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## THE BIDUAL OF A RADICAL OPERATOR ALGEBRA CAN BE SEMISIMPLE

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The paper of Sidney (Denny) L. Gulick (“Commutativity and ideals in the biduals of topological algebras”, *Pacific J. Math.* 18, 1966) contains some good mathematics, but it also contains an error. It claims that for a Banach algebra  $A$ , the intersection of the Jacobson radical of  $A^{**}$  with  $A$  is precisely the radical of  $A$  (this is claimed for either of the Arens products on  $A^{**}$ ). In this paper we begin with a simple counterexample to that claim, in which  $A$  is a radical operator algebra, but not every element of  $A$  lies in the radical of  $A^{**}$ . We then develop a more complicated example  $\mathcal{A}$ , which, once again, is a radical operator algebra, but  $\mathcal{A}^{**}$  is semisimple. So  $\text{rad } \mathcal{A}^{**} \cap \mathcal{A}$  is zero, but  $\text{rad } \mathcal{A} = \mathcal{A}$ . We conclude by examining the uses Gulick’s paper has been put to since 1966 (at least 8 subsequent papers refer to it), and we find that most authors have used the correct material from that paper, and avoided using the wrong result. We reckon, then, that we are not the first to suspect that the result  $\text{rad } A^{**} \cap A = \text{rad } A$  was wrong; but we believe we are the first to provide “neat” counterexamples as described.

### 1. Introduction

The theorem in which Gulick [1966] makes the claim  $\text{rad } A^{**} \cap A = \text{rad } A$  is Theorem 4.6. We believe that the place where his proof breaks down is nearby, in the proof of Lemma 4.5, the seventh line: “note that  $M_E$  is once again a maximal regular left ideal in  $E$ ”. We could not see why this should be so, and Theorem 4.6 is definitely false; this introductory section contains a counterexample.

We shall always be working with operator algebras (norm-closed subalgebras of the algebra  $B(H)$  of all operators on a Hilbert space  $H$ ), so the question of which Arens product is involved need never be addressed, for as is well known, every operator algebra is Arens regular — the two products coincide.

Let us conclude this introduction with the simpler counterexample mentioned in the abstract.

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Let  $H$  be a Hilbert space with orthonormal basis  $(e_i)_{i \in \mathbb{N}}$ . Let  $T_0 : H \rightarrow H$  be the operator with

$$(1) \quad T_0 e_i = \begin{cases} e_{i+1} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

For  $n \in \mathbb{N}$ , let  $T_n : H \rightarrow H$  be the rank-1 operator with

$$(2) \quad T_n e_i = \begin{cases} e_{i+1} & \text{if } i = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $A$  denote the operator algebra (the norm-closed subalgebra of  $B(H)$ ) generated by  $\{T_n : n \in \mathbb{N}_0\}$ .

**Lemma 1.1.** *A is radical.*

*Proof.* First,  $T_0^2 = 0$  and each  $T_n$  ( $n \geq 1$ ) has rank 1, so everything in  $A$  is of form  $\lambda T_0 + K$ , where  $\lambda \in \mathbb{C}$  and  $K$  is a compact operator. Second, the subspaces  $E_k = \overline{\text{lin}}\{e_i : i > k\} \subset H$  are invariant for every  $T_n$  (and hence for every  $T \in A$ ); indeed, every  $T \in A$  maps  $E_k$  into  $E_{k+1}$  ( $k \in \mathbb{N}_0$ ). So, let  $T = \lambda T_0 + K \in A$ , with  $\lambda \in \mathbb{C}$  and  $K \in K(H)$ . It is enough to show that  $T$  is quasinilpotent. Since  $K$  is compact, the norms  $\varepsilon_n = \|K|_{E_n}\|$  tend to zero as  $n \rightarrow \infty$ . Furthermore, since  $T_0^2 = 0$ , we have

$$(3) \quad \|T^2|_{E_n}\| = \|\lambda T_0 K + \lambda K T_0 + K^2|_{E_n}\| \leq 2|\lambda|\varepsilon_n + \varepsilon_n^2 = \delta_n,$$

with  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Now  $T^{2k} = T^2|_{E_{2k-2}} T^2|_{E_{2k-4}} \dots T^2|_{E_2} T^2|_{E_0}$ ; hence

$$\|T^{2k}\| \leq \prod_{j=0}^{k-1} \delta_{2j},$$

so  $\|T^{2k}\|^{1/k} \rightarrow 0$ . Plainly  $T^2$ , and hence  $T$  itself, is quasinilpotent. □

**Theorem 1.2.**  $T_0 \notin \text{rad } A^{**}$ , so  $A = \text{rad } A \subsetneq A \cap \text{rad } A^{**}$ .

*Proof.* Now  $A \subset B(H)$ , and  $B(H)$  is of course a dual Banach algebra, so there is a natural projection from  $B(H)^{**}$  (the third dual of the Banach space of trace class operators on  $H$ ) onto  $B(H)$ . This projection is an algebra homomorphism, so when we restrict it to  $A^{**} \subset B(H)^{**}$ , we get a representation of  $A^{**}$  acting on  $H$ , such that the canonical image  $A \subset A^{**}$  acts on  $H$  in its usual way, and the representation of  $A^{**}$  consists of the weak- $*$  closure of  $A$  in  $B(H)$ .

Among the operators in this weak- $*$  closure is the weak- $*$  convergent sum  $T = \sum_{n=1}^{\infty} T_n$ , with

$$(4) \quad T e_i = \begin{cases} e_{i+1} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

The product  $TT_0$  has  $TT_0e_i = e_{i+2}$  (if  $i$  is odd) or  $TT_0e_i = 0$  (if  $i$  is even); so  $\|(TT_0)^k\| = 1$  for all  $k$ , and indeed 1 is in the spectrum of  $TT_0$ . If  $\tau \in A^{**}$  is any element represented as  $T$  by this representation, then  $1 \in \text{Sp}(\tau T_0)$ . So  $T_0$  does not lie in the Jacobson radical of  $A^{**}$ , by a well-known characterization of that radical.  $\square$

Note that the proof given above does not depend on the faithfulness (injectivity) of the natural representation of  $A^{**}$  in  $B(H)$ . However, when we give the more complicated counterexample — when we make the claim that the bidual of our radical algebra  $\mathcal{A}$  is semisimple — we will have to show that the analogous representation for the bidual of that algebra is indeed faithful.

### 2. The main construction

We now seek to develop the example given in the introduction into an example  $\mathcal{A}$  where  $\mathcal{A}$  is radical but  $\mathcal{A}^{**}$  is semisimple.

**Definition 2.1.** Let  $S$  denote the free unital semigroup on two generators  $g, h$ . If  $s \in S$  with  $s = \gamma_n \gamma_{n-1} \dots \gamma_2 \gamma_1 = \prod_{j=0}^{n-1} \gamma_{n-j}$ , and each  $\gamma_i \in \{g, h\}$ , we define the length  $l(s) = n$  and the depth  $\rho(s) = \#\{i : 1 \leq i \leq n, \gamma_i = h\}$ ; and if  $n > 0$  (that is, if  $s \neq 1$ , the unit), we define the predecessor  $p(s) = \prod_{j=1}^{n-1} \gamma_{n-j}$ . We define  $S^- = S \setminus \{1\}$ .

We define the Cayley graph  $G$  of  $S$  to be an abstract directed graph with vertex set  $S$ , and a directed edge  $p(s) \rightarrow s$  for each  $s \in S^-$ .

Note that  $G$  is an infinite tree with root vertex 1, such that every vertex  $s \in S$  has two outward edges leaving it (the edges  $s \rightarrow gs$  and  $s \rightarrow hs$ ) and every vertex  $s \in S^-$  has a single edge entering it (the edge  $p(s) \rightarrow s$ ). If  $l(s) = k$ , the unique directed path from 1 to  $s$  consists of  $k + 1$  vertices  $1 \rightarrow p^{k-1}(s) \rightarrow p^{k-2}(s) \rightarrow \dots \rightarrow p(s) \rightarrow s$ .

**Definition 2.2.** For  $s \in S$  we define the weight  $w(s) = 2^{-\rho(s)}$ , and if  $l(s) = l$  we define

$$(5) \quad W(s) = \prod_{j=0}^{l-1} w(p^j s).$$

We define a Hilbert space  $\mathcal{H} = l^2(S, W)$  to be the collection of all formal sums  $\mathbf{x} = \sum_{s \in S} x_s \cdot s$  with  $x_s \in \mathbb{C}$  and

$$(6) \quad \|\mathbf{x}\|^2 = \sum_{s \in S} W(s)^2 |x_s|^2 < \infty.$$

We define a particular subset  $\mathcal{C} \subset S^-$ , the *colour set*

$$(7) \quad \mathcal{C} = \{g^k : k \in \mathbb{N}\} \cup \{g^k h s : k \in \mathbb{N}_0, s \in S, 1 + l(s) \mid k\}$$

(here and elsewhere we use “ $1 + l(s) \mid k$ ” for “ $1 + l(s)$  divides  $k$ ”).

We define a *colour map*  $\mu : S^- \rightarrow \mathcal{C}$  recursively as follows:

$$(8) \quad \mu(s) = \begin{cases} s & \text{if } s \in \mathcal{C}, \\ \mu(p^{n-k'}y) & \text{if } s = g^khy, y \in S, l(y) = n, 1 \leq k' \leq n, k \equiv k' \pmod{n+1}. \end{cases}$$

Note that (8) really “works” as a recursive definition, because if  $s \notin \mathcal{C}$ , we necessarily have  $s = g^khy$  for some  $k \in \mathbb{N}$  such that  $1 + l(y) \nmid k$ ; so writing  $n = l(y)$ , there is a unique  $k' \in [1, n]$  such that  $k' \equiv k \pmod{n+1}$ . The iterated predecessor  $p^{n-k'}y$  will not be equal to 1 because  $k' > 0$  and  $l(y) = n$ , so  $\mu(p^{n-k'}y)$  will be (recursively) defined. Note that for  $s \in S$ , the colour  $\mu(hs)$  is always equal to  $hs$ , while the colour  $\mu(gs)$  is either  $gs$  itself, or one of the iterated predecessors of  $gs$ . So we never have  $\mu(gs) = \mu(hs)$  for any  $s \in S$ .

**Definition 2.3.** For each colour  $c \in \mathcal{C}$ , we define a linear map  $T_c \in B(\mathcal{H})$  by its action on the basis  $S$ : for each  $s \in S$ , we define

$$(9) \quad T_c(s) = \begin{cases} gs & \text{if } \mu(gs) = c, \\ hs & \text{if } \mu(hs) = c, \\ 0 & \text{otherwise.} \end{cases}$$

Each  $T_c$  is a weighted shift operator (for  $S$  is an orthogonal, though not an orthonormal, basis of  $H$ ). Writing  $e_s = W(s)^{-1} \cdot s$  ( $s \in S$ ) for the corresponding orthonormal basis, and giving due regard to the fact that  $W(s)/W(p(s)) = w(s)$  for each  $s \in S^-$ , we have

$$(10) \quad T_c(e_s) = \begin{cases} w(gs)e_{gs} & \text{if } \mu(gs) = c, \\ w(hs)e_{hs} & \text{if } \mu(hs) = c, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that for each  $c \in \mathcal{C}$ ,

$$(11) \quad \|T_c\| = \max\{w(x) : \mu(x) = c\} = w(c) = 2^{-\rho(c)}.$$

**Definition 2.4.** We define two families of *coordinatewise* orthogonal projections on  $\mathcal{H}$ . For  $n \in \mathbb{N}_0$ ,  $P_n$  is the orthogonal projection onto  $\overline{\text{lin}\{s \in S : \rho(s) = n\}}$ , and  $\bar{P}_n = \sum_{i=0}^n P_i$ ; while  $\pi_n$  is the orthogonal projection onto  $\text{lin}\{s \in S : l(s) = n\}$ , and  $\bar{\pi}_n = \sum_{i=0}^n \pi_i$ .

We also define, for  $n \in \mathbb{N}_0$ , a subgraph  $G^{(n)}$  of  $G$ , obtained from  $G$  by deleting some of the edges. Specifically,  $G^{(n)}$  is a graph with vertex set  $S$  and a directed edge  $p(s) \rightarrow s$  for every  $s \in S$  such that the *colour depth*  $\rho\mu(s)$  is no greater than  $n$ . (Equivalently, we obtain  $G^{(n)}$  by deleting from  $G$  every edge  $p(s) \rightarrow s$  such that

the colour depth  $\rho\mu(s)$  is greater than  $n$ ). If  $K \subset G^{(n)}$  is a connected component, we define the coordinatewise projection  $Q_{n,K}$  by

$$(12) \quad Q_{n,K}(s) = \begin{cases} s & \text{if } s \in K, \\ 0 & \text{otherwise} \end{cases} \quad (s \in S).$$

We define  $H_{n,K} = Q_{n,K}(\mathcal{H})$ .

Note that while  $\pi_n$  has finite rank  $2^n$ , the projection  $P_n$  always has infinite rank (even when  $n = 0$ , when it is the orthogonal projection onto  $\overline{\text{lin}}\{g^k : k \geq 0\}$ ).

**Definition 2.5.** We define an algebra  $\mathcal{A}_0 \subset B(\mathcal{H})$ .  $\mathcal{A}_0$  is the nonunital subalgebra of  $B(\mathcal{H})$  generated by the operators  $T_c$  ( $c \in \mathcal{C}$ ). We define the operator algebra  $\mathcal{A} = \overline{\mathcal{A}_0}$ , the norm closure of  $\mathcal{A}_0$  in  $B(\mathcal{H})$ . We define  $\mathcal{A}^{(n)} \subset \mathcal{A}_0$  to be the linear span of products  $T = T_{c_k} T_{c_{k-1}} \dots T_{c_2} T_{c_1} = \prod_{i=0}^{k-1} T_{c_{k-i}}$  such that  $c_i \in \mathcal{C}$  and  $\max\{\rho(c_i) : 1 \leq i \leq k\} = n$ . We define  $\overline{\mathcal{A}}^{(n)} = \sum_{r=0}^n \mathcal{A}^{(r)}$ , the subalgebra of  $\mathcal{A}_0$  generated by maps  $T_c$  ( $c \in \mathcal{C}$ ) with  $\rho(c) \leq n$ .

For  $n, r \geq 0$ , let  $S_{n,r} = \{s \in S : \text{the path from } 1 \text{ to } s \text{ in } G \text{ contains exactly } r \text{ edges } p(u) \rightarrow u \text{ with colour depth } \rho\mu(u) > n\}$ . Let  $P_{n,r}$  be the orthogonal projection onto  $\overline{\text{lin}}(S_{n,r})$ , and let  $\overline{P}_{n,r} = \sum_{t=0}^r P_{n,t}$ .

Note that  $S_{n,0} = \{s \in S : \rho(s) \leq n\}$ , so  $P_{n,0} = \overline{P}_n$  for each  $n \in \mathbb{N}_0$ .

**Lemma 2.6.** (a) For each  $n \in \mathbb{N}_0$ , the subspaces  $\ker \overline{P}_n, \ker \overline{\pi}_n \subset \mathcal{H}$  are invariant for  $\mathcal{A}$ . Further,  $\mathcal{A}$  maps  $\ker \overline{\pi}_n$  into  $\ker \overline{\pi}_{n+1}$  for each  $n$ .

(b) For each component  $K$  of  $G^{(n)}$ , the subspace  $H_{n,K}$  is invariant for  $\overline{\mathcal{A}}^{(n)}$  and also for the hermitian conjugate  $(\overline{\mathcal{A}}^{(n)})^*$ . The component of  $G^{(n)}$  containing 1 is  $S_{n,0}$ , and the associated projection is  $\overline{P}_n$ .

(c) Every map  $T_c$  with  $\rho(c) > n$  maps  $\mathcal{H}$  into  $\ker \overline{P}_n$ .

(d) For  $T \in \mathcal{A}_0$ , the decomposition  $T = \sum_{n=1}^{\infty} T^{(n)}$ , with  $T^{(n)} \in \mathcal{A}^{(n)}$ , is unique and continuous; writing  $\overline{T}^{(n)} = \sum_{i=0}^n T^{(i)}$ , we have  $\|\overline{T}^{(n)}\| \leq \|T\|$  for every  $n$  and  $T$ ; in fact  $\overline{T}^{(n)} = \sum_{r=0}^{\infty} P_{n,r} T P_{n,r}$  in the strong operator topology, while  $T - \overline{T}^{(n)} = \sum_{r=0}^{\infty} (1 - \overline{P}_{n,r}) T P_{n,r}$ .

(e) For all  $s \in S$  we have  $\rho\mu(s) \leq \rho(s)$ , with equality if  $s \in hS$ .

*Proof.* (a) is obvious because the generating maps  $T_c$  all map an element  $s \in S$  to  $gs, hs$ , or zero; and we have  $\rho(gs) \geq \rho(s)$ ,  $\rho(hs) \geq \rho(s)$  and  $l(gs) = l(s) + 1$ ,  $l(hs) = l(s) + 1$  for all  $s \in S$ .

For  $c \in \mathcal{C}$ , we have  $\langle T_c s, t \rangle \neq 0$  ( $s, t \in S$ ) only when there is an edge  $s \rightarrow t$  in  $G$ , and  $\mu(t) = c$ . So if  $T$  is in  $\overline{\mathcal{A}}^{(n)}$ , the algebra generated by maps  $T_c$  with  $\rho(c) \leq n$ , and if  $\langle Ts, t \rangle \neq 0$ , then there is a path from  $s$  to  $t$  in  $G$ , and each edge  $p(u) \rightarrow u$  in that path has  $\rho\mu(u) \leq n$ , so the edge  $p(u) \rightarrow u$  is present in the graph  $G^{(n)}$ . Thus  $s, t$  belong to the same component of  $G^{(n)}$ . So for a connected component

$K \subset G^{(n)}$ , the associated subspace  $H_{n,K}$  is invariant for both  $\mathcal{A}^{(n)}$  and  $(\mathcal{A}^{(n)})^*$ , establishing the first part of (b).

The component of  $G^{(n)}$  containing 1 is the set of  $s \in S$  such that the path from 1 to  $s$  in  $G$  contains only edges  $p(u) \rightarrow u$  with  $\rho\mu(u) \leq n$ . Now for any  $u \in S$ ,  $\mu(u)$  is either  $u$  itself or one of the iterated predecessors  $p^i(u)$ ; taking predecessors cannot increase the depth  $\rho(u)$ , so  $\rho\mu(u) \leq \rho(u)$  for all  $u$ . If  $s \in S$  with  $\rho(s) \leq n$ , then every edge  $p(u) \rightarrow u$  in the path from 1 to  $s$  has colour depth  $\rho\mu(u) \leq n$  also, so  $s$  lies in the component of  $G^{(n)}$  containing 1. Conversely, if  $\rho(s) > n$  then we have  $s = g^k ht$  for some  $t \in S$  and  $k \in \mathbb{N}_0$ ; the edge  $t \rightarrow ht$  is part of the path from 1 to  $s$ , and  $ht \in \mathcal{C}$  by (7), so the colour depth satisfies  $\rho\mu(ht) = \rho(ht) = \rho(s) > n$ , and therefore  $s$  is not in the connected component of  $G^{(n)}$  containing 1. Therefore that component is precisely  $\{s : \rho(s) \leq n\}$ , and the associated coordinatewise projection is  $\bar{P}_n$ . Thus we have established the second part of (b), and also part (e).

For part (c), note that  $T_c$  maps  $\mathcal{H}$  into  $\overline{\text{lin}}\{x \in S : \mu(x) = c\}$ ; if  $\rho(c) > n$  then this subspace is contained in  $\overline{\text{lin}}\{x \in S : \rho\mu(x) > n\} \subset \overline{\text{lin}}\{x \in S : \rho(x) > n\} \subset \ker \bar{P}_n$ , where the first  $\subset$  depends on part (e).

To prove part (d), we note that the edges of  $G^{(n)}$  include the edge  $p(u) \rightarrow u$  only if  $\rho\mu(u) \leq n$ ; hence the set  $S_{n,r}$  is a union of some of the components  $K$  of  $G^{(n)}$ . So by part (b) of this lemma, each image  $P_{n,r}\mathcal{H}$  is  $\mathcal{A}^{(n)}$ -invariant; but for  $c \in \mathcal{C}$  with  $\rho(c) > n$ ,  $T_c$  maps  $P_{n,r}\mathcal{H}$  into  $P_{n,r+1}\mathcal{H}$  because  $\langle T_c s, t \rangle \neq 0$  ( $s, t \in S$ ) only when  $s = p(t)$  and the colour depth  $\rho\mu(t)$  is greater than  $n$ . Now take any  $T \in \mathcal{A}_0$  and write  $T = \sum_i T^{(i)}$  with each  $T^{(i)} \in \mathcal{A}^{(i)}$ . We have  $\bar{T}^{(n)} = \sum_{i=0}^n T^{(i)} \in \bar{\mathcal{A}}^{(n)}$ , so each  $P_{n,r}\mathcal{H}$  is a  $\bar{T}^{(n)}$ -invariant subspace; but  $T - \bar{T}^{(n)}$  maps  $P_{n,r}\mathcal{H}$  into  $\bigoplus_{i=r+1}^\infty P_{n,i}\mathcal{H}$ . Therefore we have

$$(13) \quad \bar{T}^{(n)} = \sum_{r=0}^\infty \bar{T}^{(n)} P_{n,r} = \sum_{r=0}^\infty P_{n,r} T^{(n)} P_{n,r} = \sum_{r=0}^\infty P_{n,r} T P_{n,r},$$

while  $T - \bar{T}^{(n)} = \sum_{r=1}^\infty (1 - \bar{P}_{n,r}) T P_{n,r}$  as required by the lemma. This shows that the decomposition  $T = \sum_{i=0}^\infty T^{(i)}$  is indeed unique, and furthermore the compression  $\bar{T}^{(n)}$  as given by (13) plainly satisfies  $\|\bar{T}^{(n)}\| \leq \|T\|$ . Thus the lemma is proved.  $\square$

**Definition 2.7.** Let us write  $\mathcal{B}^{(n)}$  ( $\bar{\mathcal{B}}^{(n)}$ ) for the norm closure of  $\mathcal{A}^{(n)}$  ( $\bar{\mathcal{A}}^{(n)}$ ) in  $B(\mathcal{H})$ . Let us write  $\Delta_n$  for the map  $\mathcal{A}_0 \rightarrow \mathcal{A}^{(n)}$  with  $\Delta_n(T)$  the unique element  $T^{(n)} \in \mathcal{A}^{(n)}$  such that  $T = \sum_{n=0}^\infty T^{(n)}$ ; and let  $\bar{\Delta}_n : \mathcal{A}_0 \rightarrow \bar{\mathcal{A}}^{(n)}$  be the map  $\sum_{i=0}^n \Delta_i$ .

The maps  $\Delta_n, \bar{\Delta}_n$  are uniformly norm-bounded by part (d) of the previous lemma; so they extend continuously to maps  $\Delta_n : \mathcal{A} \rightarrow \mathcal{B}^{(n)}$  and  $\bar{\Delta}_n : \mathcal{A} \rightarrow \bar{\mathcal{B}}^{(n)}$ ; and because of the uniform bound on  $\|\bar{\Delta}_n\|$  (each  $\bar{\Delta}_n$  is contractive), we have  $T = \sum_{n=0}^\infty \Delta_n T = \sum_{n=0}^\infty T^{(n)}$ , with  $T^{(n)} \in \mathcal{B}^{(n)}$ , for all  $T \in \mathcal{A}$ . The formulae  $\bar{\Delta}_n T = \bar{T}^{(n)} = \sum_{r=0}^\infty P_{n,r} T P_{n,r}$  and  $T - \bar{T}^{(n)} = \sum_{r=0}^\infty (1 - \bar{P}_{n,r}) T P_{n,r}$  remain true in the strong operator topology.



### 3. $\mathcal{A}$ is radical

In order to prove that our algebra  $\mathcal{A}$  is radical, the main theorem we need is the following:

**Theorem 3.1.** *Every  $T \in \mathcal{A}^{(n)}$ , or the norm closure thereof, satisfies*

$$(14) \quad (1 - \bar{\pi}_k)\bar{P}_n T \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

*Indeed,  $\bar{P}_n T$  is a compact operator. Furthermore, every  $T \in \overline{\mathcal{A}^{(n)}}$  satisfies*

$$(15) \quad \|T\| = \|\bar{P}_n T \bar{P}_n\|.$$

*Proof of Theorem 3.1, first part.* From Definition 2.2, we find that if  $s \neq c$  but  $\mu(s) = c$ , then we must have  $s = g^k h y c$  for some  $k \in \mathbb{N}_0$  and  $y \in S$ . In particular,  $\rho(s) > \rho(c)$ . So if  $\rho(c) = n$ , then the map  $\bar{P}_n T_c$  in fact has rank 1; it maps  $p(c)$  to  $c$ , and all other  $s \in S$  to zero. Any product  $T = \prod_{i=0}^{k-1} T_{c_{k-i}}$  with  $c_i \in \mathcal{C}$  and  $\max\{\rho(c_i) : 1 \leq i \leq k\} = n$  accordingly satisfies  $\bar{P}_n T = \prod_{i=0}^{k-1} \bar{P}_n T_{c_{k-i}}$  (because  $\ker \bar{P}_n$  is an invariant subspace for each  $T_{c_j}$ ) so the rank of  $\bar{P}_n T$  is at most 1.  $\mathcal{A}^{(n)}$  is the linear span of such maps, so any  $T \in \mathcal{A}^{(n)}$ , or its norm closure, will have  $\bar{P}_n T$  a compact operator; hence  $\|(1 - \bar{\pi}_k)\bar{P}_n T\| \rightarrow 0$  as  $k \rightarrow \infty$ .  $\square$

To prove the second part of the theorem, we need certain preliminaries, which we give in the following two lemmas, the first of which is rather elementary:

**Lemma 3.2.** *Let  $M \in M_{m+1}(\mathbb{C})$  be a strictly lower triangular matrix, and let  $\|\cdot\|$  and  $\|\cdot\|'$  be two norms on  $\mathbb{C}^{m+1}$ , with  $\|\lambda_0, \lambda_1, \dots, \lambda_m\| = (\sum_{i=0}^m \omega_i^2 |\lambda_i|^2)^{1/2}$  and  $\|\lambda_0, \lambda_1, \dots, \lambda_m\|' = (\sum_{i=0}^m (\omega'_i)^2 |\lambda_i|^2)^{1/2}$  for positive constants  $\omega_i, \omega'_i$  (with  $i = 1, \dots, m$ ). Suppose we have*

$$(16) \quad \frac{\omega'_{i+1}}{\omega'_i} \leq \frac{1}{2} \cdot \frac{\omega_{i+1}}{\omega_i}$$

for each  $i = 0, \dots, m - 1$ . Then

$$(17) \quad \|M\|' \leq \|M\|.$$

*Proof.* Let  $(e_i)_{i=0}^m$  be the unit vectors of  $\mathbb{C}^{m+1}$ , and write  $M e_i = \sum_{j>i} M_{j,i} e_j$ . We may assume  $\|M\| = 1$ , in which case  $|M_{j,i}| \leq \|e_i\|/\|e_j\| = \omega_i/\omega_j$  for all  $i$  and  $j$ . For  $k \in [1, m]$ , the weighted shift matrix  $M^{(k)}$  with

$$(18) \quad M^{(k)} e_i = \begin{cases} M_{i+k,i} e_{i+k} & \text{if } i+k \leq m, \\ 0 & \text{if } i+k > m \end{cases}$$

satisfies

$$\|M^{(k)}\|' = \max_{i \in [0, m-k]} |M_{i+k,i}| \frac{\omega'_{i+k}}{\omega'_i} \leq \max_{i \in [0, m-k]} \left( \frac{\omega_i}{\omega_{i+k}} \right) \cdot \frac{\omega'_{i+k}}{\omega'_i} \leq 2^{-k},$$

by (16). Then  $M = \sum_{k=1}^m M^{(k)}$ , so  $\|M\|' \leq \sum_{k=1}^m 2^{-k} < 1$ . □

**Lemma 3.3.** (a) *Let  $K$  be a connected component of  $G^{(n)}$ . Then either  $K$  equals  $S_{n,0}$ , the component which contains 1, or  $K$  consists of a path*

$$y \rightarrow gy \rightarrow g^2y \rightarrow \dots \rightarrow g^m y$$

for some  $y \in S$  and  $m \in \mathbb{N}$  such that the colour depths satisfy  $\rho\mu(y) > n$  and  $\rho\mu(g^{m+1}y) > n$ , but  $\rho\mu(g^i y) \leq n$  for  $i \in [1, m]$ . Furthermore, there is a path  $s_0 \rightarrow s_1 \rightarrow \dots \rightarrow s_m$  in the component  $S_{n,0}$  such that the colours satisfy  $\mu(s_i) = \mu(g^i y)$  for each  $i \in [1, m]$ .

(b) For every  $T \in \mathcal{A}$ , we have  $\|(1 - \bar{P}_n)T\| \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* (a) Suppose  $K \neq S_{n,0}$ . Since  $K$  cannot meet  $S_{n,0}$ , every vertex  $x \in K$  must have  $\rho(x) > n$ . But if  $x \rightarrow x'$  is an edge in  $K$ , we must have  $\rho\mu(x') \leq n$ , and therefore  $\mu(x') \neq x'$ , so  $x' \notin \mathcal{C}$ , so  $x' = g^k hz$  for some  $z \in S$  and  $k > 0$  with  $\rho(hz) = \rho(x') > n$ . Indeed, we must have  $1 + l(z) \nmid k$ . Every edge of  $K$  must be of form  $x \rightarrow gx$  rather than  $x \rightarrow hx$ , so  $K$  does indeed consist of a path (finite or infinite) of form  $g^r hz \rightarrow g^{r+1} hz \rightarrow g^{r+2} hz \rightarrow \dots$ , for some  $r \geq 0$ . But we have the condition  $1 + l(z) \nmid k$  for any  $k$  such that  $k > r$  and  $g^k hz$  is in the path; so the path is finite. Its last vertex must be  $g^t hz$  for some  $t$  with  $t - r \leq 1 + l(z)$ . Writing  $m = t - r$  and  $y = g^r hz$ , we see that  $K = \{g^i y : i = 0, \dots, m\}$ .

If  $r > 0$ , we must have  $\rho\mu(y) = \rho\mu(g^r hz) > n$  or we could continue the path in  $K$  backwards to include the vertex  $g^{r-1} hz$ . If  $r = 0$ , we have  $\mu(y) = \mu(hz) = hz$ , so  $\rho\mu(y) > n$  anyway. Also, we must have  $\rho\mu(g^{t+1} hz) > n$  or we could include the vertex  $g^{t+1} hz$  in our component  $K$ . For  $i \in (r, t]$  we have  $\rho\mu(g^i hz) \leq n$  because the edge  $g^{i-1} hz \rightarrow g^i hz$  lies in  $K$ . Thus the component  $K$  is as described in part (a) of this lemma.

To complete the proof of part (a), we claim that there is a sequence  $s_0 \rightarrow s_1 \rightarrow \dots \rightarrow s_m \in S_{n,0}$  such that  $\mu(s_i) = \mu(g^i y)$  for each  $i \in [1, m]$ . This is proved by induction on  $l(y) = \min\{l(u) : u \in K\}$ . If  $l(y) \leq n$ , there is nothing to prove because the component is  $S_{n,0}$  after all. If the component  $K$  is not  $S_{n,0}$ , write  $K = \{g^i hz : r \leq i \leq r + m\}$ . We return to (8) to compute the colours  $\mu(g^i hz)$  for  $i \in (r, r + m]$ . Writing  $l = l(z)$  and  $z = \prod_{j=0}^{l-1} z_j$  ( $z_j \in \{g, h\}$ ), we find that if  $i' \in [1, l]$  is the unique integer with  $i' \equiv i \pmod{1+l}$ , then  $\mu(g^i hz) = \mu(\prod_{j=0}^{i'-1} z_{i-j}) = \mu(p^{l-i'} z)$ . If  $r_0 \in [0, l]$  satisfies  $r_0 \equiv r \pmod{1+l}$ , then the sequence  $\mu(g^i y)$  ( $i = 1, \dots, m$ ) is the sequence  $\mu(p^{l-r_0-i} z)$  ( $i = 1, \dots, m$ ). The vertices  $(p^{l-r_0-i} z)_{i=0}^m$  form a path in  $G$  which, since it involves the same colours for  $i > 0$ , is also a path in  $G^{(n)}$ . So this path is part of a component  $K'$  of  $G^{(n)}$ . If  $K' = S_{n,0}$  we are done; if not, we note that the minimum length of an element of  $K'$  is strictly less than  $l(y)$ , so the result follows by induction hypothesis.

(b) Let  $c \in \mathcal{C}$ . From (10), for  $s \in S$  we have

$$(1 - \bar{P}_n)T_c e_s = \begin{cases} w(gs)e_{gs} & \text{if } \mu(gs) = c \text{ and } \rho(gs) > n, \\ w(hs)e_{hs} & \text{if } \mu(hs) = c \text{ and } \rho(hs) > n, \\ 0 & \text{otherwise.} \end{cases}$$

But  $w(x) = 2^{-\rho(x)}$ , so  $\|(1 - \bar{P}_n)T_c\| \leq 2^{-n-1}$ . We will also have  $\|(1 - \bar{P}_n)T\| \rightarrow 0$  for any operator  $T$  in the norm-closed right ideal generated by the operators  $T_c$ . But this right ideal is the entire algebra  $\mathcal{A}$ .  $\square$

*Proof of Theorem 3.1, second part.* By Lemma 2.6(b), when  $T \in \bar{\mathcal{A}}^{(n)}$  we have  $T = \sum_K Q_{n,K} T Q_{n,K}$ , where the sum is taken over the connected components  $K$  of  $G^{(n)}$ . So

$$(19) \quad \|T\| = \sup_K \|Q_{n,K} T Q_{n,K}\|.$$

If  $K$  equals  $S_{n,0}$ , the component containing 1, then the norm  $\|Q_{n,K} T Q_{n,K}\|$  equals  $\|\bar{P}_n T \bar{P}_n\|$ . If  $K$  is any other component, we claim that the norm is at most  $\|\bar{P}_n T \bar{P}_n\|$ . By Lemma 3.3(a), we can write  $K = \{g^i y : 0 \leq i \leq m\}$  for suitable  $y \in S$  and  $m$ ; writing  $\gamma_i$  for the colour  $\mu(g^i y)$ , there is also a set  $\kappa = \{s_i : 0 \leq i \leq m\} \subset S_{n,0}$  such that the colour  $\mu(s_i)$  equals  $\gamma_i$  for  $i \in [1, m]$ . Let  $q$  denote the orthogonal (coordinatewise) projection onto  $\text{lin}(\kappa)$ . If  $c_1, c_2, \dots, c_r \in \mathcal{C}$ , then the compression  $\tau_1 = Q_{n,K} T_{c_r} T_{c_{r-1}} \dots T_{c_1} Q_{n,K}$  sends  $g^i y$  to  $g^{i+r} y$ , if  $i+r \leq m$  and  $c_i = \gamma_{r+i}$  for each  $i = 1, \dots, r$ ; otherwise, we have  $\tau_1 g^i y = 0$ . Similarly, the compression  $\tau_2 = q T_{c_r} T_{c_{r-1}} \dots T_{c_1} q$  sends  $s_i$  to  $s_{i+r}$  if  $i+r \leq m$  and  $c_i = \gamma_{r+i}$  for each  $i = 1, \dots, r$ ; otherwise, we have  $\tau_2 s_i = 0$ . So the compressions  $\tau_1$  and  $\tau_2$  are intertwined by the map  $\eta$  sending  $g^i y$  to  $s_i$  for each  $i$ . Indeed, if  $T \in \bar{\mathcal{A}}^{(n)}$ , the compressions  $\tau = Q_{n,K} T Q_{n,K}$  and  $\tau' = q T q$  are intertwined, with  $\eta \tau = \tau' \eta$ . So  $\tau$  has the same  $(m+1) \times (m+1)$  matrix  $M$  with respect to the basis  $(g^i y)_{i=0}^m$  of  $Q_{n,K} \mathcal{H}$ , as  $\tau'$  has with respect to the basis  $(s_i)_{i=0}^m$  of  $q \mathcal{H}$ .  $M$  is strictly lower triangular, because all such compressions  $q T q$  map  $s_i$  into  $\text{lin}\{s_j : j > i\}$  for each  $i$ . The norm on  $q \mathcal{H}$  is given by  $\|\sum_{i=0}^m \lambda_i s_i\| = (\sum_{i=0}^m \omega_i^2 |\lambda_i|^2)^{1/2}$ , where  $\omega_i = W(s_i)$ . The norm on  $Q_{n,K} \mathcal{H}$  is likewise given by  $\|\sum_{i=0}^m \lambda_i g^i y\| = (\sum_{i=0}^m (\omega'_i)^2 |\lambda_i|^2)^{1/2}$ , where  $\omega'_i = W(g^i y)$ . For  $0 \leq i < m$ , the ratio  $\omega_{i+1}/\omega_i$  equals  $W(s_{i+1})/W(s_i) = w(s_{i+1})$  because there is an edge  $s_i \rightarrow s_{i+1}$  in  $G$ ; and  $w(s_{i+1}) \geq 2^{-n}$  because  $s_{i+1} \in S_{n,0}$  so  $\rho(s_{i+1}) \leq n$ . On the other hand, the ratio  $\omega'_{i+1}/\omega'_i$  equals  $W(g^{i+1} y)/W(g^i y) = w(g^{i+1} y) \leq 2^{-n-1}$ , because  $g^{i+1} y \notin S_{n,0}$  so  $\rho(g^{i+1} y) \geq n+1$ . We deduce that  $\omega'_{i+1}/\omega'_i \leq \frac{1}{2} \cdot \omega_{i+1}/\omega_i$ . By Lemma 3.2, we have  $\|Q_{n,K} T Q_{n,K}\| = \|\tau\| \leq \|\tau'\|$ , and of course  $\|\tau'\| \leq \|\bar{P}_n T \bar{P}_n\|$  because the orthogonal projection satisfies  $q \leq \bar{P}_n$ . By (19), the norm of  $T$  is the supremum of  $\|\bar{P}_n T \bar{P}_n\|$  and the norms  $\|Q_{n,K} T Q_{n,K}\|$  for all other connected components  $K \subset G^{(n)}$ ; so  $\|T\| = \|\bar{P}_n T \bar{P}_n\|$  as claimed by the theorem.  $\square$

We can now prove the main theorem of this section:

**Theorem 3.4.**  $\mathcal{A}$  is radical.

*Proof.* If not, let  $T \in \mathcal{A}$  have spectral radius at least 1. By Lemma 3.3, there is an  $n \in \mathbb{N}$  such that  $\|(1 - \bar{P}_n)T\| \leq \frac{1}{2}$ . We claim that the spectral radius of the compression  $\bar{P}_n T \bar{P}_n$  is at least 1. For by Lemma 2.6(a), for each  $k \in \mathbb{N}$  we have  $T^k = \bar{P}_n T^k \bar{P}_n + (1 - \bar{P}_n)T^k$  (any  $k \in \mathbb{N}$ ) because  $\ker \bar{P}_n$  is an invariant subspace for  $\mathcal{A}$ ; indeed,  $T^k = (\bar{P}_n T \bar{P}_n)^k + (1 - \bar{P}_n)T^k$ , because the compression map  $T \rightarrow \bar{P}_n T \bar{P}_n$  is an algebra homomorphism on  $\mathcal{A}$ . So for all  $k > 0$ ,  $T^k = (\bar{P}_n T \bar{P}_n)^k + (1 - \bar{P}_n)T \cdot T^{k-1}$ , and hence

$$\begin{aligned} \|T^k\| &\leq \|(\bar{P}_n T \bar{P}_n)^k\| + \frac{1}{2} \cdot \|T^{k-1}\| \\ &\leq \|(\bar{P}_n T \bar{P}_n)^k\| + \frac{1}{2} \cdot \|(\bar{P}_n T \bar{P}_n)^{k-1}\| + \frac{1}{4} \cdot \|T^{k-2}\| \\ &\leq \dots \leq 2^{-k} + \sum_{j=0}^{k-1} 2^{-j} \|(\bar{P}_n T \bar{P}_n)^{k-j}\|. \end{aligned}$$

If the spectral radius of  $\bar{P}_n T \bar{P}_n$  is less than 1, we can find  $r < 1$  and  $C > 0$  such that  $\|(\bar{P}_n T \bar{P}_n)^j\| \leq Cr^j$  for all  $j \in \mathbb{N}$ , so we have  $1 \leq \|T^k\| \leq 2^{-k} + \sum_{j=0}^{k-1} C \cdot 2^{-j} \cdot r^{k-j} \leq 2^{-k} + kC \max(\frac{1}{2}, r)^k$  for all  $k \in \mathbb{N}$ . This is a contradiction for large  $k$ , so the spectral radius of the compression  $\bar{P}_n T \bar{P}_n$  must be at least 1.

It is thus sufficient to show that for each  $T \in \mathcal{A}$  and  $n \in \mathbb{N}$ , the compression  $\bar{P}_n T \bar{P}_n$  is quasinilpotent. Let us prove this by induction on  $n$ , beginning with the not-quite-trivial case  $n = 0$ .

By Lemma 2.6(d) (and its generalisation to  $T \in \mathcal{A}$  rather than  $T \in \mathcal{A}_0$  as discussed after Definition 2.7), we have  $\bar{P}_0 T \bar{P}_0 = \bar{P}_0 \bar{T}^{(0)} \bar{P}_0 = \bar{P}_0 T^{(0)} \bar{P}_0$  for any  $T \in \mathcal{A}$ ; and  $T^{(0)} \in \mathcal{B}^{(0)}$ . By Theorem 3.1, we have  $(1 - \bar{\pi}_k) \bar{P}_0 T^{(0)} \rightarrow 0$ , and by Lemma 2.6(a),  $T^{(0)}$  maps  $\ker \bar{\pi}_k$  into  $\ker \bar{\pi}_{k+1}$  for every  $k$ . Writing  $\varepsilon_k = \|(1 - \bar{\pi}_k) \bar{P}_0 T^{(0)}\|$ , we have  $\varepsilon_k \rightarrow 0$ , and

$$\begin{aligned} (\bar{P}_0 T \bar{P}_0)^k &= (\bar{P}_0 T^{(0)} \bar{P}_0)^k \\ &= (1 - \bar{\pi}_{k-1}) \bar{P}_0 T^{(0)} (1 - \bar{\pi}_{k-2}) \bar{P}_0 T^{(0)} (1 - \bar{\pi}_{k-3}) \\ &\quad \dots \bar{P}_0 T^{(0)} (1 - \bar{\pi}_0) \bar{P}_0 T^{(0)} \bar{P}_0, \end{aligned}$$

so  $\|(\bar{P}_0 T \bar{P}_0)^k\| \leq \prod_{j=0}^{k-1} \varepsilon_j$ , and hence  $\bar{P}_0 T \bar{P}_0$  is indeed quasinilpotent.

Proceeding to the case of a general  $n \in \mathbb{N}$ , we note that for  $T \in \mathcal{A}$ ,  $\bar{P}_n T \bar{P}_n = \bar{P}_n \bar{T}^{(n)} \bar{P}_n = \bar{P}_n (T^{(n)} + \bar{T}^{(n-1)}) \bar{P}_n$ , where  $T^{(n)} \in \mathcal{B}^{(n)}$  and  $\bar{T}^{(n-1)} \in \mathcal{B}^{(n-1)}$ .

Writing  $\tau = \bar{T}^{(n-1)}$ , we have  $\tau^k \in \mathcal{B}^{(n-1)}$  for all  $k$ , so by Theorem 3.1,  $\|\tau^k\| = \|\bar{P}_{n-1} \tau^k \bar{P}_{n-1}\|$  for all  $k$ . But  $\ker \bar{P}_{n-1}$  is an invariant subspace for  $\mathcal{A}$ , so

$$\bar{P}_{n-1} \tau^k \bar{P}_{n-1} = (\bar{P}_{n-1} \tau \bar{P}_{n-1})^k;$$

and our induction hypothesis tell us that  $\bar{P}_{n-1}\tau\bar{P}_{n-1}$  is quasinilpotent. Thus  $\|\tau^k\|^{1/k} \rightarrow 0$  as  $k \rightarrow \infty$ , and also  $\|(\bar{P}_n\tau\bar{P}_n)^k\|^{1/k} = \|\bar{P}_n\tau^k\bar{P}_n\|^{1/k} \rightarrow 0$  as  $k \rightarrow \infty$ . So  $\bar{P}_n\bar{T}^{(n-1)}$  is quasinilpotent.

Meanwhile  $\sigma = \bar{P}_nT^{(n)}$  is a compact operator by Theorem 3.1, satisfying  $\varepsilon_k = \|(1 - \bar{\pi}_k)\sigma\| \rightarrow 0$  as  $k \rightarrow \infty$ ; and both  $\sigma$  and  $\tau$  map  $\ker \bar{\pi}_k$  into  $\ker \bar{\pi}_{k+1}$  for each  $k$ .

Let us pick an arbitrary  $\delta > 0$  and choose  $C > 0$  such that  $\|(\bar{P}_n\bar{T}^{(n-1)})^k\| \leq C \cdot \delta^k$  for all  $k \in \mathbb{N}_0$ . Then for any  $k \in \mathbb{N}$ , we have

$$\begin{aligned} (\bar{P}_nT\bar{P}_n)^k &= (\bar{P}_nT^{(n-1)} + \sigma)^k \bar{P}_n \\ &= \sum_{r=0}^k \sum_{\substack{i_0+i_1+\dots+i_r=k-r \\ i_j \in \mathbb{N}_0}} (\bar{P}_nT^{(n-1)})^{i_0} \cdot \prod_{j=1}^r \sigma \cdot (\bar{P}_nT^{(n-1)})^{i_j} \cdot \bar{P}_n, \end{aligned}$$

and writing  $u_j = \sum_{t=j}^r (1 + i_t) - 1$ , the product from  $j = 1$  to  $r$  is equal to  $\prod_{j=1}^r (1 - \bar{\pi}_{u_j})\sigma(\bar{P}_nT^{(n-1)})^{i_j}$ ; so

$$\|(\bar{P}_nT\bar{P}_n)^k\| \leq \sum_{r=0}^k \sum_{\substack{i_0+i_1+\dots+i_r=k-r \\ i_j \in \mathbb{N}_0}} C^{r+1} \delta^{k-r} \cdot \prod_{j=1}^r \varepsilon_{u_j}.$$

Now  $u_j \geq j - 1$  in all cases, so writing  $\eta_j = \prod_{j=1}^r \varepsilon_{j-1}$ , we have

$$\|(\bar{P}_nT\bar{P}_n)^k\| \leq \sum_{r=0}^k \sum_{\substack{i_0+i_1+\dots+i_r=k-r \\ i_j \in \mathbb{N}_0}} C^{r+1} \delta^{k-r} \eta_r = \sum_{r=0}^k \binom{k}{r} C^{r+1} \delta^{k-r} \eta_r.$$

But  $\eta_r^{1/r} \rightarrow 0$ , so we can choose  $D > 0$  such that  $\eta_r \leq D \cdot (\delta/C)^r$  for all  $r$ ; substituting this in the previous equation, we find that  $\|(\bar{P}_nT\bar{P}_n)^k\| \leq \sum_{r=0}^k \binom{k}{r} CD \delta^k = CD \cdot (2\delta)^k$ . So the spectral radius of  $\bar{P}_nT\bar{P}_n$  is at most  $2\delta$ ; but  $\delta > 0$  was arbitrary, so  $\bar{P}_nT\bar{P}_n$  is quasinilpotent. Therefore every  $T \in \mathcal{A}$  is quasinilpotent;  $\mathcal{A}$  is a radical Banach algebra. □

#### 4. $\bar{\mathcal{A}}^{w*}$ is semisimple

We wish to prove the second half of our main result, namely that the bidual  $\mathcal{A}^{**}$  is semisimple. We shall do this by showing that the weak- $*$  closure  $\bar{\mathcal{A}}^{w*}$  of  $\mathcal{A}$  in  $B(\mathcal{H})$  is semisimple, and then show that the natural representation  $\theta : \mathcal{A}^{**} \rightarrow B(\mathcal{H})$ , whose image is  $\bar{\mathcal{A}}^{w*}$ , is faithful, so that  $\mathcal{A}^{**}$  itself is semisimple. (Our “natural representation” is the restriction to  $\mathcal{A}^{**}$  of the natural projection  $\mathcal{T}^{***} \rightarrow \mathcal{T}^*$ , where  $\mathcal{T}$  are the trace-class operators on  $\mathcal{H}$ , and  $\mathcal{T}^* = B(\mathcal{H})$ ,  $\mathcal{T}^{***} = B(\mathcal{H})^{**}$ ).

In this section, we show that  $\bar{\mathcal{A}}^{w*}$ , very unlike  $\mathcal{A}$  itself, is semisimple.

**Definition 4.1.** Let  $\mathcal{C}^{<\infty}$  be the collection of all finite sequences  $(c_1, c_2, \dots, c_m)$  of colours  $c_i \in \mathcal{C}$ , for  $m \in \mathbb{N}$  (we exclude  $m = 0$ ). For  $\mathbf{c} = (c_1, c_2, \dots, c_m) \in \mathcal{C}^{<\infty}$ , let  $T_{\mathbf{c}}$  denote the operator  $\prod_{i=1}^m T_{c_i} \in \mathcal{A}_0$ . Let  $S_{\mathcal{A}} \subset \mathcal{C}^{<\infty}$  be the set of  $\mathbf{c} \in \mathcal{C}$  such that  $T_{\mathbf{c}} \neq 0$ .

We think of  $S_{\mathcal{A}}$  as the ‘‘support’’ of  $\mathcal{A}$ , because clearly every  $T \in \mathcal{A}_0$  is equal to a sum

$$(20) \quad T = \sum_{\mathbf{c} \in S_{\mathcal{A}}} \lambda_{\mathbf{c}} \cdot T_{\mathbf{c}},$$

the coefficients  $\lambda_{\mathbf{c}} \in \mathbb{C}$  being finitely nonzero.

**Lemma 4.2.** *Given  $T \in \mathcal{A}_0$ , the coefficients  $\lambda_{\mathbf{c}}(T)$  such that  $T = \sum_{\mathbf{c} \in S_{\mathcal{A}}} \lambda_{\mathbf{c}}(T) \cdot T_{\mathbf{c}}$  are unique, and they are weak- $*$  continuous linear functionals of  $T$ .*

*Proof.* For  $c \in \mathcal{C}$ , (10) tells us that  $\langle T_{\mathbf{c}} e_s, e_t \rangle \neq 0$  if and only if  $s = p(t)$  and the colour  $\mu(t) = c$ , in which case it is equal to  $w(t)$ . Any easy induction then tells us that for  $\mathbf{c} = (c_1, c_2, \dots, c_m) \in S_{\mathcal{A}}$ ,  $\langle T_{\mathbf{c}} e_s, e_t \rangle \neq 0$  if and only if  $s = p^m(t)$  and, for each  $i = 1, \dots, m$ , the colour  $\mu(p^{i-1}t)$  equals  $c_i$ . In that case,  $\langle T_{\mathbf{c}} e_s, e_t \rangle = \prod_{i=0}^{m-1} w(p^i t) = W(t)/W(s)$ . So for fixed  $s, t$ , the colour sequence  $\mathbf{c} \in S_{\mathcal{A}}$  such that  $\langle T_{\mathbf{c}} e_s, e_t \rangle \neq 0$  is unique if it exists; and since  $T_{\mathbf{c}} \neq 0$  for  $\mathbf{c} \in S_{\mathcal{A}}$ , for fixed  $\mathbf{c} \in S_{\mathcal{A}}$  there is at least one pair  $s, t \in S$  such that  $\langle T_{\mathbf{c}} e_s, e_t \rangle \neq 0$ .

Given  $T \in \mathcal{A}_0$ ,  $T = \sum_{\mathbf{c} \in S_{\mathcal{A}}} \lambda_{\mathbf{c}} \cdot T_{\mathbf{c}}$ , we therefore have

$$(21) \quad \lambda_{\mathbf{c}} = \lambda_{\mathbf{c}}(T) = \frac{W(s)}{W(t)} \langle T e_s, e_t \rangle,$$

where  $s, t$  is any pair such that  $\langle T_{\mathbf{c}} e_s, e_t \rangle \neq 0$ . Now  $\lambda_{\mathbf{c}}$  is indeed uniquely determined by  $T$ , and it is indeed a weak- $*$  continuous function of  $T$ ; (21) even equates  $\lambda_{\mathbf{c}} \in B(H)_*$  with an element of  $\mathcal{T}$  of rank 1.  $\square$

Given two elements  $\mathbf{c} = (c_1, \dots, c_m), \mathbf{d} = (d_1, \dots, d_n)$  in  $\mathcal{C}$ , we can define the product  $\mathbf{c} \cdot \mathbf{d}$  to be the sequence  $(c_1, \dots, c_m, d_1, \dots, d_n)$ . From (20), we see that for  $T, T' \in \mathcal{A}_0$ , we have

$$(22) \quad \lambda_{\mathbf{c}}(TT') = \sum_{\substack{\mathbf{d}, \mathbf{e} \in S_{\mathcal{A}}, \\ \mathbf{d} \odot \mathbf{e} = \mathbf{c}}} \lambda_{\mathbf{d}}(T) \cdot \lambda_{\mathbf{e}}(T'),$$

where the product  $\mathbf{d} \odot \mathbf{e}$  denotes concatenation of sequences. The sum is always finite (it has  $m - 1$  terms when  $\mathbf{c} = (c_1, \dots, c_m)$ ), so (22) remains true even when we extend  $\lambda_{\mathbf{c}}$  to the weak- $*$  closure  $\overline{\mathcal{A}}^{w*}$  of  $\mathcal{A}_0$ .

Now for each  $c \in \mathcal{C}$ , (10) tells us that the left support projection  $l(T_c)$  for the operator  $T_c$  is the orthogonal projection onto  $\overline{\text{lin}}\{e_t : t \in S^-, \mu(t) = c\}$ . We also have  $\|T_c\| = w(c) = 2^{-\rho(c)} \leq 1$ . These left support projections are mutually orthogonal for different colours  $c$ . The corresponding right support projection  $r(T_c)$  is the

projection onto  $\overline{\text{lin}}\{e_s : s \in S, s = p(t), \mu(t) = c\}$ . These right support projections are not mutually orthogonal, but nevertheless, for each  $s \in S$  there are only two  $t \in S^-$  such that  $s = p(t)$ , so the norm of any sum  $\sum_{c \in S_{\mathcal{A}}} \lambda_c r(T_c)$  is at most  $2 \cdot \sup\{|\lambda_c| : c \in S_{\mathcal{A}}\}$ . Hence for any sequence  $\mathbf{x} \in l^\infty(S_{\mathcal{A}})$ , the formal sum

$$(23) \quad T = \sum_{c \in S_{\mathcal{A}}} \frac{x_c}{w_c} \cdot T_c$$

satisfies

$$T^*T = \sum_{c,d \in S_{\mathcal{A}}} \frac{x_c^* x_d}{w_c w_d} T_c^* l(T_c) l(T_d) T_d = \sum_{c \in S_{\mathcal{A}}} \frac{|x_c|^2}{w_c^2} T_c^* T_c \leq \sum_{c \in S_{\mathcal{A}}} |x_c|^2 r(T_c),$$

in particular  $\|T^*T\| \leq 2 \cdot \|\mathbf{x}\|_\infty^2$ . So the sum  $T$  in fact converges in the weak-\* topology to an element of  $\overline{\mathcal{A}}^{w*}$  of norm at most  $\sqrt{2} \cdot \|\mathbf{x}\|_\infty$ .

**Theorem 4.3.**  $\overline{\mathcal{A}}^{w*}$  is semisimple.

*Proof.* Let  $T \in \overline{\mathcal{A}}^{w*}$ ,  $T \neq 0$ . We claim that  $T \notin \text{rad } \overline{\mathcal{A}}^{w*}$ . Let us choose  $s, t \in S$  such that  $\langle T e_s, e_t \rangle \neq 0$ .

Suppose first that  $s \neq 1$ . Let  $l_0 = l(s) > 0$ , and for  $i = 1, \dots, l_0$ , write  $d_i = \mu(p^{i-1}s) = \mu(p^{i+m-1}t)$ . Writing  $\mathbf{d} = (d_1, \dots, d_{l_0}) \in \mathcal{C}^{<\infty}$ , we will have  $T_{\mathbf{d}}(1) = s$ , so  $\mathbf{d} \in S_{\mathcal{A}}$  and the product  $T' = T \cdot T_{\mathbf{d}}$  satisfies  $\langle T' e_1, e_t \rangle \neq 0$ . Furthermore, in order to show  $T \notin \text{rad } \overline{\mathcal{A}}^{w*}$  it is enough to show that  $T' \notin \text{rad } \overline{\mathcal{A}}^{w*}$ , because the radical is an ideal. So, we can replace  $T$  with  $T'$  if necessary, and assume that  $\langle T e_1, e_t \rangle \neq 0$ .

Then  $\lambda_{\mathbf{c}}(T) \neq 0$ , where  $\mathbf{c} = (c_1, c_2, \dots, c_l) \in S_{\mathcal{A}}$  is the unique sequence such that  $l = l(t)$  (so  $1 = p^l t$ ), and the colours  $\mu(p^{i-1}t)$  ( $i = 1, \dots, l$ ) are  $c_i$ . Write  $\xi_m = g^{(m-1)(l+1)} h t$ , and let  $E \subset \mathcal{C}$  be the collection  $\{\xi_m : m \in \mathbb{N}_0\}$  (noting from (7) that these elements are truly elements of the colour set  $\mathcal{C}$ ). Let us also note that the weight  $w_{\xi_m} = 2^{-\rho(\xi_m)} = 2^{-(1+\rho(t))}$  is independent of  $m$ . So  $U = \sum_{c \in E} T_c \in \overline{\mathcal{A}}^{w*}$  (for  $U$  is a weak-\* convergent sum like  $T$  in (23)). We claim that the product  $U \cdot T \in \overline{\mathcal{A}}^{w*}$  is not quasinilpotent, so  $UT$  and  $T$  itself are not in the radical of  $\overline{\mathcal{A}}^{w*}$ . To prove this, we compute the inner product  $\langle (UT)^m e_1, e_{\xi_m} \rangle$  for every  $m \in \mathbb{N}$ . Obviously  $\lambda_{\mathbf{d}}(U) = 1$  (if  $\mathbf{d} \in E$ ) or zero otherwise.

Now the length  $L$  equals  $l(\xi_m) = m(1+l)$ , and the colour sequence  $\mu(p^{i-1}\xi_m)$  (with  $i = 1, \dots, L$ ) is obtained from (8) as follows: if  $1+l \mid i-1$ , we have  $p^{i-1}\xi_m = g^{(m-1-r)(l+1)} h t$  (with  $r = (i-1)/(1+l) \in [0, m)$ ), and  $\mu(p^{i-1}\xi_m) = p^{i-1}\xi_m = \xi_{m-r} \in E$ . But if  $1+l \nmid i-1$ , then writing  $i-1 = r(1+l) + j$  (with  $r \in [0, m)$ ,  $j \in [1, l]$ ), if  $r = m-1$  we have  $p^{i-1}\xi_m = p^j h t = p^{j-1} t$ , so  $\mu(p^{i-1}\xi_m) = c_j$ ; but if  $r < m-1$  we have  $p^{i-1}\xi_m = g^{(m-2-r)(l+1)+l+1-j} h t$  and the recursive definition in (8) tells us that  $\mu(p^{i-1}\xi_m) = \mu(p^{l-(l+1-j)} t) = \mu(p^{j-1} t) = c_j$  also. So for all  $i$ ,

$1 \leq i \leq L$ , we have

$$(24) \quad \mu(p^{i-1}\xi_m) = \begin{cases} \xi_{m-r} \in E & \text{if } i - 1 = r(1+l), \\ c_j \notin E & \text{if } i - 1 \equiv j \pmod{l+1}, 1 \leq j \leq l. \end{cases}$$

The full sequence  $(\mu(p^{i-1}\xi_m))_{i=1}^L \in S_{\mathcal{A}}$  is the concatenation  $\bigodot_{r=0}^{m-1}(\xi_{m-r} \odot \mathbf{c})$ , where we slightly abuse notation by writing  $\xi_{m-r}$  for the sequence of length 1 in  $\mathcal{C}^{<\infty}$ . Now from (21), we have the inner product

$$(25) \quad \langle (UT)^m e_1, e_{\xi_m} \rangle = W(\xi_m) \cdot \lambda_{\xi_m}((UT)^m);$$

and using (22)  $2m$  times, we have

$$\lambda_{\xi_m}((UT)^m) = \sum_{\substack{\mathbf{d}^{(1)}, \mathbf{c}^{(1)}, \dots, \mathbf{d}^{(m)}, \mathbf{c}^{(m)} \in S_{\mathcal{A}}, \\ \bigodot_{i=1}^m (\mathbf{d}^{(i)} \odot \mathbf{c}^{(i)}) = \bigodot_{r=0}^{m-1} (\xi_{m-r} \odot \mathbf{c})}} \prod_{i=1}^m \lambda_{\mathbf{d}^{(i)}}(U) \lambda_{\mathbf{c}^{(i)}}(T).$$

But the coefficient  $\lambda_{\mathbf{d}}(U)$  can only be nonzero if the sequence  $\mathbf{d}$  has length 1 and consists of one of the colours  $\xi_j \in E$  (in which case the coefficient is equal to 1). There are only  $m$  such colours in the sequence  $\bigodot_{r=1}^m (\xi_{m-r} \odot \mathbf{c})$ , and the rest of the sequence consists precisely of  $m$  copies of  $\mathbf{c}$ , so in fact

$$(26) \quad \lambda_{\xi_m}((UT)^m) = \lambda_{\mathbf{c}}(T)^m.$$

Equation (26) makes the rest of the proof rather straightforward. Substituting it in (25), we have  $\|(UT)^m\| \geq |\langle (UT)^m e_1, e_{\xi_m} \rangle| = |\lambda_{\mathbf{c}}(T)|^m \cdot W(\xi_m)$ ; where writing  $L = m(1+l)$  as usual, we have  $W(\xi_m) = \prod_{j=1}^L w(p^{j-1}\xi_m) = 2^{-\sum_{j=1}^L \rho(p^{j-1}\xi_m)}$ , from Definition 2.2. But  $\xi_m = g^{(m-1)(l+1)}ht$ , so  $\rho(\xi_m) = 1 + \rho(t)$ . And  $\rho(p^i \xi_m) \leq \rho(\xi_m)$  for all  $i \geq 0$ , so for all  $m \in \mathbb{N}$ ,

$$\|(UT)^m\| \geq |\lambda_{\mathbf{c}}(T)|^m \cdot 2^{-L(1+\rho(t))} = |\lambda_{\mathbf{c}}(T)|^m \cdot 2^{-m(1+l)(1+\rho(t))}.$$

Accordingly  $UT \in \overline{\mathcal{A}}^{w*}$  is not a quasinilpotent operator, and  $T \notin \text{rad } \overline{\mathcal{A}}^{w*}$ . □

### 5. $\mathcal{A}^{**}$ is semisimple

Let  $\theta_0 : \mathcal{T}^{***} \rightarrow \mathcal{T}^* = B(\mathcal{H})$  be the natural projection, which is an algebra homomorphism, and let  $\theta = \theta_0|_{\mathcal{A}^{**}}$  be the restriction, which is a representation of  $\mathcal{A}^{**}$ . If  $\tau \in \mathcal{A}^{**}$  is a weak- $*$  limit of operators  $T_\alpha$  in  $\mathcal{A}$ , then for each  $\eta, \zeta \in \mathcal{H}$ , we have  $\langle \theta(\tau)\eta, \zeta \rangle = \lim_\alpha \langle T_\alpha \eta, \zeta \rangle$ , so  $\theta(\tau)$  is the  $\sigma(B(\mathcal{H}), \mathcal{T})$ -limit of the operators  $T_\alpha$ , and the image  $\theta(\mathcal{A}^{**})$  is contained in the weak- $*$  closure  $\overline{\mathcal{A}}^{w*}$  of  $\mathcal{A}$  in  $B(\mathcal{H})$ . Conversely, the image of the unit ball of  $\mathcal{A}^{**}$ , being the weak- $*$  continuous image of a weak- $*$  compact set, is weak- $*$  compact, and therefore contains the weak- $*$  closure  $\overline{B}^{w*}$  of the unit ball of  $\mathcal{A}$ . It is a consequence of the Hahn–Banach theorem that  $\overline{\mathcal{A}}^{w*}$  is equal to the union  $\bigcup_{n=1}^\infty n \cdot \overline{B}^{w*}$ , so we have  $\theta(\mathcal{A}^{**}) = \overline{\mathcal{A}}^{w*}$ , which by Theorem 4.3



is semisimple. To deduce that  $\mathcal{A}^{**}$  is semisimple, we need only prove that  $\theta$  is a faithful (injective) representation.

**Theorem 5.1.** *The representation  $\theta : \mathcal{A}^{**} \rightarrow B(H)$  is faithful.*

*Proof.* Let  $\tau \in \mathcal{A}^{**}$  with  $\|\tau\| = 1$ . We claim that  $\theta(\tau) \neq 0$ . To establish this, we first prove the following lemma:

**Lemma 5.2.** *If  $\tau \in \mathcal{A}^{**}$  with  $\|\tau\| = 1$ , then for every  $\varepsilon > 0$ , there are  $n \in \mathbb{N}$  and  $\phi \in B(H)^*$  with  $\|\phi\| = 1$ , such that the compression  $\phi_n = \bar{P}_n \cdot \phi \cdot \bar{P}_n$  satisfies  $|\langle \tau, \phi_n \rangle| > 1 - \varepsilon$ .*

*Proof.* If  $a, b \in B(H)$  and  $Q$  is an orthogonal projection, then simple calculations yield the inequalities

$$\begin{aligned} \|aQ + b(1 - Q)\| &\leq \sqrt{\|aQ\|^2 + \|b(1 - Q)\|^2}, \\ \|Qa + (1 - Q)b\| &\leq \sqrt{\|Qa\|^2 + \|(1 - Q)b\|^2}. \end{aligned}$$

When these are dualized, the directions of the inequalities are reversed: if  $\phi, \psi \in B(H)^*$ , then

$$\begin{aligned} \|\phi \cdot Q + \psi \cdot (1 - Q)\| &\geq \sqrt{\|\phi \cdot Q\|^2 + \|\psi \cdot (1 - Q)\|^2}, \\ (27) \quad \|Q \cdot \phi + (1 - Q) \cdot \psi\| &\geq \sqrt{\|Q \cdot \phi\|^2 + \|(1 - Q) \cdot \psi\|^2}. \end{aligned}$$

For every  $\eta > 0$  there is a  $\phi \in \mathcal{A}^*$  such that  $\|\phi\| = 1$  and  $\langle \tau, \phi \rangle > 1 - \eta$ . There is also a witness  $T \in \mathcal{A}$  such that  $\|T\| = 1$  and  $\langle \phi, T \rangle > 1 - \eta$ . By Lemma 3.3(b) there is an  $n \in \mathbb{N}$  such that  $\|(1 - \bar{P}_n)T\| < \eta$ . Hence,  $|\langle \phi - \phi_n, T \rangle| \leq \|(1 - \bar{P}_n)T\| + \|\bar{P}_n T(1 - \bar{P}_n)\| = \|(1 - \bar{P}_n)T\| < \eta$  also (because  $\ker \bar{P}_n$  is an invariant subspace for  $\mathcal{A}$ ), and so  $\|\phi_n\| \geq |\langle \phi_n, T \rangle| > 1 - 2\eta$ . By (27) we therefore have  $\|(1 - \bar{P}_n) \cdot \phi\|, \|\phi \cdot (1 - \bar{P}_n)\| < \sqrt{1 - (1 - 2\eta)^2} < 2\sqrt{\eta}$ , and hence  $\|\phi - \phi_n\| < 4\sqrt{\eta}$ . Since  $\langle \tau, \phi \rangle > 1 - \eta$ , we have  $|\langle \tau, \phi_n \rangle| > 1 - \eta - \|\phi - \phi_n\| \geq 1 - \eta - 4\sqrt{\eta}$ . Appropriate choice of  $\eta > 0$  yields  $|\langle \tau, \phi_n \rangle| > 1 - \varepsilon$  as required.  $\square$

We now prove Theorem 5.1. Let  $\tau \in \mathcal{A}^{**}$  with  $\|\tau\| = 1$ , and assume towards a contradiction that  $\theta(\tau) = 0$ . Write  $\gamma_n = \sup\{|\langle \bar{P}_n \cdot \phi \cdot \bar{P}_n, \tau \rangle| : \phi \in B(H)^*, \|\phi\| = 1\}$ . The sequence  $\gamma_n$  is nondecreasing, and by Lemma 5.2 we have  $\gamma_n \rightarrow 1$ . Pick then  $N \in \mathbb{N}$  such that  $\gamma_N > 0$ , and let  $n \leq N$  be the least natural number such that  $\gamma_n = \gamma_N$ . For each  $\varepsilon > 0$  we can find  $\phi \in B(H)^*$ ,  $\|\phi\| = 1$  such that  $\langle \bar{P}_n \cdot \phi \cdot \bar{P}_n, \tau \rangle \geq \gamma_n - \varepsilon$ .

Given such an  $\varepsilon > 0$  and  $\phi$ , we write  $\phi_1$  for a weak-\* accumulation point of the functionals  $\bar{\pi}_k \cdot \phi$ ; but actually, we claim that  $\phi_1$  is the norm-convergent limit of  $\bar{\pi}_k \cdot \phi$ . For the norms  $\|\bar{\pi}_k \cdot \phi\|$  are a nondecreasing sequence tending to a limit  $l$ ; (27), with  $Q = \bar{\pi}_k$  and  $\psi = \bar{\pi}_m \cdot \phi$ , tells us that for  $m > k$  we have  $\|\bar{\pi}_k \cdot \phi\|^2 + \|(\bar{\pi}_m - \bar{\pi}_k) \cdot \phi\|^2 \geq \|\bar{\pi}_m \cdot \phi\|^2$ , so  $\|(\bar{\pi}_m - \bar{\pi}_k) \cdot \phi\| \rightarrow 0$  as  $k, m \rightarrow \infty$ ;

so the sequence  $(\bar{\pi}_k \cdot \phi)_{k \in \mathbb{N}}$  satisfies the Cauchy criterion and is norm-convergent. Each projection  $\bar{\pi}_k$  is of finite rank, so  $\bar{\pi}_k \cdot \phi$  belongs to the trace-class operators  $\mathcal{T}$ . Therefore,  $\phi_1 \in \mathcal{T}$ . But the difference  $\phi - \phi_1 = \lim_k (1 - \bar{\pi}_k) \cdot \phi$  will annihilate any compact operator.

We therefore claim that  $n > 1$ . For by Theorem 3.1, whenever  $T \in \mathcal{A}$  the operator  $P_1 T = P_1 T^{(1)}$  is a compact operator, so  $\langle P_1 T P_1, \phi \rangle = \langle P_1 T P_1, \phi_1 \rangle$ . We may write  $\tau$  as a weak-\* convergent limit  $\tau = \lim_{w^*} T_\alpha$  for  $T_\alpha \in \mathcal{A}$  with  $\|T_\alpha\| = 1$ . Then  $\gamma_N - \varepsilon \leq \langle P_1 \cdot \phi \cdot P_1, \tau \rangle = \lim_\alpha \langle T_\alpha, P_1 \cdot \phi \cdot P_1 \rangle = \lim_\alpha \langle P_1 T_\alpha P_1, \phi \rangle = \lim_\alpha \langle P_1 T_\alpha P_1, \phi_1 \rangle = \lim_\alpha \langle T_\alpha, P_1 \cdot \phi_1 \cdot P_1 \rangle = \langle \tau, P_1 \cdot \phi_1 \cdot P_1 \rangle$ . For small  $\varepsilon$  this implies  $\langle \tau, P_1 \cdot \phi_1 \cdot P_1 \rangle \neq 0$ . But  $P_1 \cdot \phi_1 \cdot P_1 \in \mathcal{T} = B(H)_*$ , so  $\theta(\tau)$  is not the zero operator in  $B(H)$ , a contradiction. Therefore we have  $n > 1$ .

Given  $n > 1$ , we again pick  $\varepsilon > 0$  and find  $\phi \in B(H)^*$ ,  $\|\phi\| = 1$  such that

$$(28) \quad \langle \bar{P}_n \cdot \phi \cdot \bar{P}_n, \tau \rangle \geq \gamma_N - \varepsilon > 0.$$

The norm limit  $\phi_1 = \lim_k \bar{\pi}_k \cdot \phi$  is again in  $\mathcal{T}$ . However, the difference  $\phi - \phi_1$  will not necessarily annihilate  $\bar{P}_n T \bar{P}_n$  for  $T \in \mathcal{A}$ , because though  $\phi - \phi_1$  annihilates  $K(H)$ , the operator  $\bar{P}_n T \bar{P}_n$  need not be compact. Rather, for  $T \in \mathcal{A}$  we have  $\bar{P}_n T \bar{P}_n = \bar{P}_n \bar{T}^{(n)} \bar{P}_n$ , where  $\bar{T}^{(n)} = \bar{\Delta}_n(T)$  as in Definition 2.7; and  $\bar{T}^{(n)} = \bar{T}^{(n-1)} + T^{(n)}$ , where the operator  $\bar{P}_n T^{(n)}$  is compact by Theorem 3.1. So  $\langle \bar{P}_n T^{(n)} \bar{P}_n, \phi - \phi_1 \rangle = 0$  for all  $T \in \mathcal{A}$ . Writing  $\tau = \lim_\alpha T_\alpha$  for a suitable net  $(T_\alpha)$  in  $\mathcal{A}$ , we have  $\langle T_\alpha^{(n)}, \bar{P}_n(\phi - \phi_1) \bar{P}_n \rangle = 0$  for all  $\alpha$ . Because  $\phi_1 \in \mathcal{T}$  and  $\theta(\tau) = 0$  by hypothesis, if we write  $\beta = \lim_\alpha \langle \bar{T}_\alpha^{(n-1)}, \bar{P}_n(\phi - \phi_1) \bar{P}_n \rangle$ , we will have  $0 = \langle \bar{P}_n \phi_1 \bar{P}_n, \tau \rangle = \langle \bar{P}_n \phi \bar{P}_n, \tau \rangle - \lim_\alpha \langle \bar{P}_n(\phi - \phi_1) \bar{P}_n, T_\alpha \rangle = \langle \bar{P}_n \phi \bar{P}_n, \tau \rangle - \beta$ . By (28), we have  $|\beta| \geq \gamma_N - \varepsilon$ .

For each  $T \in \mathcal{A}$  and  $n > 1$ , the norms of  $\bar{T}^{(n-1)}$  and  $\bar{P}_{n-1} \bar{T}^{(n-1)} \bar{P}_{n-1} = \bar{P}_{n-1} T \bar{P}_{n-1}$  are the same by (15). Thus there is a unique map  $\eta : \bar{P}_{n-1} \cdot \mathcal{A} \cdot \bar{P}_{n-1} \rightarrow \bar{\mathcal{A}}^{(n-1)}$  which is a right inverse to the compression  $p : \mathcal{A} \rightarrow \bar{P}_{n-1} \cdot \mathcal{A} \cdot \bar{P}_{n-1}$  with  $p(T) = \bar{P}_{n-1} T \bar{P}_{n-1}$  ( $T \in \mathcal{A}$ ); and  $\|\eta\| = 1$ . We will have  $\eta \cdot p = \bar{\Delta}_{n-1}$ . Let us write  $\psi = (\bar{P}_n(\phi - \phi_1) \bar{P}_n) \circ \eta$ . Then  $\psi \in (\bar{P}_{n-1} \cdot \mathcal{A} \cdot \bar{P}_{n-1})^*$  with  $\|\psi\| \leq 1$ .

By the Hahn–Banach theorem, we can extend  $\psi$  to  $\bar{P}_{n-1} \cdot B(H) \cdot \bar{P}_{n-1}$  with the same norm; and then extend to all of  $B(H)$  so that  $\psi = \psi \circ p$  (where we abuse notation slightly by writing  $p$  for the compression  $B(H) \rightarrow \bar{P}_{n-1} \cdot B(H) \cdot \bar{P}_{n-1}$  also).

Then for  $T \in \mathcal{A}$  we have  $\psi(T) = \psi \circ \eta p(T) = \psi(\bar{\Delta}_{n-1}(T))$ ; so

$$|\langle \psi, \tau \rangle| = \lim_\alpha |\langle \psi, \bar{\Delta}_{n-1} T_\alpha \rangle| = \lim_\alpha |\langle \bar{P}_n(\phi - \phi_1) \bar{P}_n, \bar{T}_\alpha^{(n-1)} \rangle| = |\beta| \geq \gamma_N - \varepsilon.$$

Since  $\psi = \psi \circ p = \bar{P}_{n-1} \cdot \psi \cdot \bar{P}_{n-1}$ , we find that

$$\gamma_{n-1} = \sup\{|\langle \bar{P}_{n-1} \cdot \phi \cdot \bar{P}_{n-1}, \tau \rangle| : \phi \in B(H)^*, \|\phi\| = 1\}$$

is at least  $\gamma_N - \varepsilon$ . But  $\varepsilon > 0$  is arbitrary, so  $\gamma_{n-1} = \gamma_N$ , and  $n$  was not the minimal integer with  $\gamma_n = \gamma_N$ , contrary to hypothesis. This contradiction proves the theorem.  $\square$

## 6. References to Gulick's paper

Having established that the result  $\text{rad } A^{**} \cap A = \text{rad } A$  of Gulick is wrong, let us look at papers which have referenced [Gulick 1966] and try to establish that no further damage has been done.

The lengthy paper of Dales and Lau [2005] refers to [Gulick 1966], but does not use the false Theorem 4.6; private communication with my colleague Garth Dales reveals a history of previous suspicion about that result, but no actual counterexamples as presented here. The paper of Daws, Haydon, Schlumprecht and White [Daws et al. 2012] refers to (the proof of) Theorem 3.3 of [Gulick 1966], which we believe to be completely correct. Likewise the paper of Bouziad and Filali [2011] quotes the proof, given by Gulick [1966, Lemma 5.2], that the radical of  $L^\infty(G)^*$  is nonseparable for any nondiscrete locally compact group  $G$ . This proof also is perfectly valid. The earlier paper of Granirer [1973] makes reference to that same, correct, lemma. Tomiuk [1981] likewise refers to Gulick's untainted Theorem 5.5. A. Ülger [1987] solves one of the problems posed by Gulick [1966]. Finally, Tomiuk and Wong [1970] make a passing reference to [Gulick 1966] in their paper on Arens products.

We have not found a case in which another author has used the false Theorem 4.6 from Gulick's paper, or anything tainted by it. This chimes with our reckoning that more than one author apart from ourselves has suspected that that theorem is false. So, the general literature on Banach algebras is not seriously harmed; but it was nonetheless high time that these counterexamples were made known so that such errors will not occur in the future.

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
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