Pacific Journal of Mathematics

MEAN VALUES OF L-FUNCTIONS OVER FUNCTION FIELDS

JEFFREY LIN THUNDER

Volume 274 No. 1 March 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 274, No. 1, 2015
dx.doi.org/10.2140/pjm.2015.274.237

MEAN VALUES OF L-FUNCTIONS OVER FUNCTION FIELDS

JEFFREY LIN THUNDER

For a fixed global function field, positive integer and complex number, we prove estimates for mean values of L-functions evaluated at the given complex number, where the averaging is done over quadratic extensions of the given function field with genus equal to the given positive integer. To accomplish this we utilize our previous results on certain quadratic character sums over function fields.

1. Introduction

In modern number theory L-series play a prominent role. They encode many deep properties of number fields and primes and are objects of intense interest. The analogous L-functions over global function fields play an equally prominent role. Here we will prove estimates for mean values of such L-functions, where the averaging is done over quadratic extensions of a fixed global function field. Our estimates cover a much wider range of cases than the similar estimates of Hoffstein and Rosen [1992] and those of Andrade and Keating (for values on the critical line) [2012]. Our methods are akin to those used by Siegel [1944], where he estimates the average number of quadratic forms with given discriminant and signature.

For a prime p, let $\mathbb{F}_p$ denote the finite field with p elements and let X be transcendental over $\mathbb{F}_p$, so that $\mathbb{F}_p(X)$ is a field of rational functions. Fix algebraic closures $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$ and $\overline{\mathbb{F}_p(X)} \supset \overline{\mathbb{F}_p}$ of $\mathbb{F}_p(X)$. In what follows, by *global function field* (or simply *function field*) we mean a finite algebraic extension $K \supseteq \mathbb{F}_p(X)$ contained in $\overline{\mathbb{F}_p(X)}$. For such a field K we have $K \cap \overline{\mathbb{F}_p} = \mathbb{F}_{q_K}$ for some finite field $\mathbb{F}_{q_K}$ with q_K elements; this field is called the field of constants of K. We write g_K for the genus of K and J_K for the number of divisor classes of degree 0. We denote the set of places of K by $M(K)$ and the divisor group (i.e., the free abelian group generated by the places) by $\operatorname{Div} K$. The reader can refer to Chapters I and V of [Stichtenoth 1993] for a thorough background on these notions. We will use capital script German letters to denote divisors $\mathfrak{A}$, $\mathfrak{B}$, etc., with the sole exception of the zero divisor 0. For any divisor $\mathfrak{A} \in \operatorname{Div} K$ we write $\mathfrak{A} = \sum \operatorname{ord}_v(\mathfrak{A}) \cdot v$, where the

MSC2010: 11M41.
Keywords: function field, L-functions.

sum is over all places $v \in M(K)$. The support $\operatorname{Supp}\mathfrak{A}$ of a divisor $\mathfrak{A} \in \operatorname{Div} K$ is the finite (possibly empty) set of places $v \in M(K)$ where $\operatorname{ord}_v(\mathfrak{A}) \neq 0$. We say $\mathfrak{A}$ is effective if $\mathfrak{A} \geq 0$, i.e., $\operatorname{ord}_v(\mathfrak{A}) \geq 0$ for all places $v \in M(K)$. The degree map on $\operatorname{Div} K$, normalized to have image $\mathbb{Z}$ (see Chapter V of [Stichtenoth 1993] again) will be denoted deg.

With the above notation, the zeta function ζ_K is given by

$$\zeta_K(s) = \sum_{\substack{\mathfrak{A}\in\operatorname{Div} K\\ \mathfrak{A}\geq 0}} q^{-s\deg\mathfrak{A}}.$$

Since

$$\sum_{\substack{\mathfrak{A}\in\operatorname{Div} K\\ \mathfrak{A}\geq 0,\ \deg\mathfrak{A}=j}} 1 = \frac{J_K}{q_K-1}(q_K^{j+1-g_K}-1) \tag{0}$$

for all integers $j \geq 2g_K - 1$ by the Riemann–Roch Theorem (see [Stichtenoth 1993, Lemma V.1.4], for example), the series defining $\zeta_K(s)$ converges for all $s \in \mathbb{C}$ with $\Re(s) > 1$. The L-function L_K is given by

$$L_K(q_K^{-s}) = (1-q_K^{-s})(1-q_K^{1-s})\zeta_K(s) = \frac{\zeta_K(s)}{\zeta_{\mathbb{F}_{q_K}(X)}(s)}.$$

It is well known that L_K is a polynomial of degree $2g_K$ in q_K^{-s} and all its zeros have $\Re(s) = \frac{1}{2}$ (see [Stichtenoth 1993, Chapter V], for example).

For a fixed function field K and integer $m \geq 0$, we will be concerned with sums over quadratic extensions of K with genus m and the same field of constants $\mathbb{F}_{q_K}$. We first denote the number of such quadratic extensions

$$N_K(m) = \sum_{\substack{[F:K]=2\\ g_F=m,\ q_F=q_K}} 1.$$

We note that $N_K(m)$ is asymptotically $q_K^{2m} 2J_K q_K^{3-5g_K}/\zeta_K(2)(q_K-1)$ as $m \to \infty$ (see Proposition 1 below). We will investigate the arithmetic mean of the set of values $L_F(q^{-s})$ over the quadratic extensions $F \supset K$ of genus m, and higher moments as well. It will prove convenient to multiply these means by the L-function value of the ground field K, Thus, for a fixed $s \in \mathbb{C}$ and integer $n \geq 1$, we set

$$M_K(s,m,n) = (N_K(m))^{-1} \sum_{\substack{[F:K]=2\\ g_F=m,\ q_F=q_K}} L_F(q^{-s})^n L_K(q^{-s})^{-n}$$

provided $N_K(m) > 0$, and set $M_K(s,m,n) = 0$ otherwise. We will prove the following estimates:

Theorem 1. *Let K be a function field with field of constants $\mathbb{F}_q$. For all positive integers n and all $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{2}$, set*

$$\sigma_K(s,n) = \prod_{v \in M(K)} \left(1 + \frac{P_n(q^{-s \deg v})}{(n-1)!(1+q^{-\deg v})(1-q^{-2s \deg v})^n}\right),$$

where $P_n(X) \in \mathbb{Z}[X]$ is given by

$$\frac{d^{n-1} X^{n+1}/(1-X^2)}{dX^{n-1}} = \frac{P_n(X)}{(1-X^2)^n}.$$

Then for all integers $m \geq 0$, all $\epsilon > 0$ and all $s \in \mathbb{C}$ with $\Re(s) > 1 + (n-1)\epsilon$ if q is odd, or $\Re(s) > 1 + n\epsilon - 1/2n$ if q is even, we have

$$|M_K(s,m,n) - \sigma_K(s,n)| \leq \begin{cases} c(\epsilon)^n (q^{-m} + q^{-4m(\Re(s)-1-(n-1)\epsilon)}) & \text{if } q \text{ is odd},\\ c(\epsilon)^n (q^{-m} + q^{-2mn(\Re(s)-1-\epsilon+1/2n)}) & \text{if } q \text{ is even}, \end{cases}$$

where the constant $c(\epsilon) > 0$ depends only on K and ϵ.

We obtain stronger estimates (i.e., better error terms) when we consider the case $n = 1$:

Theorem 2. *Let K be a function field with field of constants $\mathbb{F}_q$ and let m be an integer with $m > g_K$. Then $M_K(s,m,1)$ is a polynomial of degree $2(m-g_K)$ in q^{-s} satisfying the same functional equation as the L-function,*

$$M_K(s,m,1) = q^{m-g_K} q^{-s2(m-g_K)} M_K(1-s,m,1).$$

Further, $M_K(s,m,1)$ is an even function in q^{-s} with

$$M_K(s,m,1) = 1 + a_2 q^{-2s} + \cdots + a_{2(m-g_K)} q^{-2s(m-g_K)}.$$

We have $a_{2(m-g_K-j)} = q^{m-g_K-2j} a_{2j}$ for all $j = 0, \ldots, m-g_K$, and, for all $\epsilon > 0$,

$$a_{2j} = \sum_{\substack{\mathfrak{C} \geq 0\\ \deg \mathfrak{C} = j}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1+q^{-\deg v})^{-1} + \begin{cases} O(q^{-m} q^{2j(5/4+\epsilon)}) & \text{if } q \text{ is odd},\\ O(q^{-m} q^{2j(1+\epsilon)}) & \text{if } q \text{ is even}, \end{cases}$$

where the implicit constants depend only on K and ϵ. Finally,

$$\sum_{\substack{\mathfrak{C} \geq 0\\ \deg \mathfrak{C} = j}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1+q^{-\deg v})^{-1} = q^j \frac{J_K q^{1-g_K} \zeta_K(2)}{q-1} \prod_{v \in M(K)} (1 - 2q^{-2\deg v} + q^{-3 \deg v}) + O(q^{\epsilon j})$$

for all $j \geq 0$, where the implicit constant depends only on K and ϵ.

Corollary 1. *Let K be a function field with field of constants $\mathbb{F}_q$ and let m be a positive integer. Then for all $\epsilon > 0$ and all $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{2}$*

$$M_K(s, m, 1) = \sigma_K(s, 1) + \begin{cases} O\left(\dfrac{q^{-m(2/3-\epsilon)(2\Re(s)-1)}}{1 - q^{1-2\Re(s)}}\right) & \text{if } q \text{ is odd,} \\ O\left(\dfrac{q^{-m(1-\epsilon)(2\Re(s)-1)}}{1 - q^{1-2\Re(s)}}\right) & \text{if } q \text{ is even,} \end{cases}$$

where the implicit constants depend only on K and ϵ. In particular, setting $s = 1$ we have

$$\begin{aligned} (N_K(m))^{-1} \sum_{\substack{[F:K]=2 \\ g_F=m,\ q_F=q}} J_F q^{-g_F} &= J_K q^{-g_K} (\zeta_K(2))^2 \prod_{v \in M(K)} (1 - q^{-2\deg v} - q^{-3\deg v} + q^{-4\deg v}) \\ &\qquad + \begin{cases} O(q^{-m(2/3-\epsilon)}) & \text{if } q \text{ is odd,} \\ O(q^{-m(1-\epsilon)}) & \text{if } q \text{ is even.} \end{cases} \end{aligned}$$

Mean values similar to those in Corollary 1 were previously considered by Hoffstein and Rosen [1992], but only in the case where the field K is a field of rational functions and only in odd characteristic. More general cases were considered by Fisher and Friedberg [2004], with further refinements by Chinta, Friedberg and Hoffstein [Chinta et al. 2006], but again only in odd characteristic. The higher moments in Theorem 1 have not been previously estimated to our knowledge. Our approach differs from those previous by utilizing more general estimates for quadratic characters over function fields, including estimates in characteristic 2 (which is clearly special when considering quadratic extensions).

Theorem 2 can also be used to estimate the "average" of $L_F(q^{-1/2})$ (cf. [Goldfeld and Hoffstein 1985, Theorem 1] for the case where K is replaced by $\mathbb{Q}$). As alluded to above, such a result is proven in [Andrade and Keating 2012], though again only in certain special cases where the ground field is a field of rational functions (specifically, for q congruent to 1 modulo 4).

Corollary 2. *Let K be a function field with field of constants $\mathbb{F}_q$ and let m be an integer with $m > g_K$. Set $m' = m - g_K$ and*

$$C(K) = \frac{J_K q^{1-g_K} \zeta_K(2)}{q-1} \prod_{v \in M(K)} (1 - 2q^{-2\deg v} + q^{-3\deg v}).$$

Then the series

$$C'(K) = \sum_{j=0}^{\infty} \Bigg(\sum_{\substack{\mathfrak{C} \geq 0 \\ \deg \mathfrak{C} = j}} q^{-\deg \mathfrak{C}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1} \Bigg) - C(K)$$

converges and, for all $\epsilon > 0$,

$$M_K\left(\tfrac{1}{2}, m, 1\right) = (m'+1)C(K) + 2C'(K) + \begin{cases} O(q^{-m(1/4-\epsilon)}) & \textit{if } q \textit{ is odd,} \\ O(q^{-m(1/2-\epsilon)}) & \textit{if } q \textit{ is even,} \end{cases}$$

where the implicit constants depend only on K *and* ϵ.

Finally, we note that Theorem 2 can also be used to give the "average" number of places of degree 1.

Corollary 3. *Let* K *be a function field with field of constants* $\mathbb{F}_q$. *Then* (*assuming* $N_K(m) \neq 0$)

$$(N_K(m))^{-1} \sum_{\substack{[F:K]=2 \\ g_F=m,\, q_F=q}} \#\{w \in M(F) : \deg w = 1\} = \#\{v \in M(K) : \deg v = 1\}.$$

One can compare this with the famous estimate due to Drinfeld and Vladut [Stichtenoth 1993, Theorem V.3.5],

$$\limsup_{m\to\infty} \max_{\substack{F\supset K \\ g_F=m,\, q_F=q}} \frac{\#\{w \in M(F) : \deg w = 1\}}{m} \leq q^{1/2} - 1.$$

2. Preparatory Results

We briefly discuss separability issues before proceeding further. If K is a function field and $F \supset K$ is a quadratic extension, then F is clearly a separable extension if q_K is odd. If q_K is even this is not necessarily the case. However, it turns out that there is exactly one inseparable quadratic extension $F \supset K$ with $q_F = q_K$ when q_K is even; it satisfies $K = \{\alpha^2 : \alpha \in F\}$ and $g_F = g_K$ by [Stichtenoth 1993, Proposition III.9.2]. Therefore we can safely ignore this inseparable extension and tacitly assume in what follows that all quadratic extensions that appear are separable extensions.

If K is a function field, $v \in M(K)$ and F is a quadratic extension of K with $q_F = q_K$, we set

$$\chi(F/v) = \begin{cases} 0 & \text{if } v \text{ ramifies in } F, \\ 1 & \text{if } v \text{ is inert in } F, \\ -1 & \text{if } v \text{ splits in } F. \end{cases}$$

This is extended to effective divisors $\mathfrak{A} \in \operatorname{Div} K$ by

$$\chi(F/\mathfrak{A}) = \prod_{v\in\operatorname{Supp}\mathfrak{A}} \left(\chi(F/v)\right)^{\operatorname{ord}(\mathfrak{A})}.$$

The following is shown in [Thunder 2013, §1]:

Lemma 1. *Let K be a function field with field with field of constants $\mathbb{F}_q$ and $F \supset K$ be a quadratic extension with $q_F = q$. Then*

$$L_F(q^{-s}) = L_K(q^{-s}) \sum_{\substack{\mathfrak{A}\in \operatorname{Div} K\\ \mathfrak{A}\geq 0}} \chi(F/\mathfrak{A}) q^{-s\deg \mathfrak{A}},$$

so that

$$N_K(m) M_K(s,m,n) = \sum_{\substack{\mathfrak{C}\in \operatorname{Div} K\\ \mathfrak{C}\geq 0}} \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=\mathfrak{C}}} q^{-s\deg \mathfrak{C}} \sum_{\substack{[F:K]=2\\ g_F=m, q_F=q_K}} \chi(F/\mathfrak{C}).$$

It turns out that the sums in Lemma 1 where $\deg \mathfrak{C}$ is odd vanish entirely and, when $\deg \mathfrak{C}$ is even, the $\mathfrak{C} \in 2\operatorname{Div} K$ dominate.

Lemma 2 [Thunder 2013, Lemma 9]. *Suppose K is a function field with field of constants $\mathbb{F}_q$ and m is a nonnegative integer. Then for all effective divisors $\mathfrak{C} \in \operatorname{Div} K$ of odd degree,*

$$\sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/\mathfrak{C}) = 0.$$

Proposition 1 [Thunder 2013, Proposition 7]. *Let K be a function field with field of constants $\mathbb{F}_q$ and m be a nonnegative integer. Set*

$$N_K'(m) = q^{2m} \frac{2J_K q^{3-5g_K}}{\zeta_K(2)(q-1)}.$$

For all effective divisors $\mathfrak{C} \in \operatorname{Div} K$ and all $\epsilon > 0$

$$\left| \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/2\mathfrak{C}) - N_K'(m) \prod_{v\in \operatorname{Supp}\mathfrak{C}} (1+q^{-\deg v})^{-1} \right| \leq \begin{cases} c'(\epsilon) q^{(1/2+\epsilon)m} q^{\epsilon \deg \mathfrak{C}} & \text{if } q \text{ is odd,}\\ c'(\epsilon)\big(q^{\epsilon m} q^{\epsilon \deg \mathfrak{C}} + q^m\big) & \text{if } q \text{ is even,}\end{cases}$$

where $c'(\epsilon) > 0$ depends only on K and ϵ. In particular,

$$|N_K(m) - N_K'(m)| \leq \begin{cases} c'(\epsilon) q^{(1/2+\epsilon)m} & \text{if } q \text{ is odd,}\\ c'(1) q^m & \text{if } q \text{ is even.}\end{cases}$$

Proposition 2 [Thunder 2013, Proposition 5]. *Suppose K is a function field with $q_K = q$ odd and let $\mathfrak{C} \in \operatorname{Div} K$ be an effective divisor with $\mathfrak{C} \notin 2\operatorname{Div} K$. Then, for all nonnegative integers m and all $\epsilon > 0$, we have*

$$\left| \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/\mathfrak{C}) \right| \leq c''(\epsilon) q^m q^{(\epsilon+1/4)\deg \mathfrak{C}},$$

where the constant $c''(\epsilon) > 0$ depends only on K and ϵ.

Proposition 3 [Thunder 2013, Proposition 6]. *Suppose K is a function field with $q_K = q$ even and let $\mathfrak{C} \in \operatorname{Div} K$ be an effective divisor with $\mathfrak{C} \notin 2\operatorname{Div} K$. Then, for all nonnegative integers m and all $\epsilon > 0$, we have*

$$\left| \sum_{\substack{[F:K]=2 \\ g_F=m,\, q_F=q}} \chi(F/\mathfrak{C}) \right| \leq c''(\epsilon) q^m q^{\epsilon \deg \mathfrak{C}},$$

where the constant $c''(\epsilon) > 0$ depends only on K and ϵ.

We also have the following elementary estimates:

Lemma 3. *Suppose K is a function field with field of constants $\mathbb{F}_q$. Let $\mathfrak{C} \in \operatorname{Div} K$ be an effective divisor. For all integers $n > 1$ and all $\epsilon > 0$,*

$$\sum_{\substack{\mathfrak{A}_i \geq 0 \\ \mathfrak{A}_1 + \cdots + \mathfrak{A}_n = \mathfrak{C}}} 1 \leq c_1(\epsilon)^{n-1} q^{(n-1)\epsilon \deg \mathfrak{C}},$$

where the constant $c_1(\epsilon) > 0$ depends only on K and ϵ. Also, for all positive integers m and all $\epsilon > 0$,

$$\sum_{\substack{\mathfrak{A} \geq 0 \\ \deg \mathfrak{A} \leq m}} q^{(\epsilon - 1)\deg \mathfrak{A}} \leq \frac{c_2}{\epsilon} q^{\epsilon m}, \qquad \sum_{\substack{\mathfrak{A} \geq 0 \\ \deg \mathfrak{A} \geq m}} q^{-(1+\epsilon)\deg \mathfrak{A}} \leq \frac{c_2}{\epsilon} q^{-\epsilon m},$$

and

$$\zeta_K(1+\epsilon) \leq \left(\frac{c_2}{\epsilon} \right)^{[K:\mathbb{F}_q(X)]},$$

where the constant $c_2 > 0$ depends only on K.

Proof. We prove the first part by induction on n. The case $n = 2$ follows directly from [Thunder 2013, Lemma 0]. Now assume $n > 2$. Then

$$\begin{aligned}
\sum_{\substack{\mathfrak{A}_i \geq 0 \\ \mathfrak{A}_1 + \cdots + \mathfrak{A}_n = \mathfrak{C}}} 1 &= \sum_{0 \leq \mathfrak{A}_n \leq \mathfrak{C}} \sum_{\substack{\mathfrak{A}_i \geq 0 \\ \mathfrak{A}_1 + \cdots + \mathfrak{A}_{n-1} = \mathfrak{C} - \mathfrak{A}_n}} 1 \\
&\leq \sum_{0 \leq \mathfrak{A}_n \leq \mathfrak{C}} c_1(\epsilon)^{n-2} q^{(n-2)\deg(\mathfrak{C} - \mathfrak{A}_n)} \\
&\leq c_1(\epsilon)^{n-2} q^{(n-2)\epsilon \deg \mathfrak{C}} \sum_{0 \leq \mathfrak{A}_n \leq \mathfrak{C}} 1 \\
&\leq c_1(\epsilon)^{n-1} q^{(n-1)\epsilon \deg \mathfrak{C}}.
\end{aligned}$$

For the next two inequalities, we see by (0) that there is a positive constant c, depending only on the field K, such that for all nonnegative integers j we have

$$\sum_{\substack{\mathfrak{A} \geq 0 \\ \deg \mathfrak{A} = j}} 1 \leq c q^j.$$

Finally, by the Euler product representation of the zeta function, we have

$$\zeta_K(s) \le (\zeta_{\mathbb{F}_q(X)}(s))^{[K:\mathbb{F}_q(X)]}$$

for all real $s > 1$. (See [Thunder and Widmer 2013, Lemma 2], for example.) Using the well-known formula

$$\zeta_{\mathbb{F}_q(X)}(s) = \frac{1}{(1-q^{-s})(1-q^{1-s})}$$

and substituting $s = 1+\epsilon$ gives

$$\zeta_K(1+\epsilon) \le \left(\frac{c'}{\epsilon}\right)^{[K:\mathbb{F}_q(X)]}$$

for some positive constant c' depending only on q. Setting c_2 to be the maximum of c and c' completes the proof. $\square$

3. Proof of Theorem 1

We first deal with the summands in Lemma 1 where $\mathfrak{C} \in 2\,\mathrm{Div}\,K$. This is done in two steps.

Lemma 4. *Suppose K is a function field with field of constants $\mathbb{F}_q$ and $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{2}$. Then, for all integers $n \ge 1$,*

$$\sum_{\mathfrak{C}\ge 0} \sum_{\substack{\mathfrak{A}_i \ge 0 \\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n = 2\mathfrak{C}}} q^{-2s\deg\mathfrak{C}} \prod_{v\in\mathrm{Supp}\,\mathfrak{C}} (1+q^{-\deg v})^{-1} = \sigma_K(s,n).$$

Proof. Set

$$\theta_n(\mathfrak{C}) = q^{-2s\deg\mathfrak{C}} \prod_{v\in\mathrm{Supp}\,\mathfrak{C}} (1+q^{-\deg v})^{-1} \sum_{\substack{\mathfrak{A}_i \ge 0 \\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n = 2\mathfrak{C}}} 1.$$

Note that $\theta_n(\mathfrak{C}+\mathfrak{D}) = \theta_n(\mathfrak{C})\theta_n(\mathfrak{D})$ for all n whenever $\mathfrak{C}, \mathfrak{D} \in \mathrm{Div}\,K$ have disjoint support. Thus

$$(1) \qquad \sum_{\mathfrak{C}\ge 0} \theta_n(\mathfrak{C}) = \prod_{v\in M(K)} \left(1 + \sum_{k=1}^{\infty} \theta_n(kv)\right).$$

For all positive integers k and all places $v \in M(K)$,

$$\theta_n(kv) = (1+q^{-\deg v})^{-1} q^{-2ks\deg v} f(2k,n),$$

where

$$f(m,n) = \sum_{\substack{i_j \ge 0 \\ i_1+\cdots+i_n = m}} 1 = \frac{(m+1)\cdots(m+n-1)}{(n-1)!}$$

for integers $m \geq 0$ and $n \geq 1$. Therefore

$$\sum_{k=1}^{\infty} \theta_n(kv) = (1+q^{-\deg v})^{-1} \sum_{k=1}^{\infty} q^{-2ks\deg v} \frac{(2k+1)\cdots(2k+n-1)}{(n-1)!}. \tag{2}$$

Differentiating term-by-term $n-1$ times yields

$$\frac{d^{n-1}\sum_{k=1}^{\infty} x^{2k+n-1}}{dx^{n-1}} = \sum_{k=1}^{\infty} x^{2k}(2k+1)\cdots(2k+n-1). \tag{3}$$

On the other hand,

$$\begin{aligned} \frac{d^{n-1}\sum_{k=1}^{\infty} x^{2k+n-1}}{dx^{n-1}} &= \frac{d^{n-1}x^{n-1}\sum_{k=1}^{\infty} x^{2k}}{dx^{n-1}} \\ &= \frac{d^{n-1}x^{n+1}\sum_{k=0}^{\infty} x^{2k}}{dx^{n-1}} \\ &= \frac{d^{n-1}x^{n+1}/(1-x^2)}{dx^{n-1}} \\ &= P_n(x)(1-x^2)^{-n}. \end{aligned} \tag{4}$$

The lemma follows from (1)–(4). □

Lemma 5. *Let K be a function field with field of constants $\mathbb{F}_q$. Suppose m is a nonnegative integer such that $N_K(m) > 0$. Then, for all $\epsilon > 0$ and all $s \in \mathbb{C}$ with $\Re(s) > (1+n\epsilon)/2$,*

$$\left| \frac{1}{N_K(m)} \sum_{\mathfrak{C} \geq 0} \sum_{\substack{\mathfrak{A}_i \geq 0 \\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n = 2\mathfrak{C}}} q^{-2s\deg \mathfrak{C}} \sum_{\substack{[F:K]=2 \\ g_F = m,\, q_F = q}} \chi(F/2\mathfrak{C}) - \sigma_K(s,n) \right| \leq \begin{cases} c_3(\epsilon)^{n+1} q^{-m(3/2-\epsilon)} & \text{if } q \text{ is odd}, \\ c_3(\epsilon)^n q^{-m} & \text{if } q \text{ is even}, \end{cases}$$

where $c_3(\epsilon) > 0$ depends only on K and ϵ.

Proof. We have

$$\begin{aligned} &\left| \sum_{\mathfrak{C} \geq 0} \sum_{\substack{\mathfrak{A}_i \geq 0 \\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n = 2\mathfrak{C}}} q^{-2s\deg \mathfrak{C}} \sum_{\substack{[F:K]=2 \\ g_F = m,\, q_F = q}} \chi(F/2\mathfrak{C}) - N_K(m)\sigma_K(s,n) \right| \\ &\leq \left| \sum_{\mathfrak{C} \geq 0} \sum_{\substack{\mathfrak{A}_i \geq 0 \\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n = 2\mathfrak{C}}} q^{-2s\deg \mathfrak{C}} \sum_{\substack{[F:K]=2 \\ g_F = m,\, q_F = q}} \chi(F/2\mathfrak{C}) - N_K'(m)\sigma_K(s,n) \right| \\ &\quad + |N_K(m) - N_K'(m)|\, |\sigma_K(s,n)|. \end{aligned} \tag{5}$$

By Lemma 3 and using $2\Re(s)-(n-1)\epsilon > 1+\epsilon$,

$$\text{(6)}\quad \left|\sum_{\mathfrak{C}\geq 0}\ \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=2\mathfrak{C}}} q^{-2s\deg\mathfrak{C}} \prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1}\right|$$
$$\begin{aligned}
&\leq \sum_{\mathfrak{C}\geq 0}\ \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=2\mathfrak{C}}} q^{-2\Re(s)\deg\mathfrak{C}}\\
&\leq c_1\left(\frac{\epsilon}{2}\right)^{n-1}\sum_{\mathfrak{C}\geq 0} q^{(-2\Re(s)+(n-1)\epsilon)\deg\mathfrak{C}}\\
&< c_1\left(\frac{\epsilon}{2}\right)^{n-1}\zeta_K(1+\epsilon)\\
&\leq c_4\epsilon^n,
\end{aligned}$$

where $c_4(\epsilon)=\max\{c_1(\epsilon/2),\ (c_2/\epsilon)^{[K:\mathbb{F}_q(X)]}\}$.

Now, by Proposition 1, Lemma 4 and (6),

$$\text{(7a)}\quad \left|\sum_{\mathfrak{C}\geq 0}\ \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=2\mathfrak{C}}} q^{-2s\deg\mathfrak{C}} \sum_{\substack{[F:K]=2\\ g_F=m,\ q_F=q}} \chi(F/2\mathfrak{C}) - N'_K(m)\sigma_K(s,n)\right|$$
$$\leq c'(\epsilon)q^{(1/2+\epsilon)m}\sum_{\mathfrak{C}\geq 0}\ \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=2\mathfrak{C}}} q^{-2\Re(s)\deg\mathfrak{C}} \leq c'(\epsilon)c_4(\epsilon)^n q^{(1/2+\epsilon)m}$$

if q is odd, and

$$\text{(7b)}\quad \left|\sum_{\mathfrak{C}\geq 0}\ \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=2\mathfrak{C}}} q^{-2s\deg\mathfrak{C}} \sum_{\substack{[F:K]=2\\ g_F=m,\ q_F=q}} \chi(F/2\mathfrak{C}) - N'_K(m)\sigma_K(s,n)\right|$$
$$\ll c'(1)q^{m}\sum_{\mathfrak{C}\geq 0}\ \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=2\mathfrak{C}}} q^{-2\Re(s)\deg\mathfrak{C}} \leq c'(1)c_4(\epsilon)^n q^{m}$$

if q is even. Also, by Proposition 1, Lemma 4 and (6)

$$\text{(8)}\quad |N_K(m)-N'_K(m)|\,|\sigma_K(s,n)| \leq \begin{cases} c'(\epsilon)c_4(\epsilon)^n q^{(1/2+\epsilon)m} & \text{if } q \text{ is odd,}\\ c'(1)c_4(\epsilon)^n q^m & \text{if } q \text{ is even.}\end{cases}$$

Finally, if $N_K(m)$ isn't zero, then

$$\text{(9)}\qquad c_5q^{2m} \leq N_K(m) \leq c_6q^{2m}$$

by Proposition 1, where $c_5, c_6 > 0$ depend only on K. The lemma follows from (5) and (7a)–(9) when q is odd, and (5), (7b), (8) and (9) when q is even. □

With the sums over the main terms done, we now turn to the sums over the error terms, i.e., the sums where $\mathfrak{A}_1+\cdots+\mathfrak{A}_n \notin 2\operatorname{Div}K$. We have the following:

Lemma 6. *Suppose K is a function field with field of constants $\mathbb{F}_q$ and m is a nonnegative integer. Fix an integer $n \geq 2$ and an $\epsilon > 0$. Suppose that $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{2} + (n-1)\epsilon$. Then, for any quadratic extension $F \supset K$ with $g_F = m$ and $q_F = q$,*

$$\sum_{\mathfrak{A}_i \geq 0} q^{-\Re(s)\deg(\mathfrak{A}_1+\cdots+\mathfrak{A}_{n-1})} \Bigg| \sum_{\substack{\mathfrak{A}_n \geq 0 \\ \deg \mathfrak{A}_n > 2m-2g_K \\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n \notin 2\,\mathrm{Div}\,K}} q^{-s\deg \mathfrak{A}_n} \chi(F/\mathfrak{A}) \Bigg| \leq c_7 \epsilon^{n+1} q^{-2m(\Re(s)-1/2)},$$

where the constant $c_7(\epsilon) > 0$ depends only on K and ϵ.

Proof. For the moment, fix $\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}$ and set $\mathfrak{B} = \mathfrak{A}_1 + \cdots + \mathfrak{A}_{n-1}$. Write $\mathfrak{B} = \mathfrak{B}' + 2\mathfrak{B}''$, where $\mathfrak{B}'$ and $\mathfrak{B}''$ are both effective divisors and $\mathrm{ord}_v(\mathfrak{B}') = 1$ for all $v \in \mathrm{Supp}\,\mathfrak{B}'$. Fix an integer $j > 2m - 2g_K$. As shown in the proof of [Thunder 2013, Lemma 25],

$$\Bigg| \sum_{\substack{\mathfrak{A}_n \geq 0 \\ \deg \mathfrak{A}_n = j \\ \mathfrak{B}+\mathfrak{A}_n \notin 2\,\mathrm{Div}\,K}} \chi(F/\mathfrak{A}) q^{-s\deg\mathfrak{A}_n} \Bigg| \leq c_8 q^{-\deg(\mathfrak{B}')/2} q^{-j(\Re(s)-1/2)} \tag{10}$$

for some $c_8 > 0$ depending only on K. Now, since $\Re(s) - \frac{1}{2} > (n-1)\epsilon$,

$$\begin{aligned} \sum_{j > 2m-2g_K} q^{-j(\Re(s)-1/2)} &< \sum_{j>2m-2g_K} q^{-j(n-1)\epsilon} \\ &\leq c_9 q^{-2m(n-1)\epsilon} \sum_{j \geq 0} q^{-j(n-1)\epsilon} \\ &= c_9 q^{-2m(\Re(s)-1/2)} (1 - q^{-(n-1)\epsilon})^{-1} \\ &\leq c_9 c_{10} q^{-2m(\Re(s)-1/2)} ((n-1)\epsilon)^{-1}, \end{aligned} \tag{11}$$

where $c_9, c_{10} > 0$ depend only on K. By Lemma 3,

$$\begin{aligned} \sum_{\mathfrak{B} \geq 0} \sum_{\substack{\mathfrak{A}_i \geq 0 \\ \mathfrak{A}_1+\cdots+\mathfrak{A}_{n-1}=\mathfrak{B}}} & q^{-\Re(s)\deg\mathfrak{B}} q^{-\deg(\mathfrak{B}')/2} \\ &\leq c_1(\epsilon)^{n-2} \sum_{\mathfrak{B}\geq 0} q^{((n-2)\epsilon - \Re(s))\deg\mathfrak{B}} q^{-\deg(\mathfrak{B}')/2} \\ &= c_1(\epsilon)^{n-2} \sum_{\mathfrak{B}\geq 0} q^{((n-2)\epsilon-1/2-\Re(s))\deg\mathfrak{B}'} q^{2((n-2)\epsilon-\Re(s))\deg\mathfrak{B}''} \\ &\leq c_1(\epsilon)^{n-2} \sum_{\mathfrak{B}'\geq 0} q^{((n-2)\epsilon-1/2-\Re(s))\deg\mathfrak{B}'} \sum_{\mathfrak{B}''\geq 0} q^{2((n-2)\epsilon-\Re(s))\deg\mathfrak{B}''} \\ &< c_1(\epsilon)^{n-2} \zeta_K(1+\epsilon)\zeta_K(1+2\epsilon) < c_{11}(\epsilon)^n, \end{aligned} \tag{12}$$

where $c_{11}(\epsilon) = \max\{c_1(\epsilon), (c_2/\epsilon)^{[K:\mathbb{F}_q(X)]}\}$. The lemma follows from (10)–(12). □

Proof of Theorem 1. Suppose first that q is odd. Since the cases $n=2$ and $n=1$ of Theorem 1 follow directly from [Thunder 2013, Theorem 1, Corollary 1], we will assume that $n \geq 3$. We may also assume that $N_K(m) > 0$. Rearranging the sums and then using Lemma 6 yields

$$\left|\frac{1}{N_K(m)} \sum_{\substack{\mathfrak{C}\geq 0\\ \mathfrak{C}\notin 2\,\mathrm{Div}\,K}} \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=\mathfrak{C}\\ \deg\mathfrak{A}_n>2m-2g_K}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} q^{-s\deg\mathfrak{C}}\chi(F/\mathfrak{C})\right| \leq c_7(\epsilon)^{n+1} q^{-2m(\Re(s)-1/2)} \tag{13}$$

whenever $\Re(s) > \frac{1}{2}+(n-1)\epsilon$.

Let $\delta > 0$, to be chosen later. Using Proposition 2 and setting $\mathfrak{B}=\mathfrak{A}_1+\cdots+\mathfrak{A}_{n-1}$ in what follows, we have

$$\begin{aligned}
&\left|\sum_{\substack{\mathfrak{C}\geq 0\\ \mathfrak{C}\notin 2\,\mathrm{Div}\,K\\ \deg\mathfrak{C}<4m}} \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=\mathfrak{C}\\ \deg\mathfrak{A}_n\leq 2m-2g_K}} q^{-s\deg\mathfrak{C}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/\mathfrak{C})\right| \\
&\leq c''(\delta) q^m \sum_{\substack{\mathfrak{A}_n\geq 0\\ \deg\mathfrak{A}_n\leq 2m-2g_K}} q^{(\delta+1/4-\Re(s))\deg\mathfrak{A}_n} \\
&\qquad\times \sum_{\substack{\mathfrak{B}\geq 0\\ \deg\mathfrak{B}<4m-\deg\mathfrak{A}_n}} \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_{n-1}=\mathfrak{B}}} q^{(\delta+1/4-\Re(s))\deg\mathfrak{B}} \\
&\leq c''(\delta)c_1(\delta)^{n-2} q^m \sum_{\substack{\mathfrak{A}_n\geq 0\\ \deg\mathfrak{A}_n\leq 2m-2g_K}} q^{(\delta+1/4-\Re(s))\deg\mathfrak{A}_n} \sum_{\substack{\mathfrak{B}\geq 0\\ \deg\mathfrak{B}<4m-\deg\mathfrak{A}_n}} q^{((n-1)\delta+1/4-\Re(s))\deg\mathfrak{B}}.
\end{aligned} \tag{14}$$

If $\Re(s) \leq \frac{5}{4}+(n-2)\epsilon$ we set $\delta=\epsilon$ above. Since $\frac{5}{4}+(n-1)\epsilon-\Re(s)\geq\epsilon$, Lemma 3 implies that

$$\begin{aligned}
&\sum_{\substack{\mathfrak{A}_n\geq 0\\ \deg\mathfrak{A}_n\leq 2m-2g_K}} q^{(\epsilon+1/4-\Re(s))\deg\mathfrak{A}_n} \sum_{\substack{\mathfrak{B}\geq 0\\ \deg\mathfrak{B}<4m-\deg\mathfrak{A}_n}} q^{((n-1)\epsilon+1/4-\Re(s))\deg\mathfrak{B}} \\
&\qquad\leq \frac{c_2}{\epsilon} q^{4m(5/4+(n-1)\epsilon-\Re(s))} \sum_{\substack{\mathfrak{A}_n\geq 0\\ \deg\mathfrak{A}_n\leq 2m-2g_K}} q^{-(1+(n-2)\epsilon)\deg\mathfrak{A}_n} \\
&\qquad< \frac{c_2}{\epsilon}\zeta_K(1+(n-2)\epsilon) q^{4m(5/4+(n-1)\epsilon-\Re(s))} \\
&\qquad\leq \frac{c_2}{\epsilon}\left(\frac{c_2}{\epsilon}\right)^{[K:\mathbb{F}_q(X)]} q^{4m(5/4+(n-1)\epsilon-\Re(s))}.
\end{aligned} \tag{15}$$

If $\Re(s) > \frac{5}{4}+(n-1)\epsilon$ we set $n\delta = \Re(s)-\frac{5}{4}$ and note that $\delta > (n-2)\epsilon/n \geq \epsilon/3$ since $\Re(s) > 1+(n-1)\epsilon$. We thus may assume that $c''(\delta)c_1(\delta)^{n-2} \leq c_{12}(\epsilon)^{n-2}$

for some $c_{12}(\epsilon) > 0$ depending only on K and ϵ. Also, by Lemma 3,

$$\begin{aligned}(16)\quad \sum_{\substack{\mathfrak{A}_n \ge 0 \\ \deg \mathfrak{A}_n \le 2m - 2g_K}} q^{(\delta + 1/4 - \Re(s)) \deg \mathfrak{A}_n} \sum_{\substack{\mathfrak{B} \ge 0 \\ \deg \mathfrak{B} < 4m - \deg \mathfrak{A}_n}} q^{((n-1)\delta + 1/4 - \Re(s)) \deg \mathfrak{B}} & \\ < \sum_{\mathfrak{A}_n \ge 0} q^{-((n-1)\delta + 1) \deg \mathfrak{A}_n} \sum_{\mathfrak{B} \ge 0} q^{-(\delta+1) \deg \mathfrak{B}} & \\ < \zeta_K(1 + (n-1)\delta) \zeta_K(1+\delta) & \\ \le \left(\frac{c_2}{\delta} \right)^{2[K:\mathbb{F}_q(X)]} & \\ < \left(\frac{3c_2}{\epsilon} \right)^{2[K:\mathbb{F}_q(X)]} & .\end{aligned}$$

When $\deg \mathfrak{C} \ge 4m$, we trivially estimate

$$\begin{aligned}(17)\quad & \left| \sum_{\substack{\mathfrak{C} \ge 0 \\ \mathfrak{C} \notin 2 \operatorname{Div} K \\ \deg \mathfrak{C} \ge 4m}} \sum_{\substack{\mathfrak{A}_i \ge 0 \\ \mathfrak{A}_1 + \cdots + \mathfrak{A}_n = \mathfrak{C} \\ \deg \mathfrak{A}_n \le 2m - 2g_K}} q^{-s \deg \mathfrak{C}} \sum_{\substack{[F:K]=2 \\ g_F = m,\, q_F = q}} \chi(F/\mathfrak{C}) \right| \\ & \le N_K(m) \sum_{\substack{\mathfrak{A}_n \ge 0 \\ \deg \mathfrak{A}_n \le 2m - 2g_K}} q^{-\Re(s) \deg \mathfrak{A}_n} \sum_{\substack{\mathfrak{B} \ge 0 \\ \deg \mathfrak{B} \ge 4m - \deg \mathfrak{A}_n}} \sum_{\substack{\mathfrak{A}_i \ge 0 \\ \mathfrak{A}_1 + \cdots + \mathfrak{A}_{n-1} = \mathfrak{B}}} q^{-\Re(s) \deg \mathfrak{B}} \\ & \le N_K(m) c_1(\epsilon)^{n-2} \sum_{\substack{\mathfrak{A}_n \ge 0 \\ \deg \mathfrak{A}_n \le 2m - 2g_K}} q^{-\Re(s) \deg \mathfrak{A}_n} \sum_{\substack{\mathfrak{B} \ge 0 \\ \deg \mathfrak{B} \ge 4m - \deg \mathfrak{A}_n}} q^{((n-2)\epsilon - \Re(s)) \deg \mathfrak{B}}.\end{aligned}$$

Since $\Re(s) > 1 + (n-1)\epsilon$ by hypothesis, Lemma 3 implies that

$$(18)\quad \sum_{\substack{\mathfrak{B} \ge 0 \\ \deg \mathfrak{B} \ge 4m - \deg \mathfrak{A}_n}} q^{((n-2)\epsilon - \Re(s)) \deg \mathfrak{B}} \le \frac{c_2}{\epsilon} q^{(4m - \deg \mathfrak{A}_n)((n-2)\epsilon + 1 - \Re(s))}$$

and also

$$(19)\quad \sum_{\substack{\mathfrak{A}_n \ge 0 \\ \deg \mathfrak{A}_n \le 2m - 2g_K}} q^{-\Re(s) \deg \mathfrak{A}_n} q^{(\Re(s) - 1 - (n-2)\epsilon) \deg \mathfrak{A}_n} < \zeta_K(1 + (n-2)\epsilon) \le \left(\frac{c_2}{\epsilon} \right)^{[K:\mathbb{F}_q(X)]}.$$

Combining (9) with (13)–(19) yields

$$\begin{aligned}(20)\quad & \left| \frac{1}{N_K(m)} \sum_{\substack{\mathfrak{C} \ge 0 \\ \mathfrak{C} \notin 2 \operatorname{Div} K}} \sum_{\substack{\mathfrak{A}_i \ge 0 \\ \mathfrak{A}_1 + \cdots + \mathfrak{A}_n = \mathfrak{C}}} \sum_{\substack{[F:K]=2 \\ g_F = m,\, q_F = q}} q^{-s \deg \mathfrak{C}} \chi(F/\mathfrak{C}) \right| \\ & \qquad \le c_{13}(\epsilon)^{n+1} \left(q^{-m} + q^{-4m(\Re(s) - 1 - (n-1)\epsilon)} \right)\end{aligned}$$

for some $c_{13}(\epsilon) > 0$ depending only on K and ϵ. The case where q is odd (and $n \geq 3$) in Theorem 1 follows from Lemma 1, Lemma 5 and (20).

Suppose now that q is even. This time we use Lemma 6 to get

$$\left|\frac{1}{N_K(m)} \sum_{\substack{\mathfrak{C}\geq 0\\ \mathfrak{C}\notin 2\,\mathrm{Div}\,K}} \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=\mathfrak{C}\\ \deg\mathfrak{A}_i>2m-2g_K \text{ for some } i}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} q^{-s\deg\mathfrak{C}}\chi(F/\mathfrak{C})\right| \leq nc_7(\epsilon)^{n+1}q^{-2m(\Re(s)-1/2)} \tag{21}$$

Let $\delta > 0$, to be chosen later. By Proposition 3,

$$\left|\sum_{\substack{\mathfrak{C}\geq 0\\ \mathfrak{C}\notin 2\,\mathrm{Div}\,K}} \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=\mathfrak{C}\\ \deg\mathfrak{A}_i\leq 2m-2g_K}} q^{-s\deg\mathfrak{C}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/\mathfrak{C})\right| \leq c''(\delta)q^m\Bigg(\sum_{\substack{\mathfrak{A}\geq 0\\ \deg\mathfrak{A}\leq 2m-2g_K}} q^{(\delta-\Re(s))\deg\mathfrak{A}}\Bigg)^n. \tag{22}$$

If $\Re(s) \leq 1+\epsilon/2$ we set $\delta = \epsilon$. Since $1+\epsilon-\Re(s) \geq \epsilon/2$, Lemma 3 implies that

$$\sum_{\substack{\mathfrak{A}\geq 0\\ \deg\mathfrak{A}\leq 2m-2g_K}} q^{(\epsilon-\Re(s))\deg\mathfrak{A}} \leq \frac{2c_2}{\epsilon}q^{(2m-2g_K)(1+\epsilon-\Re(s))} \leq c_{14}(\epsilon)q^{-2m(\Re(s)-1-\epsilon)}, \tag{23}$$

where $c_{14}(\epsilon) > 0$ depends only on K and ϵ. If $\Re(s) > 1+\epsilon/2$ then we set $\delta = \epsilon/4$. We now have $c''(\delta) = c_{15}(\epsilon)$ and, by Lemma 3,

$$\sum_{\substack{\mathfrak{A}\geq 0\\ \deg\mathfrak{A}\leq 2m-2g_K}} q^{(\delta-\Re(s))\deg\mathfrak{A}} < \zeta_K(1+\epsilon/4) \leq \left(\frac{4c_2}{\epsilon}\right)^{[K:\mathbb{F}_q(X)]}. \tag{24}$$

Combining (9) and (21)–(24) gives

$$\left|\frac{1}{N_K(m)} \sum_{\substack{\mathfrak{C}\geq 0\\ \mathfrak{C}\notin 2\,\mathrm{Div}\,K}} \sum_{\substack{\mathfrak{A}_i\geq 0\\ \mathfrak{A}_1+\cdots+\mathfrak{A}_n=\mathfrak{C}}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} q^{-s\deg\mathfrak{C}}\chi(F/\mathfrak{C})\right| \leq c_{16}(\epsilon)^n(q^{-m}+q^{-2mn(\Re(s)-1-\epsilon+1/2n)}) \tag{25}$$

for some $c_{16}(\epsilon) > 0$ depending only on K and ϵ. The case where q is even in Theorem 1 follows from Lemma 1, Lemma 5 and (25). □

4. Proof of Theorem 2 and Corollaries

Proof of Theorem 2. We know that $M_K(s, m, 1)$ is a polynomial in q^{-s} thanks to a theorem of Weil (see [Rosen 2002, Theorem 9.16B]). It's an even function of q^{-s} by Lemma 2 and Lemma 1. Also, $a_0 = 1$ since $\chi(F/0) = 1$ by definition. The functional equation for $M_K(s, m, 1)$ follows directly from the functional equations for $L_K(q^{-s})$ and $L_F(q^{-s})$ for all quadratic extensions $F \supset K$. The identity

$$a_{2(m-g_K-j)} = q^{m-g_K-2j} a_{2j}, \quad j = 0, \ldots, m - g_K,$$

follows immediately from the functional equation.

Similar to the proof of Lemma 4, for an effective divisor $\mathfrak{C} \in \operatorname{Div} K$ set

$$\theta(\mathfrak{C}) = q^{-s \deg \mathfrak{C}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1}$$

and set $f(s) = \sum_{\mathfrak{C} \geq 0} \theta(\mathfrak{C})$. Since $\theta(\mathfrak{C} + \mathfrak{D}) = \theta(\mathfrak{C})\theta(\mathfrak{D})$ whenever $\mathfrak{C}$ and $\mathfrak{D}$ have disjoint support,

$$\begin{aligned}
f(s) &= \prod_{v \in M(K)} \left(1 + \sum_{k=1}^{\infty} \theta(kv)\right) \\
&= \prod_{v \in M(K)} \left(1 + \frac{q^{-s \deg v}}{(1 + q^{-\deg v})(1 - q^{-s \deg v})}\right) \\
&= \prod_{v \in M(K)} \left(1 + \frac{q^{-s \deg v}(1 - q^{-\deg v})}{(1 + q^{-2 \deg v})(1 - q^{-s \deg v})}\right) \\
&= \zeta_K(2)\zeta_K(s) \prod_{v \in M(K)} \big((1 - q^{-2 \deg v})(1 - q^{-s \deg v}) + q^{-s \deg v}(1 - q^{-\deg v})\big) \\
&= \zeta_K(2)\zeta_K(s) \prod_{v \in M(K)} (1 - q^{-2 \deg v} - q^{-(s+1) \deg v} + q^{-(s+2) \deg v}).
\end{aligned}$$

For any $\epsilon > 0$, $f(s)$ is holomorphic on $\{s \in \mathbb{C} : \Re(s) \geq \epsilon, -\pi/\log q \leq \Im(s) < \pi/\log q\}$ except for a simple pole at $s = 1$, where the residue is

$$\begin{aligned}
\operatorname{Res}_{s=1} f(s) &= \zeta_K(2) \prod_{v \in M(K)} (1 - 2q^{-2 \deg v} + q^{-3 \deg v}) \operatorname{Res}_{s=1} \zeta_K(s) \\
&= \frac{J_K q^{1-g_K} \zeta_K(2)}{(q-1) \log q} \prod_{v \in M(K)} (1 - 2q^{-2 \deg v} + q^{-3 \deg v})
\end{aligned}$$

(see [Weil 1974, Chapter VII], for example, for the residue of the zeta function). Now by a Tauberian argument (see [Rosen 2002, Theorem 17.1], for example)

$$\begin{aligned}
&\sum_{\substack{\mathfrak{C} \geq 0 \\ \deg \mathfrak{C} = j}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1} \\
&\qquad = q^j \frac{J_K q^{1-g_K} \zeta_K(2)}{q-1} \prod_{v \in M(K)} (1 - q^{-2 \deg v} + q^{-3 \deg v}) + O(M q^{\epsilon j}),
\end{aligned}$$

where the implicit constant is absolute and

$$M = \max_{\Re(s)=\epsilon} |f(s)|,$$

which is clearly bounded above by a constant that depends only on K and ϵ. Therefore

$$\sum_{\substack{\mathfrak{C}\geq 0\\ \deg \mathfrak{C}=j}} \prod_{v\in \operatorname{Supp}\mathfrak{C}} (1+q^{-\deg v})^{-1} = q^j \frac{J_K q^{1-g_K}\zeta_K(2)}{q-1} \prod_{v\in M(K)} (1-q^{-2\deg v}+q^{-3\deg v}) + O(q^{\epsilon j}), \tag{26}$$

where the implicit constant depends only on K and ϵ.

We may assume that $N_K(m) > 0$. For the remainder of the proof, all implicit constants depend only on K and ϵ. Fix an index j between 0 and $m - g_K$ and an $\epsilon > 0$. Then by Lemma 1 (separating out those divisors of degree $2j$ that are twice an effective divisor and those that aren't)

$$\begin{aligned} N_K(m)a_{2j} &= \sum_{\substack{\mathfrak{C}\geq 0\\ \deg \mathfrak{C}=2j}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/\mathfrak{C}) \\ &= \sum_{\substack{\mathfrak{C}\geq 0\\ \deg \mathfrak{C}=j}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/2\mathfrak{C}) + \sum_{\substack{\mathfrak{C}\geq 0\\ \mathfrak{C}\notin 2\operatorname{Div}K\\ \deg \mathfrak{C}=2j}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/\mathfrak{C}). \end{aligned} \tag{27}$$

Now, by (0) and Proposition 1,

$$\begin{aligned} \sum_{\substack{\mathfrak{C}\geq 0\\ \deg \mathfrak{C}=j}} \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}} \chi(F/2\mathfrak{C}) &= N'_K(m) \sum_{\substack{\mathfrak{C}\geq 0\\ \deg \mathfrak{C}=j}} \prod_{v\in \operatorname{Supp}\mathfrak{C}} (1+q^{-\deg v})^{-1} \\ &\quad + \begin{cases} O(q^{(1/2+\epsilon)m}q^{(1+\epsilon)j}) & \text{if } q \text{ is odd,}\\ O(q^m q^j + q^{\epsilon m}q^{(1+\epsilon)j}) & \text{if } q \text{ is even.}\end{cases} \end{aligned} \tag{28}$$

Using the estimate for $|N'_K(m) - N_K(m)|$ in Proposition 1 and (26), we get

$$\begin{aligned} &N'_K(m) \sum_{\substack{\mathfrak{C}\geq 0\\ \deg \mathfrak{C}=j}} \prod_{v\in \operatorname{Supp}\mathfrak{C}} (1+q^{-\deg v})^{-1} \\ &\quad = N_K(m) \sum_{\substack{\mathfrak{C}\geq 0\\ \deg \mathfrak{C}=j}} \prod_{v\in \operatorname{Supp}\mathfrak{C}} (1+q^{-\deg v})^{-1} + \begin{cases} O(q^{(1/2+\epsilon)m}q^j) & \text{if } q \text{ is odd,}\\ O(q^m q^j) & \text{if } q \text{ is even.}\end{cases} \end{aligned} \tag{29}$$

Combining (28), (29) and (9) yields

$$(30)\quad (N_K(m))^{-1}\sum_{\substack{\mathfrak{C}\geq 0\\ \deg\mathfrak{C}=j}}\ \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}}\chi(F/2\mathfrak{C})$$
$$=\sum_{\substack{\mathfrak{C}\geq 0\\ \deg\mathfrak{C}=j}}\ \prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1}+\begin{cases}O(q^{-(3/2-\epsilon)m}q^{(1+\epsilon)j}) & \text{if } q \text{ is odd,}\\ O(q^{-m}q^{j}) & \text{if } q \text{ is even.}\end{cases}$$

Using (0) in conjunction with Propositions 2 and 3, we get

$$\sum_{\substack{\mathfrak{C}\geq 0\\ \mathfrak{C}\notin 2\operatorname{Div}K\\ \deg\mathfrak{C}=2j}}\ \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}}\chi(F/\mathfrak{C})=\begin{cases}O(q^{m}q^{(5/4+\epsilon)2j}) & \text{if } q \text{ is odd,}\\ O(q^{m}q^{(1+\epsilon)2j}) & \text{if } q \text{ is even.}\end{cases}$$

Combining this with (9) yields

$$(31)\quad (N_K(m))^{-1}\sum_{\substack{\mathfrak{C}\geq 0\\ \mathfrak{C}\notin 2\operatorname{Div}K\\ \deg\mathfrak{C}=2j}}\ \sum_{\substack{[F:K]=2\\ g_F=m,\, q_F=q}}\chi(F/\mathfrak{C})=\begin{cases}O(q^{-m}q^{(5/4+\epsilon)2j}) & \text{if } q \text{ is odd,}\\ O(q^{-m}q^{(1+\epsilon)2j}) & \text{if } q \text{ is even.}\end{cases}$$

Finally, by (27), (30) and (31),

$$a_{2j}=\sum_{\substack{\mathfrak{C}\geq 0\\ \deg\mathfrak{C}=j}}\ \prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1}+\begin{cases}O(q^{-m}q^{(5/4+\epsilon)2j}) & \text{if } q \text{ is odd,}\\ O(q^{-m}q^{(1+\epsilon)2j}) & \text{if } q \text{ is even.}\end{cases}$$

This completes the proof of Theorem 2. □

Proof of Corollary 1. Set $m'=m-g_K$ in what follows for notational convenience. We first note that $a_{2j}=O(q^{j})$ for all $j=0,\dots,m'$ by Theorem 2, where the implicit constant depends only on K. Let $x\leq 1$ to be chosen later. Then, whenever $\Re(s)>\frac{1}{2}$,

$$(32)\quad M_K(s,m,1)=\sum_{j\leq xm'}a_{2j}q^{-2sj}+O\Bigl(\sum_{j>xm'}q^{-j(2\Re(s)-1)}\Bigr)$$
$$=\sum_{j\leq xm'}a_{2j}q^{-2sj}+O\Bigl(\frac{q^{-xm'(2\Re(s)-1)}}{1-q^{1-2\Re(s)}}\Bigr),$$

where the implicit constants depend only on K. Also, by Theorem 2, for any $\delta>0$,

$$(33)\quad \sum_{j\leq xm'}a_{2j}q^{-2sj}=\sum_{j\leq xm'}\ \sum_{\substack{\mathfrak{C}\geq 0\\ \deg\mathfrak{C}=j}}q^{-2s\deg\mathfrak{C}}\prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1}$$
$$+\begin{cases}O\bigl(q^{-m}\sum_{j\leq xm'}q^{2j(5/4+\delta-\Re(s))}\bigr) & \text{if } q \text{ is odd,}\\ O\bigl(q^{-m}\sum_{j\leq xm'}q^{2j(1+\delta-\Re(s))}\bigr) & \text{if } q \text{ is even,}\end{cases}$$

$$= \sum_{j \le xm'} \sum_{\substack{\mathfrak{C} \ge 0 \\ \deg \mathfrak{C} = j}} q^{-2s \deg \mathfrak{C}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1} + \begin{cases} O(q^{-m}(1 + q^{2xm'(5/4+\delta-\Re(s))})) & \text{if } q \text{ is odd,} \\ O(q^{-m}(1 + q^{2xm'(1+\delta-\Re(s))})) & \text{if } q \text{ is even,} \end{cases}$$

where the implicit constants depend only on K and δ. We may assume that $\epsilon \le \frac{1}{6}$. We now choose x and δ such that

$$x = \begin{cases} \dfrac{1}{3/2+2\delta} = \dfrac{2}{3} - \epsilon & \text{if } q \text{ is odd,} \\ \dfrac{1}{1+2\delta} = 1 - \epsilon & \text{if } q \text{ is even,} \end{cases}$$

so that

$$-xm'(2\Re(s) - 1) = \begin{cases} -m' + 2xm'\left(\frac{5}{4} + \delta - \Re(s)\right) & \text{if } q \text{ is odd,} \\ -m' + 2xm'(1 + \delta - \Re(s)) & \text{if } q \text{ is even.} \end{cases}$$

Then, by (32), (33) and the definition of m',

$$M_K(s, m, 1) = \sum_{j \le xm'} \sum_{\substack{\mathfrak{C} \ge 0 \\ \deg \mathfrak{C} = j}} q^{-2s \deg \mathfrak{C}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1} + \begin{cases} O\left(\dfrac{q^{-m(2/3-\epsilon)(2\Re(s)-1)}}{(1 - q^{1-2\Re(s)})}\right) & \text{if } q \text{ is odd,} \\ O\left(\dfrac{q^{-m(1-\epsilon)(2\Re(s)-1)}}{(1 - q^{1-2\Re(s)})}\right) & \text{if } q \text{ is even,} \end{cases} \tag{34}$$

where the implicit constants depend only on K and ϵ. Also, by Theorem 2,

$$\begin{aligned} &\sum_{j \le xm'} \sum_{\substack{\mathfrak{C} \ge 0 \\ \deg \mathfrak{C} = j}} q^{-2s \deg \mathfrak{C}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1} \\ &\quad = \sum_{\mathfrak{C} \ge 0} q^{-2s \deg \mathfrak{C}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1} + O\left(\sum_{j > xm'} q^{-j(2\Re(s)-1)}\right) \\ &\quad = \sum_{\mathfrak{C} \ge 0} q^{-2s \deg \mathfrak{C}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1} \\ &\qquad + \begin{cases} O\left(\dfrac{q^{-m(2/3-\epsilon)(2\Re(s)-1)}}{(1 - q^{1-2\Re(s)})}\right) & \text{if } q \text{ is odd,} \\ O\left(\dfrac{q^{-m(1-\epsilon)(2\Re(s)-1)}}{(1 - q^{1-2\Re(s)})}\right) & \text{if } q \text{ is even.} \end{cases} \end{aligned} \tag{35}$$

Finally, by Lemma 3,

$$\sum_{\mathfrak{C} \ge 0} q^{-2s \deg \mathfrak{C}} \prod_{v \in \operatorname{Supp} \mathfrak{C}} (1 + q^{-\deg v})^{-1} = \sigma_K(s, 1). \tag{36}$$

Corollary 1 follows from (34)–(36). □

Proof of Corollary 2. Set $m' = m - g_K$ again. By Theorem 2,

$$\sum_{j\ge m'/2}\Bigg(\sum_{\substack{\mathfrak{C}\ge 0\\ \deg\mathfrak{C}=j}} q^{-\deg\mathfrak{C}}\prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1}\Bigg)-C(K) = O\Bigg(\sum_{j\ge m'/2} q^{-j(1-2\epsilon)}\Bigg) = O(q^{-m'(1/2-\epsilon)}), \tag{37}$$

where the implicit constant depends only on K and ϵ. This shows that the series $C'(K)$ converges. Also, by Theorem 2,

$$M_K\big(\tfrac12, m, 1\big) = \begin{cases} 2\sum_{j<m'/2} a_{2j}q^{-j} & \text{if } m' \text{ is odd,}\\ 2\sum_{j<m'/2} a_{2j}q^{-j} + a_{m'}q^{-m'/2} & \text{if } m' \text{ is even,}\end{cases} \tag{38}$$

and

$$\begin{aligned} 2\sum_{j<m'/2} a_{2j}q^{-j} &= 2\sum_{j<m'/2}\sum_{\substack{\mathfrak{C}\ge 0\\ \deg\mathfrak{C}=j}} q^{-\deg\mathfrak{C}}\prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1} + O\Bigg(\sum_{j<m'/2} q^{-m'}q^{j(3/2+2\epsilon)}\Bigg)\\ &= 2\sum_{j<m'/2}\sum_{\substack{\mathfrak{C}\ge 0\\ \deg\mathfrak{C}=j}} q^{-\deg\mathfrak{C}}\prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1} + O(q^{-m'(1/4-\epsilon)}) \end{aligned} \tag{39a}$$

if q is odd. If q is even, similar estimates give

$$2\sum_{j<m'/2} a_{2j}q^{-j} = 2\sum_{j<m'/2}\sum_{\substack{\mathfrak{C}\ge 0\\ \deg\mathfrak{C}=j}} q^{-\deg\mathfrak{C}}\prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1} + O(q^{-m'(1/2-\epsilon)}). \tag{39b}$$

Now, by (37),

$$\begin{aligned} &2\sum_{j<m'/2}\sum_{\substack{\mathfrak{C}\ge 0\\ \deg\mathfrak{C}=j}} q^{-\deg\mathfrak{C}}\prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1}\\ &= 2C'(K) + 2\sum_{j<m'/2} C(K)\\ &\qquad - 2\sum_{j\ge m'/2}\Bigg(\sum_{\substack{\mathfrak{C}\ge 0\\ \deg\mathfrak{C}=j}} q^{-\deg\mathfrak{C}}\prod_{v\in\operatorname{Supp}\mathfrak{C}}(1+q^{-\deg v})^{-1}\Bigg) - C(K)\\ &= 2C'(K) + O(q^{-m'(1/2-\epsilon)}) + \begin{cases} m'C(K) & \text{if } m' \text{ is even,}\\ (m'+1)C(K) & \text{if } m' \text{ is odd.}\end{cases} \end{aligned} \tag{40}$$

Finally, if m' is even, Theorem 2 gives

$$a_{m'}q^{-m'/2} = C(K) + O(q^{-m'(1/2-\epsilon)}). \tag{41}$$

The remainder of Corollary 2 follows from (38)–(41). □

Proof of Corollary 3. It is well known that $\#\{v \in M(F) : \deg v = 1\} - q - 1$ is equal to the coefficient of q^{-s} in the polynomial $L_F(q^{-s})$ for all function fields F with $q_F = q$. (See [Stichtenoth 1993, Theorem V.1.15], for example.) By Theorem 2, the coefficient of q^{-s} in the polynomial $L_K(q^{-s})M_K(s, m, 1)$ is just the coefficient of q^{-s} in the polynomial $L_K(q^{-s})$. □

References

[Andrade and Keating 2012] J. C. Andrade and J. P. Keating, "The mean value of $L(\frac{1}{2}, \chi)$ in the hyperelliptic ensemble", *J. Number Theory* **132**:12 (2012), 2793–2816. MR 2965192 Zbl 1278.11082

[Chinta et al. 2006] G. Chinta, S. Friedberg, and J. Hoffstein, "Multiple Dirichlet series and automorphic forms", pp. 3–41 in *Multiple Dirichlet series, automorphic forms, and analytic number theory* (Bretton Woods, NH, 2005), edited by S. Friedberg et al., Proc. Sympos. Pure Math. **75**, American Mathematical Society, Providence, RI, 2006. MR 2008c:11072 Zbl 1124.11023

[Fisher and Friedberg 2004] B. Fisher and S. Friedberg, "Double Dirichlet series over function fields", *Compos. Math.* **140**:3 (2004), 613–630. MR 2005a:11183 Zbl 1082.11053

[Goldfeld and Hoffstein 1985] D. Goldfeld and J. Hoffstein, "Eisenstein series of $\frac{1}{2}$-integral weight and the mean value of real Dirichlet L-series", *Invent. Math.* **80**:2 (1985), 185–208. MR 86m:11029 Zbl 0564.10043

[Hoffstein and Rosen 1992] J. Hoffstein and M. Rosen, "Average values of L-series in function fields", *J. Reine Angew. Math.* **426** (1992), 117–150. MR 93c:11022 Zbl 0754.11036

[Rosen 2002] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics **210**, Springer, New York, 2002. MR 2003d:11171 Zbl 1043.11079

[Siegel 1944] C. L. Siegel, "The average measure of quadratic forms with given determinant and signature", *Ann. of Math.* (2) **45** (1944), 667–685. MR 7,51a Zbl 0063.07007

[Stichtenoth 1993] H. Stichtenoth, *Algebraic function fields and codes*, Graduate Texts in Mathematics **254**, Springer, Berlin, 1993. MR 94k:14016 Zbl 0816.14011

[Thunder 2013] J. L. Thunder, "On sums involving products and quotients of L-functions over function fields", preprint, 2013. arXiv 1310.8572

[Thunder and Widmer 2013] J. L. Thunder and M. Widmer, "Counting points of fixed degree and given height over function fields", *Bull. Lond. Math. Soc.* **45**:2 (2013), 283–300. MR 3064414 Zbl 06160957

[Weil 1974] A. Weil, *Basic number theory*, 3rd ed., Die Grundlehren der Mathematischen Wissenschaften **144**, Springer, New York, 1974. MR 55 #302 Zbl 0326.12001

Received December 10, 2013. Revised July 10, 2014.

JEFFREY LIN THUNDER
DEPARTMENT OF MATHEMATICAL SCIENCES
NORTHERN ILLINOIS UNIVERSITY
DEKALB, IL 60115-2874
UNITED STATES

jthunder@math.niu.edu

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2015 is US \$420/year for the electronic version, and \$570/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 274 No. 1 March 2015

0030-8730(201503)274:1;1-3