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THE WELL-POSEDNESS OF NONLINEAR SCHRÖDINGER EQUATIONS IN TRIEBEL-TYPE SPACES

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Sufficient and necessary conditions for embeddings between $F^s_{p,q}$ (Triebel spaces) and $N^s_{p,q}$ (new spaces constructed by combining frequency-uniform decomposition with $L^p(\ell^q)$) are obtained. Moreover, we study the Cauchy problem for generalized nonlinear Schrödinger equations in $L^r(0, T, N^s_{p,q})$.

1. Introduction and notation

We study NLS (nonlinear Schrödinger equations) by using frequency-uniform decomposition techniques. Suppose that Q_k is the unit cube centered at k and $\{Q_k\}_{k\in\mathbb{Z}^n}$ is a decomposition of $\mathbb{R}^n$. Such decompositions go back to the work of N. Wiener [1932], and we call them Wiener decompositions of $\mathbb{R}^n$. We can write

$$\Box_k \sim F^{-1}\chi_{Q_k}F \quad \text{for } k\in\mathbb{Z}^n, \tag{1}$$

where χ_E denotes the characteristic function on the set E. Since Q_k is just a translation of Q_0, the $\Box_k$ have the same localized structures in the frequency space, and are called the frequency-uniform decomposition operators.

Compared with the dyadic decomposition, the frequency-uniform decomposition has many advantages for the Schrödinger semigroup. It is known that

$$S(t) = e^{it\Delta} : L^p \to L^p$$

if and only if $p = 2$. This is one of the main reasons that we can not solve NLS in $L^p (p \neq 2)$. However, if we consider the frequency-uniform decomposition, we have

$$\|\Box_k S(t) f\|_{L^p} \lesssim (1+|t|)^{n|1/2-1/p|}\|\Box_k f\|_{L^p},$$

which enables us to solve NLS in frequency-uniform decomposition spaces.

Roughly speaking, combining dyadic decomposition operators with function spaces $L^p(\ell^q)$, we can introduce Triebel spaces [Triebel 1992]. Combining frequency-uniform decomposition operators with function spaces $L^p(\ell^q)$, we can introduce

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Triebel-type spaces. From a PDE point of view, the combination of frequency-uniform decomposition operators and Banach function spaces $L^p(\ell^q)$ seems to be important in making nonlinear estimates, which contains an automatic decomposition on high-low frequencies.

We now give an exact definition on frequency-uniform decomposition operators. Since χ_{Q_k} is not differentiable, one needs to replace χ_{Q_k} in (1) by a smooth cut-off function. We first denote $|\xi|_\infty := \max_{i=1,\dots,n} |\xi_i|$. Let $\rho \in S(\mathbb{R}^n) : \mathbb{R}^n \to [0,1]$ be a smooth radial bump function adapted to $\{\xi : |\xi_i|_\infty \leqslant 1\}$ with $\rho(\xi) = 1$ if $|\xi|_\infty \leqslant \frac{1}{2}$, and $\rho(\xi) = 0$ if $|\xi|_\infty \geqslant 1$. Let ρ_k be a translation of ρ:

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^n.$$

Since $\rho_k(\xi) = 1$ in the unit closed cube $Q_k : \{\xi \in \mathbb{R}^n : |\xi - k|_\infty \leqslant \frac{1}{2}\}$ and $\{Q_k\}_{k \in \mathbb{Z}^n}$ is a covering of $\mathbb{R}^n$, one has that for all $\xi \in \mathbb{R}^n$, $\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geqslant 1$. Set

$$\sigma_k(\xi) = \rho_k(\xi) \left(\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n.$$

Then we have

$$(\sharp) \qquad \begin{cases} |\sigma_k(\xi)| \geqslant c & \xi \in Q_k, \\ \operatorname{supp} \sigma_k \subset \{\xi : |\xi - k|_\infty \leq 1\}, & \\ \sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) \equiv 1 & \text{for all } \xi \in \mathbb{R}^n \\ |D^\alpha \sigma_k(\xi)| \leqslant C_{|\alpha|}, & \text{for all } \xi \in \mathbb{R}^n,\ \alpha \in (N \cup \{0\})^n. \end{cases}$$

Hence, the set

$$\Upsilon_n = \{\{\sigma_k\}_{k \in \mathbb{Z}^n} : \{\sigma_k\}_{k \in \mathbb{Z}^n} \text{ satisfies } (\sharp)\}$$

is nonempty. Let $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon_n$ be a function sequence. We call $\{\Box_k\}_{k \in \mathbb{Z}^n}$ the frequency-uniform decomposition operators, where $\Box_k := F^{-1} \sigma_k F$, $k \in \mathbb{Z}^n$. If we combine these decompositions with $L^p(\ell^q)$, we can introduce a new type of function spaces as follows. For any $k \in \mathbb{Z}^n$, we set $|k| = |k_1| + \cdots + |k_n|$, $\langle k \rangle = 1 + |k|$. If $0 < p < \infty$, $0 < q \leqslant \infty$, for any $s \in \mathbb{R}$, we denote

$$N^s_{p,q}(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{N^s_{p,q}} = \left\| \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} |\Box_k f|^q \right)^{1/q} \right\|_p < \infty \right\}.$$

If $p = \infty$, $0 < q \leqslant \infty$, for any $s \in \mathbb{R}$, we set

$$(3) \quad N^s_{\infty,q}(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) : \exists \{f_k(x)\}_{k=0}^\infty \subset L^\infty(\mathbb{R}^n) \text{ such that} \\ f = \sum_{k=0}^\infty F^{-1} \sigma_k F f_k \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \|\langle k \rangle^s f_k\|_{L^\infty(\mathbb{R}^n, \ell^q)} < \infty \Big\},$$

and

$$\|f\|_{N^s_{\infty,q}(\mathbb{R}^n)} = \inf \|\langle k \rangle^s f_k\|_{L^\infty(\mathbb{R}^n, \ell^q)},$$

where the infimum is taken over all admissible representations of f in the sense of (3).

One purpose of this paper is to study the semilinear estimates, dual estimates, Strichartz estimates and time-space estimates in the Triebel-type spaces. Furthermore, from the definitions, we see that Triebel spaces and Triebel-type spaces are rather similar; both of them are the combinations of frequency decomposition operators and function spaces $L^p(\ell^q)$. In fact, we have the following inclusion results:

Theorem 1.1. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s_1, s_2 \in \mathbb{R}$.

(a) *If* $s_1 > s_2 + \sigma(p,q)$ (*when* $\sigma(p,q) = 0$, $s_1 \geqslant s_2$), *then* $N^{s_1}_{p,q} \subset F^{s_2}_{p,q}$, *where*

$$\sigma(p,q) = \max\Big\{0, n\Big(\frac{1}{p}-\frac{1}{q}\Big), n\Big(1-\frac{1}{q}-\frac{1}{p}\Big)\Big\}.$$

(b) *If* $s_1 > s_2 + \tau(p,q)$ (*when* $\tau(p,q) = 0$, $s_1 \geqslant s_2$), *then* $F^{s_1}_{p,q} \subset N^{s_2}_{p,q}$, *where*

$$\tau(p,q) = \max\Big\{0, n\Big(\frac{1}{q}-\frac{1}{p}\Big), n\Big(\frac{1}{q}+\frac{1}{p}-1\Big)\Big\}.$$

Theorem 1.2. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s_1, s_2 \in \mathbb{R}$.

(a) *If* $F^{s_1}_{p,q} \subset N^{s_2}_{p,q}$, *then* $s_1 \geqslant s_2 + \tau(p,q)$, *where*

$$\tau(p,q) = \max\Big\{0, n\Big(\frac{1}{q}-\frac{1}{p}\Big), n\Big(\frac{1}{q}+\frac{1}{p}-1\Big)\Big\}.$$

(b) *If* $N^{s_1}_{p,q} \subset F^{s_2}_{p,q}$, *then* $s_1 \geqslant s_2 + \sigma(p,q)$, *where*

$$\sigma(p,q) = \max\Big\{0, n\Big(\frac{1}{p}-\frac{1}{q}\Big), n\Big(1-\frac{1}{q}-\frac{1}{p}\Big)\Big\}.$$

In recent decades, a large amount of work has been devoted to the study of well-posedness in Besov and modulation spaces (for example, see [Kato 1987; 1989; 1995; Cazenave and Weissler 1989; 1990; Wang 1993; Kenig et al. 1995; Pecher 1997; Nakamura and Ozawa 1998; Cazenave 2003; Wang et al. 2006; 2009; 2011; Wang and Huang 2007; Wang and Hudzik 2007; Chen and Fan 2011]). Our second goal is to explore solutions of NLS in the Triebel-type spaces. We will use the smooth effect estimates, together with the frequency-uniform decomposition techniques, to study NLS, and we show that it is locally well-posed in a class of Triebel-type spaces.

Theorem 1.3. *Assume* $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$, $2 \leqslant p \leqslant k+2$, $r \geqslant k+2$; *then for any initial data* $u_0 \in H^s$, $s > \frac{1}{2}n$, *there exists* $T^* := T^*(\|u_0\|_{H^s}) > 0$ *such that the initial value problem*

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau)f(u(\tau))\,d\tau$$

has a unique solution

$$u \in C(0, T^*, H^s) \cap L^r_{\mathrm{loc}}(0, T^*; N^s_{p,2}).$$

Moreover, if $T^* < \infty$, then

$$\|u\|_{C(0,T^*,H^s)\cap L^r(0,T^*;N^s_{p,2})} = \infty,$$

where $S(t) = F^{-1}e^{-it|\xi|^2}F$.

Theorem 1.4. *Assume $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$, $2 \leqslant p \leqslant k+2$, $r \geqslant k+2$, $p' \leqslant q \leqslant p$, then for any initial data $u_0 \in N^s_{p',q}$, $s > n(1-\frac{1}{q})$, there exists T such that the initial value problem*

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau) f(u(\tau))\,d\tau$$

has a unique solution

$$u \in L^r_{\mathrm{loc}}(0, T; N^s_{p,q}).$$

The rest of this paper is divided into five sections and an appendix. In Section 2 we will state some properties of $N^s_{p,q}$, which are useful to establish the embedding inclusions between Triebel spaces and $N^s_{p,q}$. In Section 3 we will prove Theorems 1.1 and 1.2. Section 4 is devoted to considering the multiplication algebra of $N^s_{p,q}$. Some dispersive smooth effects for the Schrödinger semigroup will be given in Section 5 and Theorems 1.3 and 1.4 will be proved in Section 6. The Appendix derives a complex interpolation in Triebel-type spaces (Theorem A.14) and shows some properties of the modulation spaces (Theorems A.1–A.8) that are used in Sections 2–5.

Notation. Throughout the paper, we set

$$L^r(\mathbb{R}, N^s_{p,q}) := \left\{ f \in \mathcal{S}' : \left(\int_{\mathbb{R}} \|f\|^r_{N^s_{p,q}}\,dt \right)^{1/r} < \infty \right\}.$$

We shall sometimes write $X \lesssim Y$ to denote the estimate $X \leqslant CY$ for some C. For any $s \in \mathbb{R}$, $0 < p, q \leq \infty$, we set

$$M^s_{p,q}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\Box_k f\|^q_{L^p} \right)^{1/q} < \infty \right\};$$

$M^s_{p,q} := M^s_{p,q}(\mathbb{R}^n)$ is called a modulation space, first introduced by Feichtinger [2003] in the case $1 \leqslant p, q \leqslant \infty$.

2. Basic properties of $N^s_{p,q}$

In order to study the Cauchy problem in $N^s_{p,q}$, we first give some properties of $N^s_{p,q}$:

Proposition 2.1. *Let* $-\infty < s < \infty$, $0 < p < \infty$, $0 < q \leqslant \infty$. *The following inclusions hold*:

(1) *Let* $q_1 \le q_2$. *Then*

$$N^s_{p,q_1} \subset N^s_{p,q_2}.$$

(2) *Let* $\varepsilon q_2 > n$. *Then*

$$N^{s+\varepsilon}_{p,q_1} \subset N^s_{p,q_2}.$$

(3) *Let* $p < \infty$. *Then*

$$M^s_{p,p\wedge q} \subset N^s_{p,q} \subset M^s_{p,p\vee q}.$$

Proof. Since $\ell^p \subset \ell^{p+a}, a \ge 0$, we can get (1) directly. Let us observe that

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq_2}|a_k|^{q_2}\Big)^{1/q_2} &= \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(s+\varepsilon)q_2}\langle k\rangle^{-\varepsilon q_2}|a_k|^{q_2}\Big)^{1/q_2}\\ &\lesssim \sup_{k\in\mathbb{Z}^n} \langle k\rangle^{s+\varepsilon}|a_k| \qquad\qquad (\varepsilon q_2 > n).\end{aligned}$$

Taking $a_k = \Box_k f$, we can show that (2) holds with the help of (1).

Finally we prove (3). Let $b_k = \langle k\rangle^s \Box_k f$. There are two cases:

Case 1. $q \le p$. In this case, we have

$$\|b_k\|_{\ell^p(L^p)} \le \|b_k\|_{L^p(\ell^q)} \le \|b_k\|_{\ell^q(L^p)}.$$

Actually, noticing that $\ell^q \subset \ell^p$, we have $\|b_k\|_{\ell^p(L^p)} = \|b_k\|_{L^p(\ell^p)} \le \|b_k\|_{L^p(\ell^q)}$. So, the first part of the above inequality holds. Moreover, by Minkowski's inequality, we have

$$\|b_k\|_{L^p(\ell^q)} = \Big\|\sum_{k=0}^{\infty} |b_k|^q\Big\|^{1/q}_{L^{p/q}} \leqslant \Big(\sum_{k=0}^{\infty} \||b_k|^q\|_{L^{p/q}}\Big)^{1/q} = \|b_k\|_{\ell^q(L^p)}.$$

This proves the second part.

Case 2. $p \le q$. By Minkowski's inequality and $\ell^p \subset \ell^q$, we have

$$\begin{aligned}\|b_k\|_{\ell^q(L^p)} &= \bigg(\Big\|\int_{\mathbb{R}^n} |b_k|^p\,dx\Big\|_{\ell^{q/p}}\bigg)^{1/p} \leqslant \Big(\int_{\mathbb{R}^n} \||b_k|^p\|_{\ell^{q/p}}\,dx\Big)^{1/p}\\ &= \|b_k\|_{L^p(\ell^q)} \leqslant \|b_k\|_{\ell^p(L^p)},\end{aligned}$$

$\square$

Proposition 2.2 (completeness). *For any $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$, we have*:

(1) *$N^s_{p,q}$ is a quasi-Banach space. Moreover, if $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, then $N^s_{p,q}$ is a Banach space.*

(2) $\mathcal{S}(\mathbb{R}^n) \subset N^s_{p,q} \subset \mathcal{S}'(\mathbb{R}^n)$.

(3) *If $0 < p, q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $N^s_{p,q}$.*

Proof.

Step 1. Thanks to [Wang and Hudzik 2007], we obtain that $\mathcal{S} \subset M^s_{p,\infty} \subset \mathcal{S}'$.

Step 2. We prove that $\mathcal{S} \subset N^s_{p,q}$. From Step 1, we have that $\mathcal{S} \subset M^{s+\varepsilon}_{p,\infty}$. By Proposition 2.1 and Theorem A.5, we know that $M^{s+\varepsilon}_{p,\infty} \subset M^s_{p,p\wedge q} \subset M^s_{p,q} \cap N^s_{p,q}$ ($\varepsilon > n/p \wedge q$). Thus, we have the desired result.

Step 3. Similarly to Step 2, we can also prove that $N^s_{p,q} \subset \mathcal{S}'$ (by $M^s_{p,p\vee q} \subset \mathcal{S}'$). We omit the details of the proof.

Step 4. $N^s_{p,q}$ is a quasinormed space. Now we prove completeness. Let $\{f_\ell\}_{\ell=1}^\infty$ be a Cauchy sequence in $N^s_{p,q}$ (with respect to a fixed quasinorm in $N^s_{p,q}$). Part (2) of the theorem shows that $\{f_\ell\}_{\ell=1}^\infty$ is also a Cauchy sequence in $\mathcal{S}'$. Because $\mathcal{S}'$ is a complete locally convex topological linear space, we can find a limit element $f \in \mathcal{S}'$. Then $F^{-1}\sigma_k F f_\ell$ converges to $F^{-1}\sigma_k F f$ in $\mathcal{S}'(\mathbb{R}^n)$ if $\ell \to \infty$. On the other hand, $\{F^{-1}\sigma_k F f_\ell\}_{\ell=1}^\infty$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$ ($N^s_{p,q} \subset M^s_{p,q\vee p}$). By Theorem A.2, it is also a Cauchy sequence in $L^\infty(\mathbb{R}^n)$. This shows that the limiting element of $\{F^{-1}\sigma_k F f_\ell\}_{\ell=1}^\infty$ in $L^p(\mathbb{R}^n)$ (which is the same as in $L^\infty(\mathbb{R}^n)$) coincides with $\{F^{-1}\sigma_k F f\}$. Now it follows by standard arguments that f belongs to $N^s_{p,q}$ and that f_ℓ converges in $N^s_{p,q}$ to f. Hence, $N^s_{p,q}$ is complete.

Step 5. We prove that if $-\infty < s < \infty$ and $0 < p, q < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $N^s_{p,q}(\mathbb{R}^n)$. Let $f \in N^s_{p,q}$; then we put

$$f_N(x) = \sum_{k=0}^{N} F^{-1}\sigma_k F f.$$

Note $f_N \in N^s_{p,q}(\mathbb{R}^n)$. Consequently (by Theorem A.6),

$$\begin{aligned}\|f - f_N\|_{N^s_{p,q}(\mathbb{R}^n)} &\leqslant c \left\| \left(\sum_{k=N}^{\infty} \sum_{r=-1}^{1} \langle k\rangle^{sq} |F^{-1}\sigma_k \sigma_{k+r} F f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\leqslant c \left\| \left(\sum_{k=N}^{\infty} \langle k\rangle^{sq} |F^{-1}\sigma_k F f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Lebesgue's bounded convergence theorem proves that the right-hand side of above inequality tends to zero if $N \to \infty$. Hence, f_N approximates f in $N^s_{p,q}(\mathbb{R}^n)$. Next, we let $\varphi \in \mathcal{S}$ with $\varphi(0)=1$ and supp $F\varphi \subset \{y : |y| \leqslant 1\}$. Let $f_\delta(x) = \varphi(\delta x) f(x)$ with $0<\delta<1$. Then $(f_N)_\delta \in \mathcal{S}(\mathbb{R}^n)$ approximates f_N in $L_p^\Omega := \{f \in L^p : \text{supp } \hat{f} \subset \Omega\}$ with $\Omega = \{y : |y| \leqslant 2^{N+2}\}$ if $\delta \to 0$. But this is also an approximation of f_N in $N^s_{p,q}(\mathbb{R}^n)$. This proves that $\mathcal{S}(\mathbb{R}^n)$ is dense in $N^s_{p,q}(\mathbb{R}^n)$. ☐

Proposition 2.3 (dual space). *Assume $-\infty < s < \infty$. The following inclusions hold:*

(a) *Let $1 \leqslant p < \infty$ and $1 < q < \infty$. Then*

$$(N^s_{p,q})^* = N^{-s}_{p',q'}.$$

(b) *Let $0 < p < 1$ and $0 < q \leqslant 1$. Then*

$$(N^s_{p,q})^* = M^{-s}_{\infty,\infty}.$$

Proof.

Step 1. For $1 \leqslant p < \infty$, $0 < q < \infty$, [Triebel 1983] showed similar results for Triebel spaces. We can prove the result similarly as for Triebel spaces, and omit the details.

Step 2. For $0 < p < 1$ and $0 < q \leqslant 1$, by the property of $N^s_{p,q}$, we have

$$M^s_{p,p\wedge q}(\mathbb{R}^n) \subset N^s_{p,q}(\mathbb{R}^n) \subset M^s_{p,p\vee q}(\mathbb{R}^n) \subset M^s_{1,1}(\mathbb{R}^n).$$

Then by Theorem A.3, we have

$$M^{-s}_{\infty,\infty}(\mathbb{R}^n) \supset (N^s_{p,q}(\mathbb{R}^n))^* \supset M^{-s}_{\infty,\infty}(\mathbb{R}^n).$$

This proves the second part of this proposition. ☐

Proposition 2.4 (equivalent norm). *Assume $\{\sigma_k\}_{k\in\mathbb{Z}^n}, \{\phi_k\}_{k\in\mathbb{Z}^n} \in \Upsilon_n$, $0 < p < \infty$, $0 < q \leqslant \infty$. Then $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ and $\{\phi_k\}_{k\in\mathbb{Z}^n}$ generate equivalent norms on $N^s_{p,q}$.*

Proof. Recall that [Feichtinger 2003; Wang and Hudzik 2007] showed the equivalence of $\|\cdot\|^{\sigma_k}_{M^s_{p,q}}$ and $\|\cdot\|^{\phi_k}_{M^s_{p,q}}$. By a similar argument as for modulation spaces and by Theorem A.6, we can obtain the claimed equivalence of $\|\cdot\|^{\sigma_k}_{N^s_{p,q}}$ and $\|\cdot\|^{\phi_k}_{N^s_{p,q}}$. ☐

Lemma 2.5. *Assume $(I-\Delta)^{s/2} f = F^{-1}(1+|\xi|^2)^{s/2} Ff$, $0 < p < \infty$, $0 < q \leqslant \infty$. Then we have*

$$\|(I-\Delta)^{s/2} f\|_{N_{p,q}} \sim \|f\|_{N^s_{p,q}}.$$

Proof. Analogously to the case of modulation spaces, by Theorem A.6 we can prove the consequence, and the details are omitted. ☐

Theorem 2.6. *Assume $1 \leqslant p_2 \leqslant p_1 < \infty$, $1 \leqslant q \leqslant \infty$. Then we have*

$$\|u\|_{N^s_{p_1,q}} \lesssim \|u\|_{N^s_{p_2,q}}.$$

Proof. By the definition of $N^s_{p,q}$, we have

$$
\begin{aligned}
\|u\|_{N^s_{p_1,q}} &= \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} |\Box_i u|^q \Big)^{1/q} \Big\|_{L^{p_1}} \\
&= \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} \Big|\Box_i \sum_{|\ell|_\infty\leqslant 1} \Box_{i+\ell} u\Big|^q \Big)^{1/q} \Big\|_{L^{p_1}} \\
&\leqslant \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} \Big| |\sigma_i^\vee| * \Big| \Big(\sum_{|\ell|_\infty\leqslant 1} \Box_{i+\ell} u\Big)\Big|\Big|^q \Big)^{1/q} \Big\|_{L^{p_1}} \\
&\lesssim \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} \Big| \Big(\sum_{|\ell|_\infty\leqslant 1} \Box_{i+\ell} u\Big)\Big|^q \Big)^{1/q} \Big\|_{L^{p_2}} \\
&\lesssim \Big\| \Big(\sum_{i\in\mathbb{Z}^n} \langle i\rangle^{qs} |\Box_i u|^q \Big)^{1/q} \Big\|_{L^{p_2}},
\end{aligned}
$$

which implies the result. In the above equation, we used Theorem A.7 and $\||\sigma_0^\vee| * f\|_{L^{p_1}} \leqslant \|f\|_{L^{p_2}}$ ($p_1 \geqslant p_2$, Young's inequality). $\square$

Theorem 2.7. *Assume $f \in L^2$. Then for any $s \in \mathbb{R}$, we have*

$$\|f\|_{H^s} \sim \|f\|_{N^s_{2,2}}.$$

Proof. First, we prove that $\|f\|_{L^2} \lesssim \|f\|_{N_{2,2}}$. By Plancherel's inequality, we have

$$
\begin{aligned}
\|f\|_{L^2} &= \Big(\int_{\mathbb{R}^n} \Big| \sum_{i\in\mathbb{Z}^n} \Box_i f \Big|^2 dx \Big)^{1/2} = \Big(\int_{\mathbb{R}^n} \Big| \sum_{i\in\mathbb{Z}^n} \sigma_i Ff \Big|^2 dx \Big)^{1/2} \\
&\lesssim \Big(\int_{\mathbb{R}^n} \sum_{i\in\mathbb{Z}^n} |\sigma_i Ff|^2 \, dx \Big)^{1/2} \\
&= \Big(\int_{\mathbb{R}^n} \Big(\Big(\sum_{i\in\mathbb{Z}^n} |F^{-1}\sigma_i Ff|^2 \Big)^{1/2} \Big)^2 dx \Big)^{1/2} = \|f\|_{N_{2,2}}.
\end{aligned}
$$

The inverse inequality can be proved similarly $\Big(\sum_{i\in\mathbb{Z}^n} |\sigma_i Ff|^2 \lesssim \Big| \sum_{i\in\mathbb{Z}^n} \sigma_i Ff \Big|^2 \Big)$. $\square$

Theorem 2.8. *Assume $s_1, s_2 \in \mathbb{R}$, $0 < q_1, q_2 \leqslant \infty$, $0 < p < \infty$. Then, for $q_2 < q_1$, $s_1 - s_2 > n/q_2 - n/q_1$, we have*

$$N^{s_1}_{p,q_1}(\mathbb{R}^n) \subset N^{s_2}_{p,q_2}(\mathbb{R}^n).$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}\|f\|_{N^{s_2}_{p,q_2}} &= \left\| \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_2 q_2} |\Box_k f|^{q_2} \Big)^{1/q_2} \right\|_{L^p} \\ &= \left\| \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(s_2-s_1)q_2} \langle k\rangle^{s_1 q_2} |\Box_k f|^{q_2} \Big)^{1/q_2} \right\|_{L^p} \\ &\leqslant \|f\|_{N^{s_1}_{p,q_1}} \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(s_2-s_1)q_1q_2/(q_1-q_2)} \Big)^{(q_1-q_2)/(q_1q_2)}.\end{aligned}$$

Then by $q_2 < q_1$, $s_1 - s_2 > n/q_2 - n/q_1$ and

$$\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(s_2-s_1)q_1q_2/(q_1-q_2)} \lesssim \sum_{i=0}^{\infty} \langle i\rangle^{(s_2-s_1)q_1q_2/(q_1-q_2)},$$

we obtain the results. □

3. Embedding between $N^s_{p,q}$ and Triebel spaces

The embedding theorem is of importance for the study of nonlinear PDEs and we give the details of the proof. We start with the embedding for the same indices p, q.

Proof of Theorem 1.1.

Step 1. For $0 < p < \infty$, $0 < q \leqslant \infty$, $\varepsilon > 0$ (a small positive number), we have

$$N_{p,q} \subset F_{p,q} \quad \text{for all } q \leqslant 1 \wedge p, \tag{4}$$

$$N^{n(1/(p\wedge 1)-1/q)+\varepsilon}_{p,q} \subset F_{p,q} \quad \text{for all } 0 < p \leqslant q. \tag{5}$$

Let $a_k = \max(0, 2^{k-1} - \sqrt{n})$, $b_k = 2^{k+1} + \sqrt{n}$. We can easily find that $\Delta_k \Box_i f = 0$ for $|i| \,\overline{\in}\, [a_k, b_k]$. First, we show the inclusion (4).

Case 1. $p \geqslant 1$, $q \leqslant 1$. By Theorem A.6, we have

$$\begin{aligned}\|\Delta_k f\|_{L^p(\ell^q)} &= \left\| \Big(\sum_k |\Delta_k f|^q \Big)^{1/q} \right\|_{L^p} \leqslant \left\| \Big(\sum_k \Big| \sum_{\substack{i\in\mathbb{Z}^n \\ |i|\in[a_k,b_k]}} \Delta_k \Box_i \Box_i f \Big|^q \Big)^{1/q} \right\|_{L^p} \\ &\leqslant C \left\| \Big(\sum_k \sum_{\substack{i\in\mathbb{Z}^n \\ |i|\in[a_k,b_k]}} (|F^{-1}(\varphi_k \sigma_i) F \Box_i f|)^q \Big)^{1/q} \right\|_{L^p} \\ &\lesssim C \left\| \Big(\sum_k \sum_{\substack{i\in\mathbb{Z}^n \\ |i|\in[a_k,b_k]}} |\Box_i f|^q \Big)^{1/q} \right\|_{L^p} \lesssim C \left\| \Big(\sum_{i\in\mathbb{Z}^n} |\Box_i f|^q \Big)^{1/q} \right\|_{L^p}.\end{aligned}$$

Case 2. $p<1,\ p=q$. By Theorem A.1, we have

$$N_{p,p}\subset M_{p,p}\subset B_{p,p}\subset F_{p,p}.$$

Then by combining Cases 1 and 2, we obtain the result (4).

Next, we show the second inclusion. By Theorem A.1, we obtain that

$$N_{p,q}^{n(1/(p\wedge 1)-1/q)+\varepsilon}\subset M_{p,q}^{n(1/p\wedge 1-1/q)+\varepsilon}\subset B_{p,q}^{\varepsilon}\subset B_{p,p}\subset F_{p,q}.$$

This proves (5). In the above discussion, we used Theorem A.15.

Step 2. We prove the following inclusions:

(a) If $0<q\leqslant 2$, $\varepsilon>0$ (a small positive number), then we have

$$F_{2,q}^{n(1/q-1/2)+\varepsilon}\subset N_{2,q}. \tag{6}$$

(b) If $p\geqslant 2$, $\varepsilon>0$ (a small positive number), then we have

$$N_{p,\infty}^{n(1-1/p)+\varepsilon}\subset F_{p,\infty}. \tag{7}$$

By Theorem A.1, one has that for $0<q\leqslant 2$,

$$B_{2,q}^{n(1/q-1/2)}\subset M_{2,q}.$$

Then by the embedding estimates in Proposition 2.1 and Theorem A.15, we have

$$F_{2,q}^{n(1/q-1/2)+2\varepsilon}\subset F_{2,\infty}^{n(1/q-1/2)+\varepsilon}\subset B_{2,\infty}^{n(1/q-1/2)+\varepsilon}\subset B_{2,q}^{n(1/q-1/2)}\subset M_{2,q}\subset N_{2,q},$$

which implies the result (6).

For the case $p\geqslant 2$, by Theorem A.1 and Proposition 2.1, we have that

$$N_{p,\infty}^{n(1-1/p)+\varepsilon}\subset M_{p,\infty}^{n(1-1/p)+\varepsilon}\subset B_{p,\infty}^{\varepsilon}\subset B_{p,p\wedge\infty}\subset F_{p,\infty}.$$

This proves (7).

Step 3. Assume $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$. Then for

$$\sigma(p,q)=\max\left(0,n\left(\frac{1}{p\wedge p'}-\frac{1}{q}\right)\right)$$

and $s_1>s_2+\sigma(p,q)$ (if $\sigma(p,q)=0$, $s_1\geqslant s_2$), we have

$$N_{p,q}^{s_1}\subset F_{p,q}^{s_2}. \tag{8}$$

Actually, thanks to (5), (7) and the dual versions of (6), we have

$$\begin{aligned} N_{p,\infty}^{n+\varepsilon}&\subset F_{p,\infty}, && 1\leqslant p<\infty,\\ N_{p,\infty}^{n(1-1/p)+\varepsilon}&\subset F_{p,\infty}, && p\geqslant 2,\\ N_{2,q}^{n(1/2-1/q)+\varepsilon}&\subset F_{2,q}, && 2\leqslant q\leqslant\infty. \end{aligned}$$

Taking $p = 1, \infty^-$ (i.e., a sufficient large number) and $q = \infty$, we have

$$N^{n+\varepsilon}_{1,\infty} \subset F_{1,\infty}, \quad N^{n/2+\varepsilon}_{2,\infty} \subset F_{2,\infty}, \quad N^{n(1-1/\infty^-)+\varepsilon}_{\infty^-,\infty} \subset F_{\infty^-,\infty}.$$

Applying the complex interpolation theorem to these three estimates, we obtain

$$N^{\max(n/p,n/p')+\varepsilon}_{p,\infty} \subset F_{p,\infty}, \quad 1 \leqslant p < \infty.$$

Moreover, by (4), we have

$$N_{p,1} \subset F_{p,1}, \quad 1 \leqslant p < \infty. \tag{9}$$

Recall that

$$N_{2,2} = F_{2,2}. \tag{10}$$

Applying the complex interpolation theorem separately to the above three estimates again, we obtain

$$N^{\sigma(p,q)+\varepsilon}_{p,q} \subset F_{p,q}.$$

(When $\sigma(p,q) = 0$, we apply the complex interpolation theorem to (9) and (10).)

Step 4. We show the sufficiency of $N^{s_1}_{p,q} \subset F^{s_2}_{p,q}$. By Step 3, we see that the conclusion holds if $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$. By Step 1, we have the result if $0 < p \leqslant 1$ or $0 < q < 1$.

Next, we prove the sufficiency of $F^{s_1}_{p,q} \subset N^{s_2}_{p,q}$. Set

$$\begin{aligned}
\mathbb{R}^2_{++} &= \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) : \frac{1}{p} > 0,\ \frac{1}{q} \geqslant 0 \right\}, \\
S_1 &= \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_{++} : \frac{1}{q} \geqslant \frac{1}{p},\ \frac{1}{p} \leqslant \frac{1}{2} \right\}, \\
S_2 &= \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathbb{R}^2_{++} : \frac{1}{q} < \frac{1}{p},\ \frac{1}{p} + \frac{1}{q} \leqslant 1 \right\}, \\
S_3 &= \mathbb{R}^2_{++} - (S_1 \cup S_2).
\end{aligned}$$

Step 1$'$. For $1 \leqslant p < \infty$, $q = \infty$, we have

$$F_{p,\infty} \subset N_{p,\infty}. \tag{11}$$

Actually, by the definition of $N_{p,\infty}$, $F_{p,\infty}$ and Theorem A.7, we have (δ_j is the Littlewood–Paley decomposition)

$$\begin{aligned}
\|f\|_{N_{p,\infty}} &= \Big\| \sup_{i \in \mathbb{Z}^n} |\Box_i f| \Big\|_{L^p} = \Big\| \sup_{i \in \mathbb{Z}^n} \Big| F^{-1}\sigma_i \sum_{\ell=-4}^{4} \delta_{j+\ell} F f \Big| \Big\|_{L^p} \\
&\leqslant \Big\| \sup_{j \in \mathbb{Z}^n} |F^{-1}\delta_j F f| \Big\|_{L^p} = \|f\|_{F_{p,\infty}}.
\end{aligned}$$

Step 2′. For $0 < p \leqslant 1$, we have

$$F^{n(1/p-1)}_{p,\infty} \subset N_{p,\infty}. \tag{12}$$

By Theorem A.6, we have that for $|i| \in [2^{j-1}, 2^j)$,

$$\begin{aligned}
\|f\|_{N_{p,\infty}} &= \Big\| \sup_{i\in\mathbb{Z}^n} |\Box_i f| \Big\|_{L^p} \\
&= \Bigg\| \sup_{i\in\mathbb{Z}^n} \Bigg| F^{-1}\sigma_i \sum_{\ell=-4}^{4} \delta_{j+\ell} F f \Bigg| \Bigg\|_{L^p} \\
&\lesssim \|2^{-js}\sigma_i(2^j\cdot)\|_{H^x} \|f\|_{F^s_{p,\infty}} && (s \geqslant n(1/p-1)) \\
&\lesssim \|f\|_{F^s_{p,\infty}} && (s \geqslant n(1/p-1),\ \|2^{-js}\sigma_i(2^j\cdot)\|_{H^x} \lesssim C),
\end{aligned}$$

which implies the result (12).

Step 3′. $(1/p, 1/q) \in S_3$, $\tau(p,q) = n(1/p+1/q-1)$. Let

$$\frac{1}{p_0} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_0} = 0, \quad \frac{1}{p_1} = \frac{1}{2}, \quad \frac{1}{q_1} = \frac{1}{p} + \frac{1}{q} - \frac{1}{2}.$$

Assume $\theta = 1/q(1/p+1/q-\frac{1}{2})^{-1}$, we have

$$\begin{gathered}
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \\
\frac{1}{p} + \frac{1}{q} - 1 = (1-\theta)\Big(\frac{1}{p_0} - 1\Big) + \Big(\frac{1}{q_1} - \frac{1}{2}\Big)\theta.
\end{gathered}$$

By (6) and (12), we have

$$F^{n(1/q_1-1/2)+\varepsilon}_{2,q_1} \subset N_{2,q_1}, \quad F^{n(1/p_0-1)}_{p_0,\infty} \subset N_{p_0,\infty}.$$

A complex interpolation yields

$$F^{n(1/p+1/q-1)+\varepsilon}_{p,q} \subset N_{p,q}.$$

Step 4′. $(1/p, 1/q) \in S_1$, $\tau(p,q) = n(1/q-1/p)$. Let $(1/p, 1/q) \in \dot{S}_1$ (inner point of s). Then $(1/p, 1/q) \in \dot{S}_1$ is a point at the line segment connecting $(1/\infty^-, 0)$ and some point $(1/p_1, 1/q_1) \in \{(1/p, 1/q) : p = 2,\ q < 2\}$. By (11), Step 3′ and complex interpolation, we have

$$F^{n(1/q-1/p)+\varepsilon}_{p,q} \subset N_{p,q}.$$

Step 5′. $(1/p, 1/q) \in S_2$, $\tau(p,q) = 0$. We can obtain the results by $F_{2,2} \subset N_{2,2}$ and the dual version of (4), i.e., for $1 \leqslant p < \infty$, $F_{p,\infty} \subset N_{p,\infty}$. $\square$

Proof of Theorem 1.2.

Step 1. For the first part, we need to show that for any $0<\eta\ll 1$, $F^{\tau(p,q)-\eta}_{p,q}\not\subset N_{p,q}$.

Case 1. $(1/p,1/q)\in S_3$, $\tau(p,q)=1/p+1/q-1$. Let $f=F^{-1}\delta_j$, $j\gg 1$. We have

$$\begin{aligned}\|f\|_{F^{\tau(p,q)-\eta}_{p,q}}&=\Big\|\Big(\sum_j 2^{j(\tau(p,q)-\eta)q}|\Delta_j f|^q\Big)^{1/q}\Big\|_{L^p}\\&\lesssim\sum_{\ell=-1}^{1}2^{j(\tau(p,q)-\eta)(j+\ell)}\|F^{-1}\delta_{j+\ell}\delta_j\|_{L^p}\lesssim 2^{jn/q-j\eta}.\end{aligned}$$

Denote

$$\begin{aligned}\Lambda_0&=\{k\in\mathbb{Z}^n:B(k,\sqrt{n})\cap\{\xi:|\xi|\in[0,2)\}\neq\varnothing\},\\ \Lambda_j&=\{k\in\mathbb{Z}^n:B(k,\sqrt{n})\cap\{\xi:|\xi|\in[2^{j-1},2^{j+1})\}\neq\varnothing\}.\end{aligned}$$

If $k\in\Lambda_j$, then $|k|\sim 2^j$. Noticing that at most $O(2^{nj})$ unit cubes intersect with Λ_j, we have

$$\begin{aligned}\|f\|_{N_{p,q}}&=\Big\|\Big(\sum_k|\Box_k f|^q\Big)^{1/q}\Big\|_{L^p}\geqslant\Big\|\Big(\sum_{k\in\Lambda_j}|\Box_k f|^q\Big)^{1/q}\Big\|_{L^p}\\&\gtrsim\Big\|\Big(\sum_{k\in\Lambda_j}|F^{-1}\sigma_k\delta_j|^q\Big)^{1/q}\Big\|_{L^p}\gtrsim 2^{jn/q}.\end{aligned}$$

Based on the above observation, we have

$$\|f\|_{N_{p,q}}\gtrsim 2^{\eta j}\|f\|_{F^{\tau(p,q)-\eta}_{p,q}},$$

which implies that $F^{\tau(p,q)-\eta}_{p,q}\not\subset N_{p,q}$.

Case 2. $(1/p,1/q)\in S_2$, $\tau(p,q)=0$. We consider the case $q=\infty$. Taking $k(j)=(2^j,0,\dots,0)$ and $f=F^{-1}\sigma_{k(j)}$, we have that

$$\|f\|_{N_{p,\infty}}\gtrsim 1\gtrsim 2^{\eta j}\|f\|_{F^{-\eta}_{p,\infty}}.$$

If $q<\infty$, we need to show that

$$N_{p,q}\not\subset F^{\varepsilon}_{p,r},\quad 1\leqslant p<\infty. \tag{13}$$

Assume $f\in\mathcal{S}$, $\operatorname{supp}\hat f\subset\{\xi:|\xi_i|<\frac12,\ i=1,\dots,n\}$. Let $N\gg 1$, $0<\varepsilon\ll 1$,

$$\begin{aligned}k(j)&=(2^{Nj},0,\dots,0)\in\mathbb{Z}^n,\\ \widehat{F}(\xi)&=\sum_{j=1}^{\infty}2^{-\varepsilon Nj}\hat f(\xi-k(j)).\end{aligned}$$

We see that

$$\begin{aligned}\|F\|_{N_{p,q}} &= \Big\| \Big(\sum_i |\Box_i F|^q \Big)^{1/q} \Big\|_{L^p} \\ &\lesssim \Big\| \Big(\sum_j \sum_{|\ell|_\infty \leqslant 1} 2^{-\varepsilon Nqj} |F^{-1} \sigma_{k(j)+\ell} \hat{f}(\cdot - k(j))| \Big)^{1/q} \Big\|_{L^p} \lesssim 1\end{aligned}$$

On the other hand, letting $s > \varepsilon$, we have

$$\begin{aligned}\|F\|_{F^s_{p,r}} &= \Big\| \Big(\sum_j 2^{jrs} |\Delta_j F|^r \Big)^{1/r} \Big\|_{L^p} \\ &= \Big\| \Big(\sum_j 2^{Njrs} |F^{-1} \delta_{Nj} 2^{-\varepsilon Nj} \hat{f}(\cdot - k(j))|^r \Big)^{1/r} \Big\|_{L^p} \\ &\gtrsim \Big\| \Big(\sum_j 2^{Njr(s-\varepsilon)} |F^{-1} \delta_{Nj} \hat{f}(\cdot - k(j))|^r \Big)^{1/r} \Big\|_{L^p} \\ &\gtrsim \Big\| \Big(\sum_j 2^{Njr(s-\varepsilon)} |F^{-1} \delta_{Nj} \chi_{Q_{k(j)}} \hat{f}(\cdot - k(j))|^r \Big)^{1/r} \Big\|_{L^p} \\ &= \Big\| \Big(\sum_j 2^{Njr(s-\varepsilon)} |F^{-1} \chi_{Q_{k(j)}} \hat{f}(\cdot - k(j))|^r \Big)^{1/r} \Big\|_{L^p} \\ &\gtrsim \infty \qquad\qquad (\text{let } \delta(\xi) = 1, \text{if } |\xi| \in [\tfrac{3}{4}, \tfrac{5}{4}]),\end{aligned}$$

which implies (13). Then by its dual version, we obtain the claimed results.

Case 3. $(1/p, 1/q) \in S_1$, $\tau(p,q) = n(1/q - 1/p)$. Let $\sigma_k(\xi) = 0$ if

$$\xi \,\overline{\in}\, \tilde{Q}_k := \{\xi : |\xi_i - k_i| \leqslant \tfrac{5}{8}, 1 \leqslant i \leqslant n\}$$

and $\delta_j(\xi) = 1$ if $\xi \in D_j := \{\xi : \frac{5}{4} 2^{j-1} \leqslant |\xi| \leqslant \frac{3}{4} 2^{j+1}\}$. Assume

$$A_j = \{k \in \mathbb{Z}^n : \widetilde{Q}_k \subset D_j\}, \quad j \gg 1.$$

Let $f \in \mathcal{S}$, $\operatorname{supp} \hat{f} \subset B(0, \frac{1}{8})$ and

$$g(x) = \sum_{k \in A_j} e^{ixk} (\tau_k f)(x), \quad \tau_k f = f(\cdot - k).$$

It is easy to see that $\operatorname{supp} \widehat{\tau_k f} \subset B(0, \frac{1}{8})$ and $\operatorname{supp} \tau_k(\widehat{\tau_k f}) \cap \operatorname{supp} \sigma_\ell = \varnothing$, if $k \neq \ell$. Then, we have

$$\begin{aligned}
\|g\|_{N_{p,q}} &= \Big\| \Big(\sum_i |\Box_i g|^q \Big)^{1/q} \Big\|_{L^p} = \Big\| \Big(\sum_i \Big| \Box_i \sum_{k \in A_j} e^{ixk} (\tau_k f)(x) \Big|^q \Big)^{1/q} \Big\|_{L^p} \\
&= \Big\| \Big(\sum_{i \in A_j} |F^{-1} \sigma_i F g|^q \Big)^{1/q} \Big\|_{L^p} = \Big\| \Big(\sum_{i \in A_j} |F^{-1} \sigma_0 (\widehat{\tau_k f})|^q \Big)^{1/q} \Big\|_{L^p} \\
&\gtrsim 2^{jn/q}.
\end{aligned}$$

On the other hand, by $\operatorname{supp} \widehat{g} \subset \{\xi : 2^{j-1} \leqslant |\xi| \leqslant 2^j\}$, we have

$$\begin{aligned}
\|g\|_{F^{n(1/q-1/p)-\eta}_{p,q}} &= \Big\| \Big(\sum_j 2^{jq(n(1/q-1/p)-\eta)} |\Delta_j g|^q \Big)^{1/q} \Big\|_{L^p} \\
&\lesssim 2^{j(n(1/q-1/p)-\eta)} \|\Delta_j g\|_{L^p} \\
&\lesssim 2^{j(n(1/q-1/p)-\eta)} \|g\|_{L^p} \\
&\lesssim 2^{j(n(1/q-1/p)-\eta)} \|g\|_{L^\infty}^{1-2/p} \|g\|_{L^2}^{2/p}.
\end{aligned}$$

By Plancherel's identity, we have

$$\|g\|_{L^2} = \|\hat{g}\|_{L^2} = \Big(\int_{\mathbb{R}^n} \sum_{k \in A_j} |\tau_k (e^{-ik\xi} \hat{f}(\xi))|^2 \, d\xi \Big)^{1/2} \lesssim 2^{nj/2}.$$

We can further assume that $f(x) = f(|x|)$ is a decreasing function on $|x|$. Then $|f(x-k)| \lesssim (1+|x-k|)^{-N}$, $N \gg 1$ and a straightforward computation will lead to $|g(x)| \lesssim 1$.

Based on the above observation, we have

$$\|g\|_{F^{n(1/q-1/p)-\eta}_{p,q}} \lesssim 2^{nj/q-\eta j}.$$

This proves the results.

Step 2. We study the second embedding estimate. Set

$$\begin{aligned}
R_1 &= \Big\{ \Big(\frac{1}{p}, \frac{1}{q} \Big) \in \mathbb{R}^2_{++} : \frac{1}{q} \geqslant \frac{1}{p}, \ \frac{1}{p} + \frac{1}{q} \geqslant 1 \Big\}, \\
R_2 &= \Big\{ \Big(\frac{1}{p}, \frac{1}{q} \Big) \in \mathbb{R}^2_{++} : \frac{1}{q} \leqslant \frac{1}{p}, \ \frac{1}{p} \geqslant \frac{1}{2} \Big\}, \\
R_3 &= \mathbb{R}^2_{++} - (R_1 \cup R_2).
\end{aligned}$$

We consider three cases:

Case 1′. $(1/p, 1/q) \in R_1$. We have discussed it in Case 2 (dual version).

Case 2′. $(1/p, 1/q) \in R_3$. For $1 \leqslant p, q < \infty$, we obtain the claimed results by Case 1. For the case $q = \infty$, letting $f = F^{-1}\delta_j$, we have

$$\|f\|_{F_{p,q}} \gtrsim 2^{jn(1-1/p)} \quad \text{and} \quad \|f\|_{N^{n(1-1/p)}_{p,\infty}} \lesssim 2^{jn(1-1/p)},$$

which implies the result.

Case 3′. $(1/p, 1/q) \in R_2$. Assume $f \in \mathcal{S}$, $f(0) = 1$ and

$$\operatorname{supp} \hat{f} \subset Q_0 := \{\xi : |\xi_i| \leqslant \tfrac{1}{2},\ 1 \leqslant i \leqslant n\}.$$

Choose $0 < a \ll 1$. Denote $f_a(x) = f(x/a)$. Then

$$\operatorname{supp} \hat{f_a} \subset Q_{0,a} := \Big\{\xi : |\xi_i| \leqslant \frac{1}{2a},\ 1 \leqslant i \leqslant n\Big\}.$$

Let

$$D_j = \{\xi : \tfrac{5}{4}2^{j-1} \leqslant |\xi| \leqslant \tfrac{3}{4}2^{j+1}\} \quad \text{and} \quad Q_{k(i),a} := k(i) + Q_{0,a}.$$

One has that $Q_{k(i),a}$ and D_j overlap at most $O(a^n 2^{jn})$ cubes. Moreover, there is a $\beta > 0$ such that $f_a(x) > \frac{1}{2}$, $x \in B(0, \beta)$. Let $A_j = \{k(i) : i = 1, \ldots, O(a^n 2^{jn})\}$ and $g(x) = \sum_{k \in A_j} e^{ixk}(\tau_k f_a)(x)$. By a straightforward computation, we obtain

$$\|g\|_{F_{p,q}} \gtrsim (a\beta)^{n/p} 2^{jn/p} \quad \text{and} \quad \|g\|_{N^{n(1/p-1/q)}_{p,q}} \lesssim 2^{jn/p}.$$

This finishes the proof. □

4. Multiplication algebra

It is well known that $B^s_{p,q}$ is a multiplication algebra if $s > n/p$; see [Cazenave and Weissler 1990]. The regularity indices, for which $N^s_{p,q}$ constitutes a multiplication algebra, are quite different from those of Besov space. Set

$$\mathbb{R}^2_{++} = \Big\{\Big(\frac{1}{p}, \frac{1}{q}\Big) : \frac{1}{p} > 0,\ \frac{1}{q} \geqslant 0\Big\}, \qquad D_1 = (0, 1] \times [0, 1],$$

$$D_2 = \Big\{\Big(\frac{1}{p}, \frac{1}{q}\Big) \in \mathbb{R}^2_{++} : \frac{1}{q} - 1 < \frac{1}{p} \leqslant \frac{1}{q},\ \frac{1}{q} > 1\Big\},$$

$$D_3 = \Big\{\Big(\frac{1}{p}, \frac{1}{q}\Big) \in \mathbb{R}^2_{++} : \frac{1}{p} \geqslant \frac{1}{q},\ \frac{1}{p} > 1\Big\}.$$

Theorem 4.1. *Assume that* $p > 0$, $1 - 1/(p+1) < q \leqslant \infty$ *and*

$$s > \begin{cases} n\Big(1 - \dfrac{1}{q}\Big) & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_1, \\ 0 & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_2, \\ np\Big(\dfrac{1}{p} - \dfrac{1}{q}\Big) & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_3. \end{cases}$$

Then $N^s_{p,q}$ *is a multiplication algebra, i.e.,*

$$\|fg\|_{N^s_{p,q}} \lesssim \|f\|_{N^s_{p,q}} \|g\|_{N^s_{p,q}}$$

holds for all $f, g \in N^s_{p,q}$.

Proof.

Step 1. $0 < p < \infty$, $q = \infty$.

Case 1. $1 \leqslant p < \infty$, $q = \infty$, $s > n$. Let $\Lambda_{i,j} = \{k \in \mathbb{Z}^n : |i - j - k| < C(n)\}$. Then it is easy to see that $\Box_i fg = \Box_i \sum_j \Box_j f \sum_{k(i,j) \in \Lambda_{i,j}} \Box_{k(i,j)} g$. Suppose $f, g \in N^s_{p,q}$, by Theorems A.6, 2.6, 2.8 and $\|a_i * b_i\|_{\ell^\infty} \leqslant \|a_i\|_{\ell^1} \|b_i\|_{\ell^\infty}$, we have

$$\begin{aligned}
\|fg\|_{N^s_{p,\infty}} &= \Big\| \sup_i \langle i \rangle^s |\Box_i fg| \Big\|_{L^p} \\
&= \Big\| \sup_i \langle i \rangle^s \Big| \Box_i \sum_j \Box_j f \sum_{k(i,j) \in \Lambda_{i,j}} \Box_{k(i,j)} g \Big| \Big\|_{L^p} \\
&\lesssim \Big\| \sup_i \langle i \rangle^s \sum_j |\Box_j f| |\Box_{k(i,j)} g| \Big\|_{L^p} \qquad (|i - j - k(i,j)| < C(n)) \\
&\lesssim \Big\| \sup_i \sum_j (\langle j \rangle^s + \langle k(i,j) \rangle^s) |\Box_j f| |\Box_{k(i,j)} g| \Big\|_{L^p} \\
&\lesssim \Big\| \sup_i \langle i \rangle^s |\Box_i g| \Big\|_{L^{p_1}} \Big\| \sum_j |\Box_j f| \Big\|_{L^{p_2}} \\
&\qquad\qquad + \Big\| \sup_i \langle i \rangle^s |\Box_i f| \Big\|_{L^{p_1}} \Big\| \sum_j |\Box_j g| \Big\|_{L^{p_2}} \\
&\lesssim \|g\|_{N^s_{p_1,\infty}} \|f\|_{N_{p_2,1}} + \|f\|_{N^s_{p_1,\infty}} \|g\|_{N_{p_2,1}} \lesssim \|f\|_{N^s_{p,\infty}} \|g\|_{N^s_{p,\infty}},
\end{aligned}$$

where $1/p = 1/p_1 + 1/p_2$.

Case 2. $0 < p < 1$, $q = \infty$, $s > n$. By the similar argument as in Case 1, we have

$$\begin{aligned}\|fg\|_{N^s_{p,\infty}} &\lesssim \Big\|\sup_i \sum_j (\langle j\rangle^s + \langle k(i,j)\rangle^s)|\Box_j f||\Box_{k(i,j)} g|\Big\|_{L^p} \\ &\lesssim \Big\|\sup_i \langle i\rangle^s |\Box_i g|\Big\|_{L^p} \Big\|\sum_j |\Box_j f|\Big\|_{L^\infty} \\ &\qquad + \Big\|\sup_i \langle i\rangle^s |\Box_i f|\Big\|_{L^p} \Big\|\sum_j |\Box_j g|\Big\|_{L^\infty}.\end{aligned}$$

On the other hand, by Theorem A.7, we have

$$\begin{aligned}\Big\|\sum_j |\Box_j f|\Big\|_{L^\infty} &= \Big\|\sum_j \Big|\Box_j \sum_{\ell=-1}^{1} \Box_{j+\ell} f\Big|\Big\|_{L^\infty} \\ &\lesssim \Big\|\sum_j |\sigma_j^\vee| * \Big|\sum_{\ell=-1}^{1} \Box_{j+\ell} f\Big|\Big\|_{L^\infty} \lesssim \Big\|\sum_j |\Box_j f|\Big\|_{L^1} = \|f\|_{N_{1,1}}.\end{aligned}$$

Then by Proposition 2.1 and Theorem A.5, we have

$$\|f\|_{N_{1,1}} = \|f\|_{M_{1,1}} \leqslant \|f\|_{M^s_{p,\infty}} \leqslant \|f\|_{N^s_{p,\infty}} \quad \text{for } s > n.$$

This finishes the proof of Case 2.

Combining Cases 1 and 2, we have $\|fg\|_{N^s_{p,\infty}} \lesssim \|f\|_{N^s_{p,\infty}} \|g\|_{N^s_{p,\infty}}$ for $s > n$.

Step 2. $0 < p = q \leqslant 1$. Suppose $f, g \in N_{p,p}$; from Proposition 2.1 and Theorem A.4, we have

$$\|fg\|_{N_{p,p}} = \|fg\|_{M_{p,p}} \leqslant \|f\|_{M_{p,p}} \|g\|_{M_{p,p}} = \|f\|_{N_{p,p}} \|g\|_{N_{p,p}}.$$

Step 3. $1 \leqslant p < \infty$, $q = 1$, $s \geqslant 0$. Suppose $f, g \in N^s_{p,q}$. By Theorems A.4, A.7, 2.6 and $\|a_i * b_i\|_{\ell^1} \lesssim \|a_i\|_{\ell^1} \|b_i\|_{\ell^1}$, we have

$$\begin{aligned}\|fg\|_{N^s_{p,1}} &= \Big\|\sum_i \langle i\rangle^s |\Box_i fg|\Big\|_{L^p} \\ &\leqslant \Big\|\sum_i \langle i\rangle^s \Big| |F^{-1}(\sigma_i)| * \Big(\sum_j \Box_j f \sum_{k(i,j)\in\Lambda_{i,j}} \Box_{k(i,j)} g\Big)\Big|\Big\|_{L^p} \\ &\lesssim \Big\|\sum_i \sum_j \Big(\langle j\rangle^s |\Box_j f| \sum_{k(i,j)\in\Lambda_{i,j}} \langle k(i,j)\rangle^s |\Box_{k(i,j)} g|\Big)\Big\|_{L^p} \\ &\lesssim \Big\|\Big(\sum_i \langle i\rangle^s |\Box_i f|\Big)\Big(\sum_i \langle i\rangle^s |\Box_i g|\Big)\Big\|_{L^p} \lesssim \|f\|_{N^s_{p,1}} \|g\|_{N^s_{p,1}},\end{aligned}$$

where we used the fact that $|i - j - k(i,j)| < c(n)$.

Step 4. Let $(1/p, 1/q) \in D_1$. It is easy to see that $(1/p, 1/q)$ is a point of the line segment connecting $(1/p, 0)$ and $(1/p, 1)$. At the point $(1/p, 0)$, in Step 1, we have shown that $N^s_{p,\infty}$ is a multiplication algebra if $s > n$. For $(1/p, 1)$, in Step 3, we have shown that $N^s_{p,1}$ is a multiplication algebra if $s \geqslant 0$. Using complex interpolation (Theorem A.14), we obtain that for $(1/p, 1/q) \in D_1$, $N^s_{p,q}$ is a multiplication algebra if $s > n(1-1/q)$.

If $(1/p, 1/q) \in D_2$, then it belongs to the segment by connecting $(1/p_0, 1)$ and $(1/\bar{p}, 1/\bar{p})$, where $1/p_0 < 1/p - 1/q + 1$ and $\bar{p} = 1 - p(1-q)(1-p_0)/(p - p_0 q)$. In Step 3, we see that for $s \geqslant 0$, $N^s_{p_0,1}$ is a multiplication algebra; in Step 2, we see that $N_{\bar{p},\bar{p}}$ is a multiplication algebra, if $s \geqslant 0$. Then complex interpolation between them gives that for $(1/p, 1/q) \in D_2$ and $s \geqslant 0$, $N^s_{p,q}$ is a multiplication algebra.

If $(1/p, 1/q) \in D_3$, then one can make a line segment connecting $(1/p, 1/p)$ and $(1/p, 0)$. For $(1/p, 1/p)$, we see that once $s \geqslant 0$, $N^s_{p,p}$ is a multiplication algebra. For $(1/p, 0)$, we see that once $s \geqslant n$, $N^s_{p,\infty}$ is a multiplication algebra. Then we use complex interpolation to obtain that $N^s_{p,q}$ is a multiplication algebra if $s > np(1/p - 1/q)$. □

Remark. Assume that $k \in \mathbb{Z}^+$, $p > 0$, $1 - 1/(p+1) < q \leqslant \infty$ and

$$s > \begin{cases} n\Big(1 - \dfrac{1}{q}\Big) & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_1, \\ 0 & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_2, \\ np\Big(\dfrac{1}{p} - \dfrac{1}{q}\Big) & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_3. \end{cases}$$

Then $N^s_{p,q}$ is a multiplication algebra, i.e.,

$$\|u^k\|_{N^s_{p,q}} \lesssim \|u\|^k_{N^s_{p,q}}$$

holds for all $u \in N^s_{p,q}$ (for $p \geqslant 1$, we obtain $\|u^k\|_{N^s_{p,q}} \lesssim \|u\|^k_{N^s_{kp,q}}$).

Proof. We obtain the result by the similar argument as for the above theorem. □

5. Smooth effects of the Schrödinger semigroup

In this section we will discuss a kind of Strichartz estimates. This kind of estimate was first introduced by R. S. Strichartz [1977], then developed by Pecher [1984], Ginibre and Velo [1995] and Wang, Han, and Huang [2011]. Set

$$S(t) = F^{-1} e^{-it|\xi|^2} F.$$

Our aim is to derive the estimates of $S(t)$ in the spaces $N^s_{p,q}$.

Theorem 5.1. *Assume* $2 \leqslant p < \infty$, $p' \leqslant q \leqslant p$, *and* $1/p + 1/p' = 1$. *Then, for any* $s \in \mathbb{R}$, *we have*

$$\|S(t)f\|_{N^s_{p,q}} \lesssim (1+|t|)^{-n(1/2-1/p)} \|f\|_{N^s_{p',q}}.$$

Proof. By Proposition 2.1 and Theorem A.8, we have

$$\begin{aligned}\|S(t)f\|_{N^s_{p,q}} \leqslant \|S(t)f\|_{M^s_{p,q}} &\leqslant (1+t)^{-n(1/2-1/p)} \|f\|_{M^s_{p',q}} \\ &\leqslant (1+t)^{-n(1/2-1/p)} \|f\|_{N^s_{p',q}}.\end{aligned}$$

In view of the estimates above, the theorem is proved. □

Theorem 5.2. *Assume* $r \geqslant 1$, $p' \leqslant q \leqslant p$, $2 \leqslant p < \infty$, *and* $Af = \int_0^t S(t-\tau) f(\tau)\, d\tau$. *Then for any* $s \in \mathbb{R}$, *we have*

$$\|Af\|_{L^r(-T,T;N^s_{p,q})} \lesssim T^{2/r} \|f\|_{L^{r'}(-T,T;N^s_{p',q})}.$$

Proof. By Theorem 5.1, we have

$$\begin{aligned}\|Af\|_{L^r(-T,T;N^s_{p,q})} &\leqslant \left\| \int_0^t \|S(t-\tau)f(\tau)\|_{N^s_{p,q}}\, d\tau \right\|_{L^r(-T,T)} \\ &\leqslant \left\| \int_0^t (1+|t-\tau|)^{-n(1/2-1/p)} \|f(\tau)\|_{N^s_{p',q}}\, d\tau \right\|_{L^r(-T,T)} \\ &\lesssim \left\| \int_0^\infty \chi_{\tau \in [0,t]} \|f(\tau)\|_{N^s_{p',q}} \right\|_{L^r(-T,T)} \\ &\lesssim T^{1/r} \left(\int_{-T}^T \|f(\tau)\|^r_{L^{r'}(-T,T,N^s_{p',q})}\, dt \right)^{1/r} \\ &\lesssim T^{2/r} \|f\|_{L^{r'}(-T,T;N^s_{p',q})}.\end{aligned}$$

□

Theorem 5.3. *Assume* $r \geqslant 1$ *and* $2 \leqslant p < \infty$. *Then we have*

$$\|S(t)f\|_{L^r(-T,T,N^s_{p,2})} \lesssim T^{1/r} \|f\|_{H^s}.$$

Proof. We show that for any $T > 0$, $I = (-T,T)$, $\varphi \in \mathcal{S}$ and $\psi \in C_c([0,T), C_c^\infty(\mathbb{R}^n))$,

$$\left| \int_{-T}^T (S(t)\varphi, \psi(t))\, dt \right| \lesssim T^{1/r} \|\varphi\|_2 \|\psi\|_{L^{r'}(I,N_{p',2})}.$$

Actually, we have

$$\left| \int_{-T}^T (S(t)\varphi, \psi(t))\, dt \right| \lesssim \|\varphi\|_2 \left\| \int_{-T}^T S(-t)\psi(t)\, dt \right\|_2.$$

Thanks to Theorem 5.2, we have

$$\begin{aligned}\left\|\int_{-T}^{T} S(-t)\psi(t)\,dt\right\|_2^2 &= \left|\int_{-T}^{T}\left(\psi(t), \int_{-T}^{T} S(t-\tau)\psi(\tau)\,d\tau\right)dt\right| \\ &\lesssim \|\psi\|_{L^{r'}(I,N_{p',2})}\left\|\int_{-T}^{T} S(t-\tau)\psi(\tau)\,d\tau\right\|_{L^r(I,N_{p,2})} \\ &\lesssim T^{2/r}\|\psi\|^2_{L^{r'}(I,N_{p',2})}.\end{aligned}$$

Then by $\|(I-\Delta)^{s/2} f\|_{N_{p,2}} \sim \|f\|_{N^s_{p,2}}$ (Lemma 2.5) and density, we are done. □

Lemma 5.4. *Assume $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$. Assume also $r \geqslant k+2$, $2 \leqslant p \leqslant k+2$, $p' \leqslant q \leqslant p$ and $A(f) = \int_0^t S(t-\tau) f(u(\tau))\,d\tau$. Then for any $s > n(1-1/q)$, we obtain*

$$\|A(f)\|_{L^r(0,T;N^s_{p,q})} \lesssim T^{1-k/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,q})}.$$

Proof. By Theorems 2.6, 5.2 and 4.1, we have

$$\begin{aligned}\|A(f)\|_{L^r(0,T;N^s_{p,q})} &= \left\|\int_0^t S(t-\tau) f(u(\tau))\,d\tau\right\|_{L^r(0,T;N^s_{p,q})} \\ &\lesssim T^{2/r}\|f\|_{L^{r'}(0,T;N^s_{p',q})} \\ &= T^{2/r}\left(\int_0^T \|f(u(\tau))\|^{r'}_{N^s_{p',q}}\,d\tau\right)^{1/r'} \\ &\lesssim T^{2/r}\left(\int_0^T \|u(\tau)\|^{(k+1)r'}_{N^s_{(k+1)p',q}}\,d\tau\right)^{1/r'} \\ &\lesssim T^{2/r}\left(\int_0^T \|u(\tau)\|^{r}_{N^s_{p,q}}\,d\tau\right)^{(k+1)/r} T^{(r-k-2)/r} \\ &\lesssim T^{1-k/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,q})}.\end{aligned}$$

□

Lemma 5.5. *Assume $f(u) = u|u|^k$, $k = 2m$, $m \in \mathbb{Z}^+$. Assume also $r \geqslant k+1$, $1 \leqslant p \leqslant 2(k+1)$ and $A(f) = \int_0^t S(t-\tau) f(u(\tau))\,d\tau$. Then for any $s > \frac{1}{2}n$, we obtain*

$$\|A(f)\|_{C(0,T;H^s_2)} \lesssim T^{1-(k+1)/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,2})}.$$

Proof. By Theorems 2.6, 2.7, 4.1 and 5.1, we have

$$\begin{aligned}\|A(f)\|_{C(0,T;H^s_2)} &\leqslant \int_0^T \|f(u(\tau))\|_{H^s_2}\,d\tau \leqslant \int_0^T \|u\|^{k+1}_{N^s_{2(k+1),2}}\,d\tau \\ &\leqslant \int_0^T \|u\|^{k+1}_{N^s_{p,2}}\,d\tau \leqslant T^{1-(k+1)/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,2})}.\end{aligned}$$

□

6. Well-posedness of nonlinear Schrödinger equations

In this section we study the well-posedness of the Schrödinger equations

$$iu_t + \Delta u = f(u), \quad u(0,x) = u_0(x).$$

The solution, $u(x,t)$, of the above Cauchy problem is given by

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau) f(u(\tau))\, d\tau, \tag{14}$$

where $S(t) = F^{-1} e^{it|\xi|^2} F$.

Proof of Theorem 1.3. Fix $T > 0$, $\delta > 0$ to be chosen later. Let

$\mathcal{D} = \{u \in C(0,T;H^s) \cap L^r(0,T;N^s_{p,2}) : \|u\|_{L^r(0,T;N^s_{p,2})} < \delta,\ \|u\|_{C(0,T;H^s)} < \delta\}$

be equipped with the metric

$$d(u,\upsilon) = \|u-\upsilon\|_{L^r(0,T;N^s_{p,2}) \cap C(0,T;H^s)}.$$

It is easy to see that $(\mathcal{D}, d)$ is a complete metric space. Now we consider the map

$$J : u(t) \to S(t)u_0 - i\int_0^t S(t-\tau) f(u(\tau))\, d\tau.$$

We shall prove that there exists $T, \delta > 0$ such that $J : (\mathcal{D}, d) \to (\mathcal{D}, d)$ is a strict contraction map.

By the nonlinear term estimate (Theorems 4.1, 5.4 and 5.5) and Theorem 5.3, we have

$$\|Ju\|_{L^r(0,T;N^s_{p,2})} \lesssim T^{1/r}\|u_0\|_{H^s} + T^{1-k/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,2})}$$

and

$$\|Ju\|_{C(0,T;H^s)} \lesssim \|u_0\|_{H^s} + T^{1-(k+1)/r}\|u\|^{k+1}_{L^r(0,T;N^s_{p,2})}.$$

Then we let

$$\delta = 2C\|u_0\|_{H^s}, \quad T < 1$$

and take T such that

$$2CT^{(1-(k+1)/r)}\delta^k \leqslant \tfrac{1}{2}.$$

Then we have

$$\|Ju\|_{C(0,T;H^s)\cap L^r(0,T;N^s_{p,2})} < \delta$$

and

$$\|Ju - J\upsilon\|_{C(0,T;H^s)\cap L^r(0,T;N^s_{p,2})} \leqslant \tfrac{1}{2}\|u-\upsilon\|_{C(0,T;H^s)\cap L^r(0,T;N^s_{p,2})}.$$

In this way, we obtain that $J : (\mathcal{D}, d) \to (\mathcal{D}, d)$ is a strict contraction map. So, there exists a $u \in \mathcal{D}$ satisfying (14). Using standard argument, we can extend the solution (considering the mapping

$$J : u(t) \to S(t-T)u_T - i \int_T^t S(t-\tau) f(u(\tau))\, d\tau, \tag{15}$$

and noticing that $u(T) \in H^s$ [Li and Chen 1989], we can use the same way as in the above to solve (15)). And we can find a maximum $T^* > 0$ which satisfies the conditions in the theorem. □

Proof of Theorem 1.4. Fix $T > 0$, $\delta > 0$ to be chosen later. Let

$$\mathcal{D} = \{u \in L^r(0, T; N^s_{p,q}) : \|u\|_{L^r(0,T;N^s_{p,q})} < \delta\},$$

which is equipped with the metric

$$d(u, v) = \|u - v\|_{L^r(0,T;N^s_{p,q})}.$$

It is easy to see that $(\mathcal{D}, d)$ is a complete metric space. Now we consider the map

$$J : u(t) \to S(t)u_0 - i \int_0^t S(t-\tau) f(u(\tau))\, d\tau.$$

By the nonlinear term estimate (Theorems 4.1 and 5.4) and Theorem 5.1, we have

$$\|Ju\|_{L^r(0,T;N^s_{p,q})} \lesssim T^{1/r} \|u_0\|_{N^s_{p',q}} + T^{1-k/r} \|u\|^{k+1}_{L^r(0,T;N^s_{p,q})}.$$

Then we let

$$\delta = 2C \|u_0\|_{N^s_{p',q}}, \quad T < 1$$

and take T such that

$$2C T^{(1-k/r)} \delta^k \leqslant \tfrac{1}{2}.$$

Then we have

$$\|Ju\|_{L^r(0,T;N^s_{p,q})} < \delta$$

and

$$\|Ju - Jv\|_{L^r(0,T;N^s_{p,q})} \leqslant \tfrac{1}{2} \|u - v\|_{L^r(0,T;N^s_{p,q})}.$$

In this way, we proved that $J : (\mathcal{D}, d) \to (\mathcal{D}, d)$ is a strict contraction map. So, there exists a $u \in \mathcal{D}$ satisfying (14). □

Appendix

We list some properties of modulation spaces and some inequalities used in this paper; most of them are well-known to those familiar with PDEs. Moreover, we sketch a proof of complex interpolation on $N^s_{p,q}$.

Theorem A.1. *Assume* $0 < p, q \leqslant \infty$ *and* $s_1, s_2 \in \mathbb{R}$. *Then we have*:

(1) $B^{s_1}_{p,q} \subset M^{s_2}_{p,q}$ *if and only if* $s_1 \geqslant s_2 + \tau(p,q)$, *where*

$$\tau(p,q) = \max\Big\{0, n\Big(\frac{1}{q} - \frac{1}{p}\Big), n\Big(\frac{1}{q} + \frac{1}{p} - 1\Big)\Big\}.$$

(2) $M^{s_1}_{p,q} \subset B^{s_2}_{p,q}$ *if and only if* $s_1 \geqslant s_2 + \sigma(p,q)$, *where*

$$\sigma(p,q) = \max\Big\{0, n\Big(\frac{1}{p} - \frac{1}{q}\Big), n\Big(1 - \frac{1}{q} - \frac{1}{p}\Big)\Big\}.$$

For details of the proof, refer to [Toft 2004; Wang et al. 2006; Sugimoto and Tomita 2007].

Theorem A.2. *Assume* $0 < p \leqslant q \leqslant \infty$. *Let* $\Omega \subset \mathbb{R}^n$ *be a compact set*, $\operatorname{diam} \Omega < 2R$. *Then there exists* $C(p,q,R) > 0$ *such that*

$$\|f\|_{L^q} \leqslant C\|f\|_{L^p} \quad \text{for all } f \in L^p_\Omega,$$

where $L^p_\Omega = \{f \in L^p : \operatorname{supp} \hat{f} \subset \Omega\}$.

For details of the proof, refer to [Wang et al. 2009; 2011].

Theorem A.3. *Suppose* $0 < p, q < \infty$. *Then we have*

$$(M^s_{p,q})^* = M^{-s}_{(1\vee p)',(1\vee q)'}.$$

For details of the proof, refer to [Han and Wang 2012].

Theorem A.4. *Assume that*

$$s > \begin{cases} n\Big(1 - 1 \wedge \dfrac{1}{q}\Big), & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_1, \\ n\Big(1 \vee \dfrac{1}{p} \vee \dfrac{1}{q} - \dfrac{1}{q}\Big), & \Big(\dfrac{1}{p}, \dfrac{1}{q}\Big) \in D_2. \end{cases}$$

Then $M^s_{p,q}$ *is a multiplication algebra*, *i.e.*,

$$\|fg\|_{M^s_{p,q}} \lesssim \|f\|_{M^s_{p,q}} \|g\|_{M^s_{p,q}}$$

holds for all $f, g \in M^s_{p,q}$.

For details of the proof, refer to [Han and Wang 2012]. Note that

$$D_1 = \Big\{\Big(\frac{1}{p}, \frac{1}{q}\Big) \in \mathbb{R}^2_+ : \frac{1}{q} \geqslant \frac{2}{p}, \frac{1}{p} \leqslant \frac{1}{2}\Big\}, \quad D_2 = \mathbb{R}^2_+ \backslash D_1.$$

Theorem A.5 (embedding). *Assume $s_1, s_2 \in \mathbb{R}$ and $0 < p_1, p_2, q_1, q_2 \leqslant \infty$. We have*:

(a) *If $s_2 \leqslant s_1$, $p_1 \leqslant p_2$, $q_1 \leqslant q_2$, then $M^{s_1}_{p_1,q_1} \subset M^{s_2}_{p_2,q_2}$.*

(b) *If $q_2 < q_1$, $s_1 - s_2 > n/q_2 - n/q_1$, then $M^{s_1}_{p,q_1} \subset M^{s_2}_{p,q_2}$.*

For details of the proof, refer to [Wang et al. 2011; Han and Wang 2012].

Theorem A.6 (further multiplier assertions). *Let $0 < p < \infty$ and $0 < q \leqslant \infty$. Let $\Omega = \{\Omega_k\}_{k=0}^{\infty}$ be a sequence of compact subsets of $\mathbb{R}^n$. Let $d_k > 0$ be the diameter of Ω_k. If $x > n(n/\min(1, p, q) - \frac{1}{2})$, then there exists a constant C such that*

$$\|F^{-1} M_k F f_k\|_{L^p(\ell^q)} \leqslant C \sup_i \|M_i(d_i \cdot)\|_{H^x} \|f_k\|_{L^p(\ell^q)}$$

holds for all systems $\{f_k\}_{k=0}^{\infty} \in L^p_\Omega(\ell^q)$ and all sequences $\{M_k(x)\}_{k=0}^{\infty} \subset H^x$.

For details of the proof, refer to [Triebel 1983]. Note that $L^p_\Omega(\ell^q)$ is defined as

$\{f : f = \{f_k\}_{k=0}^{\infty} \subset S', \operatorname{supp} F f_k \subset \Omega_k$ if $k = 0, 1, 2, \dots$ and $\|f_k\|_{L^p(\ell^q)} < \infty\}$.

Theorem A.7. *Let $0 < p, q \leqslant \infty$ and (X, μ), (Y, υ) be two measure spaces. Let T be a positive linear operator mapping $L^p(X)$ into $L^q(Y)$ (resp. $L^{q,\infty}(Y)$) with norm A. Let B be a Banach space. Then T has a B-valued extension $\vec{T}$ that maps $L^p(X, B)$ into $L^q(Y, B)$ (resp. $L^{q,\infty}(Y, B)$) with the same norm.*

For details of the proof, refer to [Grafakos 2004].

Theorem A.8. *Assume $s \in \mathbb{R}$, $2 \leqslant p < \infty$, $0 < q < \infty$ and $1/p + 1/p' = 1$. Then we have*

$$\|\Box_k S(t) f\|_{L^p} \lesssim (1 + |t|)^{-n(1/2-1/p)} \|\Box_k f\|_{L^{p'}},$$
$$\|S(t) f\|_{M^s_{p,q}} \lesssim (1 + |t|)^{-n(1/2-1/p)} \|f\|_{M^s_{p',q}}.$$

For details of the proof, refer to [Wang et al. 2011].

We start with some abstract theory about complex interpolation on quasi-Banach spaces. Let $\mathcal{S} = \{z : 0 < \operatorname{Re} z < 1\}$ be a strip in the complex plane. Its closure $\{z : 0 \leqslant \operatorname{Re} z \leqslant 1\}$ is denoted by $\bar{\mathcal{S}}$. We say that $f(z)$ is an $\mathcal{S}'$-analytic function in $\mathcal{S}$ if the following properties are satisfied:

(a) For every fixed $z \in \bar{\mathcal{S}}$, $f(z) \in \mathcal{S}'(\mathbb{R}^n)$.

(b) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $F^{-1}\varphi F f(x, z)$ is a uniformly continuous and bounded function in $\mathbb{R}^n \times \bar{\mathcal{S}}$.

(c) For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $F^{-1}\varphi F f(x, z)$ is an analytic function in $\mathcal{S}$ for every fixed $x \in \mathbb{R}^n$.

We denote the set of all $\mathcal{S}'$-analytic functions in $\mathcal{S}$ by $A(\mathcal{S}'(\mathbb{R}^n))$. The idea we used here is due to [Triebel 1983; Han and Wang 2012].

Definition A.9. Let A_0 and A_1 be quasi-Banach spaces, and $0 < \theta < 1$. We define

$$F(A_0, A_1) = \Big\{\varphi(z) \in A(\mathcal{S}'(\mathbb{R}^n)) : \varphi(\ell + it) \in A_\ell,\ \ell = 0, 1,\ \text{for all } t \in \mathbb{R}, \\ \|\varphi(z)\|_{F(A_0,A_1)} = \max_{\ell=0,1} \sup_{t\in\mathbb{R}} \|\varphi(\ell + it)\|_{A_\ell}\Big\}$$

and

$$(A_0, A_1)_\theta = \Big\{f \in \mathcal{S}' : \exists\varphi(z) \in F(A_0, A_1), \\ \text{such that } f = \varphi(\theta),\ \|f\|_{(A_0,A_1)_\theta} = \inf_{\varphi} \|\varphi(z)\|_{F(A_0,A_1)}\Big\},$$

where the infimum is taken over all $\varphi(z) \in F(A_0, A_1)$ such that $\varphi(\theta) = f$.

The following three propositions are essentially known in [Triebel 1983; Han and Wang 2012].

Proposition A.10. *Adopt the notation in Definition A.9; then*

$$((A_0, A_1)_\theta, \|\cdot\|_{(A_0,A_1)_\theta})$$

is a quasi-Banach space.

Proposition A.11. *Adopt the notation in Definition A.9; then we have*

$$\|f\|_{(A_0,A_1)_\theta} = \inf_{\varphi}\Big(\sup_{t\in\mathbb{R}} \|\varphi(it)\|_{A_0}^{1-\theta} \sup_{t\in\mathbb{R}} \|\varphi(1 + it)\|_{A_1}^{\theta}\Big),$$

where the infimum is taken over all $\varphi(z) \in F(A_0, A_1)$ such that $\varphi(\theta) = f$.

Proposition A.12. *Let T be a continuous multilinear operator from*

$$A_0^{(1)} \times A_0^{(2)} \times \cdots \times A_0^{(m)}$$

to B_0 and from $A_1^{(1)} \times A_1^{(2)} \times \cdots \times A_1^{(m)}$ to B_1, satisfying

$$\|T(f^{(1)}, f^{(2)}, \dots, f^{(m)})\|_{B_0} \leqslant C_0 \prod_{j=1}^{m} \|f^{(j)}\|_{A_0^{(j)}},$$

$$\|T(f^{(1)}, f^{(2)}, \dots, f^{(m)})\|_{B_1} \leqslant C_1 \prod_{j=1}^{m} \|f^{(j)}\|_{A_1^{(j)}},$$

$$f^{(j)} \in A_0^{(j)} \cap A_1^{(j)}.$$

Then T is continuous from $(A_0^{(1)}, A_1^{(1)})_\theta \times (A_0^{(2)}, A_1^{(2)})_\theta \times \cdots \times (A_0^{(m)}, A_1^{(m)})_\theta$ to $(B_0, B_1)_\theta$ with norm at most $C_0^{1-\theta} C_0^{\theta}$, provided $0 \leqslant \theta \leqslant 1$.

Theorem A.13 (complex interpolation). *Suppose* $0<\theta<1$, $0<p_0,p_1<\infty$, $0<q_0,q_1\leqslant\infty$ *and*

$$s=(1-\theta)s_0+\theta s_1,\quad \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},$$

then we have

$$(N^{s_0}_{p_0,q_0},N^{s_1}_{p_1,q_1})_\theta=N^s_{p,q}.$$

Sketch of proof. Let $g\in N^s_{p,q}(\mathbb{R}^n)$ and $g_k(x)=F^{-1}\sigma_k Fg$. Let also $\psi_k(x)=\sum_{\ell=-1}^1\sigma_{k+\ell}(x)$ for $k=0,1,2,\dots$ (with $\sigma_{-1}=0$). In particular, $\psi_k(x)=1$ if $x\in\operatorname{supp}\sigma_k$.

Set

$$g_k^*(x)=\sup_{x\in\mathbb{R}^n}\frac{|g_k(x-y)|}{1+|\langle k\rangle y|^a},\quad x\in\mathbb{R}^n,\ a>\frac{n}{\min(p,q)}.$$

For $z\in\bar{\mathcal{S}}$, we write

$$\begin{cases} a_1(z)=sq\Big(\frac{1-z}{q_0}+\frac{z}{q_1}\Big)-(1-z)s_0-zs_1,\\ a_2(z)=p\Big(\frac{1-z}{p_0}+\frac{z}{p_1}\Big)-q\Big(\frac{1-z}{q_0}+\frac{z}{q_1}\Big),\\ a_3(z)=1-p\Big(\frac{1-z}{p_0}+\frac{z}{p_1}\Big),\\ a_4(z)=q\Big(\frac{1-z}{q_0}+\frac{z}{q_1}\Big).\end{cases}$$

We put

$$f(z)=\sum_{k=0}^\infty F^{-1}\psi_k F\left[\langle k\rangle^{a_1(z)}\Big(\sum_{\ell=0}^k|\langle\ell\rangle^s g_\ell^*|^q\Big)^{\frac{a_2(z)}{q}}\|\langle\ell\rangle^s g_\ell^*\|^{a_3^z}_{L^p(\mathbb{R}^n,\ell^q)}g_k^{a_4(z)}(x)\right].$$

Obviously, $f(z)\in A(\mathcal{S}')$ and $f(\theta)=g$. Direct calculation shows

$$\|f(\ell+it)\|_{N^{s_\ell}_{p_\ell,q_\ell}}\lesssim\|g\|_{N^s_{p,q}},\quad \ell=0,1.$$

This proves that $N^s_{p,q}\subset(N^{s_0}_{p_0,q_0},N^{s_1}_{p_1,q_1})_\theta$.

Conversely, let $f\in F(N^{s_0}_{p_0,q_0},N^{s_1}_{p_1,q_1})$. If $\varphi\in A(\mathcal{S}')$ such that $\varphi(\theta)=f$, for some $\theta\in(0,1)$, we can find two positive functions $\mu_0(\theta,t)$ and $\mu_1(\theta,t)$ in $(0,1)\times\mathbb{R}$ satisfying

$$|f(z)|^r\leqslant\left(\frac{1}{1-\theta}\int_{\mathbb{R}}|f(it)|^r\mu_0(\theta,t)\,dt\right)^{1-\theta}\left(\frac{1}{\theta}\int_{\mathbb{R}}|f(1+it)|^r\mu_1(\theta,t)\,dt\right)^{\theta},$$

with $1/(1-\theta)\int_{\mathbb{R}}\mu_0(\theta,t)\,dt = 1/\theta\int_{\mathbb{R}}\mu_1(\theta,t)\,dt = 1$. Taking the $N^s_{p,q}$ norm of both sides and then applying Minkowski's inequality imply that

$$\begin{aligned}\|f\|_{N^s_{p,q}} &\leqslant \sup_{t\in\mathbb{R}}\|\langle k\rangle^{s_0}F^{-1}\sigma_k F\varphi(it)\|^{1-\theta}_{L^{p_0}(\mathbb{R},q_0)}\\ &\qquad\qquad\qquad\times\sup_{t\in\mathbb{R}}\|\langle k\rangle^{s_1}F^{-1}\sigma_k F\varphi(1+it)\|^{1-\theta}_{L^{p_1}(\mathbb{R},q_1)}\\ &\leqslant \|f\|_{F(N^{s_0}_{p_0,q_0},N^{s_1}_{p_1,q_1})}.\end{aligned}$$

This proves that $(N^{s_0}_{p_0,q_0}, N^{s_1}_{p_1,q_1})_\theta \subset N^s_{p,q}$. □

Theorem A.14 (complex interpolation). *Let* $-\infty < s_0, s_1 < \infty$, $0 < p_0^{(j)}, p_1^{(j)} < \infty$, $0 < q_0^{(j)}, q_1^{(j)} \leqslant \infty$, $j = 1,\dots,m$. *If* T *is a continuous multilinear mapping from*

$$N^{s_0^{(1)}}_{p_0^{(1)},q_0^{(1)}}\times\cdots\times N^{s_0^{(m)}}_{p_0^{(m)},q_0^{(m)}}$$

to $N^{s_0}_{p_0,q_0}$ *with norm* M_0, *and also continuous multilinear from*

$$N^{s_1^{(1)}}_{p_1^{(1)},q_1^{(1)}}\times\cdots\times N^{s_1^{(m)}}_{p_1^{(m)},q_1^{(m)}}$$

to $N^{s_1}_{p_1,q_1}$ *with norm* M_1, *then* T *is continuous and multilinear from*

$$N^{s^{(1)}}_{p^{(1)},q^{(1)}}\times\cdots\times N^{s^{(m)}}_{p^{(m)},q^{(m)}}$$

to $N^s_{p,q}$ *with norm at most* $M_0^{1-\theta}M_1^\theta$, *provided* $0\leqslant\theta\leqslant 1$, *and*

$$s^{(j)} = (1-\theta)s_0^{(j)} + \theta s_1^j,$$
$$\frac{1}{p^{(j)}} = \frac{1-\theta}{p_0^{(j)}} + \frac{\theta}{p_1^{(j)}},\qquad \frac{1}{q^{(j)}} = \frac{1-\theta}{q_0^{(j)}} + \frac{\theta}{q_1^{(j)}},\qquad j = 1,\dots,m$$

This theorem is a natural consequence of Proposition A.12 and Theorem A.13.

Theorem A.15. *Let* $s\in R$, $0 < p, q \leqslant \infty$. *We have*:

(1) *If* $\varepsilon > 0$, *then*

$$F^{s+\varepsilon}_{p,q_1}\subset F^s_{p,q_2},\qquad B^{s+\varepsilon}_{p,q_1}\subset B^s_{p,q_2}.$$

(2) *If* $p < \infty$, *then*

$$B^s_{p,p\wedge q}\subset F^s_{p,q}\subset B^s_{p,p\vee q}.$$

For details of the proof, refer to [Triebel 1983].

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References

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10**, American Mathematical Society, Providence, RI, 2003. MR 2004j:35266 Zbl 1055.35003

[Cazenave and Weissler 1989] T. Cazenave and F. B. Weissler, "Some remarks on the nonlinear Schrödinger equation in the critical case", pp. 18–29 in *Nonlinear semigroups, partial differential equations and attractors* (Washington, DC, 1987), edited by T. L. Gill and W. W. Zachary, Lecture Notes in Math. **1394**, Springer, Berlin, 1989. MR 91a:35149 Zbl 0694.35170

[Cazenave and Weissler 1990] T. Cazenave and F. B. Weissler, "The Cauchy problem for the critical nonlinear Schrödinger equation in H^s", *Nonlinear Anal.* **14**:10 (1990), 807–836. MR 91j:35252 Zbl 0706.35127

[Chen and Fan 2011] J. Chen and D. Fan, "Multilinear estimates on modulation spaces and applications", preprint, 2011, http://wenku.baidu.com/view/57caf058ad02de80d4d8408d.html. In press.

[Feichtinger 2003] H. G. Feichtinger, "Modulation spaces on locally compact abelian groups", pp. 99–140 in *Wavelets and their application* (Chennai, 2002), edited by M. Krishna et al., Allied Publishers, New Delhi, 2003.

[Ginibre and Velo 1995] J. Ginibre and G. Velo, "Generalized Strichartz inequalities for the wave equation", *J. Funct. Anal.* **133**:1 (1995), 50–68. MR 97a:46047 Zbl 0849.35064

[Grafakos 2004] L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Upper Saddle River, NJ, 2004. MR 2449250 Zbl 1148.42001

[Han and Wang 2012] J. Han and B. Wang, "α-Modulation spaces (I)", preprint, 2012. arXiv 1108.0460v3

[Kato 1987] T. Kato, "On nonlinear Schrödinger equations", *Ann. Inst. H. Poincaré Phys. Théor.* **46**:1 (1987), 113–129. MR 88f:35133 Zbl 0632.35038

[Kato 1989] T. Kato, "Nonlinear Schrödinger equations", pp. 218–263 in *Schrödinger operators* (Sønderborg, 1988), edited by H. Holden and A. Jensen, Lecture Notes in Phys. **345**, Springer, Berlin, 1989. MR 91d:35202 Zbl 0698.35131

[Kato 1995] T. Kato, "On nonlinear Schrödinger equations, II: H^s-solutions and unconditional well-posedness", *J. Anal. Math.* **67** (1995), 281–306. MR 98a:35124a Zbl 0848.35124

[Kenig et al. 1995] C. E. Kenig, G. Ponce, and L. Vega, "On the IVP for the nonlinear Schrödinger equations", pp. 353–367 in *Harmonic analysis and operator theory* (Caracas, 1994), edited by S. A. M. Marcantognini et al., Contemp. Math. **189**, American Mathematical Society, Providence, RI, 1995. MR 96e:35071 Zbl 0846.35128

[Li and Chen 1989] D. Li and Y. Chen, *Nonlinear evolution equation*, Science Press, Beijing, 1989. In Chinese.

[Nakamura and Ozawa 1998] M. Nakamura and T. Ozawa, "Nonlinear Schrödinger equations in the Sobolev space of critical order", *J. Funct. Anal.* **155**:2 (1998), 364–380. MR 99c:35225 Zbl 0909.35126

[Pecher 1984] H. Pecher, "Nonlinear small data scattering for the wave and Klein–Gordon equation", *Math. Z.* **185**:2 (1984), 261–270. MR 85h:35165 Zbl 0538.35063

[Pecher 1997] H. Pecher, "Solutions of semilinear Schrödinger equations in H^s", *Ann. Inst. H. Poincaré Phys. Théor.* **67**:3 (1997), 259–296. MR 98k:35180 Zbl 0888.35101

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR 58 #23577 Zbl 0372.35001

[Sugimoto and Tomita 2007] M. Sugimoto and N. Tomita, "The dilation property of modulation spaces and their inclusion relation with Besov spaces", *J. Funct. Anal.* **248**:1 (2007), 79–106. MR 2009a:46064 Zbl 1124.42018

[Toft 2004] J. Toft, "Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I", *J. Funct. Anal.* **207**:2 (2004), 399–429. MR 2004j:35312 Zbl 1083.35148

[Triebel 1983] H. Triebel, *Theory of function spaces*, Monographs in Mathematics **78**, Birkhäuser, Basel, 1983. MR 86j:46026 Zbl 0546.46027

[Triebel 1992] H. Triebel, *Theory of function spaces, II*, Monographs in Mathematics **84**, Birkhäuser, Basel, 1992. MR 93f:46029 Zbl 0763.46025

[Wang 1993] B. Wang, *The Cauchy problem for the nonlinear Schrodinger equation, nonlinear Klein–Gordon equation and their coupled equations*, thesis, Institute of Applied Physics and Computational Mathematics, Beijing, 1993.

[Wang and Huang 2007] B. Wang and C. Huang, "Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations", *J. Differential Equations* **239**:1 (2007), 213–250. MR 2008e:35190 Zbl 1219.35289

[Wang and Hudzik 2007] B. Wang and H. Hudzik, "The global Cauchy problem for the NLS and NLKG with small rough data", *J. Differential Equations* **232**:1 (2007), 36–73. MR 2007j:35215 Zbl 1121.35132

[Wang et al. 2006] B. Wang, L. Zhao, and B. Guo, "Isometric decomposition operators, function spaces $E^{\lambda}_{p,q}$ and applications to nonlinear evolution equations", *J. Funct. Anal.* **233**:1 (2006), 1–39. MR 2007b:35187 Zbl 1099.46023

[Wang et al. 2009] B. Wang, L. Han, and C. Huang, "Global well-posedness and scattering for the derivative nonlinear Schrödinger equation with small rough data", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:6 (2009), 2253–2281. MR 2011a:35502 Zbl 1180.35492

[Wang et al. 2011] B. Wang, Z. Huo, C. Hao, and Z. Guo, *Harmonic analysis method for nonlinear evolution equations, I*, World Scientific, Hackensack, NJ, 2011. MR 2848761 Zbl 1254.35002

[Wiener 1932] N. Wiener, "Tauberian theorems", *Ann. of Math.* (2) **33**:1 (1932), 1–100. MR 1503035 Zbl 0004.05905

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