defined for those values of r that lie between

the first and second root of the polynomial $q(r, a, C)$ and the third and fourth root of the polynomial $q(r, a, C)$ respectively.

- (iii) Every point (a, C, ω) in the set β_1 represents a circular helicoidal rotating drop. This cylinder is the same for all values of ω .
- (iv) Every point (a, C, ω) in the set β_2 represents two helicoidal rotating drops: a circular cylinder and a noncircular cylinder with bounded length of its fundamental piece. The circular cylinder is the same for all values of ω .
- (v) Every point in the set β_3 represents three helicoidal rotating drops. One is a circular cylinder, which is the same for all values of ω . The second one has a treadmill sled parametrized by ρ defined for those values of r that lie between the first and second root of the polynomial $q(r, a, C)$. Recall that the second root has multiplicity two. The third surface has a treadmill sled parametrized by ρ defined for those values of r that lie between the second and third root of the polynomial $q(r, a, C)$. The second and third surfaces are not properly immersed and their profile curves have a circle as a limit cycle and they have infinite winding number with respect to a point interior to this circle. Solutions whose profile curves possess a circle as a limit cycle will be called helicoidal drops of exceptional type.
- (vi) Points of the form $(a, C, \omega) = \left(\frac{8}{27}, -\frac{9}{8}, \omega\right)$ represent two helicoidal rotating drops: a circular cylinder, which is the same for all values of ω , and one helicoidal drop of exceptional type.
- (vii) Up to a rigid motion, every helicoidal drop falls into one of the cases above.
- (viii) Every helicoidal drop that is not exceptional is either properly immersed (when $\Delta\tilde{\theta}(a, C, \omega)/\pi$ is a rational number) or it is dense in the region bounded by two round cylinders (when $\Delta\tilde{\theta}(a, C, \omega)/\pi$ is an irrational number).

Proof. We already know that the treadmill sled of the profile curve of any helicoidal rotating drop satisfies the equation

$$G(\xi_1, \xi_2) = \frac{2\xi_2}{\sqrt{1 + \omega^2\xi_1^2}} + \Lambda_0(\xi_1^2 + \xi_2^2) - \frac{a}{4}(\xi_1^2 + \xi_2^2)^2 = C.$$

We also know that, up to rigid motions, the treadmill sled of a curve determines the curve; see [Perdomo 2012]. Since any level set of G can be parametrized using the map ρ given in Definition 2.3, and every parametrization of a level set of G is defined for values of r where the polynomial q is positive, it then follows from Lemma 2.13 that every helicoidal rotating drop can be represented as one of the cases (i), (ii), (iii), (iv), (v) and (vi). It is worth recalling (see Remark 2.4) that the parametrization ρ only covers half of the level set of the map G . Each one of these

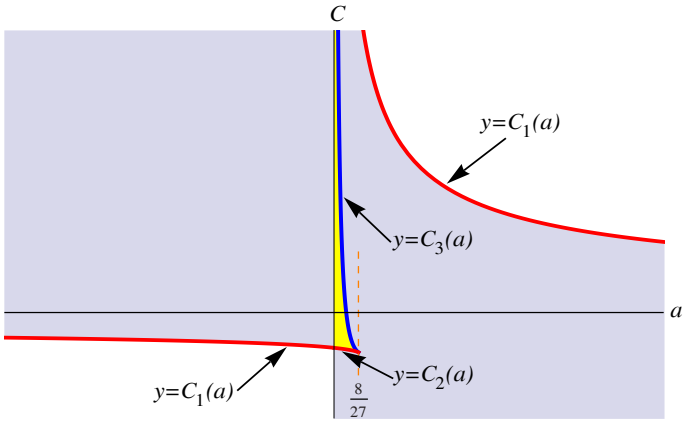


Figure 4. Graphs of the functions C_1 , C_2 and C_3 ; these functions are used in [Theorem 2.14](#) to describe the moduli space of all helicoidal rotational drops with $\Lambda_0 = 1$.

level sets is symmetric with respect to the ξ_2 -axis, and the parametrization ρ covers the half on the right.

Notice that when the profile curve is a circle, the level set $G = C$ reduces to a point. In this case we will take the parametrization ρ to be defined just in a point, a root with multiplicity two of the polynomial q .

When case (i) occurs, q has only two simple roots, x_1 and x_2 , with $x_1 < x_2$. We can check that the derivative of q at x_1 is positive while the derivative of q at x_2 is negative, so the length of the fundamental piece, according to (2-7), reduces to

$$\int_{x_1}^{x_2} \sqrt{\frac{64 + (4C + r(-4\Lambda_0 + ar))^2 \omega^2}{q(r, a, C)}} dr,$$

which converges. Therefore the length of the fundamental piece is finite.

For values of C , ω and a that fall into case (ii), the polynomial q has four roots $x_1 < x_2 < x_3 < x_4$, and q is positive from x_1 to x_2 and from x_3 to x_4 . Also, the level set of G has two connected components. Half of each connected component of $G = C$ can be parametrized using the map ρ . One half of the connected component of $G = C$ uses the domain (x_1, x_2) for ρ and the half of the other connected component of $G = C$ uses the domain (x_3, x_4) for ρ . The proof that the length of the fundamental piece of each surface is finite follows as in the proof in case (i).

For values of (a, C, ω) that satisfy the case (iii), the polynomial q has only one root $x_1 = r_1$ with multiplicity two. We take $R = \sqrt{x_1}$. A direct calculation shows that if $a > 0$, then $R_1(a) = -R$ and if we consider the profile curve

$$\alpha(s) = \left(R \sin \frac{s}{R}, -R \cos \frac{s}{R} \right),$$

then

$$\xi_1 = 0, \quad \xi_2 = R \quad \text{and} \quad G(\xi_1, \xi_2) = 2R + \Lambda_0 R^2 - \frac{1}{4}aR^4.$$

Using the definition of C_1 and the fact that $R_1(a) = -R$, we can check that the expression $G(\xi_1, \xi_2)$ reduces to $C = C_1(a)$, which was our goal in order to show that the point (a, C, ω) represents a round cylinder. Similarly, a direct verification shows that if $a < 0$, then $R_1(a) = R$ and if we consider the profile curve

$$\alpha(s) = \left(R \sin \frac{s}{R}, R \cos \frac{s}{R} \right),$$

then $\xi_1 = 0, \xi_2 = -R$ and $G(\xi_1, \xi_2) = -2R + \Lambda_0 R^2 - \frac{1}{4}aR^4$. Using the definition of C_1 and the fact that $R_1(a) = R$, we can check that the expression $G(\xi_1, \xi_2)$ reduces to $C = C_1(a)$. Since r_1 is independent of w these cylinders are independent of the value of w . This finish the proof of part (iii).

For values of (a, C, ω) that fall into case (iv), the polynomial q has three roots $x_1 < x_2 < x_3$, where $x_1 = r_1$ has multiplicity two and x_2 and x_3 are simple. The polynomial q is positive for $r \in (x_2, x_3)$. In this case the level set $G = C$ is the union of the point $(0, -\sqrt{x_1})$ and a closed curve. If we consider the cylinder of radius $\sqrt{x_1}$ oriented by the inward-pointing normal, then a direct computation shows that its mean curvature is $2H = 1 - \frac{1}{2}ar_1$. Therefore this circular cylinder is a helicoidal rotating drop for the given parameters. Note that this cylinder is independent of w . The treadmill sled of the profile curve of the other rotating drop is the level closed curve component of $G = C$; half of this part can be parametrized by the map ρ with domain (x_2, x_3) .

For values of (a, C, ω) in case (v), the polynomial q has only three roots $x_1 < x_2 < x_3$, where $x_2 = r_2$ has multiplicity two and x_1 and x_3 are simple. The polynomial q is positive for $r \in (x_1, x_2) \cup (x_2, x_3)$. In this case the level set $G = C$ is connected but it self-intersects at the point $(0, -\sqrt{x_2})$. Any part of a curve that crosses the ξ_2 -axis nonhorizontally cannot be the treadmill sled of a regular curve (see Proposition 2.11 in [Perdomo 2013]). Therefore the correct way to view the level set $G = C$ in this case is not as a connected closed curve that self-intersects but as the union of two curves and a point. Figure 5 shows one of these level sets.

One can check that the circular cylinder with radius $\sqrt{r_2}$, oriented by the inward-pointing normal, satisfies the equation $2H = 1 - \frac{1}{2}ar_2$; therefore this circular cylinder is a helicoidal rotating drop for the given parameters. The treadmill sled associated with the profile curve of this round cylinder reduces to the point $(0, -\sqrt{x_2})$. The set $G = C \setminus \{(0, -\sqrt{x_2})\}$ has two connected components. One of these connected components can be parametrized using the map ρ with $r \in (x_1, x_2)$ and the other using the map ρ with $r \in (x_2, x_3)$. Each of these connected components is the treadmill sled of the fundamental curve for a rotating helicoidal drop whose length

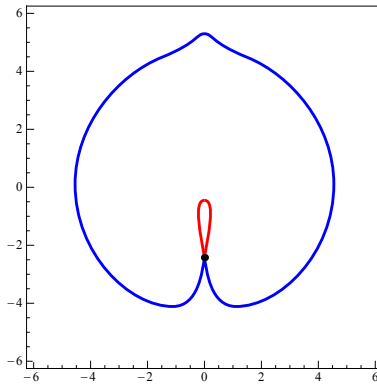


Figure 5. The level set $G = C$ when $a = 0.2$, $\omega = 3$, $\Lambda_0 = 1$ and $C = C_3(0.2)$. For helicoidal drops in case (v) of [Theorem 2.14](#), the level set of G should be regarded as the union of two curves and a point. Each curve is the treadmill sled of an exceptional helicoidal rotating drop and the point is the treadmill sled of a circular cylinder.

is unbounded. Specifically, their lengths are given by the divergent integrals

$$\int_{x_1}^{x_2} \sqrt{\frac{64 + (4C + r(-4\Lambda_0 + ar))^2 \omega^2}{q(r, a, C)}} dr$$

and

$$\int_{x_2}^{x_3} \sqrt{\frac{64 + (4C + r(-4\Lambda_0 + ar))^2 \omega^2}{q(r, a, C)}} dr,$$

respectively. Moreover, using the definition of treadmill sled, we notice that the function giving the distance to the origin of the profile curve, $|(x(s), y(s))|$, agrees with the function giving the distance to the origin of the level set $G = C$ given by $|(\xi_1(s), \xi_2(s))| = |\rho(\sigma^{-1}(s))|$. Therefore, as r approaches r_2 , $s = \sigma(r)$ goes to $-\infty$ and the function $|(x(s), y(x))|$ approaches $\sqrt{r_2}$. Since polar angle of the profile curve can be calculated by integrating the expression in (2-8), we conclude that $\tilde{\theta}(r)$ also goes to $-\infty$ as r approaches r_2 . We conclude that the profile curve has a circle of radius $\sqrt{r_2}$ as a limit cycle and it has infinite winding number with respect to a point interior to this circle.

For values of (a, C, w) that satisfy the case (vi), the polynomial q has only two roots $x_1 < x_2$, where $x_1 = \frac{9}{4}$ has multiplicity three and $x_2 = \frac{81}{4}$ is simple. The polynomial q is positive for $r \in (x_1, x_2)$. In this case the level set $G = C$ is connected but it has a singularity at the point $(0, -\frac{3}{2})$. We can check that the circular cylinder with radius $\frac{3}{2}$ oriented by the inward-pointing normal satisfies the equation $2H = 1 - \frac{1}{2}ar_1$; therefore this circular cylinder is a helicoidal rotating drop.

The treadmill sled associated to the profile curve of this round cylinder reduces to the point $(0, -\frac{3}{2})$. The set $G = C$ minus the point $(0, -\frac{3}{2})$ is connected and half of it can be parametrized using the map ρ with $r \in (\frac{9}{4}, \frac{81}{4})$. This part of the set $G = C$ is the treadmill sled of the fundamental curve of a rotating helicoidal drop whose length is not bounded.

Since we know that the profile curve of every rotating helicoidal drop satisfies the integral equation $G = C$ and cases (i)–(vi) cover all the possibilities for the level sets of G , then every rotating helicoidal drop falls into one of the first six cases of this proposition. This proves (vii).

In order to prove (viii) we notice that when a helicoidal rotating drop is not exceptional, it has a fundamental piece with finite length whose treadmill sled is a closed regular curve (a connected component of the set $G = C$). By the properties of the treadmill sled operator (in particular, the one that states that the treadmill sled inverse is unique up to rotations about the origin), we have that the whole profile curve is a union of rotations of the fundamental piece. The angle of rotation is given by $\Delta\tilde{\theta} = \Delta\tilde{\theta}(C, a, \omega, x_1, x_2)$. The profile curve is invariant under the group \mathbb{G} of rotations of the form

$$(2-12) \quad (y_1, y_2) \mapsto (\cos(n\Delta\tilde{\theta})y_1 + \sin(n\Delta\tilde{\theta})y_2, -\sin(n\Delta\tilde{\theta})y_1 + \cos(n\Delta\tilde{\theta})y_2),$$

where $n \in \mathbb{Z}$. It is clear that if $\Delta\tilde{\theta}/\pi$ is a rational number then the group \mathbb{G} is finite and the helicoidal surface is properly immersed. Moreover, if $\Delta\tilde{\theta}/\pi$ is not a rational number, then the group \mathbb{G} is not finite and the helicoidal drop is dense in the region bounded by the two cylinders of radius $\sqrt{r_1}$ and $\sqrt{r_2}$. A more detailed explanation of this last statement can be found in [Perdomo 2012]. □

Embedded and properly embedded examples. In this subsection we will find some embedded examples and we will show their profile curves. As pointed out before, when the helicoidal drop is not exceptional, its profile curve is a union of rotations of fundamental pieces that ends up being invariant under the group \mathbb{G} of rotations defined by (2-12). It is not difficult to see that a necessary condition for the helicoidal drop to be embedded is that $\Delta\tilde{\theta} = 2\pi/m$ for some integer m . We will show that this condition is not sufficient. In order to catch the potentially embedded examples we need to understand the function $\Delta\tilde{\theta}(C, a, \omega, x_1, x_2)$. As a very elementary technique to solve the equation $\Delta\tilde{\theta} = 2\pi/m$, we will use the intermediate value theorem. We know that for any a , there is a first (or last) value of C , $C_0(a, x_1, x_2)$, for which the function $\Delta\tilde{\theta}$ is defined. We will compute the limit of $\Delta\tilde{\theta}$ when C goes to C_0 using Lemma 2.9. The graphs shown in this paper were generated using the software Mathematica 8.

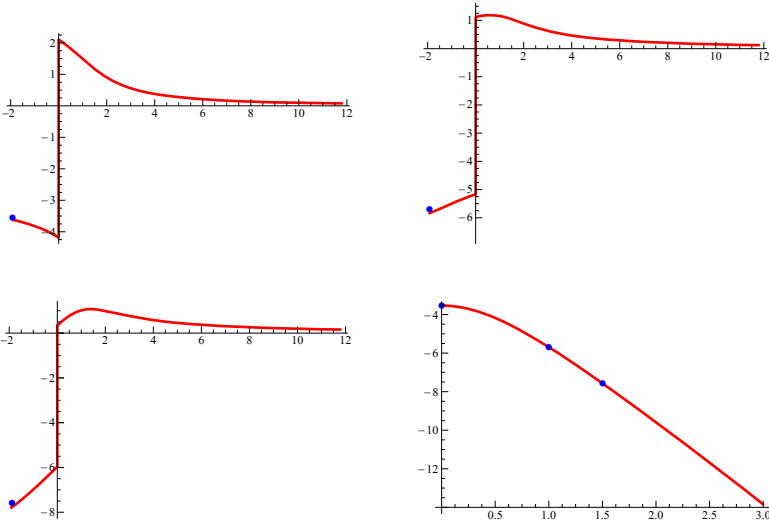


Figure 6. Top left, top right, bottom left: graphs of $\Delta\tilde{\theta}$ in terms of C when $a = -1$, $\Lambda_0 = 1$; $\omega = 0$, $\omega = 1$, and $\omega = 1.5$, respectively. Bottom right: graph of B in terms of ω when $a = -1$, $\Lambda_0 = 0$. This shows how the beginnings of the other three graphs change when ω changes. The highlighted points in this graph ($\omega = 0, 1$ and 1.5) correspond to the highlighted points in the other three graphs. For $\omega = 0$ there is no solution of the equation $\Delta\tilde{\theta} = -2\pi/m$ with negative values of C . We see that, for some values of ω , the equation $\Delta\tilde{\theta} = -2\pi$ has a solution with C negative, which is responsible for the existence of embedded examples with $\Lambda_0 = 0$ and $a = -1$.

Embedded examples with $\Lambda_0 = 0$ and $a = -1$. From Lemma 2.8 we know that the polynomial p has two nonnegative roots if and only if $C > C_0 = -3/\sqrt[3]{4}$. A direct application of Lemma 2.9 shows:

Proposition 2.15. *If $\Lambda_0 = 0$, $a = -1$ and for any $C > C_0$, x_1 and x_2 denote the two roots of the polynomial $q(r, C) = -16C^2 + 64r + 8Cr^2 - r^4$, then*

$$\begin{aligned} \lim_{C \rightarrow C_0^+} \Delta\theta(C, \omega, x_1, x_2) &= \int_{x_1}^{x_2} \frac{(4C - r^2)\sqrt{1 + r\omega^2}}{r\sqrt{q(r, C)}} dr \\ &= B(\omega) = -\frac{2\pi\sqrt{1 + \sqrt[3]{4}\omega^2}}{\sqrt{3}}. \end{aligned}$$

Using the intermediate value theorem, we can numerically solve the equation $\Delta\tilde{\theta} = -2\pi$ for values of w and C . The images in Figures 7 and 8 show some of

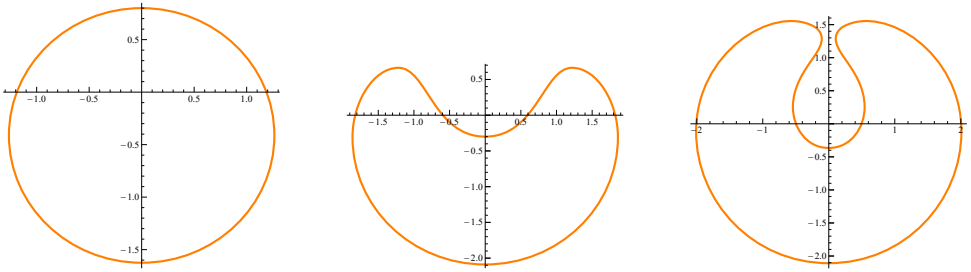


Figure 7. Profile curves of some embedded helicoidal rotating drops when $\Lambda_0 = 0$ and $a = -1$. Left: $C = -1.5$, $\omega = 1.17684$. Middle: $C = 0.6$, $\omega = 3.04646$. Right: $C = 0.735$, $\omega = 4.65615$. When C is close to the critical value C_0 , the embedded examples are close to a round cylinder. As C increases, the shape develops a self-intersection.

the resulting profile curves and the corresponding surfaces. They also show the values of C and w that solve the equation $\Delta\tilde{\theta} = -2\pi$.

Embedded examples with $\Lambda_0 = 1$ and $a \neq 0$. We now show some embedded examples in this case. Again, the intermediate value theorem is used to numerically solve the equation $\Delta\tilde{\theta} = 2\pi/m$. A direct application of Lemma 2.9 shows:

Proposition 2.16. *Let r_1, r_2 and r_3 be as in Definition 2.12 and let C_1, C_2 and C_3 be as in Lemma 2.13. Let us define the two bounds*

$$b_i(a, w) = \frac{\pi(4C_i + ar_i^2 - 4r_i)\sqrt{1 + r_i w^2}}{r_i\sqrt{16 + 8a(C - 3r_i) + 6a^2r_i^2}}, \quad i = 1, 2.$$

- (a) *If $a < 0$ then $\lim_{C \rightarrow C_1(a)^+} \Delta\tilde{\theta}(C, a, x_1, x_2) = b_1(a)$, where x_1 and x_2 are the first two roots of the polynomial $q(r, C, a)$*
- (b) *If $a > 0$ then $\lim_{C \rightarrow C_1(a)^-} \Delta\tilde{\theta}(C, a, x_1, x_2) = b_1(a)$, where x_1 and x_2 are the first two roots of the polynomial $q(r, C, a)$*

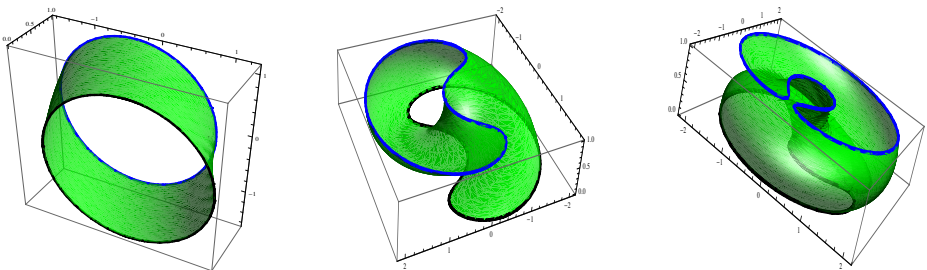


Figure 8. Surfaces associated with the profile curve in Figure 7.

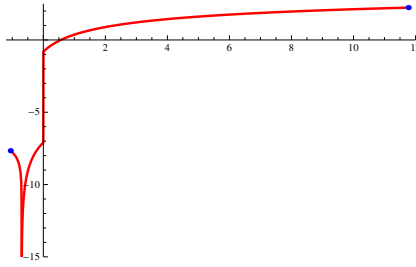


Figure 9. The graph of the function $\Delta\tilde{\theta}$ when $a = 0.2$ and $\omega = 0.15$. In this case $C_2 \approx -1.065$, $b_2 \approx -7.66$, $C_3 \approx -0.698$, $C_1 \approx 11.76$ and $b_1 \approx 2.23$. The points (C_2, b_2) and (C_1, b_1) have been highlighted.

(c) If $a > 0$ then $\lim_{C \rightarrow C_2(a)^+} \Delta\tilde{\theta}(C, a, x_1, x_2) = b_2(a)$, where x_1 and x_2 are the first two roots of the polynomial $q(r, C, a)$.

Proof. Since

$$\Delta\tilde{\theta} = \int_{x_1}^{x_2} \frac{(4C + ar^2 - 4r)\sqrt{1 + r\omega^2}}{r\sqrt{q}} dr,$$

in every case, when C approaches the limit value, the two roots approach r_i ($i = 1, 2$), which is a root of q with multiplicity two. Therefore [Lemma 2.9](#) applies and the proposition follows. Notice that the value A in [Lemma 2.9](#) is given by

$$A = -\frac{1}{2}q''(r_i) = 16 + 8a(C - 3r_i) + 6a^2r_i^2. \quad \square$$

Remark 2.17. When $0 < a < \frac{8}{27}$, the domain of $\Delta\tilde{\theta}$ is $(C_2(a), C_1(a))$, and

$$\lim_{C \rightarrow C_2(a)^+} \Delta\tilde{\theta}(C) = b_2(a, w), \quad \lim_{C \rightarrow C_1(a)^-} \Delta\tilde{\theta}(C) = b_1(a, w).$$

There is a vertical asymptote at $C = C_3(a)$ and a jump discontinuity at $C = 0$.

Taking a look at [Figure 9](#), we notice that, when $\omega = 0.15$ and $a = 0.2$, and for any integer $m > 2$, the equation $\Delta\tilde{\theta} = 2\pi/m$ has a solution. We have numerically solved this equation for $m = 4$ and $m = 8$. [Figures 10](#) and [11](#) provides a picture of the profile curves of the properly immersed examples.

If we decrease the value of a while keeping the value of ω constant, we can again solve the equation $\Delta\tilde{\theta} = \pi/4$, but this time the helicoidal rotational drop is embedded. See [Figure 12](#).

Finally, we would like to show that if we increase ω then it is possible to solve the equation $\Delta\tilde{\theta} = 2\pi$ (see [Figure 13](#)).

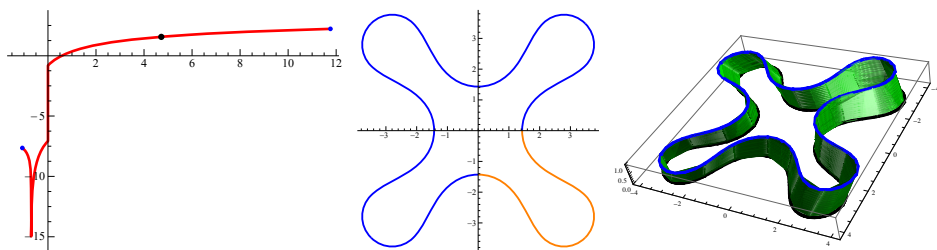


Figure 10. The embedded helicoidal rotational drop obtained by solving the equation $\Delta\tilde{\theta} = \frac{\pi}{2}$ when $a = 0.2$ and $\omega = 0.15$. In the middle graph, $\Lambda_0 = 1$ and $C = 4.72283$.

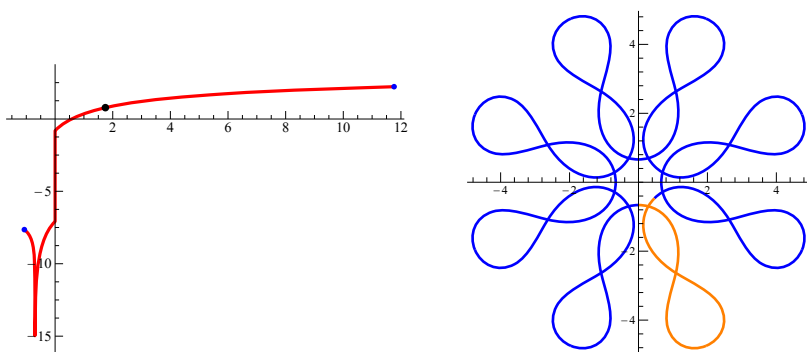


Figure 11. The properly immersed helicoidal rotational drop obtained by solving the equation $\Delta\tilde{\theta} = \frac{\pi}{4}$ when $a = 0.2$ and $\omega = 0.15$. In the graph on the right, $\Lambda_0 = 1$ and $C = 1.7453$.

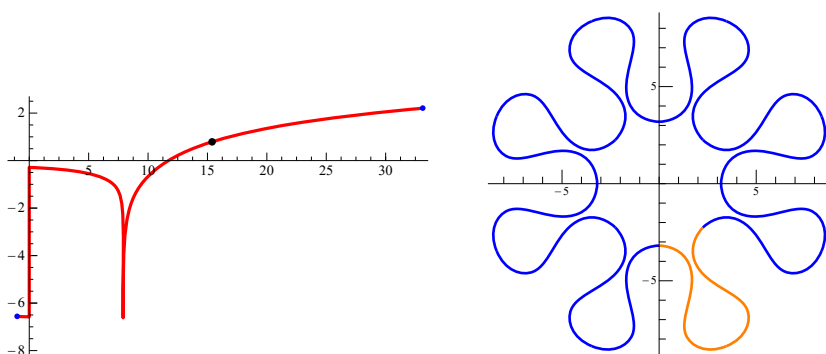


Figure 12. The embedded helicoidal rotational drop obtained by solving the equation $\Delta\tilde{\theta} = \frac{\pi}{4}$ when $a = 0.05$ and $\omega = 0.15$. In the graph on the right, $\Lambda_0 = 1$ and $C = 15.3877$.

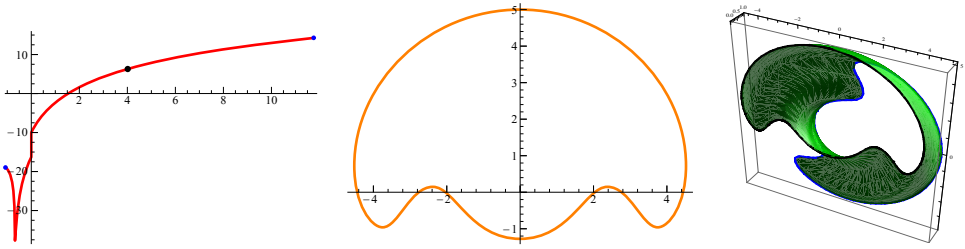


Figure 13. The embedded helicoidal rotational drop obtained by solving the equation $\Delta\tilde{\theta} = 2\pi$ when $a = 0.2$ and $\omega = 2$. In the middle graph, $\Lambda_0 = 1$ and $C = 4.0134$.

3. Second variation

For any sufficiently smooth surface, we define an invariant $\ell = 2H + \frac{1}{2}aR^2$. The first variation formula (1-4) restricted to compactly supported variations can then be expressed as

$$\delta\mathcal{E}_{a,\Lambda_0} = - \int_{\Sigma} (\ell - \Lambda_0)\psi \, d\Sigma,$$

where $\psi := \delta X \cdot \nu$. We assume that the surface is in equilibrium, so that $\ell - \Lambda_0 \equiv 0$ holds. The second variation is thus

$$\delta^2\mathcal{E}_{a,\Lambda_0} = - \int_{\Sigma} \psi(\delta\ell) \, d\Sigma.$$

A well-known formula for the pointwise variation of the mean curvature is

$$(3-1) \quad 2\delta H = \hat{L}[\psi] + 2\nabla H \cdot \delta X,$$

where $\hat{L} = \Delta + |dv|^2$. Also

$$\delta R^2 = 2 \sum_{i=1,2} x_i \delta X \cdot E_i = 2 \sum_{i=1,2} x_i (\psi \nu_i + (\delta X)^T \cdot E_i) = 2\psi \hat{Q} + 2\nabla' R^2 \cdot (\delta X)^T,$$

where $\hat{Q} = x_1 \nu_1 + x_2 \nu_2$. Combining this with (3-1), we have

$$(3-2) \quad \delta\ell = L[\psi] + \nabla\ell \cdot T,$$

where $L[\psi] = \Delta\psi + (|dv|^2 + a\hat{Q})\psi$. Since we are assuming $\ell \equiv \Lambda_0 = \text{constant}$, the second term above vanishes and the second variation formula for variations vanishing on $\partial\Sigma$ then reads

$$(3-3) \quad \begin{aligned} \delta^2\mathcal{E}_{a,\Lambda_0} &= - \int_{\Sigma} \psi L[\psi] \, d\Sigma = - \int_{\Sigma} \psi (\Delta\psi + (|dv|^2 + a\hat{Q})\psi) \, d\Sigma \\ &= \int_{\Sigma} |\nabla\psi|^2 - (|dv|^2 + a\hat{Q})\psi^2 \, d\Sigma. \end{aligned}$$

This formula can be found in [López 2010]. As usual, an equilibrium surface will be called stable if the second variation is nonnegative for all compactly supported variations satisfying the additional condition

$$(3-4) \quad \int_{\Sigma} \psi \, d\Sigma = 0.$$

This is just the first-order condition which is necessary and sufficient for the variation to be volume-preserving.

For a part of the surface of the form $\alpha \times [-h/2, h/2]$, this can be written

$$(3-5) \quad \delta^2 \mathcal{E}_{a, \Lambda_0} = \int_{\alpha} \int_{-h/2}^{h/2} \left(\frac{1}{\sqrt{1 + \omega^2 \xi_1^2}} ([1 + \omega^2 R^2] \psi_s^2 - 2\omega \xi_2 \psi_s \psi_t + \psi_t^2) - (4H^2 - 2K + a\xi_2) \sqrt{1 + \omega^2 \xi_1^2} \psi^2 \right) dt \, ds,$$

where K denotes the Gaussian curvature. Choosing $\psi = \sin(2\pi t/h)$ gives

$$\int_{\Sigma} |\nabla \psi|^2 \, d\Sigma = \frac{2\pi^2}{h} \int_{\alpha} \frac{1}{\sqrt{1 + \omega^2 \xi_1^2}} \, ds.$$

In addition, for this choice of ψ , we have $\psi \equiv 0$ on the boundary and the mean value of ψ on $\alpha \times [-h/2, h/2]$ is zero.

Lemma 3.1. *We have*

$$\int_{\alpha} K \sqrt{1 + \omega^2 \xi_1^2} \, ds = 0,$$

and hence

$$\int_{\alpha \times [-h/2, h/2]} K \, d\Sigma = 0.$$

Proof. From calculations found in [Perdomo 2012], one finds

$$K = \frac{-\omega^2(1 + \kappa \xi_2)}{(1 + \omega^2 \xi_1^2)^2} = \frac{-\omega^2(\xi_1)_s}{(1 + \omega^2 \xi_1^2)^2},$$

so

$$\int_{\alpha} K \sqrt{1 + \omega^2 \xi_1^2} \, ds = \int_{\alpha} \frac{-\omega^2(\xi_1)_s}{(1 + \omega^2 \xi_1^2)^{3/2}} \, ds = 0,$$

since the last integrand is the s -derivative of a function of ξ_1 . □

Proposition 3.2. *A necessary condition for the stability of $\alpha \times [-h/2, h/2]$ for the fixed boundary problem is that*

$$(3-6) \quad \frac{2\pi^2}{h^2} \int_{\alpha} \frac{1}{\sqrt{1 + \omega^2 \xi_1^2}} \, ds \geq \int_{\alpha} 4H^2 \sqrt{1 + \omega^2 \xi_1^2} \, ds + a\mathcal{A}$$

holds. Equivalently, this can be expressed as

$$(3-7) \quad \frac{2\pi^2}{h^2} \int_{\alpha} \frac{1}{\sqrt{1 + \omega^2 \xi_1^2}} ds \geq \int_{\alpha \times [-h/2, h/2]} 4H^2 d\Sigma + a^3V(\alpha \times [-h/2, h/2]).$$

Proof. We choose $\psi = \sin(2\pi t/h)$ in the second variation formula. For this choice of ψ , we have $\psi \equiv 0$ on the boundary and the mean value of ψ on $\alpha \times [-h/2, h/2]$ is zero. The result then follows directly from (3-5) and the previous lemma. \square

The bound (3-6) gives a condition on the maximum height of a stable helicoidal surface in terms of the geometry of the generating curve.

There is no possible way to obtain a positive lower bound for the right-hand side of (3-6). For a round cylinder of radius R , the equation

$$\frac{2\xi_2}{\sqrt{1 + \omega^2 \xi_1^2}} + \Lambda_0 R^2 - \frac{aR^4}{4} = c$$

becomes

$$2R + \Lambda_0 R^2 - \frac{aR^4}{4} = c,$$

so for arbitrary a , we can simply define c by this equation, and hence the cylinder will be an equilibrium surface. For a cylinder, the potential in the second variation formula is

$$4H^2 - 2K + a\hat{Q} = \frac{1}{R^2} + aR,$$

so for $a \ll 0$ the potential is nonpositive and the cylinder is stable for arbitrary heights.

We will now give an upper bound for the height of a stable helicoidal equilibrium surface which is valid for any such surface which is not a cylinder over a planar curve. This upper bound will only depend on the generating curve. In [Palmer and Perdomo 2014], this estimate is modified so that it applies to noncircular cylindrical equilibrium surfaces as well.

Theorem 3.3. *For a helicoidal surface which is not a round cylinder, a necessary condition for the stability of the part of the surface between horizontal planes separated by a distance h is that*

$$(3-8) \quad \frac{4\pi^2 e^4}{h^2} \geq \frac{\omega^2 \oint_{\alpha} \frac{(1 + \omega^2 R^2)(1 + \kappa \xi_2)^2}{(1 + \omega^2 \xi_1^2)^{7/2}} ds}{\oint_{\alpha} \frac{1}{1 + \omega^2 \xi_1^2} ds} (\geq \omega^2).$$

The result also holds true if $a = 0$, i.e., if the surface has constant mean curvature.

Remark. In [Palmer and Perdomo 2014] a similar estimate is given for cylindrical surfaces which are not round cylinders.

Proof. To begin, note that the third component of the normal v_3 satisfies $L[v_3] = 0$ since vertical translation is a symmetry of the normal. Also, this function will vanish identically if and only if the surface is a cylinder.

The function v_3 can be written as [Perdomo 2012]

$$v_3 = \frac{\omega \xi_1}{\sqrt{1 + \omega^2 \xi_1^2}},$$

so v_3 is a function of s only. Using local coordinate expressions found in [Perdomo 2012], we can write

$$\begin{aligned} 0 = L[v_3] &= \frac{1}{\sqrt{g}} (\sqrt{g} g^{11} (v_3)_s)_s + (|dv|^2 + a\xi_2) v_3 \\ &= \frac{1}{\sqrt{1 + \omega^2 \xi_1^2}} \left[\frac{1 + \omega^2 R^2}{\sqrt{1 + \omega^2 \xi_1^2}} (v_3)_s \right]_s + (|dv|^2 + a\xi_2) v_3 =: \mathcal{L}[v_3]. \end{aligned}$$

Note that $(|dv|^2 + a\xi_2)$ only depends on s . For any smooth function $u = u(s)$, there holds

$$\mathcal{L}[e^u] = e^u (\mathcal{L}[u] + g^{11} u_s^2) = e^u (\mathcal{L}[u] + (1 + \omega R^2) u_s^2).$$

If we now take $\psi = e^{v_3(s)} \sin(2\pi t/h)$, then (3-4) holds, and from (3-5) we get

$$\delta^{2c} \mathcal{E}_{a, \Lambda_0} = \frac{2\pi^2}{h} \oint_{\alpha} \frac{e^{2v_3}}{1 + \omega^2 \xi_1^2} ds - \frac{h}{2} \oint_{\alpha} \frac{e^{2v_3} (1 + \omega^2 R^2)}{\sqrt{1 + \omega^2 \xi_1^2}} ((v_3)_s)^2 ds.$$

Using $-1 \leq v_3 \leq 1$ and using

$$(v_3)_s = (\omega \xi_1 (1 + \omega^2 \xi_1^2)^{-1/2})_s = (\xi_1)_s \omega (1 + \omega^2 \xi_1^2)^{-3/2} = \omega (1 + \kappa \xi_2) (1 + \omega^2 \xi_1^2)^{-3/2}$$

yields the result. \square

Appendix

We assume that Σ is contained in a three-dimensional region Ω and that $\partial\Sigma$ is contained in a supporting surface S which is part of $\partial\Omega$. We assume that there is a (not necessarily connected) domain $S_1 \subset S$ such that $\Sigma \cup S_1$ bounds a subregion $\Omega_1 \subset \Omega$. The volume of Ω_1 will be denoted by \mathcal{V} .

Let ϕ be a solution of $\Delta' \phi = 1$ in Ω with $\nabla' \phi \cdot N = 0$ on S , where N is the outward-pointing normal to S . This boundary value problem is underdetermined and is solvable provided S is not closed.

We subject the surface to a variation that keeps $\partial\Sigma$ on S . We write $\delta X =: T + \psi \nu \perp N$ along $\partial\Sigma$, and

$$\mathcal{V} = \int_{\Sigma} \nabla' \phi \cdot \nu d\Sigma.$$

We have

$$\begin{aligned}
 \delta \mathcal{V} &= \int_{\Sigma} \nabla'_{T+\psi\nu} \nabla' \phi \cdot \nu + \nabla' \phi \cdot \delta \nu \, d\Sigma + \int_{\Sigma} \nabla' \phi \cdot \nu (\nabla \cdot T - 2H\psi) \, d\Sigma \\
 &= \int_{\Sigma} \psi \nabla'_\nu \nabla' \phi \cdot \nu - 2H\psi \nabla' \phi \cdot \nu - \nabla \phi \cdot \nabla \psi \, d\Sigma + \oint_{\partial\Sigma} (\nabla' \phi \cdot \nu) T \cdot n \, ds \\
 &= \int_{\Sigma} \psi \nabla'_\nu \nabla' \phi \cdot \nu - 2H\psi \nabla' \phi \cdot \nu + \psi \Delta \phi \, d\Sigma + \oint_{\partial\Sigma} ((\nabla' \phi \cdot \nu) T - \psi \nabla \phi) \cdot n \, ds.
 \end{aligned}$$

A well-known formula relating the Laplacian on a submanifold to the Laplacian on the ambient space gives $\Delta' \phi = \nabla'_\nu \nabla' \phi \cdot \nu - 2H \nabla' \phi + \Delta \phi$. Therefore we obtain

$$\delta \mathcal{V} = \int_{\Sigma} \psi \, d\Sigma + \oint dX \times \nabla' \phi \cdot \delta X.$$

However, all of dX , $\nabla' \phi$ and δX are perpendicular to N on $\partial\Sigma$, so the line integral above vanishes.

To obtain (1-2), we let W be a vector field on Ω satisfying $\nabla' \cdot W = R^2$ and $W \cdot N = 0$ along S . This boundary value problem is underdetermined and is solvable provided S is not closed. Then, by the divergence theorem,

$$\int_{\Omega_1} R^2 \, d^3x = \int_{\Sigma} W \cdot \nu \, d\Sigma,$$

so that

$$\begin{aligned}
 \delta \int_{\Omega_1} R^2 \, d^3x &= \int_{\Sigma} \nabla'_{T+\psi\nu} W \cdot \nu + W \cdot \delta \nu \, d\Sigma + \int_{\Sigma} W \cdot \nu (\nabla \cdot T - 2H\psi) \, d\Sigma \\
 &= \int_{\Sigma} \psi \nabla'_\nu W \cdot \nu - W \cdot \nabla \psi - 2H\psi W \cdot \nu \, d\Sigma + \oint_{\partial\Sigma} (W \cdot \nu) T \cdot n \, ds \\
 &= \int_{\Sigma} \psi \nabla' \cdot W \, d\Sigma + \oint_{\partial\Sigma} ((W \cdot \nu) T - \psi W) \cdot n \, ds \\
 &= \int_{\Sigma} \psi R^2 \, d\Sigma + \oint_{\partial\Sigma} dX \times W \cdot \delta X.
 \end{aligned}$$

Again, all of dX , W and δX are perpendicular to N along $\partial\Sigma$ so the line integral will vanish.

If the pair (ϕ, W) used above are replaced by another pair $(\underline{\phi}, \underline{W})$ satisfying the same equations ($\Delta' \underline{\phi} = 1$ and $\nabla' \underline{W} = R^2$), the divergence theorem yields

$$\int_{\Sigma} \nabla' \underline{\phi} \cdot \nu \, d\Sigma = \mathcal{V} + c_1 \quad \text{and} \quad \int_{\Sigma} \underline{W} \cdot \nu \, d\Sigma = \int_{\Omega_1} R^2 \, d^3x + c_2$$

for constants c_1 and c_2 . Thus, these replacements will not affect the variational formulas for these integrals.

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