Pacific Journal of Mathematics

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JIEZHU LIN AND XUANMING YE

Volume 273 No. 2 February 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 273, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.273.461

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Let X be a compact complex manifold, and let $\pi : \mathscr{X} \to B$ be a small deformation of X, the dimensions of the Bott–Chern cohomology groups $H^{p,q}_{\mathrm{BC}}(X(t))$ and Aeppli cohomology groups $H^{p,q}_{\mathrm{A}}(X(t))$ may vary under this deformation. In this paper, we will study the deformation obstructions of a (p,q) class in the central fiber X. In particular, we obtain an explicit formula for the obstructions and apply this formula to the study of small deformations of the Iwasawa manifold.

1. Introduction

Let X be a compact complex manifold and $\pi : \mathscr{X} \to B$ be a family of complex manifolds such that $\pi^{-1}(0) = X$, where $\mathscr{X}$ is a complex manifold and B is a neighborhood of the origin. Let $X_t = \pi^{-1}(t)$ denote the fiber of π over the point $t \in B$. In [Ye 2008], the author studied the jumping phenomenon of Hodge numbers $h^{p,q}$ of X by studying the deformation obstructions of a (p,q) class in the central fiber X. In particular, the author obtained an explicit formula for the obstructions and applied it to the study of small deformations of the Iwasawa manifold. Besides the Hodge numbers, the dimensions of Bott–Chern cohomology groups and the dimensions of Aeppli cohomology groups are also important invariants of complex structures. D. Angella [2013] studied the small deformations of the Iwasawa manifold and found that the dimensions of Bott–Chern and Aeppli cohomology groups are not deformation invariants.

In this paper, we will study the Bott–Chern and Aeppli cohomologies by studying the hypercohomology of the complex $\mathscr{B}^{\bullet}_{p,q}$ constructed in [Schweitzer 2007]. M. Schweitzer [2007] proved that

$$H^{p,q}_{\mathrm{BC}}(X) \cong \mathbb{H}^{p+q}(X, \mathscr{B}^{\bullet}_{p,q}) \quad \text{and} \quad H^{p,q}_{\mathrm{A}}(X) \cong \mathbb{H}^{p+q+1}(X, \mathscr{B}^{\bullet}_{p+1,q+1}).$$

This work was supported by National Natural Science Foundation of China (Grant No. 11201491 and 11201090), Doctoral Fund of Ministry of Education of China (Grant No. 20124410120001 and 20120171120009) and the Foundation of Research Funds for Young Teachers Training Project (Grant No. 34000-31610248).

MSC2010: 14B12, 14C30, 14F05, 14F40, 18G40.

Keywords: Bott–Chern cohomology, Aeppli cohomology, deformation, obstruction, Kodaira–Spencer classes.

As in [Ye 2008], we will study the jumping phenomenons from the viewpoint of obstruction theory. More precisely, for a certain small deformation $\mathscr{X}$ of X parameterized by a base B and a certain class $[\theta]$ of the hypercohomology group $\mathbb{H}^l(X, \mathscr{B}^\bullet_{p,q})$, we will try to find out the obstruction to extend it to an element of the relative hypercohomology group $\mathbb{H}^l(\mathscr{X}, \mathscr{B}^\bullet_{p,q;\mathscr{X}/B})$. We will call those elements which have nontrivial obstruction the obstructed elements. And then we will see that these elements will play an important role when we study the jumping phenomenon, because we will see that the existence of obstructed elements is a sufficient condition for the variation of the dimensions of Bott–Chern and Aeppli cohomologies.

In Section 2 we will summarize the results of M. Schweitzer about Bott–Chern and Aeppli cohomologies, from which we can define the relative Bott–Chern and Aeppli cohomologies on X_n, where X_n is the n-th order deformation of $\pi : \mathscr{X} \to B$. We will also introduce some important maps which will be used in the calculation of the obstructions in Section 4. In Section 3 we will try to explain why we need to consider the obstructed elements. The relation between the jumping phenomenon of the dimensions of Bott–Chern and Aeppli cohomologies and the obstructed elements is the following.

Theorem 3.1. *Let $\pi : \mathscr{X} \to B$ be a small deformation of the central fiber compact complex manifold X. Now we consider $\dim \mathbb{H}^l(X(t), \mathscr{B}^\bullet_{p,q;t})$ as a function of $t \in B$. It jumps at $t = 0$ if there exists an element $[\theta]$ either in $\mathbb{H}^l(X, \mathscr{B}^\bullet_{p,q})$ or in $\mathbb{H}^{l-1}(X, \mathscr{B}^\bullet_{p,q})$ and a minimal natural number $n \geq 1$ such that the n-th order obstruction is nonzero*:

$$\mathrm{o}_n([\theta]) \neq 0.$$

In Section 4 we will get a formula for the obstruction to the extension we mentioned above.

Theorem 4.4. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X. For arbitrary $[\theta]$ belongs to $\mathbb{H}^l(X, \mathscr{B}^\bullet_{p,q})$, suppose we can extend $[\theta]$ to order $n-1$ in $\mathbb{H}^l(X_{n-1}, \mathscr{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. Denote such element by $[\theta_{n-1}]$. The obstruction of the extension of $[\theta]$ to n-th order is given by*

$$\mathrm{o}_n([\theta]) = -\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ \partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]) - \bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ \bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]),$$

where κ_n is the n-th order Kodaira–Spencer class and $\bar\kappa_n$ is the n-th order Kodaira–Spencer class of the deformation $\bar\pi : \bar{\mathscr{X}} \to \bar B$. The maps $\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}$ and $\bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}$ are defined in Section 2.

In Section 5 we will use this formula to study carefully the example given by Iku Nakamura and D. Angella, that is, the small deformation of the Iwasawa manifold and discuss some phenomena.

2. The relative Bott–Chern and Aeppli cohomologies of X_n and the representation of their cohomology classes

2A. ***The Bott–Chern and Aeppli cohomologies and hypercohomologies.*** All the details of this subsection can be found in [Schweitzer 2007]. Let X be a compact complex manifold. The Dolbeault cohomology groups $H_{\bar\partial}^{p,q}(X)$, and more generally the terms $E_r^{p,q}(X)$ in the Frölicher spectral sequence [Frölicher 1955], are well-known finite dimensional invariants of the complex manifold X. On the other hand, the Bott–Chern and Aeppli cohomologies define additional complex invariants of X given, respectively, by [Bott and Chern 1965; Aeppli 1965]

$$H_{\mathrm{BC}}^{p,q}(X)=\frac{\ker\{d\colon \mathscr{A}^{p,q}(X)\to\mathscr{A}^{p+q+1}(X)\}}{\operatorname{im}\{\partial\bar\partial\colon \mathscr{A}^{p-1,q-1}(X)\to\mathscr{A}^{p,q}(X)\}},$$

and

$$H_{\mathrm{A}}^{p,q}(X)=\frac{\ker\{\partial\bar\partial\colon \mathscr{A}^{p,q}(X)\to\mathscr{A}^{p+1,q+1}(X)\}}{\operatorname{im}\{\partial\colon \mathscr{A}^{p-1,q}(X)\to\mathscr{A}^{p,q}(X)\}+\operatorname{im}\{\bar\partial\colon \mathscr{A}^{p,q-1}(X)\to\mathscr{A}^{p,q}(X)\}}.$$

By the Hodge theory developed in [Schweitzer 2007], all these complex invariants are also finite dimensional, and $H_{\mathrm{A}}^{p,q}(X)\cong H_{\mathrm{BC}}^{n-q,n-p}(X)$. Notice that $H_{\mathrm{BC}}^{q,p}(X)$ is isomorphic to $H_{\mathrm{BC}}^{p,q}(X)$ by complex conjugation. For any $r\geq 1$ and for any p,q, there are natural maps

$$H_{\mathrm{BC}}^{p,q}(X)\to E_r^{p,q}(X)\quad\text{and}\quad E_r^{p,q}(X)\to H_{\mathrm{A}}^{p,q}(X).$$

Recall that $E_1^{p,q}(X)$ is isomorphic to $H_{\bar\partial}^{p,q}(X)$ and that the terms for $r=\infty$ provide a decomposition of the de Rham cohomology of X: $H_{\mathrm{dR}}^k(X,\mathbb{C})\cong\oplus_{p+q=k}E_\infty^{p,q}(X)$. From now on we shall denote by $h_{\mathrm{BC}}^{p,q}(X)$ the dimension of the cohomology group $H_{\mathrm{BC}}^{p,q}(X)$. The Hodge numbers will be denoted simply by $h^{p,q}(X)$ and the Betti numbers by $b_k(X)$. For any given $p\geq 1$, $q\geq 1$, we define the complex of sheaves $\mathscr{L}_{p,q}^{\bullet}$ by

$$\mathscr{L}_{p,q}^{k}=\bigoplus_{\substack{r+s=k\\ r<p,s<q}}\mathscr{A}^{r,s}\quad\text{if } k\leq p+q-2,\qquad \mathscr{L}_{p-1,q-1}^{k-1}=\bigoplus_{\substack{r+s=k\\ r\geq p,s\geq q}}\mathscr{A}^{r,s}\quad\text{if } k\geq p+q,$$

and the differential

$$\mathscr{L}_{p,q}^{0}\xrightarrow{\mathrm{pr}_{\mathscr{L}_{p,q}^{1}}\circ d}\mathscr{L}_{p,q}^{1}\xrightarrow{\mathrm{pr}_{\mathscr{L}_{p,q}^{2}}\circ d}\cdots\longrightarrow\mathscr{L}_{p,q}^{k-2}\xrightarrow{\partial\bar\partial}\mathscr{L}_{p,q}^{k-1}\xrightarrow{d}\mathscr{L}_{p,q}^{k}\xrightarrow{d}\cdots,$$

where $\mathscr{A}^{r,s}$ are the sheaves of smooth (r,s)-forms and pr is the projection operator.

Then by the above construction, we have the following isomorphisms:

$$H_{\mathrm{BC}}^{p,q}(X)=H^{p+q-1}(\mathscr{L}_{p,q}^{\bullet}(X))\cong\mathbb{H}^{p+q-1}(X,\mathscr{L}_{p,q}^{\bullet}),$$

$$H_{\mathrm{A}}^{p,q}(X)=H^{p+q}(\mathscr{L}_{p+1,q+1}^{\bullet}(X))\cong\mathbb{H}^{p+q}(X,\mathscr{L}_{p+1,q+1}^{\bullet}),$$

because $\mathscr{L}^k_{p,q}$ are soft.

We define a sub complex $\mathscr{S}^\bullet_{p,q}$ of $\mathscr{L}^\bullet_{p,q}$ by

$$(\mathscr{S}'^\bullet_p, \partial) : \mathbb{O} \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0, \quad (\mathscr{S}''^\bullet_q, \bar\partial) : \bar{\mathbb{O}} \to \bar\Omega^1 \to \cdots \to \bar\Omega^{q-1} \to 0,$$

$$\mathscr{S}^\bullet_{p,q} = \mathscr{S}'^\bullet_p + \mathscr{S}''^\bullet_q : \mathbb{O} + \bar{\mathbb{O}} \to \Omega^1 \oplus \bar\Omega^1 \to \cdots \to \Omega^{p-1} \oplus \bar\Omega^{p-1} \to \bar\Omega^p \to \cdots \to \bar\Omega^{q-1} \to 0.$$

Note that the inclusion $\mathscr{S}^\bullet \subset \mathscr{L}^\bullet$ is a quasiisomorphism [Schweitzer 2007]. There is another complex $\mathscr{B}^\bullet_{p,q}$ used in [Schweitzer 2007], defined by

$$\mathscr{B}^\bullet_{p,q} : \mathbb{C} \xrightarrow{(+,-)} \mathbb{O} \oplus \bar{\mathbb{O}} \to \Omega^1 \oplus \bar\Omega^1 \to \cdots \to \Omega^{p-1} \oplus \bar\Omega^{p-1} \to \bar\Omega^p \to \cdots \to \bar\Omega^{q-1} \to 0.$$

and the following morphism from $\mathscr{B}^\bullet_{p,q}$ to $\mathscr{S}^\bullet_{p,q}[1]$ is a quasiisomorphism [Schweitzer 2007]:

$$\begin{array}{ccccccc} \mathbb{C} & \xrightarrow{(+,-)} & \mathbb{O} \oplus \bar{\mathbb{O}} & \to & \Omega^1 \oplus \bar\Omega^1 & \to & \cdots \\ \downarrow & & \downarrow + & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{O} + \bar{\mathbb{O}} & \to & \Omega^1 \oplus \bar\Omega^1 & \to & \cdots . \end{array}$$

Therefore we have

$$H^{p,q}_{\mathrm{BC}}(X) \cong \mathbb{H}^{p+q}(X, \mathscr{L}^\bullet_{p,q}[1]) \cong \mathbb{H}^{p+q}(X, \mathscr{S}^\bullet_{p,q}[1]) \cong \mathbb{H}^{p+q}(X, \mathscr{B}^\bullet_{p,q}),$$

and

$$H^{p,q}_{\mathrm{A}}(X) \cong \mathbb{H}^{p+q}(X, \mathscr{L}^\bullet_{p+1,q+1}) \cong \mathbb{H}^{p+q}(X, \mathscr{S}^\bullet_{p+1,q+1}) \cong \mathbb{H}^{p+q+1}(X, \mathscr{B}^\bullet_{p+1,q+1}).$$

2B. ***The relative Bott–Chern and Aeppli cohomologies of X_n.*** Here we make some definitions in order to construct the relative Bott–Chern and Aeppli cohomologies of X_n. Suppose X is a compact complex manifold.

- Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$.
- For every integer $n \geq 0$, set $B_n = \operatorname{Spec} \mathbb{O}_{B,0}/m_0^{n+1}$ — the n-th order infinitesimal neighborhood of the closed point 0 of the base B.
- Let $X_n \subset \mathscr{X}$ be the complex space over B_n.
- Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X, and denote $\pi^*(m_0)$ by $\mathscr{M}_0$.
- Complex conjugation gives another complex structure of the differential manifold of $\mathscr{X}$; we denote this manifold by $\bar{\mathscr{X}}$, and π induces a deformation $\bar\pi : \bar{\mathscr{X}} \to \bar B$ of $\bar X$. Then we have $\bar X_n$ and $\bar\pi_n : \bar X_n \to \bar B_n$.
- Let $\mathscr{C}^\omega_B$ be the sheaf of $\mathbb{C}$-valued real analytic functions on B.
- Set $\mathbb{O}^\omega_{\mathscr{X}} = \pi^*(\mathscr{C}^\omega_B)$, $\bar{\mathbb{O}}^\omega_{\mathscr{X}} = \bar\pi^*(\mathscr{C}^\omega_B)$; let m^ω_0 be the maximal ideal of $\mathscr{C}^\omega_{B,0}$ and let $\mathscr{M}^\omega_0 = \pi^*(m^\omega_0)$, $\bar{\mathscr{M}}^\omega_0 = \bar\pi^*(m^\omega_0)$.
- For any sheaf of $\mathbb{O}_{\mathscr{X}}$- (resp. $\bar{\mathbb{O}}_{\mathscr{X}}$-) modules $\mathscr{F}$, set $\mathscr{F}^\omega = \mathscr{F} \otimes_{\mathbb{O}_{\mathscr{X}}} \mathbb{O}^\omega_{\mathscr{X}}$ (resp. $\bar{\mathscr{F}}^\omega = \mathscr{F} \otimes_{\bar{\mathbb{O}}_{\mathscr{X}}} \bar{\mathbb{O}}^\omega_{\mathscr{X}}$).
- Let $\mathbb{O}^\omega_{X_n} = \mathbb{O}^\omega_{\mathscr{X},0}/(\mathscr{M}^\omega_0)^{n+1}$ and $\mathbb{O}^\omega_{\bar X_n} = \bar{\mathbb{O}}^\omega_{\mathscr{X},0}/(\bar{\mathscr{M}}^\omega_0)^{n+1}$.

- For any sheaf of $\mathcal{O}_{X_n}$ (resp. $\mathcal{O}_{\overline{X}_n}$) modules $\mathcal{F}$, set $\mathcal{F}^\omega = \mathcal{F} \otimes_{\mathcal{O}_{X_n}} \mathcal{O}^\omega_{X_n}$ (resp. $\overline{\mathcal{F}}^\omega = \mathcal{F} \otimes_{\mathcal{O}_{\overline{X}_n}} \mathcal{O}^\omega_{\overline{X}_n}$).
- For any given $p \geq 1, q \geq 1$, we define the complex $\mathcal{S}^\bullet_{X_n/B_n} = \mathcal{S}^\bullet_{p,q;X_n/B_n}$ by

$$(\mathcal{S}'^\bullet_{p;X_n/B_n}, \partial_{X_n/B_n}) : \mathcal{O}^\omega_{X_n} \to \Omega^{1;\omega}_{X_n/B_n} \to \cdots \to \Omega^{p-1;\omega}_{X_n/B_n} \to 0,$$

$$(\mathcal{S}''^\bullet_{q;X_n/B_n}, \bar\partial_{X_n/B_n}) : \mathcal{O}^\omega_{\overline{X}_n} \to \overline{\Omega}^{1;\omega}_{X_n/B_n} \to \cdots \to \overline{\Omega}^{q-1;\omega}_{X_n/B_n} \to 0,$$

$$\begin{aligned}\mathcal{S}^\bullet_{p,q;X_n/B_n} = \mathcal{S}'^\bullet_{p;X_n/B_n} + \mathcal{S}''^\bullet_{q;X_n/B_n} : \mathcal{O}^\omega_{X_n} + \mathcal{O}^\omega_{\overline{X}_n} &\to \Omega^{1;^\omega}_{X_n/B_n} \oplus \overline{\Omega}^{1;\omega}_{X_n/B_n} \to \cdots \\ &\to \Omega^{p-1;\omega}_{X_n/B_n} \oplus \overline{\Omega}^{p-1;\omega}_{X_n/B_n} \to \overline{\Omega}^{p;\omega}_{X_n/B_n} \to \cdots \to \overline{\Omega}^{q-1;\omega}_{X_n/B_n} \to 0.\end{aligned}$$

- Finally, define $\mathcal{B}^\bullet_{p,q;X_n/B_n}$ by

$$\begin{aligned}\mathcal{B}^\bullet_{p,q;X_n/B_n} : \mathbb{C}^\omega_{B_n} \xrightarrow{(+,-)} \mathcal{O}^\omega_{X_n} \oplus \mathcal{O}^\omega_{\overline{X}_n} &\to \Omega^{1;\omega}_{X_n/B_n} \oplus \overline{\Omega}^{1;\omega}_{X_n/B_n} \to \cdots \\ &\to \Omega^{p-1;\omega}_{X_n/B_n} \oplus \overline{\Omega}^{p-1;\omega}_{X_n/B_n} \to \overline{\Omega}^{p;\omega}_{X_n/B_n} \to \cdots \to \overline{\Omega}^{q-1;\omega}_{X_n/B_n} \to 0,\end{aligned}$$

  where $\mathbb{C}^\omega_{B_n} = \pi^{-1}(\mathcal{C}^\omega_{B,0}/(m^\omega_0)^{n+1})$.

Now we are ready to define the relative Bott–Chern and Aeppli cohomologies of X_n:

$$H^{p,q}_{\mathrm{BC}}(X_n/B_n) \cong \mathbb{H}^{p+q}(X, \mathcal{S}^\bullet_{p,q;X_n/B_n}[1]) \cong \mathbb{H}^{p+q}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n}),$$

and

$$H^{p,q}_{\mathrm{A}}(X_n/B_n) \cong \mathbb{H}^{p+q}(X, \mathcal{S}^\bullet_{p+1,q+1;X_n/B_n}) \cong \mathbb{H}^{p+q+1}(X_n, \mathcal{B}^\bullet_{p+1,q+1;X_n/B_n}).$$

2C. ***Representation of the relative Bott–Chern and Aeppli cohomology classes.*** In this subsection we will follow [Schweitzer 2007] to construct a hypercocycle in $\check{Z}^{p+q}(X, \mathcal{B}^\bullet_{p,q})$ to represent the relative Bott–Chern cohomology classes. Let $[\theta]$ be an element of $H^{p,q}_{\mathrm{BC}}(X)$, represented by a closed (p,q)-form θ. It is defined in $\mathbb{H}^{p+q}(X, \mathcal{L}_{p,q}[1]^\bullet)$ by a hypercocycle, still denoted by θ and defined by $\theta^{p,q} = \theta|_{U_j}$ and $\theta^{r,s} = 0$ otherwise. For given $p \geq 1$ and $q \geq 1$, there exists a hypercocycle $w = (c; u^{r,0}; v^{0,s}) \in \check{Z}^{p+q}(X, \mathcal{B}^\bullet_{p,q})$ and an hypercochain $\alpha = (\alpha^{r,s}) \in \check{C}^{p+q-1}(X, \mathcal{L}_{p,q}[1]^\bullet)$ such that $\theta = \check{\boldsymbol{\delta}}\alpha + w$. We represent the data in the following table:

$$\theta \longleftrightarrow \left[\begin{array}{c|ccc} \theta_v^{0,q-1} & & & \\ \vdots & & \alpha^{r,s} & \\ \theta_v^{0,0} & & & \\ \hline \theta_c & \theta_u^{0,0} & \cdots & \theta_u^{p-1,0} \end{array}\right].$$

The equality $\theta = \check{\delta}\alpha + w$ corresponds to the relations

$$\begin{aligned} \theta^{p,q} &= \partial\bar{\partial}\alpha^{p-1,q-1} \\ (-1)^{r+s}\check{\delta}\alpha^{r,s} &= \bar{\partial}\alpha^{r,s-1} + \partial\alpha^{r-1,s} \\ (-1)^{s}\check{\delta}\alpha^{0,s} &= \bar{\partial}\alpha^{0,s-1} + \theta_v^{0,s} \\ (-1)^{r}\check{\delta}\alpha^{r,0} &= \theta_u^{r,0} + \partial\alpha^{r-1,0} \\ \check{\delta}\alpha^{0,0} &= \theta_u^{0,0} + \theta_v^{0,0} \\ \check{\delta}\theta_u^{0,0} &= \theta_c, \end{aligned}$$

where $1 \le r \le p-1$ and $1 \le s \le q-1$. Note that these relations involve relations of the hypercocycles for θ_u and θ_v:

$$(-1)^r\check{\delta}\theta_u^{r,0} = \partial\theta_u^{r-1,0}, \quad (-1)^s\check{\delta}\theta_v^{0,s} = \bar{\partial}\theta_v^{0,s-1},$$

with the same conditions on r and s. If $q = 0$, we simply have

$$\theta \longleftrightarrow \left(\theta_c, \theta_u^{0,0}, \dots, \theta_u^{p-1,0}\right)$$

with the relations

$$\theta^{p,0} = \partial\theta_u^{p-1,0}, \quad (-1)^r\check{\delta}\theta_u^{r,0} = \partial\theta_u^{r-1,0}, \quad \check{\delta}\theta_u^{0,0} = \theta_c,$$

for $1 \le r \le p-1$. Similarly, if $p = 0$, we have

$$\theta \longleftrightarrow \left(\theta_c, \theta_v^{0,0}, \dots, \theta_v^{0,q-1}\right)$$

with the relations (where $1 \le s \le q-1$)

$$\theta^{0,q} = -\bar{\partial}\theta_v^{0,q-1}, \quad (-1)^s\check{\delta}\theta_v^{0,s} = \bar{\partial}\theta_v^{0,s-1}, \quad -\check{\delta}\theta_v^{0,0} = \theta_c.$$

Similarly, let $[\theta]$ be an element of $H^{p,q}_{\mathrm{BC}}(X_n/B_n)$, then it can be represented by a Čech hypercocycle θ_u, θ_v and θ_c of $\check{Z}^{p+q}(X, \mathcal{B}^{\bullet}_{p,q;X_n/B_n})$ with the relations

$$\begin{aligned} (-1)^r\check{\delta}\theta_u^{r,0} &= \partial\theta_u^{r-1,0} & (-1)^s\check{\delta}\theta_v^{0,s} &= \bar{\partial}\theta_v^{0,s-1} \\ \check{\delta}\theta_u^{0,0} &= \theta_c, & -\check{\delta}\theta_v^{0,0} &= \theta_c, \end{aligned}$$

where $1 \le r \le p-1$ and $1 \le s \le q-1$; while for an element $[\theta]$ of $H^{p,q}_{\mathrm{A}}(X_n/B_n)$, it can be represented by a Čech hypercocycle θ_u and θ_v of $\check{Z}^{p+q+1}(X, \mathcal{B}^{\bullet}_{p+1,q+1;X_n/B_n})$ with the relations

$$\begin{aligned} (-1)^r\check{\delta}\theta_u^{r,0} &= \partial\theta_u^{r-1,0} & (-1)^s\check{\delta}\theta_v^{0,s} &= \bar{\partial}\theta_v^{0,s-1} \\ \check{\delta}\theta_u^{0,0} &= \theta_c, & -\check{\delta}\theta_v^{0,0} &= \theta_c, \end{aligned}$$

where $1 \le r \le p$ and $1 \le s \le q$.

Before the end of this section, we will introduce some important maps which will be used in the computation in Section 4.

Define

$$\partial^{\bar\partial,\mathcal{B}}_{X_n/B_n} : H^\bullet(X_n, \Omega^{p-1;\omega}_{X_n/B_n}) \to \mathbb{H}^{\bullet+p}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$$

as follows. Let $[\theta]$ be an element of $H^\bullet(X_n, \Omega^{p-1;\omega}_{X_n/B_n})$ then θ can be represented by a cocycle of $\check{Z}^\bullet(X, \Omega^{p-1;\omega}_{X_n/B_n})$, we define $\partial^{\bar\partial,\mathcal{B}}_{X_n/B_n}([\theta])$ to be the cohomology class associated to the hypercocycle in $\check{Z}^{p+\bullet}(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ given by

$$\theta_u^{p-1,0}=\theta, \quad \theta_u^{r,0}=0 \text{ for } 0\le r\le p-2 \quad \theta_v^{0,s}=0 \text{ for } 0\le s\le q-1, \quad \text{and} \quad \theta_c=0.$$

When $\bullet < 0$, $\partial^{\bar\partial,\mathcal{B}}_{X_n/B_n}$ is defined to be 0.

Lemma 2.1. *The map $\partial^{\bar\partial,\mathcal{B}}_{X_n/B_n}$ is well defined.*

Proof. It is easy to check that the hypercochain given by θ_u, θ_v and θ_c is a hypercocycle. On the other hand, suppose there exists a cochain α' in $\check{C}^{\bullet-1}(X, \Omega^{p-1;\omega}_{X_n/B_n})$ such that $\check\delta\alpha' = \theta$. Then if we take a hypercochain α in $\check{C}^{p+\bullet-1}(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ given by $\alpha_u^{p-1,0} = (-1)^{p-1}\alpha'$, $\alpha_u^{r,0} = 0$ for $0 \le r \le p-2$, $\alpha_v^{0,s} = 0$ for $0 \le s \le q-1$, and $\alpha_c = 0$, we have $\check{\boldsymbol{\delta}}\alpha = \partial^{\bar\partial,\mathcal{B}}_{X_n/B_n}([\theta])$. Therefore $\partial^{\bar\partial,\mathcal{B}}_{X_n/B_n}([\theta]) = 0$. □

Similarly, we can define

$$\bar\partial^{\partial,\mathcal{B}}_{X_n/B_n} : H^\bullet(\overline{X}_n, \overline{\Omega}^{q-1;\omega}_{X_n/B_n}) \to \mathbb{H}^{\bullet+q}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$$

as follows. Let $[\theta]$ be an element of $H^\bullet(\overline{X}_n, \overline{\Omega}^{q-1;\omega}_{X_n/B_n})$. Then θ can be represented by a cocycle of $\check{Z}^\bullet(\overline{X}, \overline{\Omega}^{q-1;\omega}_{X_n/B_n})$; we define $\bar\partial^{\partial,\mathcal{B}}_{X_n/B_n}([\theta])$ to be the cohomology class associated to the hypercocycle in $\check{Z}^{q+\bullet}(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ given by $\theta_v^{0,q-1} = \theta$, $\theta_v^{0,r} = 0$ for all $0 \le r \le q-2$, $\theta_u^{r,0} = 0$ for all $0 \le r \le p-1$, and $\theta_c = 0$ (when $\bullet < 0$, this map is defined to be 0). This map is also well defined and the proof is just as Lemma 2.1.

Define

$$\partial^{\mathcal{B},\bar\partial}_{X_n/B_n} : \mathbb{H}^{\bullet+p}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n}) \to H^\bullet(X_n, \Omega^{p;\omega}_{X_n/B_n})$$

as follows. Let $[\theta]$ be an element of $\mathbb{H}^{\bullet+p}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$. Then θ can be represented by a hypercocycle of $\check{Z}^{p+\bullet}(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$, and we define $\partial^{\mathcal{B},\bar\partial}_{X_n/B_n}([\theta])$ to be the cohomology class associated to the cocycle in $\check{Z}^\bullet(X, \Omega^{p;\omega}_{X_n/B_n})$ given by $\partial_{X_n/B_n}\theta_u^{p-1,0}$(when $\bullet < 0$, this map is defined to be 0).

Lemma 2.2. *The map $\partial^{\mathcal{B},\bar\partial}_{X_n/B_n}$ is well defined.*

Proof. First we check that the cochain given by $\partial_{X_n/B_n}\theta_u^{p-1,0}$ is a cocycle. In fact, since θ is a hypercocycle in $\check{Z}^{p+\bullet}(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$, we have

$$(-1)^{p-1}\check\delta\theta_u^{p-1,0} = \partial_{X_n/B_n}\theta_u^{p-2,0},$$

therefore

$$\check\delta\partial_{X_n/B_n}\theta_u^{p-1,0} = (-1)^p\partial_{X_n/B_n} \circ \partial_{X_n/B_n}\theta_u^{p-2,0} = 0.$$

On the other hand, suppose there exists a hypercochain $\alpha \in \check{C}^{p+\bullet-1}(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ such that $\check{\delta}\alpha = \theta$. Then if we take a cochain $\alpha' \in \check{C}^{\bullet-1}(X, \Omega^{p;\omega}_{X_n/B_n})$ given by $\alpha' = (-1)^p \partial_{X_n/B_n} \alpha_u^{p-1,0}$, we have

$$\check{\delta}\alpha' = (-1)^p \check{\delta}\partial_{X_n/B_n}\alpha_u^{p-1,0} = (-1)^{p+1}\partial_{X_n/B_n}\check{\delta}\alpha_u^{p-1,0}$$
$$= (-1)^{p+1+p-1}\partial_{X_n/B_n}\theta_u^{p-1,0} = \partial^{\mathcal{B},\bar{\partial},}_{X_n/B_n}([\theta]).$$

Therefore $\partial^{\bar{\partial},\mathcal{B}}_{X_n/B_n}([\theta]) = 0$. □

Similarly, we can define

$$\bar{\partial}^{\mathcal{B},\partial}_{X_n/B_n} : \mathbb{H}^{\bullet+q}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n}) \to H^\bullet(\overline{X}_n, \Omega^{q;\omega}_{X_n/B_n}).$$

Let $[\theta]$ be an element of $\mathbb{H}^{\bullet+q}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ then θ can be represented by a hypercocycle of $\check{Z}^{q+\bullet}(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$, we define $\bar{\partial}^{\mathcal{B},\partial}_{X_n/B_n}([\theta])$ to be the cohomology class associated to the cocycle in $\check{Z}^\bullet(X, \overline{\Omega}^{q;\omega}_{X_n/B_n})$ given by $\bar{\partial}_{X_n/B_n}\theta_u^{0,q-1}$(when $\bullet < 0$, this map is defined to be 0). This map is also well defined and the proof is just as Lemma 2.2.

Remark 2.3. The natural maps from $H^{p,q}_{\mathrm{BC}}(X_n/B_n)$ to $H^q(X_n, \Omega^{p;\omega}_{X_n/B_n})$ and from $H^q(X_n, \Omega^{p;\omega}_{X_n/B_n})$ to $H^{p,q}_{\mathrm{A}}(X_n/B_n)$ mentioned in Section 2A respectively are exactly the map

$$\partial^{\mathcal{B},\bar{\partial}}_{X_n/B_n} : \mathbb{H}^{q+p}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})(\cong H^{p,q}_{\mathrm{BC}}(X_n/B_n)) \to H^q(X_n, \Omega^{p;\omega}_{X_n/B_n}),$$
$$\partial^{\bar{\partial},\mathcal{B}}_{X_n/B_n} : H^q(X_n, \Omega^{p;\omega}_{X_n/B_n}) \to \mathbb{H}^{q+p+1}(X_n, \mathcal{B}^\bullet_{p+1,q+1;X_n/B_n})(\cong H^{p,q}_{\mathrm{A}}(X_n/B_n)),$$

and we denote these maps by $r_{BC,\bar{\partial}}$ and $r_{\bar{\partial},A}$.

We also denote the maps

$$\partial^{\bar{\partial},\mathcal{B}}_{X_n/B_n} : H^q(X_n, \Omega^{p-1;\omega}_{X_n/B_n}) \to \mathbb{H}^{q+p}(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})(\cong H^{p,q}_{\mathrm{BC}}(X_n/B_n)),$$
$$\partial^{\mathcal{B},\bar{\partial}}_{X_n/B_n} : \mathbb{H}^{q+p+1}(X_n, \mathcal{B}^\bullet_{p+1,q+1;X_n/B_n})(\cong H^{p,q}_{\mathrm{A}}(X_n/B_n)) \to H^q(X_n, \Omega^{p+1;\omega}_{X_n/B_n})$$

by $\partial^{\bar{\partial},BC}_{X_n/B_n}$ and $\partial^{A,\bar{\partial}}_{X_n/B_n}$.

The following lemma is an important observation which will be used for the computation in Section 4.

Lemma 2.4. *Let $[\theta]$ be an element of $\mathbb{H}^l(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ which is represented by an element $\theta \in \check{Z}^l(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ given by θ_u, θ_v and θ_c. Then $\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})$ is a hypercoboundary.*

Proof. The hypercochain $\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})$ is given by $(\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0}))_u^{r,0} = \partial_{X_n/B_n}\theta_u^{r-1,0}$ for $0 < r \le p-1$, $(\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0}))_u^{0,0} = 0$, $(\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0}))_v^{0,s} = 0$ for $0 \le s \le q-1$, and $(\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0}))_c = 0$. Let α be the hypercochain in

$\check{C}^l(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ given by $\alpha_u^{r,0} = -r\theta_u^{r,0}$ for $0 \le r \le p-1$, $\alpha_v^{0,s} = 0$ for $0 \le s \le q-1$, and $\alpha_c = 0$. It is easy to see that

$$\begin{aligned}(\check{\boldsymbol{\delta}}\alpha)_u^{r,0} &= (-1)^r \check{\delta}\alpha_u^{r,0} + \partial_{X_n/B_n}\alpha_u^{r-1,0}\\ &= (-1)^r \check{\delta}(-r)\theta_u^{r,0} - \partial_{X_n/B_n}(r-1)\theta_u^{r-1,0}\\ &= \partial_{X_n/B_n}\theta_u^{r-1,0}\\ &= \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})_u^{r,0}, \quad \text{for } 0 < r \le p-1,\end{aligned}$$

$$\begin{aligned}(\check{\boldsymbol{\delta}}\alpha)_u^{0,0} &= 0 = \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})_u^{0,0},\\ (\check{\boldsymbol{\delta}}\alpha)_v^{0,s} &= 0 = \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})_v^{0,s}, \quad \text{for } 0 \le s \le q-1 \quad \text{and}\\ (\check{\boldsymbol{\delta}}\alpha)_c &= 0 = \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})_c.\end{aligned}$$

Therefore $\check{\boldsymbol{\delta}}\alpha = \partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})$, and $\partial_{X_n/B_n}(\theta - \theta_u^{p-1,0})$ is a hypercoboundary. □

The following lemma can be proven similarly.

Lemma 2.5. *Let* $[\theta]$ *be an element of* $\mathbb{H}^l(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ *which is represented by an element* θ *in* $\check{Z}^l(X, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ *given by* θ_u, θ_v *and* θ_c, *then* $\bar{\partial}_{X_n/B_n}(\theta - \theta_v^{0,q-1})$ *is a hypercoboundary.*

3. The jumping phenomenon and obstructions

There is a Hodge theory also for Bott–Chern and Aeppli cohomologies, see [Schweitzer 2007]. More precisely, for a fixed Hermitian metric on X,

$$H^{\bullet,\bullet}_{\mathrm{BC}}(X) \simeq \ker \tilde{\Delta}_{\mathrm{BC}} \quad \text{and} \quad H^{\bullet,\bullet}_{\mathrm{A}}(X) \simeq \ker \tilde{\Delta}_{\mathrm{A}},$$

where

$$\begin{aligned}\tilde{\Delta}_{\mathrm{BC}} &:= (\partial\bar{\partial})(\partial\bar{\partial})^* + (\partial\bar{\partial})^*(\partial\bar{\partial}) + (\bar{\partial}^*\partial)(\bar{\partial}^*\partial)^* + (\bar{\partial}^*\partial)^*(\bar{\partial}^*\partial) + \bar{\partial}^*\bar{\partial} + \partial^*\partial,\\ \tilde{\Delta}_{\mathrm{A}} &:= \partial\partial^* + \bar{\partial}\bar{\partial}^* + (\partial\bar{\partial})^*(\partial\bar{\partial}) + (\partial\bar{\partial})(\partial\bar{\partial})^* + (\bar{\partial}\partial^*)^*(\bar{\partial}\partial^*) + (\bar{\partial}\partial^*)(\bar{\partial}\partial^*)^*\end{aligned}$$

are 4-th order elliptic self-adjoint differential operators. In particular,

$$\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\sharp}(X) < +\infty \;\text{ for } \sharp \in \{\bar{\partial}, \partial, BC, A\}.$$

Let $\pi : \mathcal{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold and B is a neighborhood of the origin in $\mathbb{C}$. Note that $h^{p,q}_{\mathrm{BC}}(X(t))$ and $h^{p,q}_{\mathrm{A}}(X(t))$ are semicontinuous functions of $t \in B$ where $X(t) = \pi^{-1}(t)$ [Schweitzer 2007]. Denote the $\tilde{\Delta}_{\mathrm{BC}}$ operator and the $\tilde{\Delta}_{\mathrm{A}}$ on $X(t)$ by $\tilde{\Delta}_{BC,t}$ and $\tilde{\Delta}_{A,t}$. From the proof of the semicontinuity of $h^{p,q}_{\mathrm{BC}}(X(t))$ (resp. $h^{p,q}_{\mathrm{A}}(X(t))$) in [Schweitzer 2007], we can see that $h^{p,q}_{\mathrm{BC}}(X(t))$ (resp. $h^{p,q}_{\mathrm{A}}(X(t))$) does not jump at the point $t = 0$ if and only if all the $\tilde{\Delta}_{BC,0}$- (resp. $\tilde{\Delta}_{A,0}$)-harmonic forms on X can be extended to relative $\tilde{\Delta}_{BC,t}$- (resp. $\tilde{\Delta}_{A,t}$)-harmonic forms on a neighborhood of $0 \in B$ which are real analytic in the direction of B, since the $\tilde{\Delta}_{BC,t}$ (resp. $\tilde{\Delta}_{A,t}$)

varies real analytically on B. The above condition is equivalent to the following: all the cohomology classes $[\theta]$ in $H^{p,q}_{\mathrm{BC}}(X)$ (resp. $H^{p,q}_{\mathrm{A}}(X)$) can be extended to a relative $d_t -$ closed (resp. $\partial_t\bar\partial_t -$ closed) forms $\theta(t)$ such that $[\theta(t)] \neq 0$ on a neighborhood of $0 \in B$ which are real analytic on the direction of B. Therefore in order to study the jumping phenomenon, we need to study the extension obstructions. So we need to study the obstructions of the extension of the cohomology classes in $\mathbb{H}^\bullet(X, \mathcal{B}^\bullet_{p,q})$ to the relative cohomology classes in $\mathbb{H}^\bullet(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$. Set

$$\begin{aligned}
&\mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}\\
&\quad = \pi^{-1}(m_0^\omega/(m_0^\omega)^{n+1}) \overset{(+,-)}{\to} \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{O}^\omega_{X_n} \oplus \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \mathcal{O}^\omega_{\bar{X}_n}\\
&\quad \to \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \Omega^{1;\omega}_{X_n/B_n} \oplus \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \bar\Omega^{1;\omega}_{X_n/B_n} \to \cdots\\
&\quad \to \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \Omega^{p-1;\omega}_{X_n/B_n} \oplus \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \bar\Omega^{p-1;\omega}_{X_n/B_n}\\
&\quad \to \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \bar\Omega^{p;\omega}_{X_n/B_n} \to \cdots \to \bar{\mathcal{M}}_0^\omega/(\bar{\mathcal{M}}_0^\omega)^{n+1} \otimes \bar\Omega^{q-1;\omega}_{X_n/B_n} \to 0.
\end{aligned}$$

Now we consider the exact sequence

$$0 \to \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}} \to \mathcal{B}^\bullet_{p,q;X_n/B_n} \to \mathcal{B}^\bullet_{p,q;X_0/B_0} \to 0,$$

which induces a long exact sequence

$$\begin{aligned}
&0 \to \mathbb{H}^0(X_n, \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}) \to \mathbb{H}^0(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})\\
&\quad \to \mathbb{H}^0(X, \mathcal{B}^\bullet_{p,q;X_0/B_0}) \to \mathbb{H}^1(X_n, \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}) \to \cdots .
\end{aligned}$$

Let $[\theta]$ be a cohomology class in $\mathbb{H}^l(X, \mathcal{B}^\bullet_{p,q;X_0/B_0})$. The obstruction for the extension of $[\theta]$ to a relative cohomology classes in $\mathbb{H}^l(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ comes from the nontrivial image of the connecting homomorphism $\delta^* : \mathbb{H}^l(X, \mathcal{B}^\bullet_{p,q;X_0/B_0}) \to \mathbb{H}^{l+1}(X_n, \mathcal{M}_0^\omega/(\mathcal{M}_0^\omega)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. We denote this obstruction by $\mathrm{o}_n([\theta])$. On the other hand, for a given real direction $\partial/\partial x$ on B, if there exits $n \in \mathbb{N}$, such that $\mathrm{o}_i([\theta]) = 0$ for all $i \leq n$ and $\mathrm{o}_n([\theta]) \neq 0$, then let θ_{n-1} be a $n-1$-st order extension of θ to a relative cohomology class in $\mathbb{H}^l(X_{n-1}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. Then $\check{\boldsymbol{\delta}}\theta_{n-1} = 0$ up to order $n-1$. Now, it is easy to check that $\check{\boldsymbol{\delta}}\theta_{n-1}/x^n$ is an extension of a nontrivial cohomology class $[\check{\boldsymbol{\delta}}\theta_{n-1}/x^n(0)]$ in $\mathbb{H}^{l+1}(X, \mathcal{B}_{p,q})$, while $[\check{\boldsymbol{\delta}}\theta_{n-1}/x^n(x_0)]$ is trivial in $X(x_0)$ as a cohomology classes in $\mathbb{H}^{l+1}(X(x_0), \mathcal{B}_{p,q;x_0})$ if $x_0 \neq 0$. The above discussion leads to the following theorem.

Theorem 3.1. *Let $\pi : \mathcal{X} \to B$ be a small deformation of the central fiber compact complex manifold X. Now we consider* $\dim \mathbb{H}^l(X(t), \mathcal{B}^\bullet_{p,q;t})$ *as a function of $t \in B$. This function jumps at $t = 0$ if there exists an element* $[\theta]$ *either in* $\mathbb{H}^l(X, \mathcal{B}^\bullet_{p,q})$ *or in* $\mathbb{H}^{l-1}(X, \mathcal{B}^\bullet_{p,q})$ *and a minimal natural number $n \geq 1$ such that the n-th order obstruction satisfies*

$$\mathrm{o}_n([\theta]) \neq 0.$$

4. The formula for the obstructions

Since the obstructions we discussed in the previous section are so important when we consider the problem of jumping phenomenon of Bott–Chern cohomology and Aeppli cohomology, in this section we try to find an explicit calculation for such obstructions. As in [Ye 2008, §3], we need some preparation. Cover X by open sets U_i such that, for arbitrary i, U_i is small enough. More precisely, U_i is Stein and the following exact sequence splits:

$$0 \to \pi_n^*(\Omega_{B_n})^{\omega}(U_i) \to \Omega^{\omega}_{X_n}(U_i) \to \Omega^{\omega}_{X_n/B_n}(U_i) \to 0,$$
$$0 \to \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega}(U_i) \to \Omega^{\omega}_{\bar{X}_n}(U_i) \to \bar{\Omega}^{\omega}_{X_n/B_n}(U_i) \to 0.$$

So we have a map $\varphi_i : \Omega^{\omega}_{X_n/B_n}(U_i) \oplus \bar{\Omega}^{\omega}_{X_n/B_n}(U_i) \to \Omega^{\omega}_{X_n}(U_i) \oplus \Omega^{\omega}_{\bar{X}_n}(U_i)$ such that

$$\varphi_i|_{\Omega^{\omega}_{X_n/B_n}(U_i)}(\Omega^{\omega}_{X_n/B_n}(U_i)) \oplus \pi_n^*(\Omega_{B_n})^{\omega}(U_i) \cong \Omega^{\omega}_{X_n}(U_i),$$
$$\varphi_i|_{\bar{\Omega}^{\omega}_{X_n/B_n}(U_i)}(\bar{\Omega}^{\omega}_{X_n/B_n}(U_i)) \oplus \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega}(U_i) \cong \Omega^{\omega}_{\bar{X}_n}(U_i).$$

This decomposition determines a local decomposition of the exterior derivative ∂_{X_n} (resp. $\bar{\partial}_{X_n}$) in $\Omega^{\bullet;\omega}_{X_n}$ (resp. $\bar{\Omega}^{\bullet;\omega}_{X_n}$) on each U_i:

$$\partial_{X_n} = \partial^i_{B_n} + \partial^i_{X_n/B_n} \quad (\text{resp. } \partial_{\bar{X}_n} = \bar{\partial}^i_{B_n} + \bar{\partial}^i_{X_n/B_n}).$$

By definition, ∂_{X_n/B_n} and $\bar{\partial}_{X_n/B_n}$ are given by $\varphi_i^{-1} \circ \partial^i_{X_n/B_n} \circ \varphi_i$ and $\varphi_i^{-1} \circ \bar{\partial}^i_{X_n/B_n} \circ \varphi_i$.

Denote the set of alternating q-cochains β with values in $\mathscr{F}$ by $\check{C}^q(\boldsymbol{U}, \mathscr{F})$, that is, to each $q+1$-tuple, $i_0 < i_1 \cdots < i_q$, β assigns a section $\beta(i_0, i_1, \dots, i_q)$ of $\mathscr{F}$ over $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q}$. Let us still use φ_i to denote the map

$$\begin{aligned}\varphi_i : \pi_n^*(\Omega_{B_n})^{\omega} \wedge \Omega^{p;\omega}_{X_n/B_n}(U_i) \oplus \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega} \wedge \bar{\Omega}^{p;\omega}_{X_n/B_n}(U_i) \\ \to \Omega^{p+1;\omega}_{X_n}(U_i) \oplus \Omega^{p+1;\omega}_{\bar{X}_n}(U_i)\end{aligned}$$
$$\begin{aligned}\omega_1 \wedge \beta_{i_1} \wedge \cdots \wedge \beta_{i_p} + \omega_2 \wedge \beta'_{j_1} \wedge \cdots \wedge \beta'_{j_p} \mapsto \omega_1 \wedge \varphi_i(\beta_{i_1}) \wedge \cdots \wedge \varphi_i(\beta_{i_p}) \\ + \omega_2 \wedge \varphi_i(\beta'_{j_1}) \wedge \cdots \wedge \varphi_i(\beta'_{j_p}).\end{aligned}$$

Define

$$\varphi : \check{C}^q\big(\boldsymbol{U}, \pi_n^*(\Omega_{B_n})^{\omega} \wedge \Omega^{p;\omega}_{X_n/B_n} \oplus \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega} \wedge \bar{\Omega}^{p;\omega}_{X_n/B_n}\big) \to \check{C}^q\big(\boldsymbol{U}, \Omega^{p+1;\omega}_{X_n} \oplus \Omega^{p+1;\omega}_{\bar{X}_n}\big)$$
$$\text{by} \quad \varphi(\beta)(i_0, i_1, \dots, i_q) = \varphi_{i_0}(\beta(i_0, i_1, \dots, i_q))$$

for all $\beta \in \check{C}^q(\boldsymbol{U}, \pi_n^*(\Omega_{B_n})^{\omega} \wedge \Omega^{p;\omega}_{X_n/B_n} \oplus \bar{\pi}_n^*(\Omega_{\bar{B}_n})^{\omega} \wedge \bar{\Omega}^{p;\omega}_{X_n/B_n})$, where $i_0 < i_1 < \cdots < i_q$. Define the total Lie derivative with respect to B_n:

$$L_{B_n} : \check{C}^q\big(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\bar{X}_n}\big) \to \check{C}^q\big(\boldsymbol{U}, \Omega^{p+1;\omega}_{X_n} \oplus \Omega^{\omega}_{X_n} \wedge \Omega^{p;\omega}_{\bar{X}_n}\big)$$
$$\text{by} \quad L_{B_n}(\beta)(i_0, i_1, \dots, i_q) = \partial^{i_0}_{B_n}(\beta(i_0, i_1, \dots, i_q))$$

for $\beta \in \check{C}^q(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\bar{X}_n})$, where $i_0 < i_1 < \cdots < i_q$ (see [Katz and Oda 1968]).

Define, for each U_i, the total interior product with respect to B_n

$$I^i : \Omega^{p;\omega}_{X_n}(U_i) \oplus \Omega^{p;\omega}_{\overline{X}_n}(U_i) \to \Omega^{p;\omega}_{X_n}(U_i) \oplus \Omega^{p;\omega}_{\overline{X}_n}(U_i)$$

by

$$\begin{aligned} &I^i(\mu_1 \partial_{X_n} g_1 \wedge \partial_{X_n} g_2 \wedge \cdots \wedge \partial_{X_n} g_p + \mu_2 \partial_{\overline{X}_n} g'_1 \wedge \partial_{\overline{X}_n} g'_2 \wedge \cdots \wedge \partial_{\overline{X}_n} g'_p) \\ &\quad = \mu_1 \sum_{j=1}^{p} \partial_{X_n} g_1 \wedge \cdots \wedge \partial_{X_n} g_{j-1} \wedge \partial^i_{B_n}(g_j) \wedge \partial_{X_n} g_{j+1} \wedge \cdots \wedge \partial_{X_n} g_p \\ &\qquad + \mu_2 \sum_{j=1}^{p} \partial_{\overline{X}_n} g'_1 \wedge \cdots \wedge \partial_{\overline{X}_n} g'_{j-1} \wedge \partial^i_{\overline{B}_n}(g'_j) \wedge \partial_{\overline{X}_n} g'_{j+1} \wedge \cdots \wedge \partial_{\overline{X}_n} g'_p. \end{aligned}$$

When $p = 0$, we put $I^i = 0$ (see [Katz and Oda 1968]). Finally, define

$$\lambda : \check{C}^q\big(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\overline{X}_n}\big) \to \check{C}^{q+1}\big(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\overline{X}_n}\big)$$

$$\text{by} \quad (\lambda\beta)(i_0, \cdots, i_{q+1}) = (I^{i_0} - I^{i_1})\beta(i_1, \cdots, i_{q+1})$$

for all $\beta \in \check{C}^q(\boldsymbol{U}, \Omega^{p;\omega}_{X_n} \oplus \Omega^{p;\omega}_{\overline{X}_n})$.

This gives the following lemma, proved identically to [Ye 2008, Lemma 3.1].

Lemma 4.1. $$\lambda \circ \varphi \equiv \delta \circ \varphi - \varphi \circ \delta.$$

With the above preparation, we are ready to study the jumping phenomenon of the dimensions of Bott–Chern or Aeppli cohomology groups. Suppose we can extend an arbitrary $[\theta] \in \mathbb{H}^l(X, \mathcal{B}_{p,q})$ to order $n-1$ in $\mathbb{H}^l(X_{n-1}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. Denote such an element by $[\theta_{n-1}]$. In what follows, we try to find the obstruction of the extension of $[\theta_{n-1}]$ to n-th order. Consider the exact sequence

$$0 \to (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0} \to \mathcal{B}^\bullet_{p,q;X_n/B_n} \to \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}} \to 0$$

which induces a long exact sequence

$$\begin{aligned} &0 \to \mathbb{H}^0\big(X_n, (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big) \to \mathbb{H}^0\big(X_n, \mathcal{B}^\bullet_{p,q;X_n/B_n}\big) \\ &\quad \to \mathbb{H}^0\big(X_{n-1}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}\big) \to \mathbb{H}^1\big(X_n, (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big) \to \cdots. \end{aligned}$$

Let $[\theta]$ be a cohomology class in $\mathbb{H}^l(X, \mathcal{B}^\bullet_{p,q;X_0/B_0})$.

The obstruction for $[\theta_{n-1}]$ comes from the nontrivial image of the connecting homomorphism

$$\delta^* : \mathbb{H}^l\big(X_{n-1}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}\big) \to \mathbb{H}^{l+1}\big(X_n, (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big).$$

Now we are ready to calculate the formula for the obstructions. Let $\tilde{\theta}$ be an element of $\check{C}^l(\boldsymbol{U}, \mathcal{B}^\bullet_{p,q;X_n/B_n})$ such that its quotient image in $\check{C}^l(\boldsymbol{U}, \mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$ is θ_{n-1}. Then $\delta^*([\theta_{n-1}])= [\check{\boldsymbol{\delta}}(\tilde{\theta})]$, which is an element of

$$\mathbb{H}^{l+1}\big(X_n, (\mathcal{M}^\omega_0)^n/(\mathcal{M}^\omega_0)^{n+1} \otimes \mathcal{B}^\bullet_{p,q;X_0/B_0}\big) \cong (\mathrm{m}^\omega_0)^n/(\mathrm{m}^\omega_0)^{n+1} \otimes \mathbb{H}^{l+1}\big(X, \mathcal{B}^\bullet_{p,q;X_0/B_0}\big).$$

Let r_{X_n} be the restriction to the space X_n^ω (the topological space X with structure sheaf $\mathcal{O}_{X_n}^\omega$) and set

$$\begin{aligned}\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}},\\ =\pi^{-1}(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\\ \xrightarrow{(-,+)}\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\otimes\mathcal{O}^\omega_{X_{n-1}}\\ \oplus\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\otimes\mathcal{O}^\omega_{\bar{X}_{n-1}}\\ \to\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\Omega^{1;\omega}_{X_{n-1}/B_{n-1}}\\ \oplus\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\bar{\Omega}^{1;\omega}_{X_{n-1}/B_{n-1}}\to\cdots\\ \to\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\Omega^{p-1;\omega}_{X_{n-1}/B_{n-1}}\\ \oplus\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\bar{\Omega}^{p-1;\omega}_{X_{n-1}/B_{n-1}}\\ \to\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\bar{\Omega}^{p;\omega}_{X_{n-1}/B_{n-1}}\to\cdots\\ \to\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\bar{\Omega}^{q-1;\omega}_{X_{n-1}/B_{n-1}}\to 0,\end{aligned}$$

where

$$\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})=\pi_{n-1}^{-1}(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\otimes_{\pi_{n-1}^{-1}(\mathcal{C}_B^\omega)}\mathcal{O}^\omega_{X_{n-1}},$$
$$\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})=\pi_{n-1}^{-1}(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\otimes_{\pi_{n-1}^{-1}(\mathcal{C}_B^\omega)}\bar{\mathcal{O}}^\omega_{X_{n-1}}.$$

In order to give the obstructions an explicit calculation, we need to consider the map

$$\begin{aligned}\rho:\mathbb{H}^l\big(X_n,(\mathcal{M}_0^\omega)^n/(\mathcal{M}_0^\omega)^{n+1}\otimes\mathcal{B}^\bullet_{p,q;X_0/B_0}\big)\\ \to\mathbb{H}^l\big(X_{n-1},\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\mathcal{B}^\bullet_{p,q;X_{n-1}/B_{n-1}}\big)\end{aligned}$$

which is defined by $\rho[\sigma]=[\varphi^{-1}\circ r_{X_{n-1}}\circ(L_{B_n}+L_{\bar{B}_n})\circ\varphi(\sigma)]$, where φ^{-1} is the quotient map

$$\begin{aligned}\check{C}^\bullet\big(\boldsymbol{U},\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\Omega^{p;\omega}_{X_n|X_{n-1}}\oplus\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\Omega^{p;\omega}_{\bar{X}_n|\bar{X}_{n-1}}\big)\\ \to\check{C}^\bullet\big(\boldsymbol{U},\pi_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\Omega^{p;\omega}_{X_{n-1}/B_{n-1}}\\ \oplus\bar{\pi}_{n-1}^*(\Omega^\omega_{B_n|B_{n-1}}+\bar{\Omega}^\omega_{B_n|B_{n-1}})\wedge\bar{\Omega}^{p;\omega}_{X_{n-1}/B_{n-1}}\big).\end{aligned}$$

And we have the following lemmas; the proofs are identical to those of [Ye 2008, Lemma 3.2 and Lemma 3.3].

Lemma 4.2. *The map ρ is well defined.*

Lemma 4.3. *Furthermore, $\rho([\check{\boldsymbol{\delta}}(\tilde{\theta})])$ is exactly $\mathrm{o}_n([\theta])$ in Section 3.*

Now consider the following exact sequence. The connecting homomorphism of the associated long exact sequence gives the Kodaira–Spencer class of order n [Voisin 1996, §1.3.2];

$$0 \to \pi_{n-1}^*(\Omega_{B_n|B_{n-1}})^\omega \to \Omega^\omega_{X_n|X_{n-1}} \to \Omega^\omega_{X_{n-1}/B_{n-1}} \to 0.$$

If we wedge the above exact sequence with $\Omega^{p-1;\omega}_{X_{n-1}/B_{n-1}}$, we get a new exact sequence. The connecting homomorphism of such an exact sequence gives us a map from $H^q(X_{n-1}, \Omega^{p;\omega}_{X_{n-1}/B_{n-1}})$ to $H^{q+1}(X_{n-1}, \pi^*(\Omega_{B_n|B_{n-1}})^\omega \wedge \Omega^{p-1;\omega}_{X_{n-1}/B_{n-1}})$.

Denote such a map by $\kappa_n\llcorner$, for it is simply the inner product with the Kodaira–Spencer class of order n. With the above preparation, we are ready to prove the main theorem of this section.

Theorem 4.4. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X. Suppose we can extend an arbitrary $[\theta] \in \mathbb{H}^l(X, \mathscr{B}^\bullet_{p,q})$ to order $n-1$ in $\mathbb{H}^l(X_{n-1}, \mathscr{B}^\bullet_{p,q;X_{n-1}/B_{n-1}})$. Denote such an element by $[\theta_{n-1}]$. The obstruction of the extension of $[\theta]$ to n-th order is given by*

$$\mathrm{o}_n([\theta]) = -\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \kappa_n\llcorner \circ \partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]) - \bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n\llcorner \circ \bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]),$$

where κ_n is the n-th order Kodaira–Spencer class and $\bar\kappa_n$ is the n-th order Kodaira–Spencer class of the deformation $\bar\pi : \bar{\mathscr{X}} \to \bar B$. The maps $\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}$ and $\bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}$ are those defined in Section 2.

Proof. Note that $\mathrm{o}_n([\theta]) = \rho \circ \delta(\tilde\theta) = [\varphi^{-1} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ \varphi \circ \delta(\tilde\theta)]$. Because $(L_{B_n} + L_{\bar B_n} + \partial_{X_n/B_n} + \bar\partial_{X_n/B_n}) \circ \check{\boldsymbol\delta} = -\check{\boldsymbol\delta} \circ (L_{B_n} + L_{\bar B_n} + \partial_{X_n/B_n} + \bar\partial_{X_n/B_n})$,

$$\begin{aligned}
&r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ \varphi \circ \check{\boldsymbol\delta}(\tilde\theta) \\
&\quad\equiv r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ (\check{\boldsymbol\delta} \circ \varphi - \lambda \circ \varphi)(\tilde\theta) \\
&\quad\equiv r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ \check{\boldsymbol\delta} \circ \varphi(\tilde\theta) \\
&\quad\equiv -r_{X_{n-1}} \circ \big(\partial^\bullet_{X_n/B_n} \circ \check{\boldsymbol\delta} + \check{\boldsymbol\delta} \circ \partial^\bullet_{X_n/B_n} + \bar\partial^\bullet_{X_n/B_n} \circ \check{\boldsymbol\delta} + \check{\boldsymbol\delta} \circ \bar\partial^\bullet_{X_n/B_n} \\
&\qquad\qquad + \check{\boldsymbol\delta} \circ (L_{B_n} + L_{\bar B_n})\big) \circ \varphi(\tilde\theta) \\
&\quad\equiv -r_{X_{n-1}} \circ \big(\partial^\bullet_{X_n/B_n} \circ \check{\boldsymbol\delta} + \check{\boldsymbol\delta} \circ \partial^\bullet_{X_n/B_n} + \bar\partial^\bullet_{X_n/B_n} \circ \check{\boldsymbol\delta} + \check{\boldsymbol\delta} \circ \bar\partial^\bullet_{X_n/B_n}\big) \circ \varphi(\tilde\theta) \\
&\qquad\qquad - \check{\boldsymbol\delta} \circ r_{X_{n-1}} \circ (L_{B_n} + L_{\bar B_n}) \circ \varphi(\tilde\theta).
\end{aligned}$$

Therefore

$$[\varphi^{-1}\circ r_{X_{n-1}}\circ(L_{B_n}+L_{\bar{B}_n})\circ\varphi\circ\delta(\tilde{\theta})]$$
$$=[-\varphi^{-1}\circ r_{X_{n-1}}\circ(\partial^{\bullet}_{X_n/B_n}\circ\check{\boldsymbol{\delta}}+\check{\boldsymbol{\delta}}\circ\partial^{\bullet}_{X_n/B_n}+\bar{\partial}^{\bullet}_{X_n/B_n}\circ\check{\boldsymbol{\delta}}+\check{\boldsymbol{\delta}}\circ\bar{\partial}^{\bullet}_{X_n/B_n})\circ\varphi(\tilde{\theta})]$$
$$=-[\partial_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})+\bar{\partial}_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})$$
$$+\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\widetilde{\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1})})+\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}(\theta_{n-1})}).$$

Since, for $0\le r\le p-1$,

$$(\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}))^{p-1,0}_u=(\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}))^{0,q-1}_v$$
$$=\check{\delta}(\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}))^{r,0}_u=0,$$

and $\check{\delta}(\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}))^{0,s}_v=0$ for $0\le s\le q-1$, we know that

$$\partial_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})+\bar{\partial}_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})$$
$$=\check{\boldsymbol{\delta}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta}).$$

Therefore, $[\partial_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})+\bar{\partial}_{X_{n-1}/B_{n-1}}\circ\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\tilde{\theta})]=0$. And from Lemma 2.4 and Lemma 2.5,

$$[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\widetilde{\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1})})]$$
$$=\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\partial_{X_{n-1}/B_{n-1}}(\theta_{n-1}-\theta^{p-1,0}_{n-1;u})}+\partial_{X_{n-1}/B_{n-1}}\theta^{p-1,0}_{n-1;u}\big)\big]$$
$$=\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\partial_{X_{n-1}/B_{n-1}}\theta^{p-1,0}_{n-1;u}}\big)\big]$$

and

$$[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}(\theta_{n-1})})]$$
$$=\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}(\theta_{n-1}-\theta^{0,q-1}_{n-1;v})}+\bar{\partial}_{X_{n-1}/B_{n-1}}\theta^{0,q-1}_{n-1;v}\big)\big]$$
$$=\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}\theta^{0,q-1}_{n-1;v}}\big)\big]$$

By the definition of the maps $\partial^{\bar{\partial},\mathscr{B}}_{X_{n-1}/B_{n-1}}$, $\partial^{\mathscr{B},\bar{\partial}}_{X_{n-1}/B_{n-1}}$ and [Ye 2008, Lemma 3.4],

$$\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\partial_{X_{n-1}/B_{n-1}}\theta^{p-1,0}_{n-1;u}}\big)\big]=\partial^{\bar{\partial},\mathscr{B}}_{X_{n-1}/B_{n-1}}\circ\kappa_n\llcorner\circ\partial^{\mathscr{B},\bar{\partial}}_{X_{n-1}/B_{n-1}}([\theta_{n-1}])$$

and similarly, we have

$$\big[\varphi^{-1}\circ r_{X_{n-1}}\circ\check{\boldsymbol{\delta}}\circ\varphi\big(\widetilde{\bar{\partial}_{X_{n-1}/B_{n-1}}\theta^{0,q-1}_{n-1;v}}\big)\big]=\bar{\partial}^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}\circ\bar{\kappa}_n\llcorner\circ\bar{\partial}^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]).$$

Finally,

$$[\varphi^{-1}\circ r_{X_{n-1}}\circ(L_{B_n}+L_{\bar{B}_n})\circ\varphi\circ\delta(\tilde{\theta})]$$
$$=-\partial^{\bar{\partial},\mathscr{B}}_{X_{n-1}/B_{n-1}}\circ\kappa_n\llcorner\circ\partial^{\mathscr{B},\bar{\partial}}_{X_{n-1}/B_{n-1}}([\theta_{n-1}])-\bar{\partial}^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}}\circ\bar{\kappa}_n\llcorner\circ\bar{\partial}^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta_{n-1}]).\quad\square$$

We apply the above theorem and Theorem 3.1 in order to study the jumping phenomenon of the dimensions of Bott–Chern(Aeppli) cohomology groups, and obtain the following theorems.

Theorem 4.5. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X. If there exists an element $[\theta^1]$ in $H^{p,q}_{\mathrm{BC}}(X)$ or an element $[\theta^2]$ in $H^{p-1,q-1}_{\mathrm{A}}(X)$ and a minimal natural number $n \geq 1$ such that the n-th order obstruction $\mathrm{o}_n([\theta^1]) \neq 0$ or $\mathrm{o}_n([\theta^2]) \neq 0$, then the $h^{\mathrm{BC}}_{p,q}(X(t))$ will jump at the point $t = 0$. The formulas for the obstructions are given by*

$$\mathrm{o}_n([\theta^1]) = -\partial^{\bar\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ r_{BC,\bar\partial}([\theta^1_{n-1}]) - \bar\partial^{\partial,\mathscr{B}}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ r_{BC,\partial}([\theta^1_{n-1}]);$$

$$\mathrm{o}_n([\theta^2]) = -\partial^{\bar\partial,BC}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ \partial^{A,\bar\partial}_{X_{n-1}/B_{n-1}}([\theta^2_{n-1}]) - \bar\partial^{\partial,BC}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ \bar\partial^{A,\partial}_{X_{n-1}/B_{n-1}}([\theta^2_{n-1}]).$$

Theorem 4.6. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Let $\pi_n : X_n \to B_n$ be the n-th order deformation of X. If there exists an element $[\theta^1]$ in $H^{p,q}_{\mathrm{A}}(X)$ or an element $[\theta^2]$ in $\mathbb{H}^{p+q}(X, \mathscr{B}^{\bullet}_{p+1,q+1})$ and a minimal natural number $n \geq 1$ such that the n-th order obstruction $\mathrm{o}_n([\theta^1]) \neq 0$ or $\mathrm{o}_n([\theta^2]) \neq 0$, then the $h^{\mathrm{A}}_{p,q}(X(t))$ will jump at the point $t = 0$. The formulas for the obstructions are given by*

$$\mathrm{o}_n([\theta^1]) = -\partial^{\bar\partial,BC}_{X_{n-1}/B_{n-1}} \circ \kappa_n \llcorner \circ \partial^{A,\bar\partial}_{X_{n-1}/B_{n-1}}([\theta^1_{n-1}]) - \bar\partial^{\partial,BC}_{X_{n-1}/B_{n-1}} \circ \bar\kappa_n \llcorner \circ \bar\partial^{A,\partial}_{X_{n-1}/B_{n-1}}([\theta^1_{n-1}]).$$

$$\mathrm{o}_n([\theta^2]) = -r_{\bar\partial,A} \circ \kappa_n \llcorner \circ \partial^{\mathscr{B},\bar\partial}_{X_{n-1}/B_{n-1}}([\theta^2_{n-1}]) - r_{\partial,A} \circ \bar\kappa_n \llcorner \circ \bar\partial^{\mathscr{B},\partial}_{X_{n-1}/B_{n-1}}([\theta^2_{n-1}]).$$

By these theorems, we can deduce the following corollaries immediately.

Corollary 4.7. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Suppose that up to order n, the maps $r_{BC,\bar\partial} : H^{p,q}_{\mathrm{BC}}(X_n/B_n) \to H^q(X_n, \Omega^{p;\omega}_{X_n/B_n})$ and $r_{BC,\partial} : H^{p,q}_{\mathrm{BC}}(X_n/B_n) \to H^p(\overline{X}_n, \overline{\Omega}^{q;\omega}_{X_n/B_n})$ are 0. Any element $[\theta] \in H^{p,q}_{\mathrm{BC}}(X)$ can be extended to order $n+1$ in $H^{p,q}_{\mathrm{BC}}(X_{n+1}/B_{n+1})$.*

Proof. This result can be shown by induction on k.

Suppose that the corollary is proved for $k-1$, then we can extend $[\theta]$ to and element $[\theta_{k-1}]$ in $H^{p,q}_{\mathrm{BC}}(X_{k-1}/B_{k-1})$. By Theorem 4.5 , the obstruction for the extension of $[\theta]$ to k-th order comes from:

$$\mathrm{o}_k([\theta]) = -\partial^{\bar\partial,\mathscr{B}}_{X_{k-1}/B_{k-1}} \circ \kappa_k \llcorner \circ r_{BC,\bar\partial}([\theta_{k-1}]) - \bar\partial^{\partial,\mathscr{B}}_{X_{k-1}/B_{k-1}} \circ \bar\kappa_k \llcorner \circ r_{BC,\partial}([\theta_{k-1}]).$$

By the assumption, $r_{BC,\bar\partial} : H^{p,q}_{\mathrm{BC}}(X_{k-1}/B_{k-1}) \to H^q(X_{k-1}, \Omega^{p;\omega}_{X_{k-1}/B_{k-1}})$ and $r_{BC,\partial} : H^{p,q}_{\mathrm{BC}}(X_{k-1}/B_{k-1}) \to H^p(\overline{X}_{k-1}, \overline{\Omega}^{q;\omega}_{X_{k-1}/B_{k-1}})$ are 0, where $k \leq n+1$. So we have $r_{BC,\partial}([\theta_{k-1}]) = 0$ and $r_{BC,\bar\partial}([\theta_{k-1}]) = 0$. So the obstruction $\mathrm{o}_k([\theta])$ is trivial which means $[\theta]$ can be extended to k-th order. □

Since

$$\partial^{A,\bar\partial}_{X_n/B_n} : H^{p,q}_{\mathrm{A}}(X_n/B_n) \to H^q(X_n, \Omega^{p+1;\omega}_{X_n/B_n})$$

is the composition of

$$\partial^{A,BC}_{X_n/B_n} : H^{p,q}_{\mathrm{A}}(X_n/B_n) \to H^{p+1,q}_{\mathrm{BC}}(X_n/B_n) \quad \text{and}$$
$$r_{BC,\bar\partial} : H^{p+1,q}_{\mathrm{BC}}(X_n/B_n) \to H^q(X_n, \Omega^{p+1;\omega}_{X_n/B_n}).$$

With the same proof of the above corollary, we have the following result and we omit the proof.

Corollary 4.8. *Let $\pi : \mathscr{X} \to B$ be a deformation of $\pi^{-1}(0) = X$, where X is a compact complex manifold. Suppose, up to order n, the maps $r_{BC,\bar\partial} : H^{p+1,q}_{\mathrm{BC}}(X_n/B_n) \to H^q(X_n, \Omega^{p+1;\omega}_{X_n/B_n})$ and $r_{BC,\partial} : H^{p,q+1}_{\mathrm{BC}}(X_n/B_n) \to H^p(\bar X_n, \bar\Omega^{q+1;\omega}_{X_n/B_n})$ is 0. Any $[\theta]$ that belongs to $H^{p,q}_{\mathrm{A}}(X)$ can be extended to order $n+1$ in $H^{p,q}_{\mathrm{A}}(X_{n+1}/B_{n+1})$.*

5. An example

In this section, we will use the formulas in Theorems 4.5 and 4.6 to study the jumping phenomenon of the dimensions of Bott–Chern and Aeppli cohomology groups $h^{p,q}_{\mathrm{BC}}$ and $h^{p,q}_{\mathrm{A}}$, respectively, of small deformations of Iwasawa manifold. It was Kodaira who first calculated small deformations of Iwasawa manifold [Nakamura 1975]. In the first part of this section, let us recall his result.

Set

$$G = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \right\} \cong \mathbb{C}^3,$$
$$\Gamma = \left\{ \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} : \omega_i \in \mathbb{Z} + \mathbb{Z}\sqrt{-1} \right\}.$$

The multiplication is defined by

$$\begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \omega_2 & \omega_3 \\ 0 & 1 & \omega_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_2+\omega_2 & z_3+\omega_1 z_2+\omega_3 \\ 0 & 1 & z_1+\omega_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$X = G/\Gamma$ is called the Iwasawa manifold. We may consider $X = \mathbb{C}^3/\Gamma$. The element $g \in \Gamma$ operates on $\mathbb{C}^3$ as follows:

$$z_1' = z_1 + \omega_1, \quad z_2' = z_2 + \omega_2, \quad z_3' = z_3 + \omega_1 z_2 + \omega_3,$$

where $g = (\omega_1, \omega_2, \omega_3)$ and $z' = z \cdot g$.

There exist holomorphic 1-forms $\varphi_1, \varphi_2, \varphi_3$ which are linearly independent at every point on X and are given by

$$\varphi_1 = dz_1, \quad \varphi_2 = dz_2, \quad \varphi_3 = dz_3 - z_1 dz_2,$$

so that

$$d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2.$$

On the other hand we have holomorphic vector fields $\theta_1, \theta_2, \theta_3$ on X given by

$$\theta_1 = \frac{\partial}{\partial z_1}, \quad \theta_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, \quad \theta_3 = \frac{\partial}{\partial z_3}.$$

It is easily seen that

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \quad [\theta_1, \theta_3] = [\theta_2, \theta_3] = 0.$$

In view of [Nakamura 1975, Theorem 3], $H^1(X, \mathbb{O}_X)$ is spanned by $\bar{\varphi}_1, \bar{\varphi}_2$. Since Θ is isomorphic to $\mathbb{O}^3$, $H^1(X, T_X)$ is spanned by $\theta_i \bar{\varphi}_\lambda$, $i = 1, 2, 3$, $\lambda = 1, 2$.

Consider the small deformation of X given by

$$\psi(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \bar{\varphi}_\lambda t - (t_{11}t_{22} - t_{21}t_{12}) \theta_3 \bar{\varphi}_3 t^2.$$

We summarize the numerical characters of deformations. The deformations are divided into the following three classes, which are characterized by the following values of the parameters (all the details can be found in [Angella 2013]):

class (i): $t_{11} = t_{12} = t_{21} = t_{22} = 0$;

class (ii): $D(\boldsymbol{t}) = 0$ and $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$;

- class (ii.a): $D(\boldsymbol{t}) = 0$ and rank $S = 1$;
- class (ii.b): $D(\boldsymbol{t}) = 0$ and rank $S = 2$;

class (iii): $D(\boldsymbol{t}) \neq 0$;

- class (iii.a): $D(\boldsymbol{t}) \neq 0$ and rank $S = 1$;
- class (iii.b): $D(\boldsymbol{t}) \neq 0$ and rank $S = 2$;

where the matrix S is defined by

$$S := \begin{pmatrix} \bar{\sigma}_{1\bar{1}} & \bar{\sigma}_{2\bar{2}} & \bar{\sigma}_{1\bar{2}} & \bar{\sigma}_{2\bar{1}} \\ \sigma_{1\bar{1}} & \sigma_{2\bar{2}} & \sigma_{2\bar{1}} & \sigma_{1\bar{2}} \end{pmatrix}$$

where $\sigma_{1\bar{1}}, \sigma_{1\bar{2}}, \sigma_{2\bar{1}}, \sigma_{2\bar{2}} \in \mathbb{C}$ and $\sigma_{12} \in \mathbb{C}$ are complex numbers depending only on $\boldsymbol{t}$ such that

$$d\varphi_{\boldsymbol{t}}^3 =: \sigma_{12} \varphi_{\boldsymbol{t}}^1 \wedge \varphi_{\boldsymbol{t}}^2 + \sigma_{1\bar{1}} \varphi_{\boldsymbol{t}}^1 \wedge \bar{\varphi}_{\boldsymbol{t}}^1 + \sigma_{1\bar{2}} \varphi_{\boldsymbol{t}}^1 \wedge \bar{\varphi}_{\boldsymbol{t}}^2 + \sigma_{2\bar{1}} \varphi_{\boldsymbol{t}}^2 \wedge \bar{\varphi}_{\boldsymbol{t}}^1 + \sigma_{2\bar{2}} \varphi_{\boldsymbol{t}}^2 \wedge \bar{\varphi}_{\boldsymbol{t}}^2.$$

$\mathbf{H}^{\bullet}_{\mathbf{dR}}$	$\mathbf{b_1}$	$\mathbf{b_2}$	$\mathbf{b_3}$	$\mathbf{b_4}$	$\mathbf{b_5}$
$\mathbb{I}_3$ and (i), (ii), (iii)	4	8	10	8	4

$\mathbf{H}^{\bullet\bullet}_{\bar\partial}$	$\mathbf{h}^{1,0}_{\bar\partial}$	$\mathbf{h}^{0,1}_{\bar\partial}$	$\mathbf{h}^{2,0}_{\bar\partial}$	$\mathbf{h}^{1,1}_{\bar\partial}$	$\mathbf{h}^{0,2}_{\bar\partial}$	$\mathbf{h}^{3,0}_{\bar\partial}$	$\mathbf{h}^{2,1}_{\bar\partial}$	$\mathbf{h}^{1,2}_{\bar\partial}$	$\mathbf{h}^{0,3}_{\bar\partial}$	$\mathbf{h}^{3,1}_{\bar\partial}$	$\mathbf{h}^{2,2}_{\bar\partial}$	$\mathbf{h}^{1,3}_{\bar\partial}$	$\mathbf{h}^{3,2}_{\bar\partial}$	$\mathbf{h}^{2,3}_{\bar\partial}$
$\mathbb{I}_3$ and (i)	3	2	3	6	2	1	6	6	1	2	6	3	2	3
(ii)	2	2	2	5	2	1	5	5	1	2	5	2	2	2
(iii)	2	2	1	5	2	1	4	4	1	2	5	1	2	2
$\mathbf{H}^{\bullet\bullet}_{\mathrm{BC}}$	$\mathbf{h}^{1,0}_{\mathrm{BC}}$	$\mathbf{h}^{0,1}_{\mathrm{BC}}$	$\mathbf{h}^{2,0}_{\mathrm{BC}}$	$\mathbf{h}^{1,1}_{\mathrm{BC}}$	$\mathbf{h}^{0,2}_{\mathrm{BC}}$	$\mathbf{h}^{3,0}_{\mathrm{BC}}$	$\mathbf{h}^{2,1}_{\mathrm{BC}}$	$\mathbf{h}^{1,2}_{\mathrm{BC}}$	$\mathbf{h}^{0,3}_{\mathrm{BC}}$	$\mathbf{h}^{3,1}_{\mathrm{BC}}$	$\mathbf{h}^{2,2}_{\mathrm{BC}}$	$\mathbf{h}^{1,3}_{\mathrm{BC}}$	$\mathbf{h}^{3,2}_{\mathrm{BC}}$	$\mathbf{h}^{2,3}_{\mathrm{BC}}$
$\mathbb{I}_3$ and (i)	2	2	3	4	3	1	6	6	1	2	8	2	3	3
(ii.a)	2	2	2	4	2	1	6	6	1	2	7	2	3	3
(ii.b)	2	2	2	4	2	1	6	6	1	2	6	2	3	3
(iii.a)	2	2	1	4	1	1	6	6	1	2	7	2	3	3
(iii.b)	2	2	1	4	1	1	6	6	1	2	6	2	3	3
$\mathbf{H}^{\bullet\bullet}_{\mathrm{A}}$	$\mathbf{h}^{1,0}_{\mathrm{A}}$	$\mathbf{h}^{0,1}_{\mathrm{A}}$	$\mathbf{h}^{2,0}_{\mathrm{A}}$	$\mathbf{h}^{1,1}_{\mathrm{A}}$	$\mathbf{h}^{0,2}_{\mathrm{A}}$	$\mathbf{h}^{3,0}_{\mathrm{A}}$	$\mathbf{h}^{2,1}_{\mathrm{A}}$	$\mathbf{h}^{1,2}_{\mathrm{A}}$	$\mathbf{h}^{0,3}_{\mathrm{A}}$	$\mathbf{h}^{3,1}_{\mathrm{A}}$	$\mathbf{h}^{2,2}_{\mathrm{A}}$	$\mathbf{h}^{1,3}_{\mathrm{A}}$	$\mathbf{h}^{3,2}_{\mathrm{A}}$	$\mathbf{h}^{2,3}_{\mathrm{A}}$
$\mathbb{I}_3$ and (i)	3	3	2	8	2	1	6	6	1	3	4	3	2	2
(ii.a)	3	3	2	7	2	1	6	6	1	2	4	2	2	2
(ii.b)	3	3	2	6	2	1	6	6	1	2	4	2	2	2
(iii.a)	3	3	2	7	2	1	6	6	1	1	4	1	2	2
(iii.b)	3	3	2	6	2	1	6	6	1	1	4	1	2	2

Table 1. Dimensions of the cohomologies of the Iwasawa manifold and of its small deformations [Angella 2013].

The first order asymptotic behavior of σ_{12}, $\sigma_{1\bar 1}$, $\sigma_{1\bar 2}$, $\sigma_{2\bar 1}$, $\sigma_{2\bar 2}$ for $\boldsymbol{t}$ near 0 in classes (i), (ii) or (iii) is

$$\sigma_{12}=-1+\mathrm{o}(|t|)\boldsymbol{t}\quad \sigma_{1\bar 1}=t_{21}+\mathrm{o}(|t|)\boldsymbol{t}\quad \sigma_{1\bar 2}=t_{22}+\mathrm{o}(|t|)\boldsymbol{t}$$
$$\sigma_{2\bar 1}=-t_{11}+\mathrm{o}(|t|)\boldsymbol{t}\quad \sigma_{2\bar 2}=-t_{12}+\mathrm{o}(|t|)\boldsymbol{t}.$$

From Table 1, we know that the jumping phenomenon happens in $h^{2,0}_{\mathrm{BC}}$, $h^{0,2}_{\mathrm{BC}}$ and $h^{2,2}_{\mathrm{BC}}$ of the Bott–Chern cohomology and symmetrically happens in $h^{3,1}_{\mathrm{A}}$, $h^{1,3}_{\mathrm{A}}$ and $h^{1,1}_{\mathrm{A}}$ of the Aeppli cohomology. Now let us explain the jumping phenomenon of the dimensions of Bott–Chern and Aeppli cohomologies by using the obstruction formula. From [Angella 2013, §4], it follows that the Bott–Chern cohomology groups in bidegree (2, 0), (0, 2), (2, 2) are

$$H^{2,0}_{\mathrm{BC}}(X)=\mathrm{Span}_{\mathbb{C}}\{[\varphi_1\wedge\varphi_2],[\varphi_2\wedge\varphi_3],[\varphi_3\wedge\varphi_1]\},$$
$$H^{0,2}_{\mathrm{BC}}(X)=\mathrm{Span}_{\mathbb{C}}\{[\bar\varphi_1\wedge\bar\varphi_2],[\bar\varphi_2\wedge\bar\varphi_3],[\bar\varphi_3\wedge\bar\varphi_1]\},$$
$$\begin{aligned}H^{2,2}_{\mathrm{BC}}(X)=\mathrm{Span}_{\mathbb{C}}\big\{&[\varphi_2\wedge\varphi_3\wedge\bar\varphi_1\wedge\bar\varphi_2],[\varphi_3\wedge\varphi_1\wedge\bar\varphi_1\wedge\bar\varphi_2],\\ &[\varphi_1\wedge\varphi_2\wedge\bar\varphi_2\wedge\bar\varphi_3],[\varphi_2\wedge\varphi_3\wedge\bar\varphi_2\wedge\bar\varphi_3],[\varphi_3\wedge\varphi_1\wedge\bar\varphi_2\wedge\bar\varphi_3],\\ &[\varphi_1\wedge\varphi_2\wedge\bar\varphi_3\wedge\bar\varphi_1],[\varphi_2\wedge\varphi_3\wedge\bar\varphi_3\wedge\bar\varphi_1],[\varphi_3\wedge\varphi_1\wedge\bar\varphi_3\wedge\bar\varphi_1]\big\},\end{aligned}$$

and the Aeppli cohomology groups in bidegree (3, 1), (1, 3), (1, 1) are

$$H^{3,1}_{\mathrm{A}}(X) = \mathrm{Span}_{\mathbb{C}}\{[\varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\bar\varphi_1], [\varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\bar\varphi_2], [\varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\bar\varphi_3]\},$$
$$H^{1,3}_{\mathrm{A}}(X) = \mathrm{Span}_{\mathbb{C}}\{[\varphi_1\wedge\bar\varphi_1\wedge\bar\varphi_2\wedge\bar\varphi_3], [\varphi_2\wedge\bar\varphi_1\wedge\bar\varphi_2\wedge\bar\varphi_3], [\varphi_3\wedge\bar\varphi_1\wedge\bar\varphi_2\wedge\bar\varphi_3]\},$$
$$H^{1,1}_{\mathrm{A}}(X) = \mathrm{Span}_{\mathbb{C}}\big\{[\varphi_1\wedge\bar\varphi_1], [\varphi_1\wedge\bar\varphi_2], [\varphi_1\wedge\bar\varphi_3], [\varphi_2\wedge\bar\varphi_1], [\varphi_2\wedge\bar\varphi_2], [\varphi_2\wedge\bar\varphi_3], [\varphi_3\wedge\bar\varphi_1], [\varphi_3\wedge\bar\varphi_2]\big\}.$$

For example, let us first consider $h^{2,0}_{\mathrm{BC}}$ under a class (ii) deformation. The Kodaira–Spencer class of the this deformation is $\psi_1(t)=\sum_{i=1}^{3}\sum_{\lambda=1}^{2} t_{i\lambda}\theta_i\bar\varphi_\lambda$, and $\overline{\psi}_1(t)=\sum_{i=1}^{3}\sum_{\lambda=1}^{2}\bar t_{i\lambda}\bar\theta_i\varphi_\lambda$, with $t_{11}t_{22}-t_{21}t_{12}=0$. It is easy to check that

$$\mathrm{o}_1(\varphi_1\wedge\varphi_2) = -\partial(\mathrm{int}(\psi_1(t))(\varphi_1\wedge\varphi_2)) - \bar\partial(\mathrm{int}(\overline{\psi}_1(t))(\varphi_1\wedge\varphi_2)) = 0,$$
$$\mathrm{o}_1(t_{11}\varphi_2\wedge\varphi_3 - t_{21}\varphi_1\wedge\varphi_3) = -\partial((t_{11}t_{22}-t_{21}t_{12})\varphi_3\wedge\bar\varphi_2) = 0,$$
$$\mathrm{o}_1(\varphi_2\wedge\varphi_3) = t_{21}\varphi_1\wedge\varphi_2\wedge\bar\varphi_1 + t_{22}\varphi_1\wedge\varphi_2\wedge\bar\varphi_2,$$
$$\mathrm{o}_1(\varphi_1\wedge\varphi_3) = t_{11}\varphi_1\wedge\varphi_2\wedge\bar\varphi_1 + t_{12}\varphi_1\wedge\varphi_2\wedge\bar\varphi_2.$$

Therefore, for an element of the subspace $\mathrm{Span}_{\mathbb{C}}\{[\varphi_1\wedge\varphi_2], [t_{11}\varphi_2\wedge\varphi_3 - t_{21}\varphi_1\wedge\varphi_3]\}$, the first order obstruction is trivial, while, since $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$, at least one of the obstructions $\mathrm{o}_1(\varphi_2\wedge\varphi_3)$, $\mathrm{o}_1(\varphi_1\wedge\varphi_3)$ is nontrivial. This partly explains why the Hodge number $h^{2,0}_{\mathrm{BC}}$ jumps from 3 to 2. For another example, let us consider $h^{1,1}_{\mathrm{A}}$ under a class (ii) deformation. It is easy to check that all the first order obstructions of the cohomology classes are trivial. However, if we want to study the jumping phenomenon, we also need to consider the obstructions that come from $\mathbb{H}^2(X, \mathcal{B}^{\bullet}_{2,2})$. It is easy to check that

$$\mathbb{H}^2(X, \mathcal{B}^{\bullet}_{2,2}) = \mathrm{Span}_{\mathbb{C}}\{[\varphi_3], [\bar\varphi_3]\},$$
$$\mathrm{o}_1(\varphi_3) = t_{11}\varphi_2\wedge\bar\varphi_1 + t_{12}\varphi_2\wedge\bar\varphi_2 - t_{21}\varphi_1\wedge\bar\varphi_1 - t_{22}\varphi_1\wedge\bar\varphi_2,$$
$$\mathrm{o}_1(\bar\varphi_3) = \bar t_{11}\bar\varphi_2\wedge\varphi_1 + \bar t_{12}\bar\varphi_2\wedge\varphi_2 - \bar t_{21}\bar\varphi_1\wedge\varphi_1 - \bar t_{22}\bar\varphi_1\wedge\varphi_2.$$

Note that the first order of S is

$$\begin{pmatrix} \bar t_{21} & -\bar t_{12} & \bar t_{22} & -\bar t_{11} \\ t_{21} & -t_{12} & -t_{11} & t_{22} \end{pmatrix}.$$

If the rank of the first order of S is 1, then there exist c_1, c_2 such that

$$\mathrm{o}_1(c_1\varphi_3 + c_2\bar\varphi_3) \neq 0.$$

If the rank of the first order of S is 2, then for all c_1, c_2

$$\mathrm{o}_1(c_1\varphi_3 + c_2\bar\varphi_3) \neq 0,$$

and exactly these obstructions make $h^{1,1}_{\mathrm{A}}$ jump from 8 to 7 in class (ii.a) and from 8 to 6 in class (ii.b).

To end the section, we give the following observation as an application of the formulas in Theorems 4.5 and 4.6.

Proposition 5.1. *Let X be a non-Kähler nilpotent complex parallelisable manifold whose dimension is more than 2, and $\pi : \mathscr{X} \to B$ be the versal deformation family of X. Then the number $h^{1,1}_{\mathrm{A}}$ will jump in any neighborhood of $0 \in B$.*

Proof. Let φ_i, for $i = 1, \dots, m = \dim_{\mathbb{C}} X$, be the linearly independent holomorphic 1-forms of X. It is easy to check that $\bar{\varphi}_i$ are $\partial\bar{\partial}$-closed and therefore each $\bar{\varphi}_i$ represents an element of $H^{0,1}_{\mathrm{A}}(X)$. On the other hand, by [Macrì 2013, Theorem 3], we know that the dimension of $H^{0,1}_{\mathrm{A}}(X)$ is less than or equal to m. Therefore $\bar{\varphi}_i$, $i = 1, \dots, n$, give us a base of $H^{0,1}_{\mathrm{A}}(X)$. So $\partial : H^{0,1}_{\mathrm{A}}(X) \to H^1(X, \Omega_X)$ is trivial. Then we know that $r_{\bar{\partial},A} : H^1(X, \Omega_X) \to H^{1,1}_{\mathrm{A}}(X)$ is injective. From the proof of [Ye 2008, Proposition 4.2], we know there exists an element $[\theta]$ in $H^0(X, \Omega_X)$ whose $\mathrm{o}_1([\theta]) \neq 0$. Since $\bar{\partial}\theta = 0$, θ also represents an element in $\mathbb{H}^2(X, \mathscr{B}^{\bullet}_{2,2})$; let us denote it by $[\theta]_{\mathscr{B}}$. By Theorem 4.6 one can see that $\mathrm{o}_1([\theta]_{\mathscr{B}}) = -r_{\bar{\partial},A}(\mathrm{o}_1([\theta]))$ in this case. From the injectivity of $r_{\bar{\partial},A}$, we know that $\mathrm{o}_1([\theta]_{\mathscr{B}}) \neq 0$. Therefore the number $h^{1,1}_{\mathrm{A}}$ will jump in any neighborhood of $0 \in B$. □

References

[Aeppli 1965] A. Aeppli, "On the cohomology structure of Stein manifolds", pp. 58–70 in *Proceedings of the Conference on Complex Analysis* (Minneapolis, MN, 1964), edited by A. Aeppli et al., Springer, Berlin, 1965. MR 36 #4588 Zbl 0166.33902

[Angella 2013] D. Angella, "The cohomologies of the Iwasawa manifold and of its small deformations", *J. Geom. Anal.* **23**:3 (2013), 1355–1378. MR 3078358 Zbl 1278.32013

[Bott and Chern 1965] R. Bott and S.-S. Chern, "Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections", *Acta Math.* **114** (1965), 71–112. MR 32 #3070 Zbl 0148.31906

[Frölicher 1955] A. Frölicher, "Relations between the cohomology groups of Dolbeault and topological invariants", *Proc. Nat. Acad. Sci. USA* **41** (1955), 641–644. MR 17,409a Zbl 0065.16502

[Katz and Oda 1968] N. M. Katz and T. Oda, "On the differentiation of de Rham cohomology classes with respect to parameters", *J. Math. Kyoto Univ.* **8** (1968), 199–213. MR 38 #5792 Zbl 0165.54802

[Macrì 2013] M. Macrì, "Cohomological properties of unimodular six dimensional solvable Lie algebras", *Differential Geom. Appl.* **31**:1 (2013), 112–129. MR 3010082 Zbl 1263.53045 arXiv 1111.5958v2

[Nakamura 1975] I. Nakamura, "Complex parallelisable manifolds and their small deformations", *J. Differential Geometry* **10** (1975), 85–112. MR 52 #14389 Zbl 0297.32019

[Schweitzer 2007] M. Schweitzer, "Autour de la cohomologie de Bott–Chern", preprint, 2007. arXiv 0709.3528v1

[Voisin 1996] C. Voisin, *Symétrie miroir*, Panoramas et Synthèses **2**, Société Mathématique de France, Paris, 1996. MR 97i:32026 Zbl 0849.14001

[Ye 2008] X. Ye, "The jumping phenomenon of Hodge numbers", *Pacific J. Math.* **235**:2 (2008), 379–398. MR 2009a:32016 Zbl 1194.32007

Received March 6, 2014. Revised July 27, 2014.

JIEZHU LIN
SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE
GUANGZHOU UNIVERSITY
KEY LABORATORY OF MATHEMATICS AND INTERDISCIPLINARY SCIENCES OF GUANGDONG
HIGHER EDUCATION INSTITUTE
NO 230, WAIHUAN ROAD WEST
GUANGZHOU, 510006
CHINA
jlin@gzhu.edu.cn

XUANMING YE
SCHOOL OF MATHEMATICS AND COMPUTATIONAL SCIENCE
SUN YAT-SEN UNIVERSITY
GUANGZHOU, 510275
CHINA
yexm3@mail.sysu.edu.cn

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 273 No. 2 February 2015