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We construct a closed Riemannian manifold (N, h) and a sequence of α-harmonic maps from S^2 into N with uniformly bounded energy such that the energy identity for this sequence is not true.

1. Introduction

Let (Σ, g) be a Riemann surface and (N, h) be an n-dimensional smooth compact Riemannian manifold which is embedded in $\mathbb{R}^K$. Usually, we denote the space of Sobolev maps from Σ into N by $W^{k,p}(\Sigma, N)$, which is defined by

$$W^{k,p}(\Sigma, N) = \{u \in W^{k,p}(\Sigma, \mathbb{R}^K) : u(x) \in N \ \text{ for a.e. } x \in \Sigma\}.$$

For $u \in W^{1,2}(\Sigma, N)$, we define locally the energy density $e(u)$ of u at $x \in \Sigma$ by

$$e(u)(x) = |\nabla_g u|^2 = g^{ij}(x) h_{\alpha\beta}(u(x)) \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.$$

The energy of u on Σ, denoted by $E(u)$ or $E(u, \Sigma)$, is defined by

$$E(u) = \frac{1}{2} \int_\Sigma e(u)\, dV_g,$$

and the critical points of E are called harmonic maps. We know that a harmonic map u satisfies

$$\tau(u) = \Delta u + A(u)(\nabla u, \nabla u) = 0,$$

where A is the second fundamental form of N in $\mathbb{R}^K$. Harmonic maps are related very closely to minimal surface. It is well known that a harmonic map from S^2 into N must be a branched conformal immersion in N.

Unfortunately, E does not satisfy the Palais–Smale condition. From the viewpoint of calculus of variation, it is difficult to show the existence of harmonic maps from a

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surface. In order to obtain harmonic maps, Sacks and Uhlenbeck [1981] introduced the so-called α-energy E_α, instead of L^2 energy E, as

$$E_\alpha(u) = \frac{1}{2}\int_\Sigma \{(1+|\nabla u|^2)^\alpha - 1\}\, dV_g,$$

where we always assume that $\alpha > 1$. It is well known that this α-energy functional E_α satisfies the Palais–Smale condition. The critical points of E_α in $W^{1,2\alpha}(\Sigma, N)$, called α-harmonic maps, satisfy

$$\Delta_g u_\alpha + (\alpha-1)\frac{\nabla_g|\nabla_g u_\alpha|^2 \nabla_g u_\alpha}{1+|\nabla_g u_\alpha|^2} + A(u_\alpha)(du_\alpha, du_\alpha) = 0. \tag{1-1}$$

The strategy of Sacks and Uhlenbeck is to employ a sequence of α-harmonic maps to approximate a harmonic map as α tends to 1. Hence, to show the existence of harmonic maps we need to study the convergence behavior of a sequence of α-harmonic maps u_α with $E_\alpha(u_\alpha) < C$ from a compact surface (Σ, g) into a compact Riemannian manifold (N, h) without boundary. Generally, such a sequence converges weakly to a harmonic map in $W^{1,2}(\Sigma, N)$ and strongly in C^∞ away from a finite set of points in Σ.

Concretely, let $\{u_{\alpha_k}\}$ be a sequence of α-harmonic maps from Σ into N with uniformly bounded α-energy, that is, $E_{\alpha_k}(u_{\alpha_k}) < \Lambda < \infty$. We assume that the sequence does not converge smoothly on Σ. By the theory of Sacks and Uhlenbeck, there exists a subsequence of $\{u_{\alpha_k}\}$, still denoted by $\{u_{\alpha_k}\}$, and a finite set $\mathcal{S} \subset \Sigma$ such that the subsequence converges to a harmonic map u_0 in $C^\infty_{\mathrm{loc}}(\Sigma \setminus \mathcal{S})$. We know that, at each point $p_i \in \mathcal{S}$, the energy of the subsequence concentrates and the blowup phenomena occurs. Moreover, there exist point sequences $\{x^l_{i_k}\}$ in Σ with $\lim_{k\to+\infty} x^l_{i_k} = p_i$ and scaling constant number sequences $\{\lambda^l_{i_k}\}$ with $\lim_{k\to+\infty} \lambda^l_{i_k} \to 0$, $l = 1, \dots, n_0$, such that

$$u_{\alpha_k}(x^l_{i_k} + \lambda^l_{i_k} x) \to v^l \quad \text{in } C^j_{\mathrm{loc}}(\mathbb{R}^2 \setminus \mathcal{A}^i),$$

where all v^i are nontrivial harmonic maps from S^2 into N, and $\mathcal{A}^i \subset \mathbb{R}^2$ is a finite set.

In order to explore and describe the asymptotic behavior of $\{u_{\alpha_k}\}$ at each blowup point, the following two problems arise naturally. The first is whether or not the energy identity holds true:

$$\lim_{\alpha_k\to 1} E_{\alpha_k}(u_{\alpha_k}, B^\Sigma_{r_0}(p_i)) = E(u_0, B^\Sigma_{r_0}(p_i)) + \sum_{l=1}^{n_0} E(v^l).$$

Here, $B^\Sigma_{r_0}(p_i)$ is a geodesic ball in Σ which contains only one blowup point p_i. The other is whether or not the necks connecting bubbles are some geodesics of finite length?

Considerable progress has been made regarding these problems; let us now recall some main results on them. Chen and Tian [1999] considered a special sequence $\{u_{\alpha_k}\}$ with uniformly bounded α-energy, for which every u_{α_k} is a minimizing α_k-harmonic map and all maps u_{α_k} belong to a fixed homotopy class. They studied the convergence behavior of such a special sequence and provided a proof on the above energy identity. Later, for the same sequence, Li and Wang [2010a] gave another constructing proof on the energy identity, which is completely different from that given in [Chen and Tian 1999].

The energy identity for a minimax sequence of α-harmonic maps has also been considered. Suppose that A is a parameter manifold. Let $h_0 : \Sigma \times A \to N$ be a continuous map, and H be such a set of continuous maps $h : \Sigma \times A \to N$ that every $h \in H$ is homotopic to h_0 and satisfies $h(t) \in W^{1,2\alpha}(\Sigma, N)$ for any fixed $t \in A$. Set

$$\beta_\alpha(H) = \inf_{h \in H} \sup_{t \in A} E_\alpha(h(\cdot\,, t)).$$

It is known that there is at least a sequence $\{u_{\alpha_k}\}$, each u_{α_k} of which attains $\beta_{\alpha_k}(H)$, satisfies the energy identity as $\alpha_k \to 1$. For more details, we refer to [Jost 1991; Lamm 2010].

On the other hand, it should be pointed out that some effective methods have been established to successfully prove the energy identity and give a detailed description of the connecting necks for the heat flow of harmonic maps from a Riemann surface, or more generally, a sequence of maps from a Riemann surface with tension fields τ bounded in the sense of L^2 [Ding 1998; Ding and Tian 1995; Qing 1995; Qing and Tian 1997].

Recently, Li and Wang [2010b] studied the above problems on the sequences of α-harmonic maps and obtained some results which can be summarized as follows. If the energy concentration phenomena appears for $\{u_{\alpha_k}\}$, one can prove a weak energy identity and a direct convergence relation between the blowup radius and the parameter α, which ensures the energy identity and no-neck property. Li and Wang also showed that the necks converge to some geodesics and gave a length formula for the neck in the case where only one bubble appears.

Motivated by an example given by Duzaar and Kuwert [1998], Li and Wang [2010b] also constructed an α-harmonic map sequence with uniformly bounded energy, for which the blowup phenomenon occurs and there exists at least a neck (geodesic) of infinite length. This answers negatively the second problem on α-harmonic map sequence.

Although some mathematicians think that the energy identity for the sequence of α-harmonic maps should also be true, up to now it has been unclear in general whether the energy identity for an α-harmonic map sequence with bounded energy holds true or not. In this short paper, we will modify the construction in [Li and Wang 2010b] to show that the energy identity is also not true.

On the other hand, a natural problem is whether the set of the values of energy for harmonic spheres in any given Riemannian manifold (N, h) is discrete or not, since the bubbles produced in the convergence of a sequence of α-harmonic maps from (Σ, g) are always harmonic spheres.

We denote this set by

$$\mathscr{E}(N, h) = \{E(u) : u \text{ is a harmonic map from } S^2 \text{ into } (N, h)\}.$$

It is well known that if (N, h) is the standard sphere S^2, we have

$$\mathscr{E}(N, h) = \{4k\pi : k = 0, 1, \dots, n, \dots\}.$$

We also know from [Valli 1988] that if (N, h) is the unitary group $U(n)$ with the standard metric, then the energy of harmonic maps $S^2 \to U(n)$ can take as values only integral multiples of 8π. Some other energy gap phenomena on unitons were discussed in [Anand 1995; Dong 2002; Uhlenbeck 1989]. Some mathematicians conjectured that $\mathscr{E}(N, h)$ is a discrete set. Here, we will also give a counterexample to show that $\mathscr{E}(N, h)$ is not discrete.

2. α-harmonic maps

Later, we will discuss the convergence behavior of some α-harmonic map sequences with uniformly bounded α-energy or L^2 energy. In fact, by discussing the convergence of α-harmonic map sequences, Sacks and Uhlenbeck developed an existence theory on minimal surfaces in [Sacks and Uhlenbeck 1981; 1982]. In particular, they established the well-known ϵ-regularity theorem on α-harmonic maps and removal singularity theorem on harmonic maps [1981], which will be used repeatedly in the present paper.

Theorem 2.1. *Let $D = D_1(0) = \{z : |z| < 1\} \subset \mathbb{C}$ be a disk with radius 1 and N be a Riemannian manifold. Assume that $u : D \to N$ satisfies Equation (1-1). Then there exists $\epsilon_0 > 0$ and $\alpha_0 > 1$ such that if $E(u, D) < \epsilon_0$ and $1 \le \alpha \le \alpha_0$, then we have*

$$\|\nabla^k u\|_{L^\infty(D_{1/2})} \le C(k) E(u, D).$$

Theorem 2.2. *Assume that $u : D \setminus \{0\} \to N$ is a harmonic map with $E(u) < +\infty$. Then u is a harmonic map from D into N.*

The above theorem tells us that, if u is a harmonic map from $\mathbb{C} \setminus \{p_i \in \mathbb{C} : i = 1, 2, \dots, l < \infty\}$ into N with $E(u) < +\infty$, then u can be viewed as a harmonic map from S^2 into N.

Now, we can state more precisely the energy concentration of $\{u_{\alpha_k}\}$. Let $B_t^{\Sigma}(x)$ denote the geodesic ball of Σ which is centered at x and has geodesic radius t. By Theorem 2.1, the finite singular set of $\{u_{\alpha_k}\}$ can be defined precisely by

$$\mathcal{S} = \left\{ x \in \Sigma : \lim_{t \to 0} \varliminf_{k \to +\infty} \int_{B_t^{\Sigma}(x)} |\nabla u_{\alpha_k}|^2 \geq \frac{\epsilon_0}{2} \right\}.$$

For any $\tilde{x}_0 \notin \mathcal{S}$, there exists $\delta > 0$ such that $E(u_{\alpha_k}, B_\delta^{\Sigma}(\tilde{x}_0)) < \epsilon_0$. Applying Theorem 2.1, $\{u_{\alpha_k}\}$ converges smoothly on any $\Omega \Subset \Sigma \setminus \mathcal{S}$. The limit map is a harmonic map from $\Sigma \setminus \mathcal{S}$ into N. Theorem 2.2 tells us that the singular points of the limit map can be removed, in other words, it is a harmonic map from Σ into N.

If $x_0 \in \mathcal{S}$, it is easy to check that

$$\|\nabla u_{\alpha_k}\|_{C^0(B_t^{\Sigma}(x_0))} \to +\infty$$

for any t. Choose $x_{\alpha_k} \in B_\delta^{\Sigma}(x_0)$ such that

$$|\nabla u_{\alpha_k}(x_{\alpha_k})| = \max_{B_\delta^{\Sigma}(x_0)} |\nabla u_{\alpha_k}|,$$

and let

$$\lambda_{\alpha_k} = \frac{1}{\max_{B_\delta^{\Sigma}(x_0)} |\nabla u_{\alpha_k}|}.$$

It is easy to see that $x_{\alpha_k} \to x_0$ as $k \to \infty$. Then, in an isothermal coordinate system around x_0, we may define

$$v_k(x) = u_{\alpha_k}(x_{\alpha_k} + \lambda_{\alpha_k} x).$$

It is well known that v_k converges in $C^\infty(D_R)$ to a harmonic map $v^1 : \mathbb{C} \to N$ for any fixed R, where $D_R = D_R(0) = \{z : |z| < R\} \subset \mathbb{C}$ is a disk with radius $R > 0$. We can regard v^1 as a harmonic map from S^2 into N. Usually, v^1 is called the first bubble. For the details on getting all the bubbles we refer to the appendix of [Li and Wang 2010b]. Moreover, in [Li and Wang 2010a] (see also [Chen and Tian 1999; Hong and Yin 2010]) we prove the following theorem which will be used later.

Theorem 2.3. *Let (Σ, g) be a closed Riemann surface and N a compact Riemannian manifold. Suppose that H is a fixed homotopy class of maps from Σ into N and u_α is a minimizer of E_α in the set $W^{1,2\alpha}(\Sigma, N) \cap H$. Then when $\alpha \to 1$ there exists a subsequence $\{u_\alpha\}$ and harmonic map u_0 such that $\{u_\alpha\}$ converges to u_0 weakly in $W^{1,2}(\Sigma, N)$ and blows up at finitely many points $\{p_i : i = 1, 2, \dots, m\}$. Moreover, associated with each $\{p_i\}$ there exist finitely many harmonic maps w_{i_j} from S^2 into N, $j = 1, 2, \dots, i_0$, such that*

$$\lim_{\alpha \to 1} E_\alpha(u_\alpha) = E(u_0) + \sum_{i=1}^{m} \sum_{j=1}^{i_0} E(w_{i_j}).$$

3. Construction of the counterexample

3A. ***Constructing the manifold* (N, h).** Let h_1 be the standard metric on

$$Y_1 = \mathbb{T}^3 = S^1 \times S^1 \times S^1 = \mathbb{R}^3/2\pi\mathbb{Z} \oplus 2\pi\mathbb{Z} \oplus 2\pi\mathbb{Z}.$$

Let $B_r(p)$ denote a geodesic ball in $\mathbb{T}^3$ with radius r and center p. Fix a point $p \in Y_1$, and set

$$X_1 = \mathbb{T}^3 \setminus B_r(p),$$

where $r < \pi/(4\sqrt{3}+2)$. It is easy to see that the injective radius of Y_1 at p is π and $B_\pi(p) \setminus B_r(p)$ is isometric to

$$\mathbb{T}_0 = \big(S^2 \times (-\log \pi, -\log r], e^{-2t}(d\mathfrak{s}^2 + dt^2)\big),$$

where $g_\mathfrak{s} = d\mathfrak{s}^2$ is the standard metric over S^2. It is also easy to check that $\mathbb{T}_0$ is isometric to

$$\mathbb{T}_0' = \Big(S^2 \times \Big[0, \log \frac{\pi}{r}\Big), e^{2t+2\log r}(d\mathfrak{s}^2 + dt^2)\Big)$$

and

$$\mathbb{T}_0'' = \Big(S^2 \times \Big(-\log \frac{\pi}{r}, 0\Big], e^{-2t+2\log r}(d\mathfrak{s}^2 + dt^2)\Big).$$

Let $(X_2, h_2) = (X_1, h_1)$. We consider the quotient space of $X_1 \cup X_2$, obtained by gluing every point $x \in \partial X_1$ with the same point $x \in \partial X_2$ together. In this way, we get a closed compact manifold N and a projection map $\phi : X_1 \cup X_2 \to N$. We set

$$M = \phi(\partial B_r(p)).$$

On $N \setminus M$, the metric $h_0 = (\phi^{-1})^*(h_1) \cup (\phi^{-1})^*(h_2)$ is well defined and can be extended to a metric g_0 over N. However, g_0 is not smooth and need to be modified. Obviously, M has a neighborhood which is isometric to

$$T = \Big(S^2 \times \Big(-\log \frac{\pi}{r}, \log \frac{\pi}{r}\Big), e^{2|t|+2\log r}(d\mathfrak{s}^2 + dt^2)\Big).$$

In fact, T is obtained by gluing $\mathbb{T}_0'$ and $\mathbb{T}_0''$ along $S^2 \times \{0\}$.

We let ψ be a smooth function defined on $\big(-\log \frac{\pi}{r}, \log \frac{\pi}{r}\big)$ which satisfies

(1) $\psi = e^{2|t|+2\log r}$ when $|t| \geq \log 2$;

(2) $\psi' < 0$ on $(-\log 2, 0)$ and $\psi' > 0$ on $(0, \log 2)$.

Note that (2) implies that 0 is the only critical point of ψ on $(-\log 2, \log 2)$.

We define a new metric h on N which is h_0 on $N \setminus T$, and $\psi(t)(d\mathfrak{s}^2 + dt^2)$ on T. It is easy to see that h is smooth on N. For convenience, we set

$$Q(a) = S^2 \times \Big(-\log \frac{a}{r}, \log \frac{a}{r}\Big) \subset T.$$

Obviously, we have

$$\phi^{-1}(Q(a)) \cap X_1 = B_a(p) \setminus B_r(p) \subset Y_1.$$

Lemma 3.1. *Let (N, h), T and $Q(a)$ be defined as above. Assume that $u : S^2 \to (N, h)$ is a nontrivial harmonic map with $u(S^2) \subset Q(\pi) = T$. Then u is a harmonic map from S^2 into M.*

Proof. Let $u = (v, f) : S^2 \to Q(\pi)$ be a harmonic map, where $v \in C^\infty(S^2, S^2)$ and $f \in C^\infty(S^2)$. The energy can be written as

$$E(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2 \, dV = \frac{1}{2}\int_{S^2} \big(|\nabla v|^2 + |\nabla f|^2\big)\psi(f)\, dV.$$

Here $dV = dV_{g_{\mathfrak{s}}}$ is the standard volume form of S^2. By a direct calculation, it is easy to see that u satisfies

$$\begin{aligned} -\nabla(\psi(f)\nabla v) + \psi(f)|\nabla v|^2 v = 0,\\ -\nabla(\psi(f)\nabla f) + \tfrac{1}{2}\big(|\nabla v|^2 + |\nabla f|^2\big)\psi'(f) = 0. \end{aligned} \tag{3-1}$$

Multiplying both sides of the second equation of (3-1) by f and then integrating the obtained identity over S^2, we get the identity

$$\int_{S^2} \Big(|\nabla f|^2 \psi(f) + \tfrac{1}{2}\big(|\nabla v|^2 + |\nabla f|^2\big)\psi'(f) f\Big)\, dV = 0.$$

Noting that $\psi'(f) f \geq 0$ always holds true, we infer from the above identity

$$\int_{S^2} |\nabla f|^2 \psi(f)\, d\mathfrak{s} = \frac{1}{2}\int_{S^2} \big(|\nabla v|^2 + |\nabla f|^2\big)\psi'(f) f \, dV = 0.$$

This implies that $\nabla f = 0$ and f is a constant. Moreover, from the above identity we also have

$$|\nabla v|^2 \psi' f \equiv 0.$$

Since u is nontrivial by assumption, there always exists a point $x_1 \in S^2$ such that $|\nabla v|(x_1) \neq 0$. Hence we conclude that $\psi'(f) f \equiv 0$ which implies $f \equiv 0$. It follows that v is a harmonic map from S^2 into M. □

Lemma 3.2. *Let (N, h) and Q be the same as in Lemma 3.1. Assume that u is a harmonic map from S^2 into (N, h) such that $u(S^2) \cap Q(2r) \neq \varnothing$ and $u(S^2) \cap \partial Q(\pi) \neq \varnothing$. Then we have*

$$E(u) \geq \pi(\pi - 2r)^2.$$

Proof. Without loss of generality, we assume $p_1 \in X_1$ is such that p_1 is in $\partial B_\pi(p)$ in Y_1 and $\phi(p_1)$ is in $u(S^2)$. First, u is a branched minimal surface since u is a harmonic map from S^2 into N. On the other hand, as h is flat on $\phi(B_{\pi-2r}(p_1))$, it is easy to check that $u(S^2)\cap\phi(B_{\pi-2r}(p_1))$ is a stationary varifold. Denote by $\mu(u(S^2)\cap B_{\pi-2r}(p_1))$ the area of $u(S^2)\cap B_{\pi-2r}(p_1)$. By the monotonicity inequality for stationary varifolds (see [Simon 1983]), we have

$$\frac{\mu(u(S^2)\cap B_{\pi-2r}(p_1))}{\pi(\pi-2r)^2}\geq 1.$$

In light of this inequality and the fact $E(u)\geq\mu(u(S^2)\cap B_{\pi-2r}(p_1))$, we derive the desired inequality

$$E(u)\geq\pi(\pi-2r)^2;$$

and the proof is complete. □

Since h is flat on $N\setminus Q(2r)$, we have the following lemma.

Lemma 3.3. *Let (N,h) and Q be the same as in Lemma 3.1. Then there is no nontrivial harmonic map $u: S^2\to(N,h)$ such that $u(S^2)\cap\overline{Q(2r)}=\varnothing$.*

By the definition of ψ, it is easy to check that

$$4\pi\psi(0)\leq 16\pi r^2<\tfrac{1}{3}\pi(\pi-2r)^2$$

when r is small enough. Using Lemma 3.2 and Lemma 3.3, we get the following result.

Corollary 3.4. *Let (N,h) and Q be the same as in Lemma 3.1. Assume that u is a nontrivial harmonic map with $E(u)<\pi(\pi-2r)^2$; then*

$$E(u)=4m\pi\psi(0)$$

where m is a positive integer.

It is easy to check that

$$12\pi\psi(0)<48\pi r^2<\pi(\pi-2r)^2,$$

if $r<\frac{\pi}{4\sqrt{3}+2}$. Therefore we know that if $E(u)<12\pi\psi(0)$ and u is a nontrivial harmonic map, then $E(u)=4\pi\psi(0)$ or $8\pi\psi(0)$.

3B. ***The homotopy class $[u_k]$.*** We have $\pi_1(Y_1)=\pi_1(\mathbb{T}^3)=\mathbb{Z}^3$. Let $\beta\in\pi_1(Y_1)$ which represents $(1,0,0)$. Let $x_1,x_2\in M$, and γ_0 be a curve in M such that $\gamma_0(0)=x_2$, and $\gamma_0(1)=x_1$. Let $\gamma_k:[0,1]\to X$ be a curve with $\gamma_k(0)=x_1,\gamma_k(1)=x_2$ and $[\gamma_k+\gamma_0]=k\beta$. Let w_0 be a diffeomorphism from S^2 onto M satisfying $w_0(0,0,1)=x_1$ and $w_0(0,0,-1)=x_2$, where $(0,0,1)$ and $(0,0,-1)$ are the north and the south poles of $S^2\subset\mathbb{R}^3$, respectively.

For the sake of convenience, we introduce the stereographic projection coordinates on S^2 with the south pole corresponding to ∞. Thus, $w_0 : S^2 \to N$ can be viewed as a map from $\mathbb{C}\cup\{\infty\}$ into N. For simplicity, we neglect the stereographic projection map $\mathfrak{S} : S^2 \to \mathbb{C}\cup\{\infty\}$ and still denote $w_0 \circ \mathfrak{S}^{-1}$ by w_0.

By the continuity of w_0, there exists a small $\delta_0 > 0$ such that $w_0(D_{\delta_0})$ is contained in a small neighborhood of x_1, where $D_{\delta_0} = \{z \in \mathbb{C} : |z| < \delta_0\}$, and a large $R_0 > 0$ such that $w_0(\mathbb{C}\setminus D_{R_0})$ is contained in a small neighborhood of x_2, where $D_{R_0} = \{z \in \mathbb{C} : |z| < R_0\}$.

In order to construct a sequence of maps, we need to define the following two smooth nonnegative functions λ and ν on $[0,\infty)$:

(1) $\lambda(s) : [0,\infty] \to [0,1]$ with $\lambda(s) \equiv 0$ as $s \in [0,\delta_0]$ and $\lambda(s) \equiv 1$ as $s \in [2\delta_0,\infty)$.

(2) $\nu(s) : [0,\infty) \to [0,1]$ with $\nu(s) \equiv 1$ as $s \in [0, R_0 - R_0^c]$ and $\nu(s) \equiv 0$ as $s > R_0$, where R_0^c is a small positive constant number.

Now we define a sequence of maps $u_k : S^2 \to N$ by

$$u_k = \begin{cases} w_0(\lambda(|z|)z) & |z| \geq \delta_0, \\ \gamma_k\left(\dfrac{\log|z| - \log R_0\epsilon_0}{\log\delta_0 - \log R_0\epsilon_0}\right) & R_0\epsilon_0 < |z| < \delta_0, \\ w_0\left(\dfrac{z}{\nu(|z|/\epsilon_0)\epsilon_0}\right) & |z| \leq \epsilon_0 R_0. \end{cases}$$

Here $\epsilon_0 > 0$ is a fixed constant number such that $R_0\epsilon_0 < \delta_0$. By the arguments in [Li and Wang 2010b], for any $i \neq j$, u_i is not homotopic to u_j. For the sequence $\{u_i\}$ constructed above, we have the following lemma:

Lemma 3.5. *Let u_k be the maps from S^2 into (N,h) constructed above and $[u_k]$ denote the class of maps in $W^{1,2}(S^2,N)\cap C(S^2,N)$, each map of which is homotopic to u_k. For any fixed k, we have*

$$\inf_{u\in[u_k]} E(u) = 8\pi\psi(0).$$

Moreover, $\inf E(u)$ *cannot be attained by a harmonic map belonging to* $[u_k]$.

Proof. First of all, we prove that for every fixed k

$$\inf_{u\in[u_k]} E(u) \leq 8\pi\psi(0). \tag{3-2}$$

Denote $z_1 = (0,0,1)$ and $z_2 = (0,0,-1) \in S^2$. Without loss of generality, we assume w_0 is a harmonic map from S^2 into M with $E(w_0) = 4\pi\psi(0)$ with $w_0(z_1) = x_1$ and $w_0(z_2) = x_2$. Let $\mathfrak{S}$ be the stereographic projection from $S^2\backslash\{z_2\}$ to $\mathbb{C}$ and

$$\hat{u}_0(z) = w_0(\mathfrak{S}^{-1}(z)) : \mathbb{C}\cup\{\infty\} \to N.$$

Choose a coordinate system (y_1, y_2, y_3) in a geodesic ball $B_\rho(x_1)$ around $x_1 \in N$ with $x_1 = (0,0,0)$ and $\{(y_1, y_2, 0) : (y_1, y_2, 0) \in B_\rho(x_1)\} \subset M$. By the continuity of w_0, there exists a small $\delta > 0$ such that $w_0(z) \in B_\rho(x_1)$ when $|z| < \delta$. We define

$$u_0' = \eta_1 \hat{u}_0,$$

where η_1 is a smooth nonnegative function which equals 1 outside $D_{2\delta}$, 0 on D_δ, and satisfies $|\nabla \eta_1| < \frac{C}{\delta}$. Here $D_{2\delta} \subset \mathbb{C}$ denotes the disk centered at the origin. Then we have

$$\int_{D_{2\delta}} |\nabla u_0'|^2 \, dx^2 \le 2 \int_{D_{2\delta}} \big(|\nabla \eta_1|^2 |\hat{u}_0|^2 + |\nabla \hat{u}_0|^2 \big) \, dx^2 \le C\delta.$$

Thus u_0' satisfies

$$\operatorname{dist}^M(u_0', \hat{u}_0) < C\delta, \quad E(u_0') < 4\pi\psi(0) + C\delta, \quad \text{and} \quad u_0'(D_\delta) = x_1.$$

Since E is conformally invariant, $\hat{u}_0(1/z)$ is also a harmonic map from $\mathbb{C} \setminus \{0\}$ into N with

$$E(\hat{u}_0(1/z), \mathbb{C}) = E(\hat{u}_0(z), \mathbb{C}).$$

Thus, $\hat{u}_0(1/z)$ can be extended smoothly to $\{0\}$. Choose a coordinate system (y_1, y_2, y_3) in a geodesic ball $B_\rho(x_2)$ around $x_2 \in N$ with $x_2 = (0,0,0)$ and $\{(y_1, y_2, 0) : (y_1, y_2, 0) \in B_\rho(x_2)\} \subset M$. By the continuity of w_0, there exists a large $R > 0$ such that $\hat{u}_0(z) \in B_\rho(x_2)$ as $|z| > R$. Then we have

$$\hat{u}_0(1/z) = O(z) \quad \text{and} \quad |\nabla \hat{u}_0(1/z)| = O(1), \quad \text{as } z \to 0.$$

Hence, we have

$$\hat{u}_0(z) = O(1/z) \quad \text{and} \quad |z^2 \nabla \hat{u}_0(z)| = O(1), \quad \text{as } z \to \infty.$$

Let

$$u_0''(z) = \eta_2(|z|) \hat{u}_0(z),$$

where $\eta_2(|z|)$ is a smooth nonnegative function which equals 0 outside D_R, 1 on $D_{R/2}$, and satisfies $|\nabla \eta_1| < \frac{C}{R}$. Then we have

$$\int_{\mathbb{C} \setminus D_{R/2}} |\nabla u_0''|^2 \, dx^2 \le 2 \int_{D_R \setminus D_{R/2}} \big(|\nabla \eta_2|^2 |\hat{u}_0|^2 + |\nabla \hat{u}_0|^2 \big) \, dx^2 \le \frac{C}{R}.$$

Thus

$$\operatorname{dist}^M(u_0'', u_0) < \frac{C}{R}, \quad E(u_0'') < 4\pi\psi(0) + \frac{C}{R}, \quad \text{and} \quad u_0''(\mathbb{C} \setminus D_R) = x_2.$$

We define

$$\phi_k = \begin{cases} u_0'(z), & |z| \geq \delta, \\ \gamma_k\left(\dfrac{\log|z| - \log R\epsilon}{\log\delta - \log R\epsilon}\right), & R\epsilon < |z| < \delta, \\ u_0''\left(\dfrac{z}{\epsilon}\right), & |z| \leq \epsilon R. \end{cases}$$

By a direct calculation, we obtain

$$\int_{D_\delta \setminus D_{R\epsilon}} |\nabla\phi_k|^2 = 2\pi \int_{R\epsilon}^{\delta} \left|\frac{\partial\gamma_k}{\partial r}\right|^2 r\,dr$$
$$< \frac{c\|\dot\gamma_k\|_{L^\infty}^2}{(-\log R\epsilon + \log\delta)^2} \int_{R\epsilon}^{\delta} \frac{dr}{r} = \frac{c\|\dot\gamma_k\|_{L^\infty}^2}{\log\delta - \log R\epsilon}.$$

Thus, for any $\epsilon_1 > 0$, we can choose suitable δ, R and ϵ such that

$$E(\phi_k) < 8\pi\psi(0) + \epsilon_1.$$

Obviously, $\varphi_k = \phi_k(\mathfrak{S}^{-1})$ is homotopic to u_k, denoted by $\varphi_k \sim u_k$. Thus, we get (3-2).

Next, we prove that $\inf_{u\in[u_k]} E(u)$ cannot be attained by a harmonic map. Assume it is attained by a harmonic map v_0. Recall that

$$8\pi\psi(0) < 12\pi\psi(0) < 48\pi r^2 < \pi(\pi - 2r)^2,$$

where $r > 0$ is small enough. By Lemma 3.2, $v_0(S^2) \subset Q(\pi)$. Thus v_0 is a harmonic map from S^2 into M. This contradicts the fact $v_0 \sim u_k$. Hence $\inf_{u\in[u_k]} E(u)$ cannot be attained by a harmonic map.

Let u_α be the α-harmonic map such that, for fixed k,

$$E_\alpha(u_\alpha) = \inf_{u\in[u_k]\cap W^{1,2\alpha}(S^2,N)} E_\alpha(u).$$

Then each map of $\{u_\alpha\}$ is minimizing and belongs to $[u_k]$. We claim that $\{u_\alpha\}$ does not converge smoothly. Otherwise, the limit map is a harmonic map from S^2 into N, which is homotopic to u_k. This contradicts the above fact that $\inf_{u\in[u_k]} E(u)$ cannot be attained by a harmonic map. Hence, the bubbles must appear in the convergence of u_α. If we denote the weak limit of $\{u_\alpha\}$ as u_0 and the bubbles as $v^1, \dots, v^m$, then, by Theorem 2.3, we have

$$\inf_{u\in[u_k]} E(u) = \lim_{\alpha\to 1} E_\alpha(u_\alpha) = E(u_0) + \sum_{i=1}^{m} E(v^i).$$

Since $E(u_0)$ and $E(v^i)$ are smaller than $\pi(\pi - 2r)^2$, $E(u_0) + \sum_{i=1}^{m} E(v^i)$ can only equal $8\pi\psi(0)$ or $4\pi\psi(0)$.

Next, we will show that the following identity does not hold true:

$$E(u_0) + \sum_{i=1}^{m} E(v^i) = 4\pi\psi(0).$$

If we assume this is true, then u_0 is trivial and u_α has only one bubble v^1. To derive a contradiction, we only need to prove $u_\alpha \sim v^1$.

Let $x_0 \in S^2$ be a blowup point. Take an isothermal coordinate system around x_0 with $x_0 = (0,0)$ on $S^2 = \mathbb{C} \cup \{\infty\}$. Let v^1 be the limit map of $u_\alpha(z_\alpha + \lambda_\alpha^1 z)$, where $z_\alpha \to 0$, $\lambda_\alpha^1 \to 0$. Then

$$v_\alpha^1(z) = u_\alpha(z_\alpha + \lambda_\alpha^1 z)$$

converges smoothly to v^1 on any $D_R = D_R(0) \subset \mathbb{C}$. Moreover, u_α converges smoothly in $\mathbb{C} \cup \{\infty\} \setminus D_{1/R}$ to a point $y_0 \in N$. For us to prove $u_\alpha \sim v^1$, it is enough to check that for any $\epsilon > 0$, there exists an $R > 0$, such that

$$\sup_{t \in [R\lambda_\alpha^1, 1/R]} \operatorname{osc}_{\partial D_t(z_\alpha)} u_\alpha < \epsilon.$$

Indeed, if this is not true then there exists a sequence of λ_α^2 with $\lambda_\alpha^2 \to 0$ and $\lambda_\alpha^2/\lambda_\alpha^1 \to +\infty$, such that

$$\operatorname{osc}_{\partial D_{\lambda_\alpha^2}(z_\alpha)} u_\alpha \to \epsilon_1 \neq 0.$$

Let

$$v_\alpha^2(z) = u_\alpha(z_\alpha + \lambda_\alpha^2 z).$$

If the sequence $\{v_\alpha^2\}$ has blowup points, then at each blowup point there exists at least a bubble of $\{v_\alpha^2\}$ which is also a bubble of u_α and is different from the previous bubble v^1. However, this is impossible since there only exists one bubble for $\{u_\alpha\}$. Hence, we infer that as $\alpha \to 1$, $\{v_\alpha^2\}$ converges smoothly on $D_{R'} \setminus D_{1/R'} \subset \mathbb{C}$ for any R'. It follows that

$$\operatorname{osc}_{\partial D_1} v_\alpha^2 \to \epsilon_1 \neq 0.$$

This means that the limit map of $\{v_\alpha^2\}$ is not trivial and the limit map is also a bubble of $\{u_\alpha\}$ which is different from v^1. This is a contradiction. Thus, we conclude

$$\inf_{u \in [u_k]} E(u) = E(u_0) + \sum_{i=1}^{m} E(v^i) = 8\pi\psi(0).$$

This completes the proof of the lemma. □

By the Sobolev embedding theorem, we know that for $\alpha > 1$,

$$W^{1,2\alpha}(S^2, N) \subset C(S^2, N).$$

For simplicity, let $[u_k]^\alpha$ denote the class of maps belonging to $W^{1,2\alpha}(S^2, N)$, each map of which is homotopic to u_k. In fact, it is easy to see that

$$[u_k]^\alpha = [u_k] \cap W^{1,2\alpha}(S^2, N).$$

From now on, we will always denote the smooth map which attains $\inf_{u\in[u_k]^\alpha} E_\alpha(u)$ by $u_{\alpha,k}$:

$$E_\alpha(u_{\alpha,k}) = \inf_{u\in[u_k]^\alpha} E_\alpha(u).$$

Lemma 3.6. *For any $\lambda_0 > 8\pi\psi(0)$, there exists a sequence $\{\alpha_k\}$ with $\alpha_k \to 1$ and a sequence $\{i_k\}$ such that $E_{\alpha_k}(u_{\alpha_k,i_k}) = \lambda_0$ for every k.*

Proof. For $\alpha \in [1, \alpha_0)$ where $\alpha_0 - 1 > 0$ is small enough, we define the following function

$$\varphi_k(\alpha) = \inf_{u\in[u_k]^\alpha} E_\alpha(u).$$

Firstly, we need to show that for any fixed $\alpha \in (1, \alpha_0)$,

$$\lim_{k\to+\infty} \varphi_k(\alpha) = +\infty. \tag{3-3}$$

If this is false, then there exists a constant C such that $\varphi_k(\alpha) \le C$ as k is large enough. We note that for any small δ and $x \in S^2$,

(3-4)
$$E(u_{\alpha,k}, B_\delta(x)) = \frac{1}{2}\int_{B_\delta(x)} |\nabla u_{\alpha,k}|^2 \le \frac{1}{2}\left(\int_{B_\delta(x)} |\nabla u_{\alpha,k}|^{2\alpha}\right)^{1/\alpha} |B_\delta(x)|^{(\alpha-1)/\alpha}.$$

Hence, we can pick a fixed δ, which is small enough, such that

$$E(u_{\alpha,k}, B_\delta(x)) < \epsilon_0.$$

Thus, by Theorem 2.1, there exists a subsequence of $u_{\alpha,k}$ which converges smoothly to a smooth map u_0 as k tends to ∞. Hence, we know that $u_{\alpha,k}$ are homotopic to u_0 for any k. This contradicts the fact that $u_{\alpha,i}$ is not homotopic to $u_{\alpha,j}$ as $i \neq j$.

Next, we want to prove φ_k is continuous on $[1, \alpha_0)$. Using (3-4) again, we can prove that, for a fixed small $\epsilon > 0$,

$$\|\nabla u_{\alpha,k}\|_{C^0(S^2)} < \Lambda(\epsilon)$$

for any $\alpha \in (1+\epsilon, \alpha_0)$. For any $\alpha, \alpha' \in (1+\epsilon, \alpha_0)$, we have

$$\begin{aligned}\varphi_k(\alpha) &\ge \tfrac{1}{2}(1+C_1^2)^{\alpha-\alpha'}\int_{S^2}(1+|\nabla u_{\alpha,k}|^2)^{\alpha'} - \tfrac{1}{2}\\ &\ge (1+C_1^2)^{\alpha-\alpha'}\varphi_k(\alpha') + \tfrac{1}{2}(1+C_1^2)^{\alpha-\alpha'} - \tfrac{1}{2},\end{aligned}$$

where

$$C_1 = \begin{cases} 0 & \text{when } \alpha > \alpha', \\ \Lambda(\epsilon) & \text{when } \alpha < \alpha'. \end{cases}$$

It follows that

$$\varliminf_{\alpha \to \alpha'} \varphi_k(\alpha) \geq \varphi_k(\alpha').$$

On the other hand, we also have

$$\varphi_k(\alpha') \geq \tfrac{1}{2}(1 + C_2^2)^{\alpha' - \alpha} \int_{S^2} (1 + |\nabla u_{\alpha',k}|^2)^{\alpha} - \tfrac{1}{2},$$

where

$$C_2 = \begin{cases} 0 & \text{when } \alpha' > \alpha, \\ \|\nabla u_{\alpha' k}\|_{L^\infty} & \text{when } \alpha' < \alpha. \end{cases}$$

It follows that

$$\varphi_k(\alpha') \geq (1 + C_2^2)^{\alpha' - \alpha} \varphi_k(\alpha) + \tfrac{1}{2}(1 + C_2^2)^{\alpha' - \alpha} - \tfrac{1}{2},$$

and

$$\varlimsup_{\alpha \to \alpha'} \varphi_k(\alpha) \leq \varphi_k(\alpha').$$

Therefore, we have

$$\lim_{\alpha \to \alpha'} \varphi_k(\alpha) = \varphi_k(\alpha'),$$

and we have shown the continuity of $\varphi_k(\alpha)$ on $(1, \alpha_0)$.

Next, we want to prove that $\varphi_k(\alpha)$ is left continuous at 1. Equivalently, we need to show

$$\lim_{\alpha \searrow 1} \varphi_k(\alpha) = \varphi_k(1). \tag{3-5}$$

Obviously, for any fixed $u \in W^{1,2}(S^2, N)$ and $\alpha_1 > \alpha_2 > 1$,

$$E_{\alpha_1}(u) \geq E_{\alpha_2}(u) \geq E(u).$$

It follows that

$$\varphi_k(\alpha_1) \geq \varphi_k(\alpha_2) \geq \varphi_k(1).$$

Hence, $\lim_{\alpha \searrow 1} \varphi_k$ exists and

$$\lim_{\alpha \searrow 1} \varphi_k(\alpha) \geq \varphi_k(1).$$

On the other hand, note that u_k is a smooth map. Then for any $\epsilon > 0$, there exists a smooth map $u'_k \in C^\infty(S^2, N)$ which is homotopic to u_k (i.e., $u'_k \sim u_k$), and satisfies

$$E(u'_k) \leq \varphi_k(1) + \epsilon.$$

Since

$$\lim_{\alpha\searrow 1} E_\alpha(u'_k) = E(u'_k) \quad \text{and} \quad \varphi_k(\alpha) \le E_\alpha(u'_k),$$

we have

$$\lim_{\alpha\searrow 1} \varphi_k(\alpha) \le \varphi_k(1) + \epsilon,$$

which implies (3-5), and shows that $\varphi_k(\alpha)$ is continuous on $[1, \alpha_0)$ for any fixed k.

By (3-3), for any given sequence $\{\alpha'_k\}$ with $\alpha'_k \to 1$, there exists a sequence $\{i_k\}$ such that $E_{\alpha'_k}(u_{\alpha'_k, i_k}) > \lambda_0$, or equivalently, $\varphi_{i_k}(\alpha'_k) > \lambda_0$. Lemma 3.5 tells us that $\varphi_{i_k}(1) = 8\pi\psi(0)$ for any i_k. By the assumption $\lambda_0 > 8\pi\psi(0)$ we have

$$\varphi_{i_k}(\alpha'_k) > \lambda_0 > \varphi_{i_k}(1).$$

Since $\varphi_k(\alpha)$ is continuous on $[1, \alpha_0)$, we conclude that for any fixed i_k there exists $\alpha_k \in (1, \alpha'_k)$ such that

$$\varphi_{i_k}(\alpha_k) = E_{\alpha_k}(u_{\alpha_k, i_k}) = \lambda_0.$$

This completes the proof. □

3C. ***The counterexample.*** By Lemma 3.6, for given $\tau \in (8\pi\psi(0), 12\pi\psi(0))$ there exist a sequence $\{\alpha_k : \alpha_k > 1, k \in \mathbb{N}\}$ with $\alpha_k \to 1$ and a sequence of minimizing α_k-harmonic maps $v_k \in W^{1,2\alpha_k}(S^2, N)$ with $v_k \sim u_{i_k}$ such that

$$\tau = E_{\alpha_k}(v_k) = \inf_{u \in [u_{i_k}]^{\alpha_k}} E_{\alpha_k}(u) \ \text{ for all } k \in \mathbb{N}.$$

Since v_i and v_j are not in the same homotopy class for any $i \ne j$, v_k must blow up as $k \to +\infty$. Let v^0 be the weak limit of $\{v_k\}$ in $W^{1,2}(S^2, N)$, and $v^1, \dots, v^m$ be all the bubbles produced in the convergence of $\{v_k\}$. Since $E(v^i) < 12\pi\psi(0)$, it follows from Corollary 3.4 that $E(v^i) = 4\pi\psi(0)$ or $8\pi\psi(0)$. Hence,

$$\frac{1}{4\pi\psi(0)}\left(E(v^0) + \sum_{i=1}^m E(v^i)\right)$$

is always an integer. However, certainly $\frac{\tau}{4\pi\psi(0)}$ is not an integer by the previous assumption. So the energy identity is not true for the sequence $\{v_k\}$:

$$\lim_{k\to\infty} E_{\alpha_k}(v_k) \ne E(v^0) + \sum_{i=1}^m E(v^i).$$

Remark 3.7. By an argument in [Li and Wang 2010b], we also have

$$\lim_{k\to\infty} E(v_k) \ne E(v^0) + \sum_{i=1}^m E(v^i).$$

4. An example of a manifold whose energy set is nondiscrete for harmonic 2-spheres

In this section, we will construct a Riemannian manifold (N, h) for which $\mathscr{E}(N, h)$ is not discrete. In other words, $\mathscr{E}(N, h)$ admits limit points.

Let $\psi(t)$ be a smooth positive function defined on $(-1, 1)$ satisfying

$$\psi(t) = e^{-1/t^2} \sin \tfrac{1}{t} + 1, \quad t \in (-\tfrac{1}{2}, \tfrac{1}{2}).$$

It is easy to check that the critical point of $\psi(t)$ satisfies the equation

$$\tan \tfrac{1}{t} = \tfrac{t}{2}.$$

Thus, we can find $t_k \to 0$, such that $\psi'(t_k) = 0$, $\psi(t_k) \neq 1$ and $\psi(t_k) \to 1$.

Let

$$h = \psi(t)(d\mathfrak{s}^2 + dt^2),$$

which is a metric over $S^2 \times (-1, 1)$. Let v be the identity map from S^2 to S^2 and

$$u_k = (v, t_k) : S^2 \to (N, h) \equiv (S^2 \times (-1, 1), h).$$

By (3-1), it is easy to see that u_k is a harmonic map from S^2 into $(S^2 \times (-1, 1), h)$ with

$$E(u_k) = 4\pi \psi(t_k).$$

Thus, 4π is not a discrete number in $\mathscr{E}(S^2 \times (-1, 1), h)$.

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