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Let f be a newform of level 1 and weight $(2\kappa - n)$ for positive even integers κ and n. We study congruence primes for the Ikeda lift of f. In particular, we consider a conjecture of Katsurada stating that primes dividing certain L-values of f are congruence primes for the Ikeda lift. Instead of focusing on a congruence to a single eigenform, we deduce a lower bound on the number of all congruences between the Ikeda lift of f and forms not lying in the space spanned by Ikeda lifts.

1. Introduction

Let κ be an integer and let χ be a Dirichlet character of conductor N satisfying $\chi(-1) = (-1)^\kappa$. One has an associated Eisenstein series $E_{\kappa,\chi}$. It is a well-known fact that for a prime $\ell \nmid N$ and a prime $\mathfrak{l}$ dividing ℓ in a suitably large extension of $\mathbb{Z}$ so that $\mathfrak{l} \mid L(1-\kappa, \chi)$ there exists a cuspidal eigenform f of level M with $N \mid M$ such that $f \equiv E_{\kappa,\chi} \pmod{\mathfrak{l}}$. Such congruences between cusp forms and Eisenstein series have been studied by many authors. For instance, one can use such congruences to make deductions on the structure of the residual Galois representation of the cusp form, which can then be used to study Selmer groups associated to the cusp form (see [Ribet 1976; Wiles 1990; Skinner and Urban 2014] for some prominent examples of this type of argument).

If we view the Eisenstein series as a "lift" of the Dirichlet character χ from GL(1) to GL(2), then we can fit the congruences mentioned above into a more general framework. Namely, one can consider more general automorphic forms and lift them to automorphic forms on other algebraic groups. This approach has also received considerable attention as it can also be used to study Selmer groups of higher-degree Galois representations; see [Skinner and Urban 2006; Klosin 2009;

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Agarwal and Brown 2014] for specific examples and [Mazur 2011] for a survey of this method. This makes classifying primes for which one will have a congruence between a lifted form and a nonlifted form a natural question to study. In this paper we investigate this problem for Ikeda lifts.

Let κ and n be positive even integers, $f \in S_{2\kappa-n}(\mathrm{SL}_2(\mathbb{Z}))$ a newform, and $I_n(f) \in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$ the Ikeda lift of f. Katsurada [2011] states a conjecture on when a prime $\mathfrak{l}$ will satisfy that there is an eigenform $F \in S_\kappa(\mathrm{Sp}_{2n}(\mathbb{Z}))$ that is not an Ikeda lift and is congruent to $I_n(f)$ modulo $\mathfrak{l}$. The conjecture is in terms of divisibilities of special values of L-functions of f by $\mathfrak{l}$. One can see Conjecture 9 for the precise statement. To provide evidence for his conjecture he proves that if a prime divides the required L-values and does not divide other L-values then one indeed does have such a congruence (see Theorem 10). In this paper we also consider Ikeda lifts, but instead of focusing on producing one congruence we introduce the Ikeda ideal. This ideal is an analogue of the Eisenstein ideal in the GL(2) case and measures congruences between $I_n(f)$ and all other eigenforms. We then show that under similar hypotheses as given in [Katsurada 2011], we can do better and bound from below the congruences between $I_n(f)$ and all other eigenforms that are not lifts. One can see Theorem 14 for the precise result.

One thing to note here is that while the Saito–Kurokawa lift is useful for studying the p-adic Bloch–Kato conjecture for the L-value $L_{\mathrm{alg}}(\kappa, f)$ due to the fact that the value $L_{\mathrm{alg}}(\kappa, f)$ "controls" the congruence between the Saito–Kurokawa lift and a nonlifted form (see [Brown 2011; Agarwal and Brown 2014] for example), the L-values that control the congruence for an Ikeda lift are given by $L_{\mathrm{alg}}(\kappa, f)\prod_{j=1}^{n/2-1} L_{\mathrm{alg}}(2j+1, \mathrm{ad}^0 f)$. This indicates that if one knew the existence of Galois representations for automorphic forms on GSp_{2n}, as well as expected properties of these representations, one could use the congruence results produced in this paper to study the ℓ-adic Bloch–Kato conjecture not only for $L_{\mathrm{alg}}(\kappa, f)$, but also for the values $L_{\mathrm{alg}}(2j+1, \mathrm{ad}^0 f)$ when $j = 1, \dots, \frac{n}{2} - 1$. This makes such congruences particularly interesting.

The structure of the paper is as follows. Section 2 recalls the basic definitions we will need throughout the paper. We recall the Ikeda lift and some necessary properties in Section 3. In Section 4 we state Katsurada's conjecture and result, introduce the Ikeda ideal, and show how Katsurada's congruence can be recovered by studying the Ikeda ideal. We then state our main result and discuss the major hypotheses in Section 5. Section 6 gives a somewhat detailed description of an Eisenstein series originally introduced by Shimura and some results needed to prove the main theorem. Finally, we conclude by proving the main theorem in Section 7.

Throughout the paper ℓ denotes an odd prime. We fix once and for all an algebraic closure $\overline{\mathbb{Q}}$ of the rationals and $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. We also fix compatible embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

2. Modular forms

In this section we recall the basics on modular forms and Siegel modular forms that will be needed throughout the rest of the paper.

2.1. ***Basic definitions.*** Given a ring R with identity, we write $\mathrm{Mat}_n(R)$ for the ring of $n \times n$ matrices with entries in R.

Set
$$J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$
and recall that the degree-n symplectic group is defined by
$$G_n = \mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} : {}^t g J_n g = \mu_n(g) J_n,\, \mu_n(g) \in \mathrm{GL}_1\}.$$
We set $\mathrm{Sp}_{2n} = \ker \mu_n$. We denote $\mathrm{Sp}_{2n}(\mathbb{Z})$ by Γ_n to ease notation. We say that $\Gamma \subset \Gamma_n$ is a congruence subgroup if it contains $\Gamma^{(n)}(N)$ as a subgroup of finite index for some integer $N \geq 1$, where
$$\Gamma^{(n)}(N) = \{\gamma \in \Gamma_n : \gamma \equiv 1_{2n} \pmod{N}\}.$$

Given a matrix $z \in \mathrm{Mat}_n(\mathbb{C})$, we can write $z = x + \sqrt{-1}y$ for $x, y \in \mathrm{Mat}_n(\mathbb{R})$. When we write $z = x + \sqrt{-1}y$, we will always mean $x, y \in \mathrm{Mat}_n(\mathbb{R})$. The Siegel upper half-space is given by
$$\mathfrak{h}_n = \{z = x + \sqrt{-1}y \in \mathrm{Mat}_n(\mathbb{C}) : {}^t z = z,\, y > 0\}.$$
We have an action of $G_n^+(\mathbb{R}) = \{g \in G_n(\mathbb{R}) : \mu_n(g) > 0\}$ on $\mathfrak{h}_n$ given by
$$gz = (a_g z + b_g)(c_g z + d_g)^{-1} \quad \text{for } g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix},$$
where $a_g, b_g, c_g, d_g \in \mathrm{Mat}_n(\mathbb{R})$.

For $g \in G_n^+(\mathbb{R})$ and $z \in \mathfrak{h}_n$, we set
$$j(g, z) = \det(c_g z + d_g).$$
Let κ be a positive integer. Given $f : \mathfrak{h}_n \to \mathbb{C}$, we define the slash operator on f by
$$(f|_\kappa g)(z) = \mu_n(g)^{n\kappa/2} j(g, z)^{-\kappa} f(gz).$$
Let $\Gamma \subset \Gamma_n$ be a congruence subgroup. We say that such an f is a genus-n Siegel modular form of weight κ and level Γ if f is holomorphic and satisfies
$$(f|_\kappa \gamma)(z) = f(z)$$
for all $\gamma \in \Gamma$. If $n = 1$ we also require that f is holomorphic at the cusps so that we recover the theory of elliptic modular forms. We denote the space of genus-n, level-Γ, and weight-κ modular forms by $M_\kappa(\Gamma)$.

Let $f \in M_\kappa(\Gamma)$ and let $\gamma \in G_n^+(\mathbb{Q})$. Then $f|_\kappa\gamma$ has a Fourier expansion of the form

$$(f|_\kappa\gamma)(z) = \sum_{T \in \Lambda_n} a_{f|_\kappa\gamma}(T) e(\mathrm{Tr}(Tz)),$$

where Λ_n is defined to be the set of $n \times n$ half-integral positive semidefinite symmetric matrices and $e(w) := e^{2\pi i w}$. We say that f is a cusp form if for all $T \in \Lambda_n$ with $\det(T) = 0$ we have $a_{f|_\kappa\gamma}(T) = 0$ for all $\gamma \in G_n^+(\mathbb{Q})$. We write $S_\kappa(\Gamma)$ for the cusp forms in $M_\kappa(\Gamma)$. Given a ring $R \subset \mathbb{C}$, we write $M_\kappa(\Gamma; R)$ for those modular forms whose Fourier coefficients all lie in R and likewise for the cusp forms.

Let $f_1, f_2 \in M_\kappa(\Gamma)$ with at least one of them a cusp form. The Petersson inner product of f_1 and f_2 is defined by

$$\langle f_1, f_2 \rangle_\Gamma = \int_{\Gamma\backslash\mathfrak{h}_n} f_1(z)\overline{f_2(z)}(\det y)^\kappa \, d\mu z,$$

where $z = x + iy$ with $x = (x_{\alpha,\beta})$, $y = (y_{\alpha,\beta}) \in \mathrm{Mat}_n(\mathbb{R})$, and

$$d\mu z = (\det y)^{-(n+1)} \prod_{\alpha \leq \beta} dx_{\alpha,\beta} \prod_{\alpha \leq \beta} dy_{\alpha,\beta},$$

with $dx_{\alpha,\beta}$ and $dy_{\alpha,\beta}$ the usual Lebesgue measure on $\mathbb{R}$. We will use the following scaled definition that is independent of the congruence subgroup considered:

$$\langle f_1, f_2 \rangle = \frac{1}{[\overline{\Gamma}_n : \overline{\Gamma}]} \langle f_1, f_2 \rangle_\Gamma,$$

where $\overline{\Gamma}_n = \Gamma_n/\{\pm 1_{2n}\}$ and $\overline{\Gamma}$ is the image of Γ in $\overline{\Gamma}_n$.

2.2. *Hecke algebras.* Let $\Gamma \subset \Gamma_n$ be a congruence subgroup. Given $g \in G_n^+(\mathbb{Q})$, we write $T(g)$ to denote the double coset $\Gamma g \Gamma$. We define the usual action of $T(g)$ on Siegel modular forms by setting

$$T(g)f = \sum_i f|_\kappa g_i,$$

where $\Gamma g \Gamma = \coprod_i \Gamma g_i$ and $f \in M_\kappa(\Gamma)$. Let p be prime and define

$$T^{(n)}(p) = T(\mathrm{diag}(1_n, p1_n)),$$

and for $i = 1, \ldots, n$, set

$$T_i^{(n)}(p^2) = T(\mathrm{diag}(1_{n-i}, p1_i, p^2 1_{n-i}, p1_i)).$$

These Hecke operators generate the local Siegel Hecke algebra at p [van der Geer 2008, Theorem 9].

Let $\mathcal{H}_{\mathbb{Z}}^{(n)}$ denote the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_\kappa(\Gamma))$ generated by $T^{(n)}(p)$ and $T_i^{(n)}(p^2)$ for $i = 1, \ldots, n$. Given any $\mathbb{Z}$-algebra A, we write $\mathcal{H}_A^{(n)}$ for $\mathcal{H}_{\mathbb{Z}}^{(n)} \otimes_{\mathbb{Z}} A$.

Let E be a finite extension of $\mathbb{Q}_\ell$ and $\mathcal{O}_E$ the ring of integers of E. Then $\mathcal{H}_{\mathcal{O}_E}^{(n)}$ is a semilocal complete finite $\mathcal{O}_E$-algebra. One has

$$\mathcal{H}_{\mathcal{O}_E}^{(n)} = \prod_{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}}^{(n)}, \tag{1}$$

where the product runs over all maximal ideals of $\mathcal{H}_{\mathcal{O}_E}^{(n)}$ and $\mathcal{H}_{\mathfrak{m}}^{(n)}$ denotes the localization of $\mathcal{H}_{\mathcal{O}_E}^{(n)}$ at $\mathfrak{m}$.

2.3. *Congruences.* Let $f, g \in M_\kappa(\Gamma_n; K)$, with $K \subseteq \overline{\mathbb{Q}}_\ell$. Let $\mathcal{O}$ denote the ring of integers of K with $\mathfrak{l}$ the prime of $\mathcal{O}$. We write

$$f \equiv g \pmod{\mathfrak{l}^b}$$

to denote

$$\mathrm{val}_{\mathfrak{l}}(a_f(T) - a_g(T)) \geq b$$

for all $T \in \Lambda_n$. We refer to this as a congruence of Fourier coefficients.

We will also use the notion of a congruence of eigenvalues. Let $f, g \in M_\kappa(\Gamma_n)$ be eigenforms and now suppose that $K/\mathbb{Q}_\ell$ is the minimal extension containing all Hecke eigenvalues of f and g. Note that this is a finite extension by [Kurokawa 1981, Theorem 1]. Furthermore, by the remark in [Mizumoto 1991, Section 2] we may assume that f, g are normalized so that the Fourier coefficients are also contained in K. We shall assume throughout the remainder of the paper that all eigenforms are normalized in this way.

In this case, if f and g are eigenforms for all $t \in \mathcal{H}_{\mathcal{O}}^{(n)}$ with eigenvalues $\lambda_f(t)$ and $\lambda_g(t)$, respectively, we write

$$f \equiv_{\mathrm{ev}} g \pmod{\mathfrak{l}^b}$$

to denote

$$\mathrm{val}_{\mathfrak{l}}(\lambda_f(t) - \lambda_g(t)) \geq b$$

for all $t \in \mathcal{H}_{\mathcal{O}}^{(n)}$.

2.4. *L-functions.* In this section we introduce the L-functions that will be needed in this paper. In the case of the relevant L-functions attached to elliptic modular forms, we also introduce the appropriate canonical periods.

Given local Euler factors $L_p(s)$ and a finite set of primes Σ, we define

$$L^\Sigma(s) = \prod_{p \notin \Sigma} L_p(s).$$

If $\Sigma = \{p \mid N\}$ we write $L^N(s)$ for $L^\Sigma(s)$. We set $L(s) = L^\varnothing(s)$.

We begin with the case of an elliptic modular form $f \in S_\kappa(\Gamma_1)$. We assume that f is a normalized eigenform with Fourier expansion

$$f(z) = \sum_{n\geq 1} a_f(n)e(nz).$$

Let $\pi_f = \bigotimes'_p \pi_{f,p}$ be the automorphic representation associated to f. For each prime p there exists a character σ_p such that $\pi_{f,p} = \pi(\sigma_p, \sigma_p^{-1})$, where $\pi(\sigma_p, \sigma_p^{-1})$ is the principal series representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. The p-Satake parameter of f is given by $\alpha_0(p; f) = \sigma_p(p)$. We will drop the f from the notation when it is clear from context. The L-function of f is given by

$$\begin{aligned} L(s, f) &= \prod_p (1-\alpha_0(p)p^{-s+(\kappa-1)/2})^{-1}(1-\alpha_0(p)^{-1}p^{-s+(\kappa-1)/2})^{-1} \\ &= \prod_p (1-a_f(p)p^{-s}+p^{\kappa-1-2s})^{-1} = \sum_{n\geq 1} a_f(n)n^{-s}. \end{aligned}$$

Given a Dirichlet character χ, we will also make use of the twisted L-function

$$L(s, f, \chi) = \sum_{n\geq 1} \chi(n)a_f(n)n^{-s}.$$

Let $\ell \geq \kappa$ be a prime and let K be a suitably large finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$. Let $f \in S_\kappa(\Gamma_1; \mathcal{O})$ be a normalized eigenform. Let $\rho_{f,\ell}$ be the ℓ-adic Galois representation associated to f and assume that the residual representation $\bar{\rho}_{f,\ell}$ is irreducible. Then we have canonical complex periods $\Omega_f^\pm$ (determined up to ℓ-units) by [Vatsal 1999]. Vatsal showed that such periods exist for level greater than 3, but using arguments in [Hida 1987] we can define $\Omega_f^\pm$ for arbitrary level. One can see [Brown 2007] for more details. Using these periods we have:

Theorem 1 [Shimura 1977; Vatsal 1999]. *Let $f \in S_\kappa(\Gamma_1; \mathcal{O})$ be as in the above discussion. There exist complex periods $\Omega_f^\pm$ such that for each integer m with $0 < m < \kappa$ and every Dirichlet character χ one has*

$$\frac{L(m, f, \chi)}{\tau(\chi)(2\pi\sqrt{-1})^m} \in \begin{cases} \Omega_f^+ \mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^m, \\ \Omega_f^- \mathcal{O}_\chi & \text{if } \chi(-1) = (-1)^{m-1}, \end{cases}$$

where $\tau(\chi)$ is the Gauss sum of χ and $\mathcal{O}_\chi$ is the extension of $\mathcal{O}$ generated by the values of χ.

With this theorem in mind we set the following notation for the algebraic part of $L(m, f, \chi)$ with $0 < m < \kappa$:

$$L_{\mathrm{alg}}(m, f, \chi) := \frac{L(m, f, \chi)}{\tau(\chi)(2\pi\sqrt{-1})^m \Omega_f^\pm},$$

where the choice of period is from the theorem.

For Siegel modular forms of genus greater than 1 there are two relevant L-functions: the standard and spinor L-functions. Let $f \in S_\kappa(\Gamma_n)$ be an eigenform. Associated to f is a cuspidal automorphic representation π_f of $\mathrm{PGSp}_{2n}(\mathbb{A})$. We can decompose π_f into local components as $\pi_f = \bigotimes' \pi_{f,p}$, with $\pi_{f,p}$ an Iwahori spherical representation of $\mathrm{PGSp}_{2n}(\mathbb{Q}_p)$. We refer the reader to [Asgari and Schmidt 2001, Section 3] for the details concerning the construction of cuspidal automorphic representations associated to Siegel cusp forms. The representation $\pi_{f,p}$ is given as $\pi(\chi_0, \chi_1, \ldots, \chi_n)$ for χ_i unramified characters of $\mathbb{Q}_p^\times$. One can see [Asgari and Schmidt 2001, Section 3.2] for the definition of this spherical representation. Let $\alpha_0(p; f) = \chi_0(p), \ldots, \alpha_n(p; f) = \chi_n(p)$ denote the p-Satake parameters of f. Note these are normalized so that

$$\alpha_0(p; f)^2 \alpha_1(p; f) \cdots \alpha_n(p; f) = 1.$$

We drop f and/or p in the notation for the Satake parameters when they are clear from context. Set $\tilde{\alpha}_0 = p^{(2n\kappa - n(n+1))/4} \alpha_0$ and

$$L_p(X, f; \mathrm{spin}) = (1 - \tilde{\alpha}_0 X) \prod_{j=1}^{n} \prod_{1 \le i_1 \le \cdots \le i_j \le n} (1 - \tilde{\alpha}_0 \alpha_{i_1} \cdots \alpha_{i_j} X).$$

The spinor L-function associated to f is given by

$$L(s, f; \mathrm{spin}) = \prod_p L_p(p^{-s}, f; \mathrm{spin})^{-1}.$$

One should note that in the case that f is an elliptic modular form the spinor L-function is exactly $L(s, f)$ defined above.

Set

$$L_p(X, f; \mathrm{st}) = (1 - X) \prod_{i=1}^{n} (1 - \alpha_i(p) X)(1 - \alpha_i(p)^{-1} X).$$

Then, we define the standard L-function associated to f by

$$L(s, f; \mathrm{st}) = \prod_p L_p(p^{-s}, f; \mathrm{st})^{-1}.$$

Given a Hecke character χ, the twisted standard L-function is given by

$$L(s, f, \chi; \mathrm{st}) = \prod_p L_p(\chi(p) p^{-s}, f; \mathrm{st})^{-1}.$$

In the case that $f \in S_\kappa(\Gamma_1; \mathcal{O})$ is an elliptic modular form the standard L-function is usually denoted by $L(s, \mathrm{ad}^0 f)$, i.e., it is the adjoint L-function. Then the corollary to [Zagier 1977, Theorem 2] gives that

$$\frac{L(m, \mathrm{ad}^0 f)}{\pi^{2m+\kappa-1} \Omega_f^+ \Omega_f^-} \in \overline{\mathbb{Q}}$$

for $m = 1, 3, \dots, \kappa - 1$ and

$$\frac{L(m, \mathrm{ad}^0 f)}{\pi^{m+\kappa-1}\Omega_f^+\Omega_f^-} \in \overline{\mathbb{Q}}$$

for $m = 2-\kappa, 4-\kappa, \dots, 0$. We will only be interested in the first case; we denote this algebraic value by $L_{\mathrm{alg}}(m, \mathrm{ad}^0 f)$.

3. The Ikeda lift

In this section we will present an introduction to the Ikeda lift. For the details the reader is referred to [Kohnen 2002] or Ikeda's original paper [2001]. The Ikeda lift can be viewed as a composition of the Shintani map from the space of elliptic modular forms to the space of half-integral weight modular forms and a map from the space of half-integral weight forms to the correct space of Siegel modular forms.

Throughout we assume κ, n to be positive even integers with $2\kappa - n > 1$. We note here that we begin with weight $2\kappa - n$ instead of 2κ as used in [Ikeda 2001; Kohnen 2002]. This normalization is more convenient for our purposes.

Recall the algebraic version of Shintani's lift that we require. One has:

Theorem 2 [Shintani 1975]. *There is a linear function*

$$\theta_{\kappa,n} : S_{2\kappa-n}(\Gamma_1) \to S^+_{\kappa-\frac{n}{2}+\frac{1}{2}}(\Gamma_0(4))$$

that is Hecke equivariant, i.e., one has $\theta_{\kappa,n}(f \mid T(p)) = \theta_{\kappa,n}(f) \mid T(p^2)$ for any prime p.

The next result will be pivotal for the algebraic construction.

Proposition 3 [Stevens 1994, Proposition 2.3.1]. *Let $f \in S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ be a Hecke eigenform, where $\mathcal{O}$ is the ring of integers of a field that can be embedded in $\mathbb{C}$. Then there is a nonzero complex number $\Omega(f) \in \mathbb{C}^\times$ so that*

$$\frac{1}{\Omega(f)}\theta_{\kappa,n}(f) \in S^+_{\kappa-\frac{n}{2}+\frac{1}{2}}(\Gamma_0(4); \mathcal{O}).$$

Moreover, if $\mathcal{O}$ is a discrete valuation ring then $\Omega(f)$ can be chosen so that at least one of the Fourier coefficients of $(1/\Omega(f))\theta_{\kappa,n}(f)$ is a unit in $\mathcal{O}$.

From now on we write $\theta^{\mathrm{alg}}_{\kappa,n}(f)$ for $(1/\Omega(f))\theta_{\kappa,n}(f)$ and will always choose the period so that $\theta^{\mathrm{alg}}_{\kappa,n}(f)$ has a unit Fourier coefficient in the case that $\mathcal{O}$ is a discrete valuation ring. We write

$$\theta^{\mathrm{alg}}_{\kappa,n}(f)(z) = \sum_{\substack{m>0 \\ m\equiv 0,1 \pmod 4}} c(m)e(mz).$$

Let $T > 0$ be in Λ_n, i.e., T is an $n \times n$ half-integral positive definite symmetric matrix. Set D_T to be the determinant of $2T$, Δ_T the absolute value of the discriminant of $\mathbb{Q}(\sqrt{D_T})$, χ_T the primitive Dirichlet character associated to $\mathbb{Q}(\sqrt{D_T})/\mathbb{Q}$, and $\mathfrak{f}_T$ the rational number satisfying $D_T = \Delta_T \mathfrak{f}_T^2$.

Let $S_n(R)$ denote the set of symmetric $n \times n$ matrices over a ring R. For a rational prime p, let $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be the unique additive character given by

$$\psi_p(x) = \exp(-\{x\}_p),$$

where $\{x\}_p \in \mathbb{Z}\left[\frac{1}{p}\right]$ is the p-adic fractional part of x. The Siegel series for T is

$$b_p(T, s) := \sum_{S \in S_n(\mathbb{Q}_p)/S_n(\mathbb{Z}_p)} \psi_p(\operatorname{Tr}(TS)) p^{-s \operatorname{ord}_p(\nu(S))} \quad \text{for } \operatorname{Re}(s) \gg 0,$$

where $\nu(S) := \det(S_1) \cdot \mathbb{Z}_p$, and S_1 is from the factorization $S = S_1^{-1} S_2$ for a symmetric coprime pair of matrices S_1, S_2. We have a factorization of the Siegel series

$$b_p(T, s) = \gamma_p(T, p^{-s}) F_p(T, p^{-s}),$$

where

$$\gamma_p(T, X) = \frac{1 - X}{1 - p^{\frac{n}{2}} \chi_T(p) X} \prod_{i=1}^{\frac{n}{2}} (1 - p^{2i} X^2),$$

and $F_p(T, X) \in \mathbb{Z}[X]$ has constant term 1 and $\deg(F_p(T, X)) = 2 \operatorname{ord}_p(\mathfrak{f}_T)$. Using this polynomial $F_p(T, X)$ we define

$$\tilde{F}_p(T, X) := X^{-\operatorname{ord}_p(\mathfrak{f}_T)} F_p(T, p^{-\frac{n}{2} - \frac{1}{2}} X).$$

For each $T > 0$ in Λ_n, define

$$a(T) = c(|\Delta_T|) \mathfrak{f}_T^{\kappa - \frac{n+1}{2}} \prod_p \tilde{F}_p(T, \alpha_0(p; f)), \tag{2}$$

and form the series

$$I_n(f)(z) = \sum_{T > 0} a(T) e(\operatorname{Tr}(Tz)),$$

where $\alpha_0(p; f)$ is the p-th Satake parameter of f. Then we have:

Theorem 4 [Ikeda 2001, Theorems 3.2 and 3.3]. *The series $I_n(f)(z)$, referred to as the Ikeda lift of f, is an eigenform in $S_\kappa(\Gamma_n)$ whose standard L-function factors as*

$$L(s, F; \mathrm{st}) = \zeta(s) \prod_{i=1}^{n} L(s + \kappa - i, f).$$

We will also need further information about the integrality of the Fourier coefficients of $I_n(f)$. In particular, the following result is essential to our applications.

Theorem 5 [Kohnen 2002, Theorem 1]. *Let $\theta_{\kappa,n}^{\mathrm{alg}}(f)$ be as above and let $a(T)$ be as in* (2). *Then*

$$a(T) = \sum_{d \mid \mathfrak{f}_T} d^{\kappa-1} \phi(d; T) c(|\Delta_T|(\mathfrak{f}_T/d)^2),$$

where $\phi(d; T)$ is an integer-valued function.

As an immediate consequence of this theorem and Proposition 3 we have:

Corollary 6. *Let $f \in S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ be a Hecke eigenform, where $\mathcal{O}$ is the ring of integers of a field that can be embedded in $\mathbb{C}$. Then $I_n(f)$ has Fourier coefficients in $\mathcal{O}$.*

We will also make use of the following result.

Proposition 7 [Katsurada 2011, Proposition 4.6]. *Let $f \in S_{2\kappa-n}(\Gamma_1)$ be a normalized eigenform with Ikeda lift $I_n(f)$. Let $\mathcal{O}$ be the ring of integers of a field that can be embedded in $\mathbb{C}$ and let $\mathfrak{l}$ be a prime in $\mathcal{O}$. If there is a fundamental discriminant D such that the D-th Fourier coefficient of $\theta_{\kappa,n}^{\mathrm{alg}}(f)$ is not divisible by $\mathfrak{l}$, then there is a Fourier coefficient of $I_n(f)$ that is not divisible by $\mathfrak{l}$. In particular, if $\mathcal{O}$ is the ring of integers of some $K \subset \overline{\mathbb{Q}}_\ell$ with prime $\mathfrak{l}$, then $I_n(f)$ has a Fourier coefficient that is a unit modulo $\mathfrak{l}$.*

Proof. The only thing to prove is the last statement, but this follows immediately from our normalization of $\theta_{\kappa,n}^{\mathrm{alg}}$. □

Let $f_1, \dots, f_r$ be an orthogonal basis of $S_{2\kappa-n}(\Gamma_1)$ consisting of normalized eigenforms. We denote the span of $I_n(f_1), \dots, I_n(f_r)$ in $S_\kappa(\Gamma_n)$ by $S_\kappa^{\mathrm{Ik}}(\Gamma_n)$. We denote the orthogonal complement of $S_\kappa^{\mathrm{Ik}}(\Gamma_n)$ in $S_\kappa(\Gamma_n)$ with respect to the Petersson product by $S_\kappa^{\text{N-Ik}}(\Gamma_n)$.

4. A conjecture of Katsurada and the Ikeda ideal

In this section we present a conjecture of Katsurada on the congruence primes of Ikeda lifts. We then introduce the Ikeda ideal and show how one can use the Ikeda ideal to study all the congruences between $I_n(f)$ and forms in $S_\kappa^{\text{N-Ik}}(\Gamma_n)$ at once. This allows us to prove a stronger congruence result under roughly the same conditions as given in [Katsurada 2011].

We fix some notation used throughout this section. Let K denote a number field, $\mathcal{O}_K$ the ring of integers of K, and $\mathfrak{l}$ a prime of $\mathcal{O}_K$ of residue characteristic ℓ. We let $\mathcal{O}$ be the completion of $\mathcal{O}_K$ at $\mathfrak{l}$ and let λ denote a uniformizer of $\mathfrak{l}$ in $\mathcal{O}$.

4.1. *A conjecture of Katsurada.*

Definition 8. Let $F \in S_\kappa(\Gamma_n; \mathcal{O})$ be an eigenform. We say $\mathfrak{l}$ is a congruence prime of F with respect to $V \subset (\mathbb{C}F)^\perp$ if there exists an eigenform $G \in V$ such that $F \equiv_{\mathrm{ev}} G \pmod{\mathfrak{l}}$. (Note that in order for this congruence to make sense we may need to extend K so that $G \in S_\kappa(\Gamma_n; \mathcal{O})$ as well.)

One should note this definition can be extended to levels other than Γ_n, but we will have no need of such a definition in this paper.

Let $f \in S_{2\kappa-n}(\Gamma_1)$ be a normalized eigenform. Katsurada's conjecture states that all of the primes dividing certain special values of L-functions of f are congruence primes for the Ikeda lift $I_n(f)$ with respect to the space $S_\kappa^{\mathrm{Ik}}(\Gamma_n)^\perp$.

Conjecture 9 [Katsurada 2011, Conjecture A]. *Let $\kappa > n$ be integers and let $f = f_1, f_2, \dots, f_r \in S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ be a basis of normalized eigenforms. Assume $\ell \nmid (2\kappa-1)!$. Then $\mathfrak{l}$ is a congruence prime of $I_n(f)$ with respect to $S_\kappa^{\mathrm{Ik}}(\Gamma_n)^\perp$ if*

$$\mathfrak{l} \,\Big|\, L_{\mathrm{alg}}(\kappa, f) \prod_{i=1}^{\frac{n}{2}-1} L_{\mathrm{alg}}(2i+1, \mathrm{ad}^0 f).$$

As evidence for this conjecture, Katsurada proves the following theorem.

Theorem 10 [ibid., Theorem 4.7]. *Let $\mathcal{O}$, f, and $\mathfrak{l}$ be as in the conjecture with $\kappa > 2n+4$. Then $\mathfrak{l}$ is a congruence prime for $I_n(f)$ with respect to $S_\kappa^{\mathrm{Ik}}(\Gamma_n)^\perp$ if the following are satisfied*:

(1)
$$\mathfrak{l} \,\Big|\, L_{\mathrm{alg}}(\kappa, f) \prod_{i=1}^{\frac{n}{2}-1} L_{\mathrm{alg}}(2i+1, \mathrm{ad}^0 f).$$

(2) *For some integer m satisfying $\frac{n}{2} < m < \frac{\kappa}{2} - \frac{n}{2}$ and some fundamental discriminant D satisfying $(-1)^{\frac{n}{2}} D > 0$,*

$$\mathfrak{l} \nmid D(m-1)!\,\zeta_{\mathrm{alg}}(2m) L_{\mathrm{alg}}\Big(\kappa - \frac{n}{2}, \chi_D\Big) \prod_{i=1}^{n} L_{\mathrm{alg}}(2m+\kappa-i, f),$$

where $\zeta_{\mathrm{alg}}(2m) = \zeta(2m)/\pi^{2m}$.

(3) *For a constant $C_{\kappa,n} := \prod_{j \le (2\kappa-n)/12} (1 + j + \dots + j^{n-1})$ if $n > 2$ and $C_{\kappa,2} = 1$,*

$$\mathfrak{l} \nmid \frac{C_{\kappa,n}\langle f, f\rangle}{\Omega_f^+ \Omega_f^-}.$$

As noted by Katsurada, the second condition allows freedom to choose m, so it is reasonable to expect that one can find an m with $\mathfrak{l} \nmid \zeta_{\mathrm{alg}}(2m) \prod_{i=1}^{n} L_{\mathrm{alg}}(2m+\kappa-i, f)$ in many cases.

4.2. ***The Ikeda ideal: definition and bounds.*** The conjecture in the previous subsection gives conditions when one will have a congruence between an Ikeda lift $I_n(f)$ and a form in $S_\kappa^{\text{N-Ik}}(\Gamma_n)$. In this section we will introduce the Ikeda ideal associated to $I_n(f)$ that will capture this information as well. In fact, the ideal captures more information as it measures all congruences between $I_n(f)$ and forms in $S_\kappa^{\text{N-Ik}}(\Gamma_n)$.

Let f be a normalized eigenform in $S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ and $I_n(f)$ the Ikeda lift of f. Recall that the Hecke algebra over $\mathcal{O}$ acting on $S_\kappa(\Gamma_n)$ is denoted by $\mathcal{H}_{\mathcal{O}}^{(n)}$.

Let $X \subseteq S_\kappa^{\mathrm{Ik}}(\Gamma_n)$ be a Hecke-stable subspace and let Y be the orthogonal complement in $S_\kappa(\Gamma_n)$ to X under the Petersson product. In particular, the examples we will be interested in are when $X = \mathbb{C} I_n(f)$ or $X = S_\kappa^{\mathrm{Ik}}(\Gamma_n)$. Let $\mathcal{H}_{\mathcal{O}}^{(n),Y}$ denote the image of $\mathcal{H}_{\mathcal{O}}^{(n)}$ in $\mathrm{End}_{\mathbb{C}}(Y)$ and let $\phi : \mathcal{H}_{\mathcal{O}}^{(n)} \to \mathcal{H}_{\mathcal{O}}^{(n),Y}$ denote the natural surjection.

We let $\mathrm{Ann}(I_n(f))$ denote the annihilator of $I_n(f)$ in $\mathcal{H}_{\mathcal{O}}^{(n)}$. We have that $I_n(f)$ induces an $\mathcal{O}$-algebra homomorphism $\mathcal{H}_{\mathcal{O}}^{(n)} \to \mathcal{O}$ by sending a Hecke operator to its eigenvalue. Since this is an $\mathcal{O}$-algebra homomorphism it is surjective and it clearly has kernel $\mathrm{Ann}(I_n(f))$. Thus, there is an isomorphism

$$\mathcal{H}_{\mathcal{O}}^{(n)} / \mathrm{Ann}(I_n(f)) \cong \mathcal{O}.$$

Using that ϕ is surjective we have that $\phi(\mathrm{Ann}(I_n(f)))$ is an ideal in $\mathcal{H}_{\mathcal{O}}^{(n),Y}$. We refer to this ideal as the *Ikeda ideal associated to $I_n(f)$ with respect to Y* and denote it by $\mathcal{I}_n^Y(f)$. We will be interested in the index of this ideal. In particular, one has that there exists an integer m such that

$$\mathcal{H}_{\mathcal{O}}^{(n),Y} / \mathcal{I}_n^Y(f) \cong \mathcal{O}/\mathfrak{l}^m \mathcal{O}.$$

We give here two elementary propositions to relate this index to Katsurada's conjecture.

Proposition 11. *With the notation as above, if there exists $G \in Y$, not necessarily an eigenform, such that*

$$I_n(f) \equiv G \pmod{\mathfrak{l}^b},$$

then $m \geq b$.

Proof. Assume that $b > m$, and consider the diagram

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{O}}^{(n)} \arrow[r, "\phi"] \arrow[d] & \mathcal{H}_{\mathcal{O}}^{(n),Y} \arrow[d] \\
\mathcal{H}_{\mathcal{O}}^{(n)} / \mathrm{Ann}(I_n(f)) \arrow[r, "\phi"] \arrow[d, "\simeq"] & \mathcal{H}_{\mathcal{O}}^{(n),Y} / \mathcal{I}_n^Y(f) \arrow[d, "\simeq"] \\
\mathcal{O} \arrow[r] & \mathcal{O}/\mathfrak{l}^m \mathcal{O}
\end{tikzcd}$$

Each map here is an $\mathcal{O}$-algebra surjection. Let $t \in \phi^{-1}(\lambda^m) \subset \mathcal{H}_{\mathcal{O}}^{(n)}$. Then by definition we have $tG = \lambda^m G$. Moreover, by the commutativity of the diagram we see that $t \in \mathrm{Ann}(I_n(f))$, so the assumed congruence gives

$$\lambda^m G \equiv 0 \pmod{\mathfrak{l}^b},$$

i.e.,

$$G \equiv 0 \pmod{\mathfrak{l}^{b-m}}.$$

However, since we are assuming $b > m$, this gives

$$I_n(f) \equiv G \equiv 0 \pmod{\mathfrak{l}}.$$

This contradicts Proposition 7, and so it must be that $b \leq m$. □

Proposition 12. *With the notation as above, suppose $m \geq 1$. Then there exists an eigenform $G \in Y$ such that*

$$I_n(f) \equiv_{\mathrm{ev}} G \pmod{\mathfrak{l}}.$$

Proof. Extend K if necessary so that $I_n(f) \in S_\kappa(\Gamma_n; \mathcal{O})$ and we have an orthogonal basis $F_1, \dots, F_r$ of Y with each $F_i \in S_\kappa(\Gamma_n; \mathcal{O})$. Suppose that there are no eigenforms $G \in Y$ eigenvalue-congruent to $I_n(f)$.

Let $\mathcal{S}$ denote the $\mathbb{C}$-vector space spanned by $I_n(f), F_1, \dots, F_r$. Let $\mathcal{H}_{\mathcal{O}}^{(n),S}$ denote the image of the Hecke algebra $\mathcal{H}_{\mathcal{O}}^{(n)}$ in $\mathrm{End}_{\mathbb{C}}(\mathcal{S})$. For each eigenform $F \in \mathcal{S}$ with eigenvalues in $\mathcal{O}$ we obtain a maximal ideal $\mathfrak{m}_F$ of $\mathcal{H}_{\mathcal{O}}^{(n),S}$ given as the kernel of the map $\mathcal{H}_{\mathcal{O}}^{(n),S} \to \mathcal{O}/\mathfrak{l}\mathcal{O} : t \mapsto \lambda_F(t) \pmod{\mathfrak{l}}$. We have that eigenforms F and G are eigenvalue-congruent modulo $\mathfrak{l}$ if and only if $\mathfrak{m}_F = \mathfrak{m}_G$.

We now use the fact that $I_n(f)$ is not congruent to any of $F_1, \dots, F_r$ to conclude that

$$\mathcal{H}_{\mathcal{O}}^{(n),S} = \mathcal{H}_{\mathfrak{m}_{I_n(f)}}^{(n),S} \times \prod_{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}}^{(n),S},$$

where the product is over the maximal ideals corresponding to $F_1, \dots, F_r$. However, this gives that $\mathcal{I}_n^Y(f) = \prod_{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}}^{(n),S}$, and this is exactly $\mathcal{H}_{\mathcal{O}}^{(n),Y}$. This contradicts the assumption that $m \geq 1$. Thus, it must be that there is a congruence as desired. □

To match the previous results with Katsurada's, simply take $X = S_\kappa^{\mathrm{Ik}}(\Gamma_n)$ and $Y = S_\kappa^{\text{N-Ik}}(\Gamma_n)$. In fact, one has that the index of the Ikeda ideal measures all congruences between forms in Y and $I_n(f)$. This follows from Proposition 13. One should note that we use the fact that the space of Ikeda lifts satisfies multiplicity one [Ikeda 2013, Theorem 7.1] in order to apply this result.

Proposition 13 [Berger et al. ≥ 2015, Proposition 4.3]. *Let X and Y be as above and let $F_1, \dots, F_r$ be a basis of Y. For each $1 \leq i \leq r$, let m_i be the largest integer so that*

$$I_n(f) \equiv_{\mathrm{ev}} F_i \pmod{\mathfrak{l}^{m_i}}.$$

Then one has

$$\frac{1}{e} \sum_{i=1}^{r} m_i \geq \mathrm{val}_\lambda\big(\#\mathcal{H}_{\mathcal{O}}^{(n),Y} / \mathcal{I}_n^Y(f)\big),$$

where e is the ramification index of $\mathcal{O}$ over $\mathbb{Z}_\ell$.

Thus, one can view results giving a lower bound on the Ikeda ideal as a strengthening of the results of [Katsurada 2011], where one is only concerned with a congruence modulo $\mathfrak{l}$ to a single eigenform.

5. Main results

We now state the main result of this paper. The proof will be given in Section 7. After stating the theorem, we discuss the main hypotheses.

Theorem 14. *Let κ and n be positive even integers with $\kappa > n+1$ and let ℓ be a prime so that $\ell > 2\kappa - n$. Assume $\ell \nmid \prod_{p \le (2\kappa-n)/12}(1+p+\cdots+p^{n-1})$. Let $f \in S_{2\kappa-n}(\Gamma_1)$ be a newform and let $\mathcal{O}$ be a suitably large finite extension of $\mathbb{Z}_\ell$ that contains all the eigenvalues of f. Let $\mathfrak{l}$ denote the prime of $\mathcal{O}$. Furthermore, assume that $\bar{\rho}_{f,\ell}$ is irreducible and $\mathrm{val}_{\mathfrak{l}}(\langle f, f\rangle/(\Omega_f^+\Omega_f^-)) = 0$. We make these assumptions:*

(1) *There exists an integer $N > 1$ prime to ℓ and a Dirichlet character χ of conductor N with $\chi(-1) = (-1)^\kappa$ such that*

$$\mathrm{val}_{\mathfrak{l}}\Big(L^N(n-\kappa+1, \chi)\prod_{j=1}^{n} L^N_{\mathrm{alg}}(n+1-j, f, \chi)\Big) = 0.$$

(2) *There exists a fundamental discriminant D prime to ℓ such that $(-1)^{n/2}D > 0$, $\chi_D(-1) = -1$, and*

$$\mathrm{val}_{\mathfrak{l}}\Big(L_{\mathrm{alg}}\Big(\kappa - \frac{n}{2}, f, \chi_D\Big)\Big) = 0.$$

(3) *We have*

$$\mathrm{val}_{\mathfrak{l}}\Big(L_{\mathrm{alg}}(\kappa, f)\prod_{j=1}^{\frac{n}{2}-1} L_{\mathrm{alg}}(2j+1, \mathrm{ad}^0 f)\Big) = b > 0.$$

Then we have

$$\mathrm{val}_{\mathfrak{l}}\big(\#\mathcal{H}_{\mathcal{O}}^{(n),Y}/\mathcal{I}_n^Y(f)\big) \ge b,$$

where Y is the orthogonal complement of $X = \mathbb{C}I_n(f)$ in $S_\kappa(\Gamma_n)$.

Corollary 15. *With the same setup and assumptions as in Theorem 14, if $F_1, \dots, F_r$ is a basis of eigenforms of $S_\kappa^{\text{N-Ik}}(\Gamma_n; \mathcal{O})$ (where we enlarge $\mathcal{O}$ if necessary) and if we let m_i be the largest integer so that*

$$I_n(f) \equiv_{\mathrm{ev}} F_i \pmod{\mathfrak{l}^{m_i}},$$

then we have

$$\frac{1}{e}\sum_{i=1}^{r} m_i \ge b,$$

where e is the ramification index of $\mathcal{O}$ over $\mathbb{Z}_\ell$.

Proof. Let $F_1, \dots, F_r$ be a basis of eigenforms of $S_\kappa^{\text{N-Ik}}(\Gamma_n; \mathcal{O})$ and $F_{r+1}, \dots, F_s$, $I_n(f)$ a basis of $S_\kappa^{\text{Ik}}(\Gamma_n)$. For $i = 1, \dots, s$ let m_i be the largest integer so that

$$I_n(f) \equiv_{\text{ev}} F_i \pmod{\mathfrak{l}^{m_i}}.$$

Theorem 14 and Proposition 13 give

$$\frac{1}{e}\sum_{i=1}^{s} m_i \geq b.$$

However, we have $m_{r+1} = \cdots = m_s = 0$ as the assumption $\text{val}_\mathfrak{l}(\langle f, f\rangle/(\Omega_f^+\Omega_f^-)) = 0$ guarantees that there are no eigenvalue congruences between $I_n(f)$ and other Ikeda lifts by the proof of [Katsurada 2011, Theorem 4.7]. Thus, we obtain the result. □

We first discuss the hypotheses that $\text{val}_\mathfrak{l}(\langle f, f\rangle/(\Omega_f^+\Omega_f^-)) = 0$. This condition is equivalent to assuming that there are no other normalized eigenforms $g \in S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ that are eigenvalue-equivalent to f modulo $\mathfrak{l}$. One can see [Hida 1981; Ribet 1983] for further discussion. For a particular f this condition can be easily checked [Bosma et al. 1997; Stein et al. 2013].

The two hypotheses we focus on are the ones concerning the $\mathfrak{l}$-indivisibility of L-values. We begin with the assumption that $\text{val}_\mathfrak{l}\big(L_{\text{alg}}\big(\kappa - \frac{n}{2}, f, \chi_D\big)\big) = 0$. Note this is a central critical value since the weight of f is $2\kappa - n$. There have been several results on the $\mathfrak{l}$-divisibility of this particular special value due to its relation with the Fourier coefficients of the half-integral weight modular form $\theta_{\kappa,n}^{\text{alg}}(f)$. For example, Corollary 3 of [Bruinier and Ono 2003] shows that for nonexceptional primes ℓ there is a period Ω of f with the property that for infinitely many fundamental discriminants D prime to ℓ with $(-1)^{n/2}D > 0$ one has

$$\text{ord}_\mathfrak{l}\left(\frac{D^{\kappa-\frac{n}{2}-\frac{1}{2}} L_{\text{alg}}\big(\kappa - \frac{n}{2}, f, \chi_D\big)}{\Omega}\right) = 0.$$

Since we assume $\bar{\rho}_{f,\ell}$ is irreducible, ℓ is automatically a nonexceptional prime for f [Swinnerton-Dyer 1973, Corollary 2]. However, we are unable to apply this result in our situation as the period Ω used is not the canonical period Ω_f^+ that we are using to normalize the L-value. We are unaware of any period relation between Ω and Ω_f^+. However, this does reduce the consideration to another period ratio; and since we have already assumed that $\mathfrak{l}$ does not divide a period ratio, this assumption is a reasonable one as well.

We next consider $L(n - \kappa + 1, \chi)$. Let p be a prime with $p \neq \ell$, $m \geq 1$ and φ be a Dirichlet character. In this setting Washington [1978] proves that for all but finitely many Dirichlet characters ψ of p-power conductor with $\varphi\psi(-1) = (-1)^m$,

$$\text{val}_\mathfrak{l}(L(1 - m, \varphi\psi)/2) = 0.$$

In our setup we can take $m = \kappa - n$, $\chi = \varphi\psi$, and observe that $\chi(-1) = (-1)^\kappa = (-1)^{\kappa-n}$ to see there are infinitely many χ so that

$$\mathrm{val}_{\mathfrak{l}}(L(n-\kappa+1, \chi)) = 0.$$

If this were the only L-value controlled by χ we would be able to remove the hypothesis regarding this L-value. However, we also require that

$$\mathrm{val}_{\mathfrak{l}}\left(\prod_{j=1}^{n} L_{\mathrm{alg}}^{N}(n+1-j, f, \chi)\right) = 0.$$

This means that we must choose a χ so that all of these L-values are simultaneously $\mathfrak{l}$-adic units. This is a much more delicate issue. We note here that we have a great deal of freedom in choosing such a χ, namely, the only restrictions concern the parity of χ and that its conductor be prime to ℓ. Thus, we have infinitely many characters to choose from so it is reasonable to expect that one can often find such a χ. In the case $n = 2$, i.e., when one considers Saito–Kurokawa lifts, one can find computational evidence supporting the existence of such a χ in [Agarwal and Brown 2013]. One can use the same methods to produce computational evidence for $n > 2$.

6. Siegel Eisenstein series

In this section we recall the definition of a Siegel Eisenstein series associated to a character. Following Shimura we then make a suitable choice of a section so that the Fourier coefficients of the Eisenstein series can be computed. Finally, we consider the pullback of our Siegel Eisenstein series and recall an inner product formula of Shimura. Throughout this section we assume that κ and n are even integers with $\kappa > n+1$.

6.1. ***Siegel Eisenstein series — general setup.*** Let P_n be the Siegel parabolic subgroup of G_n given by $P_n = \{g \in G_n : c_g = 0\}$. We have that P_n factors as $P_n = N_{P_n} M_{P_n}$, where N_{P_n} is the unipotent radical

$$N_{P_n} = \left\{ n(x) = \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix} : {}^t x = x, x \in \mathrm{Mat}_n \right\}$$

and M_{P_n} is the Levi subgroup

$$M_{P_n} = \left\{ \begin{pmatrix} A & 0_n \\ 0_n & \alpha({}^t A)^{-1} \end{pmatrix} : A \in \mathrm{GL}_n, \alpha \in \mathrm{GL}_1 \right\}.$$

Let $\mathbb{A}$ denote the rational adeles. Fix an idele class character χ and consider the induced representation

$$I(\chi) = \mathrm{Ind}_{P_n(\mathbb{A})}^{G_n(\mathbb{A})}(\chi) = \bigotimes_{v} I_v(\chi_v)$$

consisting of smooth functions $\mathfrak{f}$ on $G_n(\mathbb{A})$ that satisfy

$$\mathfrak{f}(pg) = \chi(\det(A_p))\mathfrak{f}(g) \quad \text{for } p = \begin{pmatrix} A_p & B_p \\ 0 & D_p \end{pmatrix} \in P_n(\mathbb{A}),\ g \in G_n(\mathbb{A}).$$

For $s \in \mathbb{C}$ and $\mathfrak{f} \in I(\chi)$ define

$$\mathfrak{f}(pg, s) = \chi(\det(A_p))|\det(A_p D_p^{-1})|^s \mathfrak{f}(g).$$

For υ a place of $\mathbb{Q}$ we define $I_\upsilon(\chi_\upsilon)$ and $\mathfrak{f}_\upsilon(pg, s)$ analogously. We associate to such a section the Siegel Eisenstein series

$$E_{\mathbb{A}}(g, s; \mathfrak{f}) = \sum_{\gamma \in P_n(\mathbb{Q})\backslash G_n(\mathbb{Q})} \mathfrak{f}(\gamma g, s). \tag{3}$$

Observe that $E_{\mathbb{A}}(g, s; \mathfrak{f})$ converges absolutely and uniformly for (g, s) on compact subsets of $G_n(\mathbb{A}) \times \{s \in \mathbb{C} : \mathrm{Re}(s) > (n+1)/2\}$. One can see [Shimura 1997, Appendix A.3] for this fact. Moreover, (3) defines an automorphic form on $G_n(\mathbb{A})$ and a holomorphic function on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$ with meromorphic continuation to $\mathbb{C}$ with at most finitely many poles. Furthermore, Langlands [1976] gives a functional equation for $E_{\mathbb{A}}(g, s; \mathfrak{f})$ relating the value at $(n+1)/2-s$ to the value at s.

6.2. ***A choice of section.*** For our applications we need to restrict the possible χ and pick a particular section $\mathfrak{f}$. Let $N > 1$ be an integer.

Let $\chi = \bigotimes_\upsilon \chi_\upsilon$ be an idele class character of $\mathbb{Q}$ that satisfies

$$\chi_\infty(x) = \left(\frac{x}{|x|}\right)^\kappa,$$
$$\chi_p(x) = 1 \quad \text{if } p \nmid \infty,\ x \in \mathbb{Z}_p^\times, \text{ and } x \equiv 1 \pmod{N}.$$

For each finite prime p, we set

$$K_{0,p}^{(n)}(N) = \{g \in G_n(\mathbb{Q}_p) : a_g, b_g, d_g \in \mathrm{Mat}_n(\mathbb{Z}_p), c_g \in \mathrm{Mat}_n(N\mathbb{Z}_p)\}.$$

From this definition it is immediate that if $p \nmid N$ we have

$$K_{0,p}^{(n)}(N) = G_n(\mathbb{Q}_p) \cap \mathrm{Mat}_{2n}(\mathbb{Z}_p).$$

At the infinite place we put

$$K_\infty^{(n)} = \{g \in \mathrm{Sp}_{2n}(\mathbb{R}) : g(i_n) = i_n\}.$$

Set

$$K_0^{(n)}(N) = \prod_p K_{0,p}^{(n)}(N).$$

We choose our section $\mathfrak{f} = \bigotimes_\upsilon \mathfrak{f}_\upsilon$ as follows.

(1) We set $\mathfrak{f}_\infty$ to be the unique vector in $I_\infty(\chi_\infty, s)$ so that

$$\mathfrak{f}_\infty(k,\kappa) = j(k,i)^{-\kappa}$$

for all $k \in K_\infty^{(n)}$.

(2) If $p \nmid N$ we set $\mathfrak{f}_p$ to be the unique $K_{0,p}^{(n)}(N)$-fixed vector so that

$$\mathfrak{f}_p(1) = 1.$$

(3) If $p \mid N$ we set $\mathfrak{f}_p$ to be the vector given by

$$\mathfrak{f}_p(k) = \chi_p(\det(a_k)) \quad \text{for all } k \in K_{0,p}^{(n)}(N),\ k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$$

and

$$\mathfrak{f}_p(g) = 0 \quad \text{for all } g \notin P_n(\mathbb{Q}_p) K_{0,p}^{(n)}(N).$$

The Eisenstein series $E_{\mathbb{A}}$ is the same as in [Shimura 1995; 1997].

Define

$$\Lambda_n^N(s,\chi) = L^N(2s,\chi) \prod_{i=1}^{[\frac{n}{2}]} L^N(4s-2i, \chi^2)$$

and normalize $E_{\mathbb{A}}$ by setting

$$\mathbf{E}_{\mathbb{A}}(g,s;\mathfrak{f}) = \pi^{-n(n+2)/4} \Lambda_n^N(s,\chi) E_{\mathbb{A}}(gJ_n^{-1}, s; \mathfrak{f}).$$

Set

$$G_\kappa^n(z;\mathfrak{f}) = \mathbf{E}_{\mathbb{A}}\left(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}, \frac{n+1}{2} - \frac{\kappa}{2}; \mathfrak{f} \right). \tag{4}$$

We have that $G_\kappa^n(z;\mathfrak{f})$ is a Siegel modular form of weight κ and level $\Gamma_0^{(n)}(N)$ [Shimura 1983], where

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n : C \equiv 0 \pmod N \right\}.$$

Write

$$G_\kappa^n(z;\mathfrak{f}) = \sum_{T \in \Lambda_n} a(T;\mathfrak{f}) e(\mathrm{Tr}(Tz)).$$

The Fourier coefficients $a(T;\mathfrak{f})$ are well known for this particular choice of section and normalization [Shimura 1997, Chapters 18 and 19]. In particular:

Theorem 16 [Brown 2007, Theorem 4.4]. *Let $\ell \geq n+1$ be an odd prime with $\ell \nmid N$. Then*

$$G_\kappa^n(z;\mathfrak{f}) \in M_\kappa\big(\Gamma_0^{(n)}(N); \mathbb{Z}_\ell\big[\chi, \sqrt{-1}^{n\kappa}\big]\big).$$

6.3. ***Pullbacks of Siegel Eisenstein series.*** Let $N > 1$ be an integer and $\ell > n+1$ a prime with $\ell \nmid N$.

Consider the diagonal embedding of $\mathfrak{h}^n \times \mathfrak{h}^n$ into $\mathfrak{h}^{2n}$ via the map

$$(z, w) \mapsto \operatorname{diag}[z, w] = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}.$$

We also have an embedding of $\Gamma_n \times \Gamma_n$ into Γ_{2n} given by

$$\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

This allows us to view the natural action of $\Gamma_n \times \Gamma_n$ on $\mathfrak{h}^n \times \mathfrak{h}^n$ as a restriction of the action of Γ_{2n} on $\mathfrak{h}^{2n}$.

We will be interested in the restriction of the Eisenstein series $G_\kappa^{2n}(Z; \mathfrak{f})$ to $\mathfrak{h}^n \times \mathfrak{h}^n$. We refer to such a restriction as a pullback. These pullbacks have been considered in [Garrett 1984; Böcherer 1985; Garrett 1992; Shimura 1995; 1997]. In general, if F is a modular form of degree $2n$, level $\Gamma_0^{(2n)}(N)$, and weight κ, then the restriction of F to $\mathfrak{h}^n \times \mathfrak{h}^n$ is a modular form of degree n, level $\Gamma_0^{(n)}(N)$, and weight κ when considered as a function of z or w.

Shimura calculates the following set of representatives for $P_{2n}\backslash G_{2n}/(G_n \times G_n)$.

Lemma 17 [Shimura 1995, Lemma 4.2]. *For $0 \le r \le n$ let τ_r denote the element of G_{2n} given by*

$$\tau_r = \begin{pmatrix} 1_{2n} & 0 \\ \rho_r & 1_{2n} \end{pmatrix}, \quad \rho_r = \begin{pmatrix} 0_n & e_r \\ {}^t e_r & 0_n \end{pmatrix}, \quad e_r = \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the τ_r form a complete set of representatives for $P_{2n}\backslash G_{2n}/(G_n \times G_n)$.

We will make use of τ_n. Let $F \in S_\kappa(\Gamma_n)$ be an eigenform. We can specialize [ibid., Equation (6.17)] to obtain

$$(5) \quad \langle (G_\kappa^{2n} \mid \tau_n)(\operatorname{diag}[z, w]; \mathfrak{f}), F^c(w) \rangle \\ = \mathcal{A}_{\kappa,n,N} \pi^{-n(n+1)/2} L(n+1-\kappa, F, \chi; \mathrm{st}) F(z),$$

where we have used $F \mid J_n = F$ since F has level Γ_n, and

$$\mathcal{A}_{\kappa,n,N} = \frac{2^{n(2\kappa-3n+2)/2}}{[\Gamma_n : \Gamma_0^{(n)}(N)]} \prod_{j=0}^{n-1} \frac{\Gamma((n-j)/2)}{\Gamma((2n+1-j)/2)}.$$

Since it will be important in the next section, we note that since $G_\kappa^{2n}(z; \mathfrak{f}) \in M_\kappa(\Gamma_0^{(2n)}(N); \mathbb{Z}_\ell[\chi])$, we have $(G_\kappa^{2n} \mid \tau_n)(z; \mathfrak{f}) \in M_\kappa(\tau_n^{-1}\Gamma_0^{(2n)}(N)\tau_n; \mathbb{Z}_\ell[\chi])$ by the q-expansion principle for Siegel modular forms [Chai and Faltings 1990, Proposition 1.5]. The Fourier expansion of $(G_\kappa^{2n} \mid \tau_n)(\mathrm{diag}[z, w]; \mathfrak{f})$ can be written as

$$(G_\kappa^{2n} \mid \tau_n)(\mathrm{diag}[z, w]; \mathfrak{f}) = \sum_{T_1, T_2 \in \Lambda_n} \left(\sum_{T \in \Lambda_{2n}(T_1, T_2)} a(T; G_\kappa^{2n} \mid \tau_n) \right) e(\mathrm{Tr}(T_1 z)) e(\mathrm{Tr}(T_2 w)),$$

where $a(T; G_\kappa^{2n} \mid \tau_n)$ is the T-th Fourier coefficient of $G_\kappa^{2n} \mid \tau_n$, and for $T_1, T_2 \in \Lambda_n$ we define

$$\Lambda_{2n}(T_1, T_2) = \left\{ T \in \Lambda_{2n} : T = \begin{pmatrix} T_1 & b \\ b & T_2 \end{pmatrix} \right\}.$$

This immediately gives that the Fourier coefficients of $(G_\kappa^{2n} \mid \tau_n)(\mathrm{diag}[z, w]; \mathfrak{f})$ lie in $\mathbb{Z}_\ell[\chi]$ as well.

7. Constructing a congruence

In this section we prove Theorem 14. We work under the hypotheses listed after the theorem. We again let $\mathcal{O}$ be a suitably large finite extension of $\mathbb{Z}_\ell$ with prime $\mathfrak{l}$ and uniformizer λ.

Our first step in constructing the congruence is to replace the Eisenstein series $(G_\kappa^{2n} \mid \tau_n)(\mathrm{diag}[z, w]; \mathfrak{f})$ with a form of level $\Gamma_n \times \Gamma_n$. To do this, we take the trace

$$\widetilde{G}_\kappa^{2n}(\mathrm{diag}[z, w]; \mathfrak{f}) = \sum_{\gamma_1, \gamma_2} (G_\kappa^{2n} \mid \tau_n)(\mathrm{diag}[z, w]; \mathfrak{f}) \mid (\gamma_1 \times \gamma_2)$$

where the sum is over $(\Gamma_n \times \Gamma_n)/(\tau_n^{-1} \Gamma_0^{(n)}(N) \tau_n \times \tau_n^{-1} \Gamma_0^{(n)}(N) \tau_n)$. We note again that this has Fourier coefficients in $\mathbb{Z}_\ell[\chi]$ by the q-expansion principle. Moreover, we know that $\widetilde{G}_\kappa^{2n}$ is a cusp form in each variable via [Brown 2011, Section 3.2].

Let $F_0 = I_n(f), F_1, \dots, F_r$ be an orthogonal basis of eigenforms for $S_\kappa(\Gamma_n)$. Note that $F_0^c, \dots, F_r^c$ is also an orthogonal basis of eigenforms for $S_\kappa(\Gamma_n)$. Applying [Shimura 1995, Equation (7.7)] we may write

$$\widetilde{G}_\kappa^{2n}(\mathrm{diag}[z, w]; \mathfrak{f}) = \sum_{\substack{0 \le i \le r \\ 0 \le j \le r}} c_{i,j} F_i(z) F_j^c(w)$$

for some $c_{i,j} \in \mathbb{C}$. Furthermore, from [Brown 2011, Lemma 5.1] we can rewrite

$$\widetilde{G}_\kappa^{2n}(\mathrm{diag}[z, w]; \mathfrak{f}) = c_0 I_n(f)(z) I_n(f)(w) + \sum_{0 < j \le r} c_j F_j(z) F_j^c(w), \tag{6}$$

where we write $c_j = c_{j,j}$ and we have used that since $f^c = f$, Corollary 6 gives $I_n(f)^c = I_n(f)$.

We now turn our attention to the constant c_0. Our goal is to show that we can write c_0 as a product of an element of $\mathcal{O}^\times$ and λ^{-m} for some $m > 0$.

Consider the inner product $\langle \widetilde{G}_\kappa^{2n}(\mathrm{diag}[z, w]; \mathfrak{f}), I_n(f)(w)\rangle$. Note that

$$\langle \widetilde{G}_\kappa^{2n}(\mathrm{diag}[z, w]; \mathfrak{f}), I_n(f)(w)\rangle = \langle (G_\kappa^{2n} \mid \tau_n)(\mathrm{diag}[z, w]; \mathfrak{f}), I_n(f)(w)\rangle,$$

where we view the forms on the left-hand side as being level Γ_n and on the right-hand side as being level $\tau_n^{-1}\Gamma_0^{(n)}(N)\tau_n$. Taking the inner product of both sides of (6) with $I_n(f)(w)$, applying (5), and solving for c_0 we obtain

$$c_0 = \frac{\mathcal{A}_{k,n,N} L^N(n-\kappa+1, I_n(f), \chi; \mathrm{st})}{\pi^{n(n+1)/2}\langle I_n(f), I_n(f)\rangle}.$$

Ikeda [2006] made a conjecture relating $\langle I_n(f), I_n(f)\rangle$ to $\langle f, f\rangle$. We have the following theorem, which proves Ikeda's conjecture assuming n is even. We rephrase their result to suit our purposes.

Theorem 18 [Katsurada and Kawamura 2013, Theorem 2.3]. *Let κ be a positive even integer and let $\ell > n+1$ be a prime. Let $f \in S_{2\kappa-n}(\Gamma_1; \mathcal{O})$ be a newform with $\mathcal{O}$ a suitably large finite extension of $\mathbb{Z}_\ell$. Assume $\mathrm{val}_{\mathfrak{l}}(\langle f, f\rangle/(\Omega_f^+\Omega_f^-)) = 0$. Let D be a fundamental discriminant such that $(-1)^{n/2}D > 0$, $\chi_D(-1) = -1$, and assume $\ell \nmid D$. Then if $I_n(f)$ is the Ikeda lift of f as given above, we have*

$$\frac{\langle I_n(f), I_n(f)\rangle}{\langle f, f\rangle^{n/2}} = u_1 \cdot \frac{\Gamma(\kappa)\prod_{j=1}^{\frac{n}{2}-1}\Gamma(2j+2\kappa-n)|c(|D|)|^2\prod_{j=1}^{\frac{n}{2}}\zeta_{\mathrm{alg}}(2j)}{\Gamma\big(\kappa-\frac{n}{2}\big)} \times \frac{L_{\mathrm{alg}}(\kappa, f)\prod_{j=1}^{\frac{n}{2}-1}L_{\mathrm{alg}}(2j+1, \mathrm{ad}^0 f)}{L_{\mathrm{alg}}\big(\kappa-\frac{n}{2}, f, \chi_D\big)},$$

where $\mathrm{val}_{\mathfrak{l}}(u_1) = 0$, $c(|D|)$ is the $|D|$-th Fourier coefficient of $\theta_{\kappa,n}^{\mathrm{alg}}(f)$ from above and we have used the assumption on $\langle f, f\rangle/(\Omega_f^+\Omega_f^-)$ to normalize the adjoint L-function to our conventions.

We now apply this result to remove the period $\langle I_n(f), I_n(f)\rangle$ in our expression for c_0 to obtain

$$c_0 = \frac{\mathcal{B}_{\kappa,n}}{|c(|D|)|^2}\cdot\frac{L^N(n-\kappa+1, I_n(f), \chi; \mathrm{st})L_{\mathrm{alg}}\big(\kappa-\frac{n}{2}, f, \chi_D\big)}{\pi^{\frac{n(n+1)}{2}}\langle f, f\rangle^{\frac{n}{2}}\zeta_{\mathrm{alg}}(n)\prod_{i=1}^{\frac{n}{2}-1}\zeta_{\mathrm{alg}}(2i)L_{\mathrm{alg}}(2i+1, \mathrm{ad}^0 f)L_{\mathrm{alg}}(\kappa, f)},$$

where

$$\mathcal{B}_{\kappa,n} = u_2 \cdot \frac{\Gamma\big(\kappa-\frac{n}{2}\big)\prod_{j=1}^{n-1}\Gamma\big(\frac{n-j}{2}\big)}{[\Gamma_n : \Gamma_0^{(n)}(N)]\Gamma(k)\prod_{j=1}^{n-1}\Gamma\big(\frac{2n+1-j}{2}\big)\prod_{j=1}^{\frac{n}{2}-1}\Gamma(2i+2k-n)},$$

where u_2 satisfies $\mathrm{val}_{\mathfrak{l}}(u_2) = 0$.

The following factorization is a direct consequence of Theorem 4:

$$L^N(n-k+1, I_n(f), \chi; \mathrm{st}) = L^N(n-k+1, \chi)\prod_{i=1}^{n} L^N(n+1-i, f, \chi). \tag{7}$$

Applying the assumption that $\mathrm{val}_{\mathfrak{l}}(\langle f, f\rangle/(\Omega_f^+\Omega_f^-)) = 0$, we can replace $\langle f, f\rangle^{n/2}$ by $u_3(\Omega_f^+\Omega_f^-)^{n/2}$ for u_3 an $\mathfrak{l}$-adic unit. Furthermore, note that if $\Omega_f^{\pm}$ is the period associated to $L(n+1-i, f, \chi)$ as in Theorem 1, then $\Omega_f^{\mp}$ is the period associated to $L(n+1-(i+1), f, \chi)$. Using this, we can rewrite our expression for c_0 as

$$c_0 = u_4 \cdot \mathcal{B}_{\kappa,n} \cdot \mathcal{C}_{D,n,\chi} \cdot \mathcal{L}_{f,\chi,D},$$

where u_4 is a $\mathfrak{l}$-adic unit, $\mathcal{B}_{\kappa,n}$ is defined as above,

$$\mathcal{C}_{D,n,\chi} = \frac{1}{|c_h(|D|)|^2 \prod_{i=1}^{\frac{n}{2}} \zeta_{\mathrm{alg}}(2i)},$$

and

$$\mathcal{L}_{f,\chi,\chi_D} = \frac{L^N(n-\kappa+1, \chi) L_{\mathrm{alg}}\big(\kappa - \frac{n}{2}, f, \chi_D\big) \prod_{j=1}^{n} L_{\mathrm{alg}}^N(n+1-j, f, \chi)}{L_{\mathrm{alg}}(\kappa, f) \prod_{j=1}^{\frac{n}{2}-1} L_{\mathrm{alg}}(2j+2, \mathrm{ad}^0 f)}.$$

Note that it is shown in [Brown 2007, Section 4.2] that $L^N(n-k+1, \chi) \in \mathbb{Z}_\ell[\chi]$. As $\mathcal{B}_{\kappa,n}$, $\mathcal{C}_{D,n,\chi}$, and $\mathcal{L}_{f,\chi,D}$ are algebraic, we may consider the $\mathfrak{l}$-divisibility of c_0. First, using that n is even and $\ell > n+1$ we have $\mathrm{val}_{\mathfrak{l}}(\mathcal{B}_{k,n}) \leq 0$.

Next we turn our attention to $\mathcal{C}_{D,n,\chi}$. Our choice of $\theta_{\kappa,n}^{\mathrm{alg}}(f)$ given in Section 3 gives that $|c(|D|)| \in \mathcal{O}$, and so $\mathrm{val}_{\mathfrak{l}}(|c(|D|)|^2) \geq 0$. Consider $\zeta_{\mathrm{alg}}(2j)$ for some $1 \leq j \leq \frac{n}{2}$. It is an immediate consequence of the Von Staudt–Clausen Theorem (see for example [Ireland and Rosen 1990, p. 233]) that $\zeta_{\mathrm{alg}}(2j)$ is in $\mathcal{O}$, and hence $\mathrm{val}_{\mathfrak{l}}(\zeta_{\mathrm{alg}}(2j)) \geq 0$. Thus, we have $\mathrm{val}_{\mathfrak{l}}(\mathcal{C}_{D,n,\chi}) \leq 0$.

By assumption we have $\mathrm{val}_{\mathfrak{l}}(\mathcal{L}_{f,\chi,\chi_D}) < 0$, so under our assumptions we have $\mathrm{val}_{\mathfrak{l}}(c_0) < 0$. We now show how this gives the desired congruence. Write $c_0 = \alpha\lambda^{-b'}$ for some $b' > 0$ and α an $\mathfrak{l}$-adic unit. Using this, we may rewrite (6) as

$$\widetilde{G}_\kappa^{2n}(\mathrm{diag}[z, w]; \mathfrak{f}) = \alpha\lambda^{-b'} I_n(f)(z) I_n(f)(w) + \sum_{0<j\leq r} c_j F_j(z) F_j^c(w). \tag{8}$$

Note that by Proposition 7 there is a T_0 so that $\mathrm{val}_{\mathfrak{l}}(a_{I_n(f)}(T_0)) = 0$. We expand (8) in terms of z and equate the T_0-th Fourier coefficients to obtain

$$\sum_{T_2\in\Lambda_n}\Bigg(\sum_{T\in\Lambda_{2n}(T_0,T_2)} a(T, G_\kappa^{2n} \mid \tau_n)\Bigg) e(\mathrm{Tr}(T_2 w))$$
$$= \alpha\lambda^{-b'} a_{I_n(f)}(T_0) I_n(f)(w) + \sum_{0<j\leq r} c_j a_{F_j}(T_0) F_j^c(w).$$

Multiply the equation by $\lambda^{b'}$ and recall that $a(T, G_\kappa^{2n}|\tau_n) \in \mathcal{O}$ for all T to see that

$$I_n(f)(w) \equiv -\frac{\lambda^{b'}}{\alpha a_{I_n(f)}(T_0)} \sum_{0<j\leq r} c_j a_{F_j}(T_0) F_j^c(w) \pmod{\mathfrak{l}^{b'}}.$$

Note that since $a_{I_n(f)}(T_0)$ is a $\mathfrak{l}$-adic unit, the form on the right-hand side of the congruence cannot be zero modulo $\mathfrak{l}^{b'}$, i.e., we have constructed a nontrivial congruence. Set

$$G(w) = -\frac{\lambda^{b'}}{\alpha a_{I_n(f)}(T_0)} \sum_{0<j\leq r} c_j a_{I_n(f)}(T_0) F_j(w).$$

We now return to the setting of Ikeda ideals. Let $X = \mathbb{C}I_n(f)$ and $Y = (\mathbb{C}I_n(f))^{\perp}$, where the notation follows that given in Section 4.2. We have constructed a congruence

$$I_n(f) \equiv G \pmod{\mathfrak{l}^{b'}}$$

for some $b' \geq 1$ and $G \in Y$. Note that it is clear from above that

$$b' \geq \mathrm{val}_{\mathfrak{l}}\Biggl(L_{\mathrm{alg}}(\kappa, f) \prod_{j=1}^{\frac{n}{2}-1} L_{\mathrm{alg}}(2j+1, \mathrm{ad}^0 f)\Biggr),$$

which is what we labeled b in the statement of Theorem 14. Thus, applying Proposition 11 concludes the proof of Theorem 14.

One thing to note here is that we do not obtain a lower bound of b' for the index in the Hecke algebra of the Ikeda ideal with respect to $X = S_\kappa^{\mathrm{Ik}}(\Gamma_n)$ and $Y = S_\kappa^{\text{N-Ik}}(\Gamma_n)$. The reason for this is that while we know $I_n(f)$ cannot be eigenvalue-congruent to any other Ikeda lifts, that does not imply that $G \in S_\kappa^{\text{N-Ik}}(\Gamma_n)$. One can use the fact that $I_n(f)$ is not congruent to any other Ikeda lifts along with (1) to conclude there is an idempotent t in the Hecke algebra $\mathcal{H}_{\mathcal{O}}^{(n)}$ that satisfies

$$tF = \begin{cases} 0 & \text{if } F \not\equiv_{\mathrm{ev}} I_n(f) \pmod{\mathfrak{l}}, \\ F & \text{if } F \equiv_{\mathrm{ev}} I_n(f) \pmod{\mathfrak{l}}. \end{cases}$$

If one acts on G by this Hecke operator one obtains a form tG in $S_\kappa^{\text{N-Ik}}(\Gamma_n)$ with $tG \equiv_{\mathrm{ev}} I_n(f) \pmod{\mathfrak{l}}$. Thus, one only obtains a lower bound of 1 for the Ikeda ideal with respect to $X = S_\kappa^{\mathrm{Ik}}(\Gamma_n)$ and $Y = S_\kappa^{\text{N-Ik}}(\Gamma_n)$. While one would like to have a stronger bound for this Ikeda ideal, Corollary 15 shows that it is not necessary for our results.

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